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Induced maps on Grothendieck groups

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1 Introduction

1.1 Motivation and main results

Let $\mathcal{A}$ be an abelian category. The Grothendieck group $\mathrm{K}_0(\mathcal{A})$ of $\mathcal{A}$ is defined as the quotient of the free abelian group on isomorphism classes $[X]$ of $\mathcal{A}$ by the relations

$$[X] - [Y] + [Z] = 0$$

for every exact sequence

$$0 \to X \to Y \to Z \to 0$$

in $\mathcal{A}$.

Let $\mathcal{A}, \mathcal{B}$ be abelian categories. If $F \colon \mathcal{A} \to \mathcal{B}$ is an exact functor, it clearly induces a group homomorphism $\mathrm{K}_0(\mathcal{A}) \to \mathrm{K}_0(\mathcal{B})$. If $F \colon \mathcal{A} \to \mathcal{B}$ is not exact or not even additive, it is sometimes still possible to associate to F a function on Grothendieck groups. We give two motivating examples.

If every object in $\mathcal{A}$ has a finite projective resolution and $F \colon \mathcal{A} \to \mathcal{B}$ is right exact, we can use the left-derived functors of F: the long exact sequence of homology shows that that there is a unique group homomorphism $\mathrm{K}_0(\mathcal{A}) \to \mathrm{K}_0(\mathcal{B})$ satisfying

$$[X] \mapsto \sum_{i \geq 0} (-1)^i [\mathrm{L}_i F X].$$

This can be dualised to the situation in which $\mathcal{A}$ has finite injective resolutions and F is left-exact.

For the second example, let G be a group, k a field and denote by $\mathcal{A}$ the category of finite dimensional G-representations over k. Consider for $n \geq 0$ the symmetric power functor $\Gamma^n \colon \mathcal{A} \to \mathcal{A}$ that sends a representation V to its symmetric power

$$\big(\underbrace{V \otimes_k \ldots \otimes_k V}_{n \text{ times}}\big)^{\mathrm{S}_n}$$

consisting of those elements in the n-th tensor power of V that are invariant under the action of the symmetric group S_n. For any $V \in \mathcal{A}$, we can map $[V] \in \mathrm{K}_0(\mathcal{A})$ to $[\Gamma^n V] \in \mathrm{K}_0(\mathcal{A})$ and with the use of Koszul complexes, it can be shown that this extends to a well-defined function $\mathrm{K}_0(\mathcal{A}) \to \mathrm{K}_0(\mathcal{A})$ given by

$$[V] - [W] \mapsto \sum_{i=0}^{n} (-1)^i [\Gamma^n V \otimes \bigwedge^i W].$$

For $n \geq 2$, this is not a group homomorphism.

In this thesis, we prove two new results on the existence and uniqueness of such induced maps on Grothendieck groups. To state and prove these results, we need a generalisition of the notion of derived functors to functors that are not necessarily additive, due to Dold and Puppe (see [DP61]). Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F\colon \mathcal{A} \to \mathcal{B}$ an arbitrary functor. We denote by the $\mathrm{Ch}_{\geq 0}\,\mathcal{A}$ the category of chain complexes in $\mathcal{A}$ in non-negative degree. Dold and Puppe use the Dold-Kan correspondence to replace chain complexes in non-negative degree by simplicial objects. Because simplicial homotopy is something purely combinatorial, no additivity of F is required to define in an analogous manner left derived functors

$$\mathrm{L}_i F\colon\ \mathrm{Ch}_{\geq 0}\,\mathcal{A} \to \mathcal{B}.$$

If F is left exact, then the functor $X \mapsto \mathrm{L}_i F(X[0])$ is the usual i-th derived functor. If F is not additive, then these left-derived functors map quasi-isomorphisms to isomorphisms (proposition 4.4), but we do not get long exact sequences of homology.

A functor between abelian categories is called reduced if it sends the zero object to the zero object. The first of the two main results is only applicable to reduced functors of finite degree (see definition 7.7), which roughly means they are as close to being additive functors as polynomials of finite degree are to being additive functions. For R a commutative ring and $n \geq 0$, the functors $-^{\otimes n}, \bigwedge^n, \Gamma^n, \mathrm{Sym}^n\colon \mathbf{Mod}_R \to \mathbf{Mod}_R$ are all of finite degree.

Dold and Puppe have proven the following proposition in [DP61, 4.23].

Proposition 1.1. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F\colon \mathcal{A} \to \mathcal{B}$ a functor of finite degree.Assume every object in $\mathcal{A}$ has a finite projective resolution. Then for every bounded complex $C \in \mathrm{Ch}_{\geq 0}\,\mathcal{A}$ we have $\mathrm{L}_i FC = 0$ for all but finitely many i.*

The first of the main results of this thesis is theorem 8.18, that we state here a bit differently.

Theorem 1.2. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be a reduced functor of finite degree. Suppose that every $A \in \mathcal{A}$ has a finite projective resolution. Then there exists a unique map $\mathrm{K}_0(F)\colon \mathrm{K}_0(\mathcal{A}) \to \mathrm{K}_0(\mathcal{B})$ such that for all $C_\bullet \in \mathrm{Ch}_{\geq 0}\,\mathcal{A}$ bounded we have*

$$\mathrm{K}_0(F)\left(\sum_{i\geq 0}(-1)^i[\mathrm{H}_i(C_\bullet)]\right) = \sum_{i\geq 0}(-1)^i[\mathrm{L}_i F C_\bullet].$$

We give an explicit formule for f and use explicit calculations taken from [SK10], [Köc01] to establish the property of f given above.

For many applications, this is not general enough. The existence of sufficiently many projective objects will often force the category $\mathcal{A}$ to be large, which will make the Grothendieck group small. For example, we need to embed the category **FAb** of finite abelian groups into the category **FGAb** of finitely generated abelian groups. The functors $\mathrm{L}_i F\colon\ \mathrm{Ch}_{\geq 0}\,\mathbf{FGAb} \to \mathbf{FGAb}$ can then often be restricted to functor $\mathrm{Ch}_{\geq 0}\,\mathbf{FAb} \to \mathbf{FAb}$; this is possible for tensor powers, for example. We have $\mathrm{K}_0(\mathbf{FAb}) \cong \mathbf{Q}^*_{\geq 0}$ and $\mathrm{K}_0(\mathbf{FGAb}) \cong \mathbf{Z}$ and the natural map $\mathrm{K}_0(\mathbf{FAb}) \to \mathrm{K}_0(\mathbf{FGAb})$ induced by the inlcusion is the zero map. Therefore we cannot simply use the theorem above to obtain an interesting map $\mathrm{K}_0(\mathbf{FAb}) \to \mathrm{K}_0(\mathbf{FAb})$.

As another example, using injectives instead of projectives, let $f\colon X \to Y$ be a proper morphism of Noetherian schemes. Then because of the properness of f, we can restrict the right derived functors of $f_*\colon \mathbf{QCoh}(X) \to \mathbf{QCoh}(Y)$ to functors on coherent sheaves. However, $\mathrm{K}_0(\mathbf{QCoh}(X))$ is typically zero, while $\mathrm{K}_0(\mathbf{Coh}(X))$ can be interesting.

If the functor $F\colon \mathcal{A} \to \mathcal{B}$ is nice enough, the second main results of this thesis, theorem 9.7, can be used in such cases. It requires $F\colon \mathcal{A} \to \mathcal{B}$ to have a system of filtrations, which roughly means that to every exact sequence $0 \to A \to B \to C \to 0$, we can functorially associate a filtration of FB of which the graded parts in a certain sense only depend on A and C. For $\mathcal{A}_0 \subset \mathcal{A}$ a weak Serre subcategory, we denote by $\mathrm{Ch}^{\mathcal{A}_0}_{\geq 0}\,\mathcal{A}$ the full subcategory of $C_\bullet \in \mathrm{Ch}_{\geq 0}\,\mathcal{A}$ with $\mathrm{H}_i(C_\bullet) \in \mathcal{A}_0$ for all i and $\mathrm{H}_i(C_\bullet) = 0$ for almost all i. Theorem 9.7 then states (see chapter 9 for details)

Theorem 1.3. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and let $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$ be weak Serre subcategories. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor with a system of filtrations. Suppose that $\mathcal{A}$ has enough projectives, $\mathbf{L}F$ maps $\mathrm{Ch}^{\mathcal{A}_0}_{\geq 0}\,\mathcal{A}$ to $\mathrm{Ch}^{\mathcal{B}_0}_{\geq 0}\,\mathcal{B}$, and the induced filtrations also live in $\mathrm{Ch}^{\mathcal{B}_0}_{\geq 0}\,\mathcal{B}$. Then there exists a unique map*

$$\mathrm{K}_0(F)\colon \mathrm{K}_0(\mathcal{A}_0) \to \mathrm{K}_0(\mathcal{B}_0)$$

such that for all $C_\bullet \in \mathrm{Ch}^{\mathcal{A}_0}_{\geq 0}\,\mathcal{A}$, we have

$$\mathrm{K}_0(F)\left(\sum_{i\geq 0}(-1)^i[\mathrm{H}_i(C_\bullet)]\right) = \sum_{i\geq 0}(-1)^i[\mathrm{L}_i F C_\bullet].$$

For R a commutative ring, $n \geq 0$, all of the functors $-^{\otimes n}, \bigwedge^n, \Gamma^n, \mathrm{Sym}^n\colon \mathbf{Mod}_R \to \mathbf{Mod}_R$ have a system of filtrations.

Theorem 9.9 states the following, among other things.

Theorem 1.4. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor of degree ≤ 2. Then F has a system of filtrations.*

In this thesis, we prove everything using chains (not cochains) and projectives, but almost everything can easily be dualized to cochain complexes and injectives.

1.2 Overview

In chapter 2 we recall some basic relevant definitions and properties of abelian categories and homological algebra. This is not meant as a thorough treatment and the reader who is familiar with the field can probably skip it and refer to it only when needed.

In chapter 3 we discuss simplicial methods, restricting the treatment to what is necessary for the Dold-Kan correspondence (theorem 3.25). This theorem states that for any abelian category $\mathcal{A}$, the category $\mathrm{Ch}_{\geq 0}\,\mathcal{A}$ is equivalent to the simplicial category $\mathrm{Simp}\,\mathcal{A}$ and that under this equivalence, chain homotopy corresponds to simplicial homotopy. An important result, alluded to in the introduction, is that for any functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories, the induced functor $F\colon\ \mathrm{Simp}\,\mathcal{A} \to \mathrm{Simp}\,\mathcal{B}$ respects simplicial homotopy (proposition 3.15).

In chapter 4 we first recall how additive functors are usually derived. In section 4.2 we use simplicial methods to derive more general functors in the way introduced by Dold and Puppe in [DP61].

Chapter 5 is about Grothendieck groups and the Euler characteristic. In the first two sections we prove some basic properties and give examples. In section 5.3, we show in detail how right-exact functors induce a function on Grothendieck groups that commutes with the Euler characteristic, which is an essential property that we want to generalise. Section 5.4 is used for theorem 8.18 (about functors of finite degree) and the remaining sections are used in the proof of theorem 9.7.

In chapter 6 we define what it means for functions from monoids to abelian groups to be of finite degreeand introduce deviations to measure the failure to be additive. In sections 6.1 and 6.2, we prove some basic properties about these deviations. In section 6.3, we define universal maps of finite degreeand use them to show the for M a commutative monoid, $G(M)$ its groupification and A an abelian group, restriction induces a bijective correspondence between functions of finite degree on $G(M)$ and functions of finite degree on M (corollary 6.14). This allows us to define the function on Grothendieck groups in the cases of functors of finite degree (theorem 8.18). Chapter 7 does for functors between abelian categories what chapter 6 does for functions, cross-effects playing the role of deviations. These results there follow quite directly from the results in chapter 7.

In chapter 8, we prove one of the main results of this thesis, theorem 8.18. In section 8.1, we very briefly summarize a result from [SK10]. In the very technical section 8.2, we do some calculations with the results from [SK10] that are used to prove that the diagram in theorem 8.18 is indeed commutative.

Chapter 9 proves the second main result of this thesis, theorem 9.7. We state and prove this theorem in section 9.1, for which we do not need any of the results in the previous three chapters. In the next two sections we show that this theorem is applicable to functors of degree ≤ 2 and to functors such as $-^{\otimes n}$, $\bigwedge^n$, Γ^n and Sym^n on (finitely generated) modules over a commutative ring.

2 Abelian categories and homological algebra

In this section we recall some basic definitions and theorems from category theory and homological algebra. Standard references are [ML98] and [Wei94].

2.1 Additive functors

Let $\mathcal{C}$ be a category. Consider the following axioms:

(A1) the hom-sets in $\mathcal{C}$ are abelian groups and composition is bilinear;

(A2) the category $\mathcal{C}$ has a zero object;

(A3) any two objects in $\mathcal{C}$ have a product in $\mathcal{C}$;

(A4) every morphism in $\mathcal{C}$ has both a kernel and a cokernel; and

(A5) all epimorphisms are cokernels and all monomorphisms are kernels.

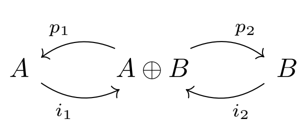
Here we mean by a zero object an object $0 \in \mathcal{C}$, such that for every object $c \in \mathcal{C}$, there is a unique morphism $0 \to c$ and a unique morphisms $c \to 0$. In other words, it is both an initial and a terminal object. It is easy to see that such a zero object is unique up to unique isomorphism.

Definition 2.1. A category $\mathcal{C}$ is *pre-additive* if it satisfies axiom (A1); *additive* if it satisfies (A1), (A2) and (A3); *pre-abelian* if it satisfies (A1) up to (A4); and *abelian* if it satisfies all of the axioms (A1) up to (A5).

Definition 2.2. A functor $F \colon \mathcal{A} \to \mathcal{B}$ between pre-additive categories is *additive*, if for any two parallel arrows $f, g \colon A \to A'$ in $\mathcal{A}$, it satisfies the identity

$$T(f+g) = Tf + Tg.$$

Definition 2.3. Let $\mathcal{A}$ be a pre-additive category. A direct sum of two objects $A, B \in \mathcal{A}$ is an object $A \oplus B \in \mathcal{A}$ together with maps p_1, p_2, i_1, i_2 as in the diagram



such that $p_1 i_1 = \mathrm{id}_A$, $p_2 i_2 = \mathrm{id}_B$ and $i_1 p_1 + i_2 p_2 = \mathrm{id}_{A \oplus B}$ hold. We call p_1, p_2 the *projection maps* and i_1, i_2 the *inclusion maps* of the direct sum.

In a pre-additive category, if the direct sum $A \oplus B$, product $A \times B$ or coproduct $A \sqcup B$ of two objects A, B exists, then all of them exist and they coincide. See [ML98, VIII.2] for a more extensive discussion, that includes the following useful proposition.

Proposition 2.4. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between pre-additive categories. Suppose that all finite products exists in $\mathcal{A}$. Then F is additive if and only if for all objects $A, A' \in \mathcal{A}$,*

$$F(A) \underset{Fi_1}{\overset{Fp_1}{\leftrightarrows}} F(A \oplus B) \underset{Fi_2}{\overset{Fp_2}{\rightleftarrows}} F(B)$$

is a direct sum.

This proposition applies in particular to additive categories.

Example 2.5. Examples of functors that are not additive are the endofunctors $-^{\otimes n}, \bigwedge^n, \Gamma^n$ and Sym_n for $n \geq 2$ of the category of modules over a non-zero commutative ring that send a module M to its n-th tensor power $M \otimes \ldots \otimes M$, its n-th exterior power $\bigwedge^n M$, the module of invariants of $M^{\otimes n}$ under the action of the n-th symmetric group S_n and the quotient of $M^{\otimes n}$ by the action of S_n, respectively.

Definition 2.6. Let $\mathcal{A}, \mathcal{B}$ be additive categories. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is *reduced* if it sends the zero object of $\mathcal{A}$ to the zero object of $\mathcal{B}$.

Definition 2.7. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is *left-exact* if it preserves all finite limits and *right-exact* if it preserves all finite colimits.

Left- and right-exact functors are also examples of additive functors.

Proposition 2.8. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is*

(a) left-exact if and only if it is additive and preserves kernels; and

(b) right-exact if and only if it is additive and preserves cokernels.

□

2.2 Chain homotopy and projective resolutions

Definition 2.9. Let $\mathcal{A}$ be an additive category and let $f, g\colon C_\bullet \to D_\bullet$ be two maps in $\mathrm{Ch}\,\mathcal{A}$. A *chain homotopy* between f and g is a collection $(s_n)_{n\in\mathbf{Z}}$ of maps $s_n\colon C_n \to D_{n+1}$ such that

$$f - g = d_{n+1}s_n + s_{n-1}d_n. \tag{1}$$

The maps f and g are said to be *chain homotopic* if such a homotopy exists. A *homotopy equivalence* between $C_\bullet$ and $D_\bullet$ are two maps $f\colon C_\bullet \to D_\bullet$ and $g\colon D_\bullet \to C_\bullet$ such that gf and fg are chain homotopic to the identity on $C_\bullet$ and $D_\bullet$ respectively.

Let now $\mathcal{A}$ be an abelian category. A map $f\colon C_\bullet \to D_\bullet$ in $\mathrm{Ch}\,\mathcal{A}$ is a *quasi-isomorphism* if it induces isomorphisms on all homology groups.

Remark 2.10. If $F\colon \mathcal{A} \to \mathcal{B}$ is a functor between additive categories with $F0 = 0$ and $C_\bullet$ is a chain complex in $\mathcal{A}$, then $FC_\bullet$ is a chain complex in $\mathcal{B}$. Thus we get a functor $F\colon \operatorname{Ch}\mathcal{A} \to \operatorname{Ch}\mathcal{B}$.

Remark 2.11. Let $f_1, f_2\colon C_\bullet \to C'_\bullet$ and $g_1, g_2\colon C'_\bullet \to C''_\bullet$ be chain maps of complexes in an additive category. If f_1 is homotopic to f_2 and g_1 to g_2, then $g_1 \circ f_1$ is homotopic to $g_2 \circ g_1$.

Definition 2.12. For an additive category $\mathcal{A}$, we define the *homotopy category of chain complexes*, denoted $\mathcal{K}(\mathcal{A})$, to be the category with chain complexes as its objects and the equivalence classes of chain maps under homotopy as morphisms. We sometimes add superscripts and subscripts (such as $\mathcal{K}^+(\mathcal{A})$) as in definitions 2.33 and 2.38 to denote restrictions on the chain complexes that form the objects.

This category is additive, but not necessarily abelian.

Remark 2.13. If $F\colon \mathcal{A} \to \mathcal{B}$ is an additive functor and $(s_n\colon C_n \to D_{n+1})_{n\in\mathbf{Z}}$ is a homotopy from $f\colon C_\bullet \to D_\bullet$ to $g\colon C_\bullet \to D_\bullet$, then $(Fs_n\colon FC_n \to FD_{n+1})_{n\in\mathbf{Z}}$ is a homotopy from Ff to Fg. Hence F induces a functor $\mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{B})$.

Remark 2.14. If $\mathcal{A}$ is abelian, then two chain homotopic maps in $\operatorname{Ch}\mathcal{A}$ induce the same maps on homology. Therefore we can consider taking the i-th homology group as a functor $\mathrm{H}_i\colon \mathcal{K}(\mathcal{A}) \to \mathbf{Ab}$. It follows that every homotopy equivalence is a quasi-isomorphism.

Definition 2.15. Let $\mathcal{A}$ be an abelian category. A *projective resolution* of a chain complex $C_\bullet \in \operatorname{Ch}^+\mathcal{A}$ is a pair $(P_\bullet, f\colon P_\bullet \to C_\bullet)$ of a chain complex $P_\bullet \in \operatorname{Ch}^+\mathcal{A}$ with P_i projective for all i and a quasi-isomorphism $f\colon P_\bullet \to C_\bullet$.

Remark 2.16. Traditionally, a projective resolution of an object A in an abelian category $\mathcal{A}$ is a chain complex $P_\bullet$ in non-negative degree together with a map $\varepsilon\colon P_0 \to A$ such that all P_i are projective and the complex

$$\ldots \to P_2 \to P_1 \to P_0 \xrightarrow{\varepsilon} A \to 0 \to \ldots$$

is exact. This is equivalent to stating that the map of chain complexes

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\varepsilon} & & \downarrow & & \\
\ldots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \ldots
\end{array}$$

is a quasi-isomorphism. Hence 2.15 is a straightforward generalisation of this concept by regarding an object as a chain complex concentrated in degree zero.

Definition 2.17. Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has *enough projectives* if for every object $A \in \mathcal{A}$ there exist a projective object $P \in \mathcal{A}$ and a surjection $P \twoheadrightarrow A$.

The following proposition shows how we can interpret taking projective resolutions as a functor.

Proposition 2.18. *Let $\mathcal{A}$ be an abelian category with enough projectives and let*

$$\mathrm{can}\colon \mathrm{Ch}^+ \mathcal{A} \to \mathcal{K}^+(\mathcal{A})$$

denote the canonical functor. There exists a functor

$$\mathrm{res}\colon \mathrm{Ch}^+ \mathcal{A} \to \mathcal{K}^+(\mathcal{A})$$

together with a natural transformation

$$\tau\colon \mathrm{res} \to \mathrm{can}$$

such that for every $A \in \mathrm{Ch}^+ \mathcal{A}$ the morphism

$$\tau_A\colon \mathrm{res}\, A \to \mathrm{can}\, A = A$$

is a projective resolution. This functor res *is unique up to unique natural isomorphism in the sense that for any other functor* $\mathrm{res}'\colon \mathrm{Ch}^+ \mathcal{A} \to \mathcal{K}^+(\mathcal{A})$ *and natural transformation* $\tau'\colon \mathrm{res}' \to \mathrm{can}$ *satisfying the given property, there exists a unique natural isomorphism* $\phi\colon \mathrm{res} \to \mathrm{res}'$ *such that*

$$\begin{array}{ccc} \mathrm{res} & & \\ \downarrow{\scriptstyle\phi} & \overset{\tau}{\searrow} & \\ & & \mathrm{can} \\ & \underset{\tau'}{\nearrow} & \\ \mathrm{res}' & & \end{array}$$

commutes. □

2.3 The mapping cone and truncations

Notation 2.19. Let $C_\bullet$ denote a chain complex in an abelian category $\mathcal{A}$ with boundary maps $(d_i)_{i\in\mathbf{Z}}$ and let $n \in \mathbf{Z}$. Then $C_\bullet[n]$ denotes the shifted chain complex in $\mathcal{A}$ with

$$(C_\bullet[n])_i = C_{n+i}$$

and boundary maps $((-1)^i d_i)_{i\in\mathbf{Z}}$.

Definition 2.20 (Mapping cone)**.** Let $\mathcal{A}$ be an abelian category and let $f\colon C_\bullet \to D_\bullet$ be a map in $\mathrm{Ch}(\mathcal{A})$. The *mapping cone* of f is the chain complex $\mathrm{cone}(f)$ with

$$\mathrm{cone}(f)_i = C_{i-1} \oplus D_i$$

and boundary maps d_i given by the matrices

$$\begin{pmatrix} -d_C & 0 \\ -f & d_D \end{pmatrix}.$$

There is a canonical exact sequence

$$0 \to D_\bullet \to \mathrm{cone}(f) \to C_\bullet[-1] \to 0$$

which in degree n is simply the inclusion $D_n \to C_{n-1} \oplus D_n$ followed by the projection $C_{n-1} \oplus D_n \to C_{n-1}$.

