Drums & Eigenvalues

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Preface

Six years ago, an enthusiastic, fairly over-confident 19-year-old entered the Mathematics Department of the University of Leiden, only to reemerge six years later as a doctorandus and, as what he hopes to be, a wiser man.
What you are about to read is my Master Thesis, which is meant to be a reflection of the experience, skills and knowledge gained over the past years, which I hope will instill the same amount of wonder into you as they did into me. With that in mind, if this treatise can conjure a smile of recognition onto your face, then my work has served its purpose.
Lastly, there are some people to whom I would like to express my heartfelt thanks:
Doctor Marcel de Jeu, for being my supervisor and convincing me that I was not expected to perform an amazing feat such as solving the Riemann Zeta Hypothesis, but to write a thesis.
Doctor Robert-Jan Kooman, for taking the time to answer my questions.
Everyone at the University of Leiden who has helped me at one time or another in better understanding mathematics.
My highschool teachers mister Piet Heideman and mister Piet Schutte, who made me want to study mathematics in the first place.

Contents

Chapter 1

Introduction

1.1 Can you hear the shape of a drum?

For the last fifty years or so, mathematicians from all over the world have been trying to solve the problem of divulging the shape of a drum from the sound it produces alone. Since we can't really think of many practical applications of solving such a conundrum, to most of us this proves yet again that mathematical research is a waste of time and tax payer's money and that mathematicians are a bunch of hitherto unsurpassed lunatics.
And although some of them fit the bill of fruitier than a fruitcake perfectly, the rest of the argument is sheer nonsense. Studying vibrations can be and has been so in the past, very beneficial. Our hearing for example, is basically just two membranes with nerves attached to them. Two drums, if you will. So by claiming that the aforementioned research is useless, we might very well put thousands of ear doctors out of a job.
Secondly, various industrial processes make use of vats containing hot, and therefore vibrating, gases. These gases can be modelled as a 'three dimensional drum' so to speak, where the vat behaves as the rim of the drum and the gas acts as the membrane. So it goes almost without saying that the study of vibrations and of drums in particular, has myriads of applications. Yet in all these examples, the shape of the drum is already known, so we might think that solving our problem is still pointless. This is not true. In the unlikely event that this problem has no applications in every day life whatsoever, solving it will still provide us with new techniques and insights that can be used in several fields of mathematics and therefore in disciplines as, for example, physics and astronomy as well.
While this may seem like a lame excuse for indulging ourselves in spending our time with solving tantalizing, yet unimportant riddles, the argument above has merit indeed.
Because most of all, mathematics is a science. Not only that, it basically is the epitome of what science should be. No other science has so many

applications to the world around us, while being at the same time composed by logic and reasoning alone. And that what is science, that what *defines* science is that what drives scientists: We want to push back the boundaries of reality, we want to solve the mysteries of the universe, we want to explore the multitide of dimensions of existence itself...
But we don't know how. And because we don't know we try what we can and we search where we are able. And so we lock ourselves in our offices, feverishly writing formulas on scraps of paper behind a desk, we lie awake at night for hours, while our half-waking mind hints at wonderful solutions, try forcing our imagination to transcend into abstractions of the world around us, until at one time an idea is formed, a point is reached where we obtain insight into the unknown, where we, if ever so briefly, experience a moment of gazing at perfection. This is what drives us. This is Science.
And sometimes, we meet science in pretty unexpected places: rumour has it that it was a falling apple that instigated Newton's theory of gravitation, that Einstein's theory of relativity started with trying to get all the clocks of a trainstation to tell the same time and Archimedes got at least one of his ideas while relaxing in a bathtub. So who knows? Perhaps there is an entire world to be discovered behind the question of hearing the shape of a drum. There is only one way to find out...

"Can you hear the shape of a drum" is not a question that appeared just out of the blue. It originates from one of the great problems of nineteenth century physics, namely finding for which values of λ the problem

$$\begin{cases} \Delta u + \lambda u = 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

where u is defined on some bounded region Ω, has nontrivial solutions and showing that these λ form a discrete sequence. This equation models the behaviour of a vibrating membrane and the λ would represent the multitude of frequencies at which the membrane could oscillate. Showing that the λ form a discrete sequence was therefore of prime importance, since there are only countably many frequencies at which a membrane can vibrate. Now, proving that the sequence was discrete was one thing, but calculating the values of λ was another and this was basically the problem posed by Lorentz in Göttingen during the Wolfskehl lectures in 1910, except for the slight difference that he was only interested i n 'large' values of λ.
One of the members of the audience, a mathematician by the name of Hermann Weyl, got intrigued and set out to solve the problem and succeeded, publishing his articles "Ueber die asymptotische Verteilung der Eigenwerte" and "Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)" in which he briliantly made use of the theory of integral equations, which his teacher Hilbert developed only a few years

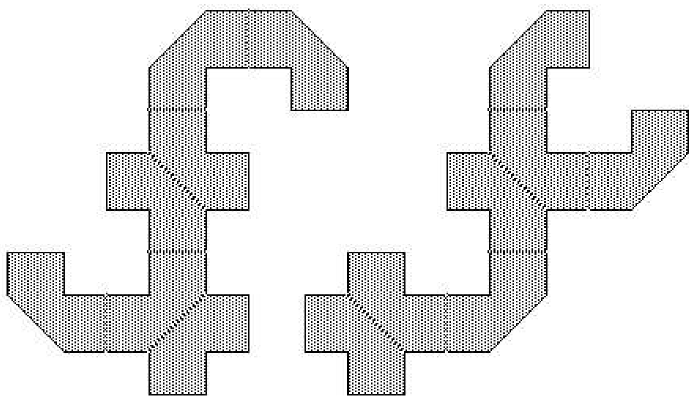
before, to prove that $\lim_{n\to\infty} \frac{n}{\lambda_n} = \frac{J}{4\pi}$, where λ_n is the nth eigenvalue of the operator $-\Delta$ and J is the area of the region on which the equation is defined. By doing so, he inadvertently showed that the eigenvalues determine the area of the membrane.

Some time after, in 1954, the Swede Åke Pleijel showed that not only do the eigenvalues determine the area, they determine the circumference as well.

Then, in 1966, Mark Kac observed that for multiply connected drums, the number of holes can be deduced from the eigenvalues also. Furthermore, Kac noted that it was possible to test wether a drum was circular, since a circle is the only object for which $L^2 = 4\pi J$ holds. Lastly, based on the aforementioned results, Kac was the first one who wondered whether one could 'hear' the shape of a drum, although he didn't deem it possible, to which he added

"I may well be wrong and I am not prepared to bet large sums either way."

As it turns out, Kac was right, since in 1992, Carolyn Gordon, David L.Webb and Scott Wolpert came up with a counterexample:



Two different shapes, albeit rather strange ones, that give identical sets of eigenvalues. So, can we hear the shape of a drum? The answer is 'no', thereby concluding our history lesson and moving on to the subject at hand, namely the contents of this thesis:

- Chapter 1 will give a short, rather intuitive introduction to the theory of integral equations and Green's functions, preparing us for

- Chapter 2, where we will find the proof from Hermann Weyl's article "Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen" (mit einer Anwendung auf die Theorie der Holraumstrahlung) and to what extent that proof can be generalized into higher dimensions.

- Chapter 3 serves as a leisurely tour past some applications of the theory of Chapter 2
- And last but not least, there is the Appendix of the Damned, which is, as you might have guessed, the appendix.

So let's not dally any longer and move on to section 1.2, the section by the title of

1.2 Integral Equations

Our main goal will be to reconstruct Weyl's proof for $\lim_{n\to\infty} \frac{n}{\lambda_n} = \frac{J}{4\pi}$ by largely following Weyl's article and generalizing it to higher dimensions, if indeed possible. This will be done in chapter 2. But as has been pointed out before, the proof makes use of Hilbert's theory of integral equations, so it would be a wise course of action to find out a bit more about integral equations first.
One should note however, that the purpose of this chapter is solely to provide the reader with a quick introduction to the subject, which is not entirely complete, but should make all background information intuitively plausible. For example, I will show that the number of eigenvalues can be countably infinite at most, yet the question wether there are eigenvalues in the first place will remain unanswered, because I am not entirely sure to what degree the reader is an adept at Functional Analysis (which you need to make the theory exact) and it would serve no purpose whatsoever to scare people silly with abstract definitions and thus tossing this thesis into a dark, dank corner, never to be read again, with the actual subject for ever untouched. However, I have included an Appendix which allows those of you who wish to absorb the finer details of the theory to do so to your heart's content.

What we plan on doing is rewriting a linear partial differential equation

$$\begin{cases} Lu = f & on \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases}$$

as $u(x) = \int\limits_{\Omega} G(x,\xi)f(\xi)\,d\xi$, where L is a linear partial differential operator and f is some function that is continuous on Ω, the bounded region in $\mathbb{R}^n$ on which the equation is defined, $\partial\Omega$ is the boundary of Ω, where $\int\limits_{\Omega}\int\limits_{\Omega} G(x,\xi)^2\,d\xi\,dx$ is finite and $G(x,\xi) = G(\xi,x)$. (On a side note; if $u(x) = \int\limits_{\Omega} K(x,y)$, then $K(x,y)$ is called the *kernel* of the integral operator). If we can find such a G, then if $\lambda \neq 0$ is an eigenvalue of L,

$Lu - \lambda u = 0$, so $Lu = \lambda u$, which implies that $u(x) = \lambda \int\limits_{\Omega} G(x,\xi)u(\xi)\,d\xi$, so $\frac{1}{\lambda}u(x) + \int\limits_{\Omega} G(x,\xi)u(\xi)\,d\xi = 0$, so $\frac{1}{\lambda}$ is an eigenvalue of G, where $Gf := \int\limits_{\Omega} G(x,\xi)f(\xi)\,d\xi$. Furthermore, L and G must have identical eigenfunctions.

What we will set out to prove is that there can be countably finite eigenvalues at most and we will show how to create a G for the case that $L = -\Delta$.

In order to expand our understanding of these operators, we will be needing a usefull tool called Bessel's inequality, but before we can verify this inequality, we will need to prove the following lemma:
(Note that when we are speaking of a Hilbert space anywhere in this thesis, we mean a real Hilbert space and not a complex one).

Lemma 1.2.1. *Let H be a Hilbert space. If a set $\{y_1, y_2, \ldots, y_m\} \subset H$ forms an orthonormal basis of an m-dimensional subspace of H, then every vector $x \in H$ can be uniquely written as $x = y + z$, where $y \in M := span\{y_1, \ldots, y_m\}$ and $z \in M^{\perp}$.*

Proof. First we will show that the representation is unique. Assume that there is an $x \in H$ which can be written as both $x = y + z$ and $x = y' + z'$ for y, $y' \in M$ and z, $z' \in M^{\perp}$. Then $y + z = x = y' + z'$, so $y - y' = z' - z$. But $y - y' \in M$ and $z' - z \in M^{\perp}$, which means that $y - y' = 0 = z' - z$, so $y = y'$ and $z = z'$, thereby proving uniqueness for the aforementioned representation of x.
Now it remains to be shown that such a representation exists. Let therefore $x \in H$, $y = \sum\limits_{i=1}^{m} \langle x, y_i \rangle y_i$ and $z = x - y$, where $\langle \cdot, \cdot \rangle$ signifies the inner product on H. If y and z are perpendicular, we are finished. This is indeed the case, since $\langle z, y_i \rangle = \langle x - y, y_i \rangle = \langle x, y_i \rangle - \langle y, y_i \rangle = \langle x, y_i \rangle - \langle x, y_i \rangle$, (because the y_j are perpendicular for $1 \leq j \leq m$).
So $\langle z, y_i \rangle = \langle x, y_i \rangle - \langle x, y_i \rangle = 0$, so y and z are perpendicular, which completes the proof.

□

Now, we can prove Bessel's inequality:

Lemma 1.2.2. *Let $y_1, y_2, \ldots, y_m$ once again form an orthonormal basis of an m-dimensional subspace of a Hilbert space H, then for every $x \in H$ the following inequality holds:*
$\sum\limits_{i=1}^{n} \langle x, y_i \rangle^2 \leq \|x\|^2$.

Proof. We know that x can be written as $x = y + z$ with $y = \sum_{i=1}^{m} \langle x, y_i \rangle y_i$, $z = x - y$ and z perpendicular to the y_i. Because the y_i are also perpendicular to one another:

$$\begin{aligned} \|x\|^2 &= \langle x, x \rangle = \langle y + z, y + z \rangle \\ &= \langle z + \sum_{i=1}^{m} \langle x, y_i \rangle y_i, z + \sum_{i=1}^{m} \langle x, y_i \rangle y_i \rangle \\ &= \langle z, z \rangle + \sum_{i=1}^{m} \langle x, y_i \rangle^2 \langle y_i, y_i \rangle^2 \\ &\geq 0 + \sum_{i=1}^{m} \langle x, y_i \rangle^2 \cdot 1 = \sum_{i=1}^{m} \langle x, y_i \rangle^2 \end{aligned}$$

giving

$$\sum_{i=1}^{n} \langle x, y_i \rangle^2 \leq \|x\|^2.$$

□

Corollary 1.2.3. *(Bessel's inequality) If $y_1, y_2, \ldots$ form a countably infinite orthonormal basis of a subspace of a Hilbert space H, then for every $x \in H$ the following inequality holds:*
$\sum_{i=1}^{\infty} \langle x, y_i \rangle^2 \leq \|x\|^2$.

With these tools in hand, we can finally make some progress on deducing some significant facts about the eigenfunctions and eigenvalues of G. First we will prove that the eigenfunctions of G are orthonormal if we define our inner product by $\langle f, g \rangle = \int_{\Omega} f(x) g(x)\, dx$.
Let ϕ_1, ϕ_2 be two eigenfunctions of G corresponding to different eigenvalues μ_1, μ_2 respectively. Since $G(x, \xi) = G(\xi, x)$,

$$\begin{aligned}\mu_1\langle\phi_1,\phi_2\rangle &= \langle\mu_1\phi_1,\phi_2\rangle \\ &= \langle G\phi_1,\phi_2\rangle = \int_\Omega\int_\Omega G(x,\xi)\phi_1(\xi)\phi_2(x)\,d\xi\,dx \\ &= \int_\Omega \phi_1(\xi)\int_\Omega G(\xi,x)\phi_2(x)\,dx\,d\xi = \langle\phi_1,G\phi_2\rangle \\ &= \langle\phi_1,\mu_2\phi_2\rangle = \mu_2\langle\phi_1,\phi_2\rangle\end{aligned}$$

resulting in

$$\mu_1\langle\phi_1,\phi_2\rangle = \mu_2\langle\phi_1,\phi_2\rangle$$

implying that

$$(\mu_1-\mu_2)\langle\phi_1,\phi_2\rangle = 0$$

from which can be deduced that

$$\langle\phi_1,\phi_2\rangle = 0$$

because $\mu_1-\mu_2\neq 0$. So ϕ_1 and ϕ_2 are orthogonal. Because eigenfunctions are always determined up to multiplication with a constant, we may assume them to be orthonormal instead.
Should a number of eigenfunctions $\phi_1,\ldots,\phi_n$ correspond to the same eigenvalue μ, we can choose these eigenfunctions to be mutually orthonormal, since if $\phi_1,\ldots,\phi_n$ are eigenfunctions corresponding to the same eigenvalue μ, a linear combination of $\phi_1,\ldots,\phi_n$ must be an eigenfunction corresponding to this eigenvalue as well, since if $c_1,\ldots,c_n\in\mathbb{R}$, then $\mu\sum\limits_{i=1}^{n}c_i\phi_i = \sum\limits_{i=1}^{n}\mu c_i\phi_i = \sum\limits_{i=1}^{n}c_i\mu\phi_i = \sum\limits_{i=1}^{n}c_iG\phi_i = G\sum\limits_{i=1}^{n}c_i\phi_i$ so $\sum\limits_{i=1}^{n}c_i\phi_i$ is an eigenfunction of G, corresponding to the eigenvalue μ, which implies that we can in this case choose our eigenfunctions to be perpendicular.
Now let $\phi_1,\ldots,\phi_m$ be eigenvectors of G corresponding to not necessarily distinct eigenvalues $\mu_1,\ldots,\mu_m$, let $g(\xi)=G(x,\xi)$ for a fixed value of x. Then for $1\leq j\leq n$

$$\begin{aligned}\langle g,\phi_j\rangle &= \int_\Omega G(x,\xi)\phi_j(\xi)\,d\xi \\ &= G\phi_j(x) = \mu_j\phi_j(x)\end{aligned}$$

providing us with

$$\langle g, \phi_j \rangle = \mu_j \phi_j(x)$$

so Lemma 1.2.2)) enables us to write

$$\sum_{i=1}^{m} \mu_i^2 |\phi_i(x)|^2 \leq \int\limits_{\Omega} |G(x,\xi)|^2 \, d\xi \tag{1.1}$$

Integrating both sides of (1.1) over Ω gives

$$\sum_{i=1}^{m} \mu_i^2 \leq \int\limits_{\Omega} \int\limits_{\Omega} |G(x,\xi)|^2 \, dx \, d\xi$$

As we've said before, $\int\limits_{\Omega}\int\limits_{\Omega} |G(x,\xi)|^2 \, dx \, d\xi$ is finite. So now we know that $\sum\limits_{i=1}^{m} \mu_i^2$ is uniformly bounded for every $n \in \mathbb{Z}_{>0}$, for every set $\mu_1, \dots, \mu_m$ of eigenvalues of G, where the μ_i for $1 \leq i \leq m$ need not be distinct. This means that there can be only finitely many eigenvalues μ_j for which $1 \leq |\mu_j|$, or $\frac{1}{2n} \leq |\mu_j| \leq \frac{1}{n}$, $n \in \mathbb{Z}_{>0}$ implying that the number of eigenvalues of G is countably infinite at most and because we pointed out that the eigenvalues need not be distinct, there can only be countably infinite eigenfunctions as well; one for every μ_j, $j \in \mathbb{Z}$.
Let's summarize this as a lemma:

Lemma 1.2.4. *The set of eigenvalues μ_i of a Green's function G is countably infinite at most and*

$$\sum_{i=1}^{\infty} \mu_i^2 \leq \int\limits_{\Omega} \int\limits_{\Omega} |G(x,\xi)|^2 \, dx \, d\xi$$

1.3 Construction of Green's function

We will conclude this chapter by showing how to create an integral kernel G corresponding to $L = -\Delta$ for the problem

$$\begin{cases} -\Delta u = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

for $f \in L^2(\Omega)$.
What will be done is proving that every function $u \in C^2 \cap C^1\left(\overline{\Omega}\right)$ can be written as follows:

$$u(x) = \int\limits_{\partial\Omega} (K(x,\xi)\frac{\partial}{\partial\nu}u(\xi) - u(\xi)\frac{\partial}{\partial\nu}K(x,\xi))dS - \int\limits_{\Omega} K(x,\xi)\Delta u(\xi) \, d\xi$$

where

$$K(x,\xi) = K(r) = \begin{cases} c_n r^{-(n-2)} & if \quad n \geq 3 \\ c_2 \log \frac{1}{r} & if \quad n = 2 \end{cases}$$

for which $x, \xi \in \mathbb{R}^n$ and $r = \|x - \xi\|$, c_2 and c_n are real constants respectively equal to $\frac{1}{2\pi}$ and $\frac{1}{(n-2)\sigma_n}$, σ_n the area of the boundary of the $n-$dimensional unit sphere, after which we will compose a function $G = K + g$ where g is a harmonic function on Ω and

$$u(x) = \int\limits_{\partial\Omega} (G(x,\xi)\frac{\partial}{\partial\nu}u(\xi) - u(\xi)\frac{\partial}{\partial\nu}G(x,\xi))dS - \int\limits_{\Omega} G(x,\xi)\Delta u(\xi)\, d\xi$$

after which will be shown that g can be chosen in such a manner that $G(x,\xi) = 0$ on $\partial\Omega$ and that $\int\limits_{\partial\Omega} G(x,\xi)\frac{\partial}{\partial\nu}u(\xi)dS = 0$ because in our case $u(x) = 0$ for $x \in \partial\Omega$.

What we will need to prove in order to do exactly that which we have just explained, is that K is harmonic in $\mathbb{R}^n/\{\xi\}$. To see this, observe that for v radially symmetric and harmonic on $\mathbb{R}^n/\{\xi\}$, $\Delta_x v = v'' + \frac{n-1}{r}v'$.

This is true, because if we assume that v is radially symmetric, then

$$\begin{aligned}
\Delta_x v(r) &= \sum_{i=1}^{n} \frac{\partial^2 v(r)}{\partial x_i^2} \\
&= \sum_{i=1}^{n} (\frac{\partial^2 v}{\partial r^2}(r)(\frac{\partial r}{\partial x_i})^2 + \frac{\partial v}{\partial r}\frac{\partial^2 r}{\partial x_i^2}) \\
&= \sum_{i=1}^{n} (v''(\frac{x_i - \xi_i}{r})^2 + v'(\frac{1}{r} + -\frac{x_i - \xi_i}{r^2}\frac{\partial r}{\partial x_i})) \\
&= v'' \sum_{i=1}^{n} \frac{(x_i - \xi_i)^2}{r^2} + v' \sum_{i=1}^{n} (\frac{1}{r} - \frac{(x_i - \xi_i)^2}{r^3}) \\
&= v'' \frac{\sum_{i=1}^{n}(x_i - \xi_i)^2}{r^2} + v'(\frac{n}{r} - \frac{\sum_{i=1}^{n}(x_i - \xi_i)^2}{r^3}) \\
&= v'' \frac{r^2}{r^2} + v'(\frac{n}{r} - \frac{1}{r}) \\
&= v'' \cdot 1 + v' \frac{n-1}{r}
\end{aligned}$$

meaning that

$$\Delta_x v = v'' \cdot 1 + v' \frac{n-1}{r}$$

So, inserting $K(x,\xi)$ into $\Delta_x v = v'' \cdot 1 + v' \frac{n-1}{r}$ for v shows that K is indeed a solution.

Now let's formulate the following theorem that will soon give us the function G that we so immensely crave:

Theorem 1.3.1. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ and let $u \in C^2 \cap C^1(\overline{\Omega})$. Then for any $x \in \Omega$*

$$u(x) = \int\limits_{\partial\Omega} (K(x,\xi)\frac{\partial u}{\partial \nu}(\xi) - u(\xi)\frac{\partial}{\partial \nu}K(x,\xi))dS - \int\limits_{\Omega} K(x,\xi)\Delta u(\xi)\, d\xi$$

where ν is a normalized vector outwardly perpendicular to $\partial\Omega$.

