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Lattice polytopes and toric varieties

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Contents

Introduction

The theory of toric varieties lies in the overlap of algebraic geometry and combinatorics. The rich interplay between these two fields in the context of the theory has led to a number of results in both areas. Some notable examples of the application of algebro-geometrical techniques to combinatorial problems include:

- Counting lattice points in convex polytopes via the Riemann-Roch theorem (see, e.g., the survey article [Bri95] by M. Brion).
- The proof, due to R.P. Stanley, of McMullen's conjectures on the number of faces of a simplicial convex polytope, obtained via Hard Lefschetz in [Sta80].

On the other hand, the combinatorial description of toric varieties allowed the proof of many important results in algebraic geometry, such as:

- The stable reduction theorem, in the area of resolution of singularities, proved in [KKMS73].
- The characterization, due to M. Demazure, of the algebraic subgroups of maximal rank of Cremona groups, in the seminal paper [Dem70].

Since the conception of the theory in the early 1970's, toric varieties have found applications in many other fields. In [CK99], for example, D.A. Cox and S. Katz explore the connections between toric geometry and mirror symmetry. The role of toric surfaces as natural generalizations of Bézier surfaces is described by R. Krasauskas in [Kra01], together with many interesting pictures. Other areas where techniques coming from the theory of toric varieties have been successfully applied include Diophantine geometry (see, e.g., [Roj16]), algebraic statistics and computational biology (see, e.g., [SS05]).

This thesis was initially motivated by the work of O. Karpenkov in the area of lattice geometry. In his paper [Kar06], he introduced functions analogous to the familiar trigonometric ones such as sine, cosine and tangent. Many of the properties holding for classical functions have their counterpart in the realm of lattice geometry. In particular, the well known fact that the sum of the (inner) angles of a triangle in the plane equals π has a lattice analogue. This proved to be a key idea for the complete classification of lattice triangles. More precisely, the results obtained by O. Karpenkov allow us to enumerate all lattice triangles of fixed lattice area up to lattice equivalence. Here by lattice equivalence we mean an affine transformation preserving the lattice. These results are useful, for example, in the study of singularities of toric varieties (see Appendix A in [Kar06]). At this point there is one natural question: is it possible to translate such properties into the language of algebraic geometry? To (or, better, begin to) answer this question one needs to set up the machinery of algebraic geometry in the context of toric varieties: this thesis serves that purpose. Moreover, in the last chapter some applications to the original problem are given.

There are several possible definitions of toric variety that can be encountered in the literature. The most common, and historically one of the earliest to be studied, starts with a convex "lattice cone" σ in some vector space. To this cone one associates an affine variety, called the *affine toric variety* of σ. When we have a collection of cones satisfying certain given properties, the corresponding affine toric varieties can be glued to form a *toric variety*. In Chapter 1 we show how to build such a collection of cones starting from a polytope and study the properties of the corresponding toric variety.

Another common construction of toric varieties starts from a lattice M in some Euclidean space $\mathbb{R}^n$ and a polytope K whose vertices lie in the lattice. Let k be an algebraically closed field of characteristic 0 and denote by $k^\times$ its group of units. Consider the finite set $\mathcal{A} = \{\alpha_0, \ldots, \alpha_m\}$ given by the intersection of K with the lattice M. If we write $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in})$ for $i = 0, \ldots, m$, then we have a map

$$\varphi_{\mathcal{A}} : (k^\times)^n \to \mathbb{P}^m_k$$

from $(k^\times)^n$ to projective m-space $\mathbb{P}^m_k$ given by

$$\varphi_{\mathcal{A}}(t_1, \cdots, t_n) = (t_1^{\alpha_{01}} \cdots t_n^{\alpha_{0n}} : \ldots : t_1^{\alpha_{m1}} \cdots t_n^{\alpha_{mn}}).$$

The closure in the Zariski topology of the image of $(k^\times)^n$ under $\varphi_{\mathcal{A}}$ is called the *projective toric variety* of the polytope K.

In the last part of Chapter 1 we show how to give the above construction for polytopes whose vertices lie in an abstract lattice, i.e. in a free finitely generated abelian group. This generalization seems to be quite natural, but it does not appear to be present explicitly in the literature.

In Chapter 2, after setting up all the machinery of divisors and invertible sheaves on toric varieties, we prove (see Theorem 2.4.3 on page 51) that the two approaches described in the first chapter are equivalent.

The last chapter is devoted to complex toric varieties. The analytical structure of these objects makes it possible to give a description of some invariants suc as the fundamental group and the cohomology groups in terms of lattice objects. In particular, we show in Section 3.2 that the Euler characteristic (the alternating sum of the dimensions of the cohomology groups) of the complex toric variety of a polytope equals the number of vertices of the polytope. This result is already present in the literature in a different form (see, e.g., [Dan78]), but without any connection to the theory of plane lattice geometry. Consider the following situation: we have a given two-dimensional toric variety (a toric surface) coming from a polygon and we want to know from which polygon it came. The aforementioned result, in the form described in this thesis, lets us distinguish whether it came from a triangle, a quadrangle, etc.

In this work we are concerned exclusively with toric varieties arising from polytopes: such objects are very particular, for example they are always projective. Arbitrary toric varieties, even though not always projective, still share many of the properties holding for polytopal toric varieties. Further generalizations are possible, and useful: for example, one could drop the assumption of separatedness and study *toric prevarieties*. In [Wło97], J. Włodarczyk proved that every normal variety admits an embedding into a toric prevariety. For the applications to lattice geometry, however, the setting of toric varieties coming from polytopes is the most natural and appropriate one.

As stated above, the motivation of this thesis is the study of lattice polygons. Since the toric variety associated to a non-degenerate lattice polygon always has dimension two (it is a so-called *toric surface*), in theory we could just work with two-dimensional lattices and vector spaces. Nonetheless, there is very little extra effort involved in setting up a description valid in higher dimensions (i.e. for polytopes instead of a polygon). This approach is the one taken in this work.

The standard textbooks on the theory of toric varieties are [Ful93], [Oda88] and [Ewa96]. In all of these, the varieties are studied over the field of complex numbers, but most of the results are valid for arbitrary algebraically closed fields of characteristic zero. The treatments of W. Fulton in [Ful93] and G. Ewald in [Ewa96] lean on the algebraic side of the theory, while T. Oda [Oda88] prefers an analytical approach. Among the three books listed above, the one by Fulton requires some prior knowledge of algebraic geometry, while the other two aim at giving an introduction to algebraic geometry through toric varieties. The first part of [Ewa96] also provides a brief but thorough introduction to convex geometry and convex polytopes. For a more advanced treatment of the theory of toric varieties, the survey article [Dan78] by V.I. Danilov is a superb starting point.

I am grateful to my supervisor Oleg Karpenkov for his constant attention to this work and to Bas Edixhoven, who answered with patience my endless questions. Particular thanks go to my wife Leonora, who always supported (and at times endured) me, and to my fellow ALGANT students.

Organization of this thesis

In Chapter 1 we start by recalling the basic results of convex geometry and algebra needed in the thesis. We introduce two constructions of the toric variety of a polytope and study its basic properties. In particular we show that such a variety is integral, separated and normal. In the last part of the chapter, we show that every toric variety contains a dense algebraic torus. This allows us to define the toric variety of a polytope lying in an abstract lattice.

In Chapter 2 we study divisors on toric varieties and their associated sheaves. We give criteria for such sheaves to be ample or very ample and apply them to show that a polytopal toric variety is always projective. In the last section we show that the two constructions defined in Chapter 1 are actually equivalent.

In Chapter 3 we study toric varieties over the field of complex numbers. We describe the topology of affine toric varieties and use this description to compute the Euler characteristic of the toric variety of a polytope.

Chapter 1

Toric varieties from polytopes

In this chapter we present two different constructions of the toric variety associated to a lattice polytope and study its basic properties. All the vector spaces throughout this work are tacitly assumed to be finite-dimensional vector spaces over the real numbers. All fields are assumed to be algebraically closed of characteristic zero.

1.1 Some convex geometry

In this section we include some basic facts about convex geometry which are used throughout the whole thesis. The book [Ewa96] contains an extensive treatment of these topics, with the applications to the theory of toric varieties in mind. The reader interested in a more general approach to convexity may consult, for example, [Roc96] or [Web94].

Let N be a lattice of rank n, that is, a finitely generated free abelian group of rank n (then $N \cong \mathbb{Z}^n$). We denote by $N_\mathbb{R}$ the associated real vector space $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ and we set $M = \mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z})$, which is isomorphic to $\mathbb{Z}^n$. If we denote by $M_\mathbb{R}$ the real vector space $M \otimes_\mathbb{Z} \mathbb{R}$, we find that $M_\mathbb{R} \cong \mathrm{Hom}(N_\mathbb{R}, \mathbb{R})$: in other words, $M_\mathbb{R}$ is isomorphic to the dual of $N_\mathbb{R}$. We denote by $\langle \cdot, \cdot \rangle$ the natural pairing $M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$. Unless otherwise specified, whenever we specify a lattice N, we always assume that its rank is n.

Let now V be a vector space. An *integral structure* on V is the datum of a lattice N such that $N_\mathbb{R} = V$. In this case $M_\mathbb{R}$ is identified with the dual $V^\vee = \mathrm{Hom}(V, \mathbb{R})$. Whenever we talk about "lattice objects" (e.g. lattice cones, lattice polygons, etc.), we assume such an integral structure to be given.

Definition 1.1.1. A (*polyhedral*) *cone* in a vector space V is the positive hull (the set of linear combinations with non-negative real coefficients) of a finite set of vectors of V, that is:

$$\sigma = \left\{ \sum_{i=1}^{m} \lambda_i s_i \;\middle|\; s_i \in M_{\mathbb{R}}, \lambda_i \in \mathbb{R}_{\geq 0} \right\}$$

for $S = \{s_1, \ldots, s_m\} \subset V$. We also write $\sigma = \mathrm{pos}(S)$ or $\sigma = \mathrm{pos}(s_1, \ldots, s_m)$. A *lattice cone* in $V = N_{\mathbb{R}}$ (or just a lattice cone, if there is no possibility of confusion) is a cone in V which can be generated by elements of N, that is, which can be written as the positive hull of a finite set of elements of N. A lattice cone is said to be *strongly convex* if it does not contain lines through the origin (non-zero linear subspaces).

In this thesis we are mainly concerned with "lattice objects", therefore we use the words "lattice cone" (resp. lattice polytope, resp. lattice polygon) and the words "cone" (resp. polytope, resp. polygon) interchangeably, always referring to the lattice object. When we want to state properties holding for general (non-lattice) objects, we always make it clear.

Strong convexity in the previous definition may seem a bit technical, but we will see in Section 1.8 that it has a very intuitive explanation and plays a fundamental role in the theory of toric varieties. For more details we refer to the aforementioned section. We now introduce the notions of face and dual lattice cone and state a useful proposition (1.1.6 below) that can be found in [Ful93], page 14.

Definition 1.1.2. The *dual cone* $\sigma^\vee$ of a lattice cone σ in $N_{\mathbb{R}}$ is the set:

$$\sigma^\vee = \{u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0 \, \forall v \in \sigma\}.$$

Remark 1. Note that, for a lattice N endowing V with an integral structure, the dual lattice M endows the dual $V^\vee$ with an integral structure as well. This means that Definition 1.1.1 and all other definitions involving "lattice objects" make sense also for cones in $V^\vee$.

Example 1.1.3. Let $N = \mathbb{Z}^2$, then $N_{\mathbb{R}} = \mathbb{R}^2$. Let $e_1 = (1, 0)$, $e_2 = (0, 1)$ be the standard basis of $\mathbb{R}^2$. Then the light blue set of Figure 1.1(a) is a lattice cone, generated by e_1 and $e_1 + e_2$.

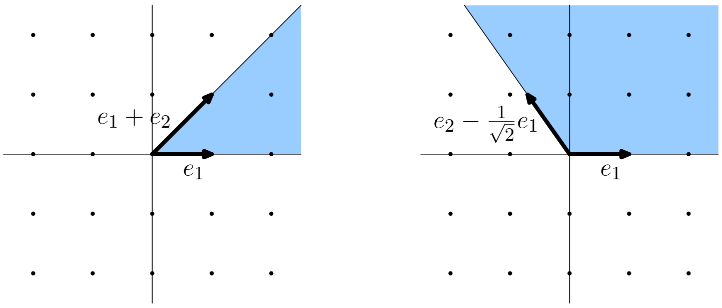
Example 1.1.4. Let $N = \mathbb{Z}^2$, then $N_{\mathbb{R}} = \mathbb{R}^2$. Let $e_1 = (1, 0)$, $e_2 = (0, 1)$ be the standard basis of $\mathbb{R}^2$. Then the light blue set of Figure 1.1(b) is a cone, generated by e_1 and $-\frac{1}{\sqrt{2}}e_1 + e_2$, but not a lattice cone.

Let H_u be the hyperplane determined by an element u of $V^\vee$:

$$H_u = \{v \in V \mid \langle u, v \rangle = 0\},$$

then H_u determines two sets, called its *closed half-spaces*, as follows:

$$H_u^+ = \{v \in V \mid \langle u, v \rangle \geq 0\}, \; H_u^- = \{v \in V \mid \langle u, v \rangle \leq 0\}.$$



(a) A lattice cone in $\mathbb{R}^2$.

(b) A cone in $\mathbb{R}^2$ which is not a lattice cone.

Figure 1.1: Examples 1.1.3 and 1.1.4

Let u be in $V^\vee$ and let $H = H_u$ be the corresponding hyperplane. We say that H is a *supporting hyperplane* of a cone σ if $\sigma \cap H \neq \emptyset$ and σ is contained in at least one of the closed half-spaces H^+ and H^- determined by H.

From now on, we always denote by $u^\perp$ the hyperplane H_u of an element u in $V^\vee$. This notation comes from the general situation of a subset S of $V^\vee$ to which we associate the set

$$S^\perp = \{v \in V \mid \langle u, v\rangle = 0 \,\forall u \in S\}.$$

Definition 1.1.5. A *face* τ of a cone σ is the intersection of σ with any of its supporting hyperplanes. We consider σ as an improper face of itself. The *dimension* of a face τ is the dimension of its linear span (recall that the linear span of a subset A of V is the intersection of all linear subspaces of V containing A). Faces of dimension zero are called *vertices* and faces of dimension one are called *rays*.

If N has rank 1, then we can enumerate explicitly all possible cones. Let e be the generator of N, then the only possible cones (see also Figure 1.2) up to translation and dilation are:

1. The trivial cone $\{0\}$, of dimension 0;
2. The ray $\text{pos}(e)$ spanned by e, of dimension 1;
3. The cone $\text{pos}(e, -e)$ spanned by e and $-e$, of dimension 1.

Note that the last cone is not strongly convex, since it contains (it is actually equal to) $N_\mathbb{R}$.

Remark 2. It is obvious from the definition of a supporting hyperplane that a face τ of a cone σ is of the form

$$\tau = \sigma \cap u^\perp = \{v \in \sigma \mid \langle u, v\rangle = 0\}$$

for some vector u of the dual cone $\sigma^\vee$.

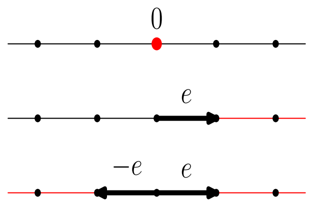


Figure 1.2: The three lattice cones of dimension one in $\mathbb{R}$

Proposition 1.1.6. Let σ be a cone in the vector space $N_\mathbb{R}$, then the following conditions are equivalent:

(1) σ contains no non-zero linear subspaces of $N_\mathbb{R}$;

(2) $\sigma \cap (-\sigma) = \{0\}$;

(3) $\{0\}$ is a face of σ;

(4) $\sigma^\vee$ spans $N_\mathbb{R}^\vee = M_\mathbb{R}$.

Remark 3. Even if a cone σ is strongly convex, its dual $\sigma^\vee$ might not be strongly convex (see Figure 1.3). Nonetheless, requiring in addition σ to be n-dimensional (which amounts to asking that σ spans $N_\mathbb{R}$), Proposition 1.1.6 ensures us that also $\sigma^\vee$ is strongly convex.

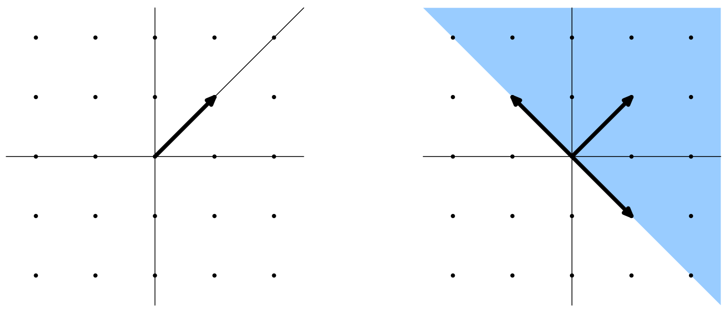


Figure 1.3: A strongly convex cone whose dual is not strongly convex.

