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The interplay between flows and C*-algebras

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Chapter 1

Introduction

In this thesis we will study the interplay between flows and C^*-algebras. We first need some preliminary definitions. Let G be a locally compact group with a unit element e, and X a compact Hausdorff space. We say that G acts on X if we have a continuous map $G \times X \to X$, such that $e \cdot x = x$ and $st \cdot x = s \cdot (t \cdot x)$ for all $x \in X$ and $s, t \in G$. The pair (G, X) is a dynamical system which we will call a *transformation group*.

Let $C(X)$ be the space of continuous complex-valued functions on X and let Aut $C(X)$ be the automorphism group of $C(X)$. Then the action of G on X gives us a homomorphism α from G to Aut $C(X)$ in the following way. Let $s \in G$, then let α_s be the map that sends a function $f \in C(X)$ to the function $x \to f(s^{-1} \cdot x)(x \in X)$. This map is an automorphism of $C(X)$. We define α to be the map that sends $s \in G$ to α_s. Then it can be shown that $s \to \alpha_s(f)$ is a continuous map from G to $C(X)$ for all $f \in C(X)$.

On the other hand, if we start with such a map $\alpha : G \to \text{Aut } C(X)$, with G a locally compact group and X a compact Hausdorff space, we can construct an action of G on X that makes (G, X) into a transformation group.

$C(X)$ is an example of a (commutative) C^*-algebra. We can now generalize the notion of dynamical systems by taking a more general C^*-algebra A instead of $C(X)$. So we let G be a locally compact group, and α a homomorphism from G to Aut A (the automorphism group of A), such that $s \to \alpha_s(a)$ is a continuous map from G to A for all $a \in A$. Then we call the triple (A, G, α) a C^*-dynamical system.

To each C^*-dynamical system, we can associate in a natural way a certain C^*-algebra, called a *crossed product C^*-algebra*. One of the reasons why this is useful is that it provides interesting examples of non-commutative C^*-algebras with certain properties. There is a well developed general theory on these crossed product C^*-algebras.

Then we can also look at specific choices of G, like $G = \mathbb{Z}$ (discrete dynamical systems) or $G = \mathbb{R}$ (flows). Of course, we can apply results coming from the general theory for crossed product C^*-algebras to these dynamical systems, however, in that context the general theory does not provide the most economic proofs, nor the strongest possible results. Also, by looking at these specific dynamical systems, certain aspects can be studied that have not been looked at in the general context, or are not meaningful there, like, for instance, recurrent points and non-wandering sets. Therefore it is useful to try to develop results on the crossed product C^*-algebras arising from these specific dynamical systems, in which the dynamics remain visible. Furthermore, in these cases it becomes interesting to try to establish *equivalences* between properties of the dynamical systems and properties of their associated crossed product C^*-algebras, instead of just using the dynamical systems to construct certain C^*-algebras. This *interplay* has been extensively studied by Tomiyama and others in the case of discrete dynamical systems. For instance, equivalence between minimality of the dynamical system and simplicity of the associated crossed product C^*-algebra has been shown. Incidentally, many results in this setting have not required X to be metrizable, which is equivalent to $C(X)$ to be separable, unlike the general theory of crossed product C^*-algebras, where separability of the C^*-algebra A is often necessary.

Much less is known on the interplay between flows and their associated crossed product C^*-algebras. In this thesis, we will associate irreducible representations of the C^*-algebra to periodic points x of the flow and irreducible representations of the isotropy subgroup at x, $\mathbb{R}_x$. This has already been studied in the case of discrete dynamical systems. We will be able to show that $x, y \in X$ are in the same orbit *and* the irreducible representations u, v of $\mathbb{R}_x$ and $\mathbb{R}_y$ are unitarily equivalent if and only if their associated irreducible representations of the crossed product C^*-algebra are unitarily equivalent.

For this we will need quite some background theory. In chapter 2 we will start with defining the C^*-dynamical systems, and try to convey the idea of the construction of the crossed product C^*-algebra associated to such a dynamical system. In chapter 3 we will give (part of) an overview on the interplay between discrete dynamical systems and C^*-algebras, and look at the construction of representations associated to periodic points in greater detail. In chapter 4 we will give an introduction to flows, and give the idea of how to construct unitary representations of groups coming from unitary representations of closed subgroups. We will use this to construct our representations of the crossed product C^*-algebras associated to periodic points of the flows. We will look at two different cases: points with period $p > 0$ and points with period $p = 0$.

In Appendix A we will give a short introduction on the theory of C^*-algebras. In Appendix B we will give the definition and properties of the Haar measure on locally compact groups. Both appendices will have no proofs. Finally, in Appendix C we will prove Schur's Lemma.

Chapter 2

Crossed Products

In this chapter we will define a C^*-dynamical system, construct the crossed product C^*-algebra, and show some relations between these two structures. We will not, however, give all the technical details of the proofs, since there are books that can be consulted on this. The goal in this chapter is to give the ideas behind the construction, and the flavour of the proofs.

2.1 C*-dynamical systems and covariant representations

Let G be a topological group with unit element e, and X a topological space. X is a *a left G-space*, if there is a continuous map

$$\phi : (s, x) \to s \cdot x$$

from $G \times X \to X$, such that for every $s, t \in G, x \in X$

$$e \cdot x = x \text{ and } s \cdot (t \cdot x) = (st) \cdot x.$$

Proposition 2.1.1. *Let X be a left G-space. Then for any $s \in G$ the map on X defined by $x \to s \cdot x$ is continuous.*

Proof. Let $s \in G$. Then the function f_s from X to $G \times X$, defined by $x \to (s, x)$ is continuous. For let U be an open subset of G, V an open subset of X, then $f_s^{-1}(U \times V) = \emptyset$ if $s \notin U$, and $f_s^{-1}(U \times V) = V$ if $s \in U$. Since the family $\{U \times V | U \subset G \text{ open}, V \subset X \text{ open}\}$ is a basis for the product topology, we can conclude that f_s is continuous.
Now, $\phi \circ f_s$ sends x to $s \cdot x$, and it is continuous, since ϕ and f_s are continuous. $\square$

Now it follows that, for every $s \in G$, the map that sends $x \in X$ to $s \cdot x$ is a homeomorphism on X, since it is continuous, and its inverse, the map that sends x to $s^{-1} \cdot x$, is also continuous.
If X is a left G-space, we call (G, X) a *transformation group*, and we say (G, X) is locally compact if both G and X are locally compact.

Let (G, X) be a locally compact transformation group. We define the *isotropy subgroup at* $x \in X$ to be $G_x := \{g \in G | g \cdot x = x\}$, which is indeed a subgroup of G. For each element $x \in X$, we can define the *orbit* of x under G as follows: $O_G(x) := \{s \cdot x | s \in G\}$.

Lemma 2.1.2. *Let (G, X) be a transformation group, then for every $x, y \in X$, either $O_G(x) = O_G(y)$ or $O_G(x) \bigcap O_G(y) = \emptyset$.*

Proof. Take $x, y \in X$ and suppose $O_G(x) \bigcap O_G(y) \neq \emptyset$. Then there are $s, t \in G$ such that $s \cdot x = t \cdot y$. Then for any $r \in G$ we have

$$r \cdot x = (rs^{-1}) \cdot (s \cdot x) = (rs^{-1}) \cdot (t \cdot y) = (rs^{-1}t) \cdot y.$$

Hence $O_G(x) \subseteq O_G(y)$. Analogously we see that $O_G(y) \subseteq O_G(x)$. $\square$

Let (G, X) be a locally compact transformation group and let $C_0(X)$ be the vector space of all continuous complex-valued functions f on X that *vanish at infinity*, i.e., with the property that for every $\epsilon > 0$ there is a compact set $K \subset X$ such that $|f(x)| < \epsilon$ whenever x is outside of K. Then $C_0(X)$ becomes a C^*-algebra with respect to the supremum norm

$$\|f\|_\infty := \sup_{x \in X} \|f(x)\| \text{ for every } f \in C_0(X),$$

pointwise multiplication and the involution

$$f^*(x) := \overline{f(x)} \text{ for all } f \in C_0(X).$$

We define the following map from G to Aut $C_0(X)$ (where Aut $C_0(X)$ is the group of *-automorphisms on $C_0(X)$).

$$\alpha_s(f)(x) := f(s^{-1} \cdot x) \text{ for every } s \in G, x \in X, f \in C_0(X).$$

Then for every $s, t \in G$, we have

$$\alpha_s(\alpha_t(f))(x) = \alpha_t(f)(s^{-1} \cdot x) = f(t^{-1}s^{-1} \cdot x) = \alpha_{st}(f)(x).$$

So $\alpha_s \circ \alpha_t = \alpha_{st}$, hence α is a homomorphism of G into Aut $C_0(X)$.
For this map the following holds.

Proposition 2.1.3. *Let (G, X) be a locally compact transformation group. Then for every $f \in C_0(X)$, $t \to \alpha_t(f)$ is a continuous map from G to $C_0(X)$.*

The proof of this proposition can be found in [21, Lemma 2.5].

This result can also be reversed. From [21, Proposition 2.7] we get that if we start with a locally compact group G, a locally compact Hausdorff space X and a homomorphism $\alpha : G \to \text{Aut } C_0(X)$, such that for every $f \in C_0(X)$, $t \to \alpha_t(f)$ is a continuous map from G to $C_0(X)$, then there is a unique transformation group (G, X), such that

$$\alpha_s(f)(x) = f(s^{-1} \cdot x) \text{ for every } s \in G, x \in X, f \in C_0(X).$$

Now, $C_0(X)$ is a commutative C^*-algebra, and in fact any commutative C^*-algebra A is isomorphic to $C_0(X)$ with X a locally compact Hausdorff space (see Appendix A). So instead of looking at a locally compact transformation group (G, X), we can also start with a homomorphism from a locally compact group G into Aut A, the group of *-automorphisms on a commutative C^*-algebra A, such that for every $f \in A$, $t \to \alpha_t(f)$ is continuous from G into A. We can now generalize this notion of transformation groups to non-commutative C^*-algebras.

Definition 2.1.4. Let A be a C^*-algebra, G a locally compact group, and α a homomorphism from G into Aut A, such that $t \to \alpha_t(a)$ is continuous from G to A for all $a \in A$. We then call the triple (A, G, α) a *C^*-dynamical system.*

Now we want to define representations of these C^*-dynamical systems. For that we first need some preliminary definitions. We denote the group of unitary operators on a Hilbert space $\mathcal{H}$ by $U(\mathcal{H})$.

Definition 2.1.5. A *unitary representation* $(u, \mathcal{H})$ of a topological group G, with $\mathcal{H}$ a Hilbert space, is a group homomorphism u from G into $U(\mathcal{H})$,

$$u : s \to u_s,$$

which is continuous in the strong topology of $B(\mathcal{H})$, i.e., for every $h \in \mathcal{H}$ the function $s \to u_s(h)$ is norm continuous.

When the Hilbert space $\mathcal{H}$ is known from the context, we will write u instead of $(u, \mathcal{H})$. We can also define representations of C^*-algebras. More on this subject can be found in Appendix A. Just as in the case of representations of C^*-algebra's, we have the following notions of equivalence and irreducibility.

Definition 2.1.6. Two unitary representations $(u, \mathcal{H})$ and $(v, \mathcal{K})$ of a topological group G are *unitarily equivalent* if there is a unitary operator $W \in B(\mathcal{H}, \mathcal{K})$ such that

$$v_s = W u_s W^* \text{ for all } s \in G.$$

$(u, \mathcal{H})$ is *irreducible* if the only closed subspaces $M \subset \mathcal{H}$, such that $u(G)M \subset M$ are the trivial ones, i.e., $M = \{0\}$ or $M = \mathcal{H}$.

The *commutant* of a set operators $D \subset B(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, is defined as

$$D' := \{T \in B(\mathcal{H}) |\ TS = ST \text{ for all } S \in D\}.$$

The following theorem will be useful when dealing with irreducible representations

Theorem 2.1.7. *(Schur's Lemma) Let $\mathcal{H}$ be a Hilbert space and $S \subset B(\mathcal{H})$, such that $S^* = S$. Then the following two statements are equivalent.*

1. *The only closed invariant linear subspaces $M \subset \mathcal{H}$ for S are the trivial ones: $\{0\}$ and $\mathcal{H}$.*

2. *$S' = \mathbb{C}I$, with I the identity operator in $B(\mathcal{H})$.*

We will give a proof of this theorem in Appendix C.

If u is a unitary representation of G, then for any $s \in G$, $u_{s^{-1}} = u_s^{-1} = u_s^*$. We shall write $u(G)$ to denote the set $\{u_s | s \in G\}$. Then $(u(G))^* = u(G)$. Hence we have the following corollary of the above theorem.

Corollary 2.1.8. *Let $(u, \mathcal{H})$ be a unitary representation of a topological group G. Then $(u, \mathcal{H})$ is irreducible if and only if the only operators in $B(\mathcal{H})$ commuting with $u(G)$ are scalar multiples of the identity.*

Definition 2.1.9. A *covariant representation* of a C^*-dynamical system (A, G, α) is a triple $(\pi, u, \mathcal{H})$ where $(\pi, \mathcal{H})$ is a representation of A, $(u, \mathcal{H})$ is a unitary representation of G, and

$$\pi(\alpha_s(a)) = u_s \pi(a) u_s^* \text{ for all } a \in A \text{ and } s \in G.$$

We say that two covariant representations $(\pi, u, \mathcal{H})$ and $(\rho, v, \mathcal{K})$ are *unitarily equivalent* if there is a unitary operator $W : \mathcal{H} \to \mathcal{K}$ such that

$$\rho(x) = W\pi(x)W^* \text{ and } v_s = W u_s W^* \text{ for all } x \in A, s \in G.$$

We call $(\pi, u, \mathcal{H})$ *irreducible*, if the only closed subspaces $M \subset \mathcal{H}$, such that $\pi(A)M \subset M$ and $u(G)M \subset M$, are the trivial ones: $\{0\}$ and $\mathcal{H}$. We say $(\pi, u, \mathcal{H})$ is *non-degenerate* if π is non-degenerate, i.e., $\pi(A)\mathcal{H}$ is a dense subset of $\mathcal{H}$.

Again, when the Hilbert space $\mathcal{H}$ is known from the context, we write (π, u) instead of $(\pi, u, \mathcal{H})$. And since $\pi(A) \cup u(G)$ is invariant under taking adjoints, we can again apply Theorem 2.1.7 to get the following corollary.

Corollary 2.1.10. *Let $(\pi, u, \mathcal{H})$ be a representation of the C^*-dynamical system (A, G, α). Then $(\pi, u, \mathcal{H})$ is irreducible if and only if the only operators in $B(\mathcal{H})$ commuting with both $\pi(A)$ and $u(G)$ are scalar multiples of the identity.*

We can easily construct trivial covariant representations of a C^*-dynamical system (A, G, α). Let $(\pi, \mathcal{H})$ be a representation of A and let u_I be the unitary representation of G that sends every element to $I_{\mathcal{H}}$. Then $(\pi, u_I, \mathcal{H})$ is a covariant representation of (A, G, α). Now let $(u, \mathcal{H})$ be a unitary representation of G and π_0 the representation of A that sends every element to the zero operator 0, then $(\pi_0, u, \mathcal{H})$ is also a covariant representation of (A, G, α).

However, these covariant representations are not very interesting. We can also construct non-trivial covariant representations of (A, G, α). Let $(\pi, \mathcal{H})$ be a representation of A. If we complete the vector space $C_c(G, \mathcal{H})$ with respect to the norm $\|.\|_2$ coming from the inner product

$$\langle h, k\rangle := \int_G \langle h(s), k(s)\rangle \, d\mu(s) \text{ for all } h, k \in C_c(G, \mathcal{H}),$$

we get a Hilbert space which we shall denote by $L^2(G, \mathcal{H})$. We can also think of $L^2(G, \mathcal{H})$ as the space of (equivalent classes of) certain $\mathcal{H}$-valued functions on G; the details on this Hilbert space can be found in [21, Appendix I]. Now we define

$$\tilde{\pi}(a)h(r) := \pi(\alpha_r^{-1}(a))(h(r)) \text{ for all } a \in A, h \in L^2(G, \mathcal{H}), r \in G,$$

and

$$u_s h(r) = h(s^{-1}r) \text{ for all } s, r \in G, h \in L^2(G, \mathcal{H}).$$

Then it can be shown that $(\tilde{\pi}, u, L^2(G, \mathcal{H}))$ is a covariant representation of (A, G, α) called the *regular representation* of (A, G, α) induced by $(\pi, \mathcal{H})$. Also, $\tilde{\pi}$ is faithful if π is faithful. And from [21, Lemma 2.17] we get that $(\tilde{\pi}, u, L^2(G, \mathcal{H}))$ is non-degenerate if and only if $(\pi, \mathcal{H})$ is non-degenerate.

2.2 Crossed products

We can now use a C^*-dynamical system to build a C^*-algebra, the so-called crossed product, whose properties are connected to that of the C^*-dynamical system.

Let (A, G, α) be a C^*-dynamical system. To make our life a little easier, we will assume G to be unimodular, i.e., its left Haar measure μ is also a right Haar measure (see Appendix B). The whole theory works fine without this assumption, but it makes some calculations a bit less technical, and in the subsequent chapters we will only be looking at abelian (and hence unimodular) groups anyway.

We will have to look at the theory of integration for functions with values in C^*-algebras, which can be complicated, but we will simplify things by only integrating continuous functions with compact support with respect to the Haar measure μ on the locally compact group G.

So, now we look at the vector space $C_c(G, A)$, consisting of all continuous functions from G into A which have compact support. It is actually a normed vector space:
For any $f \in C_c(G, A)$, the function $s \to \|f(s)\|$ belongs to $C_c(G)$, and

$$\|f\|_1 := \int_G \|f(s)\| \, d\mu(s)$$

satisfies $\|f\|_1 \leq \|f\|_\infty \mu(\text{supp } f) < \infty$. Then $\|.\|_1$ is a norm on $C_c(G, A)$, which we shall call the L^1-*norm*.

We want to turn $C_c(G, A)$ into a *-algebra, i.e., define a multiplication and involution that satisfies the necessary conditions. For this multiplication we will need to be able to integrate. After we have shown how to do this, we will see that we can find a certain C^*-algebra in which $C_c(G, A)$ lies dense; this will be the crossed product C^*-algebra, associated to our C^*-dynamical system.

