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The hyperbolic plane and a regular 24-faced polyhedron

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1. Introduction

There are five known platonic solids: the tetrahedron, the cube, the octahedron and the dodecahedron and the icosahedron. The cube and the octahedron are dual to each other: every face on the cube corresponds to a vertex on the octahedron and vice versa. The same holds for the dodecahedron and the icosahedron, while the tetrahedron is its own dual. Thus, up to duality, there are only three types: the tetrahedron, which consists of four triangles; the cube, which consists of six squares; and the dodecahedron, which consists of twelve pentagons. They share the property that each vertex lies on three different faces, but the number of vertices on each faces grows. A natural question now arises: can we not continue this trend and create more platonic solids?

First we have to specify what we mean by "regular". We can do that with the following definitions.

Definition 1.1 (flag)**.** We define a *flag* of a polyhedron to be a triple (f, e, v) with f a face of that polyhedron, e an edge on f and v a vertex on e.

Definition 1.2 (regular)**.** We say a polyhedron is *regular*, if its symmetry group acts transitively on the set of all flags.

Suppose we wanted to construct a regular polyhedron with hexagons as its faces. Because each vertex of the hexagon has an interior angle of $\frac{2\pi}{3}$, this cannot work: if we put three hexagons together at a vertex, the hexagons do not curve around but lie in a plane, because three times an angle of $\frac{2\pi}{3}$ adds up to an angle of 2π. If we were to continue putting together hexagons in the hope that we could eventually fit them together, we would end up tiling the plane with regular hexagons.

Another problem arises if we try the same for a regular heptagon, the vertices of which have interior angles of $\frac{5\pi}{7}$: we can not bend the faces in such a way that they fit together, because three times $\frac{5\pi}{7}$ adds up to more than 2π. We can't construct a polygon with heptagonal faces in Euclidean space. Note that in contrast to the hexagon and heptagon, the triangle, square and pentagon, which are the building blocks of the known platonic solids, have interior angles which are less than a third of 2π.

Though this attempt seems to have failed, the hexagonal case suggests another way of thinking about these platonic solids: in relation to tilings of a plane. The known platonic solids can in fact be regarded as tilings as well. They form tilings of a sphere if we project them from their centre to a circumscribed sphere. In other words, if we imagine that we place a light bulb at the centre of a tetrahedron, cube or dodecahedron with a sphere around it, touching its vertices, then the shadows of the faces tile the sphere.

However, this is not a tiling of Euclidean space. The sphere has another kind of geometry. If we say the lines on the spheres are the great circles (i.e., the intersection of a sphere with a plane through its centre) and vaguely define triangles to be the interior of three different lines, then the interior angles of a triangle do not add up to π, but to more. This also means that an interior angle of a "square" or "pentagon" is in fact more than respectively $\frac{\pi}{2}$ or $\frac{3\pi}{5}$. As a result, there can be triangles, squares and pentagons with interior angles of $2\pi/3$, so that it is possible to put three of them together at one vertex. This is what happens in the spherical tilings that the tetrahedron, cube and dodecahedron induce via a projection from the centre.

A regular heptagon, on the other hand, has the problem that its angles are more than a third of 2π instead of less. That problem is not be solved in a sphere where the angles of a triangle add up to more than π, since that only increases the angles of a regular heptagon. Therefore, we need a geometry in which the sum of the angles of a triangle is less than π.

The hyperbolic plane satisfies this requirement. First, we will construct a model of the hyperbolic plane, so that some more rigorous mathematics can be done. When we have done so, we can tile it with a suitably chosen triangle and from that tiling, we can finally create a "successor" to the known platonic solids.

2. A model for hyperbolic geometry

The purpose of this section is to construct a Riemannian manifold that can serve as a model for hyperbolic geometry. Our first model will be based on the upper half plane $\mathbf{H} = \{z \in \mathbf{C} : \Im(z) > 0\}$ (with $\Im(z)$ the imaginary part of z). Afterwards, we will see that the unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ is in fact complex analytically isomorphic to the upper half plane, so that we can transfer this structure from one to the other and in this way get two models, each with its own advantages and disadvantages.

We will at first concern ourselves with determining what could be a suitable group of automorphisms in our model of hyperbolic geometry. This group will turn out to be isomorphic to $\mathrm{PSL}_2(\mathbf{R})$: they are the fractional linear transformations or Möbius transformations. We will first look at such maps in a broader context than the upper half plane.

2.1. **The Riemann sphere and its automorphisms.**

Definition 2.1 (Riemann sphere)**.** The *Riemann sphere* is the projective line over $\mathbf{C}$: a two-dimensional vector space over $\mathbf{C}$ without the origin, divided out by scalars:

$$\mathbf{P} := \big(\mathbf{C}^2 \setminus \{(0,0)\}\big)/\mathbf{C}^\times$$

with $\mathbf{C}^\times = \mathbf{C} \setminus \{0\}$. It is endowed with the quotient topology.

Remark 2.2. The equivalence class of a point $(a,b) \in \mathbf{C}^2 \setminus \{(0,0)\}$ is denoted by $(a : b)$ and consists of all pairs $(c,d) \in \mathbf{C}^2 \setminus \{(0,0)\}$ with $ad = bc$.

Remark 2.3. We can consider the Riemann sphere $\mathbf{P}$ as the complex plane $\mathbf{C}$ together with a point at infinity, denoted by ∞, in the following way: every point $(z_1 : z_2) \in \mathbf{P}$ with $z_2 \neq 0$ is equal to $(z_1/z_2 : 1)$ and can be identified with $z_1/z_1 \in \mathbf{C}$; the only remaining point is $\infty := (1 : 0)$. One can visualize the Riemann sphere by viewing the point ∞ as a point added to the complex plane in such a way that the total becomes, topologically speaking, a sphere, whence the name. Note that the sphere is compact, because it is the union of the images of the two compact subsetsets $\{(z, 1) \in \mathbf{C}^2 : z \leq 1\}$ and $\{(1, z) \in \mathbf{C}^2 : z \leq 1\}$ of $\mathbf{C}^2 \setminus \{(0, 0)\}$ under the (continuous) quotient map from $\mathbf{C}^2 \setminus \{(0, 0)\}$ to $\mathbf{P}^1$.

The following notations and definitions will be used throughout the thesis.

Definition 2.4. Let R be a commutative ring. By $\mathrm{Mat}_n(R)$ we denote the matrix algebra of n by n matrices with coefficients in R. Then we define the following groups:

$$\mathrm{GL}_n(R) := \{\, M \in \mathrm{Mat}_n(R) : \det M \in R^\times \,\}$$
$$\mathrm{SL}_n(R) := \{\, M \in \mathrm{Mat}_n(R) : \det M = 1 \,\}$$

Now let k be a field or $\mathbf{Z}$. Then we define

$$\mathrm{PGL}_n(k) := \{\, M \in \mathrm{Mat}_n(k) : \det M \in k^\times \,\}/k^\times$$
$$\mathrm{PSL}_n(k) := \{\, M \in \mathrm{Mat}_n(k) : \det M = 1 \,\}/(k^\times \cap \mathrm{SL}_n(k))$$

These groups are called the *general linear group*, *projective linear group*, *special linear group* and *projective special linear group* respectively.

Remark 2.5. These definitions do not correspond entirely with the definitions often used in algebraic geometry and also for arbitrary rings, these definitions require a different generalisation.

Remark 2.6. If k is an algebraically closed field, then $\mathrm{PGL}_n(k) \cong \mathrm{PSL}_n(k)$ for all $n \geq 1$, because any matrix $M \in \mathrm{Mat}_n(k)$ is in the same equivalence class as $\frac{1}{\sqrt[n]{\det M}} M$ (which has determinant 1) in $\mathrm{PGL}_n(k)$.

Now we can define the fractional linear transformations. The following theorem is stated in some generality, although we only need it for $k = \mathbf{C}$ or $k = \mathbf{R}$ and $n = 2$.

Theorem 2.7. *Let k be a field and $n \in \mathbf{Z}_{\geq 2}$. The group $\mathrm{GL}_n(k)$ acts on the projective space $\mathbf{P}^{n-1}(k) := (k^n \setminus \{0\})/k^\times$ by the action induced by the natural action of $\mathrm{GL}_n(k)$ on vector spaces. This induces a faithful action of the quotient group $\mathrm{PGL}_n(k) := \mathrm{GL}_n(k)/k^\times$.*

Proof. We can lift an element of $\mathbf{P}^{n-1}(k)$ to $k^n \setminus \{(0, 0)\}$ and consider the image of the lift under a matrix $A \in \mathrm{GL}_n(k)$. Because A has non-zero determinant, its kernel is trivial, and since the lift is not $(0, 0)$, the image of the lift is again in $k^n \setminus \{(0, 0)\}$. Because matrices act as linear maps, this is independent of the lift: one lift is a non-zero multiple of another lift, so their images are also non-zero multiples. Thus the matrix A does indeed induce a mapping from $\mathbf{P}^{n-1}(k)$ to itself.

$$\begin{tikzcd}
k^n \setminus \{(0,0)\} \arrow[r, "A \in \mathrm{GL}_n(k)"] \arrow[d, two heads] & k^n \setminus \{(0,0)\} \arrow[d, two heads] \\
\mathbf{P}^{n-1}(k) \arrow[r] & \mathbf{P}^{n-1}(k)
\end{tikzcd}$$

Because $\mathrm{GL}_n(k)$ acts on $k^n \setminus \{(0,0)\}$, it is now clear that $\mathrm{GL}_n(k)$ acts on the projective line $\mathbf{P}^{n-1}(k)$.

We easily see that every matrix λI with $\lambda \in k^\times$ (and I the identity matrix) acts trivially on $\mathbf{P}^{n-1}(k)$. For the converse that every matrix that acts trivially on $\mathbf{P}^{n-1}(k)$ is of that form, assume that $A \in \mathrm{GL}_n(k)$ acts trivially. Then there exists for every $i \in \{1, 2, \ldots, n\}$ a non-zero scalar $\lambda_i \in k^\times$ such that $Ae_i = \lambda_i e_i$ (with $e_i \in k^n$ the vector with a one on the i-th position and zeroes elsewhere). Also, there exists a $\lambda \in k^\times$ with $A\left(\sum_{i=1}^n e_i\right) = \lambda\left(\sum_{i=1}^n e_i\right)$, but by linearity of A this is equal to $\sum_{i=1}^n Ae_i = \sum_{i=1}^n \lambda_i e_i$ which implies that $\lambda_i = \lambda$ holds for every $i \in \{1, 2, \ldots, n\}$. We conclude $A = \lambda I$.

Hence we can divide $\mathrm{GL}_n(k)$ out by the normal subgroup $k^\times = \{\lambda I : \lambda \in k^\times\}$ (normal because it is in the centre) and get a faithful action of $\mathrm{GL}_n(k)/k^\times = \mathrm{PGL}_n(k)$. □

Definition 2.8. The bijective maps that elements of $\mathrm{GL}_2(\mathbf{C})$ induce in this way, are called *fractional linear transformations* or *Möbius transformations*.

Remark 2.9. If we decompose the Riemann sphere as $\mathbf{C} \cup \{\infty\}$ (as explained in remark 2.3), this action takes the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

for $z \in \mathbf{C}$ with $cz + d \neq 0$. If we make the convention that dividing by zero gives ∞ and that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \infty = \frac{a}{c},$$

then this fraction of two linear maps describes the map on the Riemann sphere entirely.

Although we will not discuss the background theory of manifolds in great detail[1], it may be useful to recall, or perhaps introduce, the following definition:

Definition 2.10 (Riemann surface)**.** A *Riemann surface* is a one-dimensional connected complex manifold: it is a two-dimensional real connected manifold M with an equivalence class of atlasses of charts $\{U_i, z_i\}_{i \in I}$ with $U_i \subset M$ open, $M = \bigcup_{i \in I} U_i$ and $z_i : U_i \to \mathbf{C}$ a homeomorphism onto an open subset of the complex plane $\mathbf{C}$, such that the transition functions

$$z_j \circ z_i^{-1} \colon z_i(U_i \cap U_j) \to z_j(U_i \cap U_j)$$

are holomorphic.

Remark 2.11. Note that any Riemann surface can also be considered as a differentiable manifold by identifying $\mathbf{C}$ with $\mathbf{R}^2$, so that the charts map to a subset of $\mathbf{R}^2$, because the holomorphic transition maps are then certainly smooth (i.e., infinitely continuously differentiable).