Definition 1.1.7. A *fan* Δ in $N_\mathbb{R}$ is a collection of strongly convex lattice cones such that:

1. If τ is a face of a cone σ, then τ is a cone of Δ;

2. If σ_1 and σ_2 are cones of Δ, then $\sigma_1 \cap \sigma_2$ is a face of both.

Definition 1.1.8. The *support* of a fan Δ in $N_{\mathbb{R}}$, denoted by $\operatorname{supp}(\Delta)$, is the union of all its cones, i.e.

$$\operatorname{supp}(\Delta) = \bigcup_{\sigma \in \Delta} \sigma.$$

At this point we introduce the notion of lattice polytope, following the same pattern as in the description of lattice cones. We need the slightly more general notion of a supporting affine hyperplane: an *affine hyperplane* H_u in a finite-dimensional real vector space V is a set of the form

$$H_u = \{v \in V \mid \langle u, v \rangle = a\},$$

for some u in $V^\vee$ and a in $\mathbb{R}$. As in the linear case, an affine hyperplane $H = H_u$ determines two closed half spaces

$$H^+ = \{v \in V \mid \langle u, v \rangle \geq a\},\ H^- = \{v \in V \mid \langle u, v \rangle \leq a\}.$$

We say that H is a *supporting affine hyperplane* of a convex set S if $S \cap H \neq \emptyset$ and S is contained in at least one of the closed half-spaces H^+ and H^- determined by H.

Definition 1.1.9. A *polytope* K in a vector space V is the convex hull of a finite set of vectors of V, that is, a set of the form:

$$K = \left\{ \sum_{i=1}^{m} \lambda_i s_i \;\middle|\; s_i \in M_{\mathbb{R}}, \lambda_i \in \mathbb{R}_{\geq 0}, \sum_{i=1}^{s} \lambda_i = 1 \right\}$$

for $S = \{s_1, \ldots, s_m\} \subset V$. We also use the notation $K = \operatorname{conv}(S)$ or $K = \operatorname{conv}(s_1, \ldots, s_m)$.

Figure 1.4: A lattice triangle and a lattice pentagon in $\mathbb{R}^2$ for $N = \mathbb{Z}^2$.

Definition 1.1.10. A *face* F of a polytope K is the intersection of K with a supporting affine hyperplane. We consider K as an improper face of itself. The *dimension* of a face is the dimension of the affine subspace of K it spans (recall that the affine span of a set A of V is the intersection of all affine subspaces of V containing A). Faces of dimension zero are called *vertices* and faces of dimension one are called *edges*.

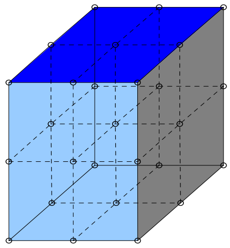


Figure 1.5: A lattice cube in a three-dimensional lattice.

Definition 1.1.11. A *lattice polytope* in $V = N_{\mathbb{R}}$ is a polytope whose vertices lie in N.

1.2 The projective toric variety of a polytope

In this section we describe the first construction of a toric variety. To do so, we have to choose an isomorphism between the ambient lattice and $\mathbb{Z}^n$. In other words, we consider lattices in Euclidean space. In Section 1.10 we show how to define a toric variety intrinsically in terms of an abstract lattice.

Consider a lattice polytope K in $N_{\mathbb{R}}$. Choose an isomorphism $N \cong \mathbb{Z}^n$ such that the basis of N corresponds to the standard one

$$e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1).$$

In other words, we identify the lattice points of $N_{\mathbb{R}}$ with n-tuples of integers. Let $\mathcal{A} = K \cap N$ be the set of lattice points of K. Let k be a field and $\mathbb{P}^m$ projective m-space over k, where $m+1$ is the cardinality of $\mathcal{A}$. Writing

$$\mathcal{A} = \{\alpha_0, \ldots, \alpha_m\} = \{(\alpha_{01}, \ldots, \alpha_{0n}), \ldots, (\alpha_{m1}, \ldots, \alpha_{mn})\},$$

we have a map

$$\varphi_{\mathcal{A}} : (k^\times)^n \to \mathbb{P}^m$$

defined by

$$\varphi_{\mathcal{A}}(t_1, \ldots, t_n) = (t_1^{\alpha_{01}} \cdots t_n^{\alpha_{0n}} : \ldots : t_1^{\alpha_{m1}} \cdots t_n^{\alpha_{mn}}).$$

For simplicity we set $t = (t_1, \ldots, t_n)$ and, for $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in})$, we set $t^{\alpha_i} = t_1^{\alpha_{i1}} t_2^{\alpha_{i2}} \cdots t_n^{\alpha_{in}}$, so we can write $\varphi_{\mathcal{A}}$ as

$$\varphi_{\mathcal{A}}(t) = (t^{\alpha_0} : \ldots : t^{\alpha_n}).$$

The Zariski closure of the image of $\varphi_{\mathcal{A}}$ is called the *projective toric variety* Y_K associated to K:

$$Y_K = \overline{\operatorname{im}(\varphi_{\mathcal{A}})}.$$

One could define in general the projective toric variety $Y_{\mathcal{A}}$ associated to any set $\mathcal{A}$ contained in $\mathbb{Z}^n$ - this is where the notation $\varphi_{\mathcal{A}}$ comes from. In contrast to the case of toric varieties from polytopes (see section 1.7), such varieties need not be normal: an immediate example is the cuspidal cubic curve $x_0x_2^2 - x_1^3 = 0$ in $\mathbb{P}^2$, which is the $Y_{\mathcal{A}}$ for $\mathcal{A} = \{0, 2, 3\} \subset \mathbb{Z}$.

1.3 The monoid algebra

To further proceed, we need to introduce some facts related to monoids and monoid algebras.

A *semigroup* $(S, +)$, or just S, is a set S together with an associative binary operation $+ : S \times S \to S$. A *monoid*[1] S is a semigroup S with an identity element, denoted by 0. In a monoid not necessarily every element (possibly no element at all, except for 0) has an inverse. We could say that "a monoid is almost a group", in the sense that if every element of a monoid S is invertible, then S is a group. A monoid $(S, +)$ is said to be *commutative* if the operation "+" is commutative. If S and T are two monoids, we say that a map $f : S \to T$ is a *monoid homomorphism* if f is compatible with the structure of the monoids, i.e. if $f(a + b) = f(a) + f(b)$ for every a and b in S and $f(0_S) = 0_T$, where 0_S and 0_T are the identity elements of S and T respectively.

A group is in a natural way, by forgetting the extra structure given by the existence of inverses, a monoid. The set I of all invertible elements of a monoid S clearly forms a group under restriction of the operation $+$ to the set $I \times I$, but it is not true that every monoid can be embedded in a group. To be more precise, given a monoid S it is not possible in general to find a group G containing S as a sub-monoid. An example is the free monoid on the set of symbols $\{a, b, c\}$ with relations $\{ab = ac\}$ (here the operation is concatenation of symbols).

Suppose now that a monoid S satisfies the *cancellation property*:

$$c + a = c + b \Rightarrow a = b \text{ and } a + c = b + c \Rightarrow a = b, \ \forall a, b, c \in S,$$

then S is said to be *cancellative*. When S is a commutative cancellative monoid, it is always possible to find an embedding in a group. Commutativity is really necessary here, since there are examples of non-commutative cancellative monoids having the property that $a + b = a$ for some elements a and b even though b is not the zero element. In such cases, could we find a group containing our monoid, it would be possible to add $-a$ to both

[1] It would be more intuitive to associate the letter M to a monoid, but, on one hand, monoids arising in the theory of toric varieties are usually denoted by S; on the other hand, we reserved the letter M for the dual lattice. Moreover, many authors call "semigroup" the object that we are calling monoid, so ours is not such a bad notation after all.

sides and arrive at a contradiction. The free monoid defined in the previous paragraph is an example of this situation.

Suppose now that we have a monoid S together with a field k, then we can form the *monoid algebra* $k[S]$, whose elements are finite formal linear combinations with coefficients in k of symbols χ^u, for u in S. This construction is completely analogous to the case when, starting from a group G and a field k, we form the group algebra $k[G]$. Elements of $k[S]$ are then of the form

$$\sum_{\text{finite}} a_u \chi^u$$

for a_u in k and u in S. Multiplication is defined on the basis $\{\chi^u\}_{u\in S}$ as

$$\chi^u \chi^v = \chi^{u+v}$$

and extended by linearity on the whole $k[S]$. This is a k-algebra with identity χ^0, which we denote by 1.

If we now set $S = \mathbb{Z}^n$, then S is a commutative cancellative monoid. There is a natural isomorphism of k-algebras between $k[\mathbb{Z}^n]$ and the algebra $k[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}]$ of Laurent polynomials in the variables $t_1, \ldots, t_n$. This isomorphism is given on the basis $\{\chi^\alpha\}_{\alpha\in\mathbb{Z}^n}$ of $k[\mathbb{Z}^n]$ by

$$\chi^\alpha \mapsto t_1^{\alpha_1} \cdots t_n^{\alpha_n},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$.

We will sometimes denote $k[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}]$ simply by $k[t, t^{-1}]$ and write t^α instead of $t_1^{\alpha_1} \cdots t_n^{\alpha_n}$, for $\alpha = (\alpha_1, \ldots, \alpha_n)$.

As stated above, a cancellative commutative monoid S can always be embedded in a group G. If such G is a finitely generated free abelian group (this will always be the case for the monoids arising in the theory of toric varieties) of rank n, then G is isomorphic to $\mathbb{Z}^n$ (as a group and thus as a monoid). It follows that the inclusion $S \hookrightarrow G$ gives rise to an injective homomorphism of k-algebras

$$k[S] \hookrightarrow k[\mathbb{Z}^n]$$

and since, by the previous discussion, $k[\mathbb{Z}^n] \cong k[t, t^{-1}]$, we have an injective homomorphism of k-algebras

$$k[S] \hookrightarrow k[t, t^{-1}].$$

We say that a monoid S is *finitely generated* if there exist elements $a_1, \ldots, a_m$ (called *generators*) such that every s in S can be written in the form

$$s = \lambda_1 a_1 + \cdots + \lambda_m a_m,\ \lambda_i \in \mathbb{Z}_{\geq 0}.$$

It is obvious that the corresponding monoid algebra $k[S]$ is then a finitely generated k-algebra.

1.4 The affine toric variety of a cone

The goal of the present section is to show how a lattice cone naturally gives rise to an affine variety. Recall that to each cone σ in the vector space $N_{\mathbb{R}}$ there corresponds a dual cone $\sigma^\vee$ in $M_{\mathbb{R}}$. To proceed further, we need the following two important lemmas.

Lemma 1.4.1 (Farkas' lemma)**.** If σ is a lattice cone in $V = N_{\mathbb{R}}$, then its dual $\sigma^\vee$ is a lattice cone in $V^\vee = M_{\mathbb{R}}$.

Proof. See, for example, [Roc96], §19 and §22. □

By Lemma 1.4.1, it makes sense to intersect (in the sense that the intersection is non-empty) the dual $\sigma^\vee$ of a lattice cone σ with M. We set $S_\sigma = \sigma^\vee \cap M$: this is evidently a monoid if we take the usual sum of vectors in $M_{\mathbb{R}}$ as operation and the zero vector as identity. Furthermore, since S_σ is contained in $M \cong \mathbb{Z}^n$, then it is also commutative and cancellative.

Let now $V = N_{\mathbb{R}}$. The usual inner product on $\mathbb{R}^n$ induces an inner product on V and therefore a norm. Being V finite-dimensional, we know from the theory of topological vector spaces that the metric topology given by this norm is the unique Hausdorff topology on V up to equivalence. When we refer to topological properties of V (e.g. closedness, compactness, etc.) we always assume the above topological structure to be given.

Lemma 1.4.2 (Gordan's lemma)**.** If σ is a lattice cone in $V = N_{\mathbb{R}}$, then S_σ is a finitely generated monoid.

Proof. By Lemma 1.4.1, the cone $\sigma^\vee$ is the positive hull of a finite number of vectors $u_i, \dots, u_m$ in M:

$$\sigma^\vee = \mathrm{pos}(u_1, \dots, u_m).$$

Consider the set

$$K = \left\{ \sum_{i=1}^{m} \lambda_i u_i \,\middle|\, \lambda_i \in \mathbb{R}, \lambda_i \in [0,1] \right\}.$$

It is clear that K is compact in $M_{\mathbb{R}}$. Since M is a discrete subgroup of $M_{\mathbb{R}}$, the intersection $K \cap M$ is a finite set. If u is an element of $S_\sigma = \sigma^\vee \cap M$, then we can express it as a linear combination with non-negative coefficients of the generators u_i of $\sigma^\vee$:

$$u = a_1 u_1 + \cdots + a_m u_m.$$

Write $\lfloor a_i \rfloor$ for the largest integer smaller than or equal to a_i. Then for each of the a_i's we have $a_i = \lfloor a_i \rfloor + b_i$, where $b_i = a_i - \lfloor a_i \rfloor$. Clearly, we have $0 \le b_i \le 1$. It follows that

$$u = a_1 u_1 + \cdots + a_m u_m = \lfloor a_1 \rfloor u_1 + \cdots + \lfloor a_m \rfloor u_m + b_1 u_1 + \cdots + b_m u_m,$$

which becomes, on setting $w = b_1u_1 + \cdots + b_mu_m$,

$$u = \lfloor a_1 \rfloor u_1 + \cdots + \lfloor a_m \rfloor u_m + w.$$

Finally, all the u_i's are in $K \cap M$ and, by construction $w = b_1u_1 + \cdots + b_mu_m$ is in $K \cap M$ as well, so that u is a combination with integer coefficients of elements of $K \cap M$. By the arbitrariness of the choice of u, we conclude that S_σ is generated as a monoid by the elements of the finite set $K \cap M$, in particular S_σ is finitely generated. □

We can now see a connection to algebraic geometry: starting from a cone σ in $N_\mathbb{R}$, we have built a monoid $S_\sigma = \sigma^\vee \cap M$. Given a field k, we can associate to S_σ (as in Section 1.3) the monoid algebra $R_\sigma = k[S_\sigma]$. Since S_σ is finitely generated by Lemma 1.4.2, R_σ will be a finitely generated k-algebra, hence we obtain an affine variety U_σ by taking the (maximal) spectrum of R_σ:

$$U_\sigma = \mathrm{Spec}(R_\sigma).$$

In this thesis we are exclusively concerned with varieties over an algebraically closed field k, so the notion of maximal spectrum is sufficient to study them. Since there is no possibility of confusion, we denote by $\mathrm{Spec}(R)$ the maximal spectrum of a finitely generated k-algebra R. The term *variety* is used in the sense of [Kem93] and [Mil92].

Definition 1.4.3. Let σ be a cone in $N_\mathbb{R}$. The affine variety U_σ defined above is called the *affine toric variety* associated to the cone σ.

Remark 4. By Lemma 1.4.1, the dual $\sigma^\vee$ of σ is generated by a finite number of vectors $u_1, \dots, u_m$ in M. It is straightforward to check that we can express σ as

$$\sigma = \{v \in N_\mathbb{R} \mid \langle u_i, v \rangle \geq 0,\ \forall i = 1, \dots, m\}.$$

This means that σ is a finite intersection of half-spaces, i.e., in the notation of Section 1.1,

$$\sigma = \bigcap_{i=1}^{m} H_{u_i}^+.$$

Remark 5. There is a bijective correspondence between points of an affine toric variety $U_\sigma = \mathrm{Spec}(k[S_\sigma])$ and monoid homomorphisms from S_σ to k, where k is considered as a multiplicative monoid.

Proof. Over an algebraically closed field k, the affine variety $\mathrm{Spec}(R)$ corresponding to a finitely generated k-algebra R can be identified with the set of k-algebra homomorphisms from R to k. To prove the assertion made in the remark, we then have to find a bijective correspondence

$$\mathrm{Hom}_{k\text{-alg.}}(k[S_\sigma], k) \leftrightarrow \mathrm{Hom}_{\text{mon.}}(S_\sigma, k)$$

between the set of k-algebra homomorphism from $k[S_\sigma]$ to k and the set of monoid homomorphisms from S_σ to k. Indeed, to a k-algebra homomorphism $f : k[S_\sigma] \to k$ we associate a map $f^* : S_\sigma \to k$ defined as

$$f^*(u) = f(\chi^u),$$

where χ^u is the basis element of $k[S_\sigma]$ corresponding to u. We have

$$f^*(u_1 + u_2) = f(\chi^{u_1+u_2}) = f(\chi^{u_1}\chi^{u_2}) = f(\chi^{u_1})f(\chi^{u_2}) = f^*(u_1)f^*(u_2)$$

and

$$f^*(0) = f(\chi^0) = f(1) = 1,$$

so f^* really is a monoid homomorphism. On the other hand, to a monoid homomorphism $\phi : S_\sigma \to k$, we associate the map $\tilde{\phi} : k[S_\sigma] \to k$ defined on the basis as

$$\tilde{\phi}(\chi^u) = \phi(u)$$

and extended by linearity. We have

$$\tilde{\phi}(\chi^{u_1}\chi^{u_2}) = \tilde{\phi}(\chi^{u_1+u_2}) = \phi(u_1 + u_2) = \phi(u_1)\phi(u_2) = \tilde{\phi}(\chi^{u_1})\tilde{\phi}(\chi^{u_2}),$$

so $\tilde{\phi}$ is a k-algebra homomorphism. It is clear that these two associations are mutually inverse. □

1.5 Normality of affine toric varieties

We would like to say more about the monoid algebra R_σ and the corresponding affine toric variety $U_\sigma = \mathrm{Spec}(R_\sigma)$. Since S_σ is contained in M, which is a finitely generated free abelian group, we identify R_σ with a subalgebra of $k[t, t^{-1}]$ as in Section 1.3. We denote by $t^u \in k[t, t^{-1}]$ the Laurent monomial corresponding to the basis element χ^u of R_σ.