From [21, Lemma 1.91] we get that there exists a unique linear map $I : C_c(G, A) \to A$, such that for any continuous linear functional φ on A, we have

$$\varphi(I(f)) = \int_G \varphi(f(s)) \, d\mu(s) \text{ for all } f \in C_c(G, A).$$

Since φ is continuous and $f \in C_c(G, A)$, the function $s \to \varphi(f(s))$ is in $C_c(G)$, hence the above integral makes sense. We now define the integral of a function $f \in C_c(G, A)$ over G as this linear map I.

$$\int_G f(s) \, d\mu(s) := I(f).$$

Proposition 2.2.1. *The map $f \to \int_G f(s)\,d\mu(s)$ defined above has the following properties.*

1. *Let $(\pi, \mathcal{H})$ be a representation of A, then*
$$\langle \pi(\int_G f(s)\,d\mu(s))h, k\rangle = \int_G \langle \pi(f(s))h, k\rangle\,d\mu(s) \text{ for all } h, k \in \mathcal{H}.$$
2. *Let $L : A \to B$ be a bounded linear map into a C^*-algebra B, then $L(\int_G f(s)\,d\mu(s)) = \int_G L(f(s))\,d\mu(s)$.*
3. *$(\int_G f(s)\,d\mu(s))^* = \int_G f(s)^*\,d\mu(s)$.*

The proof of these results can be found in [21, Section 1.5].

Let $a \in A$ and $f \in C_c(G, A)$. Then clearly the map on A sending b to ba is linear and bounded, hence Proposition 2.2.1 2. implies that

$$\int_G f(s)a\,d\mu(s) = \int_G f(s)\,d\mu(s)a. \tag{2.1}$$

Likewise we get

$$\int_G af(s)\,d\mu(s) = a\int_G f(s)\,d\mu(s). \tag{2.2}$$

We also have the following proposition.

Proposition 2.2.2. *Let $f \in C_c(G, A)$, then for each $r \in G$ we have*

$$\int_G f(sr)\,d\mu(s) = \int_G f(s)\,d\mu(s), \tag{2.3}$$

$$\int_G f(rs)\,d\mu(s) = \int_G f(s)\,d\mu(s), \tag{2.4}$$

and

$$\int_G f(s^{-1})\,d\mu(s) = \int_G f(s)\,d\mu(s). \tag{2.5}$$

Proof. Since μ is a unimodular Haar measure of G, we see that for any linear functional φ on A the following holds.

$$\int_G \varphi(f(s))\,d\mu(s) = \int_G \varphi(f(sr))\,d\mu(s) = \int_G \varphi(f(rs))\,d\mu(s).$$

Hence by definition we get that

$$\varphi(\int_G f(s)\,d\mu(s)) = \varphi(\int_G f(sr)\,d\mu(s)) = \varphi(\int_G f(rs))\,d\mu(s).$$

Hence $\int_G f(sr)\,d\mu(s) = \int_G f(rs)\,d\mu(s) = \int_G f(s)\,d\mu(s)$.
Since we also have

$$\int_G \varphi(f(s^{-1}))\,d\mu(s) = \int_G \varphi(f(s))\,d\mu(s),$$

we get the second statement analogously.

□

We also want to be able to interchange the order of integration in some way. With ordinary scalar-valued integrals this can be done using Fubini's Theorem. Using this theorem, an analogous result for the vector-valued integration is proven in [21, Proposition 1.105].

Proposition 2.2.3. *Suppose that $F \in C_c(G \times G, A)$. Then*

$$s \to \int_G F(s,r)\,d\mu(r) \text{ and } r \to \int_G F(s,r)\,d\mu(s),$$

are in $C_c(G, A)$ and the iterated integrals

$$\int_G \int_G F(s,r)\,d\mu(s)\,d\mu(r) \text{ and } \int_G \int_G F(s,r)\,d\mu(r)\,d\mu(s)$$

have a common value.

Now we want to construct a multiplication and involution on the normed vector space $C_c(G, A)$. For this we finally need the map α, coming from the C^*-dynamical system (A, G, α). Let $f, g \in C_c(G, A)$, then $(s, r) \to f(r)\alpha_r(g(r^{-1}s))$ is in $C_c(G \times G, A)$, which can be easily shown. Then the first statement in Proposition 2.2.3 guarantees that

$$f * g(s) := \int_G f(r)\alpha_r(g(r^{-1}s))\,d\mu(r)$$

actually defines an element of $C_c(G, A)$, which is the *convolution* of f and g. For every $f \in C_c(G, A)$, we define

$$f^*(s) := \alpha_s(f(s^{-1})^*) \text{ for every } s \in G.$$

Then f^* is also in $C_c(G, A)$. So we have a map * from $C_c(G, A)$ to $C_c(G, A)$, sending f to f^*.

It is easy to show that the convolution is a bilinear map and that * is a conjugate linear map. Furthermore, the following properties are satisfied. Let $f, g, h \in C_c(G, A)$, then

$$f * (g * h) = (f * g) * h,$$

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1,$$
$$(f * g)^* = g^* * f^*,$$
$$(f^*)^* = f.$$

This means that $C_c(G, A)$ becomes a normed *-algebra, by taking the convolution as multiplication, and the map * as involution.

It is not hard to prove the properties above by using the earlier mentioned properties of vector-valued integration. To give an idea of how to do this, we shall prove the associativity of the multiplication.

Proposition 2.2.4. *Let $f, g, h \in C_c(G, A)$, then $(f * g) * h = f * (g * h)$.*

Proof.

$$\begin{aligned}(f * g) * h(s) &= \int_G (f * g)(r) \alpha_r(h(r^{-1}s))\, d\mu(r) \\ &= \int_G \int_G f(t) \alpha_t(g(t^{-1}r))\, d\mu(t) \alpha_r(h(r^{-1}s))\, d\mu(r).\end{aligned}$$

Since $\alpha_r(h(r^{-1}s))$ does not depend on t, we can bring it inside the inner integral, by (2.1), and then we can interchange the integrals by Proposition 2.2.3. So we get

$$\int_G \int_G f(t) \alpha_t(g(t^{-1}r)) \alpha_r(h(r^{-1}s))\, d\mu(r)\, d\mu(t).$$

Then, by using (2.3), we can replace r by tr inside the inner integral, without changing its value. This gives us

$$\int_G \int_G f(t) \alpha_t \left(g(r) \alpha_r(h(r^{-1}t^{-1}s)) \right)\, d\mu(r)\, d\mu(t).$$

Since $f(t)$ does not depend on r, we can bring it out of the inner integral, by (2.2). Also, for every $t \in G$, $\alpha_t \in \mathrm{Aut}\, A$, hence α_t is a bounded linear map on A, hence by Proposition 2.2.1 2. we can also bring α_t outside the inner integral. Then we get

$$\int_G f(t) \alpha_t \int_G g(r) \alpha_r(h(r^{-1}t^{-1}s))\, d\mu(r)\, d\mu(t).$$

Now, since $(g * h)(t^{-1}s) = \int_G g(r)\alpha_r(h(r^{-1}t^{-1}s))\, d\mu(r)$, we finally arrive at

$$\int_G f(t)\alpha_t(g * h)(t^{-1}s)\, d\mu(t) = f * (g * h)(s).$$

□

$C_c(G, A)$ is not yet a C^*-algebra. In fact, it need not even be complete with respect to its norm. In order to be able to define an 'enveloping' C^*-algebra of $C_c(G, A)$, we will need the notion of representations on $C_c(G, A)$. These are defined just like representations of C^*-algebras.

Definition 2.2.5. A **-representation* of $C_c(G, A)$ is a pair $(\pi, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and $\pi : C_c(G, A) \to B(\mathcal{H})$ a *-homomorphism. We say $(\pi, \mathcal{H})$ is non-degenerate if

$$\{\pi(f)h | f \in C_c(G, A), h \in \mathcal{H}\}$$

spans a dense subset of $\mathcal{H}$. If $\|\pi(f)\| \leq \|f\|_1$ for all $f \in C_c(G, A)$, we call $(\pi, \mathcal{H})$ *L^1-norm decreasing.*

When $\mathcal{H}$ is known from the context, we will write π instead of $(\pi, \mathcal{H})$. Now we want to construct *-representations of $C_c(G, A)$, coming from covariant representations of (A, G, α). Let $(\pi, u, \mathcal{H})$ be a covariant representation, and $f \in C_c(G, A)$. Then for each $s \in G$ we get $\pi(f(s))u_s \in B(\mathcal{H})$, so we get a function from G to $B(\mathcal{H})$. Now we would want to integrate this function over G, just like we did before, which should give us an element of $B(\mathcal{H})$.

Then we could define a map from $C_c(G, A)$ to $B(\mathcal{H})$ by sending f to $\int_G \pi(f(s))u_s\, d\mu(s)$. The problem is that the theory we introduced above won't quite work here, for we would need that the function $s \to \pi(f(s))u_s$ is in $C_c(G, B(\mathcal{H})$. Since $f \in C_c(G, A)$, and π is a continuous map from A to $B(\mathcal{H})$, the function $s \to \pi(f(s))$ *is* in $C_c(G, B(\mathcal{H}))$, but we can claim no such thing for the function $s \to u_s$. This function is continuous in the strong operator topology and it maps into $U(\mathcal{H})$, hence into $B(\mathcal{H})$, but it is not norm continuous, which is what we want.

We can still make sense of this integral by looking at sesquilinear forms.

Definition 2.2.6. A *sesquilinear form* $[\,.\,,\,.\,]$ on a Hilbert space $\mathcal{H}$ is a function from $\mathcal{H} \times \mathcal{H}$ to $\mathbb{C}$, sending $(h, k) \in \mathcal{H} \times \mathcal{H}$ to $[h, k]$, such that for all $h, k, m \in \mathcal{H}$ and $\lambda, \mu \in \mathbb{C}$

1. $[\lambda h + \mu k, m] = \lambda[h, m] + \mu[k, m]$

2. $[h, \lambda k + \mu m] = \bar{\lambda}[h, k] + \bar{\mu}[h, m]$

It is *bounded* if there exists a constant M such that $|[h,k]| \le M\|h\|\|k\|$ for all $h,k \in \mathcal{H}$.

If $A \in B(\mathcal{H})$ then it is easy to see that the function defined by $[h,k] := \langle Ah, k\rangle$ is a bounded sesquilinear form on $\mathcal{H}$. The converse is also true.

Theorem 2.2.7. *For every bounded sesquilinear form* $[\,.\,,\,.\,]$ *on* $\mathcal{H}$ *with bound* M*, there is a unique operator* $A \in B(\mathcal{H})$ *such that*

$$[h,k] = \langle Ah, k\rangle \textit{ for all } h,k \in \mathcal{H}$$

and $\|A\| \le M$*.*

The proof is not difficult and can be found in [2, Proposition 2.1.1].

Now let $(\pi, u, \mathcal{H})$ be a covariant representation of (A, G, α) and $f \in C_c(G, A)$. We define

$$[h,k] = \int_G \langle \pi(f(s))u_s h, k\rangle \, d\mu(s) \text{ for all } h,k \in \mathcal{H}. \tag{2.6}$$

Since $s \to \pi(f(s))u_s$ is continuous in the strong operator topology, it is also continuous in the weak operator topology, i.e., the function $s \to \langle \pi(f(s))u_s h, k\rangle$ is continuous for all $h,k \in \mathcal{H}$. It has compact support, because f has compact support, hence it is in $C_c(G)$. Therefore the integral in (2.6) makes sense.

Lemma 2.2.8. *The function* $[\,.\,,\,.\,] : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ *defined above is a bounded sesquilinear form on* $\mathcal{H}$*.*

Proof. It is easy to verify that $[\,.\,,\,.\,]$ is a sesquilinear form, since both u_s and $\pi(f(s))$ are linear, and the inner product is linear in the first variable and conjugate linear in the second variable. We will now show that this sesquilinear form is bounded. Let $h,k \in \mathcal{H}$, then

$$\begin{aligned} \left|\int_G \langle \pi(f(s))u_s h, k\rangle \, d\mu(s)\right| &\le \int_G |\langle \pi(f(s))u_s h, k\rangle| \, d\mu(s) \\ &\le \int_G \|\pi(f(s))\|\|u_s\|\|h\|\|k\| \, d\mu(s), \end{aligned}$$

by the Cauchy-Schwarz inequality. Since u_s is a unitary operator, $\|u_s\| = 1$, and $\|\pi(f(s))\| \le \|f(s)\|$, hence we have

$$\int_G \|\pi(f(s))\|\|u_s\|\|h\|\|k\| \, d\mu(s) \le \|h\|\|k\| \int_G \|f(s)\| \, d\mu(s) = \|h\|\|k\|\|f\|_1.$$

□

If we then apply Theorem 2.2.7 to the bounded sesquilinear form defined in the above lemma, we get that there is a unique operator in $B(\mathcal{H})$, which we shall denote by $\pi \rtimes u(f)$, such that

$$[h,k] = \langle \pi \rtimes u(f)h, k\rangle = \int_G \langle \pi(f(s))u_s h, k\rangle \, d\mu(s).$$

Now we can make sense of the integral we mentioned before and define

$$\int_G \pi(f(s))u_s \, d\mu(s) := \pi \rtimes u(f).$$

Then we can define a map $\pi \rtimes u$ from $C_c(G,A)$ to $B(\mathcal{H})$ by sending f to $\pi \rtimes u(f)$. Now we wish to show that $\pi \rtimes u$ actually is a *-representation of $C_c(G,A)$. First of all it is linear.

$$\int_G \langle \pi(\lambda f(s)+\mu g(s)h, k\rangle \, d\mu(s) = \lambda \int_G \langle \pi(f(s))u_s h, k\rangle \, d\mu(s) + \mu \int_G \langle \pi(g(s))u_s h, k\rangle \, d\mu(s),$$

hence

$$\pi \rtimes u(\lambda f + \mu g) = \lambda \pi \rtimes u(f) + \mu \pi \rtimes u(g).$$

Furthermore we need to show that for every $f \in C_c(G,A)$, $\pi \rtimes u(f^*) = (\pi \rtimes u(f))^*$. This is true if and only if $\langle \pi \rtimes u(f^*)h, k\rangle = \langle h, \pi \rtimes u(f)k\rangle$ for all $h,k \in \mathcal{H}$.

$$\langle \pi \rtimes u(f^*)h, k\rangle = \int_G \langle \pi(f^*(s))u_s h, k\rangle \, d\mu(s) = \int_G \langle \pi(\alpha_s(f(s^{-1})^*))u_s h, k\rangle \, d\mu(s).$$

The covariance of (π, u) gives us that $\pi(\alpha_s(a))u_s = u_s \pi(a)$ for all $a \in A$. Hence we have

$$\begin{aligned}
\int_G \langle \pi(\alpha_s(f(s^{-1})^*))u_s h, k\rangle \, d\mu(s) &= \int_G \langle u_s \pi(f(s^{-1})^*)h, k\rangle \, d\mu(s) \\
&= \int_G \langle \pi(f(s^{-1}))^* h, u_s^* k\rangle \, d\mu(s) \\
&= \int_G \langle h, \pi(f(s^{-1}))u_s^* k\rangle d\mu(s) \\
&= \int_G \langle h, \pi(f(s^{-1}))u_{s^{-1}} k\rangle \, d\mu(s)
\end{aligned}$$

Since G is unimodular, we can replace s with s^{-1} without affecting the outcome. Then we get

$$\int_G \langle h, \pi(f(s^{-1}))u_{s^{-1}} k\rangle \, d\mu(s) = \int_G \langle h, \pi(f(s))u_s k\rangle d\mu(s) = \langle h, \pi \rtimes u(f)k\rangle,$$

as we wanted.

Lemma 2.2.9. *Let* $(\pi, u, \mathcal{H})$ *be a covariant representation of* (A, G, α). *Then* $\pi \rtimes u(f * g) = \pi \rtimes u(f) \circ \pi \rtimes u(g)$ *for all* $f, g \in C_c(G, A)$.

Proof. Let f, g be in $C_c(G, A)$. If we can show that

$$\langle \pi \rtimes u(f * g)h, k\rangle = \langle(\pi \rtimes u(f) \circ \pi \rtimes u(g)h, k\rangle$$

for all $h, k \in \mathcal{H}$, then we are done (by Theorem 2.2.7). So let h, k be in $\mathcal{H}$. By definition we get

$$\begin{aligned} \langle \pi \rtimes u(f * g)h, k\rangle &= \int_G \langle \pi((f * g)(s))u_s h, k\rangle \, d\mu(s) \\ &= \int_G \langle \pi(\int_G f(t)\alpha_t(g(t^{-1}s)) \, d\mu(t))u_s h, k\rangle \, d\mu(s). \end{aligned}$$

We can bring π inside the inner integral by Proposition 2.2.1 2. Since u_s is a constant with respect to t, we can bring it inside as well by using (2.1). So we get

$$\int_G \langle \int_G \pi(f(t))\pi(\alpha_t(g(t^{-1}s)))u_s \, d\mu(t)h, k\rangle \, d\mu(s).$$

The map $t \to \pi(f(t))\pi(\alpha_t(g(t^{-1}s)))u_s$ is in $C_c(G, B(\mathcal{H})$, so we can apply Proposition 2.2.1 1. to get

$$\int_G \int_G \langle \pi(f(t))\pi(\alpha_t(g(t^{-1}s)))u_s h, k\rangle \, d\mu(t) \, d\mu(s).$$

The map from $G \times G$ to $B(\mathcal{H})$ sending (s, t) to $\pi(f(t))\pi(\alpha_t(g(t^{-1}s)))u_s$ is continuous in the strong operator topology. Therefore the map from $G \times G$ to $\mathbb{C}$ sending (s, t) to $\langle \pi(f(t))\pi(\alpha_t(g(t^{-1}s)))u_s h, k\rangle$ is continuous (with compact support). So we can apply Fubini's Theorem for scalar valued functions to interchange the two integrals. Now, in the new inner integral, we can replace s by ts. This gives us

$$\int_G \int_G \langle \pi(f(t))\pi(\alpha_t(g(s)))u_t u_s h, k\rangle \, d\mu(s) \, d\mu(t).$$

By covariance of (π, u) and applying Fubini's Theorem a second time, we arrive at

$$\begin{aligned} \int_G \int_G \langle \pi(f(t))u_t \pi(g(s))u_s h, k\rangle \, d\mu(t) \, d\mu(s) &= \int_G \langle \pi \rtimes u(f)\pi(g(s))u_s h, k\rangle \, d\mu(s) \\ &= \int_G \langle \pi(g(s))u_s h, (\pi \rtimes u(f))^* k\rangle \, d\mu(s) \end{aligned}$$

We can then bring the integral inside the inner product, by using the definition. This gives us

$$\begin{aligned} \langle \int_G \pi(g(s))u_s \, d\mu(s)h, (\pi \rtimes u(f))^* k\rangle &= \langle \pi \rtimes u(g)h, (\pi \rtimes u(f))^* k\rangle \\ &= \langle \pi \rtimes u(f) \circ \pi \rtimes u(g)h, k\rangle, \end{aligned}$$

which is what we wanted to show. $\square$

$\pi \rtimes u$ is also L^1-norm decreasing, since, for every $f \in C_c(G, A)$, $\|f\|_1$ is a bound for the sesquilinear form $[h, k] = \int_G \langle \pi(f(s))u_s h, k\rangle d\mu(s)$, and then we get from Theorem 2.2.7 that $\|\pi \rtimes u(f)\| \leq \|f\|_1$. Furthermore, it is proven in [21, page 52] that $\pi \rtimes u$ is non-degenerate if π is non-degenerate.