Straightforward examples of Riemann surfaces that will appear in this thesis include the upper half plane and the unit disc, both of which can be given a Riemann surface structure by taking the inclusion map as the only chart. Another example that is important for us, is given in the following proposition.

[1]For a general introduction into manifolds, we refer readers to [Jän01]; Riemann surfaces are treated in more detail in [FK80]; and a concise treatment of (real and complex) manifolds can be found in the first chapter of [Var84].

Proposition 2.12. *The Riemann sphere can be made into a Riemann surface. On this structure, elements from* $\mathrm{PSL}_2(\mathbf{C})$ *act as automorphisms.*

Proof. We can take the following two charts:

$$(\mathbf{P} \setminus \{(1:0)\}, (z_1 : z_2) \mapsto z_1/z_2) \text{ and } (\mathbf{P} \setminus \{(0:1)\}, (z_1 : z_2) \mapsto z_2/z_1).$$

Both charts are easily seen to be homeomorphisms onto their image $\mathbf{C}$. The two transition maps are $z \mapsto \frac{1}{z}$, which is a holomorphic map wherever the transition map should be defined (i.e., $\mathbf{C} \setminus \{0\}$).

It still remains to show that matrices in $\mathrm{PSL}_2(\mathbf{C})$ act as automorphisms. So let $f \colon \mathbf{P} \to \mathbf{P}$ be the map induced by the class of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{C})$. We already know that the inverse of f is also of this form (and induced by the matrix A^{-1}), so all we need to show is that the induced map is holomorphic with respect to the charts. Depending on the charts we take, the map looks like one of

$$z \mapsto \frac{az+b}{cz+d}, \; z \mapsto \frac{cz+d}{az+b}, \; z \mapsto \frac{a+bz}{c+dz} \text{ or } z \mapsto \frac{c+dz}{a+bz}.$$

It is evident that around each point, we can take a small enough neighborhood so that not both the numerator and denominator of one of the above fraction is zero (since the determinant is non-zero) and by taking the charts in such a way that the denominator is not zero, we see that these maps are indeed holomorphic. □

The complex structure of a Riemann surface allows us to define angles between two tangent vectors. For doing this, the following definition for tangent vectors of Riemann surfaces is practical.

Definition 2.13. Let M be a Riemann surface. We define a *tangent vector* at a point $p \in M$ to be a map

$$v \colon \mathcal{D}_p(M) \to \mathbf{C}$$

with $\mathcal{D}_p(M)$ the set of charts around p, such that for any two charts $(U, h), (V, k) \in \mathcal{D}_p(M)$ we have

$$v(V, k) = \mathrm{d}(k \circ h^{-1})_{h(p)} v(U, h)$$

with d the differential operator. This makes the tangent space into a one-dimensional $\mathbf{C}$-vector space.

Definition 2.14. Let X be a Riemann surface, $p \in X$ a point and $v_1, v_2 \in T_pX$ be non-zero tangent vectors at p (i.e., not every chart is mapped to zero) as in definition 2.13. Note that we can canonically identify $\mathbf{R}^2$ with $\mathbf{C}$. Let (U, h) be a chart around p. The *angle* of the ordered pair (v_1, v_2) is defined to be

$$\frac{v_1(U,h)/v_2(U,h)}{\|v_1(U,h)/v_2(U,h)\|}. \tag{1}$$

This is independent of the chart, because the differentials of the transition maps (which are holomorphic) are multiplication by a complex number.

Proposition 2.15. *Let* X, Y *be Riemann surfaces and let* $f \colon X \to Y$ *be a morphism of Riemann surfaces (i.e., it is holomorphic with respect to the charts). Then* f *preserves angles wherever its differential is unequal to zero.*

Proof. This is because the differential of f at any point is multiplication by a non-zero complex number, under which (1) is clearly invariant. □

In particular this means that angles in the Riemann sphere are preserved if we apply elements from $\mathrm{PSL}_2(\mathbf{C})$. The following proposition is useful for getting a feeling for what fractional linear transformations do.

Proposition 2.16. *Let k be a field. The group $\mathrm{SL}_2(k)$ is generated by matrices of the form*

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with $a \in k^\times$. In other words, every fractional linear transformation is a composition of maps $z \mapsto a^2 z$, $z \mapsto z + a$ and $z \mapsto -\frac{1}{z}$ with $a \in k^\times$. The group $\mathrm{GL}_2(k)$ is generated by matrices of the form given above or of the form

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

for $a \in k^\times$.

Proof. This is a straightforward calculation. First we prove our statement for $\mathrm{SL}_2(k)$ and use that to prove the result for $\mathrm{GL}_2(k)$. Suppose we have

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(k).$$

If $c = 0$, then $\det A = 1$ implies $ad = 1$, so we have $d = a^{-1}$. In this case we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & ab \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

Now suppose $c = -1$. (We will soon see how we can reduce the case $c \neq 0$ to the case $c = -1$.) The requirement that the determinant is 1 then gives $ad + b = 1$ or $b = 1 - ad$. We thus have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 1 - ad \\ -1 & d \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -d \\ 0 & 1 \end{pmatrix}.$$

Suppose, lastly, that $c \neq 0$. We then see that the matrix

$$B = A \begin{pmatrix} -c^{-1} & 0 \\ 0 & -c \end{pmatrix}$$

has a coefficient of -1 in the lower left corner, so the matrix B is a product of the given generators. This part of the proof is now completed by recognizing that A is the product of B with (on the right) the matrix $\begin{pmatrix} -c & 0 \\ 0 & -c \end{pmatrix}$, which is also a generator.

Let $A \in \mathrm{GL}_n(k)$. Then the matrix

$$B := \begin{pmatrix} (\det A)^{-1} & 0 \\ 0 & 1 \end{pmatrix} A$$

is in $\mathrm{SL}_2(k)$ (its determinant is clearly 1) and is thus a product of the given generators. We now see that A is a product of (on the left) the matrix $\begin{pmatrix} \det A & 0 \\ 0 & 1 \end{pmatrix} B$, which is a generator, and on the right the matrix B. This completes the proof. $\square$

One of the reasons why this group happens to be a suitable candidate for the automorphism group of our model, is that they preserve the following generalization of circles and lines.

Definition 2.17 (Generalized circle)**.** A *generalized circle* on the Riemann sphere $\mathbf{P} = \mathbf{C} \cup \{\infty\}$ is a circle in $\mathbf{C}$, seen as a subset of $\mathbf{P}$, or a straight line in $\mathbf{C}$ together with the point at infinity ∞.

Theorem 2.18. *Under the action of* $\mathrm{PSL}_2(\mathbf{C})$*, generalized circles of the Riemann sphere are mapped to generalized circles.*

One way to prove this theorem is to confirm it for the generators of $\mathrm{PSL}_2(\mathbf{C})$ (see proposition 2.16). We will prove it in another way, though, with the following lemma.

Lemma 2.19. *The subsets of the form*

$$\{(x_1 : x_2) \in \mathbf{P} : |ax_1 + bx_2| = |cx_1 + dx_2|\} \tag{2}$$

for complex matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with non-zero determinant, are well-defined and consitute exactly all the generalized circles.*

Proof. Note first that the given condition does not depend on the choice of representatives for the points $(x_1 : x_2) \in \mathbf{P}$ (scaling by a non-zero factor does not change the validity of the equation). The subsets are thus well-defined.

If we restrict ourselves to the affine points (i.e., $\mathbf{P} \cap \mathbf{C} = \{(a : b) = (a/b : 1) \in \mathbf{P} : b \neq 0\}$), then by identifying the point $(x_1 : x_2)$ with $(x_1/x_2 : 1) =: (z : 1)$, the equations can be written as $|az+b| = |cz+d|$. We will use this to show the validity of our statement for all affine points. The case of the only remaining point, $(1 : 0)$, is then easily checked.

First we will consider the circles in $\mathbf{C}$ and show they can be written in this way. The proof uses Apollonius' description of circles in the Euclidean plane: any circle is equal to a set

$$\{z \in \mathbf{C} : d(z, a) = sd(z, b)\}$$

for two fixed points $a, b \in \mathbf{C}$ with $a \neq b$ and a positive real number $s \in \mathbf{R}_{>0}$ with $s \neq 1$; and conversely, any such set is a circle. (This is easily proven by simply rewriting this equation into the well-known equation for a circle.) We can also denote such a set by

$$\{z \in \mathbf{C} : |z - a| = |sz - sb|\}$$

which can be projectively extended to

$$\{(x : y) \in \mathbf{P} : |x - ay| = |sx - sby|\}.$$

Now note that we have $1 \cdot (-sb) - (-a) \cdot s = s(a - b) \neq 0$ because both $s \neq 0$ and $a \neq b$ hold, so the determinant of the matrix corresponding to this equality as in equation (2) is non-zero. We verify that the point at infinity does not lie in this subset, because of $|s| \neq 1$.

A line, too, can be described in this way, because, as most of the readers will remember from their high school geometry classes, a line can be given as the set of points that have equal distance to two distinct points. For any line, there exist $a, b \in \mathbf{C}$ with $a \neq b$ such that the line equals the subset of all $z \in \mathbf{C}$ satisfying

$$|z - a| = |z - b|.$$

Conversely, any such subset is clearly a line. Projectively, this subset becomes

$$\{\, (x : y) \in \mathbf{P} : |x - ay| = |x - by| \,\}.$$

The determinant of the corresponding matrix is $1 \cdot (-b) - (-a) \cdot 1 = a - b \neq 0$. We immediately see that the point at infinity $(1 : 0)$ also lies in this subset, as it should.

Now suppose that we have been given a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with non-zero determinant $ad - bc$. One of a and c must be non-zero. Because the conditions $|az+b| = |cz+d|$ and $ad - bc = 0$ are symmetric with respect to switching the pairs (a, b) and (c, d), we can assume $a \neq 0$. Then we can rewrite the condition $|az + b| = |cz + d|$ as

$$\left| z - \left(-\frac{b}{a} \right) \right| = \frac{1}{|a|} \left| cz - (-d) \right|.$$

Now note that if $c = 0$ holds, then the right hand side is constant, so we have a circle around the point $-\frac{b}{a}$. If $c \neq 0$ holds, then we can write

$$\left| z - \left(-\frac{b}{a} \right) \right| = \left| \frac{c}{a} \right| \left| z - \left(-\frac{d}{c} \right) \right|.$$

From $ad - bc \neq 0$ follows $\frac{b}{a} \neq \frac{d}{c}$, so this is then an Apollonius circle if $\left| \frac{c}{a} \right|^2 \neq 1$ and a line otherwise. It is evident that the point at infinity does not lie on anything that becomes a circle, while it does lie on all of the lines. □

We can now prove theorem 2.18:

Proof. Let, for any matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{C})$, C_A denote the generalized circle $\{(x_1 : x_2) \in \mathbf{P} : |ax_1 + bx_2| = |cx_1 + dx_2|$ corresponding to A via the correspondence given in the proof of lemma 2.19. Note that we can also describe this set as

$$\{ \, (x_1 : x_2) \in \mathbf{P} : A(x_1 : x_2) \text{ is a zero of } |X| - |Y| \, \}.$$

If we apply a matrix B with non-zero determinant on all these points, we get the set

$$\begin{aligned} &\{ \, B(x_1 : x_2) \in \mathbf{P} : A(x_1 : x_2) \text{ is a zero of } |X| - |Y| \, \} \\ =&\{ \, (x_1 : x_2) \in \mathbf{P} : AB^{-1}(x_1 : x_2) \text{ is a zero of } |X| - |Y| \, \} = C_{AB^{-1}} \end{aligned}$$

which is again a generalized circle. □

Proposition 2.20. *The action of* $\mathrm{PGL}_2(k)$ *on* $\mathbf{P}^1(k)$ *is sharply 3-transitive: it is free and transitive on the set of triples of distinct points in* $\mathbf{P}^1(k)$*.*

Proof. Take a triple of three pairwise distinct points $(z_1, z_2, z_3) = ((x_1 : y_1), (x_2 : y_2), (x_3 : y_3)) \in \mathbf{P}^1(k)$. These correspond to one-dimensional linear subspaces in k^2. We will show that there is, up to scaling, exactly one bijective linear map $\phi : k^2 \to k^2$ that induces a map from $\mathbf{P}^1(k)$ to $\mathbf{P}^1(k)$ that sends the triple $((0 : 1), (1 : 1), (1 : 0))$ to the triple (z_1, z_2, z_3), thereby proving the proposition.