Proposition 2.21. *With notation as above, the connecting homomorphism of the long exact sequence induced by the exact sequence* $0 \to D_\bullet \to \operatorname{cone}(f) \to C_\bullet[-1] \to 0$ *is the map* $H_\bullet(f)\colon H_\bullet(C_\bullet) \to H_\bullet(D_\bullet)$.

Proof. See [Wei94, lemma 1.5.3]. □

Corollary 2.22. *A map* $f\colon C_\bullet \to D_\bullet$ *of chain complexes is a quasi-isomorphism if and only if* $\operatorname{cone}(f)$ *is exact.*

Proof. See [Wei94, corollary 1.5.4]. □

Definition 2.23. Let $C_\bullet$ denote a chain complex and let $n \in \mathbf{Z}$. The (good) *truncation* $\tau_{\leq n} C_\bullet$ is the chain complex with

$$(\tau_{\leq n} C_\bullet)_i = \begin{cases} 0 & \text{if } i > n \\ \operatorname{coker}(d_{n+1}\colon C_{n+1} \to C_n) & \text{if } i = n \\ C_i & \text{if } i < n \end{cases}$$

and boundary maps the maps induced by the boundary maps of $C_\bullet$.

The homology groups of $\tau_{\leq n} C_\bullet$ are equal to those of $C_\bullet$ in degree at most n and equal to zero in degree greater than n. There is a natural map $C_\bullet \to \tau_{\leq n} C_\bullet$ that induces the identity on all homology groups in degree $i \leq n$ and the zero map $\mathrm{H}_i(C_\bullet) \to \mathrm{H}_i(\tau_{\leq n} C_\bullet) = 0$ for all $i > n$. Similarly, a chain map $f\colon C_\bullet \to D_\bullet$ induces a map $\tau_{\leq n} f\colon \tau_{\leq n} C_\bullet \to \tau_{\leq n} D_\bullet$ with $\mathrm{H}_i(\tau_{\leq n} f) = \mathrm{H}_i(f)$ in degree $i \leq n$ and $\mathrm{H}_i(\tau_{\leq n} f) = 0$ in degree $i > n$.

2.4 Split exact sequences

Definition 2.24. A morphism $f\colon A \to B$ in an abelian category is said to be *injective* if its kernel is zero and *surjective* if its cokernel is zero.

Definition 2.25. Let $f\colon A \to B$ be a morphism in an abelian category. A *retraction* of f is a morphism $r\colon B \to A$ with $r \circ f = \mathrm{id}_A$. A *section* of f is a morphism $s\colon B \to A$ with $f \circ s = \mathrm{id}_B$. A morphism $f\colon A \to B$ is said to be *split injective* if it has a retraction and *split surjective* if it has a section.

Remark 2.26. Split surjections are surjections and split injections are injections. If $F\colon \mathcal{A} \to \mathcal{B}$ is a functor between abelian categories and $f\colon A \to B$ is a split injection (resp. split surjection) in $\mathcal{A}$, then $F(f)\colon FA \to FB$ is a split injection (resp. split surjection) in $\mathcal{B}$.

Proposition 2.27. *Let* $\mathcal{A}$ *be an abelian category. Let*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence in $\mathcal{A}$ *. The following are equivalent:*

(a) g is split surjective;

(b) f is split injective; and

(c) there exists an isomorphism $\phi\colon B \to A \oplus C$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\phi} & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow{i_1} & A \oplus C & \xrightarrow{p_2} & C & \longrightarrow & 0 \end{array}$$

with $i_1\colon A \to A \oplus C$ the natural injection and $p_2\colon A \oplus C \to C$ the natural projection, commutes.

□

Definition 2.28. An exact sequence

$$0 \to A \to B \to C \to 0$$

in an abelian category is said to be *split exact* if it satisfies one of the equivalent properties in proposition 2.27.

Remark 2.29. If $F\colon \mathcal{A} \to \mathcal{B}$ is a functor between abelian categories with $F0 = 0$ and

$$0 \to A \to B \to C \to 0$$

is a split exact sequence in $\mathcal{A}$, then by 2.26, the complexes

$$0 \to FA \to FB \qquad \text{and} \qquad FB \to FC \to 0$$

are exact. If F is additive, then $0 \to FA \to FB \to FC \to 0$ is split exact, as can be seen by applying F to the commutative diagram in proposition 2.27.

2.5 Exact categories

In abelian categories, every map has a kernel and a well-behaved image, so that we can define the notion of exactness (see [ML98, VII.3]). If we have a full additive subcategory $\mathcal{C}$ of a category $\mathcal{A}$ that is not necessarily abelian, then we can still define a notion of short exact sequences in $\mathcal{C}$, by simply calling a sequence in $\mathcal{C}$ exact if it is in $\mathcal{A}$. The following definition captures that idea and adds an additional requirement.

Definition 2.30 (Exact category)**.** An *exact category* is a pair $(\mathcal{C}, \mathcal{E})$ of an additive category $\mathcal{C}$ and a family $\mathcal{E}$ of pairs (i, j) of composable morphisms $A \xrightarrow{i} B \xrightarrow{j} C$ in $\mathcal{C}$, such that there exists an abelian category $\mathcal{A}$ and an embedding of $\mathcal{C}$ as a full subcategory of $\mathcal{A}$ with the following properties:

(i) a composable pair (i, j) of morphisms $i\colon A \to B, j\colon B \to C$ is in $\mathcal{E}$ if and only if $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is exact in $\mathcal{A}$;

(ii) $\mathcal{C}$ is closed under extensions in $\mathcal{A}$: if $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathcal{A}$ with $A, C \in \mathcal{C}$, then the object B is also an object of $\mathcal{C}$.

For an additive $\mathcal{C}$ with a given embedding $\mathcal{C} \subset \mathcal{B}$ of $\mathcal{C}$ as a full subcategory of an abelian category $\mathcal{B}$, we say that $\mathcal{C}$ is an exact subcategory of $\mathcal{B}$ if we can take $\mathcal{A} = \mathcal{B}$ in the above definition. Note that this fully determines $\mathcal{E}$. In this situation, we will say that a complex $C_\bullet$ in $\mathcal{C}$ is *exact* if it is so in $\mathcal{B}$.

Proposition 2.31. *Let $\mathcal{A}$ be an abelian category and let $\mathcal{P}$ denote the full subcategory of $\mathcal{A}$ consisting of all projective objects in $\mathcal{A}$. Then $\mathcal{P}$ is an exact subcategory of $\mathcal{A}$. Moreover, the kernel in $\mathcal{A}$ of a surjection in $\mathcal{P}$ lies in $\mathcal{P}$.*

Proof. First note that $\mathcal{P}$ is indeed an additive subcategory of $\mathcal{A}$, because a finite product of projective objects is projective, the zero object is projective and the requirements on hom-sets and the composition immediately carry over from $\mathcal{A}$. Any exact sequence

$$0 \to Q \to X \to P \to 0$$

in $\mathcal{A}$ with Q, P projective, splits, and since a direct sum of projectives is projective X is projective as well. This proves that $\mathcal{P}$ is an exact subcategory of $\mathcal{A}$.

To prove the last statement, suppose that we have an exact sequence

$$0 \to A \to P \to Q \to 0$$

in $\mathcal{A}$ with $P, Q \in \mathcal{C}$. Then because Q is projective, this exact sequence splits and P is isomorphic to $A \oplus Q$. Now we conclude A is projective, because it is a direct summand of a projective object. □

2.6 Categories of chain complexes

Definition 2.32. A *chain complex* $C_\bullet$ in an additive category $\mathcal{A}$ is a sequence of objects $(C_i)_{i \in \mathbf{Z}}$ in $\mathcal{A}$ together with a sequence of maps $(d_i \colon C_i \to C_{i-1})_{i \in \mathbf{Z}}$ called the boundary maps such that for every $i \in \mathbf{Z}$ the composition $d_i d_{i-1}$ is the zero morphism. A *chain map* $f \colon C_\bullet \to C'_\bullet$ between two chain complexes $C_\bullet, C'_\bullet$ is a collection $(f_i \colon C_i \to C'_i)_{i \in \mathbf{Z}}$ of maps in $\mathcal{C}$ that commute with the boundary maps: $d_i f_i = f_{i-1} d_i$ for all $i \in \mathbf{Z}$. The *category of chain complexes* in $\mathcal{A}$, denoted $\mathrm{Ch}(\mathcal{A})$, has chain complexes in $\mathcal{A}$ as its objects and chain maps as its arrows.

Definition 2.33. Let $\mathcal{A}$ be an additive category. We define the following full subcategories of $\mathrm{Ch}\,\mathcal{A}$:

- the category $\mathrm{Ch}^+ \mathcal{A}$ of *bounded below* chain complexes that consists of those chain complexes $C_\bullet$ with $C_i = 0$ for all i small enough;
- the category $\mathrm{Ch}^- \mathcal{A}$ of *bounded above* chain complexes that consists of those chain complexes $C_\bullet$ with $C_i = 0$ for all i large enough;

- the category $\mathrm{Ch}^{\mathrm{b}} \mathcal{A}$ of *bounded* chain complexes that consists of those chain complexes $C_\bullet$ with $C_i = 0$ for all but finitely many i; and
- the category $\mathrm{Ch}_{\geq 0} \mathcal{A}$ of chain complexes in non-negative degree that consists of those chain complexes $C_\bullet$ with $C_i = 0$ for all $i < 0$.

Definition 2.34. Let $\mathcal{A}$ be an abelian category. For each chain complex $C_\bullet$ in $\mathcal{A}$, we define the *i-th homology group* of $C_\bullet$ as

$$\mathrm{H}_i(C_\bullet) := \frac{\ker(d_i \colon C_i \to C_{i-1})}{\mathrm{im}(d_{i+1} \colon C_{i+1} \to C_i)}.$$

Definition 2.35 (see [Sta14, Tag 02MO])**.** Let $\mathcal{A}$ be an abelian category. A *Serre subcategory* of $\mathcal{A}$ is a non-empty full subcategory $\mathcal{C} \subset \mathcal{A}$ such that if

$$A \to B \to C$$

is an exact sequence in $\mathcal{A}$ with $A, C \in \mathcal{C}$, then B also lies in $\mathcal{C}$.

A *weak Serre subcategory* of $\mathcal{A}$ is a non-empty full subcategory $\mathcal{C}$ of $\mathcal{A}$ such that if

$$A_1 \to A_2 \to A_3 \to A_4 \to A_5$$

is an exact sequence in $\mathcal{A}$ with $A_1, A_2, A_4, A_5 \in \mathcal{C}$, then A_3 also lies $\mathcal{C}$.

Remark 2.36. A weak Serre subcategory is easily seen to be closed under isomorphisms. (In other words, it is a strictly full subcategory.)

Example 2.37. The category of finitely generated modules over a ring R is a Serre subcategory of the category of modules over R. If X is a scheme, then the category of quasi-coherent sheaves on X is a weak Serre subcategory of the category of $\mathcal{O}_X$-modules. [Sta14, Tag 01LA]

Definition 2.38. Let $\mathcal{A}$ be an abelian category and let $\mathcal{A}_0 \subset \mathcal{A}$ be a weak Serre subcategory. We define the following full subcategories of $\mathrm{Ch}\,\mathcal{A}$:

- the category $\mathrm{Ch}^{\mathrm{hb}} \mathcal{A}$ of *homologically bounded* chain complexes that consists of those chain complexes $C_\bullet$ with $\mathrm{H}_i(C_\bullet) = 0$ for all but finitely many i; and
- the category $\mathrm{Ch}^{\mathcal{A}_0} \mathcal{A}$ of chain complexes with bounded homology in $\mathcal{A}_0$ that consists of those chain complexes $C_\bullet$ with $\mathrm{H}_i(C_\bullet) \in \mathcal{A}_0$ for all i and $\mathrm{H}_i(C_\bullet) = 0$ for all but finitely many i.

We will sometimes combine the subscripts and superscripts in these definitions, for example as in $\mathrm{Ch}^{\mathcal{A}_0,\mathrm{b}} \mathcal{A}$ or $\mathrm{Ch}^{\mathcal{A}_0}_{\geq 0} \mathcal{A}$.

Proposition 2.39. *Let $\mathcal{A}$ be an abelian category and $\mathcal{A}_0 \subset \mathcal{A}$ a weak Serre subcategory. Then $\mathrm{Ch}\,\mathcal{A}$ and all other categories mentioned in 2.33 and 2.38 are abelian categories.*

Proof. This is easy to check, since kernels and cokernels can be taken degreewise. For the category $\mathrm{Ch}^{\mathcal{A}_0}\,\mathcal{A}$, note that the condition stated in the definition of a weak Serre subcategory allows us to conclude from the induced long exact sequences that kernels and cokernels also have homology in $\mathcal{A}$. $\square$

3 Simplicial methods

Some general references for this section are [GJ09], [Wei94, Ch. 8] and [Sta14, Tag 0162].

3.1 Simplicial objects

Definition 3.1. The *simplex category* Δ is the category whose objects are the sets $[n] = \{0, 1, 2, \ldots, n\}$ for all integers $n \geq 0$ and whose morphisms are the non-decreasing maps, i.e., maps $\alpha\colon [n] \to [m]$ such that $i \leq j$ implies $\alpha(i) \leq \alpha(j)$. Note that the empty set is not an object of Δ.

Definition 3.2. Let $\mathcal{C}$ be a category. The category of *simplicial objects* in $\mathcal{C}$, denoted $\mathrm{Simp}(\mathcal{C})$, is the functor category $[\Delta^{\mathrm{op}}, \mathcal{C}]$. This means that its objects are the functors $X\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ and its morphisms are the natural transformations between these functors. Given a simplicial object X, we denote $X([n])$ by X_n.

Definition 3.3. A *simplicial set* is a simplicial object in the category of sets. Similarly, a *simplicial (abelian) group* is a simplicial object in the category of (abelian) groups.

Definition 3.4. For each $n \geq 0$ and $0 \leq i \leq n$, we define the map

$$\delta_i^n\colon [n-1] \to [n], \qquad j \mapsto \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

that skips the value i and the map

$$\sigma_i^n\colon [n+1] \to [n], \qquad j \mapsto \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

that doubles the value i. If X denotes a simplicial object, we write

$$d_i^n = X(\delta_i^n)\colon X_n \to X_{n-1}, \qquad s_i^n = X(\sigma_i^n)\colon X_n \to X_{n+1}.$$

The d_i^n are called *face maps* and the s_i^n are called *degeneracy maps*. We will often drop the superscripts.

Remark 3.5. All maps in Δ can be written as compositions of maps of the form δ_i^n and σ_i^n.

Example 3.6. For $n \geq 0$, the *simplicial standard n-simplex* is defined as the simplicial set

$$\Delta[n] := \mathrm{Hom}_\Delta(\,-\,, [n]).$$

Any map $[n] \to [m]$ induces a map of simplicial sets $\Delta[n] \to \Delta[m]$ by postcomposition.

Example 3.7. For $n \geq 0$, we denote by Δ^n the standard topological n-simplex $\{(i_0, \ldots, i_n) \in \mathbf{R}_{\geq 0}^{n+1} : i_0 + \ldots + i_n = 1\}$ in $\mathbf{R}^{n+1}$. By labelling the vertices of all the Δ^n, we can associate to any $\alpha\colon [m] \to [n]$ in Δ a map of $\Delta^\alpha\colon \Delta^m \to \Delta^n$ in a natural way, namely by mapping the vertices as specified by α and then extending linearly. In this way we get a functor from Δ to the category **Top** of topological spaces.

Let $\mathrm{F}\colon \mathbf{Set} \to \mathbf{Ab}$ denote the functor that sends a set to the free abelian group on that set. To any topogical space S, we can associate the simplicial abelian group

$$\Delta^{\mathrm{op}} \to \mathbf{Top}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathbf{Top}}(\,-\,,S)} \mathbf{Set} \xrightarrow{\mathrm{F}} \mathbf{Ab}.$$

In example 3.19, we will see how this simplicial group can naturally be turned into the chain complex used to define singular homology.

Proposition 3.8. *Let $\mathcal{A}$ be an abelian category. The category* $\operatorname{Simp}\mathcal{A}$ *is also abelian, and a sequence of simplicial objects in* $\operatorname{Simp}(\mathcal{A})$

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact at B if and only if for every degree $n \geq 0$ the sequence

$$A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n$$

is exact at B_n.

Proof. This is true for every functor category $[T, \mathcal{A}']$ with T small and $\mathcal{A}'$ abelian, because a limit in a functor category $[\mathcal{C}, \mathcal{D}]$ of a certain shape can be computed pointwise if all limits of that shape exist in $\mathcal{D}$. ☐

Remark 3.9. Let $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be categories. Then

$$\operatorname{Simp}(\mathcal{C}_1 \times \ldots \times \mathcal{C}_n) = \operatorname{Simp}\mathcal{C}_1 \times \ldots \times \operatorname{Simp}\mathcal{C}_n.$$

Indeed, a functor $X\colon \Delta^{\mathrm{op}} \to \mathcal{C}_1 \times \ldots \times \mathcal{C}_n$ corresponds to the n-tuple of functors $(X_i\colon \Delta^{\mathrm{op}} \to \mathcal{C}_i)_{i=1}^n$ with X_i obtained by postcomposing X with the projection $\mathcal{C}_1 \times \ldots \times \mathcal{C}_n \to \mathcal{C}_i$, and conversely, an n-tuple of functors $(X_i\colon \Delta^{\mathrm{op}} \to \mathcal{C}_i)_{i=1}^n$ corresponds to the composition

$$\Delta^{\mathrm{op}} \xrightarrow{\mathrm{diag}} \Delta^{\mathrm{op}} \times \ldots \times \Delta^{\mathrm{op}} \xrightarrow{(X^1,\ldots,X^n)} \mathcal{C}_1 \times \ldots \times \mathcal{C}_n$$

where $\mathrm{diag}\colon \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \ldots \times \Delta^{\mathrm{op}}$ is the diagonal.

Definition 3.10. Let $F\colon \mathcal{C} \to \mathcal{D}$ denote a functor. We get an induced functor

$$F\colon \operatorname{Simp}\mathcal{C} \to \operatorname{Simp}\mathcal{D}$$

by mapping objects $X\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ in $\operatorname{Simp}\mathcal{C}$ to the composition $F \circ X\colon \Delta^{\mathrm{op}} \to \mathcal{D}$. By the previous remark 3.9, we can also apply this to a functor

$$F\colon \mathcal{C}_1 \times \ldots \times \mathcal{C}_n \to \mathcal{D}$$

from the product of categories $\mathcal{C}_1, \dots, \mathcal{C}_n$ to a category $\mathcal{D}$: such a functor induces a functor

$$F\colon \operatorname{Simp}(\mathcal{C}_1) \times \dots \times \operatorname{Simp}(\mathcal{C}_n) \to \operatorname{Simp}(\mathcal{D})$$

that maps an object $(X^1, \dots, X^n)$ to the composite functor

$$\Delta^{\mathrm{op}} \xrightarrow{\text{diag}} \Delta^{\mathrm{op}} \times \dots \times \Delta^{\mathrm{op}} \xrightarrow{(X^1,\dots,X^n)} \mathcal{C}_1 \times \dots \times \mathcal{C}_n \xrightarrow{F} \mathcal{D}$$

In degree i, this is simply

$$F(X^1, \dots, X^n)_i = F(X^1_i, \dots, X^n_i)$$

where X^j_i denotes the degree i part of the j-th object X^j.

3.2 Simplicial homotopy

Definition 3.11. Let $\mathcal{C}$ be a category, X, Y objects of $\operatorname{Simp}(\mathcal{C})$ and let $f, g\colon X \to Y$ be two simplicial maps. Then a *simplicial homotopy* between f and g is a collection of maps $h_{n,i}\colon X_n \to Y_{n+1}$ for all $n \geq 0$ and i with $0 \leq i \leq n$, satisfying for all suitable indices the relations

$$\begin{aligned}
d_0^{n+1} h_{n,0} &= f_n, \\
d_{n+1}^{n+1} h_{n,n} &= g_n, \\
d_i^{n+1} h_{n,j} &= \begin{cases} h_{n-1,j-1} d_i^n & \text{if } i < j \\ d_i^{n+1} h_{n,i-1} & \text{if } i = j \neq 0 \\ h_{n-1,j} d_{i-1}^n & \text{if } i > j+1 \end{cases}, \qquad \text{and} \\
s_i^{n+1} h_{n,j} &= \begin{cases} h_{n+1,j+1} s_i^n & \text{if } i \leq j \\ h_{n+1,j} s_{i-1}^n & \text{if } i > j \end{cases}.
\end{aligned}$$

If there exists such a simplicial homotopy, f and g are said to be homotopic.

Remark 3.12. Simplicial homotopy is not in general an equivalence relation on simplicial maps with the same domain and codomain. It is, however, for every abelian category [Wei94, exc. 8.3.6].

This definition is very general but in certain cases we can give a nicer and more intuitive equivalent definition. To state it, we first need the following definition.

Definition 3.13. Let $\mathcal{C}$ be a category in which all coproducts exist. Then there exists a canonical copower functor

$$\otimes\colon \mathbf{Set} \times \mathcal{C} \to \mathcal{C}$$

given on objects by

$$(I, C) \mapsto \coprod_{i \in I} C.$$

A pair of maps $(s\colon I \to J, f\colon C \to D)$ is sent to the map $\bigsqcup_{i\in I} C \to \bigsqcup_{j\in J} D$ that maps the component corresponding to i to the component corresponding to $s(i)$ with the map f, mimicking the situation where you can actually take a product $I \times C$.

Let now S be a simplicial set and let X be a simplicial object in $\mathcal{C}$. Then we define the simplicial object $S \times X$ in $\mathcal{C}$ as the composition of functors

$$\Delta^{\mathrm{op}} \xrightarrow{\text{diag}} \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \xrightarrow{(S,X)} \mathbf{Set} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}.$$

This means that in degree n we get

$$(S \times X)_n = \bigsqcup_{s\in S_n} X_n.$$

If the category $\mathcal{C}$ has all finite coproducts, then this construction still works for simplicial sets S such that S_n is finite for all $n \geq 0$.

The following proposition shows that the definition of homotopy given above is analogous to the topological definition of homotopy. Here, the standard simplex $\Delta[1]$ plays the role of the unit interval in topology.

Proposition 3.14. *Let $\mathcal{C}$ be a category with finite coproducts, let X, Y be simplicial objects in* $\mathrm{Simp}(\mathcal{C})$ *and let $f, g\colon X \to Y$ be two simplicial maps. Then there is a simplicial homotopy from f to g if and only if there exists a map $h\colon \Delta[1] \times X \to Y$ such that we have a commutative diagram*

$$\begin{array}{ccccc} X = \Delta[0] \times X & \xrightarrow{\varepsilon_0 \times \mathrm{id}} & \Delta[1] \times X & \xleftarrow{\varepsilon_1 \times \mathrm{id}} & \Delta[0] \times X = X \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle h} & \swarrow{\scriptstyle g} & \\ & & Y & & \end{array}$$

The $\Delta[i]$ are defined in 3.6 and ε_0 and ε_1 are the simplicial maps induced by the two maps $[0] \to [1]$.

Proof. See [Wei94, 8.3.12] or [Sta14, Tag 019L]. (Note that the $h_{n,i}$ in [Sta14, Tag 019L] are defined differently than here.) □

Proposition 3.15. *Let $\mathcal{C}, \mathcal{D}$ be categories and $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Then the induced functor $F\colon \mathrm{Simp}\,\mathcal{C} \to \mathrm{Simp}\,\mathcal{D}$ preserves homotopies: if f, g are homotopic maps in $\mathrm{Simp}\,\mathcal{C}$, then Ff and Fg are homotopic in $\mathrm{Simp}\,\mathcal{D}$.*

Proof. This follows immediately from the definition: the induced map on simplicial objects is simply postcomposition with F, so that the relations in definition 3.11 are preserved. Hence for a homotopy $(h_{n,i})$ from f to g, the maps $Fh_{n,i}$ form a simplicial homotopy from Ff to Fg. (Note that for $X \in \mathrm{Simp}\,\mathcal{C}$ with face maps d_i^n, the simplicial object FX has face maps $F(d_i^n)$ by construction, and similarly for the degeneracies.)