So we get the following proposition.

Proposition 2.2.10. *Suppose that $(\pi, u, \mathcal{H})$ is a (possible degenerate) covariant representation of (A, G, α). Then*

$$\pi \rtimes u(f) := \int_G \pi(f(s))u_s \, d\mu(s)$$

defines a L^1-norm decreasing *-representation of $C_c(G, A)$ on $\mathcal{H}$ called the integrated form of $(\pi, u, \mathcal{H})$. Furthermore, $\pi \rtimes u$ is non-degenerate if π is non-degenerate.

We will use these *-representations of $C_c(G, A)$ to define another norm, under which the completion of $C_c(G, A)$ becomes a C^*-algebra. This will be the crossed product that we are after. Let $f \in C_c(G, A)$, then we define

$$\|f\| := \sup\{\|\pi \rtimes u(f)\| | (\pi, u) \text{ is a covariant representation of } (A, G, \alpha)\}.$$

Since by Proposition 2.2.10 every *-representation of the form $\pi \rtimes u$ satisfies $\|\pi \rtimes u(f)\| \leq \|f\|_1$ for all $f \in C_c(G, A)$, it can be concluded that $\|f\| \leq \|f\|_1 < \infty$ for all $f \in C_c(G, A)$.

It is proven in [21, Lemma 2.26] that there exists a covariant representation (ρ, u) of (A, G, α), such that $\rho \rtimes u$ is a faithful *-representation of $C_c(G, A)$. Then for every non-zero $f \in C_c(G, A)$ we have $\|\rho \rtimes u(f)\| > 0$, hence $\|f\| > 0$.

It is now easy to see that $\|.\|$ is actually a norm of $C_c(G, A)$, called the *universal norm.* Furthermore, since for every covariant representation (π, u) of (A, G, α) and every $f \in C_c(G, A)$ the following holds.

$$\|\pi \rtimes u(f^* * f)\| = \|(\pi \rtimes u(f))^*(\pi \rtimes u(f))\| = \|\pi \rtimes u(f)\|^2,$$

we have that $\|f^* * f\| = \|f\|^2$. Hence the completion of $C_c(G, A)$ with respect to $\|.\|$ is a C^*-algebra, which we will call the *crossed product* of A by G and denote by $A \rtimes_\alpha G$.

By definition $C_c(G, A)$ is a dense sub-*-algebra of $A \rtimes_\alpha G$. Let ρ be a *-representation of $C_c(G, A)$, such that $\|\rho(f)\| \leq \|f\|$ for all $f \in C_c(G, A)$, then ρ extends to a representation of $A \rtimes_\alpha G$, which we shall also denote by ρ. On the other hand, let ρ be a representation of $A \rtimes_\alpha G$, then its restriction to $C_c(G, A)$ must be a *-representation of $C_c(G, A)$, and $\|\rho(f)\| \leq \|f\|$ for all $f \in C_c(G, A)$. So by taking a covariant representation (π, u) of (A, G, α), we get a *-representation $\pi \rtimes u$ of $C_c(G, A)$ and hence a representation of $A \rtimes_\alpha G$, which we shall also denote by $\pi \rtimes u$.

Summarizing the theorems above, we get the following theorem.

Theorem 2.2.11. *Let (A, G, α) be a C^*-dynamical system. The vector space $C_c(G, A)$ is a normed *-algebra with norm (called L^1-norm)*

$$\|f\|_1 = \int_G \|f(s)\| \, d\mu(s),$$

multiplication

$$f * g(s) = \int_G f(r)\alpha_r(g(r^{-1}s)) \, d\mu(r),$$

and involution

$$f^*(s) = \alpha_s(f(s^{-1})^*).$$

Let $(\pi, u, \mathcal{H})$ be a covariant representation of (A, G, α). Then

$$\pi \rtimes u(f) = \int_G \pi(f(s))u_s \, d\mu(s)$$

defines a L^1-norm decreasing *-representation of $C_c(G, A)$ on $\mathcal{H}$ called the integrated form of $(\pi, u, \mathcal{H})$. On $C_c(G, A)$ we can define the universal norm.

$$\|f\| := \sup\{\|\pi \rtimes u(f)\| | (\pi, u) \text{ is a covariant representation of } (A, G, \alpha)\}.$$

The completion of $C_c(G, A)$ with respect to this norm is a C^*-algebra, the crossed product of A by G, which we will denote by $A \rtimes_\alpha G$.

2.3 Bijective correspondence between representations

In the previous section we have seen that we can construct a representation of the crossed product C^*-algebra from each covariant representation of the corresponding C^*-dynamical system. The following theorem tells us that we actually get *every* representation of the crossed product this way.

Theorem 2.3.1. *Let (A, G, α) be a C^*-dynamical system. Then the map sending a covariant pair (π, u) to its integrated form $\pi \rtimes u$ is a one-to-one correspondence between non-degenerate covariant representations of (A, G, α) and non-degenerate representations of $A \rtimes_\alpha G$. This correspondence preserves irreducibility and equivalence.*

The complete proof can be found in [21, Proposition 2.40]. In this section we will try to convey the ideas of the proof.
Except in special cases, the crossed product $A \rtimes_\alpha G$ does not contain a copy of either A or G. However, we can find a larger C^*-algebra which contains $A \rtimes_\alpha G$ as an ideal *and* both A and G. This is the so-called *multiplier algebra* of $A \rtimes_\alpha G$, denoted by $M(A \rtimes_\alpha G)$. Now, if we have a non-degenerate representation L of $A \rtimes_\alpha G$, we can uniquely extend it to a non-degenerate representation $\bar{L}$ of $M(A \rtimes_\alpha G)$. Then, by restricting to A and G we get a representation π of A, and a unitary representation u of G, which together turn out to be a covariant representation of (A, G, α), and such that $L = \pi \rtimes u$.

Remark 2.3.2. Above we used the result that any non-degenerate representation of $A \rtimes_\alpha G$ has a unique extension to $M(A \rtimes_\alpha G)$. This is more general true for any non-degenerate representation $(\pi, \mathcal{H})$ of an ideal I of a C^*-algebra A. This can be uniquely extended to a representation of A (see [7, Proposition 5.8.1]).

We will start with a slight detour over the theory of multiplier algebras, in order to convey the idea of how we can actually embed A and G into the crossed product. For that we need some preliminary definitions.

Definition 2.3.3. An ideal I of a C^*-algebra A is called *essential* if I has non-zero intersection with every other non-zero ideal A.

Definition 2.3.4. A *unitization* of a C^*-algebra A is a unital C^*-algebra B and an injective homomorphism $i : A \to B$ such that $i(A)$ is an essential ideal of B.

It is easy to see that any unitization of a unital C^*-algebra A is A itself. The multiplier algebra of a C^*-algebra A, called $M(A)$, is the *maximal* unitization of A, in the sense that every other unitization of A can be embedded into $M(A)$. We can actually construct a multiplier algebra for each C^*-algebra, and show that it is unique up to isomorphism. There are several approaches to the construction; we will follow the one given in [16, Section 2.3]. This construction makes use of the theory of Hilbert modules, of which details can also be found in [16, Section 2.1], but since most of this theory is not needed to explain the ideas of the construction of the multiplier algebra, we will not discuss it here.

Definition 2.3.5. Let A be a C^*-algebra. A function $T : A \to A$ is called *adjointable* if there is a function $T^* : A \to A$ such that $T(a)^*b = a^*T^*(b)$. We shall write $\mathcal{L}(A)$ to denote the set of all adjointable functions on A.

It is then shown in [16] that every adjointable function on A is bounded and linear. Then it is straightforward to check that if $T \in \mathcal{L}(A)$, then T^* is unique, $T^* \in \mathcal{L}(A)$ and $(T^*)^* = T$. Furthermore, $\mathcal{L}(A)$ is a subalgebra of the Banach algebra $B(A)$ of bounded linear operators on A, and $T \to T^*$ is an involution on $\mathcal{L}(A)$. Even better, $\mathcal{L}(A)$ is a C^*-algebra with respect to the operator norm, and it turns out to be a maximal unitization of A.

Let $a \in A$, then we define the following map $L_a(b) := ab$ for every $b \in B$. This map is adjointable, with adjoint L_{a^*}, since

$$L_a(b)^*c = b^*a^*c = b^*L_{a^*}(c).$$

Hence $L_a \in \mathcal{L}(A)$, and the map $L : a \to L_a$ is an injective homomorphism from A into $\mathcal{L}(A)$. To see why $\mathcal{L}(A)$ is a maximal unitization of A requires some work, which we shall not carry out here. Again, see [16] for further details. So we can then define the multiplier algebra $M(A)$ of a C^*-algebra A to be $\mathcal{L}(A)$. We will call elements of $M(A)$ *multipliers*. We will call a multiplier T *unitary* if it has an inverse and $T^* = T^{-1}$. We denote the unitary group of $M(A)$ consisting of unitary multipliers by $UM(A)$.

This ends our detour on multiplier algebras, and we return to the crossed products. Now, if we view $C_c(G, A)$ as a sub-*-algebra of $M(A \rtimes_\alpha G)$, a multiplier $T \in M(A \rtimes_\alpha G)$ need not necessarily map $C_c(G, A)$ into itself.

However, if we actually want to find a multiplier of $A \rtimes_\alpha G$, we could start with defining a map T from $C_c(G, A)$ to itself, and show that it is bounded with respect to the universal norm. Then it extends to a map from $A \rtimes_\alpha G$ to itself, which we will also call T. Then T defines a multiplier if we can find an adjoint T^*. This is the way we will define our embeddings from A and G into $M(A \rtimes_\alpha G)$ in what follows.

For every $a \in A, f \in C_c(G, A)$ and $s \in G$ we define

$$i_A(a)f(s) = af(s).$$

Then i_A is a map from $C_c(G, A)$ to itself. Let (π, u) be a covariant representation of (A, G, α), then

$$\pi \rtimes u(i_A(a)f) = \int_G \pi(af(s))u_s \, d\mu(s) = \pi(a) \circ \int_G \pi(f(s))u_s \, d\mu(s) = \pi(a) \circ \pi \rtimes u(f).$$

Since $\pi(a) \leq \|a\|$ for every $a \in A$, we get

$$\|i_A(a)f\| \leq \|a\|\|f\|,$$

hence $i_A(a)$ is bounded with respect to the universal norm, so we can extend it to a map from $A \rtimes_\alpha G$ to itself, which we shall also denote by $i_A(a)$. Now, to find the adjoint we will compute $(i_A(a)f)^* * g$. First we have

$$(i_A(a)f)^*(r) = (af)^*(r) = \alpha_r((af(r^{-1}))^*) = \alpha_r(f(r^{-1})^* a^*) = f^*(r)\alpha_r(a^*).$$

Hence

$$\begin{aligned}(i_A(a)f)^* * g(s) &= \int_G f^*(r)\alpha_r(a^*)\alpha_r(g(r^{-1}s))\, d\mu(r) \\ &= \int_G f^*(r)\alpha_r(a^* g(r^{-1}s))\, d\mu(r) \\ &= f^* * i_A(a^*)g(s),\end{aligned}$$

for every $f, g \in C_c(G, A)$ and $r, s \in G$. So $i_A(a)$ has an adjoint $i_A(a^*)$. Hence $i_A(a) \in M(A \rtimes_\alpha G)$.

It is easy to check that i_A is a homomorphism from A into the C^*-algebra $M(A \rtimes_\alpha G)$.

Now, for every $r, s \in G$ and $f \in C_c(G, A)$ we define

$$i_G(r)f(s) = \alpha_r(f(r^{-1}s)).$$

In a similar manner we can prove that $i_G(r) \in UM(A \rtimes_\alpha G)$. In fact the following results are proven in [21, Proposition 2.34].

Theorem 2.3.6. *Let (A, G, α) be a C^*-dynamical system and let i_A and i_G be as defined above. Then i_A is a faithful homomorphism from A into $M(A \rtimes_\alpha G)$, and i_G is an injective unitary valued homomorphism from G into $UM(A \rtimes_\alpha G)$. Furthermore (i_A, i_G) is covariant in the sense that*

$$i_A(\alpha_r(a)) = i_G(r)i_A(a)i_G(r)^*.$$

If (π, u) is a non-degenerate covariant representation of (A, G, α), then the extension of $\pi \rtimes u$ as representation of $M(A \rtimes_\alpha G)$, denoted by $\overline{\pi \rtimes u}$, satisfies

$$(\overline{\pi \rtimes u})(i_A(a)) = \pi(a) \text{ and } (\overline{\pi \rtimes u})(i_G(s)) = u_s.$$

Now we can complete the outline of the proof of Theorem 2.3.1. Let L be a (non-degenerate) representation of $A \rtimes_\alpha G$, and $\bar{L}$ the extension of L to $M(A \rtimes_\alpha G)$, then we can define u and π by putting

$$u_s := \bar{L}(i_G(s)) \text{ and } \pi(a) := \bar{L}(i_A(a)) \text{ for all } s \in G, a \in A.$$

It is shown in [21, (Proposition 2.39] , that u is a unitary representation of G and π a non-degenerate representation of A. We also have for every $a \in A$ and $s \in G$

$$\pi(\alpha_s(a)) = \bar{L}(i_A(\alpha_s(a))) = \bar{L}(i_G(s)i_A(a)i_G(s)^*) = u_s\pi(a)u_s^*.$$

Hence (π, u) is a non-degenerate covariant representation of (A, G, α). Then it is proven in [21] that $L = \pi \rtimes u$, by showing that the two coincide on a dense subset of $A \rtimes_\alpha G$. So we get indeed a one-to-one correspondence between the non-degenerate covariant representations of (A, G, α) and the non-degenerate representations of $A \rtimes_\alpha G$. This correspondence preserves equivalence and irreducibility.

This correspondence will be very useful. We still don't know much about $A \rtimes_\alpha G$, but an important tool in studying C^*-algebras is the representation theory. If we want to find representations of $A \rtimes_\alpha G$ with certain properties, we just need to construct covariant representations of (A, G, α) with those properties, which is in general easier to do, since we understand the C^*-dynamical system better than its associated crossed product C^*-algebra.

For instance, further on in the thesis we shall consider certain kinds of C^*-dynamical systems (like topological dynamical systems and flows), and construct irreducible representations of their crossed product C^*-algebras, by first constructing them for the dynamical system, which boils down to finding covariant representations $(\pi, u, \mathcal{H})$, such that every operator $V \in B(\mathcal{H})$ that commutes with $\pi(a)$ for every $a \in A$ and with u_s for every $s \in G$ is a scalar multiple of the identity.

Chapter 3

The interplay between discrete dynamical systems and C*-algebras

In this chapter we shall look at a more specific kind of dynamical systems: the discrete dynamical systems. In this case much is known about the interplay with C^*-algebras, and we will give an overview of some of the results on this interplay. We will take a closer look at the periodic points of the dynamical system, and the representations of the crossed product C^*-algebra we can associate with these points, for this will be the main topic in the next chapter about flows.

3.1 Discrete topological dynamical systems in compact Hausdorff spaces

Definition 3.1.1. A *discrete dynamical system* is a pair $\Sigma = (X, \sigma)$, such that X is a compact Hausdorff space, and σ a homeomorphism on X.

Let (X, σ) be a topological dynamical system. We define the following function.

$$\phi : \mathbb{Z} \times X \to X, (n, x) \to n \cdot x = \sigma^n(x).$$

We will show that this function is continuous, hence X becomes a left $\mathbb{Z}$-space, where $\mathbb{Z}$ is endowed with the discrete topology.

Lemma 3.1.2. *ϕ is a continuous map from $\mathbb{Z} \times X$ to X.*

Proof. Let U be an open subset of X. Then

$$\phi^{-1}(U) = \{(n,x) \in \mathbb{Z} \times X | \sigma^n(x) \in U\} = \bigcup_{n\in\mathbb{Z}} \{n\} \times \sigma^{-n}(U).$$

Since every subset of $\mathbb{Z}$ is open, and $\sigma^{-n}(U)$ is open in X by continuity of σ^n, $\phi^{-1}(U)$ is a union of open sets, hence open. So ϕ is continuous. $\square$

Since $\mathbb{Z}$ is locally compact, and X compact, the transformation group $(\mathbb{Z}, X)$ is locally compact. Hence, as we have seen in section 2.1.1, we get a C^*-dynamical system $(C(X), \mathbb{Z}, \alpha)$, where α is the function from $\mathbb{Z}$ into Aut $C(X)$ defined by

$$\alpha_n f(x) := f(-n \cdot x) = f(\sigma^{-n}(x)).$$

Note that $C_0(X) = C(X)$, since X is compact.

For the orbit of an element $x \in X$ we shall write $O_\sigma(x)$ instead of $O_\mathbb{Z}(x)$.

The *period* of an element $x \in X$ is the smallest $n \in \mathbb{N}$ such that $\sigma^n(x) = x$, if such n exists. If there is no such n, then we define the period to be ∞. Notice that the period of x is equal to the number of elements in the orbit of x.
If the period of x is less than infinity, we call x a *periodic point.* A *periodic orbit* is an orbit with finitely many elements. If x is not a periodic point, we call it an *aperiodic point*, and an *aperiodic orbit* is an orbit with infinitely many elements.

For the unit circle in $\mathbb{C}$ we shall write $\mathbb{T} := \{z \in \mathbb{C} | \|z\| = 1\}$.

Example 3.1.3. On the compact Hausdorff space $\mathbb{T}$, we can define a homeomorphism

$$\sigma_\theta : e^{2\pi i x} \to e^{2\pi i(x+\theta)},$$

depending on parameter $\theta \in \mathbb{R}$. If $\theta \in \mathbb{Q}$, every point in $\mathbb{T}$ is periodic, and if $\theta \notin \mathbb{Q}$, every point is aperiodic, and every orbit is in fact dense in that case.

From the C^*-dynamical system $(C(X), \mathbb{Z}, \alpha)$ we can construct the crossed product $C(X) \rtimes_\alpha \mathbb{Z}$, which we will denote by $A(\Sigma)$.

Since $\mathbb{Z}$ is a discrete group, the Haar measure μ on $\mathbb{Z}$ is simply the counting measure, i.e., for a subset H of $\mathbb{Z}$, $\mu(H)$ is the number of elements in H. Then the *-algebra $C_c(\mathbb{Z}, C(X))$ consists of all $C(X)$-valued functions f on $\mathbb{Z}$ with compact, hence finite, support. And in this case, unlike the general case, both $C(X)$ and $\mathbb{Z}$ are contained within $C_c(\mathbb{Z}, C(X))$ and hence within $A(\Sigma)$. This embedding can be realized as follows.