The idea is that the images of $(1, 0)$ and $(0, 1)$ are now fixed up to a multiple (they have to end up in the proper one-dimensional linear subspace), but the ratio of those multiples can be chosen in only one way such that $(1, 1)$ is mapped to the linear subspace spanned by (x_2, y_2). The details now follow.

For any $(\lambda, \mu) \in k^2 \setminus \{0\}$, the matrix

$$A_{\lambda,\mu} := \begin{pmatrix} \lambda x_1 & \mu x_3 \\ \lambda y_1 & \mu y_3 \end{pmatrix}$$

represents a linear map that sends the linear subspace corresponding to $(1 : 0)$ to the linear subspace corresponding to $(x_1 : y_1)$ and that of $(0 : 1)$ to that of

$(x_3 : y_3)$. Note that its determinant is non-zero, because the columns span different linear subspaces by the assumption $z_1 \neq z_3$.

The image of $(1,1)$ under $A_{\lambda,\mu}$ is $(\lambda x_1 + \mu x_3, \lambda y_1 + \mu y_3) = \lambda(x_1, y_1) + \mu(x_3, y_3)$. Because $((x_1, y_1), (x_3, y_3))$ is a basis for k^2, there exist $(\lambda, \mu) \in k^2 \setminus \{0\}$ such that

$$(\lambda x_1 + \mu x_3, \lambda y_1 + \mu y_3) = (x_2, y_2)$$

holds. Taking such (λ, μ), we see that $A_{\lambda,\mu}$ induces a mapping that sends the triple $((0:1), (1:1), (1:0))$ to the triple (z_1, z_2, z_3). Hence so do its multiples.

We easily see that the multiples of $A_{\lambda,\mu}$ are the only maps sending the triple $((0 : 1), (1 : 1), (1 : 0))$ to the triple (z_1, z_2, z_3). Such a map is of the form $A_{\lambda',\mu'}$ with $(\lambda', \mu') \in k^2 \setminus \{0\}$, because the images of $(1,0)$ and $(0,1)$ (i.e., the first and second column) lie in the subspaces spanned by (x_1, y_1) and (x_3, y_3), so they are non-zero multiples of (x_1, y_1) and (x_3, y_3) respectively. If (λ', μ') is not a multiple of (λ, μ), then the images $A_{\lambda,\mu}(1,1) = \lambda(x_1, y_1) + \mu(x_3, y_3)$ does not lie in the same subspace as $A_{\lambda',\mu'}(1,1) = \lambda'(x_1, y_1) + \mu'(x_3, y_3)$ $(((x_1, y_1), (x_3, y_3))$ is a basis). □

Remark 2.21. It is very easy to give such a map that sends one triple of distinct points to another. Let $(z_1, z_2, z_3) \in \mathbf{P}^3$ be a triple such that z_1, z_2, z_3 are pairwise distinct. The maps

$$z \mapsto \frac{z_2 - z_3}{z - z_3}, \qquad z \mapsto \frac{z - z_1}{z - z_3}, \qquad z \mapsto \frac{z - z_1}{z_2 - z_1}$$

do the trick if $z_1 = \infty$ or $z_2 = \infty$ or $z_3 = \infty$ respectively. The matrix of the matrix corresponding to the first map is $z_3 - z_2$; that of the matrix corresponding to the second is $z_1 - z_3$; and for the third we find $z_2 - z_1$. All of these are easily seen to be non-zero because we chose all points pairwise different.

If none of z_1, z_2, z_3 is equal to ∞, then the map

$$z \mapsto \frac{(z - z_1)(z_2 - z_3)}{(z_2 - z_1)(z - z_3)} = \frac{(z_2 - z_3)z - z_1(z_2 - z_3)}{(z_2 - z_1)z - z_3(z_2 - z_1)}$$

is easily seen to send the triple (z_1, z_2, z_3) to the triple $(0, 1, \infty)$. The determinant of the corresponding matrix is $-(z_2 - z_3) \cdot z_3(z_2 - z_1) + z_1(z_2 - z_3) \cdot (z_2 - z_1) = (z_1 - z_3)(z_2 - z_3)(z_2 - z_1)$ and since z_1, z_2, z_3 were pairwise distinct, this is non-zero. Using this, for any pair of triples of distinct points, we can easily determine a matrix that sends the first triple to the second.

2.2. **The upper half plane.** Because we want the upper half plane as our model for hyperbolic geometry, we are looking for automorphisms that leave the upper half plane (a subset of the Riemann sphere) intact. A good choice is the subgroup $\mathrm{PSL}_2(\mathbf{R}) \subset \mathrm{PSL}_2(\mathbf{C})$, as the following proposition shows.

Proposition 2.22. *The upper half plane is mapped to itself under the action of elements from* $\mathrm{PSL}_2(\mathbf{R}) \subset \mathrm{PSL}_2(\mathbf{C})$. *Hence,* $\mathrm{PSL}_2(\mathbf{R})$ *acts on the upper half plane.*

Proof. One way to prove this is by straightforward computation, but a more insightful approach is as follows.

Take any point z in the upper half plane $\mathbf{H}$. Note that $\mathrm{PSL}_2(\mathbf{R})$ acts on the projective line $\mathbf{P}^1(\mathbf{R}) = \mathbf{R} \sqcup \{\infty\}$, which is a subset of the Riemann sphere, by theorem 2.7. As a consequence, the set $\mathrm{PSL}_2(\mathbf{R})z$ of the images of z under the elements from $\mathrm{PSL}_2(\mathbf{R})$ does not intersect $\mathbf{P}^1(\mathbf{R})$.

We can make $\mathrm{SL}_2(\mathbf{R})$ into a topological space by using the four coefficients to consider it as a subset of $\mathbf{R}^4$. The group $\mathrm{PSL}_2(\mathbf{R})$ can then be given the quotient topology. Because $\mathrm{SL}_2(\mathbf{R})$ is path-connected[2], so is the quotient $\mathrm{PSL}_2(\mathbf{R})$ (being the continuous image of the path-connected space $\mathrm{SL}_2(\mathbf{R})$ under the quotient map). Since evaluating in z is a continuous map from $\mathrm{PSL}_2(\mathbf{R})$ to the Riemann sphere $\mathbf{P}$, we see that $\mathrm{PSL}_2(\mathbf{R})z$ is a path-connected space. Seeing furthermore that $\mathrm{PSL}_2(\mathbf{R})z$ does not intersect with $\mathbf{P}^1(\mathbf{R})$ and contains the point z in the upper half plane, we can conclude that $\mathrm{PSL}_2(\mathbf{R})z$ is a subset of the upper half plane. □

Remark 2.23. From proposition 2.12 we can conclude that elements of the group $\mathrm{PSL}_2(\mathbf{R})$ act as complex analytic automorphisms and as automorphisms of $\mathbf{H}$ as a Riemann surface.

Definition 2.24 (hyperbolic lines)**.** *Hyperbolic lines* in the upper half plane are defined to be the generalized circles that are perpendicular to the real axis, intersected with the upper half plane.

Proposition 2.25. *Under the action of the group* $\mathrm{PSL}_2(\mathbf{R})$*, hyperbolic lines in the upper half plane are mapped to hyperbolic lines in the upper half plane.*

Proof. We see that the hyperbolic lines are exactly the generalized circles in the Riemann sphere that are orthogonal to the projective real line $\mathbf{P}^1(\mathbf{R})$, intersected with the upper half plane. Let h be any hyperbolic line and let c be the corresponding generalized circle. Because angles are preserved by proposition 2.15 and $\mathbf{P}^1(\mathbf{R})$ is mapped to itself, the image of c (which is again a generalized circle by theorem 2.18) is orthogonal to $\mathbf{P}^1(\mathbf{R})$. Hence, the image of h, which is equal to the image of c intersected with the upper half plane because of 2.22, is again a hyperbolic line. □

Lemma 2.26. *The group* $\mathrm{PSL}_2(\mathbf{R})$ *acts transitively on* $\mathbf{H}$.

Proof. We will show that every $z \in \mathbf{H}$ can be mapped to i. Write $z = x+iy$ ($x \in \mathbf{R}$, $y \in \mathbf{R}_{>0}$). We can map z to iy with the map $z \mapsto z - x$ (corresponding to the matrix $\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$). Then we only have to divide by y, which operation corresponds to the matrix $\begin{pmatrix} \sqrt{y}^{-1} & 0 \\ 0 & \sqrt{y} \end{pmatrix}$. (Note that this last matrix is in $\mathrm{SL}_2(\mathbf{R})$, because y was assumed to be positive, so the root exists.) □

Lemma 2.27. *The stabilizer of* $\mathrm{PSL}_2(\mathbf{R})$ *at* i *is*

$$\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbf{R}, a^2 + b^2 = 1 \right\} =: \mathrm{SO}_2(\mathbf{R}).$$

Proof. Suppose we have a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbf{R})$ with $Ai = \frac{ai+b}{ci+d} = i$. Then multiplying by $ci + d$ gives $ai + b = di - c$, so we find $a = d$ and $b = -c$. □

2.3. **The unit disk.** Another often used model for hyperbolic geometry is the unit disk

$$\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}.$$

Its symmetry can make it more advantageous than the upper half plane in certain situations. The following proposition clarifies why it does not really matter if we consider the upper half plane or the unit disk.

[2]The proof of this uses that one can easily give a path from one matrix to that matrix with one column or row added $\lambda \in \mathbf{R}$ times to another column or row.

Proposition 2.28. *The unit disk and the upper half plane are complex analytically isomorphic.*

Proof. The map

$$\rho\colon \mathbf{H} \to \mathbf{D},\ z \mapsto \frac{z-i}{z+i} \tag{3}$$

which is the restriction to the upper half plane of the automorphism of the Riemann sphere induced by $\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \in \mathrm{GL}_2(\mathbf{C})$, clearly maps the upper half plane into the unit disk, since any $z \in \mathbf{H}$ lies strictly closer to i than to $-i$, so that the numerator is strictly smaller in absolute value than the denominator.

The inverse of the matrix $\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ that gives this fractional linear transformation is $\frac{1}{2i}\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}$, so the inverse mapping is (note that the factor $\frac{1}{2i}$ does not matter)

$$z \mapsto \frac{(z+1)i}{1-z}.$$

The imaginary part of the image of an element $z \in \mathbf{D}$ is the real part of $\frac{z+1}{1-z}$ and is equal to

$$\frac{1}{2}\left(\frac{z+1}{1-z} + \overline{\frac{\bar{z}+1}{1-z}}\right) = \frac{1}{2}\left(\frac{(z+1)(1-\bar{z}) + (\bar{z}+1)(1-z)}{|1-z|^2}\right) = \frac{1-|z|^2}{|1-z|^2}.$$

Since $|z| < 1$ holds, this is positive. This shows that the inverse maps $\mathbf{D}$ into $\mathbf{H}$ and hence $z \mapsto \frac{z-i}{z+i}$ is an complex analytic isomorphism from $\mathbf{H}$ to $\mathbf{D}$. $\square$

Obviously this allows us to transport many properties from the upper half plane to the unit disk. First we transport the group of complex analytic automorphisms $\mathrm{PSL}_2(\mathbf{R})$ that operates on the upper half plane.

Proposition 2.29. *We have*

$$\rho\,\mathrm{PSL}_2(\mathbf{R})\rho^{-1} = \mathrm{SU}(1,1)/\{\pm I\}$$

with

$$\rho = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \qquad \text{and} \qquad \mathrm{SU}(1,1) := \left\{\begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix} : a, b \in \mathbf{C}, |a|^2 - |b|^2 = 1\right\}.$$

Proof. This is a straightforward computation: Note

$$\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}^{-1} = -\frac{1}{2}i\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}.$$

Now take any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbf{R})$. Then we find

$$-\frac{1}{2}i\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (a+d)+(b-c)i & (a-d)-(b+c)i \\ (a-d)+(b+c)i & (a+d)-(b-c)i \end{pmatrix}.$$

Since $\det(\rho A \rho^{-1}) = \det(A) = 1$, it is clear that this matrix is in $\mathrm{SU}(1,1)/\{\pm I\}$.