The above identification allows us to see some properties of the affine toric variety U_σ. Let us be more precise: the algebra $k[t, t^{-1}]$ of Laurent polynomials is the localization of the ring of polynomials $k[t] = k[t_1, \dots, t_n]$ at the element $t_1 t_2 \cdots t_n$. In particular, being the localization of an integral domain, $k[t, t^{-1}]$ is an integral domain itself and therefore so is R_σ. In geometrical terms, this means that U_σ is integral as an affine variety (i.e., reduced and irreducible). Furthermore, since $k[t]$ is a unique factorization domain, so is every localization of it, in particular $k[t, t^{-1}]$. Since every unique factorization domain is integrally closed, we conclude that $k[t, t^{-1}]$ is integrally closed.

An important property of affine toric varieties (which carries over to general toric varieties, as we will see in the next section) is that they are normal. A general affine variety $V = \mathrm{Spec}(R)$ is said to be *normal* if it is irreducible and its local rings $\mathcal{O}_{V,\mathfrak{p}}$ at each $\mathfrak{p}$ of V are integrally closed. It is well known that this last condition is equivalent to the k-algebra R being integrally closed (see, e.g., [Har83]).

In the case of the affine toric variety $U_\sigma = \operatorname{Spec}(R_\sigma)$ of a cone σ, we have shown in the first two paragraphs that U_σ is irreducible. Normality of U_σ then follows from the next proposition.

Proposition 1.5.1. Let σ be a cone in $N_\mathbb{R}$, then its corresponding monoid algebra R_σ is an integrally closed ring.

Proof. By Remark 4 $\sigma^\vee = \bigcap_{i=1}^m H_{v_i}^+$, where $v_i \in (\sigma^\vee)^\vee = \sigma$. If we set $\tau_i = \operatorname{pos}(v_i) = \{\lambda_i v_i : \lambda_i \in \mathbb{R}_{\geq 0}\}$, then

$$S_\sigma = \sigma^\vee \cap M = \left(\bigcap_{i=1}^m H_{v_i}^+\right) \cap M = \bigcap_{i=1}^m \left(H_{v_i}^+ \cap M\right) = \bigcap_{i=1}^m S_{\tau_i}$$

and therefore

$$R_\sigma = k[S_\sigma] = k\left[\bigcap_{i=1}^m S_{\tau_i}\right] = \bigcap_{i=1}^m k[S_{\tau_i}] = \bigcap_{i=1}^m R_{\tau_i}.$$

Now, each of the R_{τ_i} is a a subalgebra of

$$k[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}]$$

of the form

$$k[t_{i_1}, \ldots, t_{i_l}, t_{j_1}^{-1}, \ldots, t_{j_m}^{-1}],$$

where the indices $i_1, \ldots, i_l, \ldots, j_1, \ldots, j_m$ are integers between zero and n. In particular each of them is a unique factorization domain and hence an integrally closed domain. Their intersection R_σ is then integrally closed. □

1.6 The toric variety of a polytope

We have seen in Section 1.2 a construction of the toric variety of a polytope. In the present section we introduce a second construction, which involves a substantially larger amount of work, but highlights the deep connection to convex geometry and combinatorics.

We fix a field k. If we start with a lattice polytope K in $M_\mathbb{R}$, then for each face F there is a lattice cone

$$\sigma_F = \{v \in N_\mathbb{R} \mid \langle u, v\rangle \leq \langle u', v\rangle \, \forall u \in F, \forall u' \in K\}. \tag{1.1}$$

Remark 6. Let K be a lattice polytope in $M_\mathbb{R}$. The set

$$\{\sigma_F \mid F \text{ is a face of } K\}$$

of cones σ_F corresponding to faces of K forms a fan Δ_K, called the *fan of* K.

Lemma 1.6.1. The fan Δ_K of a polytope K covers the whole vector space $N_{\mathbb{R}}$, i.e.

$$\mathrm{supp}(\Delta_K) = N_{\mathbb{R}}.$$

Proof. Let v be a vector in $N_{\mathbb{R}}$ and set $\alpha = \inf\{\langle u, v\rangle | u \in K\}$. Since a convex polytope is the convex hull of its vertices, we can just take the infimum as u varies in the vertices of K, i.e.

$$\alpha = \inf\{\langle u, v\rangle \mid u \in \mathrm{vert}(K)\}.$$

The set $\mathrm{vert}(K)$ is finite, hence α is the minimum of the set and it is attained for some vertex $\tilde{u}$, i.e.

$$\alpha = \langle \tilde{u}, v\rangle.$$

For any other vector u in the polytope we thus have

$$\langle u, v\rangle \geq \alpha = \langle \tilde{u}, v\rangle,$$

which means, according to (1.1), that v is in the cone $\sigma_{\tilde{u}}$ of the vertex $\tilde{u}$ □

For a polytope K in $M_{\mathbb{R}}$ we define the *codimension* of any face F to be

$$\mathrm{codim}(F) = \mathrm{rank}(M) - \dim(F).$$

According to the previous section, to each of the lattice cones σ_F in Δ_K there corresponds the affine toric variety U_{σ_F} over k. The structure of a convex polytope is such that these varieties satisfy the conditions needed to glue them and obtain a new variety: this will be the *toric variety* X_K associated to K.

Remark 7. If we rewrite the cone associated to a face F of a lattice polytope K as

$$\sigma_F = \{v \in N_{\mathbb{R}} \mid \langle u' - u, v\rangle \geq 0\ \forall u \in F, \forall u' \in K\},$$

we immediately see that the previous construction is translation-invariant and dilation-invariant. Specifically, if we consider another lattice polytope K' of the form $K' = u + K$ for $u \in M$, i.e. $K' = \{u + w | w \in K\}$, the fans Δ_K and $\Delta_{K'}$ coincide. Furthermore, if we consider the dilated polytope $mK = \{mu | u \in K\}$, where m is a positive integer, this is still a lattice polytope. The fans Δ_K and Δ_{mK} clearly coincide.

In terms of toric varieties, this implies that all translations and dilations of a polygon K give rise to the same variety.

Lemma 1.6.2. Let K be a polytope in $M_{\mathbb{R}}$. The map

$$F \to \sigma_F$$

sending each face to the cone σ_F of (1.1) is a one-to-one correspondence between faces of K and cones of Δ_K. This correspondence satisfies

$$\dim(\sigma_F) = \mathrm{codim}(F).$$

Proof. According to the previous remark, dilating the polytope K by an integer and translating it by a lattice vector does not change the fan Δ_K. We can therefore assume that K contains the origin in its interior (see Chapter 3 for the definition). This means that Δ_K can be obtained as the face fan of the so-called polar polytope K°. The lemma then follows from the properties of this fan (see, e.g., [Ewa96] or [Zie95]). □

We want to be more precise on how to glue affine toric varieties, but first we need to discuss one fact from convex geometry.

Lemma 1.6.3. Let σ be a lattice cone and $\tau = \sigma \cap u^\perp$ for $u \in \sigma^\vee$ one of its faces, then $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-u)$.

Proof. Let $v \in S_\tau = \tau^\vee \cap M$, we claim the following:

$$\exists p \in \mathbb{R}_{\geq 0} : v + pu \in \sigma^\vee,$$

which can be written as

$$\exists p \in \mathbb{R}_{\geq 0} : \langle v + pu, w\rangle \geq 0 \ \forall w \in \sigma. \tag{1.2}$$

It is enough to check this for the generators $u_1, \ldots, u_s$ of S_σ. Indeed, suppose that for each generator u_i there exists a real number p_i satisfying (1.2). If we set $p = \max_{i=1}^n(p_i)$, then (1.2) holds for every vector v in S_τ by bilinearity of the natural pairing.

Let u_i be any one of the generators. Suppose now that $\langle u, u_i\rangle = 0$, then u_i is already in $\sigma \cap u^\perp = \tau$. Since $v \in \tau^\vee$, it follows that $\langle v, u_i\rangle \geq 0$ and therefore

$$\langle v + pu, u_i\rangle = \langle v, u_i\rangle + p\langle u, u_i\rangle = \langle v, u_i\rangle \geq 0.$$

If instead $\langle u, u_i\rangle > 0$, we can simply choose $p_i = -\frac{\langle v,u_i\rangle}{\langle u,u_i\rangle}$ so that

$$\langle v + p_i u, u_i\rangle = \langle v, u_i\rangle + p_i\langle u, u_i\rangle \geq 0.$$

We have thus shown that a real number p as in (1.2) exists. For such p, set $l = \lceil p \rceil$ (i.e., l is the smallest integer greater than or equal to p), then $v + lu \in \sigma^\vee \cap M = S_\sigma$ and

$$v = (v + lu) + l(-u) \in S_\sigma + \mathbb{Z}_{\geq 0}(-u).$$

Conversely, let $v \in S_\sigma + \mathbb{Z}_{\geq 0}(-u)$, then $v = w + l(-u)$ for some integer $l \geq 0$ and for any $t \in \tau$ we have

$$\langle v, t\rangle = \langle w + l(-u), t\rangle = \langle w, t\rangle - l\langle u, t\rangle.$$

Since t lies in $\tau = \sigma \cap u^\perp$, then $\langle u, t\rangle = 0$ and since $w \in S_\sigma$, then $\langle w, t\rangle \geq 0$, so that $\langle v, t\rangle \geq 0$, that is, $v \in \tau^\vee$. Clearly v also lies in M and thus $v \in S_\tau$. □

If τ is a face of a lattice cone σ, then we obviously have $\sigma^\vee \subset \tau^\vee$, so that $\sigma^\vee \cap M \subset \tau^\vee \cap M$. Since $\sigma^\vee \cap M = S_\sigma$ and $\tau^\vee \cap M = S_\tau$ by definition, we can write $S_\sigma \subset S_\tau$ and consequently $R_\sigma \subset R_\tau$. This inclusion of k-algebras induces a morphism of affine varieties $\iota_{\tau,\sigma} : U_\tau \to U_\sigma$.

Proposition 1.6.4. Let σ be a lattice cone and τ a face of σ, then the induced morphism $\iota_{\tau,\sigma} : U_\tau \to U_\sigma$ embeds U_τ as a principal open subset of U_σ.

Proof. In general, let U be an affine variety $U = \mathrm{Spec}(A)$. For every f in A there is an isomorphism of affine varieties between the principal open subset $D(f)$ of $\mathrm{Spec}(A)$ and $\mathrm{Spec}(A_f)$, where A_f is the localization of A at f (i.e., the localization of A at the multiplicative subset $\{f^n : f \in A, n \geq 0\}$). Thus it is enough to show that R_τ is the localization of R_σ at some element of R_σ.

Let $\tau = \sigma \cap u^\perp$ for some $u \in \sigma^\vee$. From Lemma 1.6.3 we have that

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-u).$$

If $v \in S_\tau$, then we can write it as $v = v' + l(-u)$ for some integer $l \geq 0$ and $v' \in S_\sigma$. In the k-algebra R_σ this means that the basis elements are of the form

$$\chi^v = \chi^{v'+l(-u)} = \frac{\chi^{v'}}{(\chi^u)^l}.$$

This shows that R_τ is the localization of R_σ at χ^u. $\square$

Remark 8. With notation as in the above proof, the principal open subset of U_σ corresponding to U_τ is $D(\chi^u)$, where $\tau = \sigma \cap u^\perp$.

We now go back to the fan $\Delta_K = \Delta$ we associated to a polytope K at the beginning of this section and consider the collection of affine varieties $\mathcal{U} = \{U_\sigma\}_{\sigma\in\Delta}$. Let σ and ρ be two cones of Δ, then, by definition of fan, their intersection $\sigma \cap \rho$ is again a cone of Δ. By Proposition 1.6.4, $\sigma \cap \rho$ is embedded as a principal open subset of U_σ via the map $\iota_{\sigma\cap\rho,\sigma}$ and as a principal open subset of U_ρ via the map $\iota_{\sigma\cap\rho,\rho}$. Setting

$$U_{\sigma,\rho} = \iota_{\sigma\cap\rho,\sigma}(U_{\sigma\cap\rho})$$

and consequently

$$U_{\rho,\sigma} = \iota_{\sigma\cap\rho,\rho}(U_{\sigma\cap\rho}),$$

we have a collection

$$\mathcal{V} = \{U_{\sigma,\rho}\}_{\sigma,\rho\in\Delta}.$$

The affine varieties $U_{\sigma,\rho}$ are Zariski open subsets of U_σ for each cone σ of the fan. Finally, by setting

$$\varphi_{\sigma,\rho} = \iota_{\sigma\cap\rho,\rho} \circ \iota^{-1}_{\sigma\cap\rho,\sigma} : U_{\sigma,\rho} \to U_{\rho,\sigma},$$

we get a collection of (iso)morphisms

$$\mathcal{P} = \{\varphi_{\sigma,\rho}\}_{\sigma,\rho\in\Delta}.$$

It is straightforward to check that the data of $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{P}$ satisfy the conditions needed to glue the affine toric varieties of $\mathcal{U}$ along the subvarieties of $\mathcal{V}$, as stated in [Sha94].

Definition 1.6.5. The object constructed above is called the *toric variety* X_K associated to the polytope K.

Remark 9. All constructions carried out until now remain valid with the prime spectrum instead of the maximal spectrum. What we have shown is that X_K is a scheme of finite type over the field k. Note that X_K is a variety in the sense of Liu ([Liu02], 2.3.47). In this scheme-theoretic setting, the next section would show that X_K is an integral separated scheme of finite type over k, i.e. a variety in the sense of Hartshorne ([Har83], II.4, page 105).

1.7 Properties of toric varieties

In this section we discuss three of the main properties of a toric variety, namely normality, separatedness and completeness.

Let X_K be the toric variety associated to a polytope K in $M_{\mathbb{R}}$, and let $(\mathcal{U}, \mathcal{V}, \mathcal{P})$ be the data from which X_K is obtained. First of all we note that every affine toric variety U_σ in $\mathcal{U}$ is irreducible and hence connected.

Proposition 1.7.1. The toric variety X_K of a polytope K is irreducible.

Proof. For each ordered pair of cones (σ, ρ) in the fan Δ_K, the intersection $\sigma \cap \rho$ is a face of both σ and ρ, hence

$$U_{\sigma,\rho} = \iota_{\sigma\cap\rho,\sigma}(U_{\sigma\cap\rho}) = \iota_{\sigma\cap\rho,\sigma}\big(\mathrm{Spec}(R_{\sigma\cap\rho})\big)$$

is non-empty. Connectedness of each affine piece U_σ then implies that X_K is connected as well. Since each affine piece is irreducible and X_K is connected, it follows that also X_K is irreducible, as desired. □

Proposition 1.7.2. The toric variety X_K of a polytope K is reduced.

Proof. The variety X_K is covered by the affine varieties U_σ, as σ varies in the fan Δ_K of K. Since each U_σ is reduced (see Section 1.5), also X_K is reduced. □

The two previous propositions show that X_K is reduced and irreducible and therefore integral. To prove that X_K is also separated we need one more result from convex geometry.

Lemma 1.7.3. Let σ and ρ be two lattice cones intersecting in a common face τ, then

$$S_\tau = S_\sigma + S_\rho$$

Proof. See, for example, [Ful93], Proposition 3, Section 1.2. $\square$

Proposition 1.7.4. The toric variety X_K of a polytope K is separated.

Proof. Set $X = X_K$. We need to show that the image of the diagonal map $\Delta : X \times X \to X$ is closed, or, equivalently, that $(X \times X) \setminus \Delta(X)$ is open. Since

$$(X \times X) \setminus \Delta(X) = \bigcup_{\sigma,\rho} \left((U_\sigma \times U_\rho) \setminus \Delta(U_{\sigma\cap\rho}) \right),$$

where the union ranges over all cones σ and ρ of the fan of K, it is sufficient to prove the statement in the affine case. We have then to prove that the image of the diagonal map $\delta : U_{\sigma\cap\rho} \to U_\sigma \times U_\rho$ is closed.