□

Proposition 3.16. *Let $\mathcal{C}$ be a category with all finite coproducts, let $X, Y, Z \in \operatorname{Simp}\mathcal{C}$ and let $f_1, f_2 \colon X \to Y$ and $g_1, g_2 \colon Y \to Z$ be maps. If f_1 is homotopic to f_2 and g_1 to g_2, then $g_1 \circ f_1$ is homotopic to $g_2 \circ g_1$.*

Proof. With the use of proposition 3.14, this can easily be proven in the same way as it can be proven for topological spaces. □

This proposition allows us to introduce the following definition.

Definition 3.17. Let $\mathcal{C}$ be a category with all finite coproducts. Then we define $\operatorname{HoSimp}\mathcal{C}$ as the category with the same objects as $\operatorname{Simp}\mathcal{C}$, but with maps up to simplicial homotopy.

3.3 Dold-Kan correspondence

In this section we state the Dold-Kan correspondence (theorem 3.25), which says that for any abelian category $\mathcal{A}$, the category $\operatorname{Simp}\mathcal{A}$ is equivalent to $\operatorname{Ch}_{\geq 0}\mathcal{A}$. We do not prove this correspondence (see for example [Wei94, 8.4.1] for a proof), but we do explain the relevant background.

From simplicial objects to chain complexes Let $\mathcal{A}$ be an abelian category. We will define two functors

$$C, N \colon \operatorname{Simp}(\mathcal{A}) \to \operatorname{Ch}_{\geq 0}\mathcal{A}$$

that map a simplicial object in $\mathcal{A}$ to an associated chain complex. Of these, the functor C is perhaps the more obvious functor, but the functor N, more or less a "normalisation" of C, is the one that is used in the Dold-Kan correspondence.

Definition 3.18. We first define the functor $C \colon \operatorname{Simp}\mathcal{A} \to \operatorname{Ch}_{\geq 0}\mathcal{A}$. A simplicial object $X \in \operatorname{Simp}(\mathcal{A})$ is sent to the complex $CX \in \operatorname{Ch}_{\geq 0}\mathcal{A}$ with $(CX)_n = X_n$ for $n \geq 0$ and with for $n \geq 1$ as boundary morphisms the alternating sum

$$\sum_{i=0}^{n} (-1)^i d_i \colon X_n \to X_{n-1}$$

of the simplicial face maps $d_i \colon X_n \to X_{n-1}$. It follows easily from combinatorial identities in the simplex category Δ that this indeed a chain complex. On maps, the functor is defined in the obvious way.

Example 3.19 (Singular homology)**.** Let S be a topological space. The chain complex associated to the simplicial abelian group

$$\Delta^{\text{op}} \to \mathbf{Top}^{\text{op}} \xrightarrow{\operatorname{Hom}_{\mathbf{Top}}(\,-\,,S)} \mathbf{Set} \xrightarrow{\text{F}} \mathbf{Ab}.$$

(see example 3.7) is precisely the chain complex that is used to define the singular homology of S.

Definition 3.20. We define the functor $N\colon \operatorname{Simp}\mathcal{A} \to \operatorname{Ch}_{\geq 0}\mathcal{A}$ as follows. We map a simplicial object X in $\operatorname{Simp}\mathcal{A}$ to the complex $NX \in \operatorname{Ch}_{\geq 0}\mathcal{A}$ that is in degree n

$$(NX)_n = \bigcap_{i=1}^{n} \ker\left(d_i\colon X_n \to X_{n-1}\right)$$

and that has as its boundary morphism the zeroeth face map $d_0\colon X_n \to X_{n-1}$. On maps, the functor is defined in the same way as before for the functor C, but restricted.

Remark 3.21. For X a simplicial object in an abelian category, the chain complex NX is often called the normalized chain complex and CX is called the associated or unnormalized chain complex. In [GJ09, III.2], CX is called the Moore complex, but in [Wei94], NX is called the Moore complex.

Remark 3.22. The chain complex NX is naturally a subcomplex of CX. Indeed, the restriction of the alternating sum $d_n = \sum_{i=0}^{n}(-1)^i d_i\colon X_n \to X_{n-1}$ to the subobject $\bigcap_{i=1}^{n} \ker\left(d_i\colon X_n \to X_{n-1}\right)$ is equal to the zeroeth face map.

Proposition 3.23. *Let A be a simplicial object in in abelian category $\mathcal{A}$. Then there exists an exact subcomplex DA of CA that satisfies the equality*

$$CA = NA \oplus DA.$$

Proof. For the subcomplex DA we can take the degenerate chain subcomplex of which the degree n part $(DA)_n$ is the image $\sum_{i=0}^{n-1} s_i A_{n-1}$ of the simplicial degeneracy maps. The direct sum decomposition is proven in [Wei94, 8.3.7] and the exactness of DA is proven in [Wei94, 8.3.8]. □

From chain complexes to simplicial objects

Definition 3.24. We define the functor $\Gamma\colon \operatorname{Ch}_{\geq 0}(\mathcal{A}) \to \operatorname{Simp}(\mathcal{A})$ as follows. Let $A_\bullet$ be a chain complex in $\operatorname{Ch}_{\geq 0}\mathcal{A}$. Then $\Gamma A_\bullet\colon \Delta^{\mathrm{op}} \to \mathcal{A}$ is the simplicial object that sends $[n]$ to

$$(\Gamma A_\bullet)_n = \bigoplus_{k=0}^{n} \bigoplus_{\substack{\eta\colon [n] \twoheadrightarrow [k] \\ \text{surjective}}} A_k.$$

For a morphism $\alpha\colon [m] \to [n]$ in Δ, we define the map $\phi := (\Gamma A_\bullet)\alpha\colon (\Gamma A_\bullet)_n \to (\Gamma A_\bullet)_m$ component-wise as follows. Let $\eta\colon [n] \twoheadrightarrow [k]$ be a surjection in Δ It is easy to see that we can factor the map $\alpha \circ \eta$ uniquely as $\delta \circ \sigma$ with σ surjective and δ injective, giving the following commutative diagram

$$\begin{array}{ccc} [m] & \xrightarrow{\alpha} & [n] \\ \sigma \downarrow & & \downarrow \eta \\ [r] & \xrightarrow[\delta]{} & [k] \end{array} .$$

Then ϕ maps the component corresponding to η to the component corresponding to σ, with one of the following maps, depending on δ:

$$\begin{cases} \mathrm{id}\colon A_k \to A_k & \text{if } \delta = \mathrm{id}_{[k]} \\ d_k\colon A_k \to A_{k-1} & \text{if } \delta = \delta_0\colon [k-1] \to [k],\ i \mapsto i+1 \\ 0 & \text{otherwise} \end{cases}$$

Now that we have described the functors Γ and N, we can state the Dold-Kan correspondence.

Theorem 3.25 (Dold-Kan correspondence)**.** *Let $\mathcal{A}$ be an abelian category. The functor $N\colon \mathrm{Simp}(\mathcal{A}) \to \mathrm{Ch}_{\geq 0}(\mathcal{A})$ is an equivalence of categories with quasi-inverse $\Gamma\colon \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathrm{Simp}(\mathcal{A})$. Both N and Γ map homotopic maps to homotopic maps.*

4 Derived functors of general functors

4.1 Derived functor of an additive functor

In this subsection, we quickly recall the normal procedure of deriving an additive functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories $\mathcal{A}, \mathcal{B}$ and obtaining a long exact sequence from it, with special attention to where the additivity of F is required.

Because F is additive, it induces a functor (see remark 2.13) $F\colon \mathcal{K}_{\geq 0}(\mathcal{A}) \to \mathcal{K}_{\geq 0}(\mathcal{B})$. This is important because projective resolutions are only unique up to unique homotopy equivalence. We can then define the total left derived functor $\mathbf{L}F$ of F as the composition

$$\mathbf{L}F\colon \operatorname{Ch}_{\geq 0}\mathcal{A} \xrightarrow{\text{res}} \mathcal{K}_{\geq 0}(\mathcal{A}) \xrightarrow{F} \mathcal{K}_{\geq 0}(\mathcal{B}).$$

The i-th derived functor $\mathrm{L}_i F$ is defined as $\mathrm{H}_i \circ \mathbf{L}F$.

Given a long exact sequence $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ in $\mathcal{A}$, we can take projective resolutions $P_\bullet \to A_\bullet$, $Q_\bullet \to B_\bullet$ and $R_\bullet \to C_\bullet$ such that the induced maps

$$0 \to P_\bullet \to Q_\bullet \to R_\bullet \to 0$$

form an exact sequence. As a consequence, the sequence

$$0 \to P_n \to Q_n \to R_n \to 0$$

is split exact for every $n \geq 0$, because every R_n is projective. Since additive functors preserve split exact sequences (remark 2.29), the sequence

$$0 \to FP_\bullet \to FQ_\bullet \to FR_\bullet \to 0$$

is exact as well. Applying the long exact sequence of homology to this exact sequence of chain complexes results in the long exact sequence

$$\ldots \to \mathrm{L}_{i+1}FC_\bullet \to \mathrm{L}_iFA_\bullet \to \mathrm{L}_iFB_\bullet \to \mathrm{L}_iFC_\bullet \to \mathrm{L}_{i-1}FA_\bullet \to \ldots$$

associated to left-derived functors. If moreover F is right-exact, then the functor L_0F is equal to F.

Remark 4.1. The functor $\mathbf{L}F$ preserves quasi-isomorphisms. Indeed, let $A_\bullet, B_\bullet$ be objects of $\operatorname{Ch}_{\geq 0}\mathcal{A}$ and suppose there exists a quasi-isomorphism $f\colon A_\bullet \xrightarrow{\sim} B_\bullet$. Let $\varepsilon\colon P_\bullet \to A_\bullet$ be a projective resolution of $A_\bullet$ and let $\eta\colon Q_\bullet \to B_\bullet$ be a projective resolution of $B_\bullet$. Then $f \circ \varepsilon\colon P_\bullet \to B_\bullet$ is also a projective resolution and since projective resolutions are unique up to unique homotopy equivalence, $P_\bullet$ and $Q_\bullet$ are homotopy equivalent. This homotopy equivalence is preserved by $F\colon \mathcal{K}_{\geq 0}(\mathcal{A}) \to \mathcal{K}_{\geq 0}(\mathcal{B})$ and therefore $\mathbf{L}FA_\bullet$ and $\mathbf{L}FB_\bullet$ are even homotopy equivalent.

4.2 Deriving functors with simplicial methods

In this section we show how simplicial methods can be used to define a derived functor of a functor that is not necessarily additive. This derived functor does not, however, give rise to a long exact sequence of homology. The approach is based on [DP61].

Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories $\mathcal{A}, \mathcal{B}$. Let N, Γ be the functors from the Dold-Kan correspondence (theorem 3.25). Because N, Γ and $F\colon \mathrm{Simp}(\mathcal{A}) \to \mathrm{Simp}(\mathcal{B})$ all map homotopic maps to homotopic maps (see 3.25 and 3.15), they induce functors on homotopy categories and we can consider the composition

$$NF\Gamma\colon \mathcal{K}_{\geq 0}(\mathcal{A}) \xrightarrow{\Gamma} \mathrm{HoSimp}(\mathcal{A}) \xrightarrow{F} \mathrm{HoSimp}(\mathcal{B}) \xrightarrow{N} \mathcal{K}_{\geq 0}(\mathcal{B}).$$

Recall that if $\mathcal{A}$ has enough projectives, we denote by $\mathrm{res}\colon \mathrm{Ch}_{\geq 0}\,\mathcal{A} \to \mathcal{K}_{\geq 0}(\mathcal{A})$ be the projective resolution functor of proposition 2.18).

Definition 4.2. Let $\mathcal{A}$ be an abelian category with enough projectives, let $\mathcal{B}$ be abelian categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor. The *total left derived functor* of F is the functor

$$\mathbf{L}F\colon \mathrm{Ch}_{\geq 0}\,\mathcal{A} \to \mathcal{K}_{\geq 0}(\mathcal{B})$$

defined as the composition

$$\mathrm{Ch}_{\geq 0}\,\mathcal{A} \xrightarrow{\mathrm{res}} \mathcal{K}_{\geq 0}(\mathcal{A}) \xrightarrow{\Gamma} \mathrm{HoSimp}(\mathcal{A}) \xrightarrow{F} \mathrm{HoSimp}(\mathcal{B}) \xrightarrow{N} \mathcal{K}_{\geq 0}(\mathcal{B}).$$

For all $i \geq 0$, we define the *i-th derived functor* $\mathrm{L}_i F\colon \mathcal{A} \to \mathbf{Ab}$ ($\mathbf{Ab}$ the category of abelian groups) as the composition

$$\mathrm{Ch}_{\geq 0}\,\mathcal{A} \xrightarrow{\mathbf{L}F} \mathcal{K}_{\geq 0}(\mathcal{B}) \xrightarrow{\mathrm{H}_i} \mathbf{Ab}$$

where $\mathrm{H}_i\colon \mathcal{K}_{\geq 0}(\mathcal{B}) \to \mathbf{Ab}$ is the functor that takes the i-th homology group of a chain complex.

Remark 4.3. Suppose that F is additive. Then $\mathbf{L}F$ coincides with the total derived functor defined in section 4.1. Indeed, the construction immediately shows $F\Gamma = \Gamma F$ and hence $NF\Gamma = N\Gamma F$, and $N\Gamma F$ is naturally isomorphic to F.

Proposition 4.4. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F\colon \mathcal{A} \to \mathcal{B}$ a functor. Then the functor $\mathbf{L}F\colon \mathrm{Ch}_{\geq 0}\,\mathcal{A} \to \mathcal{K}_{\geq 0}(\mathcal{B})$ preserves quasi-isomorphisms.*

Proof. The proof is the same as in remark 4.1: the projective resolutions of quasi-isomorphic complexes in $\mathrm{Ch}_{\geq 0}\,\mathcal{A}$ are homotopy equivalent and this is preserved by $NF\Gamma$.

□

5 Grothendieck groups

A good reference, on which some of the proofs are based, is [Wei13][Ch. II].

5.1 Definition and examples

Definition 5.1. A category $\mathcal{C}$ is *skeletally small* if the class of isomorphism classes of $\mathcal{C}$ forms a set.

Definition 5.2. Let $\mathcal{C}$ be a skeletally small, exact category. The *Grothendieck group* $\mathrm{K}_0(\mathcal{C})$ of $\mathcal{C}$ is the abelian group generated by the isomorphism classes $[X]$ of objects $X \in \mathcal{C}$ satisfying the relations

$$[X'] - [X] + [X''] = 0$$

for all short exact squences

$$0 \to X' \to X \to X'' \to 0$$

in $\mathcal{C}$.

An obvious consequence of the definition of the Grothendieck group is the equality

$$[A \oplus B] = [A] + [B]. \tag{2}$$

Clearly the class of the zero object in $\mathcal{C}$ is the zero element in the group $\mathrm{K}_0(\mathcal{C})$.

The Grothendieck group has the following universal property.

Proposition 5.3 (Universal property)**.** *Let $\mathcal{C}$ be an exact category and let L be an abelian group. Then any function $f \colon \mathcal{C} \to L$ from the objects of $\mathcal{C}$ to L that for all exact sequences*

$$0 \to X' \to X \to X'' \to 0$$

in $\mathcal{C}$ satisfies the equality

$$f(X) = f(X') + f(X'')$$

factors uniquely as the composition of the canonical map $\mathcal{C} \to \mathrm{K}_0(\mathcal{C})$ given by $X \mapsto [X]$ and a group homomorphism $\mathrm{K}_0(\mathcal{C}) \to L$. □

Example 5.4 (Finite-dimensional vector spaces)**.** Let k be a field and let $\mathbf{FVect}_k$ denote the category of finite dimensional vector spaces over k. Then there exists a unique isomorphism

$$\mathrm{K}_0(\mathbf{FVect}_k) \to \mathbf{Z} \qquad \text{mapping } [V] \text{ to } \dim V.$$

This is easy to prove using the rank-nullity theorem.

The following example shows that some finiteness conditions will be necessary if we want the Grothendieck group to be in any way informative.

Example 5.5 (Infinite dimensional vector spaces)**.** Let $\aleph$ be an infinite cardinal, k a field and let $\mathbf{Vect}_{k,\aleph}$ denote the category of vector spaces over k of dimension at most $\aleph$. The bound on the dimension ensures that $\mathbf{Vect}_{k,\aleph}$ is skeletally small. Then $\mathrm{K}_0(\mathbf{Vect}_{k,\aleph})$ is isomorphic to the zero group: if V_∞ denotes a vector space of dimension $\aleph$, and V is a vector space of dimension at most $\aleph$, then the direct sum $V \oplus V_\infty$ is isomorphic to V_∞ (as their bases have the same cardinality). Hence $[V] + [V_\infty] = [V_\infty]$ holds in $\mathrm{K}_0(\mathbf{Vect}_{k,\aleph})$, which implies $[V] = 0$.

Example 5.6 (Modules over a PID)**.** Example 5.4 can easily be generalized to the situation of the category $\mathcal{M}$ of finitely generated modules over a principal ideal domain R. There exists a unique isomorphism $\phi\colon \mathrm{K}_0(\mathcal{M}) \to \mathbf{Z}$ that sends the class $[M]$ to the rank $r(M)$ of M. This can be proven with the structure theorem, together with the remark that because of the exact sequence

$$0 \to R \xrightarrow{x\cdot} R \to R/xR \to 0$$

for any $x \in R$, the class $[R/xR]$ is zero in $\mathrm{K}_0(\mathcal{M})$.

Example 5.7 (Finite abelian groups)**.** Let $\mathbf{Ab}$ be the category of finite abelian groups. Then there is a unique isomorphism $\phi\colon \mathrm{K}_0(\mathbf{Ab}) \to \mathbf{Q}^*_{>0}$ that sends the equivalence class $[A]$ to the order $\#A$ of A.

5.2 The Euler characteristic

Recall that $\mathrm{Ch}^{\mathrm{hb}}\,\mathcal{A}$ denotes the category of chain complexes $C_\bullet$ in $\mathcal{A}$ such that $\mathrm{H}_i(C_\bullet)$ is zero for all but finitely many i,

Definition 5.8. Let $\mathcal{A}$ be an abelian category. Let $C_\bullet \in \mathrm{Ch}^{\mathrm{hb}}\,\mathcal{A}$ be a homologically bounded chain complex. Then we define the *Euler characteristic* of $C_\bullet$ as

$$\chi(C_\bullet) := \sum_{i \in \mathbf{Z}} (-1)^i [\mathrm{H}_i(C_\bullet)] \in \mathrm{K}_0(\mathcal{A}).$$

Proposition 5.9. *Let $\mathcal{A}$ be an abelian category and let $C_\bullet$ be in $\mathrm{Ch}^{\mathrm{b}}\,\mathcal{A}$. Then in $\mathrm{K}_0(\mathcal{A})$ we have the identity*

$$\chi(C_\bullet) = \sum_{i \in \mathbf{Z}} (-1)^i [C_i].$$

Proof. We can split the sequence $C_\bullet$ into short exact sequences

$$0 \to \ker d_i \to C_i \to \operatorname{im} d_i \to 0$$

that impose the relations

$$[C_i] = [\operatorname{im} d_i] + [\ker d_i] \qquad \text{in } \mathrm{K}_0(\mathbf{A})$$

that hold for all $i \in \mathbf{Z}$ (though by the assumption of boundedness on $C_\bullet$, all but finitely many of these relations will be trivial). The homology group $\mathrm{H}_i(C_\bullet)$ fit into short exact sequences

$$0 \to \operatorname{im} d_{i+1} \to \ker d_i \to \mathrm{H}_i(C_\bullet) \to 0$$

so that we see for all $i \in \mathbf{Z}$

$$[\mathrm{H}_i(C_\bullet)] = [\ker d_i] - [\operatorname{im} d_{i+1}] \qquad \text{in } \mathrm{K}_0(\mathcal{A}).$$

Taking the alternating sum of all the C_i (which is possible by the boundedness) and reindexing the alternating sum, we find

$$\begin{aligned}\sum_{i\in\mathbf{Z}}(-1)^i[C_i] &= \sum_{i\in\mathbf{Z}}(-1)^i\left([\operatorname{im} d_i] + [\ker d_i]\right)\\ &= \sum_{i\in\mathbf{Z}}(-1)^i\left([\ker d_i] - [\operatorname{im} d_{i+1}]\right) = \sum_{i\in\mathbf{Z}}(-1)^i[\mathrm{H}_i(C_\bullet)].\end{aligned}$$

□

We see that the proof of the previous proposition still works in the following situation, which we will need later on.

Proposition 5.10. *Let $\mathcal{C}$ be an exact subcategory of an abelian category $\mathcal{A}$ and suppose that $\mathcal{C}$ is closed under kernels in $\mathcal{A}$ of surjections in $\mathcal{C}$. If $C_\bullet \in \mathrm{Ch}^b\,\mathcal{C}$ is a complex whose homology groups $\mathrm{H}_i(C_\bullet)$ all lie in $\mathcal{C}$, then in $\mathrm{K}_0(\mathcal{C})$ we have the equality*

$$\sum_{i\in\mathbf{Z}}(-1)^i[\mathrm{H}_i(C_\bullet)] = \sum_{i\in\mathbf{Z}}(-1)^i[C_i].$$

Proof. The only difficulty in generalizing the proof lies in showing that all the objects under consideration still lie in $\mathcal{C}$. Since $C_\bullet$ is bounded, there exists a $k \in \mathbf{Z}$ such that $C_i = 0$ for all $i < k$ and hence $C_i, \ker d_i$ are in $\mathcal{C}$ for all $i < k$ and $\operatorname{im} d_i$ is in $\mathcal{C}$ for all $i \leq k$. We can now prove by induction that for all $i \in \mathbf{Z}$, the objects $\ker d_i$ and $\operatorname{im} d_i$ are in $\mathcal{C}$.

Suppose that for a certain $i \in \mathbf{Z}$, $\ker d_i$ lies in $\mathcal{C}$. Then $\operatorname{im} d_{i+1}$ is the kernel of the canonical surjection $\ker d_i \to \mathrm{H}_i(C_\bullet)$ that lies in $\mathcal{C}$, so by assumption $\operatorname{im} d_{i+1}$ is also in $\mathcal{C}$. Since $\ker d_{i+1}$ is the kernel of the surjection $C_{i+1} \to \operatorname{im} d_{i+1}$ that we now know to be a surjection in $\mathcal{C}$, we can conclude that $\ker d_{i+1}$ lies in $\mathcal{C}$ too. By induction we see that all $\ker d_i$ and $\operatorname{im} d_i$ lie in $\mathcal{C}$, so that the proof of 5.9 is seen to be valid in this case as well. □

Corollary 5.11. *Let $\mathcal{C}$ be an exact category and let $C_\bullet \in \mathrm{Ch}^b\,\mathcal{C}$ be exact. Then*

$$\sum_{i\in\mathbf{Z}}(-1)^i[C_i] = 0$$

holds in in $\mathrm{K}_0(\mathcal{C})$.