We also see that this gives every matrix $\begin{pmatrix} u & \bar{v} \\ v & \bar{u} \end{pmatrix} \mathrm{SU}(1,1)/\{\pm I\}$: take $a + bi = \frac{1}{2}(u+v)$ and $d - ci = \frac{1}{2}(u-v)$. Then clearly equality holds. Also, $ad - bc = \Re\left((a+bi)\overline{(d-ci)}\right) = \Re(\frac{1}{2}(u+v)\cdot\frac{1}{2}(\bar{u}-\bar{v})) = \frac{1}{4}\Re(u\bar{u} - v\bar{v} + \bar{u}v - u\bar{v}) > 0$ because

$u\bar{u} - v\bar{v} = |u|^2 - |v|^2 = 1$ and $\bar{u}v - u\bar{v} = \bar{u}v - \overline{\bar{u}v} \in \mathbf{R} \cdot i$. It follows that we can scale such that the determinant is 1. $\square$

Remark 2.30. Proposition 2.29 implies that the elements of the group $\mathrm{SU}(1,1)/\{\pm I\}$ are complex analytic automorphisms of the unit disk, because we had already remarked that $\mathrm{PSL}_2(\mathbf{R})$ consists of complex analytic automorphisms of the upper half plane (see remark 2.23). We will see shortly (in proposition 2.33) that they are in fact *all* complex analytic automorphisms of the unit disk, which by this proposition also shows that $\mathrm{PSL}_2(\mathbf{R})$ are all complex analytic automorphisms of the upper half plane.

Remark 2.31. We will use on several occasions later that multiplication by a complex number $z \in \mathbf{C}$ with $|z| = 1$ is in $\mathrm{SU}(1,1)/\{\pm I\}$: if $u \in \mathbf{C}$ is a square root of z, the multiplication map is induced by the matrix $\begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix}$ because $\frac{u}{\bar{u}} = \frac{u^2}{\bar{u}u} = u^2$.

For the next result, we first need to following lemma.

Lemma 2.32 (Scharwz lemma)**.** *Let* $f\colon \mathbf{D} \to \mathbf{C}$ *be analytic with* $f(\mathbf{D}) \subset \mathbf{D}$ *and* $f(0) = 0$*. Then* $|f'(0)| \leq 1$*. If* $f'(0) = 1$*, then* $f = \mathrm{id}_{\mathbf{D}}$*.*

Proof. A proof of about half a page in length can be found in [GK06], proposition 5.5.1. $\square$

Proposition 2.33. *The complex analytic automorphisms of the unit disk induced by elements of* $\mathrm{SU}(1,1)/\{\pm I\}$ *are all the complex analytic automorphisms of the unit disk. Similarly, the automorphisms* $\mathrm{PSL}_2(\mathbf{R})$ *are all the complex analytic automorphisms of the upper half plane.*

Proof. In remark 2.30 we already saw that the elements of $\mathrm{SU}(1,1)/\{\pm I\}$ are indeed complex analytic automorphisms. It only remains to show that these are all automorphisms.

Suppose that $f : \mathbf{D} \to \mathbf{D}$ is a complex analytic automorphism of the unit disk. Then let $a \in \mathrm{SU}(1,1)/\{\pm I\}$ be such that $af(0) = 0$. Such a map exists by the transitivity that we have proven for $\mathbf{H}$, but carries over to $\mathbf{D}$. Now let $b \in \mathrm{SU}(1,1)/\{\pm I\}$ be the map given by

$$z \mapsto \frac{z}{(af)'(0)}$$

which is an automorphism in $\mathrm{SU}(1,1)/\{\pm I\}$ because af and $(af)^{-1}$ are complex analytic automorphisms that map zero to zero, so by the inequality on the derivative at zero in the Schwarz lemma we can conclude $|(af)'(0)| = 1$. (Remark 2.31 explains that multiplication (and also division) by $(af)'(0)$ is an automorphism.) Then the map $z \mapsto bfz$ satisfies both $(baf)(0) = 0$ and $(baf)'(0) = 1$, so by Schwarz lemma, it is the identity. This gives $f = a^{-1}b^{-1} \in \mathrm{SU}(1,1)/\{\pm I\}$.

This shows that $\mathrm{PSL}_2(\mathbf{R})$ are all complex analytic automorphisms of the upper half plane: for any complex analytic automorphism $\phi\colon \mathbf{H} \to \mathbf{H}$, the function $\rho\phi\rho^{-1}$ (with ρ the isomorphism given in (3)) is an automorphisms of the unit disk and hence an element of $\mathrm{SU}(1,1)/\{\pm I\}$. It follows that ϕ is an element of $\rho^{-1}\,\mathrm{SU}(1,1)/\{\pm I\}\rho = \mathrm{PSL}_2(\mathbf{R})$ (by proposition 2.29). $\square$

We will now introduce a definition and continue to derive some useful properties of the automorphism group of the unit disk.

Definition 2.34. We define *hyperbolic lines* of the unit disk to be the generalized circles of the Riemann sphere that intersect the unit circle orthogonally, intersected with the unit disk.

Proposition 2.35. *The image of the hyperbolic lines in the upper half plane under the isomorphism between* $\mathbf{H}$ *and* $\mathbf{D}$ *given in* (3) *is the set of hyperbolic lines in the unit disk.*

Proof. We prove this for the generalized circles in the Riemann sphere that give rise to the hyperbolic lines; since the isomorphism is a bijection, this is sufficient to show that the generalized circles intersected with the upper half plane are mapped to the generalized circles intersected with the unit disk.

We see that under the action of $\rho = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$, which induces the isomorphism, 0 is mapped to -1, ∞ to 1 and 1 too is mapped to the unit circle $\{z \in \mathbf{C} : |z| = 1\}$. This implies that the real axis is mapped to the unit circle, because there is exactly one generalized circle through every three different points (this is a well-known fact for three points in the Euclidean plane; if one point is ∞, take the straight line through the other two together with ∞) and theorem 2.18 tells us generalized circles are mapped to generalized circles.

We conclude by proposition 2.15 that the image under ρ of a generalized circle is orthogonal to the unit disk if and only if it is orthogonal to the real axis. This completes the proof. □

Corollary 2.36. *Hyperbolic lines in the unit disk are mapped to hyperbolic lines in the unit disk under the action of* $\mathrm{SU}(1,1)/\{\pm I\}$.

Proof. Let $g \in \mathrm{SU}(1,1)/\{\pm I\}$. Note $g = \rho \circ (\rho^{-1} \circ g \circ \rho) \circ \rho^{-1}$ with $\rho \colon \mathbf{H} \to \mathbf{D}$ the isomorphism in (3). We have $\rho^{-1} \circ g \circ \rho \in \mathrm{PSL}_2(\mathbf{R})$. Now remark that a hyperbolic line in the unit disk is first sent to a hyperbolic line in the upper half plane by ρ^{-1}, then again to a hyperbolic line in the upper half plane by $\rho^{-1} \circ g \circ \rho$ and finally to a hyperbolic line in the unit disk by ρ. □

Proposition 2.37. *The stabilizer subgroup of* $\mathrm{SU}(1,1)/\{\pm I\}$ *at 0 is*

$$\left\{ \begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix} : u \in \mathbf{C}, |u|^2 = 1 \right\} \subset \mathrm{SU}(1,1)/\{\pm I\}.$$

This corresponds to the map

$$z \mapsto \frac{u}{\bar{u}} z.$$

Proof. Take any $\begin{pmatrix} u & \bar{v} \\ v & \bar{u} \end{pmatrix} \in \mathrm{SU}(1,1)/\{\pm I\}$. The image of 0 under the action of this matrix is $\bar{v}/\bar{u}$. This is equal to zero if and only if $\bar{v} = 0$ (since $\bar{v} = 0$ implies $\bar{u} \neq 0$). □

We conclude this section with some results about both the upper half plane and the unit disk, which are most easily derived by considering first the unit disk and then transporting the results to the upper half plane.

Corollary 2.38. *The image of a point* $p \in \mathbf{D} \backslash \{0\}$ *under the stabilizer of* $\mathrm{SU}(1,1)/\{\pm I\}$ *at 0 is a (Euclidean) circle with centre 0. Similarly, the image of a point* $p \in \mathbf{H} \backslash \{0\}$ *under the stabilizer of* $\mathrm{PSL}_2(\mathbf{R})$ *at* i *is a (Euclidean) circle contained in* $\mathbf{H}$.

Proof. We first prove the first statement. The maps in the stabilizer are the maps of the form $z \mapsto \frac{u}{\bar{u}}z$ for $u \in \mathbf{C}$ (proposition 2.37) and since the norm of $\frac{u}{\bar{u}}$ is one, it is clear the images of z lie on a circle. For the opposite direction: the circle through p with the centre at 0 can be parametrized as $\{zp : z \in \mathbf{C}, |z| = 1\}$ and because multiplication by $z \in \mathbf{C}$ with $|z| = 1$ is an automorphism (see remark 2.31), we see that the image of p under the stabilizer is indeed the entire circle.

Now the second statement. Let $H \subset \mathrm{SU}(1,1)/\{\pm I\}$ be the stabilizer of 0. Note that the inverse of the isomorphism $\rho : \mathbf{H} \to \mathbf{D}$ in (3) maps 0 to i. We can hence transport the stabilizer and also the image of z under the stabilizer from one to the the other:

$$\rho^{-1}Hp = (\rho^{-1}H\rho)\rho^{-1}p$$

which shows that the image of the stabilizer in $\mathbf{D}$ is the image of the stabilizer around i in $\mathbf{H}$. This, of course, also works the other way around. Because generalized circles are mapped to generalized circles and we have already shown that Hz is a generalized circle, we can conclude that the image of a point under the stabilizer of i is also a generalized circle. Since these generalized circles are entirely contained within $\mathbf{H}$ we can even conclude that the generalized circle must be a (Euclidean) circle. □

Proposition 2.39. *Through every two different points in either* $\mathbf{H}$ *or* $\mathbf{D}$*, there is exactly one hyperbolic line (in* $\mathbf{H}$ *or* $\mathbf{D}$*, respectively).*

Proof. We prove this for $\mathbf{D}$, though we first use an argument in $\mathbf{H}$ that is useful in the proof of uniqueness.

Note that the isomorphism $\rho\colon \mathbf{H} \to \mathbf{D}$ in (3) sends the imaginary axis to the real axis (since 0 is mapped to -1, i is mapped to 0 and ∞ is mapped 1; three different points determine a generalized circle). Take any two different points on the imaginary axis. It is immediately evident by definition that there is only one hyperbolic line in the upper half plane through both these points: the imaginary axis. Hence we see that there is also one one hyperbolic line in the unit disk through two different points on the real axis.

Take two points $z, z' \in \mathbf{D}$ with $z \neq z'$. There exists an automorphism $g \in \mathrm{PSL}_2(\mathbf{R})$ with $gz = 0$. Because the image of gz' under all automorphisms in stabilizer of 0 is a circle with centre 0, there exists an automorphism h in that stabilizer such that hgz' lies on the real axis. Define $l = \mathbf{R} \cap \mathbf{D}$: it is the only hyperbolic line in the unit disk through $hgz = gz = 0$ and hgz'. Then $g^{-1}h^{-1}l$ is a hyperbolic line through z and z'. It is also the only, because if there were another, call it l', then hgl' would be another hyperbolic line through the two points on the real axis. □

Proposition 2.40. *The action of the groups* $\mathrm{PSL}_2(\mathbf{R})$ *or* $\mathrm{SU}(1,1)/\{\pm I\}$ *on the set of hyperbolic lines in the upper half plane or unit disk, is transitive.*

Proof. We prove this for the unit disk, by mapping any hyperbolic line in the unit disk to the real axis intersected with the unit disk. Let l be a hyperbolic line in the unit disk. Take two different points on l and map them to the real axis, as was done in the proof of proposition 2.39. The image of l is a hyperbolic line through two points on the real axis, so it is the real axis intersected with the unit disk. This proves transitivity. □

2.4. **Riemannian manifolds.** We have already seen that the upper half plane is in fact a Riemann surface. However, it is also essential for our model that we give it the structure of a Riemannian manifold.