Proof. Before taking on the problem where $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ we will at first assume u to be in $C^2(\overline{\Omega})$.
Write B_ϵ for the ball with radius $\epsilon > 0$ around x and $S_\epsilon := \partial B_\epsilon$. Then, choose ϵ small enough for B_ϵ to fit inside Ω. This can be done, since $x \in \Omega$. We will write $\Omega_\epsilon = \Omega \backslash \overline{B_\epsilon}$.
Both $u(\xi)$ and $K(x,\xi)$ belong to $C^2(\Omega_\epsilon)$, allowing us to use Green's second identity, which states that if f, g are two functions in $\in C^2 \cap C^1(\overline{\Omega})$ then

$$\int\limits_{\Omega} (f\Delta g - g\Delta f)\, d\Omega = \int\limits_{\partial\Omega} \left(f\frac{\partial g}{\partial \nu} - g\frac{\partial f}{\partial \nu}\right) d\partial\Omega$$

so in the case of u and K:

$$\int\limits_{\Omega_\epsilon} (K\Delta u - u\Delta K)\, d\xi = \int\limits_{\partial\Omega} (K\frac{\partial u}{\partial \nu} - u\frac{\partial K}{\partial \nu})dS + \int\limits_{\partial\Omega_\epsilon} (K\frac{\partial u}{\partial \nu} - u\frac{\partial K}{\partial \nu})dS_\epsilon$$

On S_ϵ, $K(x,\xi) = c_n\epsilon^{2-n}$, $\frac{\partial K}{\partial \nu} = (2-n)c_n\epsilon^{1-n}$ and $dS_\epsilon = \epsilon^{n-1}dS_1$. Before we go any further: everything that will be done in the remainder of this proof holds analogously for the case $n = 2$.
Therefore for $\xi = x + \epsilon\theta$ and $\theta \in S_1$

$$\int\limits_{\partial\Omega_\epsilon} (K\frac{\partial u}{\partial \nu} - u\frac{\partial K}{\partial \nu})dS_\epsilon = c_n \int\limits_{S_1} (\epsilon\frac{\partial}{\partial \nu}u(x+\epsilon\theta) - (n-2)u(x+\epsilon\theta))dS_1 \quad (1.2)$$

Which means that if we take the limit of (1.2) where ϵ tends to 0, because

$$\lim_{\epsilon \downarrow 0} \int\limits_{S_1} \epsilon u(x+\epsilon\xi)dS_1 = 0$$

and

$$\lim_{\epsilon\downarrow 0}\int\limits_{S_1} u(x+\epsilon\xi)dS_1 = u(x)\sigma_n$$

that

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\int\limits_{\Omega_\epsilon} &(K\Delta u - u\Delta K)\,d\xi\\ &= \lim_{\epsilon\downarrow 0}\int\limits_{\partial\Omega}(K\frac{\partial u}{\partial\nu} - u\frac{\partial K}{\partial\nu})dS + \int\limits_{\partial\Omega_\epsilon}(K\frac{\partial u}{\partial\nu} - u\frac{\partial K}{\partial\nu})dS_\epsilon\\ &= \int\limits_{\partial\Omega}(K\frac{\partial u}{\partial\nu} - u\frac{\partial K}{\partial\nu})dS + 0 - u(x)\end{aligned}$$

and since $\Delta K = 0$ on Ω_ϵ we get

$$u(x) = \int\limits_{\partial\Omega}(K\frac{\partial u}{\partial\nu} - u\frac{\partial K}{\partial\nu})dS - \int\limits_{\Omega} K\Delta u\,d\xi$$

If we now assume u to be in $C^2(\Omega)\cap C^1\left(\overline{\Omega}\right)$, then $u\in C^2(\Omega')$ for $\Omega'\subset\Omega$, where Ω' has a smooth boundary, leading to the fact that

$$u(x) = \int\limits_{\partial\Omega'}(K\frac{\partial u}{\partial\nu} - u\frac{\partial K}{\partial\nu})dS - \int\limits_{\Omega'} K\Delta u\,d\xi$$

Because the smoothness of both Ω' and Ω, it is possible to let Ω' tend to Ω and by the continuity of u and ∇u we may therefore conclude that

$$u(x) = \int\limits_{\partial\Omega}(K\frac{\partial u}{\partial\nu} - u\frac{\partial K}{\partial\nu})dS - \int\limits_{\Omega} K\Delta u\,d\xi$$

for $u\in C^2\cap C^1\left(\overline{\Omega}\right)$.

□

With this formula in hand, observe that if $g\in C^2(\overline{\Omega\times\Omega})\cap C^1(\overline{\Omega\times\Omega})$ is an harmonic function in ξ, then

$$0 = \int\limits_{\partial\Omega}\left(g\frac{\partial}{\partial\nu}u - u\frac{\partial}{\partial\nu}g\right)\,dS - \int\limits_{\Omega} g\Delta u\,d\xi$$

since Green's second identity tells us that

$$\int\limits_{\partial\Omega}(g\frac{\partial}{\partial\nu}u - u\frac{\partial}{\partial\nu}g)\,dS = \int\limits_{\Omega}(g\Delta u - u\Delta g)\,d\xi = \int\limits_{\Omega}(g\Delta u - 0)\,d\xi = \int\limits_{\Omega}(g\Delta u)\,d\xi$$

If we now define $G(x,\xi) = K(x,\xi) + g(x,\xi)$, we get

$$u(x) = \int_{\partial\Omega} (G(x,\xi)\frac{\partial}{\partial\nu}u(\xi) - u(\xi)\frac{\partial}{\partial\nu}G(x,\xi))dS - \int_{\Omega} G(x,\xi)\Delta u(\xi)\, d\xi$$

So let's remember what our problem was again; to find the solution of

$$-\Delta u = f$$

where

$$u = 0$$

on the boundary of Ω.
Applying this data to

$$u(x) = \int_{\partial\Omega} (G(x,\xi)\frac{\partial}{\partial\nu}u(\xi) - u(\xi)\frac{\partial}{\partial\nu}G(x,\xi))dS - \int_{\Omega} G(x,\xi)\Delta u(\xi)\, d\xi$$

we find

$$u(x) = \int_{\partial\Omega} (G(x,\xi)\frac{\partial}{\partial\nu}u(\xi) - 0 \cdot \frac{\partial}{\partial\nu}G(x,\xi))dS + \int_{\Omega} G(x,\xi)f(\xi)\, d\xi$$

and choosing $\frac{\partial}{\partial\nu}g(x,\xi) = -\frac{\partial}{\partial\nu}K(x,\xi)$ for all $\xi \in \partial\Omega$ we finally obtain

$$u(x) = \int_{\Omega} G(x,\xi)f(\xi)\, d\xi$$

Thereby proving that our function G indeed exists.
...Or not. There is still the small matter of proving that we can indeed choose a harmonic function g the way we did. That this is indeed the case can be verified from *"Elliptic Partial Differential Equations of Second Order"* by D. Gilbarg and N.S. Trudinger, page 170, Theorem 8.3. For the general existence of Green's functions I'd recommend the reader to look for *"Linear Differential Operators"* by Cornelius Lanczos, D. Van Nostrand Company, 1961 and to address the problem of the existence of eigenvalues, we refer the reader to the Appendix of the Damned (see Chapter 4).
But we digress. And our true purpose has been waiting for far too long.

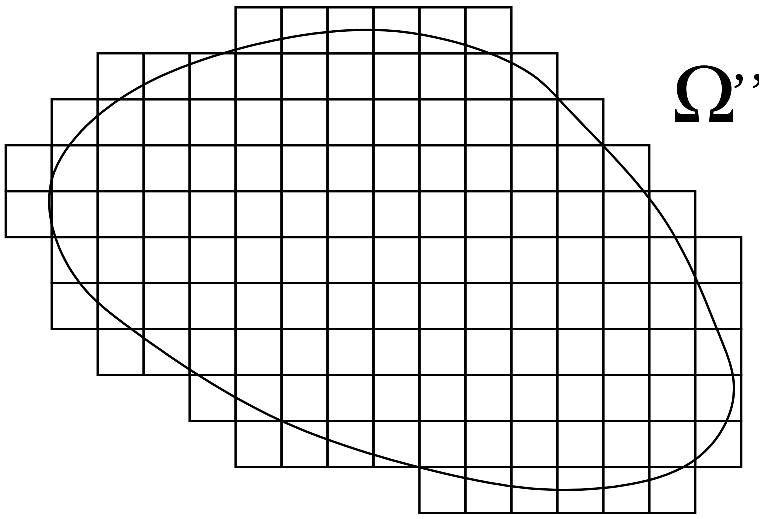
Chapter 2

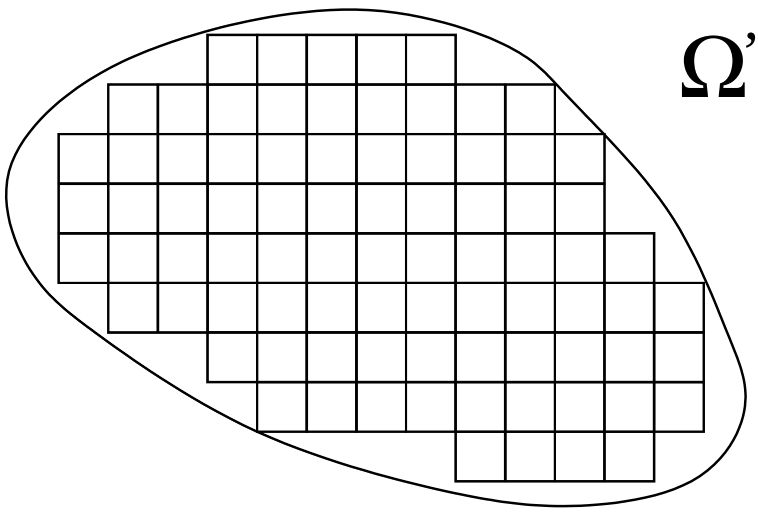
Can One hear the Area of a Drum?

2.1 Introduction

What we'll set out to derive in this chapter is that the eigenvalues λ_n of $L = -\Delta$ behave, for large n, similarly to $(\frac{2^m n\pi^m}{B_m J})^{\frac{2}{m}}$, where J is the area of the region on which our operator is defined, B_m is the volume of the $m-$dimensional unit sphere and $m = 2$, or $m = 3$. Of course, the boundary conditions for functions u within the domain of L are once again that $u(x) = 0$ if $x \in \partial\Omega$.
Our plan of attack will be the following:
First of all, we will work with values $\mu_n = \frac{1}{\lambda_n}$ instead of λ_n, because they represent the eigenvalues of the kernel K corresponding to a differential operator L. In the previous chapter we have learned that our Green's function G can be written as $G = K + g$, where g was harmonic and K was harmonic everywhere in the region except for one point. What we will show is that for differentiable kernels A, $\lim_{n\to\infty} n^{\frac{1}{2}+\frac{1}{m}}\mu_n = 0$, where μ_n is the nth eigenvalue of A and that when K, K' and K'' are kernels with $K = K' + K''$ and $\lim_{n\to\infty} n^s\mu_n = 0$ for μ''_n the nth eigenvalue of K'', then $\lim_{n\to\infty} n^s\mu_n = \lim_{n\to\infty} n^s\mu'_n$ with μ_n and μ'_n the nth eigenvalues of K and K' respectively and s some real number. This we will then use to argue that if we want to calculate, or rather approximate, the nth eigenvalue of G, for $s = \frac{2}{m}$, we might as well ignore g, since it has no influence concerning the limit values whatsoever.
And just when you think things couldn't get any better, it'll turn out that not only it suffices to solve our problem for $G = K$, we may also restrict ourselves to the case that Ω is an m-dimensional cube of volume 1.
Well... not really. But it's still rather amazing. What we will do is indeed solve our problem for the case that Ω is an m-dimensional cube of volume 1. We will then use this information to solve our problem without much trouble for the general case where Ω is just some randomly picked region

in $\mathbb{R}^m$ by approximating Ω by a region Ω'' and a region Ω', $\Omega' \subset \Omega \subset \Omega''$ constructed of m-dimensional cubes.





Trying to find a solution whilst working with Ω' instead of Ω is significantly easier, since the corresponding eigenvalues are the eigenvalues for the case that Ω is one of the cubes of which Ω' is constructed times the number of cubes in Ω'. Using the fact that the less Ω' and Ω differ, the less the corresponding sets of eigenvalues differ, we will be able to use a limit argument to finally validate that we can hear the area of a drum... to which we should probably add that in this chapter it will seem that the above mentioned argument also works for dimensions higher than 3. But this is not quite true. Which means that for higher dimensions we will have to be smart. Or nasty. Whatever gets the job done. The problem is that you probably need a lot of extra theory to make the argument stick for higher dimensions... Oh

well, too bad...
So, to summarize: we are going to find the behaviour of the eigenvalues λ_n of a differential operator L for large n for $L = -\Delta$, with nothing but straightforward reasoning. Onwards! Our destiny is at hand!

2.2 Facts about integral kernels

Again we will assume that $K(x, \xi) = K(\xi, x)$ and that $\int\limits_\Omega \int\limits_\Omega K(x, \xi)^2 \, dx \, d\xi < \infty$. Eigenvalues will be symbolized with the letter μ and ordened in such a way that $|\mu_1| \geq |\mu_2| \geq \ldots$ and for eigenfunctions of K we will use ϕ. If we want to speak of the nth positive eigenvalue, or the corresponding nth eigenfunction, we will speak of $\mu_n^{(+)}$ or $\phi_n^{(+)}$ respectively. In the same way, negative eigenvalues will be written down as $\mu^{(-)}$ and $\phi^{(-)}$ will be its corresponding eigenfunction. Lastly (we'll be needing this) just as in the finite dimensional case, Kf can be written as a linear combination of the eigenfunctions of K, if f lies within the domain of K (see Theorem 4.3.18 in the Appendix of the Damned). Therefore, if we take some fixed x, $K(x, \xi) = \sum\limits_{i=1}^{\infty} c_i(x)\phi_i(\xi)$ for some unknown series of func tions $\{c_k(x)\}_{k=1}^{\infty}$. What we did was saying that if x is fixed, then K is a function that depends only on ξ, making it possible for K to be written as a linear combination of the $\phi_i(\xi)$. Only the constants we use in this linear combination will very likely depend on x, since if we take some other fixed x, the entire linear combination might differ.
Taking the inner product defined by $\langle f, g \rangle = \int\limits_\Omega f(\xi)g(\xi) \, d\xi$ with ϕ_j on both sides gives

$$\int\limits_\Omega K(x, \xi)\phi(\xi) \, d\xi = \sum_{i=1}^{\infty} c_i(x) \int\limits_\Omega \phi_i(\xi)\phi_j(\xi) \, d\xi$$

As we recall from chapter 1, the eigenfunctions are orthonormal, so

$$\int\limits_\Omega K(x, \xi)\phi_j(\xi) \, d\xi = \sum_{i=1}^{\infty} c_i(x) \int\limits_\Omega \phi_i(\xi)\phi_j(\xi) \, d\xi = c_j(x) \cdot 1$$

thus

$$\int\limits_\Omega K(x, \xi)\phi_j(\xi) \, d\xi = c_j(x)$$

leading to

$$\mu_j \phi_j(x) = c_j(x)$$

and because we know that

$$K(x, \xi) = \sum_{i=1}^{\infty} c_i(x)\phi_i(\xi)$$

we get

$$K(x,\xi) = \sum_{i=1}^{\infty} \mu_i \phi_i(x) \phi_i(\xi)$$

Squaring and then integrating by x and ξ over Ω gives, once more because of the orthonormality of the ϕ_i:

$$K^2(x,\xi) = (\sum_{i=1}^{\infty} \mu_i \phi_i(x) \phi_i(\xi))^2$$

thus

$$\begin{aligned} K(x,\xi)^2 &= \sum_{i=1}^{\infty} \mu_i^2 \phi_i(x)^2 \phi_i(\xi)^2 \\ &+ 2 \sum_{i>j} \mu_i \mu_j \phi_i(x) \phi_j(x) \phi_i(\xi) \phi_j(\xi) \end{aligned}$$

so

$$\begin{aligned} \int_\Omega \int_\Omega K(x,\xi)^2 \, dx \, d\xi &= \sum_{i=1}^{\infty} \int_\Omega \int_\Omega \mu_i^2 \phi_i(x)^2 \phi_i(\xi)^2 \, dx \, d\xi \\ &+ \sum_{i>j} \int_\Omega \int_\Omega \mu_i \mu_j \phi_i(x) \phi_j(x) \phi_i(\xi) \phi_j(\xi) \, dx \, d\xi \end{aligned}$$

implying that

$$\begin{aligned} \int_\Omega \int_\Omega K(x,\xi)^2 \, dx \, d\xi &= \sum_{i=1}^{\infty} \mu_i^2 \int_\Omega \phi_i(x)^2 \, dx \int_\Omega \phi_i(\xi)^2 \, d\xi \\ &+ 2 \sum_{i>j} \mu_i \mu_j \int_\Omega \phi_i(x) \phi_j(x) \, dx \int_\Omega \phi_i(\xi) \phi_j(\xi) \, d\xi \end{aligned}$$

meaning that

$$\int_\Omega \int_\Omega K(x,\xi)^2 \, dx \, d\xi = \sum_{i=1}^{\infty} \mu_i^2 \cdot 1 + 2 \sum_{i>j} \mu_i \mu_j \cdot 0 = \sum_{i=1}^{\infty} \mu_i^2$$

giving

$$\int_\Omega \int_\Omega K(x,\xi)^2 \, dx \, d\xi = \sum_{i=1}^{\infty} \mu_i^2$$

So

$$K(x,\xi) = \sum_{i=1}^{\infty} \mu_i \phi_i(x)\phi_i(\xi)$$

and

$$\int_\Omega \int_\Omega K(x,\xi)^2 \, dx \, d\xi = \sum_{i=1}^{\infty} \mu_i^2$$

Summarizing these results gives

Theorem 2.2.1. *If $K(x,\xi)$ is an integral kernel with eigenvalues $\{\mu_n\}_{n=1}^{\infty}$ and corresponding eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$, then*

$$K(x,\xi) = \sum_{i=1}^{\infty} \mu_i \phi_i(x)\phi_i(\xi)$$

and

$$\int_\Omega \int_\Omega K(x,\xi)^2 \, dx \, d\xi = \sum_{i=1}^{\infty} \mu_i^2$$

Now let's get down to business.

Lemma 2.2.2. *If K is a kernel with eigenvalues $\{\mu_n\}_{n=1}^{\infty}$ and eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ and f is some function for which $\int_\Omega f(x)^2 \, dx \le 1$, then*

$$\int_\Omega \int_\Omega K(x,\xi) f(x) f(\xi) \, dx \, d\xi \le \mu_1^{(+)}$$

Proof. We know that for a Hilbert space H with a linear subspace $M = span\{y_i\}_{i=1}^{\infty}$, a span of orthonormal vectors, every $x \in H$ can be uniquely written as $x = y + z$ with $y \in M$ and $z \in M^{\perp}$, where $y = \sum_{i=1}^{\infty} \langle x, y_i \rangle y_i$.

So our function f can be written as

$$f = \sum_{i=1}^{\infty} \langle f, \phi_i \rangle \phi_i + z$$

where $z \in (span\{\phi_i\}_{i=1}^{\infty})^{\perp}$. This implies that

$$\begin{aligned}
&\int_\Omega \int_\Omega K(x,\xi) f(x) f(\xi)\, dx\, d\xi \\
= &\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \int_\Omega \int_\Omega \mu_i \phi_i(x)\phi_i(\xi) \langle f, \phi_j\rangle \langle f, \phi_k\rangle \phi_j(x)\phi_k(\xi)\, dx\, d\xi \\
&+ \sum_{i=1}^{\infty} \mu_i \int_\Omega \int_\Omega \phi_i(x)\phi_i(\xi) z(x) z(\xi)\, dx\, d\xi \\
= &\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \mu_i \int_\Omega \langle f, \phi_j\rangle \phi_i(x)\phi_j(x)\, dx \int_\Omega \langle f, \phi_k\rangle \phi_i(\xi)\phi_k(\xi)\, d\xi \\
&+ \sum_{i=1}^{\infty} \mu_i \int_\Omega \phi_i(x) z(x)\, dx \int_\Omega \phi_i(\xi) z(\xi)\, d\xi \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \mu_i \langle f, \phi_j\rangle \langle \phi_i, \phi_j\rangle \langle f, \phi_k\rangle \langle \phi_i, \phi_k\rangle \\
&+ \sum_{i=1}^{\infty} \mu_i \langle \phi_i, z\rangle \langle \phi_i, z\rangle \\
&= \sum_{i=1}^{\infty} \mu_i \langle f, \phi_i\rangle \langle \phi_i, \phi_i\rangle \langle f, \phi_i\rangle \langle \phi_i, \phi_i\rangle + \sum_{i=1}^{\infty} 0 \cdot 0
\end{aligned}$$

because of the orthonormality of the ϕ_i and z being orthogonal to all ϕ_i. This leads to

$$\begin{aligned}
\int_\Omega \int_\Omega K(x,\xi) f(x) f(\xi)\, dx\, d\xi &= \sum_{i=1}^{\infty} \mu_i \langle f, \phi_i\rangle \langle \phi_i, \phi_i\rangle \langle f, \phi_i\rangle \langle \phi_i, \phi_i\rangle \\
&= \sum_{i=1}^{\infty} \mu_i \langle f, \phi_i\rangle^2 \langle \phi_i, \phi_i\rangle^2 = \sum_{i=1}^{\infty} \mu_i \langle f, \phi_i\rangle^2 \cdot 1^2 \\
&= \sum_{i=1}^{\infty} \mu_i \langle f, \phi_i\rangle^2 \le \sum_{i=1}^{\infty} \mu_1^{(+)} \langle f, \phi_i\rangle^2 \\
&= \mu_1^{(+)} \sum_{i=1}^{\infty} \langle f, \phi_i\rangle^2 \le \mu_1^{(+)} \|f\|^2
\end{aligned}$$

because of Bessel's inequality. Thus

$$\int_\Omega \int_\Omega K(x,\xi) f(x) f(\xi)\, dx\, d\xi \leq \mu_1^{(+)} \|f\|^2 \leq \mu_1^{(+)} \cdot 1 = \mu_1^{(+)}$$

and therefore $\int_\Omega \int_\Omega K(x,\xi) f(x) f(\xi)\, dx\, d\xi \leq \mu_1^{(+)}$

□

Lemma 2.2.3. *If* $k_n(x,\xi) = \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} \Phi_i(x) \Phi_j(\xi)$ *where the* Φ_i *are quadratically integrable nonzero functions on* Ω*, then the first positive eigenvalue of* $K - k_n$ *will not be less than the* $(n+1)^{th}$ *positive eigenvalue of* K*.*

Proof. Define a function $f(x)$ by

$$f(x) = c_1 \phi_1^{(+)}(x) + c_2 \phi_2^{(+)}(x) + \ldots + c_{n+1} \phi_{n+1}^{(+)}(x)$$

where the $c_1, \ldots, c_{n+1}$ are constants chosen in such a way that

$$\int_\Omega f(x) \Phi_j(x)\, dx = 0 \text{ for all } j \in \{1, \ldots, n\}$$

This can always be done, since $f \in span\{\phi_1, \ldots, \phi_{n+1}\}$, which is an $n+1$-dimensional space on account of the ϕ being orthonormal and $span\{\Phi_1, \ldots, \Phi_n\}$ is n-dimensional at most.
Furthermore, we will normalize f by choosing

$$\int_\Omega f(x)^2\, dx = c_1^2 + \ldots + c_{n+1}^2 = 1$$

because until now, f was defined up to a constant.