Proof. Because of the exactness, all the homology groups are zero. □

Corollary 5.12. *Let $\mathcal{C}$ be an exact subcategory of an abelian category $\mathcal{A}$ and let $C_\bullet, C'_\bullet \in \mathrm{Ch}^b\,\mathcal{C}$. Suppose that $f\colon C_\bullet \to C'_\bullet$ is a quasi-isomorphism. Then the equality*

$$\sum_{i\in\mathbf{Z}}(-1)^i[C_i] = \sum_{i\in\mathbf{Z}}(-1)^i[C'_i]$$

holds in $\mathrm{K}_0(\mathcal{C})$.

Proof. To circumvent the difficulty that the homology groups of $C_\bullet$ and $C'_\bullet$ might not lie in $\mathcal{C}$, consider the mapping cone of f (see defintion 2.20), denoted $\operatorname{cone} f$. Because f is a quasi-isomorphism, the homology groups of $\operatorname{cone}(f)$ are zero. Therefore we can apply corollary 5.11 to find

$$\sum_{i\in\mathbf{Z}}(-1)^i[(\operatorname{cone} f)_i] = 0.$$

By definition, the degree i part of the mapping cone of f is $C_{i-1} \oplus C'_i$ and we can complete the proof by noting that this implies

$$\sum_{i\in\mathbf{Z}}(-1)^i[(\operatorname{cone} f)_i] = \sum_{i\in\mathbf{Z}}(-1)^i[C'_i] - \sum_{i\in\mathbf{Z}}(-1)^i[C_i].$$

□

Proposition 5.13. *Let $\mathcal{A}$ be an abelian category and let*

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

be a short exact sequence in $\mathrm{Ch}^{\mathrm{hb}}\,\mathcal{A}$. Then in $\mathrm{K}_0(\mathcal{A})$ we have the identity

$$\chi(B_\bullet) = \chi(A_\bullet) + \chi(C_\bullet).$$

Proof. This follows from applying 5.11 to the long exact sequence of homology induced by $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ (note that this is indeed a bounded exact sequence) and then taking all terms $[H_i(B_\bullet)]$ to the other side of the equality sign. □

5.3 Maps induced by right-exact functors

In this section we show how right-exact functors induce maps on Grothendieck groups. We will show these maps fit into a commutative diagram (proposition 5.16). The main objective of this thesis is to generalise that proposition.

Proposition 5.14. *Let $\mathcal{A}$ be an abelian category with enough projectives, let $\mathcal{B}$ be an abelian category let $F\colon \mathcal{A} \to \mathcal{B}$ be a right-exact functor. Suppose that* *for all $A \in \mathcal{A}$, the images $\mathrm{L}_i FA$ are zero for all but finitely many i. Then there exists a unique map*

$$\mathrm{K}_0(F)\colon \mathrm{K}_0(\mathcal{A}) \to \mathrm{K}_0(\mathcal{B})$$

that is given on equivalence classes by

$$[A] \mapsto \sum_{i \geq 0} (-1)^i [\mathrm{L}_i FA].$$

Proof. Given exact sequence

$$0 \to A \to B \to C \to 0$$

we get an induced long exact sequence

$$\ldots \to \mathrm{L}_2 FZ \to \mathrm{L}_1 FX \to \mathrm{L}_1 FY \to \mathrm{L}_1 FZ \to X \to Y \to Z \to 0$$

that is bounded by the assumption on F. Applying corollary 5.11, we find

$$\sum_{n \geq 0} (-1)^n [\mathrm{L}_i FX] + \sum_{n \geq 0} (-1)^n [\mathrm{L}_i FZ] = \sum_{n \geq 0} (-1)^n [\mathrm{L}_i FY].$$

Hence, by the universal property, such a map exists. □

Recall that for a weak Serre subcategory $\mathcal{A}_0$ of an abelian category $\mathcal{A}$, we denote by $\mathrm{Ch}_{\geq 0}^{\mathcal{A}_0} \mathcal{A}$ the chain complexes in non-negative degree with bounded homology in $\mathcal{A}_0$.

Lemma 5.15. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories, let $\mathcal{A}_0 \subset \mathcal{A}$, $\mathcal{B}_0 \subset \mathcal{B}$ be weak Serre subcategories and let $F\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor. Suppose that $\mathbf{L}F$ restricts to a functor $\mathbf{L}F\colon \mathrm{Ch}_{\geq 0}^{\mathcal{A}_0} \mathcal{A} \to \mathcal{K}_{\geq 0}^{\mathcal{B}_0}(\mathcal{B})$. Then for every exact sequence*

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

in $\mathrm{Ch}_{\geq 0}^{\mathcal{A}_0} \mathcal{A}$, we have

$$\chi(\mathbf{L}FB_\bullet) = \chi(\mathbf{L}FA_\bullet) + \chi(\mathbf{L}FC_\bullet)$$

in $\mathrm{K}_0(\mathcal{B}_0)$.

Proof. This follows from applying corollary 5.11 to the long exact sequence associated to the left derived functor □

Proposition 5.16. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories, let $\mathcal{A}_0 \subset \mathcal{A}$, $\mathcal{B}_0 \subset \mathcal{B}$ be weak Serre subcategories and let $F\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor. Suppose that $\mathbf{L}F$ restricts to a functor $\mathbf{L}F\colon \mathrm{Ch}_{\geq 0}^{\mathcal{A}_0} \mathcal{A} \to \mathcal{K}_{\geq 0}^{\mathcal{B}_0}(\mathcal{B})$. Then the diagram*

$$\begin{array}{ccc} \mathrm{Ch}_{\geq 0}^{\mathcal{A}_0}(\mathcal{A}) & \xrightarrow{\mathbf{L}F} & \mathcal{K}_{\geq 0}^{\mathcal{B}_0}(\mathcal{B}) \\ \chi \downarrow & & \downarrow \chi \\ \mathrm{K}_0(\mathcal{A}_0) & \xrightarrow[\mathrm{K}_0(F)]{} & \mathrm{K}_0(\mathcal{B}_0) \end{array}$$

where the vertical maps are the Euler characteristics and the lower horizontal map is the map defined above, is commutative.

Proof. For a chain complex $C_\bullet$, recall that we denote by $\tau_{\leq n}C_\bullet$ its good truncation, defined in 2.23. Take any $A_\bullet \in \mathrm{Ch}^{\mathcal{A}_0}_{\geq 0}\mathcal{A}$ Let $n \in \mathbf{Z}_{\geq 0}$ be such that $\mathrm{H}_k(A_\bullet)$ is zero for all $k > n$. The kernel K of the surjective map $A_\bullet \twoheadrightarrow \tau_{\leq n}A_\bullet$ is acyclic. Then by lemma 5.15, we have $\chi(\mathbf{L}FA_\bullet) = \chi(\mathbf{L}F(\tau_{\leq n}A_\bullet)) + \chi(\mathbf{L}FK)$. Because K is quasi-isomorphic to the zero complex and because $\mathbf{L}F$ preserves quasi-isomorphism, we have $\chi(\mathbf{L}FK) = 0$. Since $\chi(A_\bullet) = \chi(\tau_{\leq n}A_\bullet)$, we have reduced the proof to the case where $A_\bullet$ is a bounded (above and below) complex. We finish the prove by induction on the length n of the chain complex $A_\bullet$.

Note that the proposition is trivial for all complexes concentrated in a single degree. Let $n \in \mathbf{Z}_{\geq 1}$ and suppose the proposition is true for all complexes $C_\bullet$ with $C_i = 0$ for $i > n$. Take $A_\bullet \in \mathrm{Ch}^{\mathcal{A}_0}_{\geq 0}\mathcal{A}$ with $A_i = 0$ for all $i > n+1$. We have an exact sequence

$$0 \to K \to A_\bullet \to \tau_{\leq n}A_\bullet \to 0$$

where K is the complex

$$\ldots \to 0 \to A_{n+1} \to \mathrm{im}(d_{n+1}\colon A_{n+1} \to A_n) \to 0 \to \ldots.$$

The natural injection of the complex $H := \ldots \to 0 \to \mathrm{H}_{n+1}(A_\bullet) \to 0 \to 0 \to \ldots$ into K is a quasi-isomorphism, and therefore we obtain

$$\chi(\mathbf{L}FK) = \chi(\mathbf{L}FH) = \mathrm{K}_0(F)(\chi(H)) = \mathrm{K}_0(F)(\chi(K)).$$

By lemma 5.15 again, we have

$$\begin{aligned}\chi(\mathbf{L}FA_\bullet) &= \chi(\mathbf{L}F(\tau_{\leq n}A_\bullet)) + \chi(\mathbf{L}FK)\\ &= \mathrm{K}_0(F)(\chi(\tau_{\leq n}A_\bullet)) + \mathrm{K}_0(\chi(\mathbf{L}FK))\\ &= \mathrm{K}_0(F)(\chi(A_\bullet))\end{aligned}$$

where we use the induction hypothesis in the second equality and $\chi(A_\bullet) = \chi(K) + \chi(\tau_{\leq n}A_\bullet)$ in the third. This completes the proof.

□

5.4 The Grothendieck group of the subcategory of projectives

Let $\mathcal{A}$ be an abelian category and suppose that $X \in \mathcal{A}$ has a finite projective resolution $0 \to P_k \to \ldots \to P_1 \to P_0 \to X \to 0$. Then by corollary 5.11, we have

$$[X] = \sum_{i=0}^{k}(-1)^i[P_i].$$

If every object in $\mathcal{A}$ has such a finite projective resolution, then every element in $\mathrm{K}_0(\mathcal{A})$ can be written as $[P] - [M]$ with $P, M \in (\mathcal{A})$ projective. In this section we prove that the inclusion from $\mathrm{K}_0(\mathcal{P})$ into $\mathrm{K}_0(\mathcal{A})$, where $\mathcal{P}$ denotes the full subcategory of $\mathcal{A}$ of all projective objects, is in fact an isomorphism.

The following theorem is theorem 7.6 in [Wei13]. By a $\mathcal{P}$-resolution of an object X we mean a resolution $P_\bullet$ of X with $P_i \in \mathcal{P}$ for all $i \geq 0$.

Theorem 5.17 (The resolution theorem)**.** *Let $\mathcal{P} \subset \mathcal{C} \subset \mathcal{A}$ be an inclusion of additive categories with $\mathcal{A}$ abelian ($\mathcal{A}$ gives the notion of exact sequence to $\mathcal{P}$ and $\mathcal{C}$). Assume that*

(a) every object C of $\mathcal{C}$ has a a finite $\mathcal{P}$-resolution ("C has finite $\mathcal{P}$-dimension"); and

(b) the full subcategory $\mathcal{C}$ is closed under taking kernels in $\mathcal{A}$ of surjections in $\mathcal{C}$.

Then the inclusion $\mathcal{P} \subset \mathcal{C}$ induces an isomorphism $\mathrm{K}_0(\mathcal{P}) \cong \mathrm{K}_0(\mathcal{C})$.

Before we begin the proof, we remark that this theorem is applicable in the case we will be interested in.

Corollary 5.18. *Let $\mathcal{A}$ be an abelian category and let $\mathcal{P}$ be the full subcategory of projective objects in $\mathcal{A}$. If every object of $\mathcal{A}$ has a finite projective resolution, then the natural inclusion $\mathrm{K}_0(\mathcal{P}) \to \mathrm{K}_0(\mathcal{A})$ is an isomorphism.*

Proof. Recall that by proposition 2.31, $\mathcal{P}$ is an exact subcategory closed under taking kernels of surjections. Applying the previous theorem (theorem 5.17) with $\mathcal{C} = \mathcal{A}$ gives the result. □

The rest of this section will be devoted to the proof of the resolution theorem (theorem 5.17).

Proof of theorem 5.17. The inclusion $\mathcal{P} \subset \mathcal{C}$ induces a map $\phi\colon \mathrm{K}_0(\mathcal{P}) \to \mathrm{K}_0(\mathcal{C})$ that sends $[P] \in \mathrm{K}_0(\mathcal{P})$ to $[P] \in \mathrm{K}_0(\mathcal{C})$. We will show that there exists an inverse map $\psi\colon \mathrm{K}_0(\mathcal{C}) \to \mathrm{K}_0(\mathcal{P})$ that sends an element $[C] \in \mathrm{K}_0(\mathcal{C})$ to the alternating sum $\sum_{i\geq 0}(-1)^i[P_i]$ of some finite resolution $P_\bullet$ of C in $\mathcal{P}$, which alternating sum will turn out not to depend on the finite resolution $P_\bullet$. It is then easily seen that ϕ and ψ are inverses: $\psi \circ \phi$ is seen to be the identity on $\mathrm{K}_0(\mathcal{P})$ by taking the complex with P in degree zero as the resolution of P; and $\phi \circ \psi$ is the identity by proposition 5.10.

Define the function

$$\bar{\psi}\colon \mathcal{C} \to \mathrm{K}_0(\mathcal{P}) \qquad \text{by} \qquad C \mapsto \sum_{i \geq 0} (-1)^i [P_i]$$

where $P_\bullet$ is any finite $\mathcal{P}$-resolution of C. We now claim that (1) this function $\bar{\psi}$ is well-defined in that it does not depend on the choice of projective resolution; and (2) for every exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, we have

$\bar{\psi}(Y) = \bar{\psi}(X) + \bar{\psi}(Z)$. Then by the universal property of the Grothendieck group there exists a map $\psi\colon \mathrm{K}_0(\mathcal{C}) \to \mathrm{K}_0(\mathcal{P})$ as described above.

Part (1) is proven in 5.20 and part (2) in 5.21, both of which depend on 5.19. □

The following is a lemma due to Grothendieck.

Lemma 5.19. *Let $\mathcal{P}, \mathcal{C}, \mathcal{A}$ be as in the resolution theorem (theorem 5.17). Let $f\colon X \to Y$ be a map in $\mathcal{C}$ and let $P_\bullet$ be a finite $\mathcal{P}$-resolution of Y. Then there exists a finite $\mathcal{P}$-resolution $Q_\bullet$ of X and a map $\bar{f}\colon Q_\bullet \to P_\bullet$ of chain complexes that induces f on the zeroth homology groups.*

Proof. Let n denote the length of the resolution $P_\bullet$ (i.e., n is the smallest non-negative integer such that $P_{n+1} = 0$). The proof goes by induction on n.

In the case $n = 0$, we have an isomorphism $\varepsilon\colon P_0 \to Y$, so we can take any finite $\mathcal{P}$-resolution $Q_\bullet$ of X and define $\bar{f}\colon Q_\bullet \to P_\bullet$ to be zero everywhere, except for the map in degree zero, which we define to be the composition $Q_0 \to X \xrightarrow{f} Y \xrightarrow{\varepsilon^{-1}} P_0$.

In the case $n > 0$, we split the resolution of Y so that we can apply the induction hypothesis, and then put them back together. We are going to construct the following commutative diagram (explanation follows):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(\varepsilon' \circ \eta) & \longrightarrow & Q_0 & \xrightarrow{\varepsilon' \circ \eta} & X & \longrightarrow & 0 \\
 & & \Big\downarrow g & & \Big\downarrow \eta & & \Big\| & & \\
 & & & & P_0 \times_Y X & \xrightarrow{\varepsilon'} & X & \longrightarrow & 0 \\
 & & & & \Big\downarrow & & \Big\downarrow f & & \\
0 & \longrightarrow & \ker \varepsilon & \longrightarrow & P_0 & \xrightarrow{\varepsilon} & Y & \longrightarrow & 0
\end{array}$$

First remark that the fibre product $P_0 \times_Y X$ of X and P_0 exists in $\mathcal{C}$, as it is the kernel of the surjection $(\varepsilon, -f)\colon P_0 \oplus X \to Y$ in $\mathcal{C}$. Let $\varepsilon'\colon P_0 \times_Y X \to X$ denote the projection. Let $\eta\colon Q_0 \to P_0 \times_Y X$ be the beginning of a $\mathcal{P}$-resolution of $P_0 \times_Y X$. Then there is a natural map $g\colon \ker(\varepsilon' \circ \eta) \to \ker \varepsilon$, where these kernels are taken in $\mathcal{A}$, but live in $\mathcal{C}$, being kernels of surjections in $\mathcal{C}$, and hence we get the commutative diagram displayed above.

Because $\ldots \to P_3 \to P_2 \to P_1 \to \ker \varepsilon \to 0$ is a $\mathcal{P}$-resolution of length $n-1$ of $\ker \varepsilon$, there exists by the induction hypothesis a finite $\mathcal{P}$-resolution $Q'_\bullet$ of $\ker(\varepsilon' \circ \eta)$ and a map $\bar{g}\colon Q'_\bullet \to P_\bullet[-1]$ that is g on the zeroth homology. Glueing this together with the commutative diagram above, we get the desired result for the map f. □

Lemma 5.20. *Let $P_\bullet$ and $Q_\bullet$ be two finite $\mathcal{P}$-resolutions of $X \in \mathcal{C}$. Then*

$$\sum_{i \geq 0} (-1)^i [P_i] = \sum_{i \geq 0} (-1)^i [Q_i].$$

holds in $K_0(\mathcal{P})$.

Proof. The idea is to find a third finite $\mathcal{P}$-resolution $R_\bullet$ and quasi-isomorphisms from $R_\bullet$ to $P_\bullet$ and $Q_\bullet$, so that we can apply corollary 5.12.

Remark that $P_\bullet \oplus Q_\bullet$ is a finite $\mathcal{P}$-resolution of $X \oplus X$. By lemma 5.19, there exists a finite $\mathcal{P}$-resolution $R_\bullet$ of C and a map $f\colon R_\bullet \to P_\bullet \oplus Q_\bullet$ that induces the diagonal map $\Delta\colon C \to C \oplus C$ on the zeroth homology.

$$\begin{array}{ccccc} R_\bullet & \longrightarrow & C & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \Delta} & & \\ P_\bullet \oplus Q_\bullet & \longrightarrow & C \oplus C & \longrightarrow & 0 \end{array}$$

Because both $\pi_1 \circ f\colon R_\bullet \to P_\bullet$ and $\pi_2 \circ f\colon R_\bullet \to Q_\bullet$ (with π_1, π_2 the projections from $P_\bullet \oplus Q_\bullet$ to $P_\bullet$ and $Q_\bullet$ respectively) are quasi-isomorphisms, we have by corollary 5.12

$$\sum_{i \geq 0} (-1)^i [P_i] = \sum_{i \geq 0} (-1)^i [R_i] = \sum_{i \geq 0} (-1)^i [Q_i]$$

which completes the proof. □

Lemma 5.21. *Let*

$$0 \to X \xrightarrow{g} Y \to Z \to 0$$

be an exact sequence in $\mathcal{C}$. *Then we have*

$$\bar{\psi}(Y) = \bar{\psi}(X) + \bar{\psi}(Z).$$

Proof. Let $P_\bullet$ be a finite $\mathcal{P}$-resolution of Y. By lemma 5.19, there exists a finite $\mathcal{P}$-resolution $Q_\bullet$ of X and a map $\bar{g}\colon Q_\bullet \to P_\bullet$ that induces g. The mapping cone $\mathrm{cone}(\bar{g})$ is now a finite $\mathcal{P}$-resolution of Z, as follows from the long exact sequence. As we saw in the proof of corollary 5.12, we have

$$\sum_{i \geq 0} (-1)^i [\mathrm{cone}(\bar{g})_i] = \sum_{i \geq 0} (-1)^i [Q_i] - \sum_{i \geq 0} (-1)^i [P_i]$$

by definition of the mapping cone. □

5.5 Waldhausen categories

Waldhausen categories are a kind of category for which we can define a Grothendieck group. They are more general than exact categories and also allow a notion of weak equivalence to be used in the Grothendieck group. In categories of chain complexes, we can take quasi-isomorphisms to be the weak equivalences.

The following definition is based on [Wei13, II.9.1.1] Recall that a zero object is an object that is both final and cofinal.

Definition 5.22. A *Waldhausen category* is a triple $(\mathcal{C}, w(\mathcal{C}), c(\mathcal{C}))$ of a category $\mathcal{C}$ and two subcategories $w(\mathcal{C}), c(\mathcal{C}) \subset \mathcal{C}$, the morphisms of which are called "weak equivalences" and "cofibrations" respectively, satisfying the following axioms:

(W1) every isomorphism is both a weak equivalence and a cofibration;

(W2) there is a distinguished zero object 0 in $\mathcal{C}$ and for every A in $\mathcal{C}$, the unique map $0 \rightarrowtail A$ is a cofibration;

(W3) if $A \rightarrowtail B$ is a cofibration and $A \to C$ is any map, then the pushout $B \cup_A C$ exists and $C \rightarrowtail B \cup_A C$ is a cofibration.

(W4) ("glueing for weak equivalences") if there is a commutative diagram

$$\begin{array}{ccccc} C & \longleftarrow & A & \rightarrowtail & B \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ C' & \longleftarrow & A' & \rightarrowtail & B' \end{array}$$

with the vertical maps weak equivalences and the right horizontal maps $A \rightarrowtail B$, $A' \rightarrowtail B'$ cofibrations, then the induced map of pushouts

$$B \cup_A C \xrightarrow{\sim} B' \cup_{A'} C'$$

is a weak equivalence.

Cofibrations are often denoted with a hooked arrow $\rightarrowtail$ and weak equivalences with the decorated arrow $\xrightarrow{\sim}$. A Waldhausen category $(\mathcal{C}, w(\mathcal{C}), c(\mathcal{C}))$ is often denoted simply by the category $\mathcal{C}$, suppressing the subcategories $w(\mathcal{C}), c(\mathcal{C})$ from the notation.

Definition 5.23 (Cofibration sequence)**.** Let $\mathcal{C}$ be a Waldhausen category and let $i \colon A \rightarrowtail B$ be a cofibration. Then the pushout $B/A := B \cup_A 0$ of $A \to 0$ along i is the cokernel of i and the sequence

$$A \rightarrowtail B \twoheadrightarrow B/A$$

is called a *cofibration sequence*.

Example 5.24 (Abelian categories)**.** Any abelian category is naturally a Waldhausen category by taking the monomorphisms as cofibrations and the isomorphisms as weak equivalences. The cofibrations are then simply the exact sequences.

Example 5.25 (Exact categories)**.** Any exact category $\mathcal{C}$ is naturally a Waldhausen category by taking as cofibrations the monomorphisms with cokernel in $\mathcal{C}$ and taking as weak equivalence the isomorphisms. Note that $\mathcal{C}$ is indeed closed under pushouts along admissable monomorphisms: if $A \rightarrowtail B$ is an admissable monomorphism and $A \to C$ is any morphism, then the sequence $0 \to C \to B \cup_A C \to B/A \to 0$ is exact and $\mathcal{C}$ is closed under extensions.

Example 5.26 (Chain complexes)**.** If $\mathcal{A}$ is an abelian category, then any abelian subcategory $\mathcal{B}$ of the category $\operatorname{Ch}\mathcal{A}$ of chain complexes in $\mathcal{A}$ can be given the structure of a Waldhausen category by taking the monomorphisms as cofibrations and the quasi-isomorphisms as weak equivalences. Indeed, the first three axioms hold for any abelian category with monomorphisms as cofibrations and the glueing for weak equivalences can be proven as follows. Let A, B, C, A', B', C' be as in axiom (W4). We then get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B \oplus C & \longrightarrow & B \cup_A C & \longrightarrow & 0 \\ & & \downarrow\sim & & \downarrow\sim & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' \oplus C' & \longrightarrow & B' \cup_{A'} C' & \longrightarrow & 0 \end{array}$$

where the map $A \to B \oplus C$ is the map $A \to B \hookrightarrow B \oplus C$ minus the map $A \to C \hookrightarrow B \oplus C$ and similarly for $A' \to B' \oplus C'$. Note that the first two vertical maps are quasi-isomorphisms. This gives rise to a chain map between the induced long exact sequences of homology groups in which the maps coming from $A \to A'$ and from $A \oplus B \to A' \oplus B'$ are isomorphisms. We can then conclude by applying the five lemma.