For each $f \in C(X)$ we define $\tilde{f}$ to be the function from $\mathbb{Z}$ to $C(X)$ that maps 0 to f and the rest to zero. And for each $s \in \mathbb{Z}$ we define δ_s to be the function from $\mathbb{Z}$ to $C(X)$ that maps s to 1 and the rest to zero.
It is easy to check that $C_c(\mathbb{Z}, C(X))$, hence $A(\Sigma)$, is unital, with unit δ_0. Thus we see that the existence of an embedding of A and G into $A(\Sigma)$ is also a consequence of Theorem 2.3.6, since $M(A(\Sigma)) = A(\Sigma)$.

3.2 Overview of some results

In this section we will give an overview of some results in the interplay between topological dynamical systems and C^*-algebras. We will give definitions where necessary, but we will omit all the proofs. The idea is that properties of the topological dynamical system are translated to properties of the associated crossed product C^*-algebra. All these results can be found in [18], [19] and [20].

To every periodic point x in $\Sigma = (X, \sigma)$ and every irreducible unitary representation u_x of $\mathbb{Z}_x$ (the isotropy subgroup at x), we can associate irreducible representations of $A(\Sigma)$, such that the representations associated to two points $x, y \in X$ and two irreducible representations u_x, u_y are equivalent if and only if x and y are in the same orbit and $u = v$. We will see that those representations are finite dimensional, and that every finite dimensional irreducible representation of $A(\Sigma)$ is unitarily equivalent to a representation associated to a periodic point x and an irreducible representation on $\mathbb{Z}_x$. We will give more details on these results in the next section.

A C^*-algebra A is *simple* if it contains no non-trivial closed ideals.
A topological dynamical system $\Sigma = (X, \sigma)$ is called *minimal* if there is no proper closed set A in X such that $\sigma(A) \subseteq A$.

Then we have the following result.

Theorem 3.2.1. *Let $\Sigma = (X, \sigma)$ be a topological dynamical system. The C^*-algebra $A(\Sigma)$ is simple if and only if Σ is minimal, provided that X has an infinite number of points.*

This theorem has first been proven by Power ([15]). More recent proofs can be found in [19, Theorem 5.3] or [5, Theorem VIII.3.9]. We call Σ *topologically free* if the aperiodic points of X are dense in X. Then we get the following theorem ([19, Theorem 5.4]).

Theorem 3.2.2. *Let $\Sigma = (X, \sigma)$ be a topological dynamical system. The following three assertions are equivalent.*

1. *Σ is topologically free.*
2. *For any closed ideal I of $A(\Sigma)$, $I \cap C(X) \neq \{0\}$ if and only if $I \neq \{0\}$.*
3. *$C(X)$ is a maximal abelian C^*-subalgebra of $A(\Sigma)$.*

We also have ([20, Theorem 12.4])

Theorem 3.2.3. *Let $\Sigma = (X, \sigma)$ be a topological dynamical system. Then Σ is topologically free if and only if the infinite dimensional irreducible representations of $A(\Sigma)$ separate its elements.*

We can state something about all the irreducible representations, once we know there are only periodic points ([20, Theorem 4.5]).

Theorem 3.2.4.

1. *Every irreducible representation of $A(\Sigma)$ is finite dimensional if and only if the system $\Sigma = (X, \sigma)$ consists of periodic points.*
2. *The finite dimensional representations of $A(\Sigma)$ separate the points of X if and only if the periodic points are dense in X.*

The proof of this theorem uses induced representations arising from periodic points. In the next section we will define these induced representations and show some properties of them.

3.3 Periodic orbits and finite dimensional representations

We first look, more generally, at induced representations arising from isotropy subgroups of discrete groups, instead of $\mathbb{Z}$. Let G be a discrete group, i.e. a group with the discrete topology, and X a compact Hausdorff space and a left G-space. Then, as discussed in section 2.1.1, we get a C^*-dynamical system $(C(X), G, \alpha)$, and thus a C^*-algebra $C(X) \rtimes_\alpha G$.

We write the left coset space $G/G_x = \{s_\beta G_x\}$ for representatives $S = \{s_\beta\} \subset G$, where $s_0 = e$ (unit of G). Now, let $(u, \mathcal{H}_u)$ be a unitary representation of G_x. Let $\{e_\beta\}$ be a fixed orthonormal basis of some Hilbert space $\mathcal{H}_0$ with cardinality equal to that of G/G_x. Now let $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_u$. We can then expand every vector $\xi \in \mathcal{H}$ as $\sum_\beta e_\beta \otimes \xi_\beta$, where the sum is ranging over a countable set of indices β for which $\xi_\beta \neq 0$.

We define the unitary representation L_u^S of G on $\mathcal{H}$ induced by u in the following way.

$$L_u^S(s)(e_\beta \otimes \xi) = e_\gamma \otimes u_t(\xi),$$

where $s \in G, t \in G_x$ and γ are such that $ss_\beta = s_\gamma t$. Also, we define the representation π_x^S of $C(X)$ on $\mathcal{H}$ by

$$\pi_x^S(f)(e_\beta \otimes \xi) = f(s_\beta \cdot x)e_\beta \otimes \xi.$$

Then $(\pi_x^S, L_u^S, \mathcal{H})$ becomes a covariant representation of $(C(X), G, \alpha)$, which gives rise to a representation of $C(X) \rtimes_\alpha G$,

$$\tilde{\pi}_{x,u}^S = \pi_x^S \rtimes L_u^S.$$

From [18, Lemma 4.1.1] we get that the representation $\tilde{\pi}_{x,u}^S$ does not depend on the choice of the representatives $S = \{s_\beta\}$ of the left coset space G/G_x, within unitary equivalence. This means that the covariant representation (π_x^S, L_u^S) also does not depend on S, within unitarily equivalence. Therefore, we shall write $\tilde{\pi}_{x,u} = \pi_x \rtimes L_u$ instead of $\tilde{\pi}_{x,u}^S$. On these representations we have the following results ([18, Proposition 4.1.2, Theorem 4.1.3]).

Theorem 3.3.1.

1. *Two representations $\tilde{\pi}_{x,u}$ and $\tilde{\pi}_{y,v}$ are unitarily equivalent if and only if $O_G(x) = O_G(y)$ and, putting $x = s_{\beta_0}y$, the representations of G_x: $t \to u_t$ and $t \to v_{s_{\beta_0}^{-1} t s_{\beta_0}}$ are unitarily equivalent.*
2. *The representation $\tilde{\pi}_{x,u}$ is irreducible if and only if the representation u of G_x is irreducible.*

If we now take $G = \mathbb{Z}$, things get a lot easier. We also only look at periodic points in X. Now let $x \in X$ a periodic point, with period p. Then $\mathbb{Z}_x = p\mathbb{Z}$.

Lemma 3.3.2. *Let G be an abelian group and $(u, \mathcal{H})$ an irreducible unitary representation. Then $\mathcal{H}$ is one dimensional, hence equal to $\mathbb{C}$.*

Proof. Since u is irreducible, the only operators in $B(\mathcal{H})$ that commute with $u(G)$ are scalar multiples of the identity. Since G is abelian, all operators in $u(G)$ commute with each other, hence they are scalar multiples of the identity. Now, we fix a point $x \in \mathcal{H}$. Let $K = \{\lambda x | \lambda \in \mathbb{C}\}$, then $u(G)K \subset K$, hence $K = \mathcal{H}$, by irreducibility of u. So $\mathcal{H}$ is one dimensional, hence equal to $\mathbb{C}$. $\square$

Proposition 3.3.3. *All irreducible representations of $p\mathbb{Z}$ with $p \in \mathbb{N}$ are of the form*

$$np \to z^n \text{ with } n \in \mathbb{Z},$$

where $z \in \mathbb{T}$.

Proof. Let u be a irreducible unitary representation of $p\mathbb{Z}$. Then, since $p\mathbb{Z}$ is abelian, the previous lemma tells us that the associated Hilbert space of u is $\mathbb{C}$, hence all operators $u(p\mathbb{Z})$ are multiplications by a scalar. So $u(p) = z$ for some $z \in \mathbb{C}$. Hence $u(np) = z^n$ for all $n \in \mathbb{Z}$. Also, since $u(-p) = (u(p))^* = \bar{z}$, and $z\bar{z} = u(1)u(-1) = u(0) = 1$, $\|z\| = 1$, hence $z \in \mathbb{T}$. $\square$

The associated Hilbert space $\mathcal{H}_u$, with u an irreducible unitary representation on $p\mathbb{Z}$, is equal to $\mathbb{C}$, hence $\mathcal{H} = \mathcal{H}_0 \otimes \mathbb{C} = \mathcal{H}_0$, hence $\mathcal{H}$ is the p-dimensional Hilbert space. Let $\{e_i\}_{i=0}^{p-1}$ be an orthonormal basis for $\mathcal{H}$. We can then, obviously, write each $\xi \in \mathcal{H}$ as $\xi = \sum_{i=0}^{p-1} \lambda_i e_i$, with $\lambda_i \in \mathbb{C}$.

Since according to the previous theorem the choice of representatives for $\mathbb{Z}/p\mathbb{Z}$ does not matter, we will choose them as follows.

$$S = \{0, 1, 2, .., p-1\}.$$

We will write L_z to denote L_u, where u is the representation on $\mathbb{Z}_x$ sending np to z^n. Then we get, for $s \in \mathbb{Z}$ and $i \in S$,

$$L_z^s(e_i) = z^n e_j,$$

where $n \in \mathbb{Z}, j \in S$ are such that $s + i = j + np$.

And for $f \in C(X)$ we have

$$\pi_x(f)(e_i) = f(i \cdot x)e_i.$$

Since $\mathbb{Z}$ is abelian, 3.3.1.1 implies that $\tilde{\pi}_{x,u}$ and $\tilde{\pi}_{y,v}$ are unitarily equivalent if and only if $O_\sigma(x) = O_\sigma(y)$ and u and v are unitarily equivalent. And it is easy to see that $u : np \to z_1^n$ and $v : np \to z_2^n$ are unitarily equivalent if and only if $z_1 = z_2$, hence if and only if u and v are the same.

So now we have associated to each periodic point x with period p and each $z \in \mathbb{T}$, an irreducible representation of $A(\Sigma)$ with dimension p.

Now, let $x \in X$ be a periodic point, and consider the linear functional μ_x on $C(X)$ that sends a function f to $f(x)$. Then μ_x is positive, since $\mu_x(f^*f) = \overline{f(x)}f(x) = |f(x)|^2 \geq 0$. Furthermore $\|\mu_x\| = 1$, hence μ_x is a state of $C(X)$ (see Appendix A). It can be shown that the state on $A(\Sigma)$ defined by

$$\varphi_{x,u}(a) := \langle \tilde{\pi}_{x,u}(a)e_0, e_0 \rangle$$

is a pure state extension of μ_x. Then it is proven in [19, Proposition 4.3] that the converse is also true. Let φ be a pure state extension of μ_x, and $\{\mathcal{H}_\varphi, \pi_\varphi, \xi\}$ the GNS-representation of φ. Then π_φ is unitarily equivalent to an induced representation $\tilde{\pi}_{x,u}$.

But this is not all; in fact we have the following.

Theorem 3.3.4. *Every p-dimensional irreducible representation of $A(\Sigma)$ is unitarily equivalent to an induced representation $\tilde{\pi}_{x,v}$ arising from a periodic point x with period p and an irreducible representation v of $p\mathbb{Z}$.*

Proof. Let $\tilde{\pi} = \pi \rtimes u$ be a representation of $A(\Sigma)$ on a Hilbert space $\mathcal{H}$ and let I_π be the kernel of π. Since the image $\pi(C(X))$ is a unital commutative C^*-algebra, it is isometrically isomorphic to $C(X'_\pi)$, where X'_π is a compact space. Also, I_π is a closed invariant ideal of $C(X)$, hence we can write $I_\pi = k(X_\pi)$ for a closed invariant subset of X_π, where $k(X_\pi)$ means the set of all functions in $C(X)$ vanishing on X_π. Then we have

$$\pi(C(X) = C(X'_\pi) \cong C(X)/I_\pi = C(X_\pi). \tag{3.1}$$

Hence we can identify X'_π with X_π, together with the action $\sigma_\pi := \sigma|_{X_\pi}$ on X_π and the action on X'_π induced from the action of $u = \tilde{\pi}(\delta)$ on $C(X'_\pi)$. With this identification, $\pi(f)$ on X'_π corresponds to the restriction of f onX_π and $\|\pi(f)\| = \|f|_{X_\pi}\|$. We denote this dynamical system by $\Sigma_\pi = (X_\pi, \sigma_\pi)$. It is then shown in [19] (Proposition 4.4) that if $\tilde{\pi}$ is irreducible, the system Σ_π is *topologically transitive*, i.e., for nonempty open sets U and V there exists an integer n such that $\sigma^n U \cap V \neq \emptyset$. Now, let $\tilde{\pi} = \pi \rtimes u$ be a p-dimensional irreducible representation of $A(\Sigma)$ on a Hilbert space $\mathcal{H}$. Then we get an irreducible covariant representation $(\pi, u, \mathcal{H})$ of $(C(X), G, \alpha)$. Since $\pi(C(X))$ is finite dimensional, its spectrum X_π must consist of finitely many isolated points, and moreover the system Σ_π is topologically transitive. Therefore there exists a periodic point $x \in X_\pi$, with period n, such that $O_\alpha(x) = X_\pi$, so

$$X_\pi = \{x, \sigma x, ..., \sigma^{n-1}x\}.$$

Now let $p_i \in C(X_\pi)$ be the characteristic function at the point $\sigma^i x$. Then $C(X_\pi)$, hence $\pi(C(X))$, is spanned by the set $\{p_0, p_1, .., p_{n-1}\}$. Now by (3.1) there is an $f_i \in C(X)$, such that $\pi(f_i) = f_i|_{X_\pi} = p_i$. Hence $f_i(\sigma^j x) = 1$ if and only $j = i$. Then $\alpha_j(f_0(\sigma^k x)) = f_i(\sigma^{k-j}x) = 1$ if and only if $k - j = 0$, so $k = j \mod n$. Then $\pi(\alpha_j(f_0)) = p_j$. So

$$u^i p_0 (u^i)^* = u^i \pi(f_0)(u^i)^* = \pi(\alpha_i(f_0)) = p_i.$$

Also $u^n p_0 (u^n)^* = p_0$. Now we define $\mathcal{H}_i = p_i\mathcal{H} = \pi(f_i)\mathcal{H}$. Then the following holds.

$$\mathcal{H}_i = \{\xi \in \mathcal{H} | \pi(f)\xi = f(\sigma^i x)\xi \text{ for all } f \in C(X)\}.$$

To see this, we first take an element in $\mathcal{H}_i$: $\pi(f_i)\xi$ for some $\xi \in \mathcal{H}$. Then it is clear that $ff_i(\sigma^i(x)) = f(\sigma^i(x))$ and $ff_i(\sigma^j(x)) = 0$ if $j \neq i$. Hence for all $f \in C(X)$ we have that

$$\pi(f)\pi(f_i)\xi = \pi(ff_i)\xi = \pi(f(\sigma^i x)f_i)\xi = f(\sigma^i x)\pi(f_i)\xi.$$

Now the other direction: Take $\xi \in \mathcal{H}$, such that $\pi(f)\xi = f(\sigma^i x)\xi$ for all $f \in C(X)$, then $\pi(f_i)\xi = f_i(\sigma^i x)\xi = \xi$, hence $\xi \in \mathcal{H}_i$. Now suppose that $\xi \in \mathcal{H}_i \cap \mathcal{H}_j$, for $i \neq j$, then for all $f \in C(X)$ $\pi(f)\xi = f(\sigma^i x)\xi = f(\sigma^j x)\xi$. Then, necessarily, $\xi = 0$.

Furthermore we have $\mathcal{H}_i = p_i\mathcal{H} = u^i p_0 (u^i)^* \mathcal{H} = u^i p_0 \mathcal{H} = u^i \mathcal{H}_0$ with $0 \leq i \leq n-1$), and $\mathcal{H}_0 = u^n \mathcal{H}_0$. We now define a unitary representation v of $\mathbb{Z}_x$ on $\mathcal{H}_0$, by sending $kn \in \mathbb{Z}_x$ to the operator u^{kn} restricted to $\mathcal{H}_0$, which makes sense, since $u^n\mathcal{H}_0 = \mathcal{H}_0$.
We want to prove that v is irreducible. Let K be a subspace of $\mathcal{H}_0$,invariant under v. Then the closed linear span $[\tilde{\pi}(A(\Sigma))K]$ is an invariant (under $\tilde{\pi}$) subspace of $\mathcal{H}$. Hence, by irreducibility of $\tilde{\pi}$, $\mathcal{H} = [\tilde{\pi}(A(\Sigma))K]$.
We can write, for every $g \in C_c(\mathbb{Z}, C(X))$, $\tilde{\pi}(g) = \sum_{s\in\mathbb{Z}} \pi(g(s))u^s$. The only parts of this sum that send g into $\mathcal{H}_0$ are at $s = kn$ with $k \in \mathbb{Z}$. However we have for every $f \in C(X)$ that $\pi(f)u^{kn}K = \pi(f)v^{kn}K \subset \pi(f)K \subset K$, since for every $\xi \in \mathcal{H}_0$, $\pi(f)\xi = f(x)\xi \in \mathcal{H}_0$. Since we want to get *all* of $\mathcal{H}$, we need to get all of $\mathcal{H}_0$, hence $K = \mathcal{H}_0$. This implies that v is irreducible, hence $\mathcal{H}_0$ is one-dimensional. Then $\mathcal{H}$ is p-dimensional, and $p = n$.
Furthermore, the Hilbert spaces $\mathcal{H}_0, ..., \mathcal{H}_{p-1}$ are orthogonal: Let $\xi \in \mathcal{H}_i$ and $\eta \in \mathcal{H}_j$, with $0 \leq i < j \leq p-1$. Now we can find an $f \in C(X)$ such that $f(\sigma^i x) = 1$ and $f(\sigma^j x) = 0$. Then

$$\langle \xi, \eta \rangle = \langle \pi(f)\xi, \eta \rangle = \langle \xi, \pi(\bar{f})\eta \rangle = \langle \xi, f(\sigma^j x)\eta \rangle = 0.$$

Therefore, if we let $e_i \in \mathcal{H}_i$, with $\|e_i\| = 1$, then $\{e_i\}_{i=0}^{p-1}$ is an orthonormal basis for $\mathcal{H}$. Then it is easy to see that $\tilde{\pi}$ is unitarily equivalent to the induced representation $\tilde{\pi}_{x,v}$. $\square$

Chapter 4

The interplay between flows and C*-algebras

In this chapter we will first give a definition and some results of flows in Hausdorff spaces. Then we will introduce induced representations of subgroups, and use these to construct representations of the crossed product C^*-algebra, associated to periodic orbits of the flow, and finally we will show some properties of these representations.

4.1 Flows in compact Hausdorff spaces

Here we will give the definition of a flow on a space X, and show some basic results of such flows. Though we use compactness of the space X in the rest of this chapter, it is actually not needed in this section.