By construction of f,

$$\int_\Omega k_n(x,\xi) f(\xi)\, d\xi = 0$$

and therefore

$$\begin{aligned}\int_\Omega\int_\Omega (K(x,\xi)-k_n(x,\xi))f(x)f(\xi)\,dx\,d\xi &= \int_\Omega\int_\Omega K(x,\xi)f(x)f(\xi)\,dx\,d\xi \\ &= \mu_1^{(+)}c_1^2+\ldots+\mu_{n+1}^{(+)}c_{n+1}^2 \\ &\geq \mu_{n+1}^{(+)}c_1^2+\ldots+\mu_{n+1}^{(+)}c_{n+1}^2 \\ &= (c_1^2+\ldots+c_{n+1}^2)\mu_{n+1}^{(+)} \\ &= 1\cdot\mu_{n+1}^{(+)} = \mu_{n+1}^{(+)}\end{aligned}$$

So now we have that $\int_\Omega\int_\Omega (K(x,\xi)-k_n(x,\xi))f(x)f(\xi)\,dx\,d\xi \geq \mu_{n+1}^{(+)}$.
Because of Lemma 2.2.2, $\mu_1^{(+)\prime} \geq \int_\Omega\int_\Omega (K(x,\xi)-k_n(x,\xi))f(x)f(\xi)\,dx\,d\xi$ where $\mu_1^{(+)\prime}$ is the first positive eigenvalue of $K-k_n$ from which can be deduced that $\mu_1^{(+)\prime} \geq \int_\Omega\int_\Omega (K(x,\xi)-k_n(x,\xi))f(x)f(\xi)\,dx\,d\xi \geq \mu_{n+1}$ and we are done.

□

From this moment on, the symbol k_n will always represent a bilinear form as described in Lemma 2.2.3 unless stated otherwise.
But let us move on to the next theorem:

Theorem 2.2.4. *If $K = K' + K''$ then*

1. $\mu_{n+m+1}^{(+)} \leq \mu_{m+1}^{(+)\prime} + \mu_{n+1}^{(+)\prime\prime}$
2. $\mu_{m+n+1}^{(-)} \geq \mu_{m+1}^{(-)\prime} + \mu_{n+1}^{(-)\prime\prime}$
3. $|\mu_{m+n+1}| \leq |\mu'_{m+1}| + |\mu''_{n+1}|$

Proof. We have already learned from Lemma 2.2.2 that for each and every f for which $\int_\Omega f(x)^2\,dx \leq 1$ holds, the following is true:

$$\int_\Omega\int_\Omega K'(x,\xi)f(x)f(\xi)\,dx\,d\xi \leq \mu_1^{(+)\prime}$$

and

$$\int_\Omega\int_\Omega K''(x,\xi)f(x)f(\xi)\,dx\,d\xi \leq \mu_1^{(+)\prime\prime}$$

and therefore

$$\begin{aligned}\int_\Omega\int_\Omega K(x,\xi)f(x)f(\xi)\,dx\,d\xi &= \int_\Omega\int_\Omega K'(x,\xi)f(x)f(\xi)\,dx\,d\xi \\ &+ \int_\Omega\int_\Omega K''(x,\xi)f(x)f(\xi)\,dx\,d\xi \\ &\le \mu_1^{(+)\prime} + \mu_1^{(+)\prime\prime}\end{aligned}$$

by which

$$\int_\Omega\int_\Omega K(x,\xi)f(x)f(\xi)\,dx\,d\xi \le \mu_1^{(+)\prime} + \mu_1^{(+)\prime\prime}$$

So if we choose $f = \phi_1^{(+)}$, then because of the orthonormality of the eigenfunctions and because $K(x,\xi) = \sum_{i=1}^{\infty} \mu_i\phi_i(x)\phi_i(\xi)$,

$$\mu_1^{(+)} = \int_\Omega\int_\Omega K(x,\xi)\phi_1^{(+)}(x)f(\xi)\,dx\,d\xi \le \mu_1^{(+)\prime\prime} + \mu_1^{(+)\prime}$$

implying

$$\mu_1^{(+)} \le \mu_1^{(+)\prime} + \mu_1^{(+)\prime\prime}$$

Now define

$$[K]_n^+(x,\xi) = \mu_1^{(+)}\phi_1^{(+)}(x)\phi_1^{(+)}(\xi)+\mu_2^{(+)}\phi_2^{(+)}(x)\phi_2^{(+)}(\xi)+\ldots+\mu_n^{(+)}\phi_n^{(+)}(x)\phi_n^{(+)}(\xi)$$

and apply

$$\mu_1^{(+)} \le \mu_1^{(+)\prime} + \mu_1^{(+)\prime\prime}$$

with
$K' - [K']_m^+$ instead of K'
and
$K'' - [K'']_n^+$ instead of K''
giving
$K - [K']_m^+ - [K'']_n^+ = (K' - [K']_m^+) + (K'' - [K'']_n^+)$
Using Lemma 2.2.3 now shows, because $[K']_m^+ + [K'']_n^+$ represents a bilinear form of $m+n$ linearly independent functions, that

$$\mu_{m+n+1}^{(+)} \le \nu_1^{(+)}$$

where $\nu_1^{(+)}$ is the first positive eigenvalue of $K - [K']_m^+ - [K'']_n^+$ and if we define the normalised eigenfunction corresponding to ν_1 to be $\Psi(x)$ then

$$\int_\Omega \int_\Omega (K'(x,\xi) - [K']_m^+(x,\xi)\Psi(x)\Psi(\xi)\,dx\,d\xi \leq \mu_{m+1}^{(+)}$$

and

$$\int_\Omega \int_\Omega (K''(x,\xi) - [K'']_n^+(x,\xi)\Psi(x)\Psi(\xi)\,dx\,d\xi \leq \mu_{n+1}^{(+)}$$

because of Lemma 2.2.2. So:

$$\begin{aligned}
\mu_{m+n+1}^{(+)} \leq \nu_1^{(+)} &= \int_\Omega \int_\Omega (K(x,\xi) - [K']_m^+ - [K'']_n^+(x,\xi)\Psi(x)\Psi(\xi)\,dx\,d\xi \\
&= \int_\Omega \int_\Omega (K'(x,\xi) - [K']_m^+(x,\xi)\Psi(x)\Psi(\xi)\,dx\,d\xi \\
&+ \int_\Omega \int_\Omega (K''(x,\xi) - [K'']_n^+(x,\xi)\Psi(x)\Psi(\xi)\,dx\,d\xi \\
&\leq \mu_{m+1}^{(+)} + \mu_{n+1}^{(+)}
\end{aligned}$$

The second statement can be verified by using statement 1 and applying it to $-K = (-K') + (-K'')$.

Finally, let us prove the third statement.
The idea will be to show that from the $m+n+1$ eigenvalues $\mu_1, \ldots, \mu_{m+n+1}$ there can be at most $m+n$ eigenvalues μ for which $|\mu| > |\mu'_{m+1}| + |\mu''_{n+1}|$ holds and therefore, because $\{|\mu_i|\}_{i=1}^\infty$ is a decreasing sequence, it must be $\mu = \mu_{m+n+1}$ for which $|\mu| \leq |\mu'_{m+1}| + |\mu''_{n+1}|$.
To understand this, assume that there exist $m^{(+)}$ positive and $m^{(-)}$ negative eigenvalues among $\mu'_1, \ldots, \mu'_m$ and in the same way $n^{(+)}$ positive and $n^{(-)}$ negative eigenvalues among $\mu''_1, \ldots, \mu''_n$. This implies that by the first statement, there can be at most $m^{(+)} + n^{(+)}$ positive eigenvalues of K that are larger than $|\mu'_{n+1}| + |\mu''_{m+1}|$ (because of the decreasing order of $\{|\mu'_i|\}_{i=1}^\infty$) and by the same reasoning the second statement implies that there can be at most $m^{(-)} + n^{(-)}$ negative eigenvalues of K that are in absolute values larger than $|\mu'_{n+1}| + |\mu''_{m+1}|$. From this we easily deduce that there are at most $m^{(+)} + n^{(+)} + m^{(-)} + n^{(-)} = m+n$ eigenvalues μ among $\mu_1, \ldots, \mu_{m+n+1}$ for which $|\mu| > |\mu'_{m+1}| + |\mu''_{n+1}|$ holds, thereby revealing, as promised, that $|\mu_{m+n+1}| \leq |\mu'_{m+1}| + |\mu''_{n+1}|$.

□

Theorem 2.2.5. *If K is an integral kernel and k_n is a bilinear form of n linearly independent functions, then*

1. *The m^{th} positive eigenvalue of $K - k_n$ is not smaller than the $(n+m)^{th}$ positive eigenvalue of K.*

2. *The m^{th} negative eigenvalue of $K - k_n$ is not greater than the $(n+m)^{th}$ negative eigenvalue of K.*

3. *The absolute value of the m^{th} eigenvalue of $K - k_n$ is not smaller than the absolute value of the $(n+m)^{th}$ eigenvalue of K.*

Proof. Choose $K' = K - k_n$ and $K'' = k_n$ and use Theorem 2.2.4, only writing $m-1$ instead of m. Furthermore, make use of the fact that the $(n+1)^{th}$ eigenvalue of k_n is equal to zero, because k_n has at most n eigenvalues. The results follow instantly.

□

On a side note: if we let $\nu_1, \nu_2, \ldots$ be the eigenvalues of $K - k_n$, then

$$\int_\Omega \int_\Omega (K(x,\xi) - k_n(x,\xi))^2 \, dx \, d\xi = \sum_{i=1}^{\infty} \nu_i^2$$

The third part of Theorem 2.2.5 now implies that $|\nu_i| \geq |\mu_{n+i}|$, which means that

$$\begin{aligned} \int_\Omega \int_\Omega (K(x,\xi) - k_n(x,\xi))^2 \, dx \, d\xi &= \sum_{i=1}^{\infty} \nu_i^2 \\ &\geq \sum_{i=1}^{\infty} \mu_{n+i}^2 \end{aligned}$$

so whatever k_n you choose, $\int_\Omega \int_\Omega (K(x,\xi) - k_n(x,\xi))^2 \, dx \, d\xi$ can never be smaller than $\sum_{i=1}^{\infty} \mu_{n+i}^2$.

We have now proven

Theorem 2.2.6. $\int_\Omega \int_\Omega (K(x,\xi) - k_n(x,\xi))^2 \, dx \, d\xi \geq \sum_{i=1}^{\infty} \mu_{n+i}^2$ *for every imaginable bilinear form k_n.*

And on we go on our merry little way, finding the first of the rather exquisite theorems that will make calculating $\lim_{n\to\infty} n^2 m\mu_n$ so much easier, but not before giving the following definition:

Definition 2.2.7. *A function $f : \Omega \to \mathbb{R}$ is said to be piecewise continuous if every curve lying on the m-dimensional surface of the graph of f is continuous in all but a finite number of points.*

As for our theorem:

Theorem 2.2.8. *Let $p(x)$ be a piecewise continuous function for which $0 \leq p_0 \leq |p(x)| \leq P_0$ where p_0 and P_0 are constants. Define K' by $K' = K(x,\xi)p(x)p(\xi)$ Then*

$$\mu^{(+)} p_0^2 \leq \mu_n^{(+)\prime} \leq \mu_n^{(+)} P_0^2$$

Proof. If the quadratic integral over $f(x)$ is smaller than or equal to 1, then the quadratic integral over $\frac{p(x)f(x)}{P_0}$ must be smaller than or equal to 1 as well and so

$$\int_\Omega \int_\Omega (K(x,\xi) - \sum_{i=1}^{n} \mu_i^{(+)} \phi_i^{(+)}(x)\phi_i^{(+)}(\xi))p(x)f(x)p(\xi)f(\xi)\, dx\, d\xi =$$

$$P_0^2 \int_\Omega \int_\Omega (K(x,\xi) - \sum_{i=1}^{n} \mu_i^{(+)} \phi_i^{(+)}(x)\phi_i^{(+)}(\xi)) \frac{p(x)f(x)}{P_0} \frac{p(\xi)f(\xi)}{P_0}\, dx\, d\xi \leq P_0^2 \mu_{n+1}^{(+)}$$

because of Lemma 2.2.2.
From the proof of Lemma 2.2.3 we learned that for any kernel K

$$\int_\Omega \int_\Omega (K(x,\xi) - k_n(x,\xi))f(x)f(\xi)\, dx\, d\xi \geq \mu_{n+1}^{(+)}$$

for a suitably chosen f with $\|f\|^2 \leq 1$. Thus, if we choose $K(x,\xi)p(x)p(\xi)$ instead of $K(x,\xi)$ and $k_n(x,\xi) = \sum_{i=1}^{\infty} \mu_i^{(+)} \phi_i^{(+)}(x)\phi_i^{(+)}(\xi))p(x)p(\xi)$, it must be true that the $(n+1)^{th}$ eigenvalue of $K(x,\xi)p(x)p(\xi)$ is smaller than, or equal to $\int_\Omega \int_\Omega (K(x,\xi) - k_n(x,\xi))p(x)f(x)p(\xi)f(\xi)\, dx\, d\xi$. Therefore, for this specific f, we get that

$$\mu_{n+1}^{(+)\prime} \leq \int_\Omega \int_\Omega (K(x,\xi) - k_n(x,\xi))p(x)f(x)p(\xi)f(\xi)\,dx\,d\xi \leq P_0^2 \mu_{n+1}^{(+)}$$

and so

$$\mu_{n+1}^{(+)\prime} \leq P_0^2 \mu_{n+1}^{(+)}$$

Now it remains to be shown that $p_0^2\mu_{n+1}^{(+)} \leq$
$mu_{n+1}^{(+)\prime}$.
What we will do is constructing a function q that has similar properties to p, for which $K(x,\xi) = K'(x,\xi)q(x)q(\xi)$, so that we can use $\mu_{n+1}^{(+)\prime} \leq P_0^2\mu_{n+1}^{(+)}$ for the case that we take $K'(x,\xi)q(x)q(\xi)$ instead of $K(x,\xi)p(x)p(\xi)$ and K instead of K', giving $\mu_{n+1}^{(+)} \leq Q_0^2\mu_{n+1}^{(+)\prime}$. If we now define q by

$$q(x) = \begin{cases} \frac{1}{p(x)} & if \quad p(x) \neq 0 \\ 0 & if \quad p(x) = 0 \end{cases}$$

So if p is never equal to zero, then the maximum of $|q|$ is $\frac{1}{p_0}$, so $Q_0 = \frac{1}{p_0}$, concluding that $\mu_{n+1}^{(+)} \leq \frac{1}{p_0^2}\mu_{n+1}^{(+)\prime}$. Multiplying both sides by $\frac{1}{p_0^2}$ then finally gives

$$p_0^2\mu_{n+1}^{(+)} \leq \mu_{n+1}^{(+)\prime}$$

If $p_0 = 0$, then the statement becomes trivial, since the n^{th} positive eigenvalue is always larger than zero. □

Theorem 2.2.9. *If $\Omega' \subset \Omega$, then the n^{th} positive eigenvalue of $K(x,\xi)$ for which $x,\xi \in \Omega'$ is smaller than or equal to the n^{th} positive eigenvalue of $K(x,\xi)$ for which $x,\xi \in \Omega$. Similarly, the n^{th} negative eigenvalue of $K(x,\xi)$ for which $x,\xi \in \Omega'$ greater than or equal to the n^{th} negative eigenvalue of $K(x,\xi)$ for which $x,\xi \in \Omega$.*

Proof. If we apply Theorem 2.2.8 for

$$p(x) = \begin{cases} 1 & if \quad x \in \Omega' \\ 0 & if \quad x \notin \Omega' \end{cases}$$

then the n^{th} positive eigenvalue of $K(x,\xi)$ for which $x,\xi \in \Omega'$ must be smaller than or equal to the n^{th} positive eigenvalue of $K(x,\xi)$ for which $x,\xi \in \Omega$.
Applying this statement to $-K$ instead of K will provide us with the second statement of the n^{th} negative eigenvalue of $K(x,\xi)$ for which $x,\xi \in \Omega'$ always being greater than or equal to the n^{th} negative eigenvalue of $K(x,\xi)$ for which $x,\xi \in \Omega$.

□

2.3 Limit behaviour of eigenvalues.

Remember that our goal was proving that one could hear the area of a drum, or rather, that $\lim_{n\to\infty} \frac{n}{\lambda_n} = \frac{|\Omega|}{4\pi}$, or equivalently, $\lim_{n\to\infty} n\mu_n = \frac{|\Omega|}{4\pi}$. A very important tool in verifying this will be the following theorem:

Theorem 2.3.1. *If K' and K'' are kernels for which*

$$\lim_{n\to\infty} n\mu_n^{(+)\prime} = 1$$

and

$$\lim_{n\to\infty} n\mu_n'' = 0$$

then the positive eigenvalues of $K = K' + K''$ will have asymptotically the same distribution as K', meaning that

$$\lim_{n\to\infty} n\mu_n^{(+)} = 1$$

Proof. Let h be some constant positive integer. Theorem 2.2.4 teaches us that

$$\mu_{hn+j+n}^{(+)} \leq \mu_{hn+j}^{(+)\prime} + \mu_{n+1}^{(+)\prime}$$

implying

$$\mu_{(h+1)n+j}^{(+)} \leq \mu_{hn+j}^{(+)\prime} + \mu_{n+1}^{(+)\prime\prime}$$

thus

$$(h+1)n\mu_{(h+1)n+j}^{(+)} \leq (h+1)n\mu_{hn+j}^{(+)\prime} + (h+1)n\mu_{n+1}^{(+)\prime}$$

so

$$\limsup_{n\to\infty}(h+1)n\mu_{(h+1)n+j}^{(+)} \leq \limsup_{n\to\infty}(h+1)n\mu_{hn+j}^{(+)\prime} + \limsup_{n\to\infty}(h+1)n\mu_{n+1}^{(+)\prime\prime}$$

meaning that

$$\limsup_{n\to\infty} n\mu_n^{(+)} \leq \limsup_{n\to\infty} \frac{(h+1)n}{hn+j}(hn+j)\mu_{hn+j}^{(+)\prime} + \limsup_{n\to\infty} \frac{(h+1)n}{n+1}(n+1)\mu_{n+1}^{(+)\prime\prime}$$

so

$$\limsup_{n\to\infty} n\mu_n^{(+)} \leq \limsup_{n\to\infty} \frac{h+1}{h}(hn+j)\mu_{hn+j}^{(+)} + \limsup_{n\to\infty}(h+1)\cdot 1\cdot (n+1)\mu_{n+1}^{(+)\prime\prime}$$

therefore

$$\limsup_{n\to\infty} n\mu_n^{(+)} \leq \limsup_{n\to\infty} \frac{h+1}{h} n\mu_n^{(+)} + \limsup_{n\to\infty}(h+1)n\mu_n^{(+)\prime\prime}$$

and so

$$\limsup_{n\to\infty} n\mu_n^{(+)} \leq \frac{h+1}{h} \cdot 1 + (h+1) \cdot 0$$

finally giving

$$\limsup_{n\to\infty} n\mu_n^{(+)} \leq \frac{h+1}{h}$$

If we now use Theorem 2.2.4 for $K' = K + (-K'')$ we get by the same argument

$$\mu_{(h+1)n+j}^{(+)\prime} \leq \mu_{hn+j}^{(+)} - \mu_{n+1}^{(-)\prime\prime}$$

and

$$\liminf_{n\to\infty} n\mu_n^{(+)} \geq \frac{h}{h+1}$$

Because h can be any positive integer

$$\lim_{n\to\infty} n\mu_n^{(+)} = 1$$

□

Observe that, by using essentially the same proof, we can generalise this theorem by inserting an n^s where an n is written, as long as $s \in \mathbb{R}_{>0}$. (Yes, I know that s might as well be smaller than or equal to zero, but we know that the eigenvalues decrease to zero, so assuming $s \leq 0$ would be rather silly). To make this generalisation exact:

Corollary 2.3.2. *If K' and K'' are kernels for which, for $s > 0$*

$$\lim_{n\to\infty} n^s \mu_n^{(+)\prime} = 1$$

and

$$\lim_{n\to\infty} n^s \mu_n'' = 0$$

then the positive eigenvalues of $K = K + K''$ will have asymptotically the same distribution as K', meaning that

$$\lim_{n\to\infty} n^s \mu_n^{(+)} = 1$$

We know from the previous chapter that our G can be written as $G = K + g$, where g is a harmonic function. What we will show is that for differentiable kernels defined on a set $\Omega \times \Omega \subset \mathbb{R}^2 \times \mathbb{R}^2$ $\lim_{n\to\infty} n\mu_n = 0$, so that by Corollary 2.3.2 we can conclude that it suffices to calculate $\lim_{n\to\infty} n\mu_n = 0$, where μ_n is the n^{th} eigenvalue of K. But before we can actually use this, there is still some work to be done. So without further ado:

Theorem 2.3.3. *Assume that $\Omega = [a, b]$ with $a, b \in \mathbb{R}$ constants. If $K(x, \xi)$ is twice continuously differentiable with respect to ξ for $\xi \in \Omega$, then $\lim_{n\to\infty} n^{\frac{3}{2}} \mu_n^{(+)} = 0$.*

Proof. Dividing both the intervals $a \leq x \leq b$ and $a \leq y \leq b$ into n equal parts, the square $[a, b] \times [a, b]$ is split into n^2 smaller squares. We will choose n large enough so that $\frac{\partial K}{\partial t}$ differs at most ϵ_n from one square to an adjacent one, where $\epsilon_n > 0$ and $\lim_{n\to\infty} \epsilon_n = 0$. We desire to approximate K by its first order Taylor polynomial around some point (x_0, ξ_0).

