On the Chern correspondence for principal fibre bundles with complex reductive structure group

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A tutti coloro
che hanno reso possibile
la felicità di questi anni:
la mia famiglia,
la mia bici,
la Regina
e le NS.

Contents

Introduction

This thesis is on certain aspects of differential geometry of principal fibre bundles and vector bundles. Our *results* are not new (some well known, others only to experts), but for many of them the *proofs* are not or not easy to find in the literature.

A first example of this is the following. Let V be a finite dimensional vector space, $P(M, G)$ a principal fibre bundle with structure group $G = \mathrm{Aut}(V)$ over a manifold M, and $\pi : E \to M$ a vector bundle with fibre V, associated to $P(M, G)$ via the standard representation of $\mathrm{Aut}(V)$ on V. Consider a connection A on $P(M, G)$ with curvature form Ω_A and let R_A be the curvature of the corresponding connection D_A on E. Both Ω_A and R_A can be viewed as forms with values in the endomorphism bundle of E, and *as such they are equal.* We give a proof of this natural fact, which we (surprisingly) were not able to find elsewhere.
Another (also quite general, but more involved) example is the *Chern correspondence*, which is the central topic of this thesis. In particular, in Section 3.2 we discuss an important formula, which is used in [19]. The proof of this formula is in [19] extremely sketchy; we present here a detailed one, for which we make use of a great portion of the results of the previous chapters.

In the following we give some comments about the contents of the thesis. For more details on the contents of the chapters, we refer also to the introductions heading each of them.

In the first chapter we gather basic definitions and results on vector and principal fibre bundles, in particular about reductions, connections and curvature. Special attention is payed to the relation between principal fibre bundles and vector bundles associated to them via representations. We try to present the material in such a way that our text could be used as a basis for an advanced course (MSc-level), compiling facts otherwise scattered about the literature. However, for the sake of brevity, where results appear in standard textbooks (mainly [14] and [13]), we mostly refer to these for proofs. In Chapter 2, besides giving some background information about complex manifolds and complex reductive Lie groups, we show how the material of the first chapter should be generalized to complex fibre bundles, and we present some results specific of the complex case. In particular, the main topics of this chapter are holomorphic and almost holomorphic structures on complex principal fibre bundles and the Chern correspondence in the vector bundle case.

In the following we discuss in some detail the contents of Section 3.1.

Let G be a complex reductive Lie group with a compact real form K and let $P\,(M,G)$ be a principal fibre bundle over a complex manifold M, with a K-reduction $Q\,(M,K)$. Denote by $\mathcal{G}^P$ and $\mathcal{G}^Q$ the groups of gauge transformations of $P\,(M,G)$ and $Q\,(M,K)$ respectively, i.e. the groups of sections of the adjoint bundles $P\times_{Ad} G$ and $Q\times_{Ad} K$. Note that $\mathcal{G}^Q$ can be regarded as a subgroup of $\mathcal{G}^P$. Consider the set $\overline{\mathcal{C}}(P)$ of almost holomorphic structures on $P\,(M,G)$ and the subset $\mathcal{C}(P)$ of integrable ones. We have a natural action of $\mathcal{G}^P$ on $\overline{\mathcal{C}}(P)$, leaving $\mathcal{C}(P)$ invariant, and ${\mathcal{C}(P)}/{\mathcal{G}^P}$ (resp. ${\overline{\mathcal{C}}(P)}/{\mathcal{G}^P}$) is the set of isomorphism classes of (almost) holomorphic structures on $P\,(M,G)$. Finally, consider the set $\mathcal{A}(Q)$ of connections on $Q\,(M,K)$ and the subset $\mathcal{A}^{1,1}(Q)$ of integrable ones, i.e. connections whose curvature is of type $(1,1)$. We have a natural action of $\mathcal{G}^Q$ on $\mathcal{A}(Q)$, leaving $\mathcal{A}^{1,1}(Q)$ invariant, and ${\mathcal{A}(Q)}/{\mathcal{G}^Q}$ (resp. ${\mathcal{A}^{1,1}(Q)}/{\mathcal{G}^Q}$) is the set of gauge equivalence classes of (integrable) connections on $Q\,(M,K)$.

The main result of Section 3.1 is the following (**Chern correspondence**).