Definition 5.27 (Grothendieck group of a Waldhausen category)**.** Let $\mathcal{C}$ be a skeletally small Waldhausen category. The *Grothendieck group* $\mathrm{K}_0(\mathcal{C})$ of $\mathcal{C}$ is the quotient of the free abelian group on the isomorphism classes $[C]$ of $\mathcal{C}$ by the relations

- $[C] = [C']$ for every weak equivalence $C \xrightarrow{\sim} C'$; and
- $[C] = [B] + [C/B]$ for every cofibration sequence $B \rightarrowtail C \twoheadrightarrow C/B$.

Remark 5.28. If $\mathcal{A}$ is an exact or abelian category, then it is also naturally a Waldhausen category (examples 5.24 and 5.25). Its Grothendieck group as an exact or abelian category is the same as its Grothendieck group as a Waldhausen category with this structure.

Remark 5.29. Let $\mathcal{A}$ be an abelian category and $\mathcal{B} \subset \operatorname{Ch}\mathcal{A}$ an abelian subcategory. Then there are two natural structures of a Waldhausen category on $\mathcal{B}$: one coming from its structure as an abelian category (example 5.24) and one coming from its inclusion into a category of chain complexes (example 5.26). The Grothendieck groups of these two diffferent Waldhausen categories do not in general coincide and in the rest of this thesis we will always use the second structure, the one given in example 5.26.

Lemma 5.30. *Let $\mathcal{A}$ an abelian category and let $\mathcal{B}$ be a subcategory of* $\operatorname{Ch}\mathcal{A}$ *that is a Waldhausen subcategory when the monomorphisms are taken as cofibrations and the quasi-isomorphisms as weak equivalences. Assume that for all $B \in \mathcal{B}$, both* $\operatorname{cone}(\mathrm{id}_B)$ *and $B[-1]$ are in $\mathcal{B}$. Then*

(a) we have $[B] = -[B[-1]]$ for all $B \in \mathcal{B}$; and

(b) every element in $\mathrm{K}_0(\mathcal{B})$ *is of the form* $[B]$ *for some* $B \in \mathcal{B}$*.*

Proof. We have an exact sequence

$$0 \to B \to \mathrm{cone}(\mathrm{id}_B) \to B[-1] \to 0$$

in $\mathcal{B}$ (see 2.20) and hence

$$[B] + [B[-1]] = [\mathrm{cone}(\mathrm{id}_B)]$$

holds in $\mathrm{K}_0(\mathcal{B})$. By 2.22, $\mathrm{cone}(\mathrm{id}_B)$ is an exact chain complex because id_B is trivially a quasi-isomorphism. This implies that the unique map $0 \to \mathrm{cone}(\mathrm{id}_B)$ is a quasi-isomorphism and therefore $[\mathrm{cone}(\mathrm{id}_B)] = [0]$ holds in $\mathrm{K}_0(\mathcal{B})$. Thus we have the identity $[B] = -[B[-1]]$. It follows that any element in $\mathrm{K}_0(\mathcal{B})$ is of the form

$$\sum_{i=1}^{m} [B_i] = [\bigoplus_{i=1}^{m} B_i]$$

for some $B_1, \ldots, B_m \in \mathcal{B}$. □

5.6 The Grothendieck group of homologically bounded chain complexes

Note that by if $\mathcal{A}_0$ is a weak Serre subcategory of an abelian category $\mathcal{A}$, then the categories $\mathrm{Ch}^{\mathcal{A}_0,\mathrm{b}}_{\geq 0}\,\mathcal{A}$ and $\mathrm{Ch}^{\mathcal{A}_0}_{\geq 0}\,\mathcal{A}$ are abelian and, as explained in example 5.26, can hence be given the structure of a Waldhausen category. The goal of this section is to prove the following theorem.

Theorem 5.31. *Let* $\mathcal{A}$ *be an abelian category and* $\mathcal{A}_0 \subset \mathcal{A}$ *a weak Serre subcategory. Then the Euler characteristic*

$$\chi\colon \mathrm{K}_0(\mathrm{Ch}^{\mathcal{A}_0}_{\geq 0}\,\mathcal{A}) \to \mathrm{K}_0(\mathcal{A}_0), \qquad [A_\bullet] \mapsto \sum_i (-1)^i [\mathrm{H}_i(A_\bullet)]$$

is a well-defined isomorphism.

We prove this by showing that both groups are isomorphic to $\mathrm{K}_0(\mathrm{Ch}^{\mathcal{A}_0,b}_{\geq 0}\,\mathcal{A})$.

Proposition 5.32. *Let* $\mathcal{A}$ *be an abelian category and* $\mathcal{A}_0 \subset \mathcal{A}$ *a weak Serre subcategory. The inclusion of Waldhausen categories*

$$\mathrm{Ch}^{\mathcal{A}_0,\mathrm{b}}_{\geq 0}\,\mathcal{A} \subset \mathrm{Ch}^{\mathcal{A}_0}_{\geq 0}\,\mathcal{A}$$

induces an isomorphism on their Grothendieck groups

$$\mathrm{K}_0(\mathrm{Ch}^{\mathcal{A}_0,\mathrm{b}}_{\geq 0}\,\mathcal{A}) \xrightarrow{\sim} \mathrm{K}_0(\mathrm{Ch}^{\mathcal{A}_0}_{\geq 0}\,\mathcal{A}).$$

Proof. We will first prove that the inclusion induces a bijection between the sets of quasi-isomorphism classes. The surjectivity on quasi-isomorphism classes is easy to see: for any $C_\bullet \in \mathrm{Ch}_{\geq 0}^{\mathcal{A}_0}$, we can take n large enough so that the natural map $C_\bullet \xrightarrow{\sim} \tau_{\leq n} C_\bullet$ is a quasi-isomorphism.

To show the injectivity, suppose that two bounded complexes A, A' are in the same equivalence class in $\mathrm{K}_0(\mathrm{Ch}_{\geq 0}^{\mathcal{A}_0} \mathcal{A})$. Then there exists a zigzag of quasi-isomorphisms

$$A \xrightarrow{\sim} B_1 \xleftarrow{\sim} B_2 \xrightarrow{\sim} \dots \xleftarrow{\sim} B_n \xrightarrow{\sim} A'$$

with $B_1, \dots, B_n$ in $\mathrm{Ch}_{\geq 0}^{\mathcal{A}_0} \mathcal{A}$. Let n be such that the homology groups in every degree $i > n$ of all the chain complexes $A, A', B_1, \dots, B_n$ vanish. Then by taking the good truncations $\tau_{\leq n}$ of all the complexes in the zigzag, we get a zigzag of quasi-isomorphisms in bounded chain complexes that shows that A and A' are also in the same equivalence class in $\mathrm{Ch}_{\geq 0}^{\mathcal{A}_0, \mathrm{b}} \mathcal{A}$. This proves the injectivity of the map on quasi-isomorphism classes.

It remains to prove that the relations imposed by the exact sequences in the respective categories are the same in both sets of quasi-isomorphism classes. The following lemma shows that every relation on the quasi-isomorphism classes of homologically bounded chain complexes induces the same relation on the corresponding quasi-isomorphism classes of bounded chain complexes. This then completes the proof. □

Lemma 5.33. *Let*

$$0 \to A \to B \to C \to 0$$

be an exact sequence of chain complexes A, B, C with bounded homology. For n large enough, the induced complex

$$0 \to \tau_{\leq n} A \to \tau_{\leq n} B \to \tau_{\leq n} C \to 0$$

is exact.

Proof. Again, it suffices to take n such that $H_k(A) = H_k(B) = 0$ for all $k > n$. This follows from the long exact sequence induced by $0 \to A \to B \to C \to 0$, as we will explain.

First remark that exactness can be checked in each degree. The exactness in degree $i > n$ is trivial and the exactness in degree $i < n$ follows from the fact that the maps in degree i are the same as the maps in degree i in $0 \to A \to B \to C \to 0$. Hence it remains to show exactness in degree n.

Consider the following brutal truncations

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{d_{n+2}} & A_{n+1} & \xrightarrow{d_{n+1}} & A_n & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{d_{n+2}} & B_{n+1} & \xrightarrow{d_{n+1}} & B_n & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{d_{n+2}} & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

The columns of this double complex are clearly exact. Because we took n large enough, the rows in this double complex are exact except possibly in degree n, where the homology is $\operatorname{coker} d_{n+1}$. Note that by definition this cokernel is exactly the degree n part of the good truncation. Thus, taking the long exact sequence of the short exact sequence of the brutal truncations, we find that the degree n maps between the good truncations form a short exact sequence. □

Proposition 5.34. *Let $\mathcal{A}$ be an abelian category and $\mathcal{A}_0 \subset \mathcal{A}$ a weak Serre subcategory. The Euler characteristic*

$$\chi\colon \mathrm{K}_0(\mathrm{Ch}_{\geq 0}^{\mathcal{A}_0,\mathrm{b}}\,\mathcal{A}) \to \mathrm{K}_0(\mathcal{A}_0), \qquad [A_\bullet] \mapsto \sum_i (-1)^i [\mathrm{H}_i(A_\bullet)]$$

is a well-defined isomorphism from the Grothendieck group of the Waldhausen category of bounded chain complexes in non-negative degree with homology in $\mathcal{A}_0$ to the Grothendieck group of $\mathcal{A}$.

Proof. By proposition 5.13 and the fact that χ clearly maps quasi-isomorphic chain complexes to the same element in $\mathrm{K}_0(\mathcal{A}_0)$, the map is well-defined.

The surjectivity of the map is clear: for every $A \in \mathcal{A}_0$, the chain complex $A[0]$ that is concentrated in degree 0 maps to $[A]$.

We prove the injectivity with a proof by contradiction. Note that by lemma 5.30, any element in $\mathrm{K}_0(\mathrm{Ch}_{\geq 0}^{\mathcal{A}_0,\mathrm{b}}\,\mathcal{A})$ can be represented by a chain complex $A \in \mathrm{Ch}_{\geq 0}^{\mathcal{A}_0,\mathrm{b}}\,\mathcal{A}$. Suppose now towards a contradiction that the kernel of the Euler characteristic is non-trivial. Let $A \in \mathrm{Ch}_{\geq 0}^{\mathcal{A}_0,\mathrm{b}}\,\mathcal{A}$ be a chain complex representing a non-zero element in the kernel that has minimal length (with respect to all other chain complexes that represent non-zero elements in the kernel). (We define the length of a chain complex B as the smallest integer $n \geq 0$ such that B_k is zero for all $k \geq n$.) Let n denote the length of A. Consider the following maps of chain complexes:

$$\begin{array}{ccccccccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & \ker d_{n-1} & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
\dots & \longrightarrow & 0 & \longrightarrow & A_{n-1} & \xrightarrow{d_{n-1}} & A_{n-2} & \xrightarrow{d_{n-2}} & \dots & \xrightarrow{d_2} & A_1 & \xrightarrow{d_1} & A_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
\dots & \longrightarrow & 0 & \longrightarrow & \operatorname{im} d_{n-1} & \longrightarrow & A_{n-2} & \xrightarrow{d_{n-2}} & \dots & \xrightarrow{d_2} & A_1 & \xrightarrow{d_1} & A_0 & \longrightarrow & 0
\end{array}$$

Note that all the chain complexes in the rows have homology in $\mathcal{A}_0$. Moreover, the columns are short exact sequences (with the zeroes at the end left out). Hence we have in $\mathrm{K}_0(\mathrm{Ch}_{\geq 0}^{\mathcal{A}_0,\mathrm{b}})$ the identity

$$[A] = \left[\ker d_{n-1}[-(n-1)]\right] + [B]$$

where B is the chain complex $0 \to \operatorname{im} d_{n-1} \to A_{n-2} \to \dots \to A_0 \to 0$ and $[\ker d_{n-1}[-(n-1)]]$ is the chain complex that has $\ker d_{n-1}$ in degree $n-1$ and zero everwhere else. By 5.30, we also have

$$[A] = (-1)^{n-1}[\ker d_{n-1}[0]] + [B]$$

where $\ker d_{n-1}[0]$ is the chain complex concentrated in degree 0.

Consider now the following map of chain complexes:

$$\begin{array}{ccccccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & \operatorname{im} d_{n-1} & \longrightarrow & A_{n-2} & \xrightarrow{d_{n-2}} & \dots & \xrightarrow{d_1} & A_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \operatorname{coker} d_{n-1} & \xrightarrow{d_{n-2}} & \dots & \xrightarrow{d_1} & A_0 & \longrightarrow & 0
\end{array}$$

Denote the lower chain complex by C. This chain map is clearly a quasi-isomorphism, so that we find

$$[B] = [C]$$

in $\mathrm{K}_0(\mathrm{Ch}_{\geq 0}^{\mathcal{A}_0,\mathrm{b}}\,\mathcal{A})$. Combining this with the previous equality, we deduce

$$[A] = (-1)^{n-1}[\ker d_{n-1}[0]] + [C]$$

If C has length 0 or 1, then the equalities $[\chi(C)] = [C]$ and $[\chi(\ker d_{n-1}[0])]$ imply

$$[A] = \chi([A]) = 0$$

which contradicts our assumption that A represents a non-zero element in the kernel. If C has length at least 2, then by replacing $[\ker d_{n-1}[0]]$ with $-[\ker d_{n-1}[-1]]$ if $n-1$ is odd, we find

$$[A] = \begin{cases} [\ker d_{n-1}[0]] + [C] = [C \oplus \ker d_{n-1}[0]] & \text{if } n-1 \text{ is even} \\ [\ker d_{n-1}[-1]] + [C] = [C \oplus \ker d_{n-1}[-1]] & \text{if } n-1 \text{ is odd} \end{cases}.$$

In both cases, the lengths of $C \oplus \ker d_{n-1}[-1]$ and $C \oplus \ker d_{n-1}[0]$ are at least one less than the length of A. This contradicts the assumed minimality of the length of A. Therefore we can conclude that the kernel of $\chi\colon \mathrm{K}_0(\mathrm{Ch}_{\geq 0}^{\mathcal{A}_0,\mathrm{b}}\,\mathcal{A}) \to \mathrm{K}_0(\mathcal{A}_0)$ is zero. Because we had already proven that χ is surjective, this completes the proof that χ is an isomorphism. ☐

We can now conclude the proof of the main theorem of this section that says the Euler characteristic gives an isomorphism $\mathrm{K}_0(\mathrm{Ch}_{\geq 0}^{\mathcal{A}_0}\,\mathcal{A}) \to \mathrm{K}_0(\mathcal{A}_0)$.

Proof of theorem 5.31. This is now a simple combination of the previous two propositions. The composition

$$\mathrm{K}_0(\mathrm{Ch}_{\geq 0}^{\mathcal{A}_0}\,\mathcal{A}) \xrightarrow{\sim} \mathrm{K}_0(\mathrm{Ch}_{\geq 0}^{\mathcal{A}_0,\mathrm{b}}\,\mathcal{A}) \xrightarrow{\sim} \mathrm{K}_0(\mathcal{A}_0)$$

of the isomorphisms from the previous two propositions (5.32 and 5.34) is indeed given by

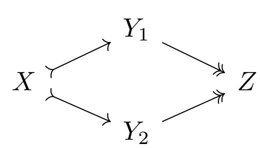
$$[A_\bullet] \mapsto \sum_{i \geq 0} (-1)^i [\mathrm{H}_i(A_\bullet)]$$

since the first isomorphism maps $[A_\bullet]$ to the equivalence class $[B_\bullet]$ of a chain complex $B_\bullet$ satisfying $\mathrm{H}_i(B_\bullet) = \mathrm{H}_i(A_\bullet)$, and the second map is simply the Euler characteristic. ☐

5.7 Presentation of $\mathrm{K}_0(\mathrm{Ch}^{\mathrm{hb}}\,\mathcal{A})$

Let $\mathcal{C}$ be a skeletally small Waldhausen category. Let $S(\mathcal{C})$ be the set of equivalence classes of objects of $\mathcal{C}$ under the equivalence relation $\sim$ generated by the following equivalences:

- $Y_1 \sim Y_2$ if there exists a weak equivalence $Y_1 \xrightarrow{\sim} Y_2$; and
- $Y_1 \sim Y_2$ if there exist X, Z and cofibration sequences



We denote the equivalence class of an object X by $[X]$.

Consider the operation

$$+\colon S(\mathcal{C}) \times S(\mathcal{C}) \to S(\mathcal{C}), \qquad ([X],[Y]) \mapsto [X \oplus Y].$$

This operation is easily seen to be well-defined and gives $S(\mathcal{C})$ the structure of a monoid with neutral element $[0]$. There is a natural well-defined additive map

$$S(\mathcal{C}) \to \mathrm{K}_0(\mathcal{C}), \qquad [X] \mapsto [X].$$

Proposition 5.35. *Let $\mathcal{C}$ be a Waldhausen category and let the relation $\sim$ and the monoid $S(\mathcal{C})$ be as above. Suppose that for every $X \in \mathcal{C}$ there exists an object $Y \in \mathcal{C}$ with $X \oplus Y \sim 0$. Then $S(\mathcal{C})$ is a group and the natural map $\phi\colon S(\mathcal{C}) \to \mathrm{K}_0(\mathcal{C})$ is an isomorphism.*

Proof. It is clear that $S(\mathcal{C})$ is a group, because the assumption implies that it has inverses.

Remark that the relations imposed on the Grothendieck group of $\mathcal{C}$ immediately imply that ϕ is well-defined. The surjectivity is obvious from the fact that $\mathrm{K}_0(\mathcal{C})$ is generated by the isomorphism class $[X] \in \mathrm{K}_0(\mathcal{C})$ with $X \in \mathcal{C}$ and these are all in the image of ϕ.

There exists a unique well-defined map $\psi\colon \mathrm{K}_0(\mathcal{C}) \to S$ that maps $[X]$ in $\mathrm{K}_0(\mathcal{C})$ to $[X]$ in S. Indeed, if

$$0 \to X \to Y \to Z \to 0$$

is an exact sequence, then $[Y] = [X \oplus Z] = [X] + [Y]$ holds in S (both Y and $X \oplus Z$ fit into the middle of a cofibration with X at the left and Z at the right), so that we can apply the universal property. Since $\psi \circ \phi$ is the identity, we conclude that ϕ is also injective. ☐

Remark 5.36. The above proposition applies in particular to the Waldhausen subcategories $\mathcal{C}$ of $\mathrm{Ch}\,\mathcal{A}$ with $\mathcal{A}$ abelian that are closed under taking cones and shifting by -1. Indeed, for every $C_\bullet \in \mathcal{C}$ the cone of the identity on $C_\bullet$, denoted $\mathrm{cone}(\mathrm{id}_{C_\bullet})$, is quasi-isomorphic to zero. Since we have two obvious exact sequences

$$\begin{array}{ccccccc} & & & C[-1] \oplus C & & & \\ & & \nearrow & & \searrow & & \\ 0 \longrightarrow & C[-1] & & & & C & \longrightarrow 0 \\ & & \searrow & & \nearrow & & \\ & & & \mathrm{cone}(\mathrm{id}_{C_\bullet}) & & & \end{array}$$

it follows that $[C_\bullet \oplus C_\bullet[-1]] = [\mathrm{cone}(\mathrm{id}_{C_\bullet})] = [0]$. In particular, if $\mathcal{A}$ is an abelian category and $\mathcal{A}_0 \subset \mathcal{A}$ is a weak Serre subcategory, which is the case we will be interested in later, the proposition gives us a presentation of $\mathrm{Ch}^{\mathcal{A}_0}_{\geq 0}\mathcal{A}$ as a set.

6 Functions of finite degree

In this section we introduce the notion of deviations to measure the failure of functions from monoids to abelian groups to be additive. This notion was introduced in [EML54], but we use a slightly different definition. In 6.3 we use an approach based on [Pas68] to extend a finite degree function defined on a monoid to its groupification. This solves one of the main problems of defining a function on Grothendieck groups.

6.1 Deviations

Recall that a monoid is a pair $(M, \cdot)$ of a set M with an associative operation $\cdot\colon M \times M \to M$ such that M has an identity element.

Definition 6.1. Let M be a monoid, A an abelian group and $f\colon M \to A$ a function. Then for $n \geq 0$, we define the n-th *deviation* $\operatorname{dev}_n f$ of f as the function

$$\operatorname{dev}_n f\colon M^n \to A$$

given by

$$(a_1, \ldots, a_n) \mapsto \sum_{\sigma \subset \{1,2,\ldots,n\}} (-1)^{n-\#\sigma} f\left(\prod_{i \in \sigma} a_i\right)$$

where the order of the integers defines the order in which the product is taken.

Note that $\operatorname{dev}_0 f\colon 1 \to A$ is the map to $f(1)$. For all $a, b, c \in M$ we have by definition $(\operatorname{dev}_1 f)(a) = f(a) - f(1)$, $(\operatorname{dev}_2 f)(a, b) = f(ab) - f(a) - f(b) + f(1)$ and $(\operatorname{dev}_3 f)(a, b, c) = f(abc) - f(ab) - f(ac) - f(bc) + f(a) + f(b) + f(c) - f(1)$. Remark that f is a homomorphism if and only if $f(1) = 0$ and $\operatorname{dev}_2 f$ is zero.

Proposition 6.2. *Let M be a monoid, A an abelian group and $f\colon M \to A$ a function. Then for all $a_1, \ldots, a_n \in M$ we have*

$$f(a_1 \cdots a_n) = \sum_{\sigma \subset \{1,2,\ldots,n\}} (\operatorname{dev}_{\#\sigma} f)(a_i, i \in \sigma).$$

Proof. This is in essence the inclusion-exclusion principle. In this form it can be proven as follows.

First note that $f(a_1 \cdots a_n)$ occurs exactly once at both sides of the equality sign. It remains to show all other summands cancel.

For any $\sigma \subset \{1, 2, \ldots, n\}$ we can naturally reindex the direct summands occurring in $(\operatorname{dev}_{\#\sigma} f)(a_i, i \in \sigma)$ by subsets of σ:

$$(\operatorname{dev}_{\#\sigma} f)(a_i, i \in \sigma) = \sum_{\tau \subset \sigma} (-1)^{\#\sigma - \#\tau} f(a_i, i \in \tau).$$

Take any strict subset $\tau \subsetneq \{1, 2, \ldots, n\}$ and $k \in \{1, 2, \ldots, n\} \setminus \tau$. Then any σ with $\tau \subset \sigma \subset \{1, 2, \ldots, n\}$ contributes $(-1)^{\#\sigma - \#\tau} f(a_i, i \in \tau)$ and all additions or subtractions of $f(a_i, i \in \tau)$ come from such a σ. Now note that each $(-1)^{\#\sigma-\#\tau} f(a_i, i \in \tau)$ coming from a σ that does not contain k, cancels against the contribution $(-1)^{\#\sigma+1-\#\tau} f(a_i, i \in \tau)$ that comes from $\sigma \cup \{k\}$. Because $\sigma \mapsto \sigma \cup \{k\}$ is a bijection from the relevant σ that do not contain k to those that do, this completes the proof.

□

Proposition 6.3. *Let M be a monoid, A an abelian group and $f\colon M \to A$ a function. For all $n \geq 1$ and $a_1, \ldots, a_n \in M$ such that one of the a_i is the identity element $1 \in M$, we have $(\operatorname{dev}_n f)(a_1, \ldots, a_n) = 0$.*

Proof. Let i be such that a_i is the identity element. The sum defining $\operatorname{dev}_n f$ is indexed by subsets of $\{1, \ldots, n\}$. These subsets can be split in two sets: those containing i and those not containing i. The map $S \mapsto S \cup \{i\}$ is a bijection between these two and we see that the summand corresponding to S cancels against that corresponding to $S \cup \{i\}$. □

The following oposition shows that the m-th deviation of the n-th deviation of f is the $m + n - 1$-th deviation.