Since $U_\sigma = \operatorname{Spec}(R_\sigma)$, $U_\rho = \operatorname{Spec}(R_\rho)$ and $U_{\sigma\cap\rho} = \operatorname{Spec}(R_{\sigma\cap\rho})$ are affine varieties, the diagonal map δ corresponds to a homomorphism of k-algebras

$$\delta^* : R_\sigma \otimes R_\rho \to R_{\sigma\cap\rho},$$

hence it is sufficient to check surjectivity of δ^*.

Explicitly, the homomorphism δ^* sends a basis element $\chi^u \otimes \chi^v$ of $R_\sigma \otimes R_\rho$ to χ^{u+v} in $R_{\sigma\cap\rho}$. Since σ and τ are cones of the fan of K, their intersection is a common face of both, hence by Lemma 1.7.3 the map

$$S_\sigma \oplus S_\rho \to S_{\sigma\cap\rho}$$

is surjective.

Finally, since $R_{\sigma\cap\rho} = k[S_{\sigma\cap\rho}]$ and $R_\sigma \otimes R_\rho = k[S_\sigma] \otimes k[S_\rho] = k[S_\sigma \oplus S_\rho]$, also δ^* is surjective, as we wanted to show. $\square$

A last important property to be discussed in this section is completeness. One general way to construct a toric variety starts by taking any fan Δ in $N_\mathbb{R}$ and building a toric variety by glueing the affine toric varieties U_σ of cones σ in Δ, exactly as we did in Section 1.6, but without the assumption that the fan comes from a polytope. Such varieties need not be complete, but the next proposition, whose proof can be found in [Ful93] or [Oda88], show that toric varieties coming from polytopes are always complete.

Proposition 1.7.5. Let Δ be a fan in $N_\mathbb{R}$. Then the corresponding toric variety is complete if and only if the support of Δ covers all of $N_\mathbb{R}$, i.e. if and only if

$$\operatorname{supp}(\Delta) = \bigcup_{\sigma\in\Delta} \sigma = N_\mathbb{R}$$

Indeed, the fan Δ_K associated to a polytope X_K has such a property by Lemma 1.6.1, so X_K is complete.

1.8 Torus actions and torus orbits

The origin of the name "toric variety" is closely related to the presence of an algebraic action of a torus on every toric variety: in this section we study this action in detail. The main point is that every toric variety contains an algebraic torus as a dense open subset. As a by-product, we obtain that the dimension of a toric variety equals the rank of the ambient lattice.

Let, as usual, N be a lattice and $N_\mathbb{R}$ its associated real vector space. Consider the set $\{0\}$ consisting of the zero vector alone: it is trivially a lattice cone, so we can study the corresponding affine toric variety $U_{\{0\}}$. The dual of $\{0\}$ is

$$\{0\}^\vee = \big\{u \in M_\mathbb{R} : \langle u, v\rangle \geq 0 \ \forall v \in \{0\}\big\} = M_\mathbb{R},$$

so that

$$S_{\{0\}} = \{0\}^\vee \cap M = M_\mathbb{R} \cap M = M$$

and therefore

$$R_{\{0\}} = k[S_{\{0\}}] = k[M].$$

We can endow $U_{\{0\}}$ with a structure of algebraic group. Since it is affine, we can just give the multiplication, inverse and identity maps at the level of k-algebras. More precisely, multiplication corresponds to

$$\begin{array}{rcl} k[M] & \to & k[M] \otimes k[M] \\ \chi^u & \mapsto & \chi^u \otimes \chi^u, \end{array} \tag{1.3}$$

taking inverses corresponds to

$$\begin{array}{rcl} k[M] & \to & k[M] \\ \chi^u & \mapsto & \chi^{-u} \end{array} \tag{1.4}$$

and the identity corresponds to:

$$\begin{array}{rcl} k[M] & \to & k \\ \chi^u & \mapsto & 1. \end{array} \tag{1.5}$$

Choosing an isomorphism $M \cong \mathbb{Z}^n$ we obtain $k[M] \cong k[t, t^{-1}]$ as in the end of Section 1.3. Then

$$U_{\{0\}} \cong \operatorname{Spec}\big(k[t, t^{-1}]\big) = \operatorname{Spec}\big(k[t_1, \dots, t_n]_{t_1 \cdots t_n}\big)$$

which, as an algebraic group, is the product of n copies of the multiplicative algebraic group $\mathbb{G}_m$ (i.e. the affine line $\mathbb{A}^1_k$ minus the origin). As a group, we have that

$$U_{\{0\}} \cong \mathbb{G}_m \times \cdots \times \mathbb{G}_m \cong k^\times \times \cdots \times k^\times = (k^\times)^n$$

where $k^\times$ is the (multiplicative) group of non-zero elements of k. In the terminology of algebraic groups, we say that $U_{\{0\}}$ is an *affine algebraic n-torus* or just an *affine algebraic torus.*

Definition 1.8.1. The affine toric variety $U_{\{0\}}$ is called the *torus* of N and it is denoted by T_N or just T.

It is straightforward to check that the multiplication map on T defines an algebraic action of T on itself. Note that, on identifying points of $U_{\{0\}}$ with monoid homomorphisms (see Remark 5), tha action of T on itself is simply given by multiplication of maps: to a pair

$$(t : M \to k, \phi : S_\sigma \to k)$$

it associates the homomorphism

$$\begin{array}{rcl} S_\sigma & \to & k \\ u & \mapsto & t(u)\phi(u). \end{array}$$

Since

$$\mathrm{Spec}(k[t, t^{-1}]) = \mathrm{Spec}(k[t_1, \ldots, t_n]_{t_1 \cdots t_n}),$$

the torus T, as an algebraic subset of k^n, can be expressed as

$$T = k^n \setminus \mathcal{V}(t_1 \cdots t_m),$$

which is nothing but $(k^\times)^n$. In this setting, the action of T on itself is given simply by componentwise multiplication.

Remark 10. One question that may arise at this point is: can we generalize the above construction to any affine toric variety U_σ? Unfortunately, there is no natural algebraic group structure on U_σ. The "best" we can do is define a multiplication map and identity map as in (1.3) and (1.5) with S_σ in place of M. In this case, however, the inverse map cannot be defined as in (1.4) because S_σ is just a monoid (and not a group).

Let now σ be a strongly convex lattice cone in $N_\mathbb{R}$, then $\{0\}$ is face of σ. We know from Section 1.6 that $U_{\{0\}}$ can be embedded as a principal open subset of U_σ via the map $\iota = \iota_{\{0\},\sigma} : U_{\{0\}} \to U_\sigma$. Under this map, T is isomorphic to $\iota(U_{\{0\}})$ as a subvariety of U_σ. Multiplication $U_\sigma \times U_\sigma \to U_\sigma$ clearly induces, by restriction, an action $t : T \times U_\sigma \to U_\sigma$. Since $k[S_\sigma] \subset k[M]$ for every strongly convex lattice cone σ, looking at the k-algebra homomorphism determined by t we see that the restriction of t to $T \times T$ gives the natural action of T on itself.

Remark 11. The above description characterizes affine toric varieties. Indeed, Oda proves in [Oda78] the following. Let X be a normal affine variety containing an algebraic torus T as a dense open subset. Suppose that, in addition, there exists an action $T \times X \to X$ extending the natural action of T on itself. Then X is of the form $X = U_\sigma$ for a lattice cone σ in some real vector space M_R associated to a lattice M.

Proposition 1.8.2. The toric variety X_K associated to an n-dimensional polytope K is n-dimensional.

Proof. We have seen that the affine toric variety U_σ associated to a cone σ contains the torus T as an open subvariety. Since U_σ is irreducible,

$$\dim(U_\sigma) = \dim(T) = n.$$

Since X_K is covered by affine toric varieties U_σ as σ varies in Δ_K, and since X_K is irreducible, the dimension of X_K is n as well. $\square$

We would like to study in detail the action of the torus on X_K. An important fact is the existence of a correspondence between orbits and faces of the polygon.

For a cone σ, there is a monoid homomorphism x_σ sending each invertible element of S_σ to 1 and every other element to 0:

$$x_\sigma(u) = \begin{cases} 1 & \text{if } -u \in S_\sigma, \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

The point corresponding to this homomorphism is called the *distinguished point* of U_σ.

We denote by O_σ the orbit of x_σ under the action of T:

$$O_\sigma = T \cdot x_\sigma.$$

To describe explicitly O_σ we note first that an element u of S_σ is invertible if and only if u belongs to $\sigma^\perp \cap M$. Indeed, if both u and $-u$ lie in $S_\sigma = \sigma^\vee \cap M$, then for any element v of σ we have $\langle u, v\rangle \geq 0$ and $\langle -u, v\rangle \geq 0$. Since $\langle u, v\rangle + \langle -u, v\rangle = \langle u - u, v\rangle = 0$, it follows that $\langle u, v\rangle = \langle -u, v\rangle = 0$. The converse implication is obvious, so we can rewrite (1.6) as

$$x_\sigma(u) = \begin{cases} 1 & \text{if } u \in \sigma^\perp \cap M, \\ 0 & \text{otherwise.} \end{cases} \tag{1.7}$$

Remark 12. Let σ be a cone in $N_\mathbb{R}$ and x_σ its distinguished point. By definition, the value of x_σ at u in $S_\sigma = \sigma^\vee \cap M$ is 1 if and only if also $-u$ lies in $\sigma^\vee \cap M$. Whenever σ is strongly convex and of dimension n, its dual $\sigma^\vee$ is strongly convex as well (see Remark 3) and thus the only vector u in $\sigma^\vee$ whose inverse $-u$ belongs to $\sigma^\vee$ is 0. In this case we can write x_σ as

$$x_\sigma(u) = \begin{cases} 1 & \text{if } u = 0, \\ 0 & \text{if } u \neq 0. \end{cases} \tag{1.8}$$

Now, for a point of the torus T corresponding to a homomorphism $t : M \to k$, the product with x_σ is given by

$$tx_\sigma(u) = \begin{cases} t(u) & \text{if } u \in \sigma^\perp \cap M, \\ 0 & \text{otherwise} \end{cases}$$

and this evidently means that the orbit O_σ can be identified with the set of monoid homomorphisms from $\sigma^\perp \cap M$ to k. We have therefore shown that

$$O_\sigma = \operatorname{Spec}(k[\sigma^\perp \cap M]).$$

Remark 13. The orbits of distinguished points actually are all the orbits of the action of T. Indeed, let σ be a cone and U_σ the corresponding affine toric variety. Point (a) of the proposition on page 54 of [Ful93] says that

$$U_\sigma = \bigcup_\tau O_\tau,$$

where τ ranges over all the faces of σ.

We have described the orbit of the distinguished point of a cone under the action of the torus. It is convenient to give an abstract description of that orbit as the torus of some toric variety. Such a description is the aim of the remaining part of this section.

Set $M(\sigma) = \sigma^\perp \cap M$. If σ has dimension l, then $M(\sigma)$ is a sublattice of M of rank $n - l$. On the other hand, the intersection of N with σ, is not in general a sublattice of N (see figure 1.6), but we can still consider the sublattice it generates (i.e. the subgroup of N it generates), given by

$$N \cap \sigma + (-N \cap \sigma) = \{u_1 + u_2 \mid u_1, -u_2 \in N \cap \sigma\}$$

and denoted by N_σ. Being it a normal subgroup of N, we can form the quotient lattice N/N_σ, denoted by $N(\sigma)$: it is straightforward to check that $N(\sigma)$ and $M(\sigma)$ are dual to each other.

Definition 1.8.3. Let Δ be a fan and let τ be a cone of Δ. The *star* of τ, denoted by $\operatorname{Star}(\tau)$ is the set of cones in Δ containing τ.

If σ is a cone in $\operatorname{Star}(\tau)$, we set

$$\bar{\sigma} = (\sigma + (N_\tau)_\mathbb{R}) / (N_\tau)_\mathbb{R},$$

which is contained in $N(\tau)_\mathbb{R}$. Note that $\bar{\sigma}$ is a lattice cone because σ is. We define the set $\Delta(\tau)$ as

$$\Delta(\tau) = \{\bar{\sigma} \mid \sigma \in \operatorname{Star}(\tau)\}.$$

In the proof of Lemma 1.8.5 we make use of the following result.

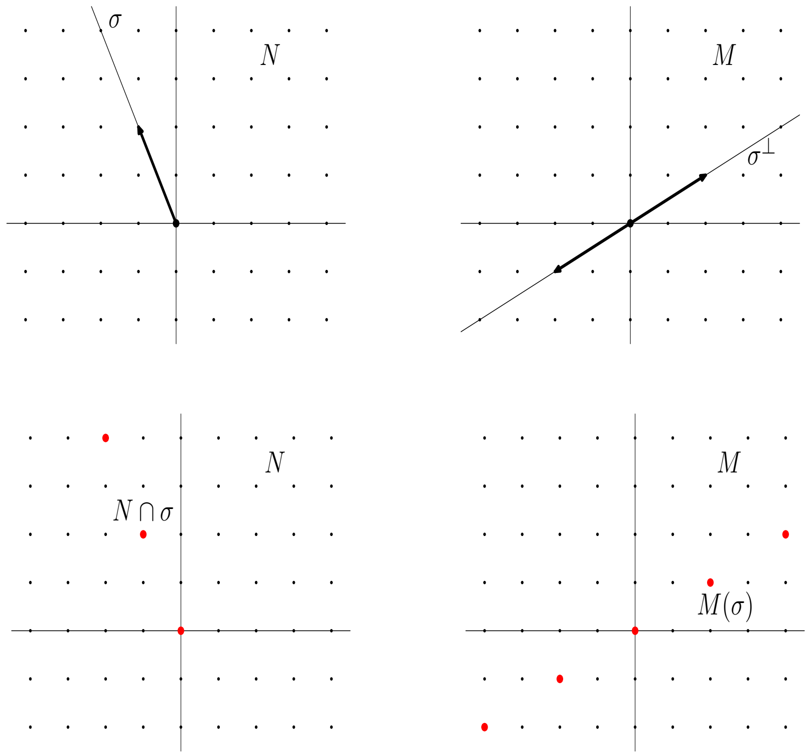


Figure 1.6: $N \cap \sigma$ and $M(\sigma)$ for a one-dimensional strongly convex cone σ in $\mathbb{R}^2$.

Lemma 1.8.4. Let τ be a face of a cone σ and v_1, v_2 two vectors in σ. Then $v_1 + v_2$ belongs to τ if and only if both v_1 and v_2 belong to τ.

Proof. One implication is obvious. For the other, assume that $v_1 + v_2$ is in τ. By Remark 2 we can write $\tau = u^\perp \cap \sigma$ for some vector u in $\sigma^\vee$. We then have

$$\langle u, v_1 + v_2 \rangle = 0,$$

so that

$$\langle u, v_1 \rangle = -\langle u, v_2 \rangle.$$

Moreover, since v_1 and v_2 are in σ, then

$$\langle u, v_1 \rangle \geq 0, \ \langle u, v_2 \rangle \geq 0$$

and therefore we must have

$$\langle u, v_1 \rangle = \langle u, v_2 \rangle = 0.$$

This shows that both v_1 and v_2 belong to $u^\perp \cap \sigma$, which is nothing but τ. $\square$

Lemma 1.8.5. The set $\Delta(\tau)$ is a fan in $N(\tau)_{\mathbb{R}}$.

Proof. We need to check that each cone in $\Delta(\tau)$ is strongly convex and that $\Delta(\tau)$ satisfies the two conditions of Definition 1.1.7.

Let $\bar{\sigma}$ be a cone of $\Delta(\tau)$. By Proposition 1.1.6, strong convexity is equivalent to $\bar{\sigma} \cap (-\bar{\sigma})$ consisting of the zero vector alone. Assume that $\bar{v}$ is in $\bar{\sigma} \cap (-\bar{\sigma})$, then v satisfies

$$v = v_1 + w_1 = -v_2 + w_2, \tag{1.9}$$

for some vectors v_1, v_2 in σ and w_1, w_2 in $(N_\tau)_{\mathbb{R}}$. If $z_1 \ldots, z_s$ are the generators of $N_\tau = \tau \cap N + (-\tau \cap N)$, then every element z of $(N_\tau)_{\mathbb{R}}$ can be written as

$$\sum_{i=1}^{s} a_i z_i - \sum_{i=1}^{s} b_i z_i,$$

where the a_i's and b_i's are non-negative real numbers. This means that every element of $(N_\tau)_{\mathbb{R}}$ can be written as a difference of elements of τ (but not of $\tau \cap N$). In particular, we have that

$$w_1 = \alpha_1 - \beta_1, \; w_2 = \alpha_2 - \beta_2,$$

for some vectors $\alpha_1, \alpha_2, \beta_1, \beta_2$ in τ. From (1.9) we obtain

$$v_1 + v_2 = w_2 - w_1 = (\alpha_2 + \beta_1) - (\alpha_1 + \beta_2)$$

and therefore that

$$v_1 + v_2 + (\alpha_1 + \beta_2) = (\alpha_2 + \beta_1).$$

Since the sum of any of two vectors of a cone lies in the cone itself, on one hand we get that $v_1 + v_2$ is in σ, while on the other hand we get that $v_1 + v_2 + (\alpha_1 + \beta_2)$ lies in τ. Since σ is a cone of $\mathrm{Star}(\tau)$, it contains τ and, since both τ and σ are cones of Δ, τ must be a face of σ. We can thus repeatedly apply Lemma 1.8.4 to obtain that v_1 and v_2 lie in τ. From this it follows that v is contained in N_τ and therefore its class $\bar{v}$ is the zero one. This proves that $\bar{\sigma}$ is strictly convex.