Therefore, we let X be Hausdorff and a left $\mathbb{R}$-space. So we have a continuous map ϕ from $\mathbb{R} \times X \to X$ that sends

$$(s, x) \to s \cdot x,$$

such that for every $s, t \in \mathbb{R}, x \in X$

$$0 \cdot x = x \text{ and } s \cdot (t \cdot x) = (s + t) \cdot x.$$

Then we call (X, ϕ) a *continuous flow*, or simply flow.

For the orbit of $x \in X$ we shall write $O_\phi(x)$ instead of $O_\mathbb{R}(x)$. A point $x \in X$ is said to be a *fixed point* if $O_\phi(x) = x$. The set of all fixed points is denoted by $F(\phi)$.
Recall that for $x \in X$, the isotropy subgroup at x is $\mathbb{R}_x := \{t \in \mathbb{R} | t \cdot x = x\}$,

consisting of all elements in $\mathbb{R}$ that fix x.
The following proposition comes from [3, Theorem 1.9].

Proposition 4.1.1. *Let (X, ϕ) be a flow, then for every $x \in X$ exactly one of the following occurs.*

1. $\mathbb{R}_x = \mathbb{R}$, *i.e., $x \in F(\phi)$,*
2. *there exists a least positive number p such that $p \cdot x = x$, and then $\mathbb{R}_x = \{np | n \in \mathbb{Z}\}$,*
3. $\mathbb{R}_x = \{0\}$*, and the function $t \to t \cdot x$ is injective.*

Proof. X is Hausdorff, hence $\{x\}$ is closed in X. Furthermore, the function $t \to t \cdot x$ is continuous, because ϕ is continuous. So the inverse of $\{x\}$ is closed under this map, and this inverse is exactly the set $\mathbb{R}_x$. Hence $\mathbb{R}_x$ is closed.
Now, let $\mathbb{R}_x^+ := \{t > 0 | t \cdot x = x\}$. Suppose $\mathbb{R}_x^+$ is not empty, but has no smallest element and let $s = \inf \mathbb{R}_x^+$. Then by assumption $s \notin \mathbb{R}_x^+$. Then there is a decreasing sequence of elements $\{r_n\}$ in $\mathbb{R}_x^+$ such that $r_n \in [s, s+\frac{1}{n}]$. Now,

$$(r_n - r_{n+1}) \cdot x = r_n(-r_{n+1} \cdot x) = x,$$

hence $r_n - r_{n+1} \in \mathbb{R}_x^+$ and $r_n - r_{n+1} \leq \frac{1}{n}$. From this we see that s has to be zero, since $\inf_n(r_n - r_{n+1}) = 0$. So, actually $r_n \in [0, \frac{1}{n}]$. Consequently the set $\{mr_n, m \in \mathbb{Z}, n \in \mathbb{N}\}$ is a dense subset of $\mathbb{R}$, and since it is also a subset of $\mathbb{R}_x$, and $\mathbb{R}_x$ is closed, we see that $\mathbb{R}_x = \mathbb{R}$, i.e., 1) is true.
Suppose on the other hand that $\mathbb{R}_x^+$ is empty, then $\mathbb{R}_x = \{0\}$. Then if

$$s \cdot x = t \cdot x,$$

we have

$$(s - t) \cdot x = x,$$

hence

$$s - t \in \mathbb{R}_x = \{0\},$$

thus $t \to t \cdot x$ is injective, hence 3) is true. If neither of the above is true, then only one case remains: $\mathbb{R}_x^+$ is not empty, and has a smallest element p. Then for every $n \in \mathbb{Z}$, $np \cdot x = x$, so $np \in \mathbb{R}_x$. Now suppose that $\mathbb{R}_x$ contains an element q that is not of the form np for some $n \in \mathbb{Z}$. Then there is an $n \in \mathbb{Z}$, such that $0 < q - np < p$. But $(q - np) \cdot x = q \cdot (-np \cdot x) = q \cdot x = x$, hence $q - np \in \mathbb{R}_x$. But this contradicts the assumption that p is the smallest element of $\mathbb{R}_x$. So $\mathbb{R}_x = \{np | n \in \mathbb{Z}\}$, hence 2) is true. □

Now we can define the *period*, i.e., the function P_ϕ from X into $[0, \infty]$, such that, for every $x \in X$,

$P_\phi(x) = 0$ if x satisfies condition 4.1.1.1,
$P_\phi(x) = p$ if x satisfies condition 4.1.1.2,
$P_\phi(x) = \infty$ if x satisfies condition 4.1.1.3.

So $P_\phi(x) = \inf \mathbb{R}_x^+$, where we take $\inf \mathbb{R}_x^+$ to be ∞ if $\mathbb{R}_x^+$ is empty.

Then $P_\phi(x)$ is constant on orbits.

We call $O_\phi(x)$ *periodic* if $P_\phi(x) < \infty$, and x a periodic point if $O_\phi(x)$ is periodic. If $P_\phi(x) = \infty$, we call $O_\phi(x)$ *aperiodic* and x an aperiodic point.

Example 4.1.2. Let $\mathbb{R}$ act on the compact Hausdorff space $\mathbb{T}$ by rotation:

$$r \cdot x = e^{ir}x, \text{ where } r \in \mathbb{R}, x \in X.$$

Then clearly $\mathbb{T}$ consists of exactly one orbit, and all elements $x \in \mathbb{T}$ are periodic, with period 2π.

Proposition 4.1.3. *Let (X, ϕ) be a flow, and x a periodic point with period $p > 0$. Then $O_\phi(x)$ is closed in X.*

Proof. It is an standard result in topology that a compact subset of a Hausdorff space is closed. Now, $[0, p]$ is a compact subset of $\mathbb{R}$, and the function $r \to r \cdot x$ is continuous by Lemma 2.1.1, hence $O_\phi(x) = [0, p] \cdot x$ is a compact subset of X. Hence $O_\phi(x)$ is closed in X.

□

If $x \in X$ is an aperiodic point, then the orbit of x need not necessarily be closed. We will give two examples to illustrate this.

Example 4.1.4. We let $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, the *extended real line.* Consider the topology on $\bar{\mathbb{R}}$ generated by the open sets in $\mathbb{R}$ and the sets $\{r \in \mathbb{R} | r > k\} \cup \{\infty\}$ and $\{r \in \mathbb{R} | r < m\} \cup \{-\infty\}$, where $k, m \in \mathbb{R}$. Then we define the following map Φ from $\mathbb{R} \cdot \bar{\mathbb{R}} \to \bar{\mathbb{R}}$.

$$r \cdot s = r + s, \text{ with } r, s \in \mathbb{R},$$

$$r \cdot \infty = \infty,$$

and

$$r \cdot -\infty = -\infty.$$

It is easy to check that this action defines a transformation group $(\mathbb{R}, \bar{\mathbb{R}})$. Then for any $x \in \mathbb{R} \subset \bar{\mathbb{R}}$, we have $O_\phi(x) = \mathbb{R}$, but $\mathbb{R}$ is not closed. The closure of $\mathbb{R}$ is $\bar{\mathbb{R}}$.

Example 4.1.5. Let X be the 2-torus: $X = \mathbb{T}^2$. Then X is a compact Hausdorff space. We let $\mathbb{R}$ act on X in the following way.

$$r \cdot (e^{2\pi i x}, e^{2\pi i y}) \to (e^{2\pi i(x+ra)}, e^{2\pi i(y+rb)}),$$

for some $a, b \in \mathbb{R}$. Then we have two cases:

1. $\frac{a}{b} \in \mathbb{Q}$. This implies that every orbit is periodic.
2. $\frac{a}{b} \notin \mathbb{Q}$. This implies that every orbit is aperiodic and it can be shown that every orbit is dense in X.

In the second case, the orbit is not the whole space; we can easily find points in X that are not on this orbit. But the closure of the orbit *is* the whole space.

Proposition 4.1.6. *Let (X, ϕ) be a flow, and x a periodic point with period $p > 0$. Then $O_\phi(x)$ is homeomorphic to $\mathbb{R}/p\mathbb{Z}$, hence to $\mathbb{T}$.*

Proof. We define $f : \mathbb{R}/p\mathbb{Z} \to O_\phi(x)$ by

$$\bar{r} \to r \cdot x.$$

Then f is well-defined, for if $\bar{r} = \bar{s}$, then $r = s + np$, for some $n \in \mathbb{Z}$, hence

$$r \cdot x = (s + np) \cdot x = s \cdot (np \cdot x) = s \cdot x.$$

This function is continuous, because of the continuity of ϕ.
It is injective, for if

$$f(\bar{r}) = f(\bar{s}),$$

then

$$r \cdot x_0 = s \cdot x, \text{ thus } (r - s) \cdot x_0 = x_0,$$

so $r - s \in \mathbb{R}_x$, hence $r - s = np$ for some $n \in \mathbb{Z}$. So $\bar{r} = \bar{s}$.
It is easy to see that f is also surjective. Now by a standard theorem in topology every continuous function from a compact space into a Hausdorff space which is surjective and injective is actually a homeomorphism. Since $\mathbb{R}/p\mathbb{Z}$ is indeed compact, and $O_\phi(x)$, as a subspace of a Hausdorff space, is Hausdorff, we are done. $\square$

4.2 Induced representations of subgroups

Now, just like the discrete case, we want to associate to each periodic point $x \in X$, and each irreducible unitary representation u of $p\mathbb{Z}$ an irreducible representation $\tilde{\pi}_{x,u}$ of $C(X) \rtimes_\alpha \mathbb{R}$, such that $\tilde{\pi}_{x,u}$ and $\tilde{\pi}_{y,v}$ are unitarily equivalent if and only if $O_\phi(x) = O_\phi(y)$ and u is unitarily equivalent to v (then $u = v$). To do this, we will first delve into the theory of induced representations of groups.

The idea of induced representations of a group G is that we use a unitary representation of a subgroup H of G to build a unitary representation of G itself. We will leave out proofs in this section; a detailed account on such induced representations can be found in [11],[8] and [10].

Let G be a locally compact group and H a closed subgroup of G. We will consider the space G/H, with the quotient map

$$\pi : x \to \pi(x) = xH = \overline{x}.$$

If we endow G/H with the quotient topology (the strongest topology that makes the quotient map continuous), it becomes a locally compact Hausdorff space. However, it does not necessarily become a topological group. If H is normal in G, then G/H becomes a locally compact group by letting $\bar{x}\bar{y} = \overline{xy}$ for all $\overline{x}, \overline{y} \in G/H$.

For $B \subset G/H$ and $x \in G$ we let $xB := \{\overline{xy} | y \in G/H\}$. Then a Borel measure μ on G/H is said to be *invariant* if

$$\mu(B) = \mu(xB) \text{ for all } x \in G \text{ and Borel sets } B \text{ of } G/H.$$

For instance, if H is normal in G, G/H is a locally compact group, and hence there exists a left Haar measure on G/H (see Appendix B for more on Haar measures). Since $\bar{x}\bar{y} = \overline{xy}$ for all $x, y \in G$, this left Haar measure is invariant.

Remark 4.2.1. Such an invariant measure does not necessarily exist on every G/H. However, there always are measures on every G/H that satisfy a weaker condition, the *quasi-invariant measures*, and these are in fact profitable to work with. However, we shall restrict ourselves to groups G and subgroups H such that G/H *does* have an invariant measure μ, which is then unique up to a constant, since this makes the notations less tedious, and we will apply this theory in the following section to $\mathbb{R}/\mathbb{R}_x$,with x a periodic point, which has such a measure, since $\mathbb{R}_x$ is a normal subgroup of $\mathbb{R}$.

Now let u be a unitary representation of H on a separable Hilbert space $\mathcal{H}$. We consider the linear space $K_u(G, \mathcal{H})$ consisting of all continuous functions $f : G \to \mathcal{H}$ that satisfy

$$f(xh) = u(h^{-1})f(x) \text{ for all } x \in G, h \in H, \tag{4.1}$$

and whose support is contained in $KH := \{kh | k \in K, h \in H\}$, for some compact set $K \subset G$.

Lemma 4.2.2. *Let $f_1, f_2 \in K_u(G, \mathcal{H})$, then*

$$\langle f_1(x), f_2(x)\rangle_{\mathcal{H}} = \langle f_1(y), f_2(y)\rangle_{\mathcal{H}} \text{ if } \bar{x} = \bar{y}.$$

Proof. If $\bar{x} = \bar{y}$, then there exists an $h \in H$, such that $x = yh$, and then

$$\langle f_1(x), f_2(x)\rangle_{\mathcal{H}} = \langle f_1(yh), f_2(yh)\rangle_{\mathcal{H}} = \langle u(h^{-1})f_1(y), u(h^{-1})f_2(y)\rangle_{\mathcal{H}},$$

and since $u(h^{-1})$ is a unitary operator,

$$\langle u(h^{-1})f_1(y), u(h^{-1})f_2(y)\rangle_{\mathcal{H}} = \langle f_1(y), f_2(y)\rangle_{\mathcal{H}}.$$

□

Now, since the support of f_1 is in K_1H, for some compact subset K_1 in G, the support of $\langle f_1(x), f_2(x)\rangle_{\mathcal{H}}$ is also contained in K_1H, for if $f_1(x) = 0$, then $\langle f_1(x), f_2(x)\rangle = 0$. So now we can view $\bar{x} \to \langle f_1(x), f_2(x)\rangle_{\mathcal{H}}$ as a continuous function on G/H with support contained in $\pi(K_1H) = \pi(K_1)$, hence compact. Now we shall define for $f_1, f_2 \in K_u(G, \mathcal{H})$,

$$\langle f_1, f_2\rangle' := \int_{G/H} \langle f_1(x), f_2(x)\rangle_{\mathcal{H}} \, d\mu(\bar{x}).$$

The integral at the right hand side is well-defined, in view of the previous paragraph. Then $\langle\, .\, ,\, .\, \rangle'$ becomes an inner product on $K_u(G, \mathcal{H})$. One might ask whether there are any non-trivial functions that satisfy (4.1), and the answer is affirmative ([10, Corollary 4.6]).

Theorem 4.2.3. *If* $\dim \mathcal{H} \neq 0$ *then* $\dim K_u(G, \mathcal{H}) \neq 0$.

Now we consider the linear space $B_u(G, \mathcal{H})$ which consists of all $\mathcal{H}$-valued functions on G which satisfy (4.1) and which are *weakly Haar measurable*, i.e., for every $h \in \mathcal{H}$ the function $x \to \langle f(x), h\rangle_{\mathcal{H}}$ is a Borel measurable function on G. Note that $K_u(G, \mathcal{H}) \subset B_u(G, \mathcal{H})$.

Let f_1 and f_2 be in $B_u(G, \mathcal{H})$. It can be shown, using the separability of $\mathcal{H}$, that $\bar{x} \to \langle f_1(x), f_2(x)\rangle_{\mathcal{H}}$ is a well-defined Borel measurable function on G/H. Hence for $f \in B_u(G, \mathcal{H})$ the definition

$$(\|f\|')^2 := \int_{G/H} \langle f(x), f(x)\rangle_{\mathcal{H}} \, d\mu(\bar{x})$$

makes sense, and $0 \leq \|f\|' \leq \infty$.

Now we let $L^2_u(G, \mathcal{H})$ be the linear space consisting of $f \in B_u(G, \mathcal{H})$ such that $\|f\|' < \infty$, where, as usual, we identify two functions f_1 and f_2 if and only if $\|f_1 - f_2\|' = 0$. Then $K_u(G, \mathcal{H}) \subset L^2_u(G, \mathcal{H})$. Furthermore, we have the following theorems, coming from [10, Proposition 4.7, Proposition 4.10].

Theorem 4.2.4. *$L^2_u(G, \mathcal{H})$ is a Hilbert space.*

Theorem 4.2.5. *$K_u(G, \mathcal{H})$ is dense in $L^2_u(G, \mathcal{H})$.*

Our induced representation will be on the Hilbert space $L^2_u(G, \mathcal{H})$; we shall first define it on $K_u(G, \mathcal{H})$. For each $f \in K_u(G, \mathcal{H})$ and $y \in G$, we define the $\mathcal{H}$-valued function $\hat{u}(y)f$ by

$$(\hat{u}(y)f)(x) := f(y^{-1}x).$$

Then this function again satisfies (4.1) and we have, for $f \in K_u(G, \mathcal{H})$ and $y, z \in G$

$$\begin{aligned} \hat{u}(yz)f &= \hat{u}(y)(\hat{u}(z)f), \\ \hat{u}(e)f &= f. \end{aligned}$$

Also, the inner product we defined on $K_u(G, \mathcal{H})$ is $\bar{u}$-invariant.

$$\begin{aligned} \langle \hat{u}(y)f_1, \hat{u}(y)f_2 \rangle' &= \int_{G/H} \langle f_1(y^{-1}x), f_2(y^{-1}x) \rangle_{\mathcal{H}} \, d\mu(\bar{x}) \\ &= \int_{G/H} \langle f_1(x), f_2(x) \rangle_{\mathcal{H}} \, d\mu(\bar{x}) \\ &= \langle f_1, f_2 \rangle', \end{aligned}$$

because of the invariance of μ.

Furthermore, it is shown in [10, Lemma 4.1] that $\hat{u}$ is weakly continuous, i.e., the function $x \to \langle \hat{u}(x)f_1, f_2 \rangle' : G \to \mathbb{C}$ is continuous for all $f_1, f_2 \in K_u(G, \mathcal{H})$. Now, for each $x \in G$, the operator $\hat{u}(x)$ has a unique extension to a unitary operator on $L^2_u(G, \mathcal{H})$. So we obtain a function from G to $U(L^2_u(G, \mathcal{H}))$, which we also denote by $\hat{u}$. The homomorphism property and the weak continuity of $\hat{u}$ are preserved under this extension. From this it follows that for every $f \in L^2_u(G, \mathcal{H})$ the function $x \to \hat{u}(x)f$ is continuous in the norm topology of $\mathcal{H}$, hence $\hat{u}$ is a unitary representation of G on $L^2_u(G, \mathcal{H})$. We will call $\hat{u}$ the representation on G induced by u.

The disadvantage of this induced representation is that the Hilbert space depends on the unitary representation u. We will now discuss a different model of this representation for which the Hilbert space is independent of u. For this model we need G to be second countable. We consider the Hilbert space of $\mathcal{H}$-valued functions on G/H that are weakly Haar measurable and square integrable with respect to μ, and denote it by $L^2(G/H, \mathcal{H}, \mu)$.

Definition 4.2.6. A *Borel cross-section* is a Borel function $s : G/H \to G$, such that $\pi \circ s = id_{G/H}$.

A fundamental lemma by Mackey tells us that if G is a locally compact second countable group and H a closed subgroup of G, then there always exists a Borel cross-section $s : G/H \to G$. Now, let s be a Borel cross-section. Then, for each $f \in B_u(G, \mathcal{H})$ we define a function g_f on G/H by

$$g_f(\bar{x}) := f(s(\bar{x})) \text{ for all } \bar{x} \in G/H.$$

Then g_f is weakly Borel on G/H, and we have the following theorem.