Definition 2.41. A *Riemannian manifold* is a pair $(M, \langle\cdot,\cdot\rangle)$ with M a differentiable manifold and a familiy

$$\langle\cdot,\cdot\rangle = \{\langle\cdot,\cdot\rangle_p\}_{p\in M}$$

of inner products (i.e., positive-definite symmetric bilinear forms) $\langle\cdot,\cdot\rangle_p$ on the tangent space T_pM of M at p, that is smooth in the following sense: for any smooth vector fields X, X', the function $p \mapsto \langle X_p, X'_p\rangle_p$ is smooth. The family of inner products $\{\langle\cdot,\cdot\rangle_p\}_{p\in M}$ is then called a *Riemannian metric.*

Now we want to give the upper half plane a Riemannian metric that is invariant under (or compatible with) the (complex analytic) automorphism group $\mathrm{PSL}_2(\mathbf{R})$: we demand that every $g \in \mathrm{PSL}_2(\mathbf{R})$ satisfies

$$\langle g'x, g'y\rangle_{gp} = \langle x, y\rangle_p \tag{4}$$

for any point $p \in \mathbf{H}$ and any tangent vectors $x, y \in T_p\mathbf{H}$ (with g' the differential of g at p). (Note that, as stated in remark 2.11, we can consider the upper half plane (a Riemann surface) as a differentiable manifold, so that we can also give it a Riemannian metric.) Because $\mathrm{PSL}_2(\mathbf{R})$ acts transitively, choosing an inner product at one point determines the inner product at all other points, as we show in the following somewhat more general lemma.

Lemma 2.42. *Let M be a connected manifold and let G be a group that acts transitively on M. Let also be given a certain point $p \in M$ and an inner product $\langle\cdot,\cdot\rangle_p$ on the tangent space T_pM and assume that this inner product is invariant under the stabilizer subgroup $G_p \subset G$ of p as in equation* (4)*. Then there exists a unique family of inner product $(\langle\cdot,\cdot\rangle_x)_{x\in M}$ on the tangent spaces of M ($\langle\cdot,\cdot\rangle_x$ is defined on T_xM) such that the collection of these inner products is invariant under G as in equation* (4)*.*

Proof. For any $x \in M$ we take an $f_x \in G$ such that $f_x x = p$ and define for any tangent vectors $v, w \in T_xM$

$$\langle v, w\rangle_x = \langle f'_x v, f'_x w\rangle_p \tag{5}$$

(with f'_x the differential of f_x at x). (Note that this is valid also for $x = p$.) We will see that this family of inner products satisfies our requirements.

Take any $g \in G$ and $x \in M$. Write $y := gx$. We can decompose g as $g = f_y^{-1} \circ (f_y \circ g \circ f_x^{-1}) \circ f_x$. Since $(f_y \circ g \circ f_x^{-1})$ is in the stabilizer of p, it leaves the inner product invariant. The inner product is also left invariant under f_x by construction of the inner product. Lastly, for any tangent vectors $v, w \in T_yM$, we have

$$\langle (df_y^{-1})_p v, (df_y^{-1})_p w\rangle_y = \langle (df_y)_y (df_y^{-1})_p v, (df_y)_y (df_y^{-1})_p w\rangle_p = \langle v, w\rangle_p$$

so we see that the construction also ensures invariance under f_y^{-1} for any $y \in M$. By our decomposition of g into maps that leave the inner product invariant, we conclude that the inner product is invariant under G.

Now note that this inner product is unique, because equation (5) necessarily holds for any G-invariant family of inner product $(\langle\cdot,\cdot\rangle_x)_{x\in M}$. □

Remark 2.43. Note that we have not in this generality proven that we get a Riemannian metric: we have not proven smoothness.

To the end of applying this lemma to our situation, we start with taking for the inner product $\langle\cdot,\cdot\rangle_i$ at the tangent space $T_i\mathbf{H}$ the standard inner product for $\mathbf{R}^2$. (Of course, we will identify $\mathbf{R}^2$ with $\mathbf{C}$ in the canonical way, which gives $\langle z, z'\rangle = \langle(z_1, z_2), (z_1', z_2')\rangle = \frac{1}{2}(z\bar{z}'+\bar{z}z')$ for two complex numbers $z = z_1+z_2 i$, $z' = z_1'+z_2' i \in \mathbf{C}$.) So far, this is compatible with the action of $\mathrm{PSL}_2(\mathbf{R})$, as the following lemma shows. Note that because we can define the Riemann surface $\mathbf{H}$ with only one chart (the inclusion in $\mathbf{C}$), we will write all tangent vectors with respect to this chart (so we will suppress the notation as in definition 2.13 by leaving out which chart we use).

Lemma 2.44. *For any $g \in \mathrm{PSL}_2(\mathbf{R})$ with $gi = i$ (i.e., in the stabilizer of i) and any tangent vectors $z, z' \in T_i\mathbf{H}$ in the tangent space at i. we have*

$$\langle z, z'\rangle = \langle g'z, g'z'\rangle$$

with g' the differential of the action of g on $\mathbf{H}$.

Proof. By lemma 2.27, we can take as a representative for g the matrix

$$A := \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

with $a, b \in \mathbf{R}$ satisfying $a^2 + b^2 = 1$. The derivative of the map $z \mapsto gz = \frac{az+b}{-bz+a}$ at the point i is easily seen to be

$$\frac{1}{(-bi+a)^2}.$$

Because a and b satisfy $a^2 + b^2$, the complex number $a - bi$ lies on the unit circle $\{z \in \mathbf{C} : |z| = 1\}$ and hence so does $\frac{1}{(-bi+a)^2}$. We conclude that $g'z$ and $g'z'$ are rotations of respectively z and z' over the same angle, from which the lemma follows. $\square$

We will now extend the inner product in the following way. Suppose we have another point $x + iy \in \mathbf{H}$. We have already seen in lemma 2.26 that we can map $x + iy$ to y by first applying the map $z \mapsto z - x$ and then $z \mapsto \frac{1}{y}z$. Choose the composition of these maps as the maps f_{x+iy} in the proof of lemma 2.42. Using the chain rule, we find that the differential of the map f_{x+iy} at any point is $z \mapsto \frac{1}{y}z$. We then define as in (5) for any $a, b \in T_{x+iy}\mathbf{H}$:

$$\langle a, b\rangle_{x+iy} := \langle f_{x+iy}'a, f_{x+iy}'b\rangle_{f_{x+iy}(x+iy)} = \langle\frac{1}{y}a, \frac{1}{y}b\rangle_i = \frac{1}{y^2}\langle a, b\rangle. \tag{6}$$

This shows how our choice for the inner product at the tangent space of i determines the entire inner product on $\mathbf{H}$. However, we still need to verify that this actually gives a Riemannian metric. Writing ∂_x for the tangent vector $(1, 0)$ (with respect to the one chart that is the inclusion) and ∂_y for the tangent vector $(0, 1)$, we can express any vector field as $p \mapsto a_1(p)\partial_x + a_2(p)\partial_y$ with a_1, a_2 smooth functions. Taking two vector fields $p \mapsto a_1(p)\partial_x + a_2(p)\partial_y$ and $p \mapsto b_1(p)\partial_x + b_2(p)\partial_y$, we see their inner product becomes $x+iy \mapsto \frac{1}{y^2}a_1(x+iy)b_1(x+iy)+\frac{1}{y^2}a_2(x+iy)b_2(x+iy)$, which is smooth in the upper half plane.

This shows that we have in this way given $\mathbf{H}$ a Riemannian metric. We can now exploit the Riemannian metric to define distance and geodesics ("lines").

Definition 2.45. Let M be a connected Riemannian manifold. On M we can define a notion of distance in the following way: for any two points x, y we define their distance $d(x, y)$ as

$$d(x, y) = \inf_{\gamma} \int_0^1 \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)}}\, dt$$

where the infimum is taken over all piecewise continuously differentiable maps $\gamma : [0, 1] \to M$. This is easily seen to be a metric.

Definition 2.46 (geodesic)**.** A *geodesic segment between two points* x, y on a Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ is a continuous, piecewise continuously differentiable function $\gamma\colon [0, 1] \to M$ with $\gamma(0) = x$, $\gamma(1) = y$ and $d(x, y) = \int_0^1 \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)}}\, dt$. A *geodesic* is a continuous, piecewise continuously differentiable function $\gamma\colon \mathbf{R} \to M$ such that it is locally a geodesic segment between two points: for every $t_0 \in \mathbf{R}$ there exists an $\epsilon \in \mathbf{R}_{>0}$ such that for all $s, s' \in (t_0 - \epsilon, t_0 + \epsilon)$ the map $\gamma'\colon [0, 1] \to M, t \mapsto \gamma(s + (s' - s)t)$ is a geodesic segment between $\gamma(s)$ and $\gamma(s')$.

Remark 2.47. Note that (1) such a geodesic segment need not be unique (even after, for example, scaling all geodesic segments in such a way that they have a constant speed); and (2) need not exist, even if M is path connected. For (1), consider for example the unit sphere with the standard inner product and the paths between two anti-podal points that go over the great circles: all such paths have the same minimal length. For (2), an example is $\mathbf{R}^2 \setminus \{(0, 0)\}$ with the standard inner product: there are infinitely many paths between $(-1, 0)$ and $(1, 0)$, but for any path you might take from one to the other, there is always a shorter path.

Proposition 2.48. *Let $(M, \langle \cdot, \cdot \rangle)$ be a connected Riemannian manifold with a Riemannian metric that is invariant under a group G that acts as automorphisms on M. Let $g \in G$, $x, y \in M$ and let γ be a piecewise continuously differentiable path between x and y. Then the length of γ is equal to the length of $g\gamma$, i.e.*

$$\int_0^1 \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)}}\, dt = \int_0^1 \sqrt{\langle g'\gamma'(t), g'\gamma'(t) \rangle_{g\gamma(t)}}\, dt.$$

and as a consequence we have

$$d(x, y) = d(gx, gy)$$

with d the metric induced by the Riemannian metric as in definition 2.45.

Proof. By definition, we have

$$d(x, y) = \inf_{\gamma} \int_0^1 \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)}} dt$$

where the infimum is taken over all piecewise continuously differentiable maps $\gamma\colon [0, 1] \to M$. Because g is an differentiable automorphism, $g\gamma$ is a path from gx to gy. It has the same length as the path γ, because the Riemannian metric is assumed to be invariant under the action of G, so that we find for all $t \in [0, 1]$

$$\sqrt{\langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)}} = \sqrt{\langle g'\gamma'(t), g'\gamma'(t) \rangle_{g\gamma(t)}}$$

and hence

$$\int_0^1 \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)}}\, dt = \int_0^1 \sqrt{\langle g'\gamma'(t), g'\gamma'(t) \rangle_{g\gamma(t)}}\, dt.$$

This also works the other way around: any path γ between gx and gy gives a path $g^{-1}\gamma$ between x and y of the same length. We conclude

$$d(gx, gy) = \inf_{\gamma \in \Gamma_1} \int_0^1 \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)}} dt = \inf_{\gamma \in \Gamma_2} \int_0^1 \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)}} dt = d(x, y)$$

where the first infimum is taken over the set Γ_1 of piecewise continuously differentiable paths from gx to gy and the second over the set Γ_2 of such paths from x to y. ☐

Corollary 2.49. *Let $(M, \langle\cdot,\cdot\rangle)$ be a connected Riemannian manifold with a Riemannian metric that is invariant under a group G that acts as automorphisms on M. Then geodesic segments are mapped to geodesic segments under G.*

Proof. Let γ be a geodesic segment between two points x, y, so the length of γ equals $d(x, y)$, and let $g \in G$. The above proposition shows that the length of $g\gamma$ equals the length of γ and that $d(x, y)$ equals $d(gx, gy)$, so the length of $g\gamma$ equals $d(gx, gy)$ and $g\gamma$ is a geodesic segment between gx and gy. ☐

Lemma 2.50. *In $\mathbf{H}$ with the Riemannian metric of* (6)*, the geodesic segment between two points on the imaginary axis is a line segment of the imaginary axis.*