So let us approximate K by its first order Taylor polynomial around points in the n^2 little squares in such a way, that they are line symmetric with respect to the diagonal $x = \xi$. Then, invoking

$$K(x,\xi) = \frac{1}{0!}K(x_0,\xi_0) + \frac{1}{1!}((x - x_0)\frac{\partial}{\partial x} + (\xi - \xi_0)\frac{\partial}{\partial \xi})K(x,\xi)|_{(x_0,\xi_0)} \quad (2.1)$$

$$+ \frac{1}{2!}((x - x_0)\frac{\partial}{\partial x} + (\xi - \xi_0)\frac{\partial}{\partial \xi})^2 K(x,\xi)|_{(x_0,\xi_0)} + \dots \quad (2.2)$$

there must exist constants A, B and C in such a way that on each of those squares

$$|K(x,\xi) - (A + B(x - x_0) + C(\xi - \xi_0)| \leq \frac{1}{2!} \cdot \epsilon_n \frac{b - a}{n}$$

Let's now define $\Phi_h(x)$, for $h \in \{1, \dots, n\}$ to be equal to 1 in the h^{th} interval of length $\frac{b-a}{n}$ and equal to 0 elsewhere and define $\Phi_{n+h}(x)$, for $h \in \{1, \dots, n\}$ to be equal to x in the h^{th} interval of length $\frac{b-a}{n}$ and equal to 0 elsewhere. Then we construct a bilinear kernel $k_{2n} = \sum_{i=1}^{2n}\sum_{j=1}^{2n} k_{ij}\Phi_i(x)\Phi_j(\xi)$ with these $2n$ functions so that

$$|K(x,\xi) - k_{2n}(x,\xi)| \leq \frac{2\epsilon_n}{n}(b - a)$$

and thus

$$\int_a^b \int_a^b (K(x,\xi) - k_{2n}(x,\xi))^2 \, dx \, d\xi \leq \frac{4\epsilon_n^2}{n^2}(b - a)^4$$

To understand why k_n has the properties we desire it to have, we note that we can choose k_{ij} for $1 \leq i \leq n$ and $1 \leq j \leq n$ in such a way that for these i, j, the terms $k_{ij}\Phi_i(x)\Phi_j(\xi)$ provide us with the constant terms A on each of the $\frac{b-a}{n}$ by $\frac{b-a}{n}$ squares, for $n + 1 \leq i \leq 2n$, $1 \leq j \leq n$ they will give the terms Bx, for $n+1 \leq j \leq 2n$, $1 \leq i \leq n$ they will give the terms $C\xi$ and for $n+1 \leq i \leq 2n$, $n+1 \leq j \leq 2n$, we define $k_{ij} = 0$.

We have already proved that whatever k_n you choose, $\int\limits_\Omega \int\limits_\Omega (K(x,\xi) - k_n(x,\xi))^2 \, dx \, d\xi$ can never be smaller than $\sum\limits_{i=1}^{\infty} \mu_{n+i}^2$ so applying this fact to this case, we get

$$\mu_{2n+1}^2 + \mu_{2n+2}^2 + \ldots \leq \int_a^b \int_a^b (K(x,\xi) - k_{2n}(x,\xi))^2 \, dx \, d\xi \leq \frac{4\epsilon_n^2}{n^2}(b-a)^4$$

meaning

$$n^2(\mu_{2n+1}^2 + \mu_{2n+2}^2 + \ldots) \leq 4\epsilon_n^2 (b-a)^4$$

implying that

$$\lim_{n\to\infty} n^2(\mu_{2n+1}^2 + \mu_{2n+2}^2 + \ldots) = 0$$

And now we can conclude that because of

$$\mu_{2n+1}^2 + \mu_{2n+2}^2 + \ldots \geq \mu_{2n+1}^2 + \mu_{2n+2}^2 + \ldots + \mu_{2n+n} \geq \\ \mu_{3n}^2 + \mu_{3n}^2 + \ldots + \mu_{3n}^2 = n\mu_{3n}^2$$

that

$$0 = \lim_{n\to\infty} n^2(\mu_{2n+1}^2 + \mu_{2n+2}^2 + \ldots) \geq \lim_{n\to\infty} n^2 \cdot n\mu_{3n}^2$$

giving

$$\lim_{n\to\infty} n^2 \cdot n\mu_{3n}^2 = 0$$

and

$$\lim_{n\to\infty} n^3 \mu_{3n}^2 = 0$$

so

$$\lim_{n\to\infty} (3n)^3 \mu_{3n}^2 = 0$$

meaning that

$$\lim_{n\to\infty} (n)^3 n^{l+2} \mu_{(lm+1)n^l}^2 \mu_n^2 = 0$$

and consequently

$$\lim_{n\to\infty} (n)^{\frac{3}{2}} \mu_n = 0$$

□

Basically using the same reasoning, we can generalize this theorem for Ω to be lying in $\mathbb{R}^2$.

First we will assume that Ω consists of a finite number, m, of squares. Each of the m squares will be cut into n^2 smaller squares, giving us m^2n^4 4-dimensional cubes in $\Omega \times \Omega$, the set on which K is defined. On each of the m^2n^4 cubes within $\Omega \times \Omega$, we will approximate K by $A + Bx_1 + Cx_2 + D\xi_1 + E\xi_2$ around points that are chosen symmetrically with respect to the 'diagonal' $(x_1, x_2) = (\xi_1, \xi_2)$ where $(x_1, x_2) \in \Omega$ and A, B, C, D, E are properly chosen constants (not to be necessarily having the same value on each cube) an. The error doesn't really differ from the one-dimensional case, being still smaller than or equal to some constant times $\frac{\epsilon_n}{n}$, since in this case our approximation becomes

$$
\begin{aligned}
K(x,\xi) = \frac{1}{0!}K(x_0,\xi_0)+ \\
\frac{1}{1!}\Big((x_1 - x_{10})\frac{\partial}{\partial x_1} + (x_2 - x_{20})\frac{\partial}{\partial x_2}+ \\
(\xi_1 - \xi_{10})\frac{\partial}{\partial \xi_1} + (\xi_2 - \xi_{20})\frac{\partial}{\partial \xi_2})\Big)K(x,\xi)|_{(x_{10},x_{20},\xi_{10},\xi_{20})}+ \\
\frac{1}{2!}\Big((x_1 - x_{10})\frac{\partial}{\partial x_1} + (x_2 - x_{20})\frac{\partial}{\partial x_2}+ \\
(\xi_1 - \xi_{10})\frac{\partial}{\partial \xi_1} + (\xi_2 - \xi_{20})\frac{\partial}{\partial \xi_2})\Big)^2 K(x,\xi)|_{(x_{10},x_{20},\xi_{10},\xi_{20})} + \ldots
\end{aligned}
$$

around some point $(x_{10}, x_{20}, \xi_{10}, \xi_{20}) \in \Omega \times \Omega$.

The only real difference with the one-dimensional case is that this time we will be needing $3mn^2$ functions Φ instead of $2n$.
Define $\Phi_{i,k}(x_1, x_2) = 1$ for $1 \le i \le n^2$,
$\Phi_{n^2+i,k}(x_1, x_2) = x_1$ for $1 \le i \le n^2$ and $\Phi_{2n^2+i,k}(x_1, x_2) = x_2$ for $1 \le i \le n^2$, $1 \le k \le m$ and you will be able to repeat the same argument as before.
Only now we will get

$$\int_\Omega \int_\Omega (K(x,\xi) - k_{3mn^2}(x,\xi))^2\, dx\, d\xi \le C \cdot \frac{\epsilon_n}{n^2}$$

where C is some positive constant, giving

$$
\begin{aligned}
\int_\Omega \int_\Omega (K(x,\xi) - k_{3mn^2}(x,\xi))^2\, dx\, d\xi &\ge \mu^2_{(3m+1)n^2} + \ldots + \mu^2_{(3m+1)n^2} \\
&\ge n^2 \mu^2_{(3m+1)n^2}
\end{aligned}
$$

from which follows that

$$n^2\mu^2_{(3m+1)n^2} \leq \int_\Omega \int_\Omega (K(x,\xi) - k_{3mn^2}(x,\xi))^2\,dx\,d\xi \leq C \cdot \frac{\epsilon_n}{n^2}$$

and thus

$$n^2\mu^2_{(3m+1)n^2} \leq C \cdot \frac{\epsilon_n}{n^2}$$

meaning that

$$n^4\mu^2_{(3m+1)n^2} \leq C \cdot \epsilon_n$$

so

$$\lim_{n\to\infty} n^4\mu^2_{(3m+1)n^2} = 0$$

and therefore

$$\lim_{n\to\infty} n^4\mu^2_{n^2} = 0$$

implying

$$\lim_{n\to\infty} n^2\mu_{n^2} = 0$$

and finally

$$\lim_{n\to\infty} n\mu_n = 0$$

If we choose Ω not to be consisting of squares alone but having a slightly more exotic shape, such as with partly curved boundaries, than still there must exist a set Ω' which does consist of squares alone, for which $\Omega \subset \Omega'$ and $|\Omega'| - |\Omega| < \epsilon$ for $\epsilon > 0$. By Theorem 3, which stated that if $p(x)$ be a piecewise continuous function for which $0 \leq p_0 \leq |p(s)| \leq P_0$ where p_0 and P_0 are constants and we define K' by $K' = K(x,\xi)p(x)p(\xi)$, then

$$\mu_n^{(+)}p_0^2 \leq \mu_n^{(+)\prime} \leq \mu_n^{(+)}P_0^2$$

we can show that in this case it is still true that

$$\lim_{n\to\infty} n\mu_n = 0$$

simply by choosing $p(x) = I_{\Omega'}$.
So if $\Omega \subset \mathbb{R}^2$ then

$$\lim_{n\to\infty} n\mu_n = 0$$

But what if $\Omega \subset \mathbb{R}^l$ for $l \geq 2$? No problem. The argument is in essence identical to the technique used for the case that $k = 2$. But this time we will be using m l-dimensional cubes of volume $\frac{1}{n^l}$ and since we will now approximate K by

$$A_1 + B_1x_1 + B_2x_2 + B_3x_3 + \ldots + B_lx_l + \\ C_1\xi_1 + C_2\xi_2 + C_3\xi_3 + \ldots + C_l\xi_l$$

we will work with a bilinear form k_{lmn^l}, so we will get

$$\begin{aligned} C \cdot \frac{\epsilon_n}{n^2} &\geq \int_\Omega \int_\Omega (K(x,\xi) - k_{lmn^l})^2 \, dx \, d\xi \\ &\geq \mu^2_{(lm+1)n^l} + \ldots + \mu^2_{(lm+1)n^l} \\ &= n^l \mu^2_{(lm+1)n^l} \end{aligned}$$

thus

$$C\epsilon_n \geq n^{l+2} \mu^2_{(lm+1)n^l}$$

and therefore

$$\lim_{n\to\infty} n^{l+2} \mu^2_{(lm+1)n^l} = 0$$

Multiplying by $(lm+1)^{\frac{l+2}{l}}$ gives

$$\lim_{n\to\infty} \left((lm+1)n^l\right)^{\frac{l+2}{l}} \mu^2_{(lm+1)n^l}$$

which can be rewritten as

$$\lim_{n\to\infty} n^{\frac{l+2}{l}} \mu_n^2 = 0$$

and finally gives

$$\lim_{n\to\infty} n^{\frac{l+2}{2l}} \mu_n = 0$$

which can be summarized as the following corollary:

Corollary 2.3.4. *If $\Omega \subset \mathbb{R}^m$ is bounded and $K(x,\xi)$ is twice continuously differentiable to ξ for $\xi \in \Omega$, then*

$$\lim_{n\to\infty} n^{\frac{1}{2}+\frac{1}{m}} \mu_n = 0$$

One more lemma and one final theorem are needed before we can move on to the next section:

Lemma 2.3.5. *For any continuously differentiable function u on an $m-$dimensional cube C with sides of length l,*

$$\int_C u^2 \, dx \leq \frac{l^2}{\pi^2} \int_C |\nabla u|^2 \, dx$$

if $\int_C u(x) \, dx = 0$.

Proof. u Is squarely integrable and therefore has a Fourier series. In the two dimensional case this series can be written as

$$\begin{aligned}
a_{00} &+ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_{nm}\sin(\frac{n\pi}{l}x_1)\cos(\frac{m\pi}{l}x_2) \\
&+ c_{nm}\cos(\frac{n\pi}{l}x_1)\sin(\frac{m\pi}{l}x_2) + d_{nm}\sin(\frac{n\pi}{l}x_1)\sin(\frac{m\pi}{l}x_2) \\
&+ \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} e_{nm}\cos(\frac{n\pi}{l}x_1)\cos(\frac{m\pi}{l}x_2) \\
&+ \sum_{m=1}^{\infty} f_m\cos(\frac{m\pi}{l}x_2)
\end{aligned}$$

Because $\int\limits_C u(x)\,dx = 0$, $a_{00} = 0$. Now define

$$v(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_{nm}\sin(\frac{n\pi}{l}x_1)\cos(\frac{m\pi}{l}x_2)$$

Then

$$\begin{aligned}
\int\limits_C v(x)^2\,dx &= \int\limits_0^l\int\limits_0^l v(x_1,x_2)^2\,dx_1\,dx_2 \\
&= \int\limits_0^l\int\limits_0^l \left(\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} b_{nm}\sin(\frac{n\pi}{l}x_1)\cos(\frac{m\pi}{l}x_2)\right)^2 \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int\limits_0^l\int\limits_0^l b_{nm}^2\sin^2(\frac{n\pi}{l}x_1)\cos^2(\frac{m\pi}{l}x_2)\,dx_1\,dx_2
\end{aligned}$$

because $\int_0^l \sin(\frac{n\pi}{l}x_1)\sin(\frac{k\pi}{l}x_1) = 0$ voor $k \neq n$ en $\int_0^l \cos(\frac{m\pi}{l}x_1)\cos(\frac{k\pi}{l}x_1) = 0$ voor $k \neq m$. Proceeding:

$$\begin{aligned}
\int\limits_C v(x)^2\,dx &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} b_{nm}^2\int\limits_0^l \sin^2(\frac{n\pi}{l}x_1)\,dx_1\int\limits_0^l \cos^2(\frac{m\pi}{l}x_2)\,dx_2 \\
&= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} b_{nm}^2(\frac{l}{2}-0)(\frac{l}{2}+0) \\
&= \frac{l^2}{4}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} b_{nm}^2
\end{aligned}$$

By the same reasoning,

$$\int\limits_C |\nabla v|^2\,dx = \frac{\pi^2}{4}\sum_{n=1}\sum_{m=0}^{\infty} b_{nm}^2(n^2+m^2)$$

implying that

$$\int_C v(x)^2\,dx \le \frac{l^2}{\pi^2}\int_C |\nabla v(x)|^2\,dx$$

Because of the orthonormality of the sines and cosines, the same recipe applies to all the terms of the Fourier series, thereby proving that

$$\int_C u(x)^2\,dx \le \frac{l^2}{\pi^2}\int_C |\nabla u|^2\,dx$$

By the same argument, the lemma is also true for dimensions higher than 2, because the only difference is larger products of sines and cosines, which doesn't really matter in the above described calculation. □

This is all right and good, but what can we do with it?
Well, prepare to be amazed:

Theorem 2.3.6. *Let K be as in Corollary 2.3.4, with the notable difference that K need not be differentiable on $\partial\Omega$. Then*

$$n^s|\mu_n| \le \frac{1}{\pi^2 n^2}\int_\Omega\int_\Omega |\nabla K|^2\,dx\,d\xi$$

with $s = \frac{1}{2} + \frac{1}{m}$.

Proof. Because of Lemma 2.3.5, for any continuously differentiable function u on an $m-$dimensional cube C with sides of lenth l,

$$\int_C u^2\,dx \le \frac{l^2}{\pi^2}\int_C |\nabla u|^2\,dx$$

if $\int_C u(x)\,dx = 0$.
So for every continuously differentiable function u on C there exists a constant u_0 for which $\int_C (u(x) - u_0)\,dx = 0$:
Simply choose u_0 to be equal to $\frac{1}{|C|}\int_C u(x)\,dx$.
We will apply Lemma 2.3.5 to use an argument similar to the proof of Corollary 2.3.4:

1. Let Ω be the region upon which K is defined and approximate it with a region Ω' consisting of M $m-$dimensional cubes glued together.

2. Divide every one of the M cubes into n^m smaller cubes as before.

3. On each of these Mn^m smaller cubes, let's call them C_n, the following equation holds:

$$\int\limits_{C_n}\int\limits_{C_n}(K(x,\xi)-K_{nM})^2\,dx\,d\xi \leq \frac{1}{n^2\pi^2}\int\limits_{C_n}\int\limits_{C_n}|\nabla K|^2\,dx\,d\xi$$

where

$$K_{nM}=\frac{1}{|C_n|}\int\limits_{C_n}\int\limits_{C_n}K(x,\xi)\,dx\,d\xi$$

4. If we now define a bilinear form k_{Mn^m} that is equal to K_{nM} on the appropriate cube, then

$$\int\limits_{\Omega}\int\limits_{\Omega}(K-k_{Mn^m})^2\,dx\,d\xi \leq \frac{1}{n^2\pi^2}\int\limits_{\Omega}\int\limits_{\Omega}|\nabla K|^2\,dx\,d\xi$$

5. Because of Theorem 2.2.6,

$$\int\limits_{\Omega}\int\limits_{\Omega}(K-k_{Mn^m})^2\,dx\,d\xi \geq \sum_{i=Mn^m+1}^{\infty}\mu_i^2 \geq Mn^m\mu_{2Mn^m}^2$$

6. Combining statements (4) and (5) gives

$$Mn^m\mu_{2Mn^m}^2 \leq \frac{1}{n^2\pi^2}\int\limits_{\Omega}\int\limits_{\Omega}|\nabla K|^2\,dx\,d\xi$$

7. We finally get that

$$\lim_{n\to\infty} n^s\mu_n \leq \frac{2}{n^2\pi^2}\int\limits_{\Omega}\int\limits_{\Omega}|\nabla K|^2\,dx\,d\xi$$

where $s=\frac{1}{2}+\frac{1}{m}$.

□

Note that we can make the bound for this limit a lot sharper: in Theorem 2.3.6 we assumed K to be differentiable within Ω, but not necessarily on $\partial\Omega$. So we can take a subset Ω' of Ω on which K is differentiable, with the extra property that K is differentiable on $\partial\Omega'$ as well. So on Ω' we have that $\lim\limits_{n\to\infty} n^s\mu_n' = 0$ because of Corollary 2.3.4, meaning that by Corollary 2.3.2

$$\lim_{n\to\infty} n^s|\mu_n| \leq \frac{2}{\pi^2}\int\limits_{\Omega\backslash\Omega'}\int\limits_{\Omega\backslash\Omega'}|\nabla K|^2\,dx\,d\xi$$

We can choose Ω' any way we want, as long it doesn't intersect with $\partial\Omega$, so we can choose

$$\int_{\Omega\backslash\Omega'}\int_{\Omega\backslash\Omega'} |\nabla K|^2 \, dx \, d\xi$$

as small as we like, thus giving $\lim_{n\to\infty} n^s \mu_n = 0$. To make things final; when splitting up K into kernels on different subsets of Ω, as we've done here, we should also check the case that $x \in \Omega'$ and $\xi \in \Omega\backslash\Omega'$ or vice versa, but since we can still choose Ω', that is rather trivial, so we don't. To summarize this argument:

Corollary 2.3.7. *If K is as in Theorem 2.3.6, then $\lim_{n\to\infty} n^s \mu_n = 0$ for $s = \frac{1}{2} + \frac{1}{m}$.*

2.4 One can hear the area of a drum

We know that the Green's function G corresponding to the system

$$\begin{cases} \Delta u = 0 \text{ within } \Omega \\ u = f \text{ on } \partial\Omega \end{cases}$$

can be written as $G = K - A$ where

$$K(x,\xi) = \begin{cases} c_2 \log \frac{1}{r} \text{ for } \Omega \subset \mathbb{R}^2 \\ c_n r^{2-m} \text{ for } \Omega \subset \mathbb{R}^m, \ m > 2 \end{cases}$$

with $r = |x - \xi|$ and A a function for which $K = A$ on $\partial\Omega$.
We'll set out to prove that we can indeed ignore A whilst calculating the limit behaviour of the eigenvalues of G.
What we would very much like to do is invoking that differentiable kernels, especially when they are harmonic, can be ignored by Corollary 2.3.4. Sadly enough, this line of reasoning is strictly forbidden, since when we proved Corollary 2.3.4, we assumed the kernel to be differentiable on the boundary of Ω as well, while A more often than not has singularities on $\partial\Omega$. Yet do not despair, since our efforts have not all been for naught: Corollary 2.3.4 will have its uses, only not yet.
So let us wipe away our tears and carry on with renewed tenacity and vigour, for the fruits of our labours are almost within reach.
A few theorems are all that remain between us and leisurely applying our newly gained knowledge.