Proposition 6.4. *Let M be a monoid, A an abelian group and $f\colon M \to A$ a function. For any $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, b_1, \ldots, b_m \in M$ we have*

$$\begin{aligned} &(\operatorname{dev}_m((\operatorname{dev}_n f)(a_1, \ldots, a_{i-1}, -, a_i, \ldots, a_n)))(b_1, \ldots, b_m) \\ &= (\operatorname{dev}_{m+n-1} f)(a_1, \ldots, a_{i-1}, b_1, \ldots, b_m, a_{i+1}, \ldots, a_n). \end{aligned}$$

Proof. It is easy to check the case with $m = 2$ and n arbitrary using proposition 6.3. Then the result follows from realizing that the operator dev_n can be seen as an iterated application of dev_2. □

Definition 6.5. Let M be a monoid, A an abelian group, and $k \in \mathbf{Z}_{\geq 1}$ an integer. A function $f\colon M \to A$ is said to be of *degree* at most k if $\operatorname{dev}_{k+1} f$ is zero. The function $f\colon M \to A$ is said to be of *finite degree* if there exists a $k \in \mathbf{Z}_{\geq 0}$ such that f is of degree at most k.

Remark 6.6. A function $f\colon M \to A$ is a homomorphism if and only if it is of degree ≤ 1 and maps $1 \in M$ to zero.

Remark 6.7. If $f\colon M \to A$ is of degree $\leq k$ (with $k \geq 1$), then for every $m \in M$, the function $(\operatorname{dev}_2 f)(m, -)\colon M \to A$ is of degree $\leq k - 1$.

Example 6.8. A polynomial $f \in \mathbf{Q}[X]$ of degree $\leq k$ is also of degree $\leq k$ as a function.

6.2 Finite-degree functions between abelian groups

We begin with some useful identities on functions of finite degree between abelian groups. Here we denote all group laws additively.

Lemma 6.9. *Let $f\colon A \to B$ be a function of finite degree between two abelian groups. For all $a \in A$, the alternating sum*

$$\sum_{i\geq 0}(-1)^i(\operatorname{dev}_i f)(a,\dots,a) \qquad \textit{is finite and equals } f(-a).$$

Proof. The finiteness follows immediately from the assumption that f has finite degree. Then we calculate

$$f(a-a) = (\operatorname{dev}_2 f)(a,-a) + (\operatorname{dev}_1 f)(a) + (\operatorname{dev}_1 f)(-a) + f(0).$$

Since $(\operatorname{dev}_1 f)(-a) + f(0) = f(-a)$, we can rewrite this as

$$f(-a) = f(0) - (\operatorname{dev}_1 f)(a) - (\operatorname{dev}_2 f)(a,-a).$$

Applying the same reasoning to the function $(\operatorname{dev}_2 f)(a, -)$, we get

$$\begin{aligned}(\operatorname{dev}_2 f)(a,-a) &= (\operatorname{dev}_2 f)(a,0) - (\operatorname{dev}_2 f)(a,a) - (\operatorname{dev}_3 f)(a,a,-a)\\ &= -(\operatorname{dev}_2 f)(a,a) - (\operatorname{dev}_3 f)(a,a,-a)\end{aligned}$$

using proposition 6.3 in the second equality. By substituting this and continuing in the same way, we find the identity that was to be proven. □

When we say that $J \subset \{1,\dots,m\}$ is a multiset, we mean that J is a multiset in which only the elements $1,\dots,m$ occur. In other words, it can be considered as a function $j\colon \{1,\dots,m\} \to \mathbf{Z}_{\geq 0}$. We denote by $\#J$ the number of elements in J, counted with multiplicity (so $\#J := \sum_{i=1}^m j(i)$).

Proposition 6.10. *Let $f\colon A \to B$ be a function of finite degree between two abelian groups. Then for all n, the n-th deviation $\operatorname{dev}_n f$ is symmetric in its arguments and for all $a_1,\dots,a_n,b_1,\dots,b_m \in A$ we have*

$$f\Big(\sum_{i=1}^n a_i - \sum_{i=1}^m b_i\Big) = \sum_{\substack{I\subset\{1,\dots,n\},\\ J\subset\{1,\dots,m\}\textit{multiset}}} (-1)^{\#J}(\operatorname{dev}_{\#I+\#J} f)(a_i, i\in I, b_j, j \in J).$$

where the sum is taken over all subsets $I \subset \{1,\dots,n\}$ and all multisets J with elements in $\{1,\dots,m\}$.

Proof. Since f is of finite degree, the sum on the right is finite. By proposition 6.2, we have

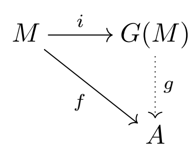
$$\begin{aligned} f(\sum_{i=1}^{n} a_i - \sum_{i=1}^{m} b_i) &= f(\sum_{i=1}^{n} a_i + \sum_{i=1}^{m}(-b_i)) \\ &= \sum_{\substack{I \subset \{1,\dots,n\}, \\ J \subset \{1,\dots,m\}}} (\mathrm{dev}_{\#I+\#J}\, f)(a_i, i \in I, -b_j, j \in J) \end{aligned}$$

Applying 6.9 gives the desired equation. ☐

6.3 Finite-degree functions on groupifications

The theory in this section are based on [Pas68]. We now turn to the question of extending a monoid morphism $f\colon M \to A$ to its groupification $G(M)$.

Definition 6.11. Let M be a commutative monoid. Then the groupification of M is an abelian group $G(M)$ with a map of monoids $i\colon M \to G(M)$, that satisfies the the universal property that for every monoid morphism $f\colon M \to A$ from M to an abelian group A, there exists a unique group homomorphism $g\colon G(M) \to A$ making the following diagram commute:



As with all constructions by universal properties, this group is unique up to unique isomorphism. We can explicitly construct $G(M)$: we take the equivalence classes of the set of formal expressions of the form $n - m$ with $n, m \in M$ under the equivalence $\sim$ defined by

$$n - m \;\sim p - q \qquad \text{iff } n + q = p + m.$$

The sum of $n - m$ and $p - q$ is $(n + p) - (m + q)$ and this addition is compatible with the equivalence relation.

Let M be a monoid and $i\colon M \to G(M)$ its groupification. Then for every abelian group A, restriction to M is a bijection from the set of homomorphisms $G(M) \to A$ to the set of homomorphisms $M \to A$. In this section we prove that restriction to M is also a bijection from the set of functions of $G(M) \to A$ of degree $\leq k$ to the set of functions $M \to A$ of degree $\leq k$, for any $k \geq 0$.

Let M be a monoid. We can define the monoid ring $\mathbf{Z}[M]$ in the same way we define a group ring: its elements are formal sums and the product is the linear extension of the product in M. We use this monoid ring in the construction of the following universal object.

Proposition 6.12 (Universal map of degree $\leq k$)**.** *Let M be a monoid and $k \geq 1$ an integer. Then there exists an abelian group $M^{\#k}$ and a map $i\colon M \to M^{\#k}$ of degree $\leq k$ such that for all maps $f\colon M \to A$ of degree $\leq k$ to an abelian group A, there exists a unique group homomorphism $g\colon M^{\#k} \to A$ making the following diagram commute*

$$\begin{array}{ccc} M & \xrightarrow{\ i\ } & M^{\#k} \\ & \searrow^{f} & \downarrow{\scriptstyle g} \\ & & A \end{array} .$$

Proof. We give an explicit construction of $M^{\#k}$. Let I be the ideal in $\mathbf{Z}[M]$ generated by all elements of the form $a - 1$ with $a \in M$. We define $M^{\#k} = \mathbf{Z}[M]/I^{k+1}$. The natural map

$$i\colon M \to \mathbf{Z}[M]/I^{k+1}$$

is easily seen to be a map of degree $\leq k$, because for all $a_1, \ldots, a_{k+1} \in M$ we see that $(\operatorname{dev}_{k+1} i)(a_1, \ldots, a_{k+1})$ is represented by the element

$$\sum_{\sigma \subset \{1,\ldots,k+1\}} (-1)^{k+1-\#\sigma} \prod_{i \in \sigma} a_i = (a_1 - 1)(a_2 - 1)\cdots(a_{k+1} - 1) \in I^{k+1}$$

and hence it is zero in $\mathbf{Z}[M]/I^{k+1}$.

Any map $f\colon M \to A$ can be extended linearly to a map $\mathbf{Z}[M] \to A$. Let J be the subgroup of $\mathbf{Z}[M]$ generated by the elements of the form $(a_1 - 1)(a_2 - 1)\cdots(a_{k+1} - 1)$ with $a_1, \ldots, a_{k+1} \in M$. However, note that J is closed under multiplication by $\mathbf{Z}[M]$, because for any $r, a_1, \ldots, a_{k+1} \in M$, we have

$$\begin{aligned} & r(a_1 - 1)(a_2 - 1)\cdots(a_{k+1} - 1) \\ &= (ra_1 - r)\cdot(a_2 - 1)\cdots(a_{k+1} - 1) \\ &= ((ra_1 - 1) - (r - 1))\cdot(a_2 - 1)\cdots(a_{k+1} - 1) \\ &= (ra_1 - 1)\cdot(a_2 - 1)\cdots(a_{k+1} - 1) - (r - 1)\cdot(a_2 - 1)\cdots(a_{k+1} - 1) \end{aligned} .$$

Hence J is an ideal and because I^{k+1} is the ideal in $\mathbf{Z}[M]$ generated by the elements of the form $(a_1 - 1)(a_2 - 1)\cdots(a_{k+1} - 1)$ with $a_1, \ldots, a_{k+1} \in M$, we conclude $J = I^{k+1}$.

Since $f\colon M \to A$ is a map of degree $\leq k$, we have $I^{k+1} = J \subset \ker f$. Hence we get an induced map $g\colon \mathbf{Z}[M]/I^{k+1} \to A$. The commutativity of the diagram is obvious. The unicity of g follows immediately from the fact that g is determined by requiring additivity and $g(i(a)) = f(a)$ for all $a \in M$. □

Theorem 6.13. *Let M be a commutative monoid and $i\colon M \to G(M)$ its groupification. The induced map*

$$i^{\#k}\colon M^{\#k} \to (G(M))^{\#k}$$

is an isomorphism.

Proof. Write $G = G(M)$. Let $I \subset \mathbf{Z}[M]$ be the ideal generated by all $a - 1$ with $a \in M$ and let $J \subset \mathbf{Z}[G]$ be the ideal generated by the $b - 1$ with $b \in G$. The induced map $i^{\#k}$ above is the map $\mathbf{Z}[M]/I^{k+1} \to \mathbf{Z}[G]/J^{k+1}$ induced by $\mathbf{Z}[M] \to \mathbf{Z}[G]$. We will show that this is an isomorphism.

Note that $\mathbf{Z}[G]$ is the localisation of $\mathbf{Z}[M]$ at the multiplicative system M. Moreover, the localisation of I is J, because for any $st^{-1} \in G$ with $s, t \in M$, we have

$$st^{-1} - 1 = t^{-1}(s - t) = t^{-1}(s - 1) - t^{-1}(t - 1) \in M^{-1}I.$$

Therefore, $\mathbf{Z}[G]/J^{k+1}$ is the localisation of $\mathbf{Z}[M]/I^{k+1}$ at M. We can now complete the proof by showing that all elements in M are already invertible in $\mathbf{Z}[M]/I^{k+1}$.

There exists a polynomial $g \in \mathbf{Z}[X]$ with $(1 - X)^{k+1} = 1 - Xg(X)$. Therefore we find for every $a \in M$ the equality $a \cdot g(a) = 1$ in $\mathbf{Z}[G]/J^{k+1}$ that shows that a is invertible. □

Corollary 6.14. *Let M be a commutative monoid, let $i\colon M \to G(M)$ be its groupification and let A be an abelian group. Then the restriction*

$$\{\text{maps } M \to A \text{ of degree} \leq k\} \to \{\text{maps } G(M) \to A \text{ of degree} \leq k\},$$
$$f \mapsto f \circ i$$

is a bijection. □

Remark 6.15. Let M a commutative monoid, A an abelian group and $f\colon M \to A$ a function of degree $\leq k$. Then proposition 6.10 gives an explicit formula for the extension of f along the groupification $M \to G(M)$. (Note that the deviations on the right hand side of the equality in 6.10 are fully determined by $f\colon M \to A$.)

7 Cross-effect functors

Recall that a functor between additive categories is reduced if it sends the zero object to the zero object. To every reduced functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories we can associate a sequence of cross-effect functors $(\mathrm{cr}_i(F)\colon \mathcal{A}^i \to \mathcal{B})_{i\in\mathbf{Z}_{\geq 1}}$ that measure the failure of the functor F to be additive in the same way the deviations defined in the previous section measure the failure of a function to be additive. The original definition of cross-effect functors can be found in [EML54].

We define the cross-effect functors explicitely in definition 7.1, but proposition 7.4 is in practice more useful to determine the cross-effects of a given functor.

Let $\mathcal{A}, \mathcal{B}$ be abelian categories and let $(A_1, \dots, A_n) \in \mathcal{A}^n$. We denote by e_i the idempotent $A_1 \oplus \cdots \oplus A_n \twoheadrightarrow A_i \hookrightarrow A_1 \oplus \cdots \oplus A_n$. Remark also that a functor $F\colon \mathcal{A} \to \mathcal{B}$ induces for every $A, B \in \mathcal{A}$ a function

$$F\colon\ \mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{B}}(FA, FB)$$

of abelian groups and hence we can take the deviations of these functions.

Definition 7.1. Let $\mathcal{A}, \mathcal{B}$ be abelian categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be a reduced functor. For $n \geq 1$, we define the n-th *cross-effect functor* of F as the functor $\mathrm{cr}_n(F)\colon \mathcal{A}^n \to \mathcal{B}$ that sends an n-tuple $(A_1, A_2, \dots, A_n)$ to the image of the map

$$(\mathrm{dev}_n F)(e_1, \dots, e_n)\colon A_1 \oplus \cdots \oplus A_n \to A_1 \oplus \cdots \oplus A_n.$$

An n-tuple $(f_1, \dots, f_n)$ of maps $f_i\colon A_i \to A'_i$ is sent to the map $\mathrm{cr}_n(F)(A_1, \dots, A_n) \to \mathrm{cr}_n(F)(A'_1, \dots, A'_n)$ that is induced by the restriction of $f_1 \oplus \cdots \oplus f_n\colon A_1 \oplus \cdots \oplus A_n \to A'_1 \oplus \cdots \oplus A'_n$. (Note that the image of this restriction is contained in $\mathrm{cr}_n(F)(A'_1, \dots, A'_n)$, so that we do indeed get such an induced map.)

For every $n \geq 1$, a natural transformation $\tau\colon F \to G$ induces a natural transformation $\tau_n\colon\ \mathrm{cr}_n(F) \to \mathrm{cr}_n(G)$ by restriction.

Proposition 7.2. *We list the following basic properties of cross-effect functors:*

(a) $\mathrm{cr}_1(F) = F$;

(b) $\mathrm{cr}_n(F)$ *is symmetric in its arguments, up to permutation of the coordinates; and*

(c) $\mathrm{cr}_n(F)(A_1, \dots, A_n)$ *is zero if one of its arguments* A_i *is zero.*

Proof. These properties follow quite directly from the definition. For the last property, remark that the map if A_i is zero, then the idempotent $e_i\colon A_1 \oplus \cdots \oplus A_n \to A_1 \oplus \cdots \oplus A_n$ is the zero map and hence so is $(\mathrm{dev}_n F)(e_1, \dots, e_n)$ by proposition 6.3. $\square$

Proposition 7.3. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F\colon \mathcal{A} \to \mathcal{B}$ a reduced functor. For every n-tuple $(A_1, \dots, A_n)$, there is an isomorphism*

$$F(A_1 \oplus \cdots \oplus A_n) \cong \bigoplus_{\emptyset \neq \sigma \subseteq \{1,2,\dots,n\}} \mathrm{cr}_{\#\sigma}(F)(A_i, i \in \sigma).$$

functorial in the A_i. If $\tau\colon F \to G$ is a natural transformations, then τ and the induced natural transformations $\tau_n\colon \mathrm{cr}_n(F) \to \mathrm{cr}_n(G)$ fit into the commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\cong} & \bigoplus_{\emptyset \neq \sigma \subseteq \{1,2,\dots,n\}} \mathrm{cr}_{\#\sigma}(F) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \bigoplus_{\emptyset \neq \sigma \subseteq \{1,2,\dots,n\}} \tau_{\#\sigma}} \\ G & \xrightarrow{\cong} & \bigoplus_{\emptyset \neq \sigma \subseteq \{1,2,\dots,n\}} \mathrm{cr}_{\#\sigma}(G) \end{array}$$

where all functors, including F and G, are considered as functors $\mathcal{A}^n \to \mathcal{B}$ in the obvious way.

Proof. The isomorphism follows from the inclusion-exclusion principle of deviations formulated in proposition 6.2: $F(A_1 \oplus \cdots \oplus A_n)$ is the image of $F(e_1 \oplus \cdots \oplus e_n) = F(\mathrm{id})$ and by the proposition, the function $F(e_1 \oplus \cdots \oplus e_n)$ can be expressed as the sum of deviations. It is not difficult to see (using, among other things, the reducedness of F) that these deviations form a set of complete orthogonal idempotents (their sum is the identity and the composition of two different idempotents is zero) and hence they induce an isomorphism of $F(A_1 \oplus \cdots \oplus A_n)$ with the direct sum of the idempotents' images, which are by definition the cross-effects. The commutative diagram follows from the fact that the τ_n are induced by restriction. □

The following proposition is very useful for inductively determining an explicit description of the cross-effect functors $\mathrm{cr}_n(F)$.

Proposition 7.4. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F\colon \mathcal{A} \to \mathcal{B}$ a reduced functor. For any $n \geq 1$ and any $n+1$-tuple $(A_1, A_1', A_2, A_3, \dots, A_n) \in \mathcal{A}^{n+1}$, there is an isomorphism*

$$\begin{aligned} \mathrm{cr}_n(F)(A_1 \oplus A_1', A_2, \dots, A_n) \cong\, & \mathrm{cr}_n(F)(A_1, A_2, \dots, A_n) \\ & \oplus \mathrm{cr}_n(F)(A_1', A_2, \dots, A_n) \\ & \oplus \mathrm{cr}_{n+1}(F)(A_1, A_1', A_2, \dots, A_n) \end{aligned}$$

natural as functors $\mathcal{A}^{n+1} \to \mathcal{B}$.

Proof. This follows from 6.4. □

Example 7.5 (Tensor powers)**.** Let R be a commutative ring, let $\mathbf{Mod}_R$ denote the category of modules over R and let $n \geq 1$ be an integer. Consider the tensor power $-^{\otimes n}\colon \mathbf{Mod}_R \to \mathbf{Mod}_R$ that sends a module M to its n-th tensor power $M^{\otimes n}$. Then for all integers $i \geq 1$, the i-th cross-effect of $-^{\otimes n}$ is given by

$$\operatorname{cr}_i(-^{\otimes n})(A_1, \ldots, A_i) = \bigoplus_{\substack{\sigma\colon \{1,\ldots,n\} \twoheadrightarrow \{1,\ldots,i\},\\ \text{surjective}}} A_{\sigma(1)} \otimes \ldots \otimes A_{\sigma(n)}.$$

This is quite easy to prove. For $-^{\otimes 2}$, this follows from

$$\begin{aligned} F(A \oplus B) &= (A \oplus B) \otimes (A \oplus B) \\ &\cong F(A) \oplus F(B) \oplus A \otimes B \oplus B \otimes A \end{aligned}$$

and the remark that $(A, B) \mapsto A \otimes B \oplus B \otimes A$ is bilinear. For the others, one can easily determine the cross-effects inductively using 7.4.

Example 7.6 (Exterior and symmetric powers)**.** Let again $\mathbf{Mod}_R$ denote the category of modules over a fixed commutative ring R. Then for any $n, k \geq 1$ and any k-tuple $(A_1, \ldots, A_k) \in \mathbf{Mod}_R^k$, we have

$$\operatorname{cr}_k(\bigwedge^n)(A_1, \ldots, A_k) \cong \bigoplus_{\substack{1 \leq i_1, \ldots i_k\\ i_1 + \ldots + i_k = n}} \bigwedge^{i_1} A_1 \otimes \ldots \otimes \bigwedge^{i_k} A_k.$$

An analogous formula holds when we replace the exterior powers by symmetric powers, either the invariants Γ^n or quotients Sym^n of the n-th tensor power under permutation of coordinates.

Definition 7.7. Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $k \geq 0$ an integer. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is said to be of *degree* $\leq k$ if $\operatorname{cr}_{k+1}(F)$ is zero. It is said to be of *finite degree* if there exists a $k \in \mathbf{Z}_{\geq 0}$ such that it is of degree $\leq k$.

Note that if $\operatorname{cr}_k(F)$ is zero for some k, then so are the higher cross-effects $\operatorname{cr}_{k'}(F)$ with $k' \geq k$. A useful reformulation is that F is of degree $\leq k$ if $\operatorname{cr}_k(F)$ is multilinear.

Example 7.8. The calculations in the previous examples show that for $n \geq 1$, the n-th tensor power, the n-th exterior power and the n-th symmetric power are all of degree $\leq n$.

Example 7.9. An example of a functor that is not of finite degree, is the functor $M \mapsto \bigoplus_{i=0}^{\infty} \bigwedge^i M$ on the category of modules over a commutative ring, even though it sends finitely generated modules to finitely generated modules.

Proposition 7.10. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F\colon \mathcal{A} \to \mathcal{B}$ a reduced functor. The n-th cross effect functor of the induced functor*

$$F\colon \operatorname{Simp} \mathcal{A} \to \operatorname{Simp} \mathcal{B}$$

(see definition 3.10,) is equal to the functor that $\operatorname{cr}_n F\colon \mathcal{A} \times \ldots \times \mathcal{A} \to \mathcal{B}$ induces on simplicial objects.

Proof. The n-th cross-effect of n simplicial sets $X^1, \ldots, X^n$ is by definition the image of the n-th deviation $\mathrm{dev}_n(e_1, \ldots, e_n)$ of the n idempotents of $X^1 \oplus \cdots \oplus X^n$. Images of simplicial objects can be calculated degree-wise and the simplicial map $\mathrm{dev}_n(e_1, \ldots, e_n)$ is in degree i the n-th deviation of the n idempotents of $X_i^1 \oplus \cdots \oplus X_i^n$ (where X_i^j denotes the degree i part of X^j). Hence this is image is simply $\mathrm{cr}_n(F)(X_i^1, \ldots, X_i^n)$ and the result follows. $\square$

8 Deriving functors with cross-effects

In this section we use cross-effects to do explicit calculations with derived functors. This culminates in theorem 8.18, where we use the calculations to define an induced map on Grothendieck groups for functors of finite degree defined on suitable categories.

8.1 Calculations in terms of cross-effect functors

Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor of finite degree between abelian categories. In this section we determine an explicit formula for $NF\Gamma$ (proposition 8.5). This summarizes section four from [SK10].