Proof. Suppose we have two points pi, qi with $p, q \in \mathbf{R}_{>0}$ and $p < q$ on the imaginary axis. Let $\gamma : [0, 1] \to \mathbf{H}$ be a path. Let $\gamma_1(t) = \Re(\gamma(t))$ be the real part of $\gamma(t)$ and $\gamma_2(t) = \Im(\gamma(t))$ the imaginary part for all $t \in [0, 1]$, so that we have $\gamma(t) = \gamma_1(t) + \gamma_2(t)i$. Suppose that there is a $t \in [0, 1]$ with $\gamma_1(t) > 0$. Because γ_1 is continuous and $\gamma_1(0) = \gamma_1(1) = 0$, this implies $\gamma_1(t) > 0$ for all t in a certain interval within $[0, 1]$. We will then show that γ_2 is in fact a strictly shorter path. This proves that the shortest path, if it exists, must lie on the imaginary axis. We do this by calculating the length:

$$
\begin{aligned}
\int_\gamma \mathrm{d}s &= \int_0^1 \sqrt{\langle \gamma'(t), \gamma'(t)\rangle_{\gamma(t)}}\,\mathrm{d}t \\
&= \int_0^1 \sqrt{\langle \gamma_1'(t) + \gamma_2'(t), \gamma_1'(t) + \gamma_2'(t)\rangle_{\gamma(t)}}\,\mathrm{d}t \\
&= \int_0^1 \sqrt{\frac{1}{\gamma_2(t)^2}(\gamma_1'(t)^2 + \gamma_2'(t)^2)}\,\mathrm{d}t \\
&> \int_0^1 \sqrt{\frac{1}{\gamma_2(t)^2}(\gamma_2'(t)^2)}\,\mathrm{d}t = \int_{\gamma_2} \mathrm{d}s.
\end{aligned} \tag{7}
$$

We see that the path that only takes the imaginary values is in fact shorter. Hence, a geodesic between these two points, if it exists, only takes purely imaginary values. This suggest that

$$\gamma : [0, 1] \to \mathbf{H}, t \mapsto (p + t(q - p))i,$$

is a path with shortest length. We see that this is true, because any other path that is purely imaginary must cross one point at least twice, so that path is necessarily longer. ☐

Theorem 2.51. *The images of the geodesics of $\mathbf{H}$ with the Riemannian metric* (6) *are exactly the hyperbolic lines.*

Proof. Let $z, z' \in \mathbf{H}$ be two different points. Take an automorphism $g \in \mathrm{Aut}(\mathbf{H})$ such that gz and gz' both lie on the imaginary axis: such an automorphism exists, because we can take a hyperbolic line through those points by proposition 2.39 and send that hyperbolic line to the hyperbolic line on the imaginary axis by 2.40. We have proven in lemma 2.50 that the geodesic between gz and gz' is a subset of the hyperbolic line on the imaginary axis. For this reason, the geodesic between z and z' is the image of this subset under g^{-1} and is is thus a part of a hyperbolic line (viz. of the image of the hyperbolic line on the imaginary axis under g^{-1}). ☐

2.5. **Some more properties.** In hyperbolic geometry, we can define triangles in the same way as in Euclidean geometry: taking three vertices not lying on one geodesic, we can connect each pair by a geodesic and call the interior a triangle.

Proposition 2.52. *Let Δ be a hyperbolic triangle with angles α, β, γ. Then*

i. the area of Δ is $\pi - (\alpha + \beta + \gamma)$.

ii. $\alpha + \beta + \gamma < \pi$ holds.

Proof. This can be computed by integration: see [Bea83], theorem 7.13.1, which proves the first part, and the corollary following it, which states the second part. □

Proposition 2.53. *Let $\alpha, \beta, \gamma \in \mathbf{R}_{>0}$ such that $\alpha + \beta + \gamma < \pi$. Then there exists a triangle with angles α, β, γ.*

Proof. See [Bea83], theorem 7.16.2. □

3. Tessellations of the hyperbolic plane

The purpose of this section is to prove that we can tile the upper half plane with a triangle with angles $(\pi/q, \pi/r, \pi/s)$ for $q, r, s \in \mathbf{Z}_{>1}$ with $\frac{1}{q} + \frac{1}{r} + \frac{1}{s} < 1$ by "flipping the triangle in its sides", so to speak. This statement can be proven for more general polygons, but we will have no need for that.

In this section, we will mostly want just a little more isometries than we have considered so far:

Definition 3.1. By $\mathrm{Aut}(\mathbf{H})$, we denote the group of isometries of the upper half plane half plane with the metric in (6) generated by those induced by $\mathrm{PSL}_2(\mathbf{R})$ and by the reflection map $z \mapsto -\bar{z}$.

This means we do not only have holomorphic maps, but anti-holomorphic maps as well.

First we make precise what we mean by a tiling of the upper half plane, using the following definition.

Definition 3.2 (Fundamental domain)**.** An open subset $D \subset \mathbf{H}$ is called a *fundamental domain* for a subgroup $\Gamma \subset \mathrm{Aut}(\mathbf{H})$ if

(1) D is open and connected;
(2) $\mathbf{H} = \bigcup_{\gamma \in \Gamma} \gamma \bar{D}$ holds; and
(3) for all $\gamma \in \Gamma \setminus \{\mathrm{id}\}$, we have $\gamma D \cap D = \emptyset$.

If D is a fundamental domain for a subgroup $\Gamma \subset \mathrm{Aut}(\mathbf{H})$, then we can say that D tiles the upper half plane. Vaguely, one could say that a fundamental domain for Γ is, together with its boundary, a nice set of representatives of $\Gamma \setminus \mathbf{H}$, such that two representatives can represent the same equivalence class only if they lie on the boundary $\partial D = \bar{D} \setminus D$ of the triangle.

It is well known that every discrete subgroup of $\mathrm{PSL}_2(\mathbf{R})$ has a fundamental domain (see e.g. [Miy89] section 1.6). Here we are interested in going in the other direction: we start with a triangle and hope to turn it into a fundamental domain for a suitably chosen subgroup of automorphisms. We consider here the group $\mathrm{Aut}(\mathbf{H})$, which we define to be the group generated by $\mathrm{PSL}_2(\mathbf{R})$ and the reflection $z \mapsto -\bar{z}$ in the imaginary axis.

Theorem 3.3. *The interior $D \subset \mathbf{H}$ of any triangle with angles $(\pi/q, \pi/r, \pi/s)$ with $q, r, s \in \mathbf{Z}_{>0}$ such that $\frac{1}{q} + \frac{1}{r} + \frac{1}{s} < 1$ is a fundamental domain for the group G generated by the reflections in its sides. Furthermore, G is isomorphic to the abstract free group*

$$G^* = \langle Q, R, S \mid Q^2 = R^2 = S^2 = (RQ)^s = (RS)^q = (SQ)^r = 1 \rangle.$$

Proof. The proof of this theorem is based on an article by Maskit [Mas71], which is much more general than we need.

Note first that such a triangle exists by proposition 2.53. Denote by Q, R, S the reflections in the side opposite to the vertices with angle π/q, π/r, π/s respectively. We see that the composition QR, RS and SQ are respectively rotations around the vertices with angles π/s, π/q and π/r over an angle of $2\pi/s$, $2\pi/q$ and $2\pi/r$.

Define the free (abstract) group

$$G^* = \langle Q, R, S \mid Q^2 = R^2 = S^2 = (RQ)^s = (RS)^q = (SQ)^r = 1 \rangle$$

generated by the abstract symbols Q, R, S satisfying only the given relations. Let $\sigma : G^* \to G \subset \operatorname{Aut}(\mathbf{H})$ be the natural homomorphism given by $Q \mapsto Q$, $R \mapsto R$, $S \mapsto S$, but now interpreting the images as automorphisms of the upper half plane, instead of formal symbols.

We now define a space X, which will turn out to be homeomorphic to the upper half plane. In the process of proving that, we will find that σ induces an isomorphism from G^* to G, which proves the final part of this theorem.

We define the topological space

$$X = (\bar{D} \times G^*)/\sim$$

with $\sim$ an equivalence relation that will be described shortly. We give the space G^* the discrete topology, $\bar{D} \times G^*$ the product topology and $(\bar{D} \times G^*)/\sim$ the quotient topology. One could think of X as consisting of disjoint pieces of $\bar{D}$ indexed by G^* that are glued together with the equivalence relation that is described below, and then laid down by a map $\bar{p}\colon \mathbf{H} \to X$ (to be defined later) to cover $\mathbf{H}$. A subset $\bar{D} \times \{g\}$ for a certain $g \in G^*$ will in this way correspond to $\sigma(g)(\bar{D})$.

We say that two points $(z, g), (z', g') \in \bar{D} \times G^*$ are equivalent if for one of the reflections $A \in \{Q, R, S\} \subset \operatorname{Aut}(\mathbf{H})$ we have

$$(z, gA) = (Az', g').$$

Note that this implies $z = z'$ lies on the border of D. This is not yet an equivalence relation, but it can be extended to an equivalence relation. Our equivalence relation $\sim$ in the definition of X is the smallest equivalence relation that extends this.

Now we define the map

$$p\colon \bar{D} \times G^* \to \mathbf{H}, \qquad (z, g) \mapsto \sigma(g)(z).$$

This map is continuous, because for $U \subset \mathbf{H}$ open we have

$$p^{-1}(U) = \bigcup_{g \in G^*} (\bar{D} \cap \sigma(g)^{-1}U) \times \{g\}$$

and every $\bar{D} \cap \sigma(g)^{-1}(U)$ is open in $\bar{D}$ since $\sigma(g)$ is an automorphism.

This map p is compatible with the action of left multiplication by G^* on itself and the action by G^* on $\mathbf{H}$ via σ, in the following sense: for every $g, h \in G^*$ and $z \in \bar{D}$ we have

$$p(z, hg) = \sigma(h)p(g).$$

This is easily seen to be true because of the fact that σ is a homomorphism.

Note that two equivalent points $(z,g) \sim (z',g')$ map to the same points under p: if there exists $A \in \{Q,R,S\}$ such that $(z,gA) = (Az',g')$, then we have $\sigma(g)(z) = \sigma(g)(Az') = \sigma(gA)(z') = \sigma(g')(z')$, and from this it follows that points equivalent in the extended relation $\sim$ also have the same image. We thus have a continuous map

$$\bar{p}\colon X \to \mathbf{H}.$$

This map is also compatible with the actions by G^* described above. We will now proceed to prove that this map $\bar{p}$ is a homeomorphism by lifting paths in $\mathbf{H}$ and using that $\mathbf{H}$ is simply connected.

Define for any $h \in G^*$ the set

$$I_h = \{g \in G^* : \exists z \in \bar{D}, (z,g) \sim (z,h)\}.$$

This definition is chosen in such a way, that for any $g \in I_h$, the triangle $\bar{p}(\bar{D} \times \{g\})$ is one of the "bordering translates" of the triangle $\bar{p}(\bar{D} \times \{h\})$: you can get from $\bar{p}(\bar{D} \times \{h\})$ to $\bar{p}(\bar{D} \times \{g\})$ by applying reflections in the sides of $\bar{p}(\bar{D} \times \{h\})$ in such a manner that after each reflection, the translated triangle still borders on the original triangle. A more direct description would be that I_1 consists of the rotations around a vertix of $\bar{D}$ and a reflection in one of the sides composed with such a rotation and then that $I_h = hI_1$ (which is easily proven using that (z,g) (z',g') implies (z,hg) (z',hg') for any $h \in G^*$).

It is easily seen that $\bar{p}$ restricted to the open set $\bar{D} \times I_1$ is a homeomorphism onto its image: on such a small scale, the translated (and possibly mirrored) triangles of the form $\bar{p}(\bar{D} \times \{g\})$ for any $g \in I_1$ fit together with overlap only on the border, but in such a way that the equivalence relation we defined ensures injectivity.

This means that we can begin lifting a part of any path γ in $\mathbf{H}$ starting at a certain point $z_0 \in D \subset \mathbf{H}$ to a path $\tilde{\gamma}$ in X starting at the point $(z_0,1)$. If the path γ reaches the end of $\bar{p}(\bar{D} \times I_1)$, then there is a certain $h \in I_1$, with $h \neq 1$, such that the path lies in the triangle $\bar{p}(\bar{D} \times \{h\})$ just before leaving $\bar{p}(\bar{D} \times I_1)$ for the first time. Then we can use that $\bar{p}$ restricted to $\bar{D} \times I_h$ is for the same reasons as before (together with the compatibility of $\bar{p}$ with the G^*-action and the identity $I_h = hI_1$) a homeomorphism onto its image to continue the lifting process. Because there is a certain minimal distance that the path has to traverse before it becomes necessary to consider a different neighborhood, this process ends and we can lift the path in its entirety. (Normally, one would prove such a lifting process ends by using the compactness of the unit interval and the existence of special open neighborhoods around every point on the path, but the last part requires surjectivity, which has only just now been proven using this path lifting process.)