Theorem 2.4.1. *(Weak Maximum Principle) Let $u : \Omega \to \mathbb{R}$ be a twice continuously differentiable function on a bounded set $\Omega \subset \mathbb{R}^n$. If $-\Delta u \leq 0$, then $\max_{x\in\Omega} u(x) = \max_{x\in\partial\Omega} u(x)$.*

Proof. Note that if $u(x)$ takes on a maximum for some $x = x_0 \in \Omega$, then $\Delta u \leq 0$ for $x = x_0$. This means that we would have a contradiction, were it not for the fact that $\Delta u \leq 0$ for $x = x_0$ instead of $\Delta u < 0$ for $x = x_0$. Our strategy therefore will be to reformulate the problem for a new function $v = u + z$, with z some function that slightly perturbs u, so to speak, leading to $-\Delta v < 0$ for $x \in \Omega$ and $\Delta v \leq 0$ for some $y \in \Omega$ should we assume that v takes on a maximum within Ω.
We need to prove that $\max_{x \in \Omega} u(x) = \max_{x \in \partial\Omega} u(x)$. If this is not true, then there exists a $y_0 \in \Omega$ for which $u(y_0) > M + \epsilon$, with $M = \max_{x \in \partial\Omega} u(x)$ and $\epsilon > 0$ some constant. Define $z(x) = \delta|x - y_0|^2$, with $\delta = \frac{\epsilon}{\rho}$ and $\rho = \max_{x \in \partial\Omega} |x - y_0|^2$. Then $v = u + z \leq M + \epsilon$ on $\partial\Omega$ and there exists a $y \in \Omega$ for which v takes on a maximum and $v(y) > M + \epsilon$. So $\Delta v \leq 0$ if $x = y$ and $-\Delta v = -\Delta u - \Delta z \leq 0 - 2n\delta < 0$ on Ω, which provides us with the needed contradiction. □

Observe that if we have two functions u_1, u_2 for which $-\Delta u_1 \leq -\Delta u_2$ on Ω and $u_1 \leq u_2$ on $\partial\Omega$, then $u_1 \leq u_2$ on Ω, because $-\Delta(u_1 - u_2) \leq 0$ on Ω and $u_1 - u_2 \leq 0$ on $\partial\Omega$. So because of the Weak Maximum Principle, $u_1 - u_2$ must be smaller than, or equal to zero on Ω. Thus

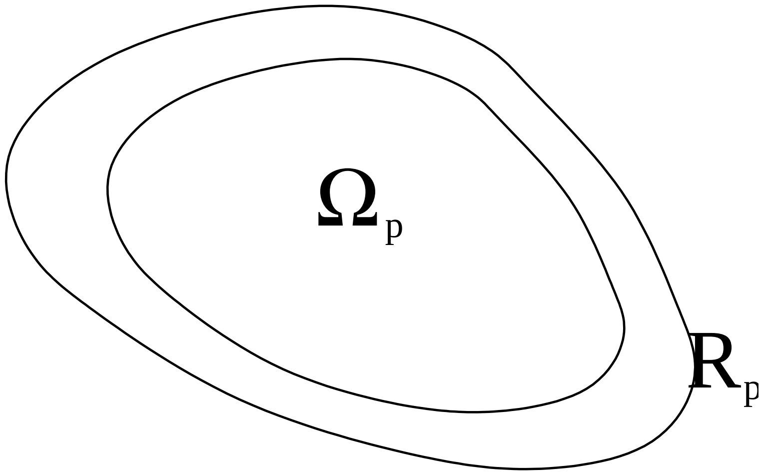
Corollary 2.4.2. *(Comparison Principle) If $u_1, u_2 \in C^2(\Omega)$, $-\Delta u_1 \leq -\Delta u_2$ on Ω and $u_1 \leq u_2$ on $\partial\Omega$, then $u_1 \leq u_2$ on Ω.*

And with that, only one obstacle remains:

Lemma 2.4.3. *If we name the eigenvalues of A to be $\{\nu_n\}_{n=1}^{\infty}$, then $\lim_{n \to \infty} n^{\frac{2}{m}} \nu_n = 0$ where m is once again either 2 or 3 and as such the dimension of the space in which we work.*

Proof. As stated before, we cannot use Corollary 2.3.4 to verify this result. At least, not directly. Should we take any subset of Ω that doesn't intersect with $\partial\Omega$, then A must be harmonic on both this new set, let's call it Ω_p and its boundary, $\partial\Omega_p$. Our hunch should be that if we define $A^{(p)}$ to be A if $x, \xi \in \Omega_p$ and zero otherwise and $\{\nu_n^{(p)}\}_{n=1}^{\infty}$ its corresponding eigenvalue spectrum, we can use some sort of limit argument to finalize our proof. It will probably not come as too great a shock that this is indeed true. That is, for $m = 2$ or $m = 3$. There will be one nagging little problem later on, but it will still prove to be beneficial to generalise what we can.
But first things first.
Let $p \in \mathbb{Z}_{>0}$ and $\lim_{p \to \infty} \Omega_p = \Omega$. Define to R_p to be $\Omega \backslash \Omega_p$, so that R_p seems like a 'rim' of Ω. It will be worth it to define a certain 'thickness' of R_p. And to define Ω_p is to define R_p, so let Ω_p be the region in Ω such that the

distance from a point in Ω to $\partial\Omega$ is larger than $\frac{1}{p}$. For the 2−dimensional case, it should look somewhat like this:



We will have to create two more kernels to effectively tackle our problem:

1. $A^{(p)*}$, which is equal to A if $x \in \Omega_p$ and $\xi \in R_p$ or vice versa and zero otherwise.

2. $A^{(p)**}$, which is equal to A if both x and ξ lie in R_p.

Now let's get things started, beginning with proving that $\lim_{p\to\infty} p^{s-\epsilon}\nu_p^{(p)} = 0$, where $s = \frac{1}{2} + \frac{1}{m}$ and $\epsilon > 0$, which, as it turns out, means that the eigenvalues of A go to zero faster than is needed, as will become apparent when calculating the eigenvalues of K. Which is, of course, much more enjoyable, since we can then take pride in going that extra mile...
Let's define

$$W(x,\xi) = \begin{cases} K(x,\xi) & for \quad r = |x-\xi| \geq \rho(x) \\ 0 & for \quad r = |x-\xi| < \rho(x) \end{cases}$$

where $\rho(x)$ is the shortest distance from x to $\partial\Omega$.
By the Dirichlet Principle (see Theorem 4.4.1),

$$\int_{\Omega_p}\int_{\Omega_p} |\nabla A|^2 \, dx \, d\xi < \int_{\Omega_p}\int_{\Omega_p} |\nabla W|^2 \, dx \, d\xi$$

and

$$\int_{\Omega_p}\int_{\Omega_p} |\nabla W|^2\, dx\, d\xi = \int_{\Omega_p}\int_{\Omega_p\setminus\{r>\rho(x)\}} |\nabla K|^2\, dx\, d\xi =$$

$$\begin{cases} \int_{\Omega_p}\int_{\Omega_p\setminus\{r>\rho(x)\}} \frac{c_2^2}{r^2}\cdot r\, dr\, dx & for \quad m=2 \\ \int_{\Omega_p}\int_{\Omega_p\setminus\{r>\rho(x)\}} c_m^2(2-m)^2 r^{2-2m}\cdot r^{m-1}\, dr\, dx & for \quad m>2 \end{cases}$$

where we have absorbed the extra constants due to the coordinate transformation into the c_m. If R is the radius of some ball that has x at its center and encloses Ω_p, then

$$\int_{\Omega_p}\int_{\Omega_p} |\nabla W|^2\, dx\, d\xi \leq \begin{cases} \int_{\Omega_p} c_2^2(\log R - \log\rho(x))\, dx & for \quad m=2 \\ \int_{\Omega_p} c_m^2(2-m)(R^{2-m} - (\rho(x))^{2-m})\, dx & for \quad m>2 \end{cases}$$

and thus

$$\int_{\Omega_p}\int_{\Omega_p} |\nabla W|^2\, dx\, d\xi \leq \begin{cases} \int_{\Omega_p} c_2^2(\log\rho(x))\, dx = & for \quad m=2 \\ \int_{\Omega_p} c_m^2(m-2)\frac{1}{\rho(x)^m}\, dx & for \quad m>2 \end{cases}$$

Considering the fact that we are integrating over Ω_p, $\rho(x)\geq\frac{1}{p}$, therefore

$$\int_{\Omega_p}\int_{\Omega_p} |\nabla W|^2\, dx\, d\xi \leq \begin{cases} \int_{\Omega_p} c_2^2(\log\rho(x))\, dx = \mathcal{O}(1) & for \quad m=2 \\ \int_{\Omega_p} c_m^2(m-2)\frac{1}{\rho(x)^m}\, dx = \mathcal{O}(\log p) & for \quad m=3 \\ \int_{\Omega_p} c_m^2(m-2)\frac{1}{\rho(x)^m}\, dx = \mathcal{O}(p^{m-3}) & for \quad m>3 \end{cases}$$

Because of the proof of Theorem 2.3.6, $n^m(\nu_{n^m}^{(p)})^2 \leq \frac{c}{n^2}\int_{\Omega_p}\int_{\Omega_p} |\nabla W|^2\, dx\, d\xi$ and therefore

$$n^m(\nu_{n^m}^{(p)})^2 = \frac{c}{n^2}\begin{cases} \mathcal{O}(1) & for \quad m=2 \\ \mathcal{O}(\log p) & for \quad m=3 \\ \mathcal{O}(p^{m-3}) & for \quad m>3 \end{cases}$$

showing that for $m=2$, the integral $\int_{\Omega_p}|\nabla A|^2$ is uniformly bounded, independent of p, giving $\lim_{p\to\infty} p^s\nu_p^{(p)} = 0$ because of Corollary 2.3.7, leaving only the case $m>2$ for us to check.
If we now choose n to be equal to p^l for some $l\in\mathbb{Z}_{>0}$, then

$$p^{lm}(\nu_{p^{lm}}^{(p)})^2 = \frac{c}{p^{2l}}\begin{cases} \mathcal{O}(\log p) & for \quad m=3 \\ \mathcal{O}(p^{m-3}) & for \quad m>3 \end{cases}$$

leading to
$p^{lm+2l-\epsilon}(\nu_{p^{lm}}^{(p)})^2 = \mathcal{O}\left(\frac{\log p}{p^\epsilon}\right)$ for $m = 3$ and
$p^{lm+2l-m+3-\epsilon}(\nu_{p^{lm}}^{(p)})^2 = \mathcal{O}(p^{-\epsilon})$ for $m > 2$ and $\epsilon > 0$

Replacing p^{lm} with p gives for $m = 3$

$$\lim_{p\to\infty} p^{\frac{1}{2}+\frac{1}{m}-\frac{\epsilon}{2lm}}|\nu_p^{(p)}| = \lim_{p\to\infty} \mathcal{O}\left(\sqrt{\frac{lm\log p}{p^{\frac{\epsilon}{lm}}}}\right) = 0$$

and for $m > 3$

$$\lim_{p\to\infty} p^{\frac{1}{2}+\frac{1}{m}-\frac{m+\epsilon}{2lm}}|\nu_p^{(p)}| = \lim_{p\to\infty} \mathcal{O}(p^{-\frac{\epsilon}{lm}}) = 0$$

Choosing l large enough finally gives $\lim\limits_{p\to\infty} p^{s-\epsilon}\nu_p^{(p)} = 0$, because $\frac{1}{2} + \frac{1}{m} = s$.
So what about $\nu_p^{(p)*}$ and $\nu_p^{(p)**}$?
By the same argument as for $p^{s-\epsilon}\nu_p^{(p)}$ we get that $\lim\limits_{p\to\infty} p^{s-\epsilon}\nu_p^{(p)*} = 0$. (Simply observe that $\int\limits_{R_p}\int\limits_{\Omega_p} |\nabla A|^2\,dx\,d\xi \le \int\limits_{\Omega}\int\limits_{\Omega_p} |\nabla A|^2\,dx\,d\xi$. FIRST calculate the integral over Ω THEN calculate the integral over Ω_p, all the while using exactly the same trick as for $p^{s-\epsilon}\nu_p^{(p)}$). And last but not least: $\nu_p^{(p)**}$.
For $m = 2$ or $m = 3$:

$$p(\nu_p^{(p)**})^2 \le \sum_{i=1}^{p}(\nu_i^{(p)**})^2 \le \sum_{i=1}^{\infty}(\nu_i^{(p)**})^2 =$$
$$\int\limits_{R_p}\int\limits_{R_p} A^2(x,\xi)\,dx\,d\xi \le \int\limits_{R_p}\int\limits_{R_p} K^2(x,\xi)\,dx\,d\xi$$

because of both the Maximum Principle and the Comparison Principle. (By the Maximum Principle $A > 0$ on Ω, because $A > 0$ on $\partial\Omega$ and by the Comparison Principle $K \ge A$ on Ω, because $K \ge A$ on $\partial\Omega$). Thus

$$p(\nu_p^{(p)**})^2 \le \int\limits_{R_p}\int\limits_{R_p} K^2(x,\xi)\,dx\,d\xi$$

which implies that for $m = 2$, or $m = 3$ $p(\nu_p^{(p)**})^2 = \mathcal{O}(\frac{1}{p^2})$ which is all perfectly within acceptable parameters.
$m > 3$ Is where the trouble starts, because now we cannot use K anymore to estimate A, which is why we prove our theorem for $m = 2$ and $m = 3$ only.
To summarize: we now know that clearly $p^{\frac{2}{m}}\nu_p^{(p)}$, $p^{\frac{2}{m}}\nu_p^{(p)*}$ and $p^{\frac{2}{m}}\nu_p^{(p)**}$ all

go to zero as p nears infinity.
Because of Lemma 2.2.4,

$$\lim_{p\to\infty} p^{\frac{2}{m}}|\nu_{3p}| \leq \lim_{p\to\infty} p^{\frac{2}{m}}|\nu_p^{(p)}| + p^{\frac{2}{m}}|\nu_p^{(p)*}| + p^{\frac{2}{m}}|\nu_p^{(p)**}| = 0$$

By which we have proven at last that for the eigenvalues ν_n of A, $\lim_{n\to\infty} n^{\frac{2}{m}}\nu_n = 0$. □

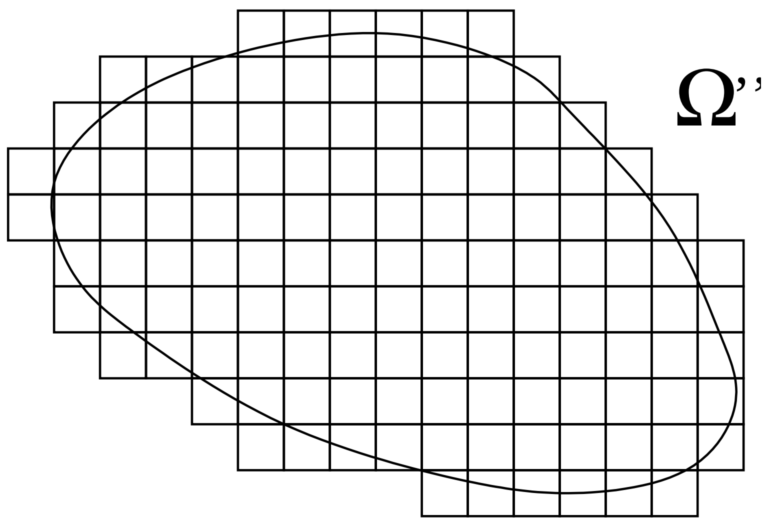
And so it begins...
Finally our patience, our perseverance, our relentless tenacity will be rewarded as at long last we have reached the theorem we have set out to prove all those pages ago:

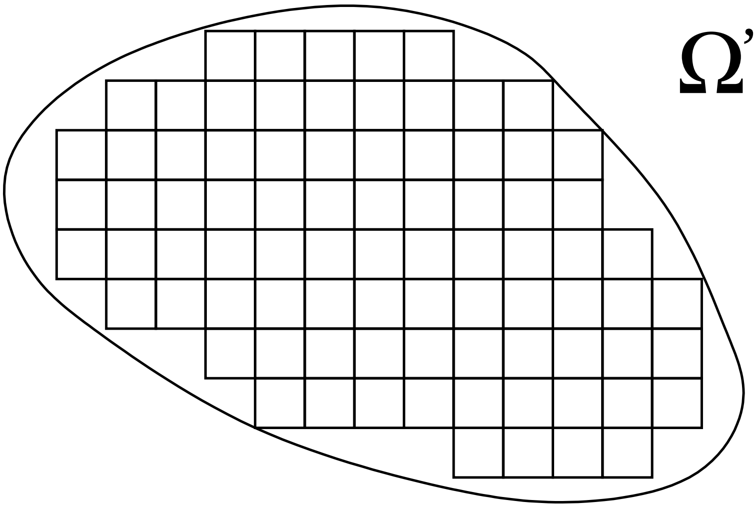
Theorem 2.4.4. *Let $m = 2$ or $m = 3$ and let $\{\lambda_n\}_{n=1}^{\infty}$ be the eigenvalues of the problem*

$$\begin{cases} -\Delta u = f & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases}$$

Then, for large n, $\lambda_n \sim (\frac{2^m n\pi^m}{B_m J})^{\frac{2}{m}}$.

Proof. Because of Theorem 2.4.3, it suffices to consider the case that our integral kernel is K instead of the Green's function G. That being said, we cover Ω in two ways with $m-$dimensional cubes with sides of length $\frac{1}{n}$. The first being the region in $\mathbb{R}m$ that is the smallest cubed space that encloses Ω. We will designate this space by Ω'. The second covering will be the largest cubed space that lies within Ω. This space will be named Ω''.

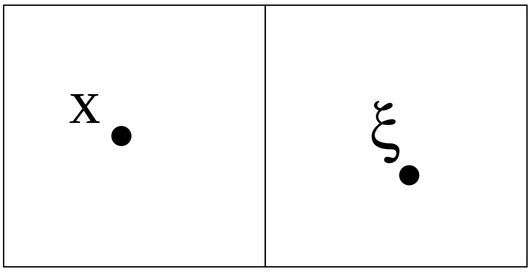




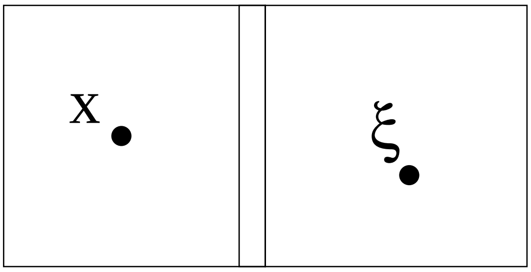
Because of Theorem 2.2.9, we now have that $\mu_i^{(+)''} \leq \mu_i^{(+)} \leq \mu_i^{(+)'}$ for $i \in \mathbb{Z}_{>0}$ and that $\mu_i^{(+)''}$, $\mu_i^{(+)}$ and $\mu_i^{(+)'}$ will coincide as n nears infinity.
So far, so good. What's next?
We've mentioned before that we want to use what eigenvalues K has on a cube. Therefore, we label the cubes q_1 to q_N, where N is the number of cubes available, and split up K into N^2 new kernels $\{K_{ij}\}_{i,j=1}^N$, where

$$\begin{cases} K_{ij} = K \text{ if } x \in q_i,\ \xi \in q_j \\ K_{ij} = 0 \text{ otherwise} \end{cases}$$

Now watch closely: if $i \neq j$ and the ith and jth cube are not adjacent, then K_{ij} is harmonic on these two cubes *including their boundaries*, which means that if $\{\mu_k^{ij}\}_{k=1}^\infty$ represent the eigenvalues of K^{ij}, then $\lim\limits_{k\to\infty} k^s \mu_k^{ij} = 0$ because of Corollary 2.3.4 and if the ith and jth cube are actually adjacent, then $\lim\limits_{k\to\infty} k^s \mu_k^{ij} = 0$ as well, by an argument similar to the one in the proof of Theorem 2.4.3 when we were observing the case that $x \in R_p$ and $\xi \in \Omega_p$ or vice versa, with the added bonus that we needn't bother about p going to zero any time soon. (When you integrate over r starting (for example) with the cube in which ξ lies, r has a minimal value, namely the distance along $x - \xi$ from x to the boundary of the two cubes and from there on you use exactly the same trick to prove that $\int\limits_\Omega \int\limits_\Omega |\nabla K|^2 \, dx \, d\xi$ is finite (see figure).



For the 3−dimensional case you split the cubes into a strip near the boundary and the remainder of the cubes (see figure) and then you let the strip fulfil the role of R_p from the proof of Theorem 2.4.3).