Assume now that $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are two cones in $\Delta(\tau)$. Let $\bar{v}$ be a vector in their intersection $\bar{\sigma}_1 \cap \bar{\sigma}_2$, then v is of the form

$$v = v_1 + w_1 = v_2 + w_2,$$

for some vectors v_1 in σ_1, v_2 in σ_2 and w_1, w_2 in $(N_\tau)_{\mathbb{R}}$. Since σ_1 and σ_2 are cones of Δ, their intersection is a common face of both. Since both contain τ, their intersection will contain τ as a face. Similarly to the previous paragraph, the fact that

$$v_1 = v_2 + w_2 - w_1, \; v_2 = v_1 + w_1 - w_2$$

gives that v_1 is in σ_2 and v_2 is in σ_1. From this it follows that

$$\bar{\sigma}_1 \cap \bar{\sigma}_2 = \overline{\sigma_1 \cap \sigma_2},$$

in particular $\bar{\sigma}_1 \cap \bar{\sigma}_2$ is in $\Delta(\tau)$.

For the last part of the proof, let $\bar{\sigma}$ be a cone of $\Delta(\tau)$. Let $u^{\perp} \cap \bar{\sigma}$ be a face of $\bar{\sigma}$ for some u in $M(\tau)$. Since $M(\tau) = \tau^{\perp} \cap M$, we see that the face $u^{\perp} \cap \bar{\sigma}$ is of the form $\bar{\rho}$ for $\rho = u^{\perp} \cap \sigma$. Since ρ is a face of σ which contains τ by construction, this means that $\bar{\rho}$ is in $\Delta(\tau)$. □

By the above lemma, $\Delta(\tau)$ is a fan in $N(\tau)_{\mathbb{R}}$ and therefore we can associate to it a variety, exactly as we did in Section 1.6. The reasons behind the following definition will be evident at the end of this section.

Definition 1.8.6. The toric variety $V(\tau)$ obtained from the fan $\Delta(\tau)$ is called the *abstract orbit closure* of τ.

Remark 14. Since $\bar{\tau}$ is trivial in $N(\tau)_{\mathbb{R}}$, its affine toric variety $U_{\bar{\tau}}$ is the torus of $V(\tau)$.

Let $\bar{\sigma}$ be a cone of $\Delta(\tau)$. Since $M(\tau) = \tau^{\perp} \cap M$, the monoid $\bar{\sigma}^{\vee} \cap M(\tau)$ can be identified with $\sigma^{\vee} \cap \tau^{\perp} \cap M$. The corresponding affine toric variety is then

$$U_{\bar{\sigma}} = \mathrm{Spec}(k[\sigma^{\vee} \cap \tau^{\perp} \cap M]).$$

Note that we have a surjective homomorphism (cfr. [Dan78], 2.5) of k-algebras

$$k[\sigma^{\vee} \cap M] \to k[\sigma^{\vee} \cap \tau^{\perp} \cap M] \tag{1.10}$$

given on the basis as

$$\chi^u \mapsto \begin{cases} \chi^u & \text{if } u \in \sigma^{\vee} \cap \tau^{\perp} \cap M, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, since $\sigma^{\vee} \cap \tau^{\perp}$ is a face of $\sigma^{\vee}$ (it is the dual face of τ), the above map is a k-algebra homomorphism. Surjectivity is obvious. We thus have a map of affine varieties

$$i_{\sigma} : U_{\bar{\sigma}} \to U_{\sigma}$$

which is a closed embedding by surjectivity of the homomorphism (1.10).

Remark 15. The cone τ is obviously in $\mathrm{Star}(\tau)$. In this case, we have the equality $k[\tau^{\vee} \cap \tau^{\perp} \cap M] = k[\tau^{\perp} \cap M]$, so it is obvious that the image of $U_{\bar{\tau}}$ under the closed embedding i is exactly the orbit O_{τ}.

Whenever $\bar{\gamma}$ is a face of $\bar{\sigma}$, then γ is a face of σ (see the proof of Lemma 1.8.5). This means that, in addition to the closed embeddings i_{σ} and i_{γ}, we also have the open embeddings $\iota_{\gamma,\sigma}$ and $\iota_{\bar{\gamma},\bar{\sigma}}$ of Proposition 1.6.4. Their compatibility is clear (we can just look at the corresponding homomorphism of coordinate rings), so we can glue the i_σ's as σ varies in $\mathrm{Star}(\tau)$. The result is a closed embedding

$$i : V(\tau) \to \bigcup_{\sigma \in \mathrm{Star}(\tau)} U_\sigma.$$

If we set

$$U = \bigcup_{\sigma \in \mathrm{Star}(\tau)} U_\sigma,$$

then we have shown that $V(\tau)$ can be regarded as a closed subset of U. Furthermore, by Remark 14 and 15, the torus of $V(\tau)$ is the orbit O_τ of the distinguished point x_τ. Since the torus is dense in $V(\tau)$ and $V(\tau)$ is closed in U, we see that $V(\tau)$ is the closure of O_τ in U. This justifies the name abstract orbit closure.

1.9 Characters and one-parameter subgroups

The main reference for this section is [Bor91], III.8. Let N be a lattice of rank n and M its dual lattice, $M = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. Let $T = T_N = \mathrm{Spec}(k[M])$ be the torus of N and $\mathbb{G}_m = \mathrm{Spec}(k[t, t^{-1}]) \cong \mathrm{Spec}(k[\mathbb{Z}]) \cong k^\times$ the multiplicative algebraic group.

Definition 1.9.1. The *group of characters* (or character group) of T is the set of morphisms of algebraic groups from T to $\mathbb{G}_m$, i.e. the set

$$\mathrm{Hom}_{\text{alg.gr.}}(T, \mathbb{G}_m).$$

The *group of one-parameter subgroups* of T is the set of morphisms of algebraic groups from $\mathbb{G}_m$ to T, i.e. the set

$$\mathrm{Hom}_{\text{alg.gr.}}(\mathbb{G}_m, T).$$

Remark 16. The group of one-parameter subgroups is canonically identified with N, while the character group of T is canonically identified with M. We do not give a complete proof here, but try to motivate the reason of this identifications.

Since T and $\mathbb{G}_m$ are algebraic groups, their coordinate rings have a natural structure of Hopf algebras. By [Bor91], 8.3, we have a canonical bijection

$$\mathrm{Hom}_{\text{alg.gr.}}(\mathbb{G}_m, T) \leftrightarrow \mathrm{Hom}_{\text{Hopf-alg.}}(k[M], k[\mathbb{Z}]),$$

where the right hand side is the set of Hopf algebra homomorphisms from $k[M]$ to $k[\mathbb{Z}]$. This latter set is in bijection with N. It is easier to see why if we consider the coordinate ring of the torus as the polynomial ring

$$k[M] \cong k[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}]$$

and the coordinate ring of the multiplicative algebraic group as

$$k[\mathbb{Z}] \cong k[u_1, u_1^{-1}].$$

The possible homomorphism from $k[t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}]$ to $k[u_1, u_1^{-1}]$ are limited by the structure of Hopf algebra. Specifically, the only homomorphisms are given on $t_1, \dots, t_n$ as

$$t_1 \mapsto u_1^{k_1}, \dots, t_n \mapsto u_1^{k_n},$$

where $k_1, \dots, k_n$ are arbitrary integers. Then $\mathrm{Hom}_{\text{Hopf-alg.}}(k[M], k[Z])$ is in bijection with $\mathbb{Z}^n$, which is isomorphic to N.

The second assertion follows immediately by [Bor91], page 115, which states that the character group and the group of 1-parameter subgroups are dual to each other.

Remark 16 shows in particular that the character group and the group of one-parameter subgroups are lattices of rank n.

If v is an element of N, then we denote by λ_v the corresponding (via the identification of the above proposition) one-parameter subgroup $\lambda_v : k^\times \to T$. If u is an element of M, we denote by χ^u the corresponding character $\chi^u : T \to k^\times$.

1.10 The projective toric variety of a polytope, revisited

In this section we generalize the construction of Section 1.2. This is a first step towards showing the equivalence between projective toric varieties and the toric varieties defined in Section 1.6.

Consider a lattice polytope K of dimension n in $M_\mathbb{R}$. Let

$$\mathcal{A} = \{u_0, \dots, u_m\}$$

be the set of lattice points of K, that is, $\mathcal{A} = K \cap M$. Let k be a field and $\mathbb{P}^m$ projective m-space over k. Each lattice point of K defines a character of the torus T of N (see Section 1.9), so we can define a map

$$\varphi_\mathcal{A} : T \to \mathbb{P}^m$$

as

$$\varphi_\mathcal{A}(x) = (\chi^{u_0}(x) : \dots : \chi^{u_m}(x)).$$

The Zariski closure of the image of T under the map $\varphi_\mathcal{A}$ is called the *projective toric variety* of K.

The choice of an isomorphism of M with $\mathbb{Z}^n$ reflects in isomorphisms $M_\mathbb{R} \cong \mathbb{R}^n$ and $T \cong (k^\times)^n$. Points u of M correspond then to elements

$\alpha = (\alpha_1, \ldots, \alpha_n)$ of $\mathbb{Z}^n$, points x of the torus T correspond to n-tuples $(t_1, \ldots, t_n)$ in $(k^\times)^n$ and characters χ^u correspond to monomials $t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ in the Laurent algebra $k[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}]$. It is now evident that the maps $\varphi_{\mathcal{A}}$ defined in this section and in Section 1.2 are exactly the same.

Chapter 2

Divisors and support functions

2.1 Weil divisors and Cartier Divisors

In this section we recall some basic facts, without proofs, about divisors on normal varieties. We follow mainly [Har83] and [Mil92]: the reader can consult these books for more details. In this section, X is a normal irreducible variety over a field k.

Recall that for an irreducible variety X, the codimension in X of a subvariety V is simply the dimension of X minus the dimension of V:

$$\mathrm{codim}_X(V) = \dim(X) - \dim(V).$$

A *prime divisor* V on X is an irreducible subvariety of X of codimension one. The *group of Weil divisors* of X, denoted by $\mathrm{Div}(X)$, is the free abelian group generated by its prime divisors. An element D of $\mathrm{Div}(X)$, called a *Weil divisor* on X (or just a divisor, if there is no possibility of confusion), is thus a finite formal sum

$$D = \sum_{\text{finite}} n_i D_i \tag{2.1}$$

of prime divisors D_i with integer coefficients. A divisor D as in (2.1) is said to be *effective* (written as $D \geq 0$) if all the coefficients n_i are non-negative.

We denote by $k(X)$ the function field of X, and by $k(X)^\times$ its multiplicative group of units. If V is a prime divisor on X, then there exists a ring

$$\mathcal{O}_V = \{f \in k(X) \,|\, f \text{ is defined on an open subset } U \text{ of } X \text{ with } U \cap V \neq \emptyset\}.$$

The ring $\mathcal{O}_V$ is a discrete valuation ring: we denote its valuation from $k(X)^\times$ to $\mathbb{Z}$ by ord_V. For a non-zero rational function f on X (i.e. $f \in k(X)^\times$), the value of $\mathrm{ord}_V(f)$ is zero for all but finitely many prime divisors V, so the sum

$$\sum \mathrm{ord}_V(f)V$$

ranging over all prime divisors on X is finite and thus defines a divisor on X. We set

$$\mathrm{div}(f) = \sum \mathrm{ord}_V(f)V$$

and call it the *principal divisor* associated to f. Any other divisor Z of the form $Z = \mathrm{div}(f)$, for some non-zero rational function f, is said to be *principal*. Two divisors D and D' are said to be *linearly equivalent*, written as $D \sim D'$, if their difference $D - D'$ is principal. The group of divisors modulo principal divisors is denoted by $\mathrm{Cl}(X)$ and called the *divisor class group* of X.

To each divisor D on X, we can associate a sheaf $\mathcal{O}_X(D)$, defined to be the sheaf such that, for every open subset U of X,

$$\Gamma(U, \mathcal{O}_X(D)) = \{f \in k(X)^\times \mid \mathrm{div}(f) + D \geq 0 \text{ on } U\} \cup \{0\}.$$

In detail, the condition $\mathrm{div}(f) + D \geq 0$ above means that, if $D = \sum n_i D_i$, then $\mathrm{ord}_{D_i}(f) + n_i \geq 0$ for every D_i whose intersection $D_i \cap U$ with U is non-empty.

Remark 17. Let $\mathcal{O}_X$ be the structure sheaf of the variety X, then:

(1) $\mathcal{O}_X(D)$ is a coherent sheaf of $\mathcal{O}_X$-modules on X,

(2) If $D = 0$, i.e. D is the trivial divisor on X, then $\mathcal{O}_X(D) = \mathcal{O}_X$,

(3) If D and D' are linearly equivalent, then $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$ as $\mathcal{O}_X$-modules.

A *Cartier divisor* on X is the data of:

1. An affine open cover $\mathcal{U} = \{U_i\}_{i \in I}$;

2. Non-zero rational function f_i on U_i, one for each i in I, such that $\frac{f_i}{f_j}$ is a non-zero regular function on the intersections $U_i \cap U_j$, for every i, j in I.

Denoting by $\mathcal{O}$ the structure sheaf of X, in sheaf-theoretic terms (cfr. [Har83], II.6) a Cartier divisor is a global section of the quotient $\mathcal{K}^\times/\mathcal{O}^\times$, where $\mathcal{K}$ is the constant sheaf with stalk $k(X)$.

Two Cartier divisors $D = \{(U_i, f_i)\}_{i \in I}$ and $D' = \{(V_j, g_j)\}_{j \in J}$ are identical when $\frac{f_i}{g_j}$ is a non-zero regular function on $U_i \cap V_j$ for every i in I and j in J. The group of all Cartier divisors forms an abelian (multiplicative) group (even though we use additive notation in analogy with the language of Weil divisors) with identity element $\{(X, 1)\}$. A Cartier divisor D is said to be *principal* if it is of the form $D = \{(X, f)\}$ for some rational function f on X. Two Cartier divisors D and D' are said to be *linearly equivalent*, written $D \sim D'$, if their difference $D - D'$ is principal.

Each Cartier divisor $D = \{(U_i, f_i)\}_{i \in I}$ determines a Weil divisor on X as follows: for each prime divisor V on X choose an index i such that the intersection $V \cap U_i$ is non-empty. The Weil divisor $[D]$ associated to D is defined to be

$$[D] = \sum \mathrm{ord}_V(f_i)V,$$

where the sum ranges over all prime divisors V on X. This definition is independent of the choice of the index i, indeed if i and j are such that $V \cap U_i \neq \emptyset$ and $V \cap U_j \neq \emptyset$, then $\frac{f_i}{f_j}$ is invertible in $U_i \cap U_j$ and therefore $\mathrm{ord}_V(\frac{f_i}{f_j}) = 0$, which immediately implies $\mathrm{ord}_V(f_i) = \mathrm{ord}_V(f_j)$.

Since X is normal, the map $D \mapsto [D]$ sending a Cartier divisor to the associated Weil divisor is injective. Under this map, principal Cartier divisors correspond to principal Weil divisors, so the group of Cartier divisors can be embedded as a subgroup of the divisor class group (see [Sha94] or [Har83]).

2.2 Divisors on toric varieties

A divisor on a toric variety which is invariant under the action of the torus admits an explicit characterization in terms of lattice objects. The aim of the present section is to develop and study such powerful description.

Let Δ be a fan in N_R and X the corresponding toric variety. Let moreover T be the torus of N. If we denote by $\Delta(1)$ the set of rays of the fan (see Definition 1.1.5), then each orbit O_ρ corresponding to a ray ρ in $\Delta(1)$ is a torus of dimension $n-1$. The orbit closure (see Definition 1.8.6) $V(\rho) = \overline{O_\rho}$ has then the same dimension $n-1$. It follows that to each ray ρ corresponds an irreducible subvariety of X of codimension 1, i.e. a prime divisor on X. We set $D_\rho = V(\rho)$.

Definition 2.2.1. A Weil divisor $D = \sum n_i D_i$ on the toric variety X_K is said to be T-*invariant* if every prime divisor D_i is invariant under the action of the torus T on X_K.

Proposition 2.2.2. The T-invariant Weil divisors are exactly the divisors of the form $\sum_{\rho \in \Delta(1)} n_\rho D_\rho$.