Now note that the space X is path connected: take a point $(z,g) \in X$ and write $g = A_n A_{n-1} \cdot \ldots \cdot A_2 A_1$ with $A_i \in \{Q,R,S\}$ for $i \in \{1,2,\ldots,n\}$. We can construct a path from a point in $\bar{D} \times \{1\}$ to a point in $\bar{D} \times \{A_n\}$, because $\bar{D}$ is path connected and $\bar{D} \times \{A_n\}$ and $\bar{D} \times \{1\}$ have an edge in common (if A_n is the reflection in side s, then $s \times \{1\} \sim s \times \{A_n\}$). From a point in $\bar{D} \times \{A_n\}$ we can construct a path to a point in $\bar{D} \times \{A_n A_{n-1}\}$ for the same reason (if A_{n-1} is reflection in the side s, they share the side $s \times \{A_{n-1}\} \sim s \times \{A_n A_{n-1}\}$), until we have constructed a path to a point in $\bar{D} \times \{g\}$, where we can complete the path to (z,g).

In fact, we can now recognize that $\bar{p} : X \to \mathbf{H}$ is a covering map. The path-lifting we have just done shows that $\bar{p}$ is surjective. Also, there is a certain distance $d > 0$ from $\bar{D}$ to the border of $\bar{p}(\bar{D} \times I_1)$ because $\bar{D}$ is compact and the border of $\bar{p}(\bar{D} \times I_1)$

does clearly not meet the border of $\bar{D}$. The same holds for any translates: there is a certain distance $d > 0$ between $g\bar{D}$ and the border of $\bar{p}(\bar{D} \times I_g)$ for any $g \in G^*$. Hence, if we take an open ball U of radius $\frac{1}{2}d$ around any point $x \in \mathbf{H}$, then the inverse image of this open ball under $\bar{p}$ consists of disjoint sets homeomorphic under $\bar{p}$ to U; this is because for every $(z, g) \in X$ with $\bar{p}(z, g) = x$, the map $\bar{p}$ restricted to $\bar{D} \times I_g$ is a local homeomorphism, so any other preimages of U that are not in $\bar{p}^{-1}(U) \cap (\bar{D} \times I_g)$ must lie outside of $\bar{D} \times I_g$ and are therefore disjoint from $\bar{p}^{-1}(U) \cap (\bar{D} \times I_g)$.

Because $\mathbf{H}$ is simply connected, X is path-connected and p is a local homeomorphism, we can conclude that p is in fact a homeomorphism. For suppose that some point $z \in \mathbf{H}$ had two different inverse images $\tilde{z}_1, \tilde{z}_2$ under p. Then let $\tilde{\gamma}_1$ be the constant path at $\tilde{z}_1$ and let $\tilde{\gamma}_2$ be a path from $\tilde{z}_1$ to $\tilde{z}_2$. The paths $p \circ \tilde{\gamma}_1$ and $p \circ \tilde{\gamma}_2$ are loops around z in $\mathbf{H}$ and are therefore path homotopic, but this leads to the contradiction that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are path homotopic (since path homotopy can be lifted). (See, for example, [Run05], corollary 5.2.5 and theorem 5.2.6.)

It still remains to show that σ is a group isomorphism. Surjectivity of σ is clear, because the three generators (which are the reflections in the sides) of G are in the image of σ. Also, suppose that $g \in G^*$ is in the kernel of σ and let $x \in \bar{D}$. Then the equation $\bar{p}(x, 1) = \sigma(1)x = \sigma(g)x = \bar{p}(x, g)$ and the injectivity of $\bar{p}$ show $g = 1$, so σ is also injective. We conclude that σ is an isomorphism. □

Remark 3.4. We found the end of the proof given by Maskit unclear if not incomplete. The end of this proof would also be applicable in that of Maskit.

In this theorem, we generated a group by reflection in the sides. For the rest of this thesis, it is more convenient to consider only orientation-preserving automorphisms. The following corollary shows how we can use the above theorem to determine the tiling for the triangle with angles $(2\pi/7, \pi/3, \pi/3)$ using only rotations. This is the tiling we will concentrate on for the remainder of this thesis.

Corollary 3.5. *Let Δ be a triangle with angles $(2\pi/7, \pi/3, \pi/3)$. Then Δ is a fundamental domain for the group N generated by the rotation around the vertex with angle $2\pi/7$ over an angle of $2\pi/7$ and the rotation over the centre of the side opposite to the vertex with angle $2\pi/7$ over an angle of π. This group N is the subgroup of orientation preserving automorphisms of the group that has as its fundamental domain the triangle Δ' with angles $(\pi/7, \pi/3, \pi/2)$ that is a half of Δ.*

Proof. We consider the triangle Δ'. Let S denote the reflection in the side opposite to the angle of size $\pi/3$ and let Q and R denote the reflections in the sides opposite to the angles of size $\pi/7$ and $\pi/2$ respectively. Then by theorem 3.3, this triangle is a fundamental domain for the group $G = \langle Q, R, S\rangle$. The theorem also states that this group is fully determined by the relations

$$Q^2 = R^2 = S^2 = (RQ)^3 = (RS)^7 = (SQ)^2 = 1.$$

Now consider the subgroup $N \subset G$ generated by RS and QS. This group also contains the elements SR and SQ, which are inverses of the above, and $RSQS = RQ$ (since $SQSQ = 1$, so $SQS = Q^{-1} = Q$) and its inverse QR. Because all generators have order two, any element in G that preserves orientation (and is therefore the product of an even number of generators) can be written as a product of these elements and hence N contains all orientation preserving elements of G. Because the generators of N are orientation preserving, this shows that N is the group of the orientation preserving elements of G. From this it also follows that N is a normal subgroup of index 2.

We now easily see that Δ, which is the interior of $\bar{\Delta}' \cup \overline{S\Delta'}$, is a fundamental domain: (1) it is clearly open and connected; (2) we have

$$\mathbf{H} = \bigcup_{g\in G} g\bar{\Delta}' = \left(\bigcup_{g\in N} g\bar{\Delta}'\right) \cup \left(\bigcup_{g\in N} gS\bar{\Delta}'\right) = \bigcup_{g\in N} g\bar{\Delta};$$

and (3) for all $g \in N \setminus \{\mathrm{id}\}$, we have $g\Delta \cap \Delta = \emptyset$.

If we consider $x := QS$ and $y := RS$ as generators of N (note that $yx = RSQS = RQ$), then N is fully determined by the relations

$$x^2 = y^7 = (yx)^3 = 1$$

as I will explain. Because of $x^{-1} = x$ and $y^{-1} = y^6$, we can write any element in N as a sequence of x's and y's. If we write this in terms of the generators Q, R and S of G, without first removing cancelling terms, then we see that R^2 and Q^2 never occur, so the relations $R^2 = Q^2 = 1$ for G do not give any relations for x and y. Also, the relation $S^2 = 1$ ensures $yx = RQ$ and also $(RQ)^3 = (yx)^3 = 1$. The other two relations for G give $x^2 = y^7 = 1$, so all relations for G have been accounted for and thus $x^2 = y^7 = (yx)^3 = 1$ are all relations for the generators x, y of N.

We also see that QS represents the rotation of Δ around the centre of the side opposite to vertex with angle $2\pi/7$ over an angle of π and RS represents the rotation around the vertex with angle $2\pi/7$ over an angle of $2\pi/7$. This completes the proof. □

4. The regular 24-polyhedron

We can now tile the upper half plane with the triangle with angles $(2\pi/7, 2\pi/3, 2\pi/3)$ described in the corollary above, and use this tiling to construct a regular polyhedron with 24 faces. To do this, we have to divide out the upper half plane by a suitable subgroup of the group that gives the tiling. Now we could simply state what this subgroup is, but it would seem to have come out of the blue if we do not first discuss some background theory, which is what we will do first.

4.1. The modular group.

A well known discrete subgroup of the automorphism group $\mathrm{PSL}_2(\mathbf{R})$ of the upper half plane, is the so-called modular group $\mathrm{PSL}_2(\mathbf{Z})$. This group gives rise to a tiling of the upper half plane, as the following proposition states.

Proposition 4.1. *The modular group* $\mathrm{SL}_2(\mathbf{Z})$ *is generated by the matrices*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Proof. See [Miy89], chapter 4. □

Proposition 4.2. *A fundamental domain (see definition 3.2) for the group* $\mathrm{PSL}_2(\mathbf{Z})$ *is the set*

$$U = \left\{ z \in \mathbf{H} : |z| > 1, |\Re(z)| < \frac{1}{2} \right\}$$

with $\Re(z)$ *the real part of* z.

Proof. See [Miy89], chapter 4. □

The border of this fundamental domain is a hyperbolic triangle, with a vertex at infinity. The interior angles of the two vertices that do not lie at infinity, are both $\cos^{-1}(\frac{1}{2}) = \pi/3$, as one can easily deduce with high school geometry. This is what relates it to the triangle with angles $(2\pi/7, \pi/3, \pi/3)$ we are studying. In this case, however, we can more easily see by which subgroup we have to divide out the upper half plane to get something that could be considered a polyhedron. These are the groups of the following form.

Definition 4.3. Let $n \in \mathbf{Z}_{>1}$ and consider the map

$$\phi_n \colon \mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/n\mathbf{Z}) \tag{8}$$

given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$$

with $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ respectively the congruence class of a, b, c, d modulo n. We then define the *principal congruence modular group* of level n to be the image of $\ker \phi_n$ under the quotient map $q \colon \mathrm{SL}_2(\mathbf{Z}) \to \mathrm{PSL}_2(\mathbf{Z})$.

Remark 4.4. The map in the definition above is well-defined, because every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$ satisfies $ad - bc = 1$ and hence $ad - bc \equiv 1 \bmod n$.

Now we can consider $\phi_7 \colon \mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/7\mathbf{Z})$ and the principal congruence modular group that is the kernel of this map: if we divide out by that kernel, then the quotient space will have "faces" consisting of seven of the fundamental domains described in proposition 4.2. We proceed with some results on $\mathrm{SL}_2(\mathbf{Z}/7\mathbf{Z})$.

Proposition 4.5. *Let $k = \mathbf{F}_q$ be a finite field. The group $\mathrm{GL}_n(k)$ consists of*

$$(q^n - 1) \cdot (q^n - q) \cdot (q^n - q^2) \cdot \ldots \cdot (q^n - q^{n-1})$$

elements. The group $\mathrm{SL}_n(k)$ consists of

$$\frac{(q^n - 1)(q^n - q)(q^n - q^2) \cdot \ldots \cdot (q^n - q^{n-1})}{q - 1}$$

elements.

Proof. We count the number of matrices by counting the number of possible column vectors. The first column vector v_1 can be any of the $q^n - 1$ non-zero vectors of length n. The second column v_2 has to be taken outside of the linear subspace of q vectors spanned by v_1, so there are $q^n - q$ possibilities. The third column vector v_3 has to be taken outside of the linear subspace spanned by v_1 and v_2, which contains q^2 elements, so there are $q^n - q^2$ possibilities. Continuing this gives the desired expression for $\# \mathrm{GL}_n(k)$.

Note that we have the exact sequence

$$1 \to \mathrm{SL}_n(k) \to \mathrm{GL}_n(k) \xrightarrow{\det} k^\times \to 1.$$

This shows $\# \mathrm{GL}_n(k) / \# \mathrm{SL}_n(k) = q$. The given expressions follows. $\square$

Remark 4.6. In particular, the number of elements of $\mathrm{SL}_2(\mathbf{Z}/7\mathbf{Z})$ is seen to be $\frac{(7^2-1)\cdot(7^2-7)}{7-1} = 336$. The only scalar matrices in $\mathrm{SL}_2(\mathbf{Z}/7\mathbf{Z})$ are the identity matrix and minus the identity matrix, so the number of elements of $\mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})$ is exactly $\frac{1}{2} \cdot 336 = 168$.