Of course, the ν_k^{ij} are the eigenvalues associated to A, in the same way as the μ_k^{ij} are associated to K.
Note that the argument for the case that q_i and q_j are not adjacent also holds for higher dimensions, because we only use the data of A which applies for all dimensions and not just $m = 2$ or $m = 3$.
And with that we have established that we can simplify our problem by replacing K with $\sum_{i=1}^{N} K_{ii}$. Because the eigenvalues of a linear partial differential equation are unchanged when Ω is translated by a constant vector, the eigenvalue spectra of all the K^{ii} are identical. The eigenvalues $\{\mu_k^{11}\}_{k=1}^{\infty}$ correspond in limit to the eigenvalues $\{\lambda_k^{11}\}_{k=1}^{\infty}$ of

$$\begin{cases} -\Delta u = \lambda u \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

where Ω is an m−dimensional cube of volume $\frac{1}{n^m}$ and without loss of generality we may assume that for this cube $0 \leq x_i \leq \frac{1}{n}$ holds, where $1 \leq i \leq m$ and the x_i are the Cartesian coordinates of the vector x.
We will now calculate the corresponding eigenvalues through the method known as Separation of Variables, which basically means that we assume that $u(x)$ can be written as $\prod_{i=1}^{m} P_i(x_i)$, where the P_i are yet to be derived and try to find a solution using this new representation:

$$-\Delta u = \lambda u$$

gives

$$-\sum_{j=1}^{m} P_j''(x_j) \prod_{i=1, i\neq j}^{m} P_i(x_i) = \lambda \prod_{i=1}^{m} P_i(x_i)$$

and dividing both sides by $\prod_{i=1}^{m} P_i(x_i)$ leads to

$$-\sum_{j=1}^{m} \frac{P_j''(x_j)}{P_j(x_j)} = \lambda$$

If we now bring $\frac{P_1''(x_1)}{P_1(x_1)}$ to one side, we get

$$\frac{P_1''(x_1)}{P_1(x_1)} = -\lambda - \sum_{j=2}^{m} \frac{P_j''(x_j)}{P_j(x_j)}$$

which means that the left hand side is independent of $x_2, \ldots, x_m$ and that the right hand side is independent of x_1, from which we can divulge that

$$\frac{P_1''(x_1)}{P_1(x_1)} = -\kappa_1 = -\lambda - \sum_{j=2}^{m} \frac{P_j''(x_j)}{P_j(x_j)}$$

where κ_1 is some real constant. So

$$\frac{P_1''(x_1)}{P_1(x_1)} = -\kappa_1$$

leading to

$$P_1''(x_1) + \kappa_1 P_1(x_1) = 0$$

meaning that

$$P_1(x_1) = A_1 \sin(\sqrt{\kappa_1} x_1) + B_1 \cos(\sqrt{\kappa_1} x_1)$$

where A_1 and B_1 are unknown real constants. Using the fact that $u = 0$ on $\partial\Omega$ and thus that $P_1(0) = P_1(\frac{1}{n}) = 0$ shows that

$$0 = A_1 \cdot \sin(0) + B_1 \cdot \cos(0) = B_1$$

leaving $P_1(x_1) = A_1 \sin(\sqrt{\kappa_1} x_1)$ and

$$0 = A_1 \cdot \sin(\sqrt{\kappa_1} \cdot \frac{1}{n})$$

which means that $\frac{\sqrt{\kappa_1}}{n} = k\pi$ for $k \in \mathbb{Z}$ or rather $\kappa_1 = k_1^2 n^2 \pi^2$ for $k_1 \in \mathbb{Z}_{>0}$, since $\kappa_1 = 0$ would mean that $u = 0$.
We can now perform the same trick for

$$\frac{P_2''(x_2)}{P_2(x_2)} = -\kappa_2 = \kappa_1 - \lambda - \sum_{j=3}^{m} \frac{P_j''(x_j)}{P_j(x_j)}$$

where κ_2 is again a constant by the same reasoning as for κ_1, giving $P_2(x_2) = A_2 \sin(\sqrt{\kappa_2} x_2)$ and $\kappa_2 = k_2^2\pi^2$. Inductively, we can keep on using this same strategy to conclude that $P_2(x_2) = A_2 \sin(\sqrt{\kappa_i} x_i)$ and $\kappa_i = k_i^2\pi^2$
for $1 \leq i \leq m-1$ giving that

$$\frac{P''_m(x_m)}{P_m(x_m)} = -\lambda + \sum_{i=1}^{m-1} \kappa_i$$

and thus, if we write $\kappa_m = \lambda - \sum_{i=1}^{m-1} \kappa_i$, $P_m(x_m) = A_m \sin(\sqrt{\kappa_m} x_m)$ and $\kappa_m = k_m^2\pi^2$, implying that $\lambda = \sum_{i=1}^{m} \kappa_i = n^2\pi^2 \sum_{i=1}^{m} k_i^2$ and finally
$\mu^{11} = \frac{1}{\sum_{i=1}^{m} \kappa_i} = \frac{1}{n^2\pi^2 \sum_{i=1}^{m} k_i^2}$.
As we can see, we may have a slight problem: We want to approximate the $i-$th eigenvalue, but can't really say which one the $i-th$ eigenvalue is. So we'll use a nifty trick:
Let R be the radius of an $m-$dimensional sphere and let $N(R)$ be the number of $m-$tuples $(k_1, \ldots, k_m)$ for which $k_1^2 + \ldots k_m^2 \leq R^2$. Then for R large, $N(R)$ acts as the volume of $\frac{1}{2^m}-$th part of an $m-$dimensional sphere of radius R, or $\frac{1}{2^m} R^m B_m$. (In the two dimensional case it is a quarter of a circle of radius R). By the same argument, $\lambda_{N(R)} \cong n^2\pi^2R^2$, so $\lim\limits_{N(R)\to\infty} \frac{N(R)}{\lambda_{N(R)}^{\frac{m}{2}}} = \frac{B_m}{2^m\pi^m n^m}$ and thus $\lim\limits_{k\to\infty} k\left(\mu_k^{11}\right)^{\frac{m}{2}} = \frac{B_m}{2^m\pi^m n^m}$. But what about μ_k? As we have seen before, the eigenvalues of K^{ii} are only defined by the cube q_i upon which the kernel is defined. And because it doesn't matter which small cube of volume $\frac{1}{n^m}$ this q_i is, we might as well arrange these cubes as one big cube with sides of length $N^{\frac{1}{m}} \cdot \frac{1}{n}$. And because (in limit) the eigenvalues of K coincide with the eigenvalues of $\sum_{i=1}^{N} K^{ii}$, they coincide with the eigenvalues of K defined upon this newly created cube-of-little-cubes. This big cube is N^m times as large as a cube of the type q_i, so in order to find the eigenvalues of the corresponding differential operator we apply a coordinate transform which replaces the coordinates $(x_1, \ldots, x_m)$ with $(x'_1, \ldots, x'_m) = (N^{\frac{1}{m}} x_1, \ldots, N^{\frac{1}{m}} x_m)$. Now let Δ_x be the Laplace operator with respect to x. Then $\Delta_x = N^{\frac{2}{m}} \Delta_{x'}$ and therefore the eigenvalues λ for the 'big cube'-case are $\frac{1}{N^{\frac{2}{m}}}$ times the eigenvalues λ for the q_i- case. Because $\mu = \frac{1}{\lambda}$, we can now conclude that $\lim\limits_{k\to\infty} k^2 m\mu_k = N^{\frac{2}{m}} \cdot \left(\frac{B_m}{2^m\pi^m n^m}\right)^{\frac{2}{m}} = \left(\left(\frac{N}{n^m}\right) \cdot \left(\frac{B_m}{2^m\pi^m}\right)\right)^{\frac{2}{m}} = \left(\frac{|\Omega| B_m}{2^m\pi^m}\right)^{\frac{2}{m}}$, since $\frac{N}{n^m} \cong |\Omega|$. And thus $\lim\limits_{k\to\infty} \frac{k^{\frac{2}{m}}}{\lambda_k} = \left(\frac{|\Omega| B_m}{2^m\pi^m}\right)^{\frac{2}{m}}$, thereby completing the proof. □

As was mentioned in the introduction of this chapter, Theorem 2.4.4 can only be proven for the cases $m = 2$ and $m = 3$. Why? Because the proof relies heavily on the fact that for all our kernels K the integral $\int\limits_\Omega \int\limits_\Omega K^2(x,\xi)\,dx\,d\xi$ is finite. Yet if we take a closer look at the proof of Theorem 2.4.4, we only really need it for the case that the cubes q_i and q_j are adjacent and the reason that the proof backfires is that we can't show (see the proof of Theorem 2.4.3) that $\int\limits_\Omega \int\limits_\Omega K^2(x,\xi)\,dx\,d\xi$ is finite *even if you are allowed to choose* Ω *as small as you like!* That's rather frustrating, isn't it? And it's even more infuriating that Weyl's article, from which this proof comes, is actually referred to on several occasions as the article that proves the higher dimensional case as well! There is bound to be some wonderful theorem, lemma or proposition or somesuch that circumvents the assumption on $\int\limits_\Omega \int\limits_\Omega K^2(x,\xi)\,dx\,d\xi$ to be finite, but it goes without saying that whatever this illustrous solution may be, it is by no means trivial!
Still, one cannot deny that mister Weyl constructed a proof that truly excels in elegance and we should take comfort in the fact that what we've read here is the solution to one of the famous Hilbert Problems and if that doesn't count for something than I don't know what does.
And with that, it's time to relax for a spell and contemplate upon the wondrous beauty of mathematics. And as we experience these blissful moments, we embark on clouds of extacy to the next chapter, where yet more jewels of mathematical thought appear at the horizon.

Chapter 3

Amusing Tidbits

3.1 Introduction

As we've reached this chapter, we cannot deny that we've absorbed a fair deal of theory and information and it would be a bloody shame and, more important, rather anti-climactic, if we had mastered all these techniques only to solve one single problem. So, without further ado, let's check some applications.

3.2 A night at the opera

Picture yourself spending a bundle to experience a night at the opera, only to find out that the people in the loge next to you can't be bothered to suppress there urge to loudly express there excitement during the ouverture, during the first act, the second act, well, during the entire show, really. Should you then be so unfortunate to be unable to shut them up, there is still some joy to be found in the fact that you can still hear the size of the various drums in the orchestra as clear as if your dreadful fellow opera visitors were blessfully dead and buried. Because

Theorem 3.2.1. *Let L be equal to $-\Delta - q$, $q \in C(\Omega)$ and let Γ be its Green's function. Then for the eigenvalues of Γ, when $m = 2$ or $m = 3$, the same limit applies as for G.*

Proof. We have defined L to be $-\Delta - q$. So $L(u) = -\Delta u - qu$, which means that if we choose $u(x)$ to be $\int\limits_\Omega \Gamma(x,\xi) f(\xi)\, d\xi$, for some function f, then

$$f(x) = -\Delta \left(\int\limits_\Omega \Gamma(x,\xi) f(\xi))\, d\xi \right) - q(x) \int\limits_\Omega \Gamma(x,\xi) f(\xi)\, d\xi$$

and if we use the Green's function of $-\Delta$ on both sides of the equation, we get

$$\int_\Omega G(x,\xi)f(\xi)\,d\xi = \int_\Omega \Gamma(x,\xi)f(\xi)\,d\xi - \int_\Omega\int_\Omega G(x,y)q(y)\Gamma(y,\xi)f(\xi)\,dy\,d\xi$$

and therefore

$$G(x,\xi) = \Gamma(x,\xi) - \int_\Omega G(x,y)q(y)\Gamma(y,\xi)\,dy$$

finally providing us with

$$\Gamma(x,\xi) - G(x,\xi) = \int_\Omega G(x,y)q(y)\Gamma(y,\xi)\,dy$$

Our next step will be to construct a bilinear form in order to obtain information about the nth eigenvalue of $\Gamma - G$:
Let $\{\mu_n\}_{n=1}^\infty$ and $\{\phi_n(x)\}_{n=1}^\infty$ be the eigenvalues and eigenfunctions of G respectively, then by Theorem 2.2.1 $G(x,\xi) = \sum_{n=1}^\infty \mu_n\phi_n(x)\phi_n(\xi)$. So if we define

$$\psi_n(\xi) = \int_\Omega \phi_n(y)q(y)\Gamma(y,\xi)\,dy$$

then

$$\Gamma(x,\xi) - G(x,\xi) = \sum_{i=1}^\infty \mu_i\phi_i(x)\psi_i(\xi)$$

if we choose our bilinear form to be $k_{2n}(x,\xi) = \sum_{i=1}^n \mu_i\phi_i(x)\psi_i(\xi)$, existing out of the n functions $\phi_1,\ldots,\phi_n$ and the n functions $\psi_1,\ldots,\psi_n$, then

$$\int_\Omega\int_\Omega \left(\Gamma(x,\xi) - G(x,\xi) - k_{2n}(x,\xi)\right)^2\,dx\,d\xi = \sum_{i=n+1}^\infty \mu_i^2 \int_\Omega \psi_i^2(\xi)\,d\xi$$

because of the orthonormality of the ϕ_i.
Observe that if we define for fixed ξ a function $F(y)$ to be equal to $q(y)\Gamma(y,\xi)$ and once again choose the inner product $\langle f,g\rangle = \int_\Omega f(x)g(x)\,dx$, then, since the ϕ_i span the function space, $F(y) = \sum_{i=1}^\infty \langle F,\phi_i\rangle\phi_i(y)$ and $\langle F,F\rangle = \sum_{i=1}^\infty \langle F,\phi_i\rangle^2$.

So

$$\sum_{i=n+1}^{\infty} \psi_i^2(\xi) \leq \sum_{i=1}^{\infty} \psi_i^2(\xi) = \sum_{i=1}^{\infty} \left(\int_{\Omega} \phi_i(y) q(y) \Gamma(y,\xi)\, dy \right)^2 =$$

$$\sum_{i=1}^{\infty} \langle \phi_i, F \rangle = \langle F, F \rangle = \int_{\Omega} q^2(y) \Gamma^2(y,\xi)\, dy$$

which means that

$$\sum_{i=n+1}^{\infty} \mu_i^2 \int_{\Omega} \psi_i^2(\xi)\, d\xi \leq \mu_{n+1}^2 \int_{\Omega} \int_{\Omega} q^2(y) \Gamma^2(y,\xi)\, dy\, d\xi$$

and because the $(n+1)$th eigenvalue of G is $\mathcal{O}\left((n+1)^{-\frac{2}{m}}\right) = \mathcal{O}\left(n^{-\frac{2}{m}}\right)$,

$$\sum_{i=n+1}^{\infty} \mu_i^2 \int_{\Omega} \psi_i^2(\xi)\, d\xi = \mathcal{O}\left(n^{-\frac{4}{m}}\right)$$

and we invoke Theorem 2.2.6 to conclude that, if ν_i represents the ith eigenvalue of $\Gamma - G$

$$n\nu_{3n}^2 \leq \sum_{i=2n+1}^{\infty} \leq \int_{\Omega} \int_{\Omega} \left(\Gamma(x,\xi) - G(x,\xi) - k_{2n}(x,\xi) \right)^2\, dx\, d\xi =$$

$$\sum_{i=n+1}^{\infty} \mu_i^2 \int_{\Omega} \psi_i^2(\xi)\, d\xi = \mathcal{O}\left(n^{-\frac{4}{m}}\right)$$

and $\nu_n = \mathcal{O}\left(n^{-\frac{1}{2}-\frac{2}{m}}\right)$ which means that the eigenvalues of $\Gamma - G$ converge to zero faster than the eigenvalues of G and because $\Gamma = (\Gamma - G) + G$, the limit behaviour of the eigenvalues of Γ is identical to the limit behaviour of the eigenvalues of G. □

And thus we have proven that disturbance through extrovert opera patrons does not obstruct us from hearing the area of a drum. Granted, we do not really know what kind of disturbance the q represents, but people who talk in theaters usually do not know how annoying they are either, so that's okay...

As ecstatic as we might become by this last result, the reader may very well be a loud and outgoing person and therefore very much displeased by our findings. Thus in order to get a 100% satisfaction ratio amongst our audience, let's find out to what ends one must go to completely muck up people's night out:

Theorem 3.2.2. *Let* $Lu = -\sum_{i=1}^{m} \frac{\partial}{\partial x_i}\left(p\frac{\partial u}{\partial x_i}\right) - qu$ *where* $q \in C(\Omega)$ *and* $p(x) \geq c > 0$ *and twice continuously differentiable for some positive constant* c*. Furthermore, let* B *represent the Green's function corresponding to* L *and* $\{\nu_n\}_{n=1}^{\infty}$ *the eigenvalues of* B*. Then for* $m = 2$ *and* $m = 3$

$$\lim_{n\to\infty} n^{\frac{2}{m}}\nu_n = \frac{B_m}{4\pi^2}\left(\int\limits_{\Omega}\left(\frac{1}{p(\xi)}\right)^{\frac{m}{2}} d\xi\right)^{\frac{2}{m}}$$

Proof. We will start with the case that $|\Omega| = 1$. Define a function v by $v = \sqrt{u}$. Then

$$\begin{aligned}
-\sum_{i=1}^{m}\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right) &= -\sum_{i=1}^{m}\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_i}\left(\frac{v}{\sqrt{p}}\right)\right)\\
&= -\sqrt{p}\Delta v - \sum_{i=1}^{m}\left(\frac{\partial\sqrt{p}}{\partial x_i} + p\frac{\partial}{\partial x_i}\left(\frac{\partial\left(\frac{1}{\sqrt{p}}\right)}{\partial x_i}\right)\right)\frac{\partial v}{\partial x_i}\\
&+ \sum_{i=1}^{m}\frac{\partial}{\partial x_i}\left(p\frac{\partial\left(\frac{1}{\sqrt{p}}\right)}{\partial x_i}\right)\\
&= -\sqrt{p}\Delta v - \sum_{i=1}^{m}\left(\frac{1}{2\sqrt{p}}\frac{\partial p}{\partial x_i} - \frac{1}{2\sqrt{p}}\frac{\partial p}{\partial x_i}\right)\frac{\partial v}{\partial x_i} + \Delta\left(\sqrt{p}\right)v\\
&= -\sqrt{p}\Delta v + \Delta\left(\sqrt{p}\right)v
\end{aligned}$$

As such, $L(u) = -\sqrt{p}\Delta v + \Delta\left(\sqrt{p}\right)v - \frac{q}{\sqrt{p}}v$ and thus

$$\frac{1}{\sqrt{p}}L(u) = -\sqrt{p}\Delta v + \left(\frac{\Delta\left(\sqrt{p}\right)}{\sqrt{p}} - \frac{p}{q}\right)v \tag{3.1}$$

An interesting remark to make would be that the Green's function corresponding to L has the same eigenvalues as $\frac{1}{\sqrt{p(x)}\sqrt{p(\xi}}\Gamma(x,\xi)$ with $\left(\frac{\Delta(\sqrt{p})}{\sqrt{p}} - \frac{p}{q}\right) := -z$ instead of q. Because

$$\begin{aligned}
\frac{1}{\sqrt{p}}L(u) &= -\Delta v + \left(\frac{\Delta\left(\sqrt{p}\right)}{\sqrt{p}} - \frac{\sqrt{p}}{q}\right)v\\
&= (-\Delta - z)(v) = (-\Delta - z)(\sqrt{p}u)
\end{aligned}$$

And if $V(u) - \lambda u = 0$ for some linear differential operator V, then $\frac{1}{\sqrt{p}}V(u) - \lambda\frac{u}{\sqrt{p}} = 0$ and thus $\frac{1}{\sqrt{p}}V(\sqrt{p}\left(\frac{u}{\sqrt{p}}\right)) - \lambda\frac{u}{\sqrt{p}} = 0$, thus $\frac{u}{\sqrt{p}}$ is the corresponding eigenfunction of V^*, defined by $V^*(u) = \frac{1}{\sqrt{p}}V(\sqrt{p}u)$, which

has the same eigenvalue as u, which implies that by Theorem 2.2.1 and the fact that multiplication of a linear differential operator with a nonzero function is of no consequence to the eigenvalues, the kernel $\frac{1}{\sqrt{p(x)}\sqrt{p(\xi}}\Gamma(x,\xi)$ has the same eigenvalues as B. And because of Theorem 3.2.1 in combination with Theorem 2.2.8 (which states that if μ_n is the nth (positive) eigenvalue of a kernel K and ν_n is the nth (positive) eigenvalue of $K(x,\xi)p(x)p(\xi)$ for some strictly positive piecewise continuous function p, then $\min_{x\in\Omega} p^2(x)\mu_n \leq \nu_n \leq \max_{x\in\Omega} p^2(x)\mu_n$) we may replace Γ with G, the Green's function of the Laplace operator.
As such, we have rewritten our problem as finding the limit behaviour of the eigenvalues of the kernel $G(x,\xi)\frac{1}{\sqrt{p(x)}\sqrt{p(\xi)}}$. So once more, as in the proof of Theorem 2.4.4, we cover Ω with N small cubes q_i with sides of length $\frac{1}{n}$ and define $G = \sum_{i=1}^{N}\sum_{i=1}^{N} G^{(ij)}$ in the same fashion as was done whilst proving Theorem 2.4.4. Because of Theorem 2.2.8, together with the reasoning of the proof of Theorem 2.4.4, we may replace $G(x,\xi)\frac{1}{p(x)p(\xi)}$ with $\sum_{i=1}^{N} G^{(ii)}(x,\xi)\frac{1}{p(x)p(\xi)}$ henceforth to be called $H(x,\xi)$. Once more because of Theorem 2.2.8, for the kth eigenvalue of H, to be represented by ν_k^N, the following must hold:

$$\lim_{k\to\infty}\sum_{i=1}^{N} k^{\frac{2}{m}}\mu_k^{(ii)}\min_{x\in q_i}\frac{1}{p(x)} \leq \lim_{k\to\infty} k^{\frac{2}{m}}\nu_k^N \leq \lim_{k\to\infty}\sum_{i=1}^{N} k^{\frac{2}{m}}\mu_k^{(ii)}\max_{x\in q_i}\frac{1}{p(x)} \quad (3.2)$$

with the $\mu_k^{(ii)}$ once more in the role of eigenvalues of G restricted to q_i. Because $\lim_{k\to\infty} k^{\frac{2}{m}}\mu_k^{(ii)} = \frac{B_m}{2^m n^m \pi^m}$, (3.2) becomes, provided we raise it to the power $\frac{m}{2}$:

$$\frac{B_m\sum_{i=1}^{N}\min_{x\in q_i}\left(\frac{1}{p(x)}\right)^{\frac{m}{2}}}{2^m\pi^m n^m} \leq \lim_{k\to\infty} k\left(\nu_k^N\right)^{\frac{m}{2}} \leq \frac{B_m\sum_{i=1}^{N}\max_{x\in q_i}\left(\frac{1}{p(x)}\right)^{\frac{m}{2}}}{2^m\pi^m n^m} \quad (3.3)$$

Some explanation is probably needed. The left hand side is a direct consequence of Newton's Binomium. The right hand side makes use of Jensen's inequality (see Theorem 4.4.2), which states that if J is a convex function $J:\mathbb{R}\to\mathbb{R}$, f is a real-valued function on Ω, $\langle f\rangle$ is defined as $\frac{1}{|\Omega|}\int_\Omega f(x)\,dx$, and $\langle J\circ f\rangle$ and $\langle f\rangle$ are both finite, then $\langle J\circ f\rangle \geq J(\langle f\rangle)$. If we choose $J(t) = t^{\frac{m}{2}}$ and $f(x) = \max_{\xi\in q_i}\frac{1}{p(\xi)}$ if $x\in q_i$ for all $i\in\{1,\ldots,N\}$, then

$\langle f \rangle = \frac{1}{\Omega}\sum_{i=1}^{N} \max_{x\in q_i} \frac{1}{p(x)}$ and $\langle J \circ f \rangle = \frac{1}{\Omega} \max_{x \in q_i} \left(\frac{1}{p(x)}\right)^{\frac{m}{2}}$, giving that

$$\left(\sum_{i=1}^{N} \max_{x\in q_i} \frac{1}{p(x)}\right)^{\frac{m}{2}} = |\Omega|^{\frac{m}{2}} \left(\frac{1}{|\Omega|}\sum_{i=1}^{N} \max_{x\in q_i} \frac{1}{p(x)}\right)^{\frac{m}{2}} \leq$$
$$|\Omega|^{\frac{m}{2}-1} \sum_{i=1}^{N} \max_{x\in q_i} \left(\frac{1}{p(x)}\right)^{\frac{m}{2}}$$

and because we have assumed $|\Omega|$ to be equal to 1, we conclude that

$$\left(\sum_{i=1}^{N} \max_{x\in q_i} \frac{1}{p(x)}\right)^{\frac{m}{2}} \leq \sum_{i=1}^{N} \max_{x\in q_i} \left(\frac{1}{p(x)}\right)^{\frac{m}{2}}$$

thereby justifying (3.3). Should we let N increase to infinity, then both the left and right hand side of (3.3) converge to

$$\frac{B_m}{2^m \pi^m} \int_\Omega \left(\frac{1}{p(x)}\right)^{\frac{m}{2}} dx$$

where $\frac{1}{n^m}$ acts as dx so to speak. This then shows that

$$\lim_{n\to\infty} n^{\frac{2}{m}} \nu_n = \frac{B_m}{4\pi^2} \left(\int_\Omega \left(\frac{1}{p(\xi)}\right)^{\frac{m}{2}} d\xi\right)^{\frac{2}{m}}$$

and a simple coordinate transformation then completes our proof. □

Clearly, it has become impossible hearing the area of the drums by using our theory, which is a plus for the aforementioned opera hooligans, I suppose. But looking at the result strongly suggests that if you'd be able to cause a disturbance of this kind, you'd be lucky to get out with all your vital functions intact, due to the infernal wrath of the audience you'd risk with this level of mayhem. Still, mission accomplished, in a self-destructive sort of way.