Theorem 1 *There is a natural 1-1 correspondence* $\overline{\mathcal{C}}(P)\xleftrightarrow{\text{1-1}}\mathcal{A}(Q)$*, equivariant with respect to the action of* $\mathcal{G}^Q$*, such that the elements of* $\mathcal{C}(P)$ *correspond precisely to the elements of* $\mathcal{A}^{1,1}(Q)$*. In particular, the* $\mathcal{G}^Q$*-action on* $\mathcal{A}(Q)$ *extends via this correspondence to a* $\mathcal{G}^P$*-action, and we get natural bijections*

$$ {\overline{\mathcal{C}}(P)}/{\mathcal{G}^P}\xleftrightarrow{\text{1-1}}{\mathcal{A}(Q)}/{\mathcal{G}^P}\quad\text{and}\quad {\mathcal{C}(P)}/{\mathcal{G}^P}\xleftrightarrow{\text{1-1}}{\mathcal{A}^{1,1}(Q)}/{\mathcal{G}^P}. $$

This result is known, and used for example in [19] and [23], but a rigorous proof seems not available in the literature. The goal of Section 3.1 is not only to present a proof of it, but also to show that the correspondence in Theorem 1 is a generalization of the classical Chern correspondence in the vector bundle case (as treated for example in [13]), i.e. the bijection between semiconnections (resp. holomorphic structures) and (integrable) h-connections for a complex vector bundle $\pi : E \to M$ (where M is a complex manifold) with an Hermitian metric h. Having this in mind, from the beginning the treatment of vector bundles is developed parallel to that of principal fibre bundles, and very much emphasis is placed throughout the thesis in the relation between principal fibre bundles and associated vector bundles (see in particular Example 1.2.10, Example 1.2.17, Proposition 1.4.9, Lemma 1.4.13, Example 1.4.22, Proposition 2.5.5 and Example 3.1.2). It should be noticed, however, that the results regarding principal fibre bundles are proved directly, without using the corresponding facts over vector bundles. The only exception of this is the proof that under the bijection $\overline{\mathcal{C}}(P)\xleftrightarrow{\textit{1-1}}\mathcal{A}(Q)$ the holomorphic structures on $P(M,G)$ correspond precisely to the integrable connections on $Q(M,K)$ (see Proposition 3.1.5). Here, by reducing to the vector bundle case via a holomorphic faithful representation of G on $\mathbb{C}^n$, we make use of a deep integrability theorem which is proved in the classical paper [1] (see Theorem 2.3.10).

Given a fixed holomorphic structure J on $P(M,G)$, we denote by $A_{J,Q}$ the extension to $P(M,G)$ of the connection on $Q(M,K)$ corresponding to J under the Chern correspondence. Note that by Theorem 1 its curvature $\Omega_{A_{J,Q}}$ is a $(1,1)$-form. One of the main goals in [19] is to show that, if M is compact, under certain conditions on J and for certain elements C in the center of the Lie algebra of G, there exists a K-reduction $Q(M,K)$ of $P(M,G)$ for which the **Hermite-Einstein equation**

$$\Lambda_g\left(\Omega_{A_{J,Q}}\right) = C$$

is satisfied, where Λ_g is the contraction operator associated to a fixed Hermitian metric g on M, mapping $(1,1)$-forms on M to 0-forms. The proof and even the precise statement of this result go far beyond the scope of this thesis. We will only be concerned, in Chapter 4, in a necessary condition on C in order to have a solution of the Hermite-Einstein equation.

Only a general knowledge of differentiable manifolds (including differential forms) and Lie groups is required to read this thesis. However, we give complete references for all non-trivial results we use. The treatment of fibre bundles and the material we need on complex manifolds and Lie groups are developed from the first principles.

Notation and conventions

Throughout the thesis, "manifold" stands for "differential (i.e. $\mathcal{C}^\infty$) manifold". The tangent bundle of a manifold M will be denoted by TM and the tangent space at a point p by T_pM. When a vector $X \in T_pM$ is regarded as an element of TM, it is denoted by (p,X). If X is a vector field on M, i.e. a smooth section $M \to TM$, then for a point p of M we denote by X_p the element $X(p)$ of T_pM, but we prefer the second notation for vector fields with too many subscripts (e.g. $\left(\widehat{X}_1{}^h\right)_A$). We denote the space of differential forms of degree r on M by $\mathcal{A}^r(M)$. Given a smooth map $f : N \to M$, we write f_* for the differential and f^* for the pullback on forms. In this thesis a smooth map $f : N \to M$ is called an embedding if it is an injective immersion; thus, with this definition, the image $f(N)$ of an embedding is not necessarily a submanifold of M (but this is the case when $f : N \to f(N)$ is a homeomorphism, where $f(N) \subset M$ has the relative topology, see for example [9, Theorem 3.1 of Chapter 1]). Given a cover (i.e. open cover) $\{\mathcal{U}_i,\, i \in I\}$ of a manifold M, we denote by $\mathcal{U}_{ij}$, for $i,j \in I$, the intersection $\mathcal{U}_i \cap \mathcal{U}_j$. All vector spaces in this thesis are real or complex finite dimensional vector spaces. Given a vector space V and its dual V^*, we denote by $\langle\,,\,\rangle$ the dual pairing $V \times V^* \to \mathbb{R}$ (resp. $\mathbb{C}$). The wedge product $\bigwedge^k V^* \times \bigwedge^l V^* \to \bigwedge^{k+l} V^*$, $(\alpha, \beta) \mapsto \alpha \wedge \beta$ is defined for us by