Notation 8.1. We denote by $\mathrm{Sur}([n],[k])$ the set of non-decreasing surjections $\phi\colon [n] \to [k]$. For $\phi \in \mathrm{Sur}([n],[k])$, we write

$$\phi^{\Delta} := \{x \in [n-1] = \{0,1,\ldots,n-1\} : \phi(x) \neq \phi(x+1)\}.$$

Intuitively, ϕ^{Δ} is the set of points in $[n]$ after which ϕ "jumps".

For any set S, we denote by $\mathcal{P}(S)$ its powerset.

Proposition 8.2. *The map*

$$\bigsqcup_{k=0}^{n} \mathrm{Sur}([n],[k]) \to \mathcal{P}([n-1]), \qquad \phi \mapsto \phi^{\Delta}$$

is a bijection. Under this bijection, maps in $\mathrm{Sur}([n],[k])$ *corresponds to subsets of cardinality* k. □

Definition 8.3. A subset $\alpha \subset \bigsqcup_{k=0}^{n} \mathrm{Sur}([n],[k])$ is called *honourable* if the set $\bigcup_{\phi\in\alpha} \phi^{\Delta}$ is equal to $\{0,1,\ldots,n-1\}$.

Definition 8.4. A *collection type* is a map $t\colon \mathbf{Z}_{\geq 0} \to \mathbf{Z}_{\geq 0}$ with finite support. The *collection type associated to* a subset $\alpha \subset \bigsqcup_{k=0}^{n} \mathrm{Sur}([n],[k])$ is the collection type $t\colon \mathbf{Z}_{\geq 0} \to \mathbf{Z}_{\geq 0}$ defined by

$$k \mapsto \#(\alpha \cap \mathrm{Sur}([n],[k])).$$

For $\alpha \subset \bigsqcup_{k=0}^{n} \mathrm{Sur}([n],[k])$, we denote by t_α its associated collection type. The following is proposition 4.4 in [SK10]:

Proposition 8.5. *Let $\mathcal{A},\mathcal{B}$ be abelian categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be a reduced functor. Then we have a canonical isomorphism for all $C_\bullet \in \mathrm{Ch}_{\geq 0}\mathcal{A}$:*

$$NF\Gamma(C_\bullet)_n \cong \bigoplus_{\substack{\alpha \subset \bigsqcup_{k=0}^{n} \mathrm{Sur}([n],[k]) \\ \alpha \text{ is honourable}}} \mathrm{cr}_{\#\alpha}(F)\big(\underbrace{C_0,\ldots,C_0}_{t_\alpha(0) \text{ times}},\ldots,\underbrace{C_n,\ldots,C_n}_{t_\alpha(n) \text{ times}}\big).$$

□

Note that the collection type determines the summand corresponding to a subset of $\operatorname{Sur}([n],[k])$.

The following is an easy but important corollary, that is also proven in [DP61, 4.23].

Corollary 8.6. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F\colon \mathcal{A} \to \mathcal{B}$ a functor of degree $\leq k$. Let $N \in \mathbf{Z}_{\geq 0}$ and $C_\bullet \in \operatorname{Ch}_{\geq 0} \mathcal{A}$ with $C_n = 0$ for all $n > N$. Then $NF\Gamma(C_\bullet)_n$ is zero for all $n > kN$.*

Proof. Since cross-effects are zero when one of the arguments is zero (proposition 7.2), in the direct sum of $NF\Gamma(C_\bullet)_n$ of the previous proposition, we only need to consider subsets $\alpha \subset \bigsqcup_{k=0}^{\max(n,N)} \operatorname{Sur}([n],[k])$; all other subsets do not contribute anything. Furthermore, we only need to consider α with $\#\alpha \leq k$ because F is of degree $\leq k$. For such an α, we see that $\bigcup_{\phi\in\alpha} \phi^\Delta$ has at most $k \cdot N$ elements. Therefore, if $n > k \cdot N$, no such α are honourable. □

8.2 Enumerating honourable subsets

This section has two goals. The first is to count the number of honourable subsets of $\operatorname{Sur}([n],[k])$ that have a given collection type t. We do this by counting the number of subsets of $\operatorname{Sur}([n],[k])$ with collection type t and using the inclusion-exclusion principle to filter out the non-honourable subsets. This is achieved in proposition 8.16.

Let $g_t(n)$ denote the number of honourable subsets of collection type t. The second goal is to determine the alternating sum $\sum_{n\geq 0}(-1)^n g_t(n)$, which is a term that appears when we take the Euler characteristic of a complex $NF\Gamma(C_\bullet)$. This is given by theorem 8.17.

The combinatorics in this section is fairly standard, but as far as the author is aware, the last two results are new.

It is convenient to extend the definition of $\binom{r}{k}$ to negative r.

Definition 8.7. For all $r, k \in \mathbf{Z}$ we define

$$\binom{r}{k} = \begin{cases} \frac{1}{k!} r(r-1)\cdot \ldots \cdot (r-k+1) & \text{if } k \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

The definition implies that for any $k \in \mathbf{Z}_{\geq 0}$ and any $r \in \mathbf{Z}$ with $0 \leq r < k$ we have

$$\binom{-1}{k} = (-1)^k \qquad \text{and} \qquad \binom{r}{k} = 0.$$

The definition also implies $\binom{r}{0} = 1$ for all $r \in \mathbf{Z}$.

Definition 8.8. Let R be a ring and A an abelian group. We define the operator Δ on the set of functions $f\colon R \to A$ by defining $\Delta f\colon R \to A$ as the function given by

$$x \mapsto f(x+1) - f(x).$$

Though the notation Δ is also used for the simplicial category, the context will always make clear what is meant.

Lemma 8.9. *Let $f\colon \mathbf{R} \to \mathbf{R}$ be any map and let $n \geq 0$. Then $\Delta^n f\colon \mathbf{R} \to \mathbf{R}$ is given by*

$$m \mapsto \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(m+k).$$

Proof. This is easy to prove using induction on n. □

Remark 8.10. The polynomial ring $\mathbf{R}[X]$ can naturally be embedded into the ring of functions $\mathbf{R} \to \mathbf{R}$. In this way we can apply the operator Δ to polynomials $f \in \mathbf{R}[X]$: we have $(\Delta f)(X) = f(X+1) - f(X)$ as polynomials in $\mathbf{R}[X]$.

We denote by $\deg(f)$ the degree of a polynomial f.

Lemma 8.11. *Let $f \in \mathbf{R}[X]$ be a polynomial. Then we have*

$$\deg(\Delta f) \leq \deg(f) - 1.$$

Proof. If f is constant, then Δf is zero and its degree is $-\infty$. If f is of degree at least one, we see that $f(X)$ and $f(X+1)$ have the same leading coefficient, and hence $(\Delta f)(X) = f(X+1) - f(X)$ is of lower degree than f. □

Notation 8.12. Let R a ring and $x \in R$. For $m \in \mathbf{Z}_{\geq 0}$, we write

$$x^{\underline{m}} = x(x-1)(x-2)\cdots(x-m+1).$$

This is often called the *falling factorial* or the *Pochhammer symbol*.

The following analogue of Taylor series expansions is well-known:

Proposition 8.13. *For any $f \in \mathbf{R}[X]$, we have for every $x, a \in \mathbf{R}$*

$$f(x+a) = \sum_{k=0}^{\infty} \frac{1}{k!} (\Delta^k f)(x) a^{\underline{k}}.$$

Proposition 8.14. *Let $f \in \mathbf{R}[X]$. Then for any $n \in \mathbf{Z}_{\geq 0}$ and any $m \in \mathbf{Z}$, we have*

$$\sum_{k=0}^{n} (-1)^k \Delta^k f(m) = (-1)^n \Delta^{n+1} f(m-1) + f(m-1)$$

Proof. First note that $\frac{1}{k!}(-1)^{\underline{k}} = (-1)^k$. Then by 8.13, we have for all $m \in \mathbf{Z}$

$$f(m-1) = \sum_{k=0}^{\infty} (\Delta^k f)(m)(-1)^k$$

and similarly

$$(\Delta^{n+1} f)(m-1) = \sum_{k=0}^{\infty} (\Delta^{n+1+k} f)(m)(-1)^k.$$

The result now immediately follows. □

If $f \in \mathbf{R}[X]$ is a polynomial, then lemma 8.11 shows that $\Delta^n f$ is zero for all n large enough.

Corollary 8.15. *Let $f \in \mathbf{R}[X]$. Take $N \in \mathbf{Z}_{\geq 0}$ with $\Delta^N f = 0$. Then for all $n \geq N$ and $m \in \mathbf{Z}$, we find*

$$\sum_{k=0}^{n} (-1)^k \Delta^k f(m) = f(m-1).$$

Proof. From the definition of the operator Δ it is evident that for all $n \geq N$, $\Delta^n f$ vanishes everywhere. Hence this is a simple application of proposition 8.14. □

Proposition 8.16. *Let $t\colon \mathbf{Z}_{\geq 0} \to \mathbf{Z}_{\geq 0}$ be a collection type. Then*

$$f_t(n) = \prod_{i \geq 0} \binom{\binom{n}{i}}{t(i)}$$

defines a well-defined map $f_t\colon \mathbf{Z} \to \mathbf{Z}$ and for all $n \in \mathbf{Z}_{\geq 0}$,

(a) $f_t(n)$ is the number of subsets $\alpha \subset \sqcup_{k=0}^{n} \operatorname{Sur}([n],[k])$ of collection type t; and

(b) $\Delta^n f_t(0)$ is the number of honourable subsets $\alpha \subset \sqcup_{k=0}^{n} \operatorname{Sur}([n],[k])$ of collection type t.

Proof. Note that f_t is well-defined, because t has finite support and $t(i) = 0$ implies that the factor corresponding to i is 1. Hence the product is in fact a finite product.

Part (a) is not difficult to prove. If $\alpha \subset \bigsqcup_{k=0}^{n} \operatorname{Sur}([n],[k])$ has collection type t, then for each i, the corresponding subset $T \subset \mathcal{P}([n-1])$ has exactly $t(i)$ subsets of size i. There are exactly $\binom{\binom{n}{i}}{t(i)}$ ways of choosing $t(i)$ subsets of size i from a set of n elements, so there are precisely $\prod_{0 \leq i} f_t(i)$ possible ways to choose T and hence as many $\alpha \subset \bigsqcup_{k=0}^{n} \operatorname{Sur}([n],[k])$ with collection type t.

To prove part (b), we use the inclusion-exclusion principle. This principle can be stated as follows: if S is some finite set and $A_1, \ldots, A_m$ are subsets of S, then

$$\#\left(\bigcup_{i=1}^{m} A_i\right) = \sum_{k=1}^{m} (-1)^{k+1} \sum_{\substack{\sigma \subset \{1,\ldots,m\} \\ \#\sigma = k}} \#\left(\bigcap_{i\in\sigma} A_i\right).$$

We can apply this principle by taking

$$S := \left\{\alpha \subset \bigsqcup_{k=0}^{n} \operatorname{Sur}([n],[k]) : \alpha \text{ has collection type } t\right\}$$

and for all $i = 0, \ldots, n-1$

$$A_i := \left\{\alpha \in S : i \notin \bigcup_{\phi\in\alpha} \phi^{\Delta}\right\}.$$

Then $\bigcup_{i=0}^{n-1} A_i$ is the set of non-honourable α in S. The inclusion-exclusion principle then gives us

$$\#(A_0 \cup \ldots \cup A_{n-1}) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} f_t(n-k)$$

because for each of the $\binom{n}{k}$ subsets $\sigma \subset \{0, \ldots, n-1\}$ of size k, the intersection $\bigcap_{i\in\sigma} A_i$ can be considered (after reindexing) as the set of $\alpha \subset \bigsqcup_{l=0}^{n-k} \operatorname{Sur}([n-k],[l])$ and we have already proven there are $f_t(n-k)$ of those. The honourable subsets are exactly those not contained in this union, so their number is equal to $\#S - \#(A_0 \cup \ldots \cup A_{n-1})$ and by the calculations above this is indeed the same as $\Delta^n f(0)$. □

Theorem 8.17. *Let $t \colon \mathbf{Z}_{\geq 0} \to \mathbf{Z}_{\geq 0}$ be a collection type and let $f_t \colon \mathbf{Z} \to \mathbf{Z}$ be the associated map as defined in 8.16 . Then we have*

$$\sum_{i=0}^{\infty} (-1)^n \Delta^n f_t(0) = \begin{cases} 0 & \text{if } t(i) > 1 \text{ for some even } i \geq 0 \\ (-1)^{\sum_{i \text{ odd}} t(i)} & \text{otherwise} \end{cases}$$

Proof. First note that this sum is in fact a finite sum, because for any n and any subset $\alpha \subset \bigsqcup_{k=0}^{n} \operatorname{Sur}([n],[k])$ of collection type t, the set $\bigcup_{\phi\in\alpha} \phi^{\Delta}$ contains at most $\sum_{i\geq 0} i \cdot t(i) < \infty$ elements. Hence for all $N > \sum_{i\geq 0} i \cdot t(i)$, there are no honourable subsets in $\bigsqcup_{k=0}^{N} \operatorname{Sur}([N],[k])$.

Because $f_t(n)$ is a polynomial in n, the function $\Delta^N f_t$ is zero for all large enough N (lemma 8.11). Hence we can apply corollary 8.15 to conclude that the alternating sum is equal to $f_t(-1)$. The rest of the proof is an easy calculation:

for all $i \in \mathbf{Z}_{\geq 0}$ with $t(i) \neq 0$, we see

$$\binom{\binom{(-1)}{i}}{t(i)} = \binom{(-1)^i}{t(i)} = \begin{cases} \binom{1}{t(i)} & \text{if } i \text{ is even} \\ \binom{-1}{t(i)} & \text{if } i \text{ is odd} \end{cases}$$
$$= \begin{cases} 0 & \text{if } i \text{ is even and } t(i) > 1 \\ 1 & \text{if } i \text{ is even and } t(i) \leq 1 \\ (-1)^{t(i)} & \text{if } i \text{ is odd} \end{cases} .$$

Taking the product over all such factors, we find

$$f_t(-1) = \begin{cases} 0 & \text{if } t(i) > 1 \text{ for some even } i \geq 0 \\ (-1)^{\sum_{i \text{ odd}} t(i)} & \text{otherwise} \end{cases} .$$

□

8.3 Induced map on Grothendieck groups

The goal of this section is to prove the following theorem.

Note that if $F\colon \mathcal{A} \to \mathcal{B}$ is a reduced functor of finite degree between two abelian categories, with $\mathcal{A}$ having enough projectives, then $\mathbf{L}F$ restricts to a functor $\mathbf{L}F\colon \mathrm{Ch}^{\mathrm{b}}_{\geq 0}\mathcal{A} \to \mathcal{K}^{\mathrm{b}}_{\geq 0}\mathcal{B}$ by 8.6.

Theorem 8.18. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories, let $k \in \mathbf{Z}_{\geq 0}$ and let $F\colon \mathcal{A} \to \mathcal{B}$ be a reduced functor of degree $\leq k$. Assume that every object in $\mathcal{A}$ has a finite resolution in projective objects. Then there exists a unique set-theoretic map $\mathrm{K}_0(F)\colon \mathrm{K}_0(\mathcal{A}) \to \mathrm{K}_0(\mathcal{B})$ such that the following diagram commutes:*

$$\begin{array}{ccc} [\mathrm{Ch}^{\mathrm{b}}_{\geq 0}\mathcal{A}] & \xrightarrow{\mathbf{L}F} & [\mathcal{K}^{\mathrm{b}}_{\geq 0}(\mathcal{B})] \\ \chi\downarrow & & \downarrow\chi \\ \mathrm{K}_0(\mathcal{A}) & \xrightarrow[\mathrm{K}_0(F)]{} & \mathrm{K}_0(\mathcal{B}) \end{array} . \tag{3}$$

The map $\mathrm{K}_0(F)$ is of degree $\leq k$ and if $P, Q \in \mathcal{A}$ are projective, then

$$\mathrm{K}_0(F)([P]-[Q]) = \sum_{n\geq 0}(-1)^n\Big([\mathrm{cr}_n(F)(P,\ldots,P)] + [\mathrm{cr}_{n+1}(F)(Q,P,\ldots,P)]\Big) \tag{4}$$

Proof. First remark that the unicity of $\mathrm{K}_0(F)$ is clear, because $\chi\colon [\mathrm{Ch}^{\mathrm{b}}_{\geq 0}\mathcal{A}] \to \mathrm{K}_0(\mathcal{A})$ is surjective.

We first construct a well-defined map $\mathrm{K}_0(F)\colon \mathrm{K}_0(\mathcal{A}) \to \mathrm{K}_0(\mathcal{B})$.

Let $\mathcal{P} \subset \mathcal{A}$ denote the full subcategory of projective objects. Corollary 5.18 tells us that the inclusion of $\mathrm{K}_0(\mathcal{P})$ into $\mathrm{K}_0(\mathcal{A})$ is an isomorphism. Because $\mathcal{P}$ is semisimple (every exact sequence is split), $\mathrm{K}_0(\mathcal{P})$ is isomorphic to the quotient of the free group on isomorphism classes of $\mathcal{P}$ modulo the relations $[P] + [N] = [P \oplus N]$ with $P, N \in \mathcal{P}$.

Let M denote the monoid that is the set of isomorphism classes of $\mathcal{P}$ equipped with the addition $[P] + [N] = [P \oplus N]$. We easily see that the natural map $i\colon M \to \mathrm{K}_0(\mathcal{P})$ is the groupification of M. Hence by corollary 6.14, it suffices to define a function $\mathrm{K}_0(F)\colon M \to \mathrm{K}_0(\mathcal{B})$ of degree $\leq k$, because it uniquely extends to a function on $\mathrm{K}_0(\mathcal{P}) \cong \mathrm{K}_0(\mathcal{A})$ of degree $\leq k$.

Define the function

$$\mathrm{K}_0(F)\colon M \to \mathrm{K}_0(\mathcal{B}) \qquad \text{by} \qquad [P] \mapsto [FP].$$

This is clearly a well-defined function of degree $\leq k$, because for all $N, P \in \mathcal{P}$ the equalities

$$\mathrm{K}_0([N] + [P]) = \mathrm{K}_0([N \oplus P]) = [F(N \oplus P)] = [FN] + [FP] + [\mathrm{cr}_2(F)(N, P)]$$

imply that the deviations of $\mathrm{K}_0(F)$ are given by

$$(\mathrm{dev}_i \mathrm{K}_0(F))([P_1], \ldots, [P_i]) = [\mathrm{cr}_n(F)(P_1, \ldots, P_i)].$$

Hence $\mathrm{K}_0(F)$ extends to a function on $\mathrm{K}_0(\mathcal{P})$ of degree $\leq k$. By 6.10, the image of $[P] - [Q]$ with $P, Q \in \mathcal{P}$ is indeed given by equation (4).

It remains to prove the commutativity of the diagram (3). Remark that this diagram is the outer square in the following diagram:

$$\begin{array}{ccccc}
[\mathrm{Ch}^{\mathrm{b}}_{\geq 0} \mathcal{A}] & \xrightarrow{\mathrm{res}} & [\mathcal{K}^{\mathrm{b}}_{\geq 0}(\mathcal{P})] & \xrightarrow{NF\Gamma} & [\mathcal{K}^{\mathrm{b}}_{\geq 0}(\mathcal{B})] \\
\chi \downarrow & & \downarrow \chi & & \downarrow \chi \\
\mathrm{K}_0(\mathcal{A}) & = & \mathrm{K}_0(\mathcal{A}) & \xrightarrow[\mathrm{K}_0(F)]{} & \mathrm{K}_0(\mathcal{B})
\end{array}$$

The small diagram on the left clearly commutes. It remains to prove the commutativity of the small diagram on the right. We calculate both routes for a complex $C_\bullet \in \mathcal{K}^{\mathrm{b}}_{\geq 0}(\mathcal{P})$.

The image of a complex $C_\bullet \in \mathcal{K}^{\mathrm{b}}_{\geq 0}(\mathcal{P})$ under

$$\mathrm{K}_0(F) \circ \chi\colon \mathcal{K}^{\mathrm{b}}_{\geq 0}(\mathcal{P}) \to \mathrm{K}_0(\mathcal{A}) \to \mathrm{K}_0(\mathcal{B})$$

is equal to

$$\sum_{\substack{I \subset 2\mathbf{Z}_{\geq 0}, \\ J \subset (1+2\mathbf{Z}_{\geq 0}) \text{ multiset} \\ \text{not both empty}}} (-1)^{|J|} \mathrm{cr}_{|I|+|J|}(F)(C_i, i \in I, C_j, j \in J).$$

Indeed, by 5.10 we have

$$\mathrm{K}_0(F)(\chi(C_\bullet)) = \mathrm{K}_0(F)\left(\sum_{k\text{ even}} C_k - \sum_{k\text{ odd}} C_k\right)$$

and an application of proposition 6.10 now immediately gives the expression above.

The image of a complex $C_\bullet \in \mathcal{K}^{\mathrm{b}}_{\geq 0}(\mathcal{P})$ under

$$\chi \circ NF\Gamma\colon \mathcal{K}^{\mathrm{b}}_{\geq 0}(\mathcal{P}) \to \mathcal{K}^{\mathrm{b}}_{\geq 0}(\mathcal{B}) \to \mathrm{K}_0(\mathcal{B})$$

is equal to

$$\sum_n (-1)^n \sum_{\substack{\alpha \subset \bigsqcup_{k=0}^n \mathrm{Sur}([n],[k]) \\ \alpha \text{ is honourable}}} \Big[\mathrm{cr}_{\#\alpha}(F)\big(\underbrace{C_0, \ldots, C_0}_{\#\alpha_0 \text{ times}}, \ldots, \underbrace{C_n, \ldots, C_n}_{\#\alpha_n \text{ times}}\big)\Big],$$

as follows from proposition 5.10 and proposition 8.5. By theorem 8.17, we see that this is in turn equal to

$$\sum_{\substack{I \subset 2\mathbf{Z}_{\geq 0}, \\ J \subset (1+2\mathbf{Z}_{\geq 0}) \text{ multiset} \\ \text{not both empty}}} (-1)^{|J|} \,\mathrm{cr}_{|I|+|J|}(F)(C_i, i \in I, C_j, j \in J).$$

Hence the diagram commutes and the proof is completed.

□

Remark 8.19. The proof shows that the function $\mathrm{K}_0(F)\colon \mathrm{K}_0(\mathcal{A}) \to \mathrm{K}_0(\mathcal{B})$ is of degree $\leq k$.