3.3 Circular Drums

While unable at this point in time to say anything about hearing the shape of a drum, we can however, as was mentioned before, hear wether a drum is circular or not, which is almost as good, because most drums we know are

circular and therefore practically, being able to hear circular drums roughly means that we can hear wether a drum is curiously shaped or not. Which is all very usefull if you ever run out of topics during a conversation at a party.
But seriously:

Theorem 3.3.1. *Any twice continuously differentiable closed curve with that encloses a region of maximal area for some fixed circumference must be a circle.*

Proof. Define a vectorspace V of twice continuously differentiable closed curves $\gamma(t) = (x(t), y(t))$ and reparametrize $\gamma(t)$ in such a way that $\|\frac{d}{dt}\gamma(t)\| = 1$ for all t. This is possible. Simply start your curve at some $t = a$, $a \in \mathbb{R}$. The length $L(t)$ of the curve at time t must then be equal to $\int_a^t \|\frac{d\gamma}{ds}\| ds$, which means that $\frac{d\gamma}{dt} = \frac{d\gamma}{dL}\frac{dL}{dt} = \frac{d\gamma}{dL}\|\frac{d\gamma}{dt}\|$ and $\frac{d\gamma}{dL} = \frac{\frac{d\gamma}{dt}}{\|\frac{d\gamma}{dt}\|}$. Note that $L(t)$ only increases if t increases, which makes the parametrization valid.
Next, fix the length of your closed curves to be C, starting at $t = 0$ in $\gamma(0) = (0, 0)$. What we want to do is maximizing the area

$$\int_0^C y(t)x'(t)\,dt \tag{3.4}$$

enclosed by a curve $(x(t), y(t))$ under the restriction that the length of the curve, $\int_0^C dt = \int(x'(t)^2 + y'(t)^2)\,dt$ is equal to C. Therefore, we define the functional

$$F(x, x', y, y') = \int_0^C (y(t)x'(t) + \lambda(x'(t)^2 + y'(t)^2))\,dt$$

where λ is some constant, that will act as a safety measure when we maximize F to find the curve with maximal area. If $\gamma(t) = (x(t), y(t))$ is this curve, then let $\phi(t) = (\xi(t), \eta(t)) \in V$ and $0 < \epsilon << 1$, so that

$$\lim_{\epsilon \to 0} \frac{F(x + \epsilon\xi, x' + \epsilon\xi', y + \epsilon\eta, y' + \epsilon\eta') - F(x, x', y, y')}{\epsilon} = 0 \tag{3.5}$$

where (3.5) represents the directional derivative of F in the 'point' (x, x', y, y') so to speak, which has to be equal to zero for all $\phi \in V$, due to the nature of γ being the curve with maximal area. Using the chain rule, (3.5) can be rewritten as, if we define $Q(x, x', y, y') = yx' + \lambda(x'^2 + y'^2)$:

$$\int_0^C \left(\left(\frac{\partial Q}{\partial x}\xi(t) + \frac{\partial Q}{\partial x'}\xi'(t)\right) + \left(\frac{\partial Q}{\partial y}\eta(t) + \frac{\partial Q}{\partial y'}\eta'(t)\right)\right) dt = 0 \tag{3.6}$$

and because (3.6) holds for all ϕ, it also holds for $\phi_2(t) = (\xi(t), -\eta(t))$, allowing us to rewrite (3.6) as

$$\int_0^C \left(\left(\frac{\partial Q}{\partial x}\xi(t) + \frac{\partial Q}{\partial x'}\xi'(t)\right) \pm \left(\frac{\partial Q}{\partial y}\eta(t) + \frac{\partial Q}{\partial y'}\eta'(t)\right)\right) dt = 0 \tag{3.7}$$

and therefore

$$\int_0^C \left(\frac{\partial Q}{\partial x}\xi(t) + \frac{\partial Q}{\partial x'}\xi'(t) \right) dt = 0 \tag{3.8}$$

and

$$\int_0^C \left(\frac{\partial Q}{\partial y}\eta(t) + \frac{\partial Q}{\partial y'}\eta'(t) \right) dt = 0 \tag{3.9}$$

which by partial integration and the fact that all curves begin and end in $(0,0)$ allows (3.8) and (3.9) to become

$$\int_0^C \left(\frac{\partial Q}{\partial x} - \frac{d}{dt}\frac{\partial Q}{\partial x'} \right) \xi(t)\, dt = 0 \tag{3.10}$$

and

$$\int_0^C \left(\frac{\partial Q}{\partial y} - \frac{d}{dt}\frac{\partial Q}{\partial y'} \right) \eta(t)\, dt = 0 \tag{3.11}$$

for all ϕ and so (3.10) and (3.11) imply that, because $Q(x, x', y, y') = yx' + \lambda(x'^2 + y'^2)$

$$0 = \frac{\partial Q}{\partial x} - \frac{d}{dt}\frac{\partial Q}{\partial x'} = 0 - y' + 2\lambda x'' \tag{3.12}$$

and

$$0 = \frac{\partial Q}{\partial y} - \frac{d}{dt}\frac{\partial Q}{\partial y'} = x' + 2\lambda y'' \tag{3.13}$$

Adding (3.12), multiplied with x' and (3.13), multiplied with y', leads to

$$0 = -(x'^2 + y'^2) + 2\lambda(x''y' - y''x') = -1 + 2\lambda\kappa$$

because of our choice of parametrization of γ and because the curvature of γ, which is symbolized as κ, is equal to $|x''y' - y''x'|$, which means that $\kappa = |\frac{1}{2\lambda}|$, proving that γ is a circle, since a curve in $\mathbb{R}^2$ is a circle only if its curvature is constant.
If you are unfamiliar with concepts such as curvature, then please study section 4.5 of the Appendix. □

Because Åke Pleiel proved that you can hear the circumference of a drum in *"A study of certain Green's functions with applications in the theory of vibrating membranes"*, Arkiv för Mathematik, Band 2, no. 29 and we know that you can hear the area of a drum, we have now ascertained that it is indeed possible to hear wether a drum is circular or not.

Chapter 4

Appendix of the Damned

4.1 Introduction

This is the Appendix of the Damned, of the Forgotten, of the theorems and lemmas noone ever bothers to check. This is the Appendix of the Mentioned, of theorems and arguments that were repeatedly used during classes and presentations, of which it was always said that you had seen them one time or another during your studies, which was seldomly the case. This is the Appendix of the Damned.

4.2 On the existence of eigenvalues

Before we discuss whether or not the eigenfunctions of a Green's function span the function space on which it has been defined, it might be prudent to find out wether these eigenfunctions exist in the first place. In that spirit, a definition is called for:

Definition 4.2.1. *Let T be a bounded linear functional on a Banach space X. Then $\sigma(T)$ is the set of $\lambda \in \mathbb{C}$ such that $(\lambda I - T)^{-1}$ doesn't exist.*

Our first question should be wether $\sigma(T)$ can be empty. This means that $(\lambda I - T)^{-1}$ exists for all $\lambda \in \mathbb{C}$. Choose some fixed λ_0 and define $R(\lambda) = (\lambda I - T)^{-1}$. Then

$$\begin{aligned} R(\lambda) &= (\lambda I - T)^{-1} = ((\lambda - \lambda_0)I + \lambda_0 I - T)^{-1} \\ &= (\lambda_0 I - T)^{-1}((\lambda - \lambda_0)(\lambda_0 I - T)^{-1} + I)^{-1} \\ &= (\lambda_0 I - T)^{-1} \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n (\lambda_0 I - T)^{-n} \\ &= \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0)^{n+1} \end{aligned}$$

which is convergent if $|\lambda_0 - \lambda| < \|R(\lambda_0\|^{-1}$. So for every $\lambda_0 \in \mathbb{C}$, there exists a region $\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \rho\}$ for some positive constant ρ on which $\|R(\lambda)\|$ is analytic. So if $R(\lambda)$ exists for all $\lambda \in \mathbb{C}$, then $\|R(\lambda)\| \geq 0$ is analytic on the entire complex plane. But for $|\lambda|$ large enough, $\|R(\lambda)\| = \|(\lambda I - T)^{-1}\| = \|\sum_{n=1}^{\infty} \frac{T^n}{\lambda^{n+1}}\| \leq \sum \frac{\|T\|^n}{|\lambda|^{n+1}} = \frac{1}{|\lambda| - \|T\|}$ which means $R(\lambda)$ tends to zero as $|\lambda|$ goes to infinity. So by Liouville's theorem, $R(\lambda) = 0$ for every λ, which gives us a contradiction, proving that $\sigma(T)$ is in fact, non-empty. Summarizing:

Theorem 4.2.2. *For any bounded linear operator T on a Banach space X, $\sigma(T)$ is non-empty.*

Note that the above mentioned argument relies heavily upon complex function theory. If you have little or no experience whatsoever in this field, I highly recommend *Complex Analysis* by L.V. Ahlfors.
In the next section we will study operators T for which $\sigma(T)$ consists of eigenvalues alone, so in that case we indeed have eigenfunctions, because by Theorem 4.2.2 $\sigma(T)$ is non-empty. As it happens, Green's functions will turn out to be examples of these very same operators. Imagine that.
Before marching on to the next section, there is one final tool that will have its merits in the theorems and lemmas to come and that is the *spectral radius*.

Definition 4.2.3. *The spectral radius $r(T)$ of an operator T is defined as $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$.*

Theorem 4.2.4. $r(T) \geq \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$.

Proof. For $\frac{\|T^n\|}{|\lambda|^{n+1}} < 1$ for all $n \in \mathbb{Z}_{>0}$, $\sum_{n=1}^{\infty} \frac{T^n}{\lambda^{n+1}}$ is convergent. For these λ, $|\lambda|^{\frac{n+1}{n}} > \|T^n\|^{\frac{1}{n}}$ and by definition, $r(T) > |\lambda|$ must hold and therefore, $r(T) \geq \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ □

4.3 Green's functions

As can be concluded from the first two chapters, there are a few loose ends that need to be tied up, among others the question wether the eigenfunctions of a linear partial differential operator really span the function space on which it is defined. We'll be needing some new tools to tackle this problem, which will be formulated shortly, but the general idea is proving that our integral operators are in fact normal compact operators (whatever they may be for now) and subsequently ascertain that normal compact operators have eigenfunctions that span the entire space on which the operator is defined (as long as the operator has kernel $\{0\}$). So here goes:

Definition 4.3.1. *A linear operator $T : X \to Y$, for X and Y Banach spaces, is said to be compact if the closure of the image TB of any bounded set $B \subset X$ is compact.*

Definition 4.3.2. *Let H be a Hilbert space with a countable orthonormal basis $\{e_n\}_{n=1}^{\infty}$ and $T : H \to H$ be a linear continuous operator. T is said to be Hilbert-Schmidt if*

$$\sum_{n=1}^{\infty} \|Te_n\|^2 < \infty$$

So why do we need to know what a Hilbert-Schmidt operator is? Because Hilbert-Schmidt operators are compact! And to prove that we will be needing some information about *adjoint* operators, first and foremost what they are.

Definition 4.3.3. *T^* is said to be the adjoint of T if $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in H$.*

And we need this definition to make some sense out of the following lemma:

Lemma 4.3.4. *An operator T is Hilbert-Schmidt if and only if its adjoint T^* is Hilbert-Schmidt as well.*

Proof. Observe that

$$\begin{aligned}\sum_{n=1}^{\infty} \|Te_n\|^2 &= \sum_{n=1}^{\infty} \langle Te_n, Te_n\rangle \\ &= \sum_{n=1}^{\infty} \langle Te_n, \sum_{m=1}^{\infty} \langle Te_n, e_m\rangle e_m\rangle \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle Te_n, e_m\rangle^2 \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle e_n, T^* e_m\rangle^2\end{aligned}$$

If we now repeat this calculation for T^* instead of T, then

$$\sum_{n=1}^{\infty} \|Te_n\|^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle Te_n, e_m\rangle^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle e_n, T^* e_m\rangle^2 = \sum_{m=1}^{\infty} \|T^* e_m\|^2$$

which completes the proof. □

With that out of the way, we can now commit ourselves to deducing the following result:

Theorem 4.3.5. *If $T : H \to H$ is Hilbert-Schmidt, then T is compact.*

Proof. We need to show that the closure of the image TB of a bounded set B is compact. Therefore we define a set V which consists of $x \in H$ for which $\|x\| \leq 1$.
Let $x, y \in V$. Then

$$\begin{aligned}\|Tx - Ty\|^2 &= \langle T(x-y), T(x-y)\rangle \\ &= \langle \left(\sum_{n=1}^{\infty} \langle T(x-y), e_n\rangle e_n\right), \left(\sum_{m=1}^{\infty} \langle T(x-y), e_m\rangle e_m\right)\rangle \\ &= \sum_{n=1}^{\infty} \langle T(x-y), e_n\rangle^2\end{aligned}$$

due to the orthonormality of the e_n.
If we now pick some $N \in \mathbb{Z}_{>0}$, then

$$\begin{aligned}\|Tx - Ty\|^2 &= \sum_{n=1}^{\infty} \langle T(x-y), e_n\rangle^2 \\ &= \sum_{n=1}^{N} \langle T(x-y), e_n\rangle^2 + \sum_{n=N+1}^{\infty} \langle T(x-y), e_n\rangle^2 \\ &= \sum_{n=1}^{N} \langle T(x-y), e_n\rangle^2 + \sum_{n=N+1}^{\infty} \langle x-y, T^* e_n\rangle^2 \\ &\leq \sum_{n=1}^{N} \langle T(x-y), e_n\rangle^2 + \sum_{n=N+1}^{\infty} \|x-y\|^2 \|T^* e_n\|^2 \\ &\leq \sum_{n=1}^{N} \langle T(x-y), e_n\rangle^2 + \sum_{n=N+1}^{\infty} (\|x\| + \|y\|)^2 \|T^* e_n\|^2 \\ &\leq \sum_{n=1}^{N} \langle T(x-y), e_n\rangle^2 + 4 \sum_{n=N+1}^{\infty} \|T^* e_n\|^2\end{aligned}$$

Next, choose some $\epsilon_N > 0$ for which $4 \sum_{n=N+1}^{\infty} \|T^* e_n\|^2 < \epsilon_N$. Because T is Hilbert-Schmidt if and only if T^* is Hilbert-Schmidt (see Lemma 4.3.4), we can choose our ϵ_N to be strictly decreasing as N goes to infinity.
And with that, we have all the ingredients to finalize our proof. Simply take some sequence $\{x^{(n)}\}_{n=1}^{\infty} \subset V$. Then for any two points $x^{(k)}$, $x^{(m)}$ from this sequence

$$\|Tx^{(k)} - Tx^{(m)}\|^2 \leq \sum_{n=1}^{N} \langle T(x^{(k)} - x^{(m)}), e_n\rangle^2 + \epsilon_N \tag{4.1}$$

and as any finite dimensional bounded set is compact, a bounded subset of the space spanned by the vectors $e_1, \ldots, e_N$ is compact as well, which means that we can find a subsequence $\{x^{(n)}_{(N)}\}_{n=1}^{\infty}$ of $\{x^{(n)}\}_{n=1}^{\infty}$ for which there exist an $M_N \in \mathbb{Z}_0$ and an $\epsilon'_N > 0$ such that $\langle T(x^{(k)} - x^{(m)}), e_n\rangle^2 < \epsilon'_N$ for all $m, k > M_N$. Note that this argument can basically be summarized by saying that the sequence one gets by replacing $\{x^{(n)}_{(N)}\}_{n=1}^{\infty}$ by its projection onto the space spanned by $e'_1 \ldots e_N$ is compact. Due to the random nature of ϵ'_N and ϵ_N, we might as well choose $\epsilon_N = \epsilon'_N$, leading, because of (4.1), to

$$\|Tx^{(k)}_{(N)} - Tx^{(m)}_{(N)}\|^2 < 2\epsilon_N \text{ for all } k, m > M_N$$

As such, take the sequence $\{x^{(n)}_{(N)}\}_{n=1}^{\infty}$ and find in the same manner a subsequence $\{x^{(n)}_{(N+1)}\}_{n=1}^{\infty}$ for which

$$\|Tx^{(k)}_{(N+1)} - Tx^{(m)}_{(N+1)}\|^2 < 2\epsilon_{N+1} \text{ for all } k, m > M_{N+1}$$

thereby, through induction, creating subsequences $\{x^{(n)}_{(N)}\}_{n=1}^{\infty} \subset \{x^{(n)}_{(N+1)}\}_{n=1}^{\infty} \subset \{x^{(n)}_{(N+2)}\}_{n=1}^{\infty} \subset \{x^{(n)}_{(N+3)}\}_{n=1}^{\infty} \subset \ldots$ for

$$\|Tx^{(k)}_{(N+L)} - Tx^{(m)}_{(N+L)}\|^2 < 2\epsilon_{N+L} \text{ for all } k, m > M_{N+L}$$

for $L \in \mathbb{Z}_{\geq 0}$, which, because of the strictly decreasing property of the ϵ_N, shows that $\{Tx^n\}_{n=1}^{\infty}$ indeed contains a convergent subsequence, proving T to be compact. □

So what does this mean for integral operators? Only all we could ever hope for:

Theorem 4.3.6. *Every integral operator that defines a Green's function is compact.*

Proof. Our Green's functions are defined on $L^2(\Omega)$ with the restriction that all functions u in their domain fulfil the boundary condition that $u(x) = 0$ for $x \in \partial\Omega$. Because of Fourier Analysis, there exists an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ that spans this space, so if we write

$$(Kf)(x) = \int_{\Omega} k(x, \xi) f(\xi) d\xi$$

and thus $(Kf)(x) = \langle k_x, f\rangle$, where $\langle \cdot, \cdot \rangle$ signifies the inner product on $L^2(\Omega)$, $f \in L^2(\Omega)$ and k_x is the function $k(x, \xi)$ for fixed x, then, due to Theorem 4.3.5, it suffices to show that $\sum_{n=1}^{\infty} \|Ke_n\|^2 < \infty$:

$$\begin{aligned}\sum_{n=1}^{\infty} \|Ke_n\|^2 &= \sum_{n=1}^{\infty} \int_{\Omega} \langle k_x, e_n \rangle^2 \, dx \\ &= \int_{\Omega} \sum_{n=1}^{\infty} \langle k_x, e_n \rangle^2 \, dx \\ &= \int_{\Omega} \|k_x\|^2 dx \\ &= \int_{\Omega} \int_{\Omega} k^2(x,\xi) \, d\xi \, dx < \infty\end{aligned}$$

which shows that

$$\sum_{n=1}^{\infty} \|Ke_n\|^2 < \infty$$

proving K to be Hilbert-Schmidt and therefore compact □

Hopefully, this gives the reader enough motivation to delve into the theory of compact operators, as we are about to do:

Theorem 4.3.7. *Let T be a compact operator on a Hilbert space H and a some positive constant. Then there are only finitely many eigenvectors of T with eigenvalues μ for which $|\mu| \geq a$.*

Proof. Suppose that the theorem is false. In that case there must exist an infinite sequence $\{x_n\}_{n=1}^{\infty}$ of linearly independent eigenvectors of T, with a corresponding sequence of eigenvalues $\{\mu_n\}_{n=1}^{\infty}$ for which $|\lambda| \geq a$. We plan on using this information to construct a bounded infinite sequence $\{z_n\}_{n=1}^{\infty}$ with the property that its image $\{Tz_n\}_{n=1}^{\infty}$ has no convergent subsequence, thereby arriving at a contradiction with the compactness of T. As such, let $H_n = span\{x_1, \ldots, x_n\}$. Because of Lemma 1.2.1, there exists a $y_n \in H_n$ for every n for which $d(y_n, H_{n-1}) = \|y_n\| = 1$. Now write $y_n = \sum_{i=1}^{n} c_i x_i$, where the c_i are constants and define $z_n = \frac{y_n}{\lambda_n}$. Then $Tz_n - y_n = \frac{1}{\lambda_n} T\left(\sum_{i=1}^{n} c_i x_i\right) - \sum_{i=1}^{n} c_i x_i = \sum_{i=1}^{n-1} c_i \left(\frac{\lambda_i}{\lambda\, n} - 1\right) x_i \in H_{n-1}$ and thus, if $n > m$, then $Tz_m \in H_m$ and $\|Tz_n - Tz_m\| \geq d(Tz_n, H_m) \geq d(y_n, H_{n-1}) = \|y_n\| = 1$, by which it is now plain as day that $\{Tz_n\}_{n=1}^{\infty}$ can't possibly have a convergent subsequence and therefore, because this contradicts the compactness of T, the theorem must be true. □

We need another theorem to verify that for a compact operator T, $\sigma(T)$ consists of eigenvalues alone:

Theorem 4.3.8. *If $\lambda \neq 0$ is not an eigenvalue of a compact operator T on a Hilbert space H, then $\lambda \notin \sigma(T)$.*

Proof. For starters, replacing T with $\frac{1}{\lambda}T$, we can assume λ to be equal to 1. Secondly, if we can show that $I - T$ is bijective then we are finished, so let's have a go at that, shall we? A first observation should be that the image of H under $I - T$, while not necessarily equal to H, is certainly closed. We should probably elaborate a bit on that: Define $V = I - T$ and as such, $Ker(V) = \{0\}$. Suppose that $\inf_{\|x\|=1} \|Vx\|$ is not bounded from below. Then there exist $\{x_n\}_{n=1}^{\infty} \subset H$, $\|x_n\| = 1$, such that Vx_n tends to zero as n grows large. Because we know that the x_n will never become zero and $V = I - T$, there must exist a subsequence $\{x'_n\}_{n=1}^{\infty}$ for which Tx'_n (T is compact) goes to some $y \neq 0$. So $\lim_{n\to\infty} x'_n = \lim_{n\to\infty}(Vx'_n + Tx'_n) = 0 + y = y$. But that means that $Vy = 0$, while $y \neq 0$, contradicting the fact that $Ker(V) = \{0\}$. So we correctly observed that the image of H under $\lambda I - T$ is closed. And this is where the fun begins. We want to prove that V is bijective. We know that VH is closed in H, so we construct subspaces H_n of H by $H_{n+1} = VH_n$ and $H_1 = H$, suspecting that it will be possible to prove that there must exist a space H_k such that $H_k = H_{k+1}$. Yet again we can commend ourselves on our amazing intuition, because had it been false, then we could have constructed a sequence $\{\xi_n\}_{n=1}^{\infty}$ with $\xi_n \in H_n \backslash H_{n+1}$ and $d(\xi_n, H_{n+1}) = \|\xi_n\| = 1$ because of Lemma 1.2.1, giving $1 = d(\xi_n, H_{n+1}) = d(V\xi_n + T\xi_n, VH_n) = d(T\xi_n, VH_n) = d(T\xi_n, H_{n+1})$, leading to $\|T\xi_n - T\xi_m\| \geq 1$ for $n > m$ and a contradiction with the compactness of T, because $\{T\xi_n\}_{n=1}^{\infty}$ should have a convergent subsequence on the grounds of $\{\xi_n\}_{n=1}^{\infty}$ being boun ded. So there exists a k such that $H_k = H_{k+1}$. Wouldn't it be great if that k equals 1? If not, then there must be an m for which $H_m \subset H_{m+1} = H_{m+2}$ and of course $H_m \neq H_{m+1}$. But this means that the image of $H_m \backslash H_{m+1}$ under $I - T$ is 0, resulting in $H_m \backslash H_{m+1} \subset Ker(V)$. But $Ker(V) = 0$, so k is indeed 1. Now let there be much rejoicing, for our theorem has been proved. ☐

And that's about all the data we need on compact operators for the moment.
Because of the *self-adjointness* of our $K(x, \xi)$, these integral operators are *normal* as well, because

Definition 4.3.9. *An operator T on a Banach space H is said to be normal if $TT^* = T^*T$ where T^* is the adjoint of T.*

Some tidbits about normal operators T on a Hilbert space H:

Lemma 4.3.10. $\|Tx\| = \|T^*x\|$.