Proof. Recall that, by definition, D_ρ is the orbit closure $V(\rho)$ of the cone ρ. By the proposition in [Ful93], page 54, we have

$$D_\rho = \bigcup_{\sigma \in \mathrm{Star}(\rho)} O_\sigma.$$

Therefore D_ρ is a union of orbits and hence T-invariant, so any sum of the form $\sum_{\rho\in\Delta(1)} n_\rho D_\rho$ is T-invariant.

On the other hand, if $D = \sum n_i D_i$ is a T-invariant Weil divisor, each D_i is invariant and thus a union of orbits. The torus T is also an orbit and, since T is dense in X_K, the intersections $T \cap D_i$ with each D_i are empty (because prime divisors have codimension 1 in X). Since $X_K \setminus T$ is a union of orbit closures corresponding to rays, i.e.

$$X_K \setminus T = \bigcup_{\rho\in\Delta(1)} D_\rho$$

then every D_i must be one of the D_ρ's. □

We turn now our attention to T-invariant Cartier divisors.

Definition 2.2.3. A Cartier divisor D on a toric variety X is said to be T-*invariant* if it corresponds to a T-invariant Weil divisor.

We start by giving a description of the Cartier divisor corresponding to a character of the torus T. Since a character χ^u is a non-zero rational function on the toric variety X_K, then $\{(X_K, \chi^u)\}$ is a Cartier divisor which we denote by $\mathrm{div}(\chi^u)$. For each ray ρ_i in $\Delta(1)$, denote by v_i the corresponding *minimal generator* (i.e. the first lattice point along the ray, starting from the vertex). The proof of the next three statements can be found in [Ful93], page 61.

Lemma 2.2.4. Let X_K be the toric variety associated to a polytope K in $M_{\mathbb{R}}$. Let u be an element of M and χ^u its corresponding character, then

$$\mathrm{ord}_{D_{\rho_i}} = \langle u, v_i\rangle$$

for every $\rho_i \in \Delta(1)$.

Corollary 2.2.5. The Weil divisor associated to the principal Cartier divisor $\{(X_K, \chi^u)\}$ is

$$\sum_{\rho_i\in\Delta(1)} \langle u, v_i\rangle D_{\rho_i}.$$

For affine toric varieties, a very strong result holds.

Theorem 2.2.6. Let U_σ be the affine toric variety of a cone σ in $N_{\mathbb{R}}$, then every T-invariant Cartier divisor on U_σ is of the form $\{(U_\sigma, \chi^u)\}$ for some character χ^u of the torus T. In particular, every T-invariant Cartier divisor on U_σ is principal.

Remark 18. Theorem 2.2.6 permits us to describe a T-invariant Cartier divisor D on a general toric variety X_K. Indeed, consider the open cover of X_K given by the affine toric varieties U_σ, as σ varies in $\Delta = \Delta_K$. By the above theorem, for each σ we can find an element $u(\sigma)$ of M such that the local equation of D on U_σ is $\chi^{-u(\sigma)}$ (the notation with the minus is used to conform to the literature), so that

$$D = \{(U_\sigma, \chi^{-u(\sigma)})\}_{\sigma \in \Delta}$$

is the description of the T-invariant Cartier divisor D.

We can make use of Theorem 2.2.6 and Corollary 2.2.5 to determine when two T-invariant Cartier divisors are the same. Since the group of Cartier divisors is embedded in the group of Weil divisors, two Cartier divisors are identical if and only if their associated Weil divisors are so. In particular, two T-invariant Cartier divisors $D = \{(U_\sigma, \chi^u)\}$ and $D' = \{(U_\sigma, \chi^{u'})\}$ (for u and u' in M) on an affine toric variety U_σ are identical if and only if $[D] = \sum_{\rho_i \in \Delta(1)} \langle u, v_i \rangle D_{\rho_i}$ and $[D'] = \sum_{\rho_i \in \Delta(1)} \langle u', v_i \rangle D_{\rho_i}$ are identical. This happens if and only if

$$\sum_{\rho_i \in \Delta(1)} \langle u, v_i \rangle D_{\rho_i} = \sum_{\rho_i \in \Delta(1)} \langle u', v_i \rangle D_{\rho_i}$$

and therefore if and only if

$$\sum_{\rho_i \in \Delta(1)} \langle u, v_i \rangle - \langle u', v_i \rangle = \sum_{\rho_i \in \Delta(1)} \langle u - u', v_i \rangle = 0$$

This last statement is equivalent to saying that $u - u'$ lies in $\sigma^\perp \cap M$, which is the sublattice of M we denoted by $M(\sigma)$ (on page 29). Therefore we have proven the following:

Theorem 2.2.7. There is a bijection between the set of T-invariant Cartier divisors on an affine toric variety U_σ and the quotient lattice $M/M(\sigma)$.

2.3 Ample sheaves and support functions

In this section we give a characterization of the sheaf associated to a T-invariant divisor. This allows us to state two criteria for such a sheaf to be ample or very ample.

Recall that (see Definition 1.1.8) the support $\mathrm{supp}(\Delta)$ of a fan Δ is defined to be the union of all its cones.

Definition 2.3.1. A function $\psi : \mathrm{supp}(\Delta) \to \mathbb{R}$ is said to be a Δ-*linear support function* if it is linear on each cone σ of Δ, i.e. on each cone it is determined by a linear function, and assumes integer values at lattice vectors, i.e.

$$\psi(\mathrm{supp}(\Delta) \cap N) \subset \mathbb{Z}.$$

If there is no possibility of confusion, we call ψ just a *support function*.

We say that a general function φ from a real vector space V to $\mathbb{R}$ is *convex* if

$$\varphi(tv_1 + (1-t)v_2) \geq t\varphi(v_1) + (1-t)\varphi(v_2) \tag{2.2}$$

for all vectors v_1, v_2 in V and all $t \in [0,1] \subset \mathbb{R}$. In convex geometry, one calls such function "concave" instead of "convex", but in the theory of toric variety is customary to use our denomination.

A function φ from a real vector space V to $\mathbb{R}$ is called *positive homogeneous* if $\varphi(\lambda v) = \lambda\varphi(v)$ for any v in V and non-negative real λ. Obviously, any linear function is positive homogeneous.

Lemma 2.3.2. A positive homogeneous function $\varphi : V \to \mathbb{R}$ is convex if and only if

$$\varphi(v_1 + v_2) \geq \varphi(v_1) + \varphi(v_2) \tag{2.3}$$

for all vectors v_1 and v_2 in V.

Proof. Suppose φ is convex. If v_1 and v_2 are vectors of V, then

$$\frac{1}{2}\varphi(v_1+v_2) = \varphi(\frac{1}{2}v_1 + \frac{1}{2}v_2) \geq \varphi(\frac{1}{2}v_1) + \varphi(\frac{1}{2}v_2) = \frac{1}{2}\varphi(v_1) + \frac{1}{2}\varphi(v_2),$$

hence $\varphi(v_1+v_2) \geq \varphi(v_1) + \varphi(v_2)$.

Conversely, suppose that φ satisfies (2.3). If v_1 and v_2 are vectors of V and t is a real number contained in $[0,1]$, then

$$\varphi(tv_1 + (1-t)v_2) \geq \varphi(tv_1) + \varphi((1-t)v_2) = t\varphi(v_1) + (1-t)\varphi(v_2),$$

proving that φ is indeed convex. □

Definition 2.3.3. A Δ-linear support function ψ is said to be *strictly convex* if it is convex and the linear functions determined by different cones are different.

Let now Δ be a fan and X the associated toric variety. Combining Remark 18 and Theorem 2.2.7, we see that a Cartier divisor is specified by the data $\{u(\sigma) \in M/M(\sigma)\}_{\sigma\in\Delta}$.

Proposition 2.3.4. There is a bijective correspondence between T-invariant Cartier divisors on a toric variety X and Δ-linear support functions.

Proof. Suppose D is the divisor determined by $\{u(\sigma)\}_{\sigma\in\Delta}$. Identifying M with the set of $\mathbb{Z}$-linear group homomorphisms $\mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$, each vector $u(\sigma)$ defines an $\mathbb{R}$-linear function from $N_{\mathbb{R}}$ to $\mathbb{R}$ (by extension of scalars) whose restriction to the cone σ depends only on the residue class modulo $M(\sigma)$. We denote this function again by $u(\sigma)$. For any two cones σ and γ in Δ, their intersection is a common face of both, so the divisors corresponding to $u(\sigma)$ and $u(\gamma)$ agree on $U_{\sigma\cap\gamma}$ and thus the corresponding linear functions agree on $\sigma \cap \gamma$. Therefore the function $\psi_D : \mathrm{supp}(\Delta) \to \mathbb{R}$ defined as

$$\psi_D(v) = \langle u(\sigma), v\rangle,$$

where σ is a cone containing v, is well-defined and satisfies the first condition of Definition 2.3.1. It is obvious that ψ_D assumes integer values at lattice vectors, since for $u \in M$ and $v \in N$ the value $\langle u, v\rangle$ is integer. This shows that ψ_D is indeed a support function.

Conversely, let $\psi : \mathrm{supp}(\Delta) \to \mathbb{R}$ be a support function. We define a Weil divisor on X by setting

$$D_\psi = \sum_{\rho\in\Delta(1)} -\psi(v_i) D_{\rho_i}, \tag{2.4}$$

where v_i is the minimal generator of the ray ρ_i. By construction, D_ψ is T-invariant. Consider the open cover of X by the affine toric varieties U_σ, for σ in Δ. Since ψ is linear on each cone, then $\psi(v_i) = \langle u(\sigma), v_i\rangle$, for some $u(\sigma)$ in M. Using equation (2.4), we see that D_ψ is given locally on U_σ as

$$\mathrm{div}(\chi^{-u(\sigma)}).$$

This $u(\sigma)$ is well defined modulo $M(\sigma)$. Indeed, suppose we have two vectors $u_1(\sigma)$ and $u_2(\sigma)$ such that

$$\psi(w) = \langle u_1(\sigma), w\rangle, \quad \psi(w) = \langle u_2(\sigma), w\rangle$$

for any vector w of σ. Then we must have

$$\langle u_1(\sigma) - u_2(\sigma), v\rangle = \langle u_1(\sigma), w\rangle - \langle u(\gamma), v\rangle = 0,$$

and this can happen if and only if $u_1(\sigma) - u_2(\sigma)$ lies in $\sigma^\perp \cap M$, which is nothing but $M(\sigma)$.

The previous lines also show that the local data determined by D_ψ is compatible, so it is clear that that D_ψ corresponds to a T-invariant Cartier divisor. $\square$

Using the description of a T-invariant Cartier divisor given above, we can see that any such divisor D defines a polytope P_D. Let ψ_D be the support function defined by D, then, identifying vectors u of $M_\mathbb{R}$ with linear functions from $N_\mathbb{R}$ to $\mathbb{R}$, we define P_D to be

$$P_D = \big\{u \in M_\mathbb{R} \mid u \geq \psi_D \text{ on } \mathrm{supp}(\Delta)\big\} \tag{2.5}$$

Identifying D with its corresponding Weil divisor $[D] = \sum n_i D_{\rho_i}$, we can rewrite (2.5) as

$$P_D = \{u \in M_\mathbb{R} \mid \langle u, v_i\rangle \geq -n_i \ \forall i\} \tag{2.6}$$

A priori, (2.6) only says that P_D is a polyhedron (an intersection of closed half spaces), but it is shown in [Ful93], page 67, that P_D is in fact bounded and therefore a polytope.

Lemma 2.3.5. Let X be a toric variety and $T = \mathrm{Spec}(k[M])$ its torus. Let D be a T-invariant Cartier divisors and $\mathcal{O}(D)$ its associated sheaf. If we denote by $\mathcal{O}$ the structure sheaf of X, then we have

$$\Gamma(T, \mathcal{O}(D)) = \Gamma(T, \mathcal{O}).$$

Proof. By definition of the sheaf $\mathcal{O}(D)$, we have

$$\Gamma(T, \mathcal{O}(D)) = \{f \in k(X)^{\times} \mid \mathrm{div}(f) + D \geq 0 \text{ on } T\} \cup \{0\}. \tag{2.7}$$

By Theorem 2.2.6, the divisor D is of the form $\mathrm{div}(\chi^u)$, for some u in M, so we can rewrite (2.7) as

$$\Gamma(T, \mathcal{O}(D)) = \{f \in k(X)^{\times} \mid \mathrm{div}(f) + \mathrm{div}(\chi^u) \geq 0 \text{ on } T\} \cup \{0\},$$

which, since $\mathrm{div}(f) + \mathrm{div}(\chi^u) = \mathrm{div}(f\chi^u)$, becomes

$$\Gamma(T, \mathcal{O}(D)) = \{f \in k(X)^{\times} \mid \mathrm{div}(f\chi^u) \geq 0 \text{ on } T\} \cup \{0\}.$$

In general, a principal divisor on a normal variety is effective if and only if it is a regular function. We thus get

$$\Gamma(T, \mathcal{O}(D)) = \{f \in k(X)^{\times} \mid f\chi^u \in k[M]\} \cup \{0\},$$

from which it follows immediately that

$$\Gamma(T, \mathcal{O}(D)) = \Gamma(T, \mathcal{O}).$$

□

Note that $k[M]$ can be expressed as a direct sum

$$k[M] = \bigoplus_{u \in M} k\chi^u,$$

so the previous lemma says that

$$\Gamma(T, \mathcal{O}(D)) = \bigoplus_{u \in M} k\chi^u.$$

We can give a similar description for a T-invariant divisor on any toric variety. Let K be a polytope in $M_{\mathbb{R}}$, $\Delta = \Delta_K$ its fan and $X = X_K$ the associated toric variety. We denote by v_i the minimal generators of the rays ρ_i of Δ. Let $D = D_\psi$ the T-invariant Cartier divisor on X corresponding to a support function ψ and $\mathcal{O}(D)$ the associated sheaf.

Theorem 2.3.6. With notation as above we have

$$\Gamma(X, \mathcal{O}(D)) = \bigoplus_{u \in P_D \cap M} k\chi^u,$$

where P_D is the polytope of (2.6).

Proof. Consider the open cover of X given by affine toric varieties U_σ corresponding to cones σ in Δ. As in the proof of Proposition 2.2.2, D is supported on the complement of T, so the domain of every rational function in $\Gamma(U_\sigma, \mathcal{O}(D))$ contains T. This means that we have an inclusion $\Gamma(U_\sigma, \mathcal{O}(D)) \subset \Gamma(T, \mathcal{O}(D))$ and therefore, on the open affine U_σ, the principal divisor $\operatorname{div}(f)$ of a rational function f on X is given locally by $\operatorname{div}(\chi^u)$, for some $u \in M$. By Corollary 2.2.5 we have

$$\operatorname{div}(\chi^u)\big|_{U_\sigma} = \sum_{\rho_i \in \Delta(1)\cap\sigma} \langle u, v_i\rangle D_{\rho_i}.$$

Furthermore, since the divisor D_ψ is by definition

$$D_\psi = \sum_{\rho_i \in \Delta(1)} -\psi(v_i) D_{\rho_i},$$

its restriction to U_σ is given by the terms of the sum corresponding to minimal generators lying in σ, i.e. :

$$D_\psi\big|_{U_\sigma} = \sum_{\rho_i \in \Delta(1)\cap\sigma} -\psi(v_i) D_{\rho_i}.$$

From what we said, we can write

$$\operatorname{div}(f) + D\big|_{U_\sigma} = \sum_{\rho_i \in \Delta(1)\cap\sigma} \big(\langle u, v_i\rangle - \psi(v_i)\big) D_{\rho_i}. \tag{2.8}$$

Set now $P_D(\sigma) = \{u \in M_\mathbb{R} \mid \langle u, v_i\rangle \geq \psi(v_i)\ \forall v_i \in \sigma\}$. Since the sections over U_σ of $\mathcal{O}(D)$ are given by

$$\Gamma(U_\sigma, \mathcal{O}(D)) = \Big\{ f \in k(X)^\times \,\Big|\, \operatorname{div}(f) + D\big|_{U_\sigma} \geq 0 \Big\} \cup \{0\},$$

according to (2.8), such sections are actually given by

$$\bigoplus_{u \in P_D(\sigma)\cap M} k\chi^u.$$

Now, the space of global sections $\Gamma(X, \mathcal{O}(D))$ is the intersection of all the subspaces $\Gamma(U_\sigma, \mathcal{O}(D))$ corresponding to the open sets of the cover, so we finally get

$$\Gamma(X, \mathcal{O}(D)) = \bigcap_{\sigma\in\Delta} \Gamma(U_\sigma, \mathcal{O}(D)) = \bigoplus_{u \in P_D\cap M} k\chi^u.$$

□

Remark 19. Let ψ be a Δ-linear support function, with Δ the fan of a polytope in $M_{\mathbb{R}}$. Assume that there exist $u(\sigma) \in M_{\mathbb{R}}$ such that $\psi(v) = \langle u(\sigma), v\rangle$ for any v in σ. In this case it is straightforward to check that ψ is convex if and only if for every maximal cone σ of Δ and v in $\operatorname{supp}(\Delta)$ we have

$$\langle u(\sigma), v\rangle \geq \psi(v).$$

In the following, we often consider the sheaf $\mathcal{O}(D)$ on the toric variety X as a line bundle. In this case, a global section is a map $s : X \to \mathcal{O}(D)$ such that the composition with the standard projection on $\mathcal{O}(D)$ is trivial. Theorems 2.3.9 and 2.3.7 give a very explicit criterion in terms of the support function ψ_D to determine when $\mathcal{O}(D)$ is ample or very ample.