$$(\alpha\wedge\beta)\,(v_1, \ldots, v_{k+l}) := \frac{1}{(k+l)!} \sum_{\sigma} (-1)^{\sigma} \alpha \left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) \beta \left(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}\right)$$

for $v_1, \ldots, v_{k+l} \in V$, where the summation is taken over all permutations σ of $(1, \ldots, k+l)$; note that the factor $\frac{1}{(k+l)!}$ does not appear in the definition given by some textbooks. Finally, we denote by $e_1, \ldots, e_n$ the canonical basis of $\mathbb{R}^n$ or $\mathbb{C}^n$.

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Chapter 1

Fibre bundles

Definition 1.0.1 *Let F and M be manifolds. A* ***fibre bundle*** *over M with typical fibre F consists of a manifold E and a smooth map $\pi : E \to M$ (*projection*) such that the condition of* local triviality *is satisfied, i.e. there exist a cover $\{\mathcal{U}_i,\, i \in I\}$ of M and diffeomorphisms*

$$\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times F$$

(local trivializations) *making the following diagram commutative.*

$$\begin{array}{ccc} \pi^{-1}(\mathcal{U}_i) & \xrightarrow{\theta_i} & \mathcal{U}_i \times F \\ & \searrow^{\pi} & \downarrow{\scriptstyle pr_1} \\ & & \mathcal{U}_i \end{array}$$

E is called the total space *of the fibre bundle and M the* base space*. Usually we will write $\pi : E \to M$ (or simply E) for a fibre bundle over M with total space E and projection $\pi : E \to M$. A* ***section*** *of a fibre bundle $\pi : E \to M$ is a smooth map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}_M$, i.e. $\sigma(p) \in E_p$ for all $p \in M$. If there is a global trivialization $\theta : E \to M \times F$, then E is called a* ***trivial bundle****. A* ***homomorphism*** *between two fibre bundles $\pi_E : E \to M$ and $\pi_F : F \to N$ consists of two smooth maps $f : E \to F$ and $f' : M \to N$ such that*

$$\pi_F \circ f = f' \circ \pi_E \,.$$

If $M = N$, $f' : M \to M$ is the identity and $f : E \to F$ is a diffeomorphism, then f is called an ***isomorphism*** *between the fibre bundles $\pi_E : E \to M$ and $\pi_F : F \to M$.*

The set $E_p := \pi^{-1}(p)$ is called the *fibre* of E over the point $p \in M$. It is a closed submanifold of E, diffeomorphic to F. In the two special cases of fibre bundles that we will consider, the fibres will have an additional structure: a linear structure in the case of vector bundles and the structure of a G-space, where G is some Lie group, in the case of principal fibre bundles.

In the first two paragraphs of this chapter we will give an outline of those aspects of vector bundles and principal fibre bundles that are needed in the rest of the thesis. In particular, we will focus on the correspondence between vector bundles and principal fibre bundles with structure group $GL(n, \mathbb{R})$.

A connection on a vector bundle is a geometric structure which enables us to differentiate sections in the direction of vector fields of the base manifold. A connection on a principal fibre bundle is a *horizontal* distribution on the total space which is invariant by the action of the structure group. In Paragraphs 1.3 and 1.4 we will treat connections on vector bundles and on principal fibre bundles and we will show that these two concepts coincide when we consider principal fibre bundles with structure group $GL(n, \mathbb{R})$ and the associated vector bundles.

Standard references for this chapter are for example [14], [11], [27], [28], [29], [22] (for 1.1. and 1.3), and [13] (for 1.3).