Proof. $\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle = \langle x, TT^*x\rangle = \langle T^*x, T^*x\rangle = \|T^*x\|^2$ □

Lemma 4.3.11. $Ker(T) = Ker(T^*)$.

Proof. Because of Lemma 4.3.10 $Tx = 0$ only if $\|Tx\| = 0$, so because of Lemma 4.3.10, $Tx = 0$ only if $\|T^*x\| = 0$. □

Lemma 4.3.12. $\|T^n\| = \|T\|^n$

Proof. Firstly, let V be Hermitian, that is $V^*V = V^2$. Then we have that

$$\|Vx\|^2 = \langle Vx, Vx\rangle = \langle x, V^*Vx\rangle = \langle x, V^2x\rangle \leq \|x\|\|V^2x\|$$

By the same argument, for $m \in \mathbb{Z}_{>0}$, $\|V\|^{2^m} \leq \|V^{2^m}\|$. For a normal operator T, the operator T^*T is Hermitian, so $\|T^*Tx\|^{2^m} \leq \|(T^*)^{2^m}T^{2^m}x\|\|x\|$ and therefore $\|T^*T\|^{2^m} \leq \|(T^*)^{2^m}\|\|T^{2^m}\|$. But $\|T^*T\| = \|T\|^2$ and $\|(T^*)^{2^m}\| = \|T^{2^m}\|$ by the same reasoning as in the proof of Lemma 4.3.10, from which we can derive that $\|T\|^{2^m} \leq \|T^{2^m}\|$. Because it goes without saying that $\|T\|^{2^m} \geq \|T^{2^m}\|$, we can write $\|T\|^{2^m} = \|T^{2^m}\|$ for $m \geq 0$. To complete our proof, let $n \in \mathbb{Z}_{>0}$ and $n \leq 2^m$ for some $m \geq 1$. Then

$$\|T^{2^m}\| = \|T^{2^m-n}T^n\| \leq \|T\|^{2^m-n}\|T^n\| \leq \|T\|^{2^m-n}\|T\|^n = \|T\|^{2^m}$$

and because $\|T^{2^m}\| = \|T\|^{2^m}$, $\|T\|^{2^m-n}\|T^n\| = \|T\|^{2^m}$. Dividing both sides by $\|T\|^{2^m-n}$ gives $\|T^n\| = \|T\|^n$. □

Lemma 4.3.13. $r(T) \geq \|T\|$.

Proof. Because of Theorem 4.2.4, $r(T) \geq \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ and by Lemma 4.3.12, the result follows. □

Lemma 4.3.14. $Ker(\lambda I - T) \perp Ker(\mu I - T)$ *for* $\lambda \neq \mu$.

Proof. Let $v_\lambda \in Ker(\lambda I - T)$ and $v_\mu \in Ker(\mu I - T)$. Then because of Lemma 4.3.11, $\lambda\langle v_\lambda, v_\mu\rangle = \langle Tv_\lambda, v_\mu\rangle = \langle v_\lambda, T^*v_\mu\rangle = \langle v_\lambda, \mu v_\mu\rangle$. (observe that $\|\mu v - Tv\|^2 = \|\mu v - Tv\|^2$ by the same argument as in the proof of Lemma 4.3.10), so $(\lambda - \mu)\langle v_\lambda, v_\mu\rangle = 0$. □

One necessary definition before we go on:

Definition 4.3.15. *A space H is said to be invariant under an operator T if $TH \subset H$.*

And one more lemma:

Lemma 4.3.16. *If H_1 and H_2 are mutually orthogonal spaces that are invariant under T with $H = H_1 \oplus H_2$ and we define T_1 as the operator T restricted to H_1 and T_2 as T restricted to H_2, then $\|T\| = \max\{\|T_1\|, \|T_2\|\}$*

Proof. Self-evidently, $\|T\| \geq \max\{\|T_1\|, \|T_2\|\}$. Because of Lemma 1.2.1, every $x \in H$ can be uniquely written as $x = x_1 + x_2$, $x_1 \in H_1$, $x_2 \in H_2$, meaning that $T_1x_1 = Tx_1 \perp Tx_2 = T_2x_2$ and

$$\begin{aligned}\|Tx\|^2 &= \|Tx_1\|^2 + 2\langle Tx_1, Tx_2\rangle + \|Tx_2\|^2 = \|T_1x_1\|^2 + 0 + \|T_2x_2\|^2 \\ &\leq \|T_1\|^2\|x_1\|^2 + \|T_2\|^2\|x_2\|^2 \leq \max\{\|T_1\|^2, \|T_2\|^2\}\left(\|x_1\|^2 + \|x_2\|^2\right) \\ &= \max\{\|T_1\|^2, \|T_2\|^2\}\|x\|^2\end{aligned}$$

giving $\|T\| \leq \max\{\|T_1\|, \|T_2\|\}$ and therefore $\|T\| = \max\{\|T_1\|, \|T_2\|\}$. □

And now for the theorem it all comes down to:

Theorem 4.3.17. *Let T be a normal, compact operator on a Hilbert space H, let $H_\lambda = Ker(\lambda I - T)$ and let P_λ be the orthogonal projection of H onto $H - \lambda$. Then T has countably many non-zero eigenvalues λ_k, the P_{λ_k} are orthogonal, id est $P_{\lambda_k}P_{\lambda_l} = 0$ for $k \neq l$ and $Tx = \sum_{k=1}^{\infty} \lambda_k P_{\lambda_k} x$ for all $x \in H$.*

Proof. Because of Theorem 4.3.7, Theorem 4.3.8 and Lemma 4.3.14, we only need to check that $T = \sum_{k=1}^{\infty} \lambda_k P_{\lambda_k}$. Thus we define $H_n = \sum_{k=1}^{n}$, $U_n = H_n^\perp$, $S_n = \sum_{k=1}^{n} \lambda_k P_{\lambda_k}$ and we choose n large enough for $|\lambda_k|$ to be smaller than some positive ϵ for all $k > n$. Both H_n and U_n are invariant under T and S_n. Therefore we define T_{H_n} and S_{H_n} to be the restrictions of respectively T and S_n to H_n and T_{U_n} and S_{U_n} to be the restrictions of respectively T and S_n to U_n. Then by Lemma 4.3.16, $\|T - S_n\| = \max\{\|T_H - S_H\|, \|T_U - S_U\|\} = \max\{0, \|T_{U_n} - 0\|\} = \|T_{U_n}\|$. By Lemma 4.3.13, $r(T_{U_n}) \geq \|T_{U_n}\|$, so $\|T - S_n\| \leq r(T_{U_n})$. The eigenvectors of T lying within U_n corr espond by construction to eigenvalues λ for which $|\lambda| < \epsilon$, thus resulting in $\|T - S_n\| \leq r(T_{U_n}) = sup\{|\lambda| : \lambda \in \sigma(T_{U_n})\} < \epsilon$ and therefore $T = \sum_{k=1}^{\infty} \lambda_k P_{\lambda_k}$ □

Note that this does not necessarily imply that the eigenfunctions of a normal, compact operator span the entire function space. For that, an extra assumption is needed, namely that for a normal, compact operator T, $Tx = 0$ only if $x = 0$. Because in that case, let $x \in H$ be written as $x = x_\lambda + x_{\lambda^\perp}$, where x_λ lies in the span of the eigenfunctions and $x_{\lambda^\perp}$ does not. Then because of Theorem 4.3.17, $Tx = Tx_\lambda + Tx_{\lambda^\perp} = Tx_\lambda + 0$. By our new assumption, $x_{\lambda^\perp} = 0$, proving that now the eigenfunctions span the function space. Coincidentally, this also proves that the eigenfunctions of Green's operators span the entire function space upon which they are defined, because Green's operators correspond with linear partial differential operators. If inserting a nonzero function into a Green's operator would give 0, then that would mean that there exists a linear partial differential operator that gives a nonzero function if 0 is inserted, which of course is impossible. So:

Theorem 4.3.18. *The eigenfunctions of a Green's operator span the entire function space upon which it has been defined.*

4.4 Two Theorems

This section is about two theorems that have no obvious relation to each other, but had to be put somewhere and this place is as good a place as any place.

Theorem 4.4.1. *(The Dirichlet Principle) Define a functional F on $C^2(\Omega)$ by $F(u) = \int_\Omega |\nabla u|^2 \, dx$, $u \in C^2(\Omega)$, with the restriction that $u = \phi$ on $\partial\Omega$ for some fixed $\phi \in C^2(\partial\Omega)$. Then $F(u)$ is minimal under this condition if and only if u is harmonic.*

Proof. Assume h to be the function that is harmonic in Ω and fulfils the boundary condition and u to be the function for which $F(u)$ is minimal and write $w = u - h$. Let $v \in C^2(\Omega)$ be a function that is equal to zero on $\partial\Omega$ and $0 < \epsilon << 1$, ϵ a constant. Then

$$F(u + \epsilon v) = F(w + h + \epsilon v) = \int_\Omega |\nabla(w + h + \epsilon v)|^2 \, dx$$

$$= \int_\Omega \left(|\nabla(w + h)|^2 + 2\epsilon \nabla(w + h) \cdot \nabla v + \epsilon^2 |\nabla v|^2\right) dx$$

which means that

$$\frac{\partial}{\partial \epsilon} F(u + \epsilon v) = \int_\Omega \left(\nabla(w + h) \cdot \nabla v + 2\epsilon |\nabla v|^2\right) dx \qquad (4.2)$$

And because $F(w + h)$ is minimal, $\frac{\partial}{\partial \epsilon} F(w + h + \epsilon v) \mid_{\epsilon=0} = 0$ from which follows that (4.2) can be rewritten as

$$0 = \int_\Omega (\nabla(w + h) \cdot \nabla v) \, dx =$$

$$0 - \int_\Omega v(x) \Delta(w + h) \, dx = - \int_\Omega v(x) \Delta w \, dx$$

which holds for all $v \in C^2(\Omega)$ that are equal to zero on $\partial\Omega$, which shows that $\Delta w = 0$ and thus, because $w = u - h$, $\Delta u = \Delta h = 0$, from which we deduce that $F(u)$ is minimal only if u is harmonic. ☐

Theorem 4.4.2. *(Jensen's inequality) Let $J : \mathbb{R} \to \mathbb{R}$ be a convex function, $f : \Omega \to \mathbb{R}$ some function for which both $\int_\Omega |f(x)|\, dx$ and $\int_\Omega |J(f(x))|\, dx$ are finite. Furthermore, define $\langle f \rangle = \frac{1}{\Omega} \int_\Omega f(x)\, dx$. Then*

$$\langle J \circ f \rangle \geq J(\langle f \rangle)$$

Proof. When a function J is convex, it means that for all $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$ $J(\lambda x + (1-\lambda)y) \leq \lambda J(x) + (1-\lambda)J(y)$. To prove Jensen's inequality, we'll be needing that J is continuous and has both a left and a right derivative in each point. Let's start with the continuity.
Define

$$\Delta_{x,y} = \frac{J(y) - J(x)}{y - x}$$

Let $x < y < z$ and $y = \lambda x + (1-\lambda)z$. Then

$$\begin{aligned} \Delta_{x,y} = \frac{J(y) - J(x)}{y - x} &\leq \frac{\lambda J(x) + (1-\lambda)J(z) - J(x)}{\lambda x + (1-\lambda)z - x} \\ &= \frac{(1-\lambda)(J(z) - J(x))}{(1-\lambda)(z - x)} = \Delta_{x,z} \end{aligned}$$

and

$$\begin{aligned} \Delta_{y,z} = \frac{J(z) - J(y)}{z - y} &\geq \frac{J(z) - \lambda J(x) - (1-\lambda)J(z)}{z - \lambda x - (1-\lambda)z} \\ &= \frac{\lambda(J(z) - J(x))}{\lambda(z - x)} = \Delta_{x,z} \end{aligned}$$

giving

$$\Delta_{x,y} \leq \Delta_{x,z} \leq \Delta_{y,z} \tag{4.3}$$

With this in mind, choose a w with $w < x < y < z$, so that because of (4.3)

$$\frac{y-x}{x-w}\,(J(x) - J(w)) \leq J(y) - J(x) \leq \frac{y-x}{z-y}\,(J(z) - J(y))$$

which means that

$$\lim_{y \downarrow x} \frac{y-x}{x-w}\,(J(x) - J(w)) \leq \lim_{y \downarrow x} J(y) - J(x) \leq \lim_{y \downarrow x} \frac{y-x}{z-y}\,(J(z) - J(y))$$

and thus $\lim_{y \downarrow x} J(y) - J(x) = 0$. By the same argument, $\lim_{x \uparrow y} J(y) - J(x) = 0$ as well, revealing J to be continuous. Once more applying (4.3) together with the continuity of J, shows that J is also left and right differentiable, because

$$\Delta_{w,y} = \lim_{x \uparrow y} \Delta w, x \leq \lim_{x \uparrow y} \Delta_{x,y} \lim_{x \uparrow y} \Delta_{x,w} = \Delta_{y,w}$$

providing us with a left derivative and by similar reasoning $\lim_{y \downarrow x} \Delta_{x,y}$ exists as well.
We now have all we need to finish the proof.
For any point $x \in \mathbb{R}$, let m_{1x} be the left derivative and m_{2x} be the right derivative of J in x. Then, once more making use of (4.3),

$$\frac{J(x) - J(w)}{x - w} \leq m_{1x} \tag{4.4}$$

and

$$\frac{J(y) - J(x)}{y - x} \geq m_{2x} \tag{4.5}$$

thus showing that by (4.4) $J(w) \geq J(x) + m_{1x}(w - x)$ and through (4.5) $J(y) \geq J(x) + m_{2x}(y - x)$, which means that for any fixed $x \in \mathbb{R}$, there exists a constant C such that $J(y) \geq J(x) + C(y - x)$ for all $y \in \mathbb{R}$. Should we at this time replace y with $f(t)$ and x with $\langle f \rangle$, integration of both sides of the inequality then results in

$$\begin{aligned} \int_\Omega J(f(t))\,dt &\geq \int_\Omega J(\langle f \rangle)\,dt + C \int_\Omega (f(t) - \langle f \rangle)\,dt \\ &= |\Omega| J(\langle f \rangle) + C(|\Omega|\langle f \rangle - |\Omega|\langle f \rangle \end{aligned}$$

from which we can deduce that $\langle J \circ f \rangle \geq J(\langle f \rangle)$. □

4.5 Curves in $\mathbb{R}^3$

This section functions as a really short crash course on twice continuously differentiable curves in $\mathbb{R}^3$. If you'd like a less concise introduction to the subject then I can warmly recommend "Elements of Differential Geometry" by Richard S. Millman and George D. Parker, Prentice Hall, 1977.
Our main purpose will be proving that all twice continuously differentiable curves in $\mathbb{R}^3$, which will be called 'curves' in the remainder of the text, are determined up to rotation and translation by *curvature* and *torsion*. But let us begin where precedence dictates us to begin: at the beginning.
As mentioned in the proof of Theorem 3.3.1, every curve $\gamma(t)$ can be parametrized so that $\|\gamma'(t)\| = 1$, where $\|\cdot\|$ represents the Euclidian norm

on $\mathbb{R}^3$ and naturally, we will be using the Euclidian inner product $\langle \cdot, \cdot \rangle$ as well. A curve parametrized in this way is called *parametrized by arc length*, which from now on we will assume all our curves to be. In that case, it is customary to write $T(t)$ for $\gamma'(t)$. Furthermore, we write $N(t) = \frac{T'(t)}{\|T'(t)\|}$ and $B(t) = T(t) \times N(t)$, so that $\|T\| = \|N\| = \|B\| = 1$, because T and N are perpendicular (as we will discover shortly). The basic idea will be to create a plane tangent to the curve γ and the first thing to do is defining a function that indicates 'how much γ locally twists whithin this plane.' This function will be called the *curvature* and it will be represented by $\kappa(t)$. Then we will define a second function that measures 'how much γ locally leaves the plane.' This function will be called the *torsion* and it will be represented by $\tau(t)$. Intuitively, it must be true that if you know how much a curve 'twists within the tangent plane' and to what extent it 'leaves this plane', there can't be much variation left for your curve. If this all sounds to vague, don't worry, it will all become clear.

We will define $\kappa(t)$ as $\|T'(t)\|$ and $\tau(t)$ as $-\langle B'(t), N(t)\rangle$. If we choose our tangent plane to be spanned by T and N, which are orthonormal vectors, as it will turn out, then κ should give an indication of 'how much the curve bends' within the $(T, N)-$plane and τ should give some idea of 'how much γ gets out of the $(T, N)-$plane. Once again, this is all rather intuitive argumentation.

So let's get more exact:

$\langle T(t), T(t)\rangle = 1$, giving $0 = \frac{d}{dt}\langle T(t), T(t)\rangle = \langle T'(t), T(t)\rangle + \langle T(t), T'(t)\rangle = 2\kappa(t)\langle T(t), N(t)\rangle$, revealing that T and N are perpendicular.

Because $T \times N = B$, the vectors T, N and B span $\mathbb{R}^3$, which means that T', N' and B' can be written as linear combinations of T, N and B: T' is easy, because $T' = \kappa N$.

Secondly, let's find an expression for N' of the type $a_1T + a_2N + a_3B$, where a_1, a_2 and a_3 are constants. Clearly, $a_1 = \langle N', T\rangle$, $a_2 = \langle N', N\rangle$ and $a_3 = \langle N', B\rangle$ (simply take the inner products of N' with T, N and B respectively). $\langle N, N'\rangle = 0$ by the same reasoning as why T and N are perpendicular.

Lastly,

$$\begin{aligned} 0 = \frac{d}{dt}\langle N(t), T(t)\rangle &= \langle N'(t), T(t)\rangle + \langle N(t), T'(t)\rangle \\ &= \langle N'(t), T(t)\rangle + \kappa(t)\langle N(t), N(t)\rangle \end{aligned}$$

providing us with $\langle N'(t), T(t)\rangle = -\kappa(t)$ and

$$\begin{aligned} 0 = \frac{d}{dt}\langle N(t), B(t)\rangle &= \langle N'(t), B(t)\rangle + \langle N(t), B'(t)\rangle \\ &= \langle N'(t), T(t)\rangle - -\langle B'(t), N(t)\rangle \end{aligned}$$

providing us with $\langle N'(t), T(t)\rangle = \tau(t)$ and more importantly
$N(t) = -\kappa(t)T(t) + \tau(t)B(t)$. Following the same techniques gives us that

$B'(t) = -\tau(t)N(t)$.
To summarize things in matrix form:

Theorem 4.5.1. *(Frenet-Serret)*

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

And with that we have existence of a curve given some $\kappa(t)$ and $\tau(t)$ by the same reasoning as for the existence of an ordinary differential equation and uniqueness up to translation, since we have uniqueness of T, which is the derivative of our curve. Or, to put it differently

Corollary 4.5.2. *Every curve in $\mathbb{R}^3$ is uniquely determined up to rotation, translation and reflection by its curvature and its torsion.*

This section served as background information to Theorem 3.3.1, giving insight into the concepts of curvature and torsion. Observe that a circle can be interpreted as a curve in $\mathbb{R}^3$ with torsion equal to zero and constant curvature, which is exactly what you want at the end of Theorem 3.3.1.

Chapter 5

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