Theorem 2.3.7. Let X be the toric variety of a polytope K and Δ its fan. Let D be the divisor associated to a support function ψ, then $\mathcal{O}(D)$ is generated by its sections if and only if ψ is convex.

Proof. We claim the following:

> The sheaf $\mathcal{O}(D)$ is generated by its sections if and only if for every maximal cone σ of Δ there is a $u(\sigma)$ in $P_D \cap M$ such that $D + \operatorname{div}(\chi^{u(\sigma)})$ is effective and its restriction to U_σ is trivial.

If $\mathcal{O}(D)$ satisfies the above condition, it is clearly generated by its sections. Conversely, suppose $\mathcal{O}(D)$ is generated by its sections and let σ be any maximal cone of Δ. Then, for any point x in U_σ there exists a global section of $\mathcal{O}(D)$ not vanishing at x. In particular, such a section exists for the distinguished point x_σ. By Theorem 2.3.6, the vector space of global sections is generated by elements of the form χ^u, for u in $P_D \cap M$. This means that there is a $u(\sigma)$ in M such that $D + \operatorname{div}(\chi^{u(\sigma)}) \geq 0$ and the support of $D + \operatorname{div}(\chi^{u(\sigma)})$ does not contain x_σ. Since x_σ is fixed by the action of the torus, it lies in every invariant subvariety of U_σ, so that the restriction of $D + \operatorname{div}(\chi^{u(\sigma)})$ to U_σ must be trivial. This shows our claim.

Denote by $v_1, \ldots, v_s$ the minimal generators of the rays $\rho_1, \ldots, \rho_s$ in Δ. Recall that D is defined as

$$D = \sum_{i=1}^{s} -\psi(v_i) D_{\rho_i}\,.$$

On U_σ, the divisor D becomes

$$D\big|_{U_\sigma} = \sum_{v_i \in \sigma} -\psi(v_i) D_{\rho_i}\,.$$

Similarly, by Corollary 2.2.5, the principal divisor $\operatorname{div}\left(\chi^{u(\sigma}\right)$ can be written as

$$\operatorname{div}(\chi^{u(\sigma)}) = \sum_{i=1}^{s} \langle u(\sigma), v_i\rangle D_{\rho_i}\,,$$

which can be expressed on U_σ as

$$\operatorname{div}(\chi^{u(\sigma)})\big|_{U_\sigma} = \sum_{v_i \in \sigma} \langle u(\sigma), v_i \rangle D_{\rho_i} .$$

We can now specify the condition of $D + \operatorname{div}(\chi^{u(\sigma)})$ being trivial on U_σ as

$$\psi(v_i) = \langle u(\sigma), v_i \rangle \text{ for every minimal generator } v_i \text{ contained in } \sigma,$$

which is equivalent to

$$\psi(v) = \langle u(\sigma), v \rangle \text{ for every } v \text{ in } \sigma.$$

Similarly, the condition of $D + \operatorname{div}(\chi^{u(\sigma)})$ being effective translates as

$$\psi(v) \leq \langle u(\sigma), v \rangle \text{ for every } v \text{ in } \operatorname{supp}(\Delta) = N_{\mathbb{R}}.$$

Finally, by Remark 19, ψ satisfies these conditions if and only if it is convex as a support function, thus proving the theorem. □

A useful reference for the last part of this section is Section II.7 of [Har83], in particular the part "Morphisms to $\mathbf{P}^n$".

Denote by $\mathbb{P}^r$ projective r-space over k. Let D be a T-invariant Cartier divisor on a toric variety X_K such that $\mathcal{O}(D)$ is generated by its sections. Choosing and ordering a basis $\{\chi^{u_i} \mid u_i \in P_D \cap M\}$ gives a morphism

$$\begin{aligned} f_D : X_K &\to \mathbb{P}^r \\ x &\mapsto (\chi^{u_0}(x), \ldots, \chi^{u_r}(x)) \end{aligned} \tag{2.9}$$

where $r + 1 = \# P_D \cap M$. Such a mapping is a closed embedding if and only if the sheaf $\mathcal{O}(D)$ is very ample. As in the previous theorem, we can give a characterization of this condition in terms of the support function ψ of D.

Proposition 2.3.8. Let D be a T-invariant Cartier divisor on a toric variety X, then $\mathcal{O}(D)$ is very ample if and only if ψ_D is strictly convex and for every maximal cone σ of Δ, S_σ is generated by $\{u - u(\sigma) | u \in P_D \cap M\}$.

Proof. We denote by $T_0, \ldots, T_r$ the homogeneous coordinates in $\mathbb{P}^r$ and by V_k the open sets of $\mathbb{P}^r$ defined by $T_k \neq 0$. Such open sets cover $\mathbb{P}^r$ and, in homogeneous coordinates, are described as

$$V_k = \operatorname{Spec}(k[T_i/T_k \mid 0 \leq i \leq r]).$$

Let σ and γ be two maximal cones of Δ with, then there are indexes j, k such that $u(\sigma) = u_j$ and $u(\gamma) = u_k$. With this notation, it is evident that $f_D(U_\sigma) = V_j \cap f_D(X_K)$. Moreover, since ψ_D is strictly convex, $u(\sigma)$ is different from $u(\gamma)$ whenever σ is different from γ, so we have that $\sigma \cap (u_j - u_k)^\perp = \sigma \cap \gamma$. Note that for $\sigma = \gamma$ this is still true, since in such case we have that $u_j = u_k$ and $(u_j - u_k)^\perp = 0^\perp = N_\mathbb{R}$, so $\sigma \cap (u_j - u_k)^\perp = \sigma \cap \sigma = \sigma$. It follows that

$$U_\sigma \cap f_D^{-1}(V_k) = U_{\sigma\cap\gamma}.$$

Now fix γ. Since the affine toric varieties U_σ cover X as σ varies in Δ, we get that

$$f_D^{-1}(V_k) = U_\gamma.$$

The corresponding morphism of affine varieties $f_D|_{U_\gamma} \to V_k$ is induced by the k-algebra homomorphism

$$\begin{aligned} k[T_i/T_k \mid 0 \leq 1 \leq r] &\to k[S_\gamma] \\ T_i/T_k &\mapsto \chi^{u_i}/\chi^{u_k} = \chi^{u_i - u(\gamma)} \end{aligned}$$

The morphism $f_D|_{U_\gamma} \to V_k$ is a closed immersion if and only if the above map is surjective, i.e. if and only if $u_i - u(\gamma)$ generates S_γ. This proves one implication.

For the other implication, assume that f_D is a closed immersion, then the inverse image $f_D^{-1}(V_j)$ of every V_j is an affine open subset of X (see [Har83], Section II.7). It being invariant under the action of the torus, it must be the affine toric variety of some cone γ of Δ. If σ is any other maximal cone, then take j to be the index such that $u_j = u(\sigma)$. Since $f_D^{-1}(V_j) \supset U_\sigma$, then we must actually have $f_D^{-1}(V_j) = U_\sigma$. This implies that $\gamma \cap (u(\sigma) - u(\gamma))^\perp = \sigma \cap \gamma$ for any two maximal cones σ and τ, which shows (as in the proof of the first implication) that ψ_D is strictly convex. $\square$

Theorem 2.3.9. Let D be a T-invariant Cartier divisor on a toric variety X, then $\mathcal{O}(D)$ is ample if and only if ψ_D is strictly convex.

Proof. By Proposition II.7.5 in [Har83], $\mathcal{O}(D)$ is ample if and only $\mathcal{O}(D)^{\otimes m}$ is very ample for some positive integer m. Since $\mathcal{O}(D)^{\otimes m} = \mathcal{O}(mD)$ and $\psi_{mD} = m\psi_D$, one implication is a corollary of Proposition 2.3.8.

Conversely, suppose that ψ_D is strictly convex. Denote by $v_1, \ldots, v_s$ the minimal generators of the rays $\rho_1, \ldots, \rho_s$ in Δ. If we denote by mP_D the dilation by a positive integer m of the polytope P_D, then

$$P_{mD} = \{u \in M_\mathbb{R} \mid \langle u, v_i\rangle \geq m\psi(v_i) \forall i = 1, \ldots, s\} = mP_D.$$

As above, since $P_{mD} = mP_D$ and $\psi_{mD} = m\psi_D$, we can apply Proposition 2.3.8. Then it is sufficient to prove that there exists a positive integer m such that S_σ is generated by

$$\{u - mu(\sigma) \mid u \in mP_D \cap M\}, \tag{2.10}$$

for every maximal cone σ in Δ.

Recall that the divisor D is given by $D = \sum_{i=1}^{s} -\psi(v_i)D_{\rho_i}$. If u lies in S_σ then $u + mu(\sigma)$ lies in mP_D for sufficiently large m. Indeed, if a maximal generator v_i lies in σ, we have $\langle u, v_i\rangle \geq 0$ and, since $u(\sigma)$ is in P_D, $\langle u(\sigma), v_i\rangle \geq \psi(v_i)$. It follows that

$$\langle u + m_i u(\sigma), v_i\rangle = \langle u, v_i\rangle + m_i\langle u(\sigma), v_i\rangle \geq m_i\langle u(\sigma), v_i\rangle$$

for any positive integer m_i. If v_i is not in σ, since ψ is strictly convex we have $\langle u(\sigma), v_i\rangle > \psi(v_i)$. If $\langle u, v_i\rangle \geq 0$, we are in the same situation as before, while if $\langle u, v_i\rangle < 0$, we can take any integer m_i such that

$$m_i > \langle u, v_i\rangle / \big(\psi(v_i) - \langle u(\sigma), v_i\rangle\big).$$

The largest of the m_i's chosen for each maximal generator v_i is the desired integer.

Finally, since the monoids S_σ are finitely generated, and $u + mu(\sigma)$ lies in mP_D for sufficiently large m, S_σ is generated by elements of the form $u - mu(\sigma)$ as u varies in $mP_D \cap M$. This proves the theorem. □

2.4 Projective toric varieties

In this section we prove that every toric variety arising from a polytope is projective. This fact makes it possible to compare the two different constructions of a toric variety we have studied, and show that they are indeed equivalent.

Let K be a polytope in $M_{\mathbb{R}}$ and Δ its fan in $N_{\mathbb{R}}$. Recall that, by Lemma 1.6.1, the support of Δ is such that $\operatorname{supp}(\Delta) = N_{\mathbb{R}}$. We define a function ψ_K from $\operatorname{supp}(\Delta)$ to $\mathbb{R}$ as

$$\psi_K(v) = \inf\{\langle u, v\rangle \mid u \in K\}. \tag{2.11}$$

We call this function the *support function* of K. For this name to make sense we need to check that ψ_K actually is a support function as defined in 2.3.1.

Proposition 2.4.1. The support function ψ_K of a polytope K is a Δ-linear support function.

Proof. We have to prove that ψ_K is linear on each cone σ of Δ and assumes integer values at lattice vectors.

Let σ be a cone of Δ and v any vector in σ, then σ is the cone σ_F, for some face F of K (see Section 1.6). By definition, a face F of K is the intersection of a supporting affine hyperplane H with the polytope K. With notation as in Section 1.1, this means

$$F = H \cap K \text{ and } F \subset H^+.$$

Here H is a supporting affine hyperplane corresponding to an element w in $N_{\mathbb{R}}$, and H^+ is the closed half-space determined by H, i.e.

$$H = \{u \in M_{\mathbb{R}} \mid \langle u, w\rangle = a\},\ H^+ = \{u \in M_{\mathbb{R}} \mid \langle u, w\rangle \geq a\},$$

where a is a real number. The choice of w is not unique, in our case we can even take w to be v. It follows that

$$\langle u, v\rangle \geq a \ \forall u \in M_{\mathbb{R}}. \tag{2.12}$$

By definition of the cone σ_F, for any vector w in σ_F the following holds for any u in K and u' in F:

$$\langle u, w\rangle \geq \langle u', w\rangle \ \forall u \in K, \forall u' \in F, \tag{2.13}$$

so that, combining (2.12) and (2.13), we get

$$\langle u, w\rangle \geq a \ \forall u \in K. \tag{2.14}$$

This last equation really says that $\inf\{\langle u, v\rangle \mid u \in K\} = a$ and hence

$$\psi_K(v) = a.$$

Moreover, since for any vector on the face F this minimum is obtained, we have

$$\psi_K(v) = \langle u(\sigma), v\rangle,$$

for some $u(\sigma)$ in F. Since the vector $u(\sigma)$ depends only on the face F, it depends only on the cone σ. The identification of vectors of $M_{\mathbb{R}}$ with linear functions from $N_{\mathbb{R}}$ to $\mathbb{R}$ proves the first part of the proposition (that is, ψ_K is linear on each cone).

For the second part, note that a polytope is the convex hull of its vertices. This means that we can actually write ψ_K as

$$\psi_K(v) = \min\{\langle u, v\rangle \mid u \in \mathrm{vert}(K)\},$$

being the set $\mathrm{vert}(K)$ of vertices of K a finite set. Note that the vertices of K lie in M because K is a lattice polytope. Since $\langle u, v\rangle$ is an integer whenever u and v are lattice vectors (i.e. $u \in M, v \in N$), then $\psi_K(v)$ is an integer whenever v is a lattice vector. This shows the second part of the proposition and concludes the proof. □

Remark 20. The support function ψ_K of a polytope K is strictly convex. Indeed, convexity follows from the very definition, since

$$\inf\{a + b \mid a \in A, b \in B\} \geq \inf\{a \mid a \in A\} + \inf\{b \mid b \in B\}$$

for arbitrary sets A and B of real numbers. Moreover, in the notation of the previous proof, the value on a cone σ_F is determined by a vector $u(\sigma_F)$ in the face F. The corresponding linear function is different from the one determined by any $u(\sigma_G)$, where G is a face of K distinct from F.

At this point it is possible to apply the results of Section 2.3 to obtain a key property of a toric variety.

Proposition 2.4.2. The toric variety of a polytope is projective.

Proof. Let K be a polytope in $M_{\mathbb{R}}$ and X_K the associated toric variety. The previous remark shows that the support function ψ_K is strictly convex. Then, by Theorem 2.3.9, ψ_K determines a divisor D on X_K whose associated sheaf $\mathcal{O}(D)$ is ample. By [Har83], II.7.5, there exists an integer such that the sheaf $\mathcal{O}(D)^{\otimes m}$ on X_K is very ample. Since X_K is complete, in particular it is proper, so X_K is a proper algebraic variety admitting a very ample sheaf. This shows that X_K is projective. $\square$

Having established that the toric variety of a polytope is projective, it makes sense to compare the different definitions of toric variety we have given in Section 1.10 and Section 1.6. With notation as in the above sections we have the following.

Theorem 2.4.3. The varieties Y_K and X_K, defined in Section 1.6 and Section 1.10 respectively, are isomorphic.

Proof. Let K be a polytope and ψ its support function, as defined in (2.11). The divisor determining the embedding in projective space of X_K is the divisor D_ψ associated to ψ. By definition, this divisor is given by

$$\sum_{\rho_i \in \Delta(1)} -\psi(v_i) D_{\rho_i},$$

hence the corresponding polytope is

$$P_{D_\psi} = \{u \in M_{\mathbb{R}} | \langle u, v_i \rangle \geq \psi(v_i) \ \forall i\},$$

which gives P_D as an intersection of half-spaces. In the proof of Proposition 2.4.1 we have seen that the values of ψ are determined by the half-spaces containing the polytope K. This means that the polytope P_D is actually K. The embedding in projective space f_D of (2.9) is determined by the lattice points of P_D, i.e. by the points of $P_D \cap M = K \cap M$. In Section 1.10 we denoted this last set by $\mathcal{A}$. Moreover, the toric variety X_K contains the torus T as a dense open subset. It is clear that f_D restricted to the torus T gives exactly $\psi_{\mathcal{A}}$. We can visualize the situation with the following diagram.

$$\begin{tikzcd} T \arrow[r, "\varphi_{\mathcal{A}}"] \arrow[d, phantom, "\cap"] & \mathbb{P}^m \\ X_K \arrow[ur, hook, "f_D"'] & \end{tikzcd}$$

It is true for any continuous map $g : Y \to Z$ between topological spaces that $\overline{g(A)} \supset g(\bar{A})$ for any subset A of X, where $\bar{A}$ is the closure of A. In our case, the embedding f_D is a map of varieties, in particular it is continuous. From this it follows that, under the embedding in projective space, X_K is isomorphic to the closure of $\mathrm{im}(\psi_{\mathcal{A}})$, which is nothing but Y_K. $\square$

Chapter 3

Complex toric varieties

One of the motivating problems for this thesis was the study of lattice polygons in the plane. In this chapter we prove that, for the complex toric variety of a polytope, the Euler characteristic equals the number of vertices of the starting polytope. Consider the following situation: we have a given two-dimensional toric variety (a toric surface) and we want to know from which polygon it came from. Computing its Euler characteristic lets us distinguish whether it came from a triangle, a quadrangle, etc.