Proposition 4.7. *The group $\mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})$ is simple.*

Proof. See [Lan02], section XIII.8. $\square$

4.2. **The construction.** Let $\Delta \subset \mathbf{H}$ be a triangle with angles $(2\pi/7, \pi/3, \pi/3)$. By corollary 3.5, the group G generated by the rotations around the vertex with angle $2\pi/7$ and around the centre of the side opposite to that vertex, has generators T and V satisfying the relations $T^7 = V^2 = (TV)^3 = 1$. Using proposition 4.1, we see that the matrices

$$T' = \begin{pmatrix} \bar{1} & \bar{1} \\ 0 & \bar{1} \end{pmatrix} \qquad \text{and} \qquad V' = \begin{pmatrix} 0 & -\bar{1} \\ \bar{1} & 0 \end{pmatrix} \qquad \text{in } \mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})$$

are generators of $\mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})$. Also, they too satisfy the relations

$$T'^7 = V'^2 = (T'V')^3 = I$$

though these relations do not completely determine the group. Therefore, we have a surjective group homomorphism

$$\phi \colon G \to \mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})$$

$$\text{given by} \qquad \begin{cases} T \mapsto T' \\ V \mapsto V' \end{cases} . \tag{9}$$

Hence we have the exact sequence

$$1 \to \Gamma \to G \xrightarrow{\phi} \mathrm{SL}_2(\mathbf{Z}/7\mathbf{Z}) \to 1 \tag{10}$$

with $\Gamma = \ker \phi$.

We will now see that we can consider $\Gamma \setminus \mathbf{H}$ as a regular polyhedron with 24 faces. The automorphisms that act on this quotient are

$$G/\Gamma \cong \mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z}).$$

(We will from now one often talk about elements of $\mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})$ as if they were elements of G/Γ.) If Δ denotes the triangle that was used to cover $\mathbf{H}$, then the quotient $\Gamma \setminus \mathbf{H}$ is the image of the equivalence classes of $\bar{\Delta}$ under elements from G/Γ, by construction. The equivalence classes of $\{\, T^n \bar{\Delta} \colon n \in \{0, 1, 2, \ldots, 6\} \,\}$ form a face; the other faces of the triangle are the images of this face under an element from G/Γ. In this way, we get faces that are regular heptagons, consisting of seven triangles with angles $(2\pi/7, \pi/3, \pi/3)$ that are the translates of Δ.

Also, the group G/Γ of (orientation-preserving) automorphisms acts transitively on the translates of Δ, which determine a pair (f, e) with f a face and e an edge on that face. If we also allow for reflection in the line through the centre of a face and the centre of an edge on that face in addition to the automorphisms in G/Γ, then we have a transitive action on the flags, which shows this polyhedron is regular.

We have already seen in remark 4.6, that $\mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})$ consists of 168 elements. Hence, $\Gamma \setminus \mathbf{H}$ consists of 168 translates of $\bar{\Delta}$ (with some overlap on the boundaries). Because every face consists of 7 of these translates, there is a total of $\frac{168}{7} = 24$ faces. This shows we have succeeded in constructing a kind of regular polyhedron with 24 faces.

4.3. **Some more properties.**

Lemma 4.8. *The genus of the regular polyhedron with 24 faces is 3.*

Proof. This is a simple counting argument, using the Euler characteristic, which equals

$$\chi = V - E + F = 2 - 2g$$

with V the number of vertices, E the number of edges and F the number of faces and g the genus of the regular polyhedron. We have $F = 24$ (see subsection

4.2). Each of the 24 faces has 7 vertices, but each vertex is shared by 3 faces, so $V = 24 \cdot 7/3 = 56$. Each of the 24 faces also has 7 edges, but each edge is shared by 2 faces, so $E = 24 \cdot 7/2 = 84$. This gives $\chi = -6$, so $g = 3$. $\square$

Using $\mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})$, we can deduce some properties of the polyhedron we constructed. For example, because the action of $\mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})$ on the set of pairs (f, e) of faces with one of its edges, is free and transitive, the flags are in bijective correspondence with the elements of $\mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})$. In the same way, the faces are in bijective correspondence with the left cosets in $\mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})/\langle T'\rangle$ (with $T' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})$), because $\langle T'\rangle$ are exactly the automorphisms that fix the face on which the triangle corresponding to the identity lies. We can thus study the faces by studying the cosets in $\mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})/\langle T'\rangle$. The following theorem gives a characterization of all those cosets.

Theorem 4.9. *The coset in* $\mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})/\langle T'\rangle$ *to which an element* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})$ *belongs, depends only on the values of* c *and* a. *Furthermore, from every equivalence class in* $\mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})$ *we can take a representative with* $c \in \{0, 1, 2, 3\}$ *and if* $c = 0$, *with* $a \in \{1, 2, 3\}$. *In this way, the cosets are in bijection with the pairs* (a, c) *in the set*

$$(\{1, 2, 3\} \times \{0\}) \cup (\{0, 1, 2, 3, 4, 5, 6\} \times \{1, 2, 3\}) .$$

Proof. Write

$$F := \langle T'\rangle$$

and note that it consists of matrices of the form

$$T'^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

with $n \in \{0, 1, \ldots, 6\}$. For the remainder of the proof, it is useful to keep in mind that multiplying by T^n on the left is the same as adding the bottom row of the matrix n times to the top row.

We will show that each coset can be gotten by multiplying F by one of the elements

$$T', S' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 2 & 0 \\ 0 & 2^{-1} \end{pmatrix}$$

Take a representative $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of an equivalence class in $\mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})$. Suppose $c = 0$. Since $a = 0$ would now imply that the determinant is zero, we have $a \neq 0$. Since multiplying by $-I$ does not change the element of $\mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})$, we can and do assume $a \in \{1, 2, 3\}$. It follows that we have $d = a^{-1}$. Hence the matrix is of the form

$$\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} .$$

It is clear that for a fixed a, all matrices of this type form a coset, because that set is equal to $A^{a-1}F$.

If c is not equal to zero, then again by multiplying by $-I$, we can and do assume $c \in \{1, 2, 3\}$. It is routinely checked that the matrix is in the coset $T^q S A^{c-1} F$ with q such that $cq = a \bmod 7$. This is easily seen by recognizing that $SA^{c-1}F$ consists of matrices of the form

$$\begin{pmatrix} 0 & -c^{-1} \\ c & x \end{pmatrix}$$

for $x \in \{0, 1, \ldots, 6\}$ and applying T^q on the left of any such matrix gives the matrix of the form

$$\begin{pmatrix} a & -c^{-1} + qx \\ c & x \end{pmatrix}.$$

Hence, all cosets are of the form

$$\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} c & * \\ a & * \end{pmatrix}$$

for $a \in \{1, 2, 3\}$ and $c \in \{0, 1, \ldots, 6\}$. (Note that the switch of the meaning of a and c is intentional: in this way $a \in \{1, 2, 3\}$ always holds.) It is evident that these are different cosets. Because they are 24 in number, these are all cosets. We can thus refer to a coset by a pair (a, c), where the a gives the upper left coordinate and the c the lower left coordinate of the matrix. $\square$

With this information, we can graphically depict the structure of the 24-polyhedron by showing how the faces are mapped to each other by either S' or T'. See figure 1.

Also, we can easily deduce which faces are connected to which other faces: the faces adjacent to the face F are exactly the faces $T^n SF$ for $n \in \{0, 1, 2, \ldots, 6\}$, so faces adjacent to the face gF with $g \in \mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})$ are exactly the faces $gT^n SF$ for $n \in \{0, 1, 2, \ldots, 6\}$. The result is shown in figure 2.

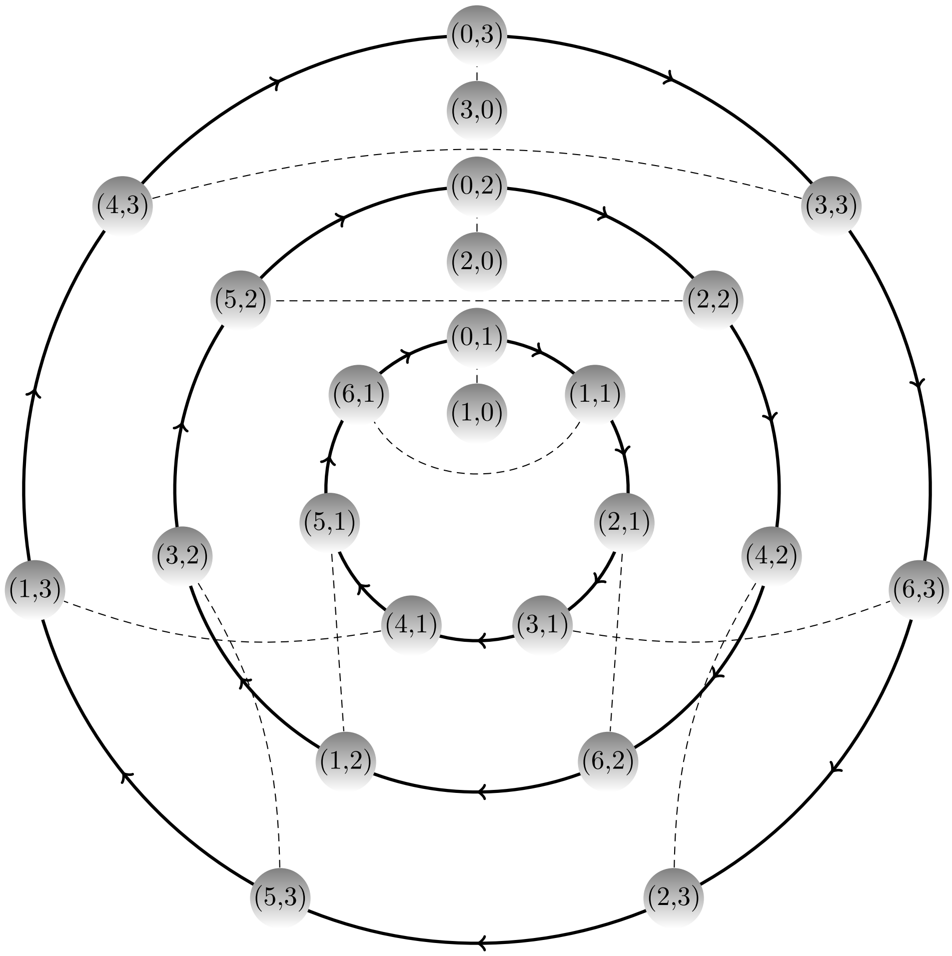

FIGURE 1. The cosets in $\mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})/\langle T' \rangle$. The numbers correspond to the pairs in theorem 4.9. The dashed lines connect cosets that are mapped to eachother when multiplied by $S' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on the left. The arrows show where cosets are sent when multiplied by $T' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on the left.

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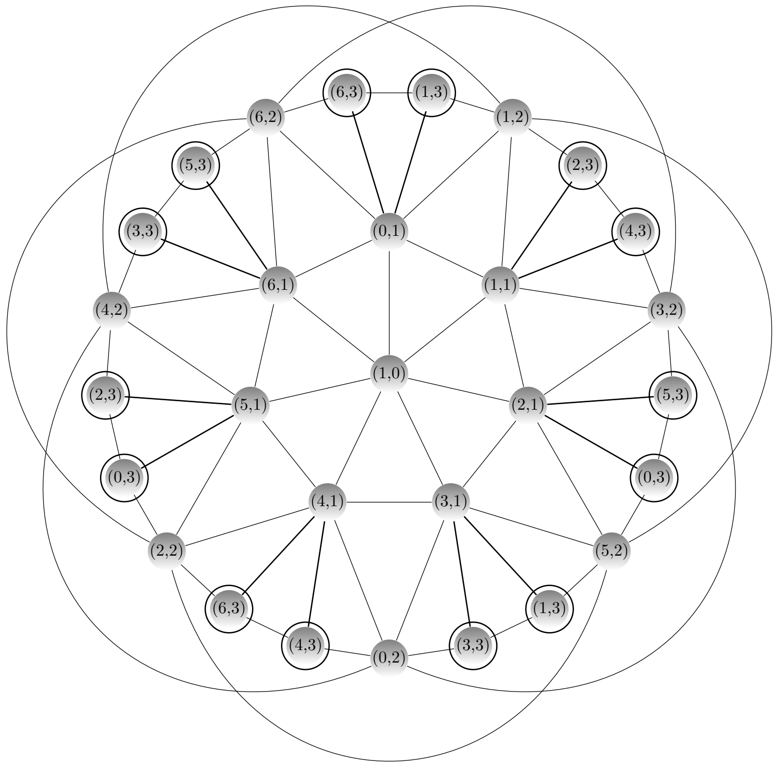

Figure 2. An overview of which faces are connected to which other faces. The pairs of numbers represent faces as they do in theorem 4.9. A line between two faces signifies an edge that both faces share. Note that the circled faces in the outer ring occur twice. The faces $(3,0)$ and $(2,0)$ are not listed: the face $(2,0)$ is connected to all encircled faces and the face $(3,0)$ is connected to all other faces on the outer ring.

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