Theorem 4.2.7. *The map $W : f \to g_f$ is an isometric isomorphism from $L^2_u(G, \mathcal{H})$ onto $L^2(G/H, \mathcal{H}, \mu)$, and the inverse is given by*

$$W^{-1}g(x) := u(x^{-1}s(\bar{x}))g(\bar{x}).$$

Hence W is a unitary operator from $L^2_u(G, \mathcal{H})$ onto $L^2(G/H, \mathcal{H}, \mu)$. Thus we can transport every unitary representation v of G on $L^2_u(G, \mathcal{H})$ to a (necessarily unitarily equivalent) unitary representation $\tilde{v}$ of G on $L^2(G/H, \mathcal{H}, \mu)$, by putting

$$\tilde{v} = WvW^{-1}.$$

When we transport our induced representation $\hat{u}$, we get the following representation.

$$(\tilde{u}(y)g)(\bar{x}) = u(s(\bar{x})^{-1}ys(y^{-1}\bar{x}))g(y^{-1}\bar{x}), \tag{4.2}$$

where $y \in G, g \in L^2(G/H, \mathcal{H}, \mu)$ and $\bar{x} \in G/H$.

So we can also look at $\tilde{u}$ as the representation induced from u, and now the associated Hilbert space is independent of u.

Remark 4.2.8. We will now look back on section 3.3 and show that the induced unitary representation we got there coincides with the definition for $\tilde{u}$ we have here. Since $\mathbb{Z}/p\mathbb{Z}$ is a group that consists of p points with the discrete topology, the Haar measure μ is the counting measure. Thus $L^2(\mathbb{Z}/p\mathbb{Z}, \mu)$ is a p-dimensional Hilbert space. It has as orthonormal basis the functions f_i $(i = 0, 1, .., p-1)$ that send $\bar{i}$ to 1 and the rest to zero. In section 3.3 we looked at how the induced unitary representation acted on e_i for $\{e_i\}_{i=0}^{p-1}$ an orthonormal basis of the Hilbert space. Here this e_i translates to the function $f_i \in L^2(\mathbb{Z}/p\mathbb{Z})$ that sends $\bar{i}$ to 1 and the rest to zero.
Let u be the unitary representation on $p\mathbb{Z}$ that sends np to z^n. Now, if we

take as Borel cross-section $s : \bar{i} \to i$, such that $0 \le i < p$, we get from (4.2) that for $t \in \mathbb{Z}$, $i \in \{0, 1, .., p-1\}$, and $\bar{x} \in \mathbb{Z}/p\mathbb{Z}$

$$\tilde{u}(t)(f_i)(\bar{x}) = z^n f_i(\overline{x-t}),$$

where

$$np = -s(\bar{x}) + t + s(\overline{x-t}).$$

Now since $f_i(\overline{x-t}) = 1$ if and only if $\overline{x-t} = \bar{i}$, we get that $\bar{x} \to f_i(\overline{x-t})$ equals the function f_j, where j is such that $\overline{j-t} = i$. Hence

$$\tilde{u}(t)(f_i)(\bar{x}) = z^n f_j(\bar{x}),$$

which is only non-zero if $\bar{x} = \bar{j}$. In that case

$$np = -s(\bar{j}) + t + s(\overline{j-t}) = -s(\bar{j}) + t + s(\bar{i}),$$

hence

$$t + i = j + np,$$

which is the same as what we had in section 3.3.

4.3 Representations associated to periodic orbits

We let (X, Φ) be a flow. We will now use the representations defined in the previous section to define representations of $C(X) \rtimes_\alpha \mathbb{R}$ associated to periodic points x, just like in the discrete case. We will do this by taking an irreducible representation of the isotropy subgroup at x and use this to construct a unitary representation u of $\mathbb{R}$ and a representation π of $C(X)$, and show that the pair (π, u) is covariant. We will then establish some nice properties of these representations, e.g., that they are irreducible. We will have to consider two different cases: x has period $p > 0$, in which case the isotropy subgroup is $\mathbb{R}_x = p\mathbb{Z}$, or x has period $p = 0$, which makes x a fixed point, and where $\mathbb{R}_x = \mathbb{R}$.

4.3.1 Case I: period $p > 0$

Let $x \in X$ be periodic, with period $p > 0$. Then we can look at $\mathbb{R}_x$, which is a subgroup of $\mathbb{R}$, and we can induce every unitary representation of $\mathbb{R}_x$ to a unitary representation of $\mathbb{R}$.

Motivated by the discrete case, we will only look at irreducible unitary representations of $\mathbb{R}_x$. Since $\mathbb{R}_x = p\mathbb{Z}$, it follows from Lemma 3.3.2 that all such representations are of the form

$$u_{x,z} : np \to z^n,$$

where $z \in \mathbb{T}$. We use this notation since u depends both on (the period of) x and on z.

So in this case the induced representations are on the Hilbert space $L^2_{u_{x,z}}(\mathbb{R}, \mathbb{C})$, which we shall denote by $L^2_{x,z}(\mathbb{R})$, where $u_{x,z}$ is the representation sending np to z^n.

Now we can make (4.1) specific.

$$g(r + np) = z^{-n} g(r) \text{ for all } r \in \mathbb{R}, n \in \mathbb{Z}. \tag{4.3}$$

Let $K := [0, p]$, a compact set in $\mathbb{R}$, then $K\mathbb{R}_x = \mathbb{R}$, since every $r \in \mathbb{R}$ can be written as $np + t$ for some $n \in \mathbb{Z}, t \in [0, p]$. So $K_{u_{x,z}}(\mathbb{R}, \mathbb{C})$ consists of *all* continuous functions from $\mathbb{R}$ to $\mathbb{C}$ that satisfy (4.3).
Also, note that weakly Haar measurable $\mathbb{C}$-valued functions are Borel measurable functions. So $B_{u_{x,z}}(\mathbb{R}, \mathbb{C})$ consists of all Borel measurable functions from $\mathbb{R}$ to $\mathbb{C}$ that satisfy (4.3).
We shall write $K_{x,z}(\mathbb{R})$ and $B_{x,z}(\mathbb{R})$ for these two spaces.

We will denote the induced unitary representation of $\mathbb{R}$ coming from $u_{x,z}$ by $\hat{u}_{x,z}$. Then we get

$$(\hat{u}_{x,z}(t)g)(r) = g(-t + r) \text{ for all } g \in L^2_{x,z}(\mathbb{R}), t, r \in \mathbb{R}.$$

We now want to construct a representation of $C(X)$ on $L^2_{x,z}(\mathbb{R})$.

Let $f \in C(X)$, then we consider the function

$$r \to f(r \cdot x) \text{ for all } r \in \mathbb{R}$$

Since $r \to r \cdot x$ is by definition continuous, this function is again continuous, hence Borel measurable.

We define for each $f \in C(X)$ and $g \in L^2_{x,z}(\mathbb{R})$ the following function from $\mathbb{R}$ to $\mathbb{C}$.

$$(\pi_{x,z}(f)g)(r) := f(r \cdot x)g(r). \tag{4.4}$$

This function is measurable and it again satisfies (4.3).

$$\begin{aligned}
(\pi_{x,z}(f)g)(r + np) &= f((r + np) \cdot x)g(r + np) \\
&= f(r \cdot (np \cdot x))z^{-n}g(r) \\
&= z^{-n}f(r)g(r) \\
&= z^{-n}(\pi_{x,z}(f)g)(r).
\end{aligned}$$

Also since f is bounded, $\pi_{x,z}(f)g$ is again in $L^2_{x,z}(\mathbb{R})$. It is easily verified that $\pi_{x,z}(f)$ is a *linear* operator on $L^2_{x,z}(\mathbb{R})$. It is also bounded, since for every $g \in L^2_{x,z}(\mathbb{R})$

$$\begin{array}{rcl}(\|\pi_{x,z}(f)g\|')^2 & = & \int_{\mathbb{R}/p\mathbb{Z}} \|f(r\cdot x)g(r)\|^2\, d\mu(\bar{r}) \\ & \leq & \|f\|^2_\infty \int_{\mathbb{R}/p\mathbb{Z}} \|g(r)\|^2\, d\mu(\bar{r}) \\ & = & \|f\|^2_\infty (\|g\|')^2,\end{array}$$

So $\pi_{x,z}$ is a function from $C(X)$ into $B(L^2_{x,z}(\mathbb{R}))$. To show π is actually a representation, we need to verify that it preserves sums, scalar products, multiplications and adjoints. So, let $g \in L^2_{x,z}(\mathbb{R})$ and $r \in \mathbb{R}$, then for all $f_1, f_2 \in C(X)$ and all $\lambda, \mu \in \mathbb{C}$ we have

$$\begin{array}{rcl}(\pi_{x,z}(\lambda f_1 + \mu f_2)g)(r) & = & (\lambda f_1 + \mu f_2)(r\cdot x)g(r) \\ & = & \lambda f_1(r\cdot x)g(r) + \mu f_2(r\cdot x)g(r) \\ & = & \lambda(\pi_{x,z}(f_1)g)(r) + \mu(\pi_{x,z}(f_2)g)(r).\end{array}$$

Also

$$\begin{array}{rcl}(\pi(f_1 f_2)g)(r) & = & (f_1 f_2)(r\cdot x)g(r) \\ & = & f_1(r\cdot x)f_2(r\cdot x)g(r) \\ & = & (\pi_{x,z}(f_1)\circ\pi_{x,z}(f_2)g)(r).\end{array}$$

Now we want to find the adjoint of $\pi_{x,z}(f)$ with $f \in C(X)$. Let $g_1, g_2 \in L^2_{x,z}(\mathbb{R})$, then

$$\begin{array}{rcl}\langle \pi_{x,z}(f)g_1, g_2\rangle' & = & \int_{\mathbb{R}/p\mathbb{Z}} f(r\cdot x)g_1(r)\overline{g_2(r)}\, d\mu(\bar{r}) \\ & = & \int_{\mathbb{R}/p\mathbb{Z}} g_1(r)\overline{\overline{f(r\cdot x)}g_2(r)}\, d\mu(\bar{r}) \\ & = & \langle g_1, \pi^*_{x,z}(f)g_2\rangle'\end{array}$$

Hence

$$(\pi^*_{x,z}(f)g)(r) = \overline{f(r\cdot x)}g(r) = (\pi_{x,z}(\overline{f})g)(r).$$

So $(\pi_{x,z}, L^2_{x,z}(\mathbb{R}))$ is a representation of $C(X)$.

Proposition 4.3.1. *$(\pi_{x,z}, \hat{u}_{x,z})$ is a covariant representation of the C^*-dynamical system $(C(X), \mathbb{R}, \alpha)$.*

Proof. We have to show that for all $f \in C(X)$ and $t \in \mathbb{R}$ the following holds.

$$\pi_{x,z}(\alpha_t(f)) = \hat{u}_{x,z}(t)\pi(f)\hat{u}^*_{x,z}(t),$$

or equivalently

$$\pi_{x,z}(\alpha_t(f))\hat{u}_{x,z}(t) = \hat{u}_{x,z}(t)\pi_{x,z}(f).$$

Let $g \in L^2_{x,z}(\mathbb{R})$, then

$$(\hat{u}_{x,z}(t)g)(r) = g(-t+r),$$

hence

$$\begin{aligned}(\pi_{x,z}(\alpha_t(f))\hat{u}_{x,z}(t)g)(r) &= (\alpha_t f)(r\cdot x)(\hat{u}_{x,z}(t)g)(r)\\ &= (\alpha_t f)(r\cdot x)g(-t+r)\\ &= f((-t+r)\cdot x)g(-t+r),\end{aligned}$$

while

$$(\hat{u}_{x,z}(t)\pi_{x,z}(f)g)(r) = (\pi_{x,z}(f)g)(-t+r) = f((-t+r)\cdot x)g(-t+r).$$

□

By Theorem 2.3.1 the covariant representation $(\pi_{x,z}, \hat{u}_{x,z})$ gives rise to a representation $\pi_{x,z} \rtimes \hat{u}_{x,z}$ of $C(X) \rtimes_\alpha \mathbb{R}$, which we will denote by $\hat{\pi}_{x,z}$.

To summarize our results: We can associate to each periodic point x in X and each irreducible unitary representation $u_{x,z}$ of $p\mathbb{Z}$, where p is the period of x, a representation $(\hat{\pi}_{x,z}, L^2_{x,z}(\mathbb{R}))$ on $C(X) \rtimes_\alpha \mathbb{R}$.

Lemma 4.3.2. *Let x be a periodic point of the flow (X,ϕ) with period p, $z \in \mathbb{T}$. If x and y have the same orbit, then $\hat{\pi}_{x,z}$ is unitarily equivalent to $\hat{\pi}_{y,z}$.*

Proof. The representation $\hat{\pi}_{x,z}$ is unitarily equivalent to $\hat{\pi}_{y,z}$ if and only if their corresponding covariant representations $(\pi_{x,z}, \hat{u}_{x,z})$ and $(\pi_{y,z}, \hat{u}_{y,z})$ are unitarily equivalent. Note that the unitary representations $\hat{u}_{x,z}$ and $\hat{u}_{y,z}$ are the same, since they only depend on the period of x and y, and those are equal. Likewise, $L^2_{x,z}(\mathbb{R}) = L^2_{y,z}(\mathbb{R})$ for the same reason. Now we need to show that there is a unitary operator $W \in B(L^2_{x,z}(\mathbb{R}))$, such that

$$W\pi_{x,z}(f) = \pi_{y,z}(f)W \text{ for all } f \in C(X), \tag{4.5}$$

and

$$W\hat{u}_{x,z}(t) = \hat{u}_{y,z}(t)W \text{ for all } t \in \mathbb{R}. \tag{4.6}$$

Let $r \in [0,p)$ such that $r\cdot x = y$. We then define $W = \hat{u}_{x,z}(-r)$. Then by covariance of $(\pi_{x,z}, \hat{u}_{x,z})$

$$\begin{aligned}W\pi_{x,z}(f) &= \hat{u}_{x,z}(-r)\pi_{x,z}(f)\\ &= \pi_{x,z}(\alpha_{-r}(f))\hat{u}_{x,z}(-r).\end{aligned}$$

And for $g \in L^2_{x,z}(\mathbb{R})$ and $t \in \mathbb{R}$, we have

$$\begin{aligned}(\pi_{x,z}(\alpha_{-r}(f))g)(t) &= (\alpha_{-r}(f))(t\cdot x)g(t)\\ &= f((t+r)\cdot x)g(t)\\ &= f(t\cdot y)g(t)\\ &= (\pi_{y,z}(f)g)(t).\end{aligned}$$

So

$$\pi_{x,z}(\alpha_{-r}(f)) = \pi_{y,z}(f),$$

And hence

$$W\pi_{x,z}(f) = \pi_{y,z}(f)\hat{u}_{x,z}(-r) = \pi_{y,z}(f)W.$$

Also,

$$\begin{aligned} W\hat{u}_{x,z}(t) &= \hat{u}_{x,z}(r)\hat{u}_{x,z}(t) = \hat{u}_{x,z}(r+t) \\ &= \hat{u}_{x,z}(t)\hat{u}_{x,z}(r) = \hat{u}_{x,z}(t)W = \hat{u}_{y,z}(t)W, \end{aligned}$$

so W satisfies both (4.5) and (4.6). Hence $\hat{\pi}_{x,z}$ and $\hat{\pi}_{y,z}$ are unitarily equivalent. □

Lemma 4.3.3. *Let x and y be periodic points of the flow (X, Φ), such that x and y do not have the same orbit, and let z_1 and z_2 be in $\mathbb{T}$. Then $\hat{\pi}_{x,z_1}$ is not unitarily equivalent to $\hat{\pi}_{y,z_2}$.*

Proof. Urysohn's Lemma ([9, 4.15]) states that for any closed disjoint subsets A and B of X there exists an $f \in C(X)$ such that $f = 0$ on A and $f = 1$ on B.

Now, since $O_\phi(x)$ and $O_\phi(y)$ are closed disjoint subsets of X, there is an $f \in C(X)$ such that $f = 0$ when restricted to $O_\phi(x)$, and $f = 1$ when restricted to $O_\phi(y)$.

Suppose that $\hat{\pi}_{x,z_1}$ is unitarily equivalent to $\hat{\pi}_{y,z_2}$. Then $\pi_{x,z_1}(f)$ is the zero operator on L^2_{x,z_1}, while $\pi_{y,z_2}(f)$ is the identity on L^2_{y,z_2}, hence π_{x,z_1} and π_{y,z_2} cannot be unitarily equivalent. □

It will also be useful, just like in the discrete case, to look at the second model of induced representations. We define the map s from $\mathbb{R}/p\mathbb{Z}$ to $\mathbb{R}$ by

$$s : \bar{r} \to r,$$

where $0 \leq r < p$. Then $\pi \circ s(\bar{r}) = \bar{r}$ for all $\bar{r} \in \mathbb{R}/p\mathbb{Z}$, where π is the projection map from $\mathbb{R}$ onto $\mathbb{R}/p\mathbb{Z}$. It is clear that s is a Borel function, hence it is a Borel cross-section.

Now, we consider the map W we introduced in the previous section. It sends a function $f \in B_{x,z}(\mathbb{R})$ to the function g_f on $\mathbb{R}/p\mathbb{Z}$, defined by

$$g_f(\bar{r}) = f(s(\bar{r})) = f(r), \text{ where } r \in [0, p).$$

Then, as stated earlier, W becomes an unitary operator in $B(L^2_{x,z}(\mathbb{R}), L^2((\mathbb{R}/p\mathbb{Z}), \mathbb{C}, \mu))$, where μ is the Haar measure on $\mathbb{R}/p\mathbb{Z}$. For convenience we will write $L^2(\mathbb{R}/p\mathbb{Z})$ instead of $L^2(\mathbb{R}/p\mathbb{Z}, \mathbb{C}, \mu)$.

When we work out (4.2), with $u = u_{x,z}$, we get the following unitary representation of $\mathbb{R}$.

$$(\tilde{u}_{x,z}(t)g)(\bar{r}) = z^{k(\bar{r},t)}g(\overline{r-t}),$$

where $g \in L^2(\mathbb{R}/p\mathbb{Z}), t \in \mathbb{R}, \bar{r} \in \mathbb{R}/p\mathbb{Z}$ and

$$k(\bar{r}, t) = \frac{1}{p}(-s(\bar{r}) + t + s(\overline{r-t})) \in \mathbb{Z}.$$

Then $\tilde{u}$ is unitarily equivalent to $\hat{u}$ by construction.