Let N be a lattice of rank n with dual M, $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ their respective real vector spaces. In this chapter we consider toric varieties over the field $\mathbb{C}$ of complex numbers. Explicitly, let K be a polytope and Δ_K its corresponding fan. We consider the variety X_K obtained by glueing the affine toric varieties $U_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$ corresponding to cones σ in Δ_K. Complex toric varieties admit a very direct description of many topological invariants, e.g. the Euler characteristic, in terms of combinatorial properties of Δ_K.

As a reference for basic facts about algebraic topology used in what follows (e.g. contractible spaces, deformation retracts of spaces, etc.), the reader may consult, for example, [Rot88] or [Hat02]. In this chapter we make

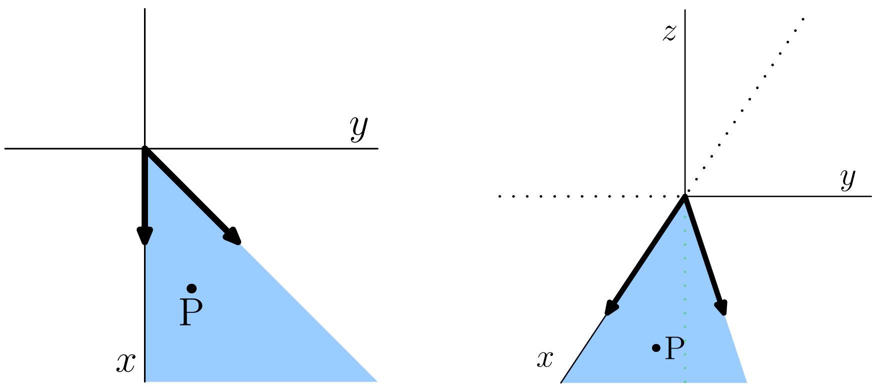


Figure 3.1: The same cone in $\mathbb{R}^2$ and $\mathbb{R}^3$.

use of the concepts of interior and relative interior of a cone. The cones we are considering lie inside a finite-dimensional real vector space $N_{\mathbb{R}}$: we give to this space the topology induced by the Euclidean one. The *interior* of a cone σ in $N_{\mathbb{R}}$, denoted by $\operatorname{int}(\sigma)$ is the interior of σ with respect to such topology. This notion does not always "behave well": let us illustrate this with a drawing. In Figure 3.1 we have a cone σ of dimension two in $\mathbb{R}^2$ and a point which lies in its interior. If we consider σ as sitting in the xy plane in $\mathbb{R}^3$, then it is still a cone, but now the topological interior is empty! To address this issue, we consider the *relative interior*, which is the interior of σ in the linear subspace it generates.

3.1 Singular cohomology of affine toric varieties

In the present section we study the topology of affine toric varieties. This is useful to determine their cohomology groups and is a key step for the computation of the Euler characteristic of a toric variety.

Proposition 3.1.1. Let σ be a strongly convex cone of dimension n in $N_{\mathbb{R}}$, then its affine toric variety U_σ is contractible.

Proof. To show that U_σ is contractible, we have to find a homotopy between the identity map of U_σ

$$\mathrm{id}_{U_\sigma} : U_\sigma \to U_\sigma,\ \phi \to \phi$$

and some constant map

$$U_\sigma \to U_\sigma,\ \phi \to P,$$

sending every element to a single point P. We take P to be the distinguished point x_σ (see (1.6) and (1.7) on page 28). Let v be a lattice vector in $\operatorname{int}(\sigma)$, i.e. a vector v in $\operatorname{int}(\sigma) \cap N$ and let I be the closed interval $[0, 1]$ in $\mathbb{R}$. Identifying points of U_σ with monoid homomorphisms from S_σ to $\mathbb{C}$, we define a map $H : U_\sigma \times I \to U_\sigma$ as

$$H(\phi, t)(u) = \begin{cases} t^{\langle u,v\rangle}\phi(u) & \text{if } t \in (0, 1], \\ x_\sigma(u) & \text{if } t = 0. \end{cases}$$

When u is zero, since v lies in the interior of σ, we have $\langle u, v\rangle = 0$, so

$$H(\phi, t)(0) = 1\,\phi(0) = 1 \ \text{ for all } t.$$

When u is non-zero, $\langle u, v\rangle > 0$ and therefore $\lim_{t\to 0} t^{\langle u,v\rangle}\phi(u) = 0$, so $\lim_{t\to 0} H(\phi, t) = x_\sigma$. To check continuity, we choose explicit generators $s_1, \ldots, s_l$ of S_σ. By ([Ful93], exercise on page 19), the affine toric variety U_σ (as an algebraic subset of $\mathbb{C}^l$) is defined by the ideal of $\mathbb{C}[y_1, \ldots, y_l]$ generated by all polynomials of the form

$$y_1^{a_1} \cdots y_l^{a_l} - y_1^{b_1} \cdots y_l^{b_l},$$

where $a_1, \dots, a_l, b_1, \dots, b_l$ are solutions of the equation

$$a_1 s_1 + \dots + a_l s_l = b_1 s_1 + \dots + b_l s_l.$$

Under this embedding of U_σ in $\mathbb{C}^l$, the distinguished point x_σ corresponds to the point $(0, \dots, 0)$. Indeed, $x_\sigma : S_\sigma \to \mathbb{C}$ corresponds to the $\mathbb{C}$-algebra homomorphism $\widetilde{x_\sigma} : \mathbb{C}[S_\sigma] \to \mathbb{C}$ given on the basis as

$$\widetilde{x_\sigma}(\chi^u) = x_\sigma(u) = \begin{cases} 1 & \text{if } u = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This corresponds in turn to the maximal ideal $\ker(\widetilde{x_\sigma})$ in $\operatorname{Spec}(\mathbb{C}[S_\sigma])$. This maximal ideal is

$$\begin{aligned} \ker(\widetilde{x_\sigma}) = \left\{ \sum_{i=1}^{n} \alpha_i \chi^{u_i} \;\middle|\; \widetilde{x_\sigma}(\chi^{u_i}) = 0,\ i = 1, \dots, n \right\} = \\ \left\{ \sum_{i=1}^{n} \alpha_i \chi^{u_i} \;\middle|\; x_\sigma(u_i) = 0,\ i = 1, \dots, n \right\} = \\ \left\{ \sum_{i=1}^{n} \alpha_i \chi^{u_i} \;\middle|\; u_i \neq 0,\ i = 1, \dots, n \right\} = \mathbb{C}[S_\sigma] \setminus \mathbb{C}. \end{aligned}$$

This is identified, as a point in $\mathbb{C}^l$, with $(0, \dots, 0)$.

It is now possible to write the map H as

$$H(\alpha, t) = (t^{\langle s_1, v \rangle} \alpha_1, \dots, t^{\langle s_l, v \rangle} \alpha_l)$$

for $\alpha = (\alpha_1, \dots, \alpha_l)$ in $U_\sigma \subset \mathbb{C}^l$ and t in $[0,1] \subset \mathbb{R}$. Since v is in the interior of S_σ, $\langle s_i, v \rangle > 0$ for all i's, showing thus continuity of H.

Summarizing, we found a continuous map $H : U_\sigma \times I \to U_\sigma$ such that $H(-,1) = \mathrm{id}_{U_\sigma}$ and $H(-,0)$ is the constant map sending each point of U_σ to x_σ. We have thus constructed the required homotopy. ☐

Recall that the orbit of the distinguished point x_σ in U_σ under the action of the torus T_N is the affine variety $O_\sigma = \operatorname{Spec}(M(\sigma)) \subset U_\sigma$. Using almost the same proof as above, we have the following:

Proposition 3.1.2. Let σ be a k-dimensional cone in $N_\mathbb{R}$ with $k < n$, then O_σ is a deformation retract of U_σ.

Proof. It is enough to find a continuous function $H : U_\sigma \times [0,1] \to U_\sigma$ such that

(1) $H(\phi, 1) = \phi$,

(2) $H(\phi, 0) \in O_\sigma$,

(3) $H(\theta, 0) = \theta$,

for all ϕ in U_σ and θ in O_σ.

We define H in a way similar to the proof of the previous proposition. In this case, σ is not n-dimensional so we need to choose a lattice vector in the relative interior of σ, i.e. a vector v in $\mathrm{relint}(\sigma) \cap N$. For $\phi : S_\sigma \to \mathbb{C}$ and t in $[0,1]$ we define $H(\phi, t)$ as

$$H(\phi,t)(u) = \begin{cases} t^{\langle u,v\rangle}\phi(u) & \text{if } t \in (0,1], \\ x_\sigma(u)\phi(u) & \text{if } t = 0. \end{cases}$$

Checking for continuity is completely analogous to what we did in the proof of Proposition 3.1.1.

Let now ϕ be in U_σ, then $H(\phi,1)(u) = \phi(u)$, showing (1). For (2) we have

$$H(\phi,0) = \phi(u)x_\sigma(u) = \begin{cases} \phi(u) & \text{if } u \in \sigma^\perp \cap M = M(\sigma), \\ 0 & \text{otherwise.} \end{cases}$$

This latter map is zero outside $M(\sigma)$ and thus can be identified with a monoid homomorphism from $M(\sigma)$ to $\mathbb{C}$. This shows that $H(\phi,0)$ belongs to $\mathrm{Spec}(M(\sigma)) = O_\sigma$.

For (3), let θ be in O_σ. Then θ is a monoid homomorphism from $M(\sigma)$ to $\mathbb{C}$ and thus can be identified with its extension by zero to a monoid homomorphism $S_\sigma \to \mathbb{C}$. It is now obvious that

$$H(\theta,0) = \theta(u)x_\sigma(u) = \begin{cases} \theta(u) & \text{if } u \in \sigma^\perp \cap M = M(\sigma), \\ 0 & \text{otherwise,} \end{cases}$$

so that the monoid homomorphism $H(\phi,0)$ can be identified with the monoid homomorphism $\theta : M(\sigma) \to \mathbb{C}$. This shows that the required function H exists, and thus concludes the proof. □

From this last proposition follows that O_σ and U_σ have the same homotopy type and thus the same integral cohomology. This is a very strong result, since by Remarks 14 and 15 the orbit O_σ is an algebraic torus. The cohomology of an algebraic torus is well-known: its cohomology groups are given by the exterior powers of the dual of the first homology group. We then have the following result.

Corollary 3.1.3. Let σ be a strongly convex cone in $N_\mathbb{R}$ and U_σ its affine toric variety. If i is a non-negative integer, then

$$H^i(U_\sigma;\mathbb{Z}) \cong \bigwedge^i M(\sigma). \tag{3.1}$$

Proof. By [Ful93], Section 3.2, the fundamental group of an affine toric variety U_σ is isomorphic to $N(\sigma)$. Its dual is $M(\sigma)$, hence the discussion above proves the corollary for σ of dimension less than n. For σ of dimension n, we just have to note that $\sigma^\perp = \{0\}$ and thus $M(\sigma) = M \cap \sigma^\perp = \{0\}$. The zeroth exterior power of $M(\sigma)$ is then $\mathbb{Z}$, and all other powers are the zero group. Since, by Proposition 3.1.1, U_σ has the same homotopy type of a space with a single point, its zeroth cohomology group is $\mathbb{Z}$ and all the others are zero. □

3.2 The Euler characteristic

The results established in the previous section for affine toric varieties permits us to compute the cohomology of general toric varieties. Let then K be a polytope in $M_\mathbb{R}$, let Δ_K be its fan in $N_\mathbb{R}$ and X_K the corresponding toric variety. By a *good cover* of a topological space X we mean an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ such that finite intersections of the U_i's are contractible. For such covers, Čech cohomology with coefficients in the constant presheaf $\mathbb{Z}_X$ (i.e. the presheaf assigning the abelian group $\mathbb{Z}$ to each open subset of X) and singular cohomology with integer coefficients coincide (see [BT82], III.15).

In our case, the most natural cover to consider is the finite cover $\{U_i\}$ of X_K given by the affine toric varieties U_i corresponding to the cones σ_i of Δ_K. The problem is that at least two of such cones intersect in the trivial face $\{0\}$. The affine toric variety $U_{\{0\}}$ is the torus T of Section 1.8, which is not contractible. We can work around this by replacing the cohomology with a particular spectral sequence converging to it (see [God73], II, 5.4).

Recall now that the *topological Euler characteristic* $\chi(X)$ of a space X is defined as the alternating sum

$$\chi(X) = \sum_{q=0}^{\infty} (-1)^q \operatorname{rank}\left(H^q(X;\mathbb{Z})\right). \tag{3.2}$$

Theorem 3.2.1. The Euler characteristic of the toric variety X_K associated to a convex polytope K is the number of n-dimensional cones in its fan Δ_K.

Proof. For any open set of X_K, let $S_q(U)$ denote the group of singular q-chains of U (see, e.g., one of the references on algebraic topology listed at the beginning of this chapter). For any non-negative integer q consider the presheaf of singular q-cochains of X_K, i.e. the sheaf sending an open subset U of X_K to the set

$$S^q(U) = \operatorname{Hom}_\mathbb{Z}(S_q(U), \mathbb{Z}).$$

We denote this presheaf by S^q.

Take the finite cover

$$\mathcal{U} = \{U_i\}$$

consisting of affine toric varieties corresponfing to cones of maximal dimension. We try to apply the method described above using this cover. Denoting by $C^\bullet$ the Čech complex of $\mathcal{U}$, we can form a double complex

$$K^{\bullet,\bullet} = C^\bullet(\mathcal{U}, S^\bullet),$$

whose (p,q)-th term is

$$K^{p,q} = C^p(\mathcal{U}, S^q) = \bigoplus_{i_0<\cdots<i_p} S^q(U_{i_0} \cap \cdots \cap U_{i_p}).$$

The corresponding spectral sequence, whose E_1 term is

$$E_1^{p,q} = \bigoplus_{i_0<\cdots<i_p} H^q(U_{i_0} \cap \cdots \cap U_{i_p}; \mathbb{Z}),$$

converges to to the singular cohomology of X_K (cfr. [God73]). In the notation of spectral sequences we write

$$E_1^{p,q} \Rightarrow H^{p+q}(X_K; \mathbb{Z}).$$

Since the spectral sequence degenerates already at the E_1 level, the Euler characteristic of X_K can now be written as

$$\chi(X_K) = \sum_{p,q} (-1)^{p+q} \operatorname{rank}(E_1^{p,q}) =$$

$$= \sum_{p,q} (-1)^{p+q} \operatorname{rank}\left(\bigoplus_{i_0<\cdots<i_p} H^q(U_{i_0} \cap \cdots \cap U_{i_p}; \mathbb{Z}) \right)$$

and thus, applying Theorem 3.1.3, as

$$\chi(X_K) = \sum_{p,q} (-1)^{p+q} \operatorname{rank}\left(\bigoplus_{i_0<\cdots<i_p} \bigwedge^q M(\sigma_{i_0} \cap \cdots \cap \sigma_{i_p}) \right). \qquad (3.3)$$

Now, for a cone σ, the q-th exterior power $\bigwedge^q M(\sigma)$ of $M(\sigma)$ has rank $\binom{k}{q}$, where k is the rank of $M(\sigma)$. Since for q greater than k, $\binom{k}{q}$ is 0, we have that

$$\sum_{q=0}^{\infty} (-1)^q \operatorname{rank}(\bigwedge^q M(\sigma)) = \sum_{q=0}^{k} (-1)^q \operatorname{rank}(\bigwedge^q M(\sigma)) =$$

$$= \sum_{q=0}^{k} (-1)^q \binom{k}{q} = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $M(\sigma) = \sigma^\perp \cap M$ has dimension 0 if and only if σ is n-dimensional, it follows that in (3.3) the terms corresponding to the n-dimensional cones contribute each by 1, while the terms corresponding to the cones of dimension less than n do not contribute to the sum. This proves that $\chi(X_K)$ equals the number of n-dimensional cones in Δ_K. □

As a corollary, we obtain a result which is very useful in the theory of lattice geometry, as explained in the introduction and at the beginning of this chapter.

Corollary 3.2.2. The Euler characteristic of the toric variety X_K associated to a convex polytope K is the number of vertices of K.

Proof. Let Δ_K the fan of K. By Lemma 1.6.2, the correspondence between faces of K and cones of Δ_K is such that

$$\dim(\sigma_F) = \operatorname{codim}(F), \tag{3.4}$$

for any face F of K. The fan Δ_K is finite and covers all of $N_{\mathbb{R}}$, so maximal cones have full dimension $n = \operatorname{rank}(N)$. By (3.4), this means that maximal cones correspond to vertices. The corollary then follows from Theorem 3.2.1. $\square$

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