8.4 An example

Let R be a commutative ring and $F = -^{\otimes 2}\colon \mathbf{Mod}_R \to \mathbf{Mod}_R$ the tensor power functor. Let $A \in \mathbf{Mod}_R$ and let

$$0 \to P \xrightarrow{f} Q \to A \to 0$$

in $\mathrm{Ch}_{\geq 0}\,\mathbf{Mod}_R$ be a projective resolution of A. Then with proposition 8.5 we see that $NF\Gamma(P \to Q)$ is the chain complex

$$0 \to P \otimes P \oplus P \otimes P \to P \otimes P \oplus (Q \otimes P \oplus P \otimes Q) \to Q \otimes Q \to 0$$

where the first non-trivial map is given by $(a, b) \mapsto (-(a+b), a, b)$ and the second by $(a, (b, c)) \mapsto a + b + c$. Consider the map of chain complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P\otimes P & \longrightarrow & Q\otimes P\oplus P\otimes Q & \longrightarrow & Q\otimes Q & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P\otimes P\oplus P\otimes P & \longrightarrow & P\otimes P\oplus (Q\otimes P\oplus P\otimes Q) & \longrightarrow & Q\otimes Q & \longrightarrow & 0
\end{array}$$

where the first vertical map is the diagonal, the second the inclusion and the third the identity. This map is a quasi-isomorphism, so we see there is a quasi-isomorphism

$$NF\Gamma(P\to Q)\xrightarrow{\sim}(P\to Q)^{\otimes 2}.$$

The non-trivial homology groups of the chain complex $(P\to Q)^{\otimes 2}$ are

$$\mathrm{H}_0(NF\Gamma(P\to Q))=A\otimes A \qquad \text{and} \qquad \mathrm{H}_1(NF\Gamma(P\to Q))=\mathrm{Tor}^1(A,A).$$

The above calculations are in particular applicable to the case $R=\mathbf{Z}$ of abelian groups, because every abelian group has a projective resolution of the form $0\to P\to Q\to 0$. Let **FGAb** denote the category of finitely generated abelian groups and let **FAb** denote the category of finite abelian groups. Then the tensor power $F\colon \mathbf{FGAb}\to\mathbf{Ab}$ restricts to $F\colon\mathbf{FAb}\to\mathbf{FAb}$. We cannot derive the functor $F\colon\mathbf{FAb}\to\mathbf{FAb}$, because **FAb** does not have any projectives, except the zero object. What we can do, is derive the functor $F\colon\mathbf{FGAb}\to\mathbf{FGAb}$ and then restrict the derived functor $\mathbf{L}F\colon\ \mathrm{Ch}^{\mathrm{hb}}_{\geq 0}\,\mathbf{FGAb}\to\mathcal{K}^{\mathrm{hb}}_{\geq 0}\mathbf{FGAb}$ to the subcategory $\mathrm{Ch}^{\mathbf{FAb}}_{\geq 0}\,\mathbf{FGAb}$. Note, however, that the natural map $\mathrm{K}_0(\mathbf{FAb})\to\mathrm{K}_0(\mathbf{FGAb})$ is the zero map (see examples 5.6 and 5.7). Therefore, if $\mathbf{L}F\,\mathrm{Ch}^{\mathrm{hb}}_{\geq 0}\,\mathbf{FGAb}\to\mathcal{K}^{\mathrm{hb}}_{\geq 0}\mathbf{FGAb}$ induces a function $\mathrm{K}_0(F)\colon\mathrm{K}_0(\mathbf{FGAb})\to\mathrm{K}_0(\mathbf{FGAb})$, its restriction to $\mathrm{K}_0(\mathbf{FGAb})$ is not very interesting. The results from the next section provide a solution for this problem.

9 Functors with filtrations

Some functors $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories have the property that for every split exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, the object FB has a canonical filtration of which the graded parts only depend on A and C, up to an isomorphism that is functorial in the exact sequence. We say that such a functor has a system of filtrations; this is made precies in definition 9.3. Examples are the functors $\otimes^n, \bigwedge^n, \Gamma^n, \mathrm{Sym}^n\colon \mathbf{Mod}_R \to \mathbf{Mod}_R$ for a commutative ring R, as we remark in section 9.3. In section 9.1, we prove that such functors induce maps on Grothendieck groups (theorem 9.7).

9.1 Induced maps on Grothendieck groups

Definition 9.1. Let $\mathcal{A}$ be an abelian category, A an object of $\mathcal{A}$ and $0 = \mathrm{Fil}_0 A \subset \ldots \subset \mathrm{Fil}_n A = A$ a filtration of A. Then for $i = 1, \ldots, n$, we define the i-th *graded part* of the filtration $\mathrm{Fil}_\bullet A$ as

$$\mathrm{gr}_i \mathrm{Fil}_\bullet A = A^i/A^{i-1}.$$

Definition 9.2. Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ an arbitrary category. There is an obvious definition for the category of short exact sequences in $\mathcal{A}$ and we define the category of split exact sequences as a full subcategory of this category. In what follows, we will associate to any split exact sequence C

$$C := \quad 0 \to X \to Y \to Z \to 0$$

an object $F(C)$ of $\mathcal{C}$. We say that this association is *functorial in the exact sequence* $0 \to X \to Y \to Z \to 0$ if it can be extended in an obvious way to a functor F from the category of split exact sequence in $\mathcal{A}$ to the category $\mathcal{C}$.

In particular, we will say this when $\mathcal{C}$ is the arrow category of another category. This implies that a commutative diagram

$$\begin{array}{llccccccccc} C := & 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow & & \downarrow & & \\ C' := & 0 & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

of split exact sequences in $\mathcal{A}$ induces a commutative diagram

$$\begin{array}{ccc} A_C & \xrightarrow{F(C)} & B_C \\ \downarrow & & \downarrow \\ A_{C'} & \xrightarrow{F(C')} & B_{C'} \end{array}.$$

The following definition makes precise what properties we require.

Definition 9.3. Let $\mathcal{A}, \mathcal{B}$ be abelian categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor. We define a *system of filtrations* on F as the following data:

- an integer $n \geq 1$;
- n functors $G_1, \ldots, G_n \colon \mathcal{A} \times \mathcal{A} \to \mathcal{B}$;
- for every split exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, a filtration

$$0 = \operatorname{Fil}_0 FB \subset \ldots \subset \operatorname{Fil}_n FB = FB$$

of length n functorial in the exact sequence $0 \to A \to B \to C \to 0$; and
- for every split exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ and every $i = 1, \ldots, n$, an isomorphism

$$\operatorname{gr}_i \operatorname{Fil}_\bullet FB \xrightarrow{\sim} G_i(A, C)$$

functorial in the exact sequence $0 \to A \to B \to C \to 0$.

If there is a system of filtrations on F, we also say that F admits a system of filtrations.

Note that while the exact sequences are split, a choice of splitting is not part of the data, so the filtrations and isomorphisms do not depend on a choice of splitting.

Definition 9.4. Let $\mathcal{A}$ be an abelian category. We say that an exact sequence

$$0 \to X \to Y \to Z \to$$

in $\operatorname{Simp} \mathcal{A}$ or in $\operatorname{Ch} \mathcal{A}$ is *degree-wise split* if for each degree n, the sequence

$$0 \to X_n \to Y_n \to Z_n \to 0$$

is split exact.

Note that split sequences in $\operatorname{Ch} \mathcal{A}$ or $\operatorname{Simp} \mathcal{A}$ are degree-wise split, but the converse is not in general true.

Lemma 9.5. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F \colon \mathcal{A} \to \mathcal{B}$ a functor with a system of filtrations. Then for every degreewise split exact sequence*

$$0 \to X \to Y \to Z \to 0$$

in $\operatorname{Simp} \mathcal{A}$*, we have a filtration* $\operatorname{Fil}_\bullet FY$ *of* FY *and for* $i = 1, \ldots, n$ *isomorphisms*

$$\operatorname{gr}_i \operatorname{Fil}_\bullet FY \xrightarrow{\sim} G_i(X, Z),$$

both functorial in the exact sequence $0 \to X \to Y \to Z \to 0$*.*

Proof. For every integer $n \geq 0$, the data gives us a filtration $\operatorname{Fil}_\bullet FY_n$ of FY_n. Because this filtration is functorial, the face and degeneracy maps respect this filtration and hence for each $i = 0, 1, \ldots, n$, the $\operatorname{Fil}_i FY_n$ form a simplicial object $\operatorname{Fil}_i FY$ in $\operatorname{Simp} \mathcal{A}$. These make up the filtration of FY. The isomorphisms in the data can then be applied degreewise to get the isomorphisms $\operatorname{gr}_i \operatorname{Fil}_\bullet FY \xrightarrow{\sim} G_i(X, Z)$. □

Lemma 9.6. *Let $\mathcal{A}$ be an abelian category, let $A \in \mathcal{A}$ be an object with a filtration $0 = \operatorname{Fil}_0 A \subset \ldots \subset \operatorname{Fil}_n A = A$. Then we have*

$$[A] = \sum_{i=1}^{n} (-1)^{n-i} [\operatorname{gr}_i \operatorname{Fil}_\bullet A] \qquad \text{in } \mathrm{K}_0(\mathcal{A}).$$

Proof. We prove this by induction on n. For $n = 0$ this is trivial.

Suppose $n > 0$ and assume the result is true for $n - 1$. Note that we have an exact sequence

$$0 \to \operatorname{Fil}_{n-1} A \to \operatorname{Fil}_n A \to \operatorname{gr}_i \operatorname{Fil}_\bullet A \to 0$$

and hence

$$[\operatorname{gr}_i \operatorname{Fil}_\bullet A] = [\operatorname{Fil}_n A] - [\operatorname{Fil}_{n-1} A].$$

Applying the induction hypothesis to the filtration $0 \subset \operatorname{Fil}_0 A \subset \ldots \subset \operatorname{Fil}_{n-1} A$ now gives the result. ☐

If $0 \to X \to Y \to Z \to 0$ is an degreewise split exact sequence in $\mathrm{Ch}^{\mathcal{A}_0}_{\geq 0} \mathcal{A}$, then $0 \to \Gamma X \to \Gamma Y \to \Gamma Z \to 0$ is easily seen to be degreewise split exact in $\operatorname{Simp} \mathcal{A}$. Hence by lemma 9.5, we get a filtration $\operatorname{Fil}_\bullet F\Gamma Y$ on Y and isomorphisms $\operatorname{gr}_i \operatorname{Fil}_\bullet F\Gamma Y \xrightarrow{\sim} G_i(\Gamma X, \Gamma Z)$ for $i = 1, \ldots, n$. Note that N, being an equivalence of abelian categories, is an exact functor, so that we get a filtration $\operatorname{Fil}_\bullet NF\Gamma Y$ and isomorphisms $\operatorname{gr}_i \operatorname{Fil}_\bullet NF\Gamma Y \xrightarrow{\sim} NG_i(\Gamma X, \Gamma Z)$ for $i = 1, \ldots, n$.

Theorem 9.7. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F \colon \mathcal{A} \to \mathcal{B}$ a functor with a system of filtrations. Let $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$ be weak Serre subcategories. Suppose that*

- *$\mathcal{A}$ has enough projectives;*
- *$\mathbf{L}F$ restricts to a map $\mathrm{Ch}^{\mathcal{A}_0}_{\geq 0} \mathcal{A} \to \mathcal{K}^{\mathcal{B}_0}_{\geq 0}(\mathcal{B})$; and*
- *for every degreewise split exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathrm{Ch}^{\mathcal{A}_0}_{\geq 0} \mathcal{A}$ with all X_n, Y_n, Z_n projective, the associated filtration $\operatorname{Fil}_\bullet NF\Gamma Y$ lives in $\mathrm{Ch}^{\mathcal{B}_0}_{\geq 0} \mathcal{B}$.*

Then there exists a unique map

$$\mathrm{K}_0(F) \colon \mathrm{K}_0(\mathcal{A}_0) \to \mathrm{K}_0(\mathcal{B}_0)$$

making the diagram

$$\begin{array}{ccc} \left[\mathrm{Ch}^{\mathcal{A}_0}_{\geq 0} \mathcal{A}\right] & \xrightarrow{\mathbf{L}F} & \left[\mathcal{K}^{\mathcal{B}_0}_{\geq 0}(\mathcal{B})\right] \\ \big\downarrow \chi & & \big\downarrow \chi \\ \mathrm{K}_0(\mathcal{A}_0) & \xrightarrow{\mathrm{K}_0(F)} & \mathrm{K}_0(\mathcal{B}_0) \end{array}$$

commute.

Proof. First note that $\mathrm{K}_0(F)$, if it exists, is unique, because $\chi\colon \left[\mathrm{Ch}^{\mathcal{A}_0}_{\geq 0}\mathcal{A}\right] \to \mathrm{K}_0(\mathcal{A}_0)$ is surjective.

It follows from theorem 5.31 that $\chi\colon \left[\mathrm{Ch}^{\mathcal{A}_0}_{\geq 0}\mathcal{A}\right] \to \mathrm{K}_0(\mathcal{A}_0)$ factors as

$$\left[\mathrm{Ch}^{\mathcal{A}_0}_{\geq 0}\mathcal{A}\right] \to \mathrm{K}_0(\mathrm{Ch}^{\mathcal{A}_0}_{\geq 0}\mathcal{A}) \xrightarrow{\chi} \mathrm{K}_0(\mathcal{A}_0)$$

where the first map is the canonical map and the last map is an isomorphism. Hence it is equivalent to find a compatible map $\mathrm{K}_0(\mathrm{Ch}^{\mathcal{A}_0}_{\geq 0}\mathcal{A}) \to \mathrm{K}_0(\mathrm{Ch}^{\mathcal{B}_0}_{\geq 0}\mathcal{B})$.

The presentation of $\mathrm{K}_0(\mathrm{Ch}^{\mathcal{A}_0}_{\geq 0}\mathcal{A})$ from 5.36 shows that $\mathbf{L}F$ induces a well-defined function $\mathrm{K}_0(\mathrm{Ch}^{\mathcal{A}_0}_{\geq 0}\mathcal{A}) \to \mathrm{K}_0(\mathrm{Ch}^{\mathcal{B}_0}_{\geq 0}\mathcal{B})$ if

- it preserves quasi-isomorphisms; and
- the image of an object Y in an exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathrm{Ch}^{\mathcal{A}_0}_{\geq 0}\mathcal{A}$ only depends on X and Z.

The first of these is proven in 4.4. (The proof is the same as for classically derived functors.)

The second we prove now. If $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathrm{Ch}^{\mathcal{A}_0}_{\geq 0}\mathcal{A}$, then we can take projective resolutions P, Q, R of X, Y, Z respectively, that fit into a degreewise split exact sequence $0 \to P \to Q \to R \to 0$ in $\mathrm{Ch}^{\mathcal{A}_0}_{\geq 0}\mathcal{A}$. Then for $i = 1, \ldots, n$ we have isomorphisms $\mathrm{gr}_i \mathrm{Fil}_\bullet NF\Gamma(X \oplus Z) \xrightarrow{\sim} \mathrm{gr}_i \mathrm{Fil}_\bullet NF\Gamma Y$ in $\mathrm{Ch}^{\mathcal{B}_0}_{\geq 0}\mathcal{B}$ (both graded parts are isomorphic to $NG_i(\Gamma X, \Gamma Z)$). Then lemma 9.6, which we can apply to the category $\mathrm{Ch}^{\mathcal{B}_0}_{\geq 0}\mathcal{B}$, shows $[NF\Gamma(X \oplus Z)] = [NF\Gamma Y]$ in $\mathrm{K}_0(\mathrm{Ch}^{\mathcal{B}_0}_{\geq 0}\mathcal{B})$. This proves that $X \oplus Z$ and Y do indeed have the same image under $\mathbf{L}F$.

□

9.2 Functors of degree 2

Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F\colon \mathcal{A} \to \mathcal{B}$ a reduced functor of degree 2. If $0 \to X \to Y \to Z \to 0$ is a split exact sequence, then $0 \to FX \to FY$ and $FY \to FZ \to 0$ are exact (see 2.29). Hence we have a canonical filtration

$$0 \subset FX \subset \ker(FY \to FZ) \subset FY.$$

In this section we prove that the graded parts are canonically isomorphic to FX, $\mathrm{cr}_2 F(X, Z)$, and FZ, respectively. The first is obvious and the last follows from the exactness of $FY \to FZ \to 0$ that we already noted above. The isomorphism of the middle graded is proven in the following proposition.

Lemma 9.8. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F\colon \mathcal{A} \to \mathcal{B}$ a reduced functor of degree 2. For every split exact sequence*

$$0 \to X \to Y \to Z \to 0$$

in $\mathcal{A}$, there is an isomorphism

$$\frac{\ker(FY \to FZ)}{FX} \xrightarrow{\sim} \operatorname{cr}_2 F(X, Z)$$

functorial in the exact sequence $0 \to X \to Y \to Z \to 0$.

Proof. Because F is assumed to be of degree 2, the map $\operatorname{cr}_2 F(X, -)$ is an additive functor. Hence (see remark 2.29) it preserves split exact sequences: the sequence

$$0 \to \operatorname{cr}_2 F(X, X) \to \operatorname{cr}_2 F(X, Y) \to \operatorname{cr}_2 F(X, Z) \to 0$$

is split exact. This shows we have a natural isomorphism

$$\frac{\operatorname{cr}_2 F(X, Y)}{\operatorname{cr}_2 F(X, X)} \xrightarrow{\sim} \operatorname{cr}_2 F(X, Z).$$

We now prove that $\frac{\operatorname{cr}_2 F(X,Y)}{\operatorname{cr}_2 F(X,X)}$ is also naturally isomorphic to $\frac{\ker(FY \to FZ)}{FX}$. There is a natural map

$$+ \circ \operatorname{cr}_2 F(f, \mathrm{id}_Y) \colon \operatorname{cr}_2 F(X, Y) \to \ker(FY \to FZ)$$

where $+ \colon \operatorname{cr}_2 F(Y, Y) \to F(Y)$ is the composition of the inclusion $\operatorname{cr}_2 F(Y, Y) \to F(Y \oplus Y)$ and the map induced by the codiagonal $Y \oplus Y \to Y$. Modding out by $F(X) \subset \ker(FY \to FZ)$, we get a surjective map

$$\operatorname{cr}_2 F(X, Y) \to \frac{\ker(FY \to FZ)}{F(X)}$$

with kernel $\operatorname{cr}_2 F(X, X)$ and hence an isomorphism

$$\frac{\operatorname{cr}_2 F(X, Y)}{\operatorname{cr}_2 F(X, X)} \xrightarrow{\sim} \frac{\ker(FY \to FZ)}{F(X)}.$$

The surjectivity and the kernel mentioned above are easy to check by taking a compatible isomorphism $Y \to X \oplus Z$ coming from a splitting and using cross-effects.

Composing the first of these isomorphisms with the inverse of the second gives the desired isomorphism. □

The filtrations and isomorphisms given above determine a system of filtrations as in definition 9.3 on any functor between abelian categories of degree 2. Hence we have the following application of the results in this section and theorem 9.7.

Theorem 9.9. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F \colon \mathcal{A} \to \mathcal{B}$ a functor of degree ≤ 2. Let $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$ be weak Serre subcategories. Suppose that $\mathcal{A}$ has enough projectives and $\mathbf{L}F$ restricts to a map $\operatorname{Ch}_{\geq 0}^{\mathcal{A}_0} \mathcal{A} \to \mathcal{K}_{\geq 0}^{\mathcal{B}_0}(\mathcal{B})$. Then there exists a unique map*

$$\mathrm{K}_0(F) \colon \mathrm{K}_0(\mathcal{A}) \to \mathrm{K}_0(\mathcal{B})$$

making the diagram

$$\begin{array}{ccc} \left[\operatorname{Ch}^{\mathcal{A}_0}_{\geq 0}\mathcal{A}\right] & \xrightarrow{\mathbf{L}F} & \left[\mathcal{K}^{\mathcal{B}_0}_{\geq 0}(\mathcal{B})\right] \\ \downarrow \chi & & \downarrow \chi \\ \mathrm{K}_0(\mathcal{A}_0) & \xrightarrow{\mathrm{K}_0(F)} & \mathrm{K}_0(\mathcal{B}_0) \end{array}$$

commute.

Proof. We have just proven that F has a system of filtrations. Note that for every degreewise split exact sequence $0 \to X \to Y \to Z \to 0$ in $\operatorname{Ch}^{\mathcal{A}_0}_{\geq 0}\mathcal{A}$ with all X_n, Y_n, Z_n projective, the associated filtration $\operatorname{Fil}_\bullet NF\Gamma Y$ is

$$0 \subset NF\Gamma X \subset N\ker(F\Gamma Y \to F\Gamma Z) = \ker(NF\Gamma Y \to NF\Gamma Z) \subset NF\Gamma Y.$$

By the assumption that $\mathbf{L}F$ restricts to a functor $\operatorname{Ch}^{\mathcal{A}_0}_{\geq 0}\mathcal{A} \to \mathcal{K}^{\mathcal{B}_0}_{\geq 0}(\mathcal{B})$, the complexes $NF\Gamma X$,$NF\Gamma Y$ and $NF\Gamma Z$ all lie in $\operatorname{Ch}^{\mathcal{B}_0}_{\geq 0}\mathcal{B}$. Since $\operatorname{Ch}^{\mathcal{B}_0}_{\geq 0}\mathcal{B}$ is abelian (proposition 2.39), the kernel in the middle part of the filtration above is also in $\operatorname{Ch}^{\mathcal{B}_0}_{\geq 0}\mathcal{B}$. Hence we can indeed apply theorem 9.7. □

9.3 Tensor powers and similar functors

Example 9.10. Let R be a commutative ring and let $\mathbf{FGMod}_R$ denote the category of finitely generated modules over R. Let $n \geq 1$ and consider the functor $F\colon \mathbf{FGMod}_R \to \mathbf{FGMod}_R$ that sends a module A to its n-th tensor power $A^{\otimes n}$. For $i = 0, \dots, n$ we define the functor $G_i\colon \mathbf{FGMod}_R \times \mathbf{FGMod}_R \to \mathbf{FGMod}_R$ by

$$G_i(A, C) := \bigoplus_{\tau \in S_n/(S_{n-i} \times S_i)} \tau(\underbrace{A \otimes \ldots \otimes A}_{n-i \text{ times}} \otimes \underbrace{C \otimes \ldots \otimes C}_{i \text{ times}}) \tag{5}$$

where the symmetric group S_n acts on the tensor products in the obvious way. Note that for all $A, C \in \mathbf{FGMod}_R$, we have $G_0(A, C) = F(A)$ and $G_n(A, C) = F(C)$.

Given a degreewise split exact sequence $0 \to A \to B \to C \to 0$ in $\mathbf{FGMod}_R$, we define for each $i = 0, \dots, n+1$

$$\operatorname{Fil}_i FB := \begin{array}{c} \text{the subobject in } FB \text{ generated by the images of} \\ \tau(\underbrace{A \otimes \ldots \otimes A}_{n-i \text{ times}} \otimes \underbrace{B \otimes \ldots \otimes B}_{i \text{ times}}) \\ \text{for each } \tau \in S_n/(S_{n-i} \times S_i). \end{array}$$

These form a filtration

$$\operatorname{Fil}_\bullet FB := 0 \subsetneq \operatorname{Fil}_0 FB \subset \ldots \subset \operatorname{Fil}_n FB = FB$$

of FB with graded parts isomorphic to the $G_i(A, C)$. By inspection, we indeed find for any two modules $A, C \in \mathbf{FGMod}_R$

$$F(A \oplus C) = \bigoplus_{i=0}^{n} G_i(A, C)$$

and the filtration above is then easily seen to be

$$0 \subset G_0(A, B) \subset G_0(A, B) \oplus G_1(A, B) \subset \dots$$
$$\dots \subset \bigoplus_{i=0}^{n-1} G_i(A, B) \subset \bigoplus_{i=0}^{n} G_i(A, B) = F(A \oplus B)$$

with graded parts isomorphic to the $G_i(A, B)$. This defines a system of filtrations on F that can be used to apply theorem 9.7.

Remark 9.11. Similar statements hold for the functors $\bigwedge^n, \Gamma^n, \mathrm{Sym}^n \colon \mathbf{FGMod}_R \to \mathbf{FGMod}_R$. For example, for a degreewise split exact sequence $0 \to A \to B \to C \to 0$ in $\mathbf{FGMod}_R$, $\bigwedge^n B$ has a filtration defined by

$$\mathrm{Fil}_i \bigwedge^n B := \text{image in } \bigwedge^n B \text{ of } \bigwedge^{n-i} A \otimes \bigwedge^i B$$

with graded parts isomorphic to

$$G_i(A, C) := \bigwedge^{n-i} A \otimes \bigwedge^i C.$$

Remark 9.12. This approach is also applicable in many similar situations, such as the finite dimensional group representations over k mentioned in the introduction, where we tensor over k.

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