In the second model we get a representation $\psi_{x,z}$ of $C(X)$ on $L^2(\mathbb{R}/p\mathbb{Z})$ by putting

$$\psi_{x,z}(f)g = W\pi_{x,z}(f)W^*(g) \text{ with } f \in C(X) \text{ and } g \in L^2(\mathbb{R}/p\mathbb{Z}),$$

so

$$(\psi_{x,z}(f)g)(\bar{r}) = f(r \cdot x)g(\bar{r}) \text{ for all } \bar{r} \in \mathbb{R}/p\mathbb{Z}, f \in C(X), g \in L^2(\mathbb{R}/p\mathbb{Z}).$$

Actually, since both $\psi_{x,z}$ and its associated Hilbert space no longer depend on z, we can simply write ψ_x for this representation. Then $(\psi_x, \tilde{u}_{x,z}, L^2(\mathbb{R}/p\mathbb{Z}))$ is a covariant representation of $(C(X), \mathbb{R}, \alpha)$ that is unitarily equivalent to $(\pi_{x,z}, \hat{u}_{x,z}, L^2_{x,z}(\mathbb{R}))$. Hence the representation $\tilde{\pi}_{x,z} := \psi_x \rtimes \tilde{u}_{x,z}$ of $C(X) \rtimes_\alpha \mathbb{R}$ is unitarily equivalent to $\hat{\pi}_{x,z}$.

What we want to do now is prove the following theorem.

Theorem 4.3.4. *Let x be a periodic point in X with period $p > 0$ and let z_1, z_2 be in $\mathbb{T}$. Then $\tilde{\pi}_{x,z_1}$ is unitarily equivalent to $\tilde{\pi}_{x,z_2}$ if and only $z_1 = z_2$. Moreover, $\tilde{\pi}_{x,z}$ is an irreducible representation for all $z \in \mathbb{T}$.*

Before we prove this, we first need some ingredients, coming from Arveson [1].

Let Y be a metric space. We define $\mathcal{B}(Y)$ to be the set of bounded complex valued Borel functions on Y. Then $\mathcal{B}(Y)$ becomes a commutative C^*-algebra with pointwise sum and multiplication and norm $\|h\| = \sup_y |h(y)|$, containing $C(Y)$ as a C^*-subalgebra. Let ν be a finite Borel measure on Y, then each function $h \in \mathcal{B}(Y)$ gives rise to a multiplication operator L_h on $L^2(Y, \nu)$, by

$$L_h g(y) = h(y)g(y), \text{ where } g \in L^2(Y, \nu) \text{ and } y \in Y.$$

Then the map $h \to L_h$ is a representation of $\mathcal{B}(Y)$ on $L^2(Y, \nu)$. We will denote the restriction of this map to $C(Y)$ by π_ν; this then becomes a representation of $C(Y)$ on $L^2(Y, \nu)$. Let $\mathcal{Z}$ denote the set of all multiplications

$L_h, h \in \mathcal{B}(Y)$, then clearly $\mathcal{Z}$ is a commutative *-algebra. Recall that the *commutant* of a set operators $D \subset B(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, is defined as

$$D' := \{T \in B(\mathcal{H}) | TS = ST \text{ for all } S \in D\}.$$

Then in [1, Theorem 2.2.1] the following theorem is proven.

Theorem 4.3.5. *With the above notation, the sets $\mathcal{Z}$, $\mathcal{Z}'$ and the strong closure of $\pi_\nu(C(Y))$ are identical.*

Also, we need the following lemma.

Lemma 4.3.6. *Let $\mathcal{H}$ be a Hilbert space, A a subset of $B(\mathcal{H})$, and $\bar{A}$ the strong closure of A. Then $A' = (\bar{A})'$.*

Proof. Since $A \subset \bar{A}$, it is clear that $(\bar{A})' \subset A'$. To prove the converse, we let $V \in A'$, and $W \in \bar{A}$, and need to show that $VW = WV$. Since $W \in \bar{A}$, there is a net $\{W_i\}_{i \in I} \subset A$, such $\lim_i W_i = W$ in the strong topology, hence for every $h \in \mathcal{H}$, $\lim_i W_i h = Wh$. Since $V \in B(\mathcal{H})$, it is continuous, hence $\lim_i V(W_i h) = V(\lim_i W_i h)$. So for every $h \in \mathcal{H}$, we have

$$W(Vh) = \lim_i W_i(Vh) = \lim_i V(W_i h) = V(\lim_i W_i h) = V(Wh).$$

So V commutes with all $W \in \bar{A}$, hence $V \in (\bar{A})'$.

$\square$

We will use this lemma and the previous theorem to prove the following lemma.

Lemma 4.3.7. *Let x be a periodic point in X, with period $p > 0$. Let V be an operator in $B(L^2(\mathbb{R}/p\mathbb{Z}))$ such that*

$$V\psi_x(f) = \psi_x(f)V \text{ for all } f \in C(X),$$

then $V = L_h$ for some $h \in \mathcal{B}(\mathbb{R}/p\mathbb{Z})$.

Proof. First of all, $Y := \mathbb{R}/p\mathbb{Z}$ is a metric space, since it is homeomorphic to $\mathbb{T}$. And $L^2(\mathbb{R}/p\mathbb{Z}) = L^2(\mathbb{R}/p\mathbb{Z}, \mu)$, where μ is a Haar measure (hence a finite Borel measure).
Claim: $\pi_\mu(C(Y)) = \psi_x(C(X))$.
Proof of claim:
Both of the spaces consist of multiplication operators on $L^2(\mathbb{R}/p\mathbb{Z}, \mu)$. The space $\pi_\mu(C(Y))$ consists of operators that multiply with functions in $C(Y)$,

whereas operators in $\psi_x(C(X))$ multiply with functions of the form $\bar{r} \to f(r \cdot x)$, where $f \in C(X)$. Now, if $f \in C(X)$, then $\bar{r} \to f(r \cdot x)$ is continuous, for it is a composition of the map $\eta : \mathbb{R}/p\mathbb{Z} \to O_\phi(x)$ that sends $\bar{r}$ to $r \cdot x$, which is a homeomorphism by Proposition 4.1.6, and the map $f|_{O_\phi(x)}$, which is continuous by continuity of f.

Now let $g \in C(Y)$. Then, by composing with the continuous map η^{-1}, we get a continuous map $g \circ \eta^{-1} \in C(O_\phi(x))$. We now use the Tietze Extension theorem ([9, 4.16]), which tells us that for any $A \subset X$ closed, and $f \in C(A)$, there exists an $F \in C(X)$ such that $F|_A = f$.
So, since $O_\phi(x)$ is closed in X, we know that there exists a $f \in C(X)$, such that $f|_{O_\phi(x)} = g \circ \eta^{-1}$, so that $\pi_\mu(g) = \psi_x(f)$. Hence $g = f|_{O_\phi(x)} \circ \eta$ sends $\bar{r} \to f(r \cdot x)$.

So $\psi_x(C(X)) = \pi_\mu(C(Y))$.

Now, by assumption V commutes with $\psi_x(C(X))$. So $V \in \psi_x(C(X))' = \pi_\mu(C(Y))'$. From Lemma 4.3.6 we see that V is in the commutant of the strong closure of $\pi_\mu(C(Y))$, which we will denote by $\overline{\pi_\mu(C(Y))}$.

Just like before, we let $\mathcal{Z}$ denote the set of all multiplications $L_h, h \in \mathcal{B}(Y)$. Then by Theorem 4.3.5 we get that $\mathcal{Z} = \mathcal{Z}' = \overline{\pi_\mu(C(Y))}$. So $V \in \overline{\pi_\mu(C(Y))}' = \mathcal{Z}' = \mathcal{Z}$. Hence $V = L_h$ for some $h \in \mathcal{B}(Y)$. □

Now we can finally prove Theorem 4.3.4.

Proof. (of Theorem 4.3.4) When $z_1 = z_2$, then $\hat{\pi}_{x,z_1} = \hat{\pi}_{x,z_2}$, hence they are unitarily equivalent. Suppose that there is an operator $V \in B(L^2(\mathbb{R}/p\mathbb{Z}))$, such that

$$V\psi_x(f) = \psi_x(f)V \text{ for all } f \in C(X),$$

and

$$V\tilde{u}_{x,z_1}(t) = \tilde{u}_{x,z_2}(t)V \text{ for all } t \in \mathbb{R}.$$

We will show that such an operator cannot be unitary if $z_1 \neq z_2$, which implies that $\tilde{\pi}_{x,z_1}$ is not unitarily equivalent to $\tilde{\pi}_{x,z_2}$.

Lemma 4.3.7 implies that $V = L_h$ for a $h \in \mathcal{B}(\mathbb{R}/p\mathbb{Z})$. Then for $g \in L^2(\mathbb{R}/p\mathbb{Z})$ and $\bar{r} \in \mathbb{R}/p\mathbb{Z})$ we have

$$V(\tilde{u}_{x,z_1}(t)g)(\bar{r}) = h(\bar{r})z_1^{k(\bar{r},t)}g(\overline{r-t}),$$

and

$$\tilde{u}_{x,z_2}(t)(V(g))(\bar{r}) = z_2^{k(\bar{r},t)}h(\overline{r-t})g(\overline{r-t}).$$

Recall that

$$k(\bar{r},t) = \frac{1}{p}(-s(\bar{r}) + t + s(\overline{r-t})).$$

So by assumption we have

$$h(\bar{r})z_1^{k(\bar{r},t)}g(\overline{r-t}) = z_2^{k(\bar{r},t)}h(\overline{r-t})g(\overline{r-t}). \tag{4.7}$$

Now, we take $t = p$. Then

$$k(\bar{r},t) = k(\bar{r},p) = \frac{1}{p}(-s(\bar{r}) + p + s(\overline{r-p})) = \frac{1}{p}(-s(\bar{r}) + p + s(\bar{r})) = 1.$$

If we put this into equation (4.7) a we get

$$h(\bar{r})z_1 g(\bar{r}) = z_2 h(\bar{r})g(\bar{r}).$$

These two must, for all $g \in L^2(\mathbb{R}/p\mathbb{Z})$, be equal, thus also for the identity $\mathbf{1}$ in $L^2(\mathbb{R}/p\mathbb{Z})$.

Then we get that $h(\bar{r})z_1 = z_2 h(\bar{r})$ for almost every $\bar{r} \in L^2(\mathbb{R}/p\mathbb{Z})$. If $z_1 \neq z_2$, then this is only true if $h(\bar{r}) = 0$ for almost every $\bar{r} \in L^2(\mathbb{R}/p\mathbb{Z})$. But then L_h is not unitary.

So $\tilde{\pi}_{x,z_1}$ and $\tilde{\pi}_{x,z_2}$ are not unitarily equivalent if and only if $z_1 \neq z_2$.

Now it remains to show that $\hat{\pi}_{x,z_1}$ is irreducible. This is the case if and only if the only operators that commute with both $\tilde{u}_{x,z_1}$ and ψ_x are scalar multiples of the identity (by Corollary 2.1.10). Let V be such an operator, then we have, again, that (4.7) has to be true, with $z_2 = z_1$. If we again take $g = \mathbf{1}$ and fix $t \in \mathbb{R}$, then we get $h(\bar{r}) = h(\overline{r-t})$ for almost every $\bar{r} \in \mathbb{R}/p\mathbb{Z}$. Hence h is constant almost everywhere, hence $V = L_h$ is a scalar multiple of the identity

So indeed we can conclude that $\hat{\pi}_{x,z_1}$ is irreducible.

□

We summarize the results in this section in the following theorem.

Theorem 4.3.8. *Let x, y be periodic points of the flow (X, Φ) and let z_1, z_2 be in $\mathbb{T}$. Then $\hat{\pi}_{x,z_1}$ is unitarily equivalent to $\hat{\pi}_{y,z_2}$ if and only if x and y have the same orbit and $z_1 = z_2$. Moreover, $\hat{\pi}_{x,z}$ is an irreducible representation for all $z \in \mathbb{T}$.*

4.3.2 Case II: period $p = 0$

Now, suppose x is a fixed point of the flow (X, Φ). Then $\mathbb{R}_x = \mathbb{R}$. We can now try to go through the same construction as before. First, let $(u, \mathcal{H})$ be an irreducible unitary representation of $\mathbb{R}_x$. Then, since $\mathbb{R}_x = \mathbb{R}$ is abelian, Lemma 3.3.2 tells us that $\mathcal{H} = \mathbb{C}$ and then $B(\mathbb{C}) = \mathbb{C}$. So u is a continuous homomorphism from $\mathbb{R}$ to $\mathbb{C}$. Then there is a $\lambda \in \mathbb{R}$, such that

$$u(r) = e^{i\lambda r}, \text{ with } r \in \mathbb{R}.$$

The proof of this can be found in [8, Theorem 4.5 a]

Since u depends only on λ, we will write u_λ to denote this representation.

The space $\mathbb{R}/\mathbb{R}_x = \{\bar{0}\}$ consists of one point. So every complex-valued function on $\{\bar{0}\}$ is of the form $f : \bar{0} \to z$, for some $z \in \mathbb{C}$. So the space $L^2(\mathbb{R}/\mathbb{R}_x, \mu)$, where μ is the Haar measure on the space $\{\bar{0}\}$, is simply $\mathbb{C}$ itself.

We get a representation π_x of $C(X)$ on $L^2(\mathbb{R}/\mathbb{R}_x, \mu)$ in the same way as before.

$$\pi_x(f)g(\bar{0}) = f(0 \cdot x)g(\bar{0}), \text{ with } f \in C(X), g \in L^2(\mathbb{R}/\mathbb{R}_x, \mu), \bar{0} \in \mathbb{R}/\mathbb{R}_x.$$

And we can write this simply as

$$\pi_x(f) = f(x), \text{ where } f \in C(X).$$

Then (π_x, u_λ) is a covariant representation: Let $r \in \mathbb{R}, f \in C(X)$, then

$$u_\lambda(r)\pi_x(f)u_\lambda(-r) = e^{i\lambda r} f(x) e^{-i\lambda r} = f(x) = f(-r \cdot x) = \pi_x(\alpha_r f).$$

Therefore we get a representation $\hat{\pi}_{x,\lambda} := \pi_x \rtimes u_\lambda$ of $C(X) \rtimes_\alpha \mathbb{R}$; this representation is automatically irreducible, since its associated Hilbert space is $\mathbb{C}$. Also we get similar results as in case I, though these do not require as much work.

Theorem 4.3.9. *Let $x, y \in X$ be fixed points and $\lambda_1, \lambda_2 \in \mathbb{R}$. Then $\hat{\pi}_{x,\lambda_1}$ is unitarily equivalent to $\hat{\pi}_{x,\lambda_2}$ if and only if $x = y$ and $\lambda_1 = \lambda_2$.*

Proof. Suppose that $\hat{\pi}_{x,\lambda_1}$ is unitarily equivalent to $\hat{\pi}_{x,\lambda_2}$. Then there is a unitary operator $W \in B(\mathbb{C})$ (so $W = z \in \mathbb{T}$), such that for every $f \in C(X)$ and $r \in \mathbb{R}$

$$W\pi_x(f) = \pi_y(f)W,$$

and

$$Wu_{\lambda_1}(r) = u_{\lambda_2}(r)W.$$

So we get

$$zf(x) = f(y)z$$

and

$$ze^{i\lambda_1 r} = e^{i\lambda_2 r}z.$$

Since $z \neq 0$, it follows that for every $f \in C(X)$ and $r \in \mathbb{R}$ $f(x) = f(y)$ and $e^{i\lambda_1 r} = e^{i\lambda_2 r}$. Then clearly $\lambda_1 = \lambda_2$ and $x = y$. $\square$

Moreover, it is also true that every one-dimensional irreducible representation of $C(X) \rtimes_\alpha \mathbb{R}$ is unitarily equivalent to an induced representation $\tilde{\pi}_{x,\lambda}$ for some fixed point $x \in X$ and $\lambda \in \mathbb{R}$.

Theorem 4.3.10. *Let $\bar{\pi} = \pi \rtimes u$ be an irreducible representation of $C(X) \rtimes_\alpha \mathbb{R}$ on the Hilbert space $\mathbb{C}$. Then $\bar{\pi} = \tilde{\pi}_{x,\lambda}$, with $x \in X$ a fixed point and $\lambda \in \mathbb{R}$.*

Proof. Since u is a one-dimensional unitary representation of $\mathbb{R}$, it is necessarily irreducible and of the form $r \to e^{i\lambda r}$ for some $\lambda \in \mathbb{R}$. And π is a non-zero *-homomorphism of $C(X)$ into $\mathbb{C}$, hence $\pi \in \widehat{C(X)}$. Then there is an $x \in X$, such that $\pi(f) = f(x)$ for all $f \in C(X)$, which follows from [2, Theorem 1.10.4]. And then for all $r \in \mathbb{R}$ we have

$$f(-r \cdot x) = \pi(\alpha_r f) = u(r)\pi(f)u(-r) = e^{i\lambda r} f(x) e^{-i\lambda r} = f(x).$$

Hence $-r \cdot x = x$ for all $r \in \mathbb{R}$, hence x is a fixed point. Now we have that $\hat{\pi} = \tilde{\pi}_{x,\lambda}$. $\square$

Chapter 5

Topics for further research

Of course, this is not all that we can tell about the interplay between flows and C^*-algebras. In fact, it is just a beginning. Here are some suggestions for further research. The results on discrete dynamical systems and their associated C^*-algebras serve as motivation.

- In the case of discrete dynamical systems, the induced representations arising from a periodic point with period p and an irreducible representation of the isotropy subgroup of that point, are p-dimensional, and in fact any irreducible p-dimensional representation of the crossed product C^*-algebra are unitarily equivalent to such an associated representation.
  In the case of flows, can we also find some characterization of the structure of the induced representations?

- In the discrete case, the GNS-representations of pure state extensions on the crossed product C^*-algebra, coming from certain states on $C(X)$, are unitarily equivalent to induced representations arising from a periodic point and an irreducible representation of the isotropy subgroup of that point. Can something similar be shown in the case of flows? Then $C(X)$ can no longer be embedded in $C(X) \rtimes_\alpha \mathbb{R}$, so we have to work with the multiplier algebra of the crossed product C^*-algebra.

- In the discrete case, the dynamical system is minimal if and only if its associated C^*-algebra is simple. Is this also true in the case of flows?

Appendix A

Very short course on C*-algebras

In this appendix we will give a short introduction on Banach algebras and C*-algebras. The basic definitions and some basic results will be stated, mostly without proof. There are numerous introductory books on this subject, which cover the basic theory of C*-algebras far better than such a short appendix, and we refer those who have not yet been immersed in the wondrous world of C*-algebras and wish to learn more, to books such as [12] by Murphy, [5] by Davidson or [1] by Arveson.

Throughout this appendix the ground field for all vector spaces and algebras is the complex field $\mathbb{C}$.

A.1 Basic definitions and examples

In this section we will give the basic definitions and discuss some relevant examples.

Definition A.1.1. A *Banach algebra* is a Banach space A together with a map (multiplication)

$$A \times A \to A,\ (a, b) \to ab,$$

such that

$$(\lambda a + \mu b)c = \lambda ac + \mu bc,$$
$$a(\lambda b + \mu c) = \lambda ab + \mu ac,$$
$$a(bc) = (ab)c,$$

and

$$\|ab\| \leq \|a\|\|b\|$$

for all $a, b, c \in A, \lambda, \mu \in \mathbb{C}$.

The last condition will ensure that multiplication is norm-continuous.

A Banach algebra is called *unital* if it has a unit element, i.e., an element *1*, such that $1a = a1 = a$ for all $a \in A$.

Definition A.1.2. A Banach *-algebra A is a Banach algebra endowed with an *involution*, i.e., a map $^* : A \to A$, sending a to $a^* \in A$, such that

$$(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^*,$$

$$(a^*)^* = a,$$

and

$$(ab)^* = b^* a^*$$

for all $a, b \in A, \lambda, \mu \in \mathbb{C}$. An element $a \in A$ is called *self-adjoint* if $a = a^*$, *normal* if $aa^* = a^*a$ and, if A is unital, *unitary* if $a^*a = aa^* = 1$.

Finally we arrive at the C^*-algebra itself.

Definition A.1.3. A C^*-algebra A is a Banach *-algebra, with the following property.

$$\|a^*a\| = \|a\|^2 \text{ for all } a \in A.$$

We will call this property the C^*-property.

Before delving into the theory of C^*-algebras, we will give some important examples.

Example A.1.4. Let X be a locally compact Hausdorff space, and $C_0(X)$ the space of continuous complex-valued functions f on X that *vanish at infinity*, i.e., with the property that for every $\epsilon > 0$ there is a compact set $K \subset X$ such that $|f(x)| < \epsilon$ whenever x is outside of K. $C_0(X)$ is a Banach space with respect to the *supremum norm.*

$$\|f\|_\infty = \sup_{x \in X} \|f(x)\| \text{ for every } f \in C_0(X).$$

We can define multiplication by *pointwise multiplication.*

$$(fg)(x) = f(x)g(x) \text{ for all } f, g \in C_0(X), x \in X,$$

and then it is easy to check it satisfies the conditions necessary for a Banach algebra. We can define an involution on $C_0(X)$ in a natural way.

$$f^*(x) = \overline{f(x)} \text{ for all } f \in C_0(X).$$

Then $C_0(X)$ satisfies all the requiring properties, including the C^*-property. This makes $C_0(X)$ a commutative C^*-algebra. Observe that $C_0(X)$ contains the identity function, sending everything to 1, if and only if X is compact, and then $C_0(X) = C(X)$.

Example A.1.5. Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ the space of bounded linear operators on $\mathcal{H}$. $B(\mathcal{H})$ is then also a Banach space, having the operator norm

$$\|S\| = \sup_{\|x\|\leq 1} \|S(x)\| \text{ for every } S \in B(\mathcal{H})$$

We can define multiplication on $B(\mathcal{H})$ by composition.

$$(ST)(x) := S(T(x)) \text{ for all } S, T \in B(\mathcal{H}), x \in \mathcal{H}.$$

This multiplication satisfies the conditions necessary for a Banach algebra. We can also look at the adjoint of an operator $A \in B(\mathcal{H})$, i.e., the (unique) operator A^* such that $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for every $x, y \in \mathcal{H}$. This involution has the required properties, and it satisfies the C^*-property, which means that $B(\mathcal{H})$ is a C^*-algebra. It is non-commutative unless $\mathcal{H}$ is one dimensional or zero dimensional.

Definition A.1.6. A *-*homomorphism* between two Banach *-algebras is a map that preserves sum, scalar multiplication, product and adjoint. A *-*isomorphism* is a bijective *-homomorphism.

When we talk about homomorphisms and isomorphisms between Banach *-algebras we will mean *-homomorphisms and *-isomorphisms.

A.2 Into the wondrous world of C*-algebras

The C^*-property may seem like just a simple equation, one small condition added to a long list of conditions necessary for a Banach *-algebra, but it is an extremely powerful and important condition. It ensures that all homomorphisms between C^*-algebras *must* be norm decreasing. Any injective homomorphism is then automatically isometric. Furthermore, the norm on a C^*-algebra is *unique*, in the sense that there can be no other norm which still makes it a C^*-algebra. The two examples we gave before are very important classes of C^*-algebras.
A very fascinating fact is that the only *commutative* C^*-algebras are of the form $C_0(X)$, by which we mean that for any commutative C^*-algebra A there is an isomorphism that maps A onto $C_0(X)$ for some locally compact Hausdorff space X. And this isomorphism can be realized in a very concrete way, as we will now explain.

Definition A.2.1. The *spectrum* $\hat{A}$ of a commutative C^*-algebra A is the set of non-zero homomorphisms of A into $\mathbb{C}$.

Under the weak*-topology, $\hat{A}$ becomes a locally compact Hausdorff space, and $\hat{A}$ is compact if and only if A is unital.

Definition A.2.2. For each a in a commutative C^*-algebra A, we define a function $\hat{a}$ on $\hat{A}$ by

$$\hat{a}(\omega) = \omega(a) \text{ for all } \omega \in \hat{A}.$$

The *Gelfand transform* on a commutative C^*-algebra A is then the map that sends $x \in A$ to $\hat{x}$.

Theorem A.2.3. (Gelfand-Naimark) *The Gelfand transform on a commutative C^*-algebra A is an isomorphism between A and $C_0(\hat{A})$.*

We also have the following theorem.

Theorem A.2.4. *Let X,Y be compact Hausdorff spaces. Then X is homeomorphic to Y if and only if $C(X)$ is isomorphic to $C(Y)$.*

An important tool for C^*-algebras is representation theory.

Definition A.2.5. A *representation* of a C^*-algebra A is a pair $(\pi, \mathcal{H})$ consisting of a Hilbert space $\mathcal{H}$ and a homomorphism π of A into the C^*-algebra $B(\mathcal{H})$. An injective representation is also called *faithful.* A representation $(\pi, \mathcal{H})$ of A is called *non-degenerate* if $\pi(A)\mathcal{H}$ is a dense subset of $\mathcal{H}$. Unless mentioned otherwise, we will assume a representation to be non-degenerate.

When the Hilbert space $\mathcal{H}$ is known from the context, we will write π instead of $(\pi, \mathcal{H})$. Since a representation of a C^*-algebra is a homomorphism between C^*-algebras, it is necessarily norm-decreasing.

Definition A.2.6. A representation $(\pi, \mathcal{H})$ of a C^*-algebra A is called *irreducible* if there is no proper closed subspace M of $\mathcal{H}$, such that $\pi(A)M \subset M$. Two representations $(\pi, \mathcal{H})$ and $(\psi, \mathcal{K})$ of A are *unitarily equivalent* if there is a unitary operator $u : \mathcal{H} \to \mathcal{K}$ such that $\psi(a) = u\pi(a)u^*$ for all $a \in A$.
A representation $(\pi, \mathcal{H})$ of A is called *cyclic* if there is an $h \in \mathcal{H}$ such that the set $\pi(A)h$ is dense in $\mathcal{H}$. Then h is called a *cyclic vector* for π. We say that a cyclic representation $(\pi, \mathcal{H})$ of A with cyclic vector h is unitarily equivalent to a cyclic representation $(\psi, \mathcal{K})$ with cyclic vector k if there is a unitary operator $u : \mathcal{H} \to \mathcal{K}$ such that $\psi(a) = u\pi(a)u^*$ for all $a \in A$ *and* $u(h) = k$.

Then, for general C^*-algebras, we have the following theorem.

Theorem A.2.7. (Gelfand-Naimark) *Let A be a C^*-algebra. Then A has a faithful representation.*

This is a very strong result. It means that A is isomorphic to $\pi(A) \subset B(\mathcal{H})$, hence *any* C^*-algebra is isomorphic to a self-adjoint, closed subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. In the proof of this theorem, the following linear functionals play an important part.

Definition A.2.8. A linear functional $\phi : A \to \mathbb{C}$ is called *positive* if $\phi(a^*a) \geq 0$ for every $a \in A$. A *state* on A is a positive linear functional of norm one.

The states form a convex set of linear functionals on A. An extreme point of this set is called a *pure state*. Hence a state ϕ is a pure state if and only if for any two states ϕ_1, ϕ_2 and every $t \in (0,1)$, the condition $\phi = t\phi_1 + (1-t)\phi_2$ implies $\phi_1 = \phi_2 = \phi$.

Example A.2.9. If $(\pi, \mathcal{H})$ is a representation of A and $h \in \mathcal{H}$ then

$$\phi(a) := \langle \pi(a)h, h \rangle \text{ for all } a \in A$$

is a positive linear functional on A. Furthermore, this is a state if and only if $\|h\| = 1$.

Conversely, states can be used to construct representations. This procedure is called the *GNS construction*, named after Gelfand, Naimark and Segal, and is a vital ingredient for the proof of Theorem A.2.7.

Theorem A.2.10. *Let A be a C^*-algebra.*

1. *To any state ϕ on A there corresponds a cyclic representation $(\pi_\phi, \mathcal{H})$ of A with cyclic unit vector h_ϕ such that*

   $$\phi(a) = \langle \pi_\phi(a)h_\phi, h_\phi \rangle \textit{ for all } a \in A.$$

   This correspondence is a bijection between the states on A and the unitary equivalence classes of cyclic representations of A with a given cyclic unit vector.

2. *With this bijection, the* pure *states correspond with the unitary equivalence classes of* irreducible *cyclic representations with a given cyclic unit vector.*

Appendix B

Haar measure and integration

The goal of this appendix is to give an overview of Haar measures and integration. We will not give any proofs in this section. The proofs, and further details, can be found in [9], [8] and [21], among others.

First we will give some preliminary definitions.
A *topological group* G is both a group, and a topological Hausdorff space such that
a) the map $(s, r) \to sr$ is continuous, and
b) the map $s \to s^{-1}$ is continuous.

A *locally compact group* is a topological group G, for which the underlying topology is locally compact, i.e., every element of G has a compact neighbourhood.

A *Borel measure* on a locally compact group G is a measure μ on the *Borel σ-algebra* on G, i.e., the σ-algebra generated by the family of open sets in G. If furthermore we have that for every open $V \subset G$,

$$\mu(V) = \sup\{\mu(C) | C \subset V \text{ and } C \text{ compact}\},$$

for every measurable $A \subset G$,

$$\mu(A) = \inf\{\mu(V) | A \subset V \text{ and } V \text{ open}\},$$

and for every compact $C \subset G$

$$\mu(C) < \infty,$$

then we call μ a *Radon measure*. We say that μ is *left-invariant* if

$$\mu(sA) = \mu(A) \text{ for all } s \in G \text{ and } A \text{ measurable}.$$

If $\mu(As) = \mu(A)$ we say that μ is *right-invariant*.

Definition B.1. A left *Haar measure* on a locally compact group G is a left-invariant Radon measure on that group. A right-invariant Radon measure is called a right Haar measure. We will use the term Haar measure to denote left-invariant Haar measure. If the Haar measure is also right-invariant, then we call it *bi-invariant*.

Remark B.2. We can easily construct right Haar measures from left Haar measures, and vice versa: if μ is a left Haar measure, then $\nu(A) := \mu(A^{-1})$ is a right Haar measure.

Then we have the following fundamental result.

Theorem B.3. *Every locally compact group G has a Haar measure which is unique up to a strictly positive scalar.*

Example B.4. If G is discrete, i.e., every subset of G is open, then the counting measure is a Haar measure of G. So if $G = \mathbb{Z}$, and f a complex-valued function on G with finite support, then

$$\int_{\mathbb{Z}} f(n)\, d\mu(n) = \sum_{n \in \mathbb{Z}} f(n).$$

Example B.5. If $G = \mathbb{R}^n$, then the Lebesgue measure restricted to the Borel σ-algebra is a Haar measure.

If μ is a Haar measure on G then for every non-empty open $V \subset G$, $\mu(V) > 0$.

These properties give us that $\|f\|_1 := \int_G \|f(s)\|\, d\mu(s)$ defines a norm on $C_c(G)$, the continuous complex valued functions on G. We will denote $L^1(G)$ for the completion of $C_c(G)$ with respect to this norm. The left-invariance of μ then gives us that for every $f \in L^1(G)$

$$\int_G f(rs)\, d\mu(s) = \int_G f(s)\, d\mu(s).$$

A left Haar measure need not be a right Haar measure. The following theorem relates them.

Theorem B.6. *Let μ be a Haar measure on a locally compact group G and $\mathbb{R}^+$ the multiplicative group of positive real numbers. Then there is a* *continuous homomorphism* $\Delta : G \to \mathbb{R}^+$ *such that*

$$\Delta(r) \int_G f(sr)\, d\mu(s) = \int_G f(s)\, d\mu(s)$$

for all $f \in L^1(G)$. *The function* Δ *is independent of choice of Haar measure and is called the modular function on* G.

From this theorem we see that a Haar measure μ on a group G is bi-invariant if and only if $\Delta \equiv 1$; such groups are called *unimodular.* It is clear that every abelian group is unimodular. Every compact group G is also unimodular: Δ is a continuous homomorphism from G to $\mathbb{R}^+$, hence $\Delta(G)$ is a compact subgroup of $\mathbb{R}^+$. The only compact subgroup of $\mathbb{R}^+$ is $\{1\}$, so $\Delta(G) = \{1\}$, hence G is unimodular.

Appendix C

Schur's Lemma

In this appendix we will prove a theorem that we did not want to prove in the main chapters of the thesis, in order to not flood the reader with unnecessary details. However, we do feel the proof is somewhat relevant to this thesis, hence this appendix.

Definition C.1. Let $\mathcal{H}$ be a Hilbert space and $S \subset B(\mathcal{H})$. Then we call a subspace $M \subset \mathcal{H}$ *invariant* for S if $SM \subset M$.

Definition C.2. The *commutant* of a set operators $D \subset B(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, is defined as

$$D' := \{T \in B(\mathcal{H}) | \, TS = ST \text{ for all } S \in D\}.$$

Our goal in this appendix is to prove the following theorem.

Theorem C.3. *(Schur's Lemma) Let $\mathcal{H}$ be a Hilbert space and $S \subset B(\mathcal{H})$, such that $S^* = S$. Then the following two statements are equivalent.*

1. *The only closed invariant linear subspaces $M \subset \mathcal{H}$ for S are the trivial ones: $\{0\}$ and $\mathcal{H}$.*
2. *$S' = \mathbb{C}I$, with I the identity operator in $B(\mathcal{H})$.*

We first need the following lemma.

Lemma C.4. *Let $S \subset B(\mathcal{H})$, such that $S = S^*$. Then a closed linear subspace $M \subset \mathcal{H}$ is invariant for S if and only if $M^\perp$ is invariant for S.*

Proof. First we assume that M is invariant for S. Let $m^\perp \in M^\perp$ and $T \in S$, then we need to show that $T(m^\perp) \in M^\perp$. Since $S = S^*$, T^* is also in S, hence $T^*(m) \in M$ for all $m \in M$. So we have

$$\langle m, T(m^\perp)\rangle = \langle T^*(m), m^\perp\rangle = 0 \text{ for all } m \in M.$$

Hence $T(m^\perp) \in M^\perp$. This is true for all $T \in S$ and $m^\perp \in M^\perp$, hence $SM^\perp \subset M^\perp$.
Now assume that $M^\perp$ is invariant for S. Then, by the above, $(M^\perp)^\perp$ is invariant for S, and since M is a closed linear subspace of $\mathcal{H}$, $(M^\perp)^\perp = M$, hence M is invariant for S.

□

We also need a consequence of the Spectral Theorem, which can be found in [17, 12.23-12.24]. We will not further go into the Spectral Theorem itself, since details can be found in [17]. We will merely state the consequences we need.

Theorem C.5. *Let T be a normal operator in $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, then T is the limit in the operator norm of linear combinations of finitely many orthogonal projections which commute with every $V \in B(\mathcal{H})$ that commutes with T (hence these projections are in T'').*

Proof. (of Theorem C.3.) First we assume C.3.1, so let $S \subset B(\mathcal{H})$, such that $S = S^*$, and let $S' = \mathbb{C}I$. Assume that there is a closed invariant subspace $M \subset \mathcal{H}$ for S. Then, by Lemma C.4, $M^\perp$ is also invariant for S. Let $P_M \in B(\mathcal{H})$ be the orthogonal projection on M.
Let $h \in \mathcal{H}$, then $h = m + m^\perp$, with $m \in M$ and $m^\perp \in M^\perp$. Then we have, for any $T \in S$, that $T(m) \in M$ and $T(m^\perp) \in M^\perp$. Hence for any $T \in S$

$$TP_M(h) = TP_M(m + m^\perp) = T(m),$$

and

$$P_M T(h) = P_M(T(m) + T(m^\perp)) = T(m).$$

Hence P_M commutes with T for any $T \in S$. So $P_M \in S' = \mathbb{C}I$, hence $P_M = \lambda I$ for some $\lambda \in \mathbb{C}$. Since P_M is a projection, $\lambda = 0$ or $\lambda = 1$. In the first case $M = \{0\}$ and in the second case $M = \mathcal{H}$. So the only closed invariant subspaces for S are the trivial ones.

Now we assume C.3.2, so let $S \subset B(\mathcal{H})$, such that $S = S^*$, and such that the only closed invariant subspaces for S are the trivial ones. Now let $T \in S'$. We can decompose T as

$$T = \frac{T + T^*}{2} + i\frac{T - T*}{2i}.$$

Now, both $\mathrm{Re}\, T := \frac{T+T^*}{2}$ and $\mathrm{Im}\, T := \frac{T-T*}{2i}$ are self-adjoint, hence normal. And since $T \in S'$ and $S^* = S$, T^* is also in S', hence $\mathrm{Re}\, T \in S'$ and

$\operatorname{Im} T \in S'$. By Theorem C.5, $\operatorname{Re} T$ and $\operatorname{Im} T$ are the limits in the norm topology of finitely many linear combinations of orthogonal projections that are in
$(\operatorname{Re} T)''$ and $(\operatorname{Im} T)''$. Since $\operatorname{Re} T \in S'$, $(\operatorname{Re} T)'' \subset S''' = S'$. Likewise, $(\operatorname{Im} T)'' \subset S'$. Now let P be an orthogonal projection in S'. Then the image of P, which we will denote by M, is a closed linear subspace. Hence for any $m \in M$, and $R \in S$, we have

$$Rm = RP(m) = P(R(m)) \subset Q.$$

Hence M is invariant for S. So, by assumption, $M = \{0\}$ or $M = \mathcal{H}$. So $P = 0$ of $P = I$.

Hence linear combination of orthogonal projections in S' are of the form λI for some $\lambda \in \mathbb{C}$. So both $\operatorname{Re} T$ and $\operatorname{Im} T$ are limits in the operator norm of elements of $\mathbb{C}I$, hence both $\operatorname{Re} T$ and $\operatorname{Im} T$ are in $\mathbb{C}I$. Then T is also in $\mathbb{C}I$. $\square$

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