1.1 Vector bundles

Definition 1.1.1 *A* ***(real) vector bundle*** *of* rank n *over a manifold* M *is a fibre bundle* $\pi : E \to M$ *with typical fibre* $\mathbb{R}^n$ *and local trivializations*

$$\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n$$

such that for all $i, j \in I$ *and for all* $p \in \mathcal{U}_{ij}$ *the map*

$$\mathbb{R}^n \xrightarrow{\cong} \{p\} \times \mathbb{R}^n \xrightarrow{\theta_i \circ \theta_j^{-1}|_{\{p\} \times \mathbb{R}^n}} \{p\} \times \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n \tag{1.1}$$

is linear, where we identify $\mathbb{R}^n$ *with* $\{p\} \times \mathbb{R}^n$ *by* $v \mapsto (p, v)$.

Linearity of (1.1) allows us to give to each fibre E_p the structure of a real vector space. We can do this by requiring the composition

$$\theta_{ip} : E_p \xrightarrow{\theta_i|_{E_p}} \{p\} \times \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n$$

to be an isomorphism of vector spaces for some (and hence for all) $i \in I$ with $p \in \mathcal{U}_i$.

Example 1.1.2 *The* ***tangent bundle*** TM *of a manifold* M *is a vector bundle over* M *with rank* $n = dim(M)$.

The set of all sections of a vector bundle $\pi : E \to M$ is denoted by $\Gamma(E)$. It has a natural structure of a real vector space [1]. Note that every vector bundle $\pi : E \to M$ has a *zero section*, i.e. the section $\sigma : M \to E$, $p \mapsto 0 \in E_p$. It is the zero vector of the vector space $\Gamma(E)$.

Definition 1.1.3 *A* ***homomorphism*** *(or* ***vector bundle map****) between two vector bundles $\pi_E : E \to M$ and $\pi_F : F \to M$ over the same base space M is a smooth map $f : E \to F$ such that $f(E_p) \subseteq F_p$ for all $p \in M$ and $f|_{E_p} : E_p \to F_p$ is linear. If $f : E \to F$ is a diffeomorphism, then f is called a* ***(vector bundle) isomorphism****.*

Lemma 1.1.4 *Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be vector bundles and let $f : E \to F$ be a vector bundle map such that $f|_{E_p} : E_p \to F_p$ is an isomorphism for all $p \in M$. Then f is a vector bundle isomorphism.*

For a proof of this, see [11, Theorem 2.5 of Chapter 3].

Denote by $\mathrm{Hom}\,(E, F)$ the set of all vector bundle maps $E \to F$. It has the structure of a real vector space in a natural way.

Let $\pi : E \to M$ be a vector bundle and let $\{\, \theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n \;,\; i \in I \,\}$ be local trivializations with respect to a cover $\{\, \mathcal{U}_i,\, i \in I \,\}$ of M. For every $p \in \mathcal{U}_{ij}$ we can define a vector space isomorphism $\theta_{ij}\,(p) : \mathbb{R}^n \to \mathbb{R}^n$ to be the composition $\theta_{ip} \circ \theta_{jp}^{-1}$. The functions $\{\, \theta_{ij} : \mathcal{U}_{ij} \to GL\,(n, \mathbb{R}) \,\}$ are called the **transition functions** of the vector bundle $\pi : E \to M$ with respect to the cover $\{\, \mathcal{U}_i,\, i \in I \,\}$ and the trivializations $\{\theta_i\}$. Note that they are smooth and that they satisfy the *cocycle condition* [2]

$$\theta_{ij}\, \theta_{jk} = \theta_{ik}\,, \tag{1.2}$$

where multiplication is in $GL\,(n, \mathbb{R})$. In particular, from (1.2) it follows that $\theta_{ii} = I$ and $\theta_{ji} = \theta_{ij}^{-1}$.

We can use transition functions to construct global objects on vector bundles by gluing together local definitions given on the domain of the trivializations. In the next examples we will show how to do this for sections and vector bundle maps.

Example 1.1.5 *Let $\pi : E \to M$ be a vector bundle with local trivializations $\{\, \theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n,\, i \in I \,\}$ with respect to a cover $\{\, \mathcal{U}_i,\, i \in I \,\}$ of M. Then any set of smooth functions $\{\, \sigma_i : \mathcal{U}_i \to \mathbb{R}^n,\, i \in I \,\}$ induces a well-defined global section $\sigma : M \to E$, provided that $\theta_{ij}\, \sigma_j = \sigma_i$: for $p \in M$, choose $i \in I$ with $p \in \mathcal{U}_i$ and define $\sigma\,(p) := \theta_{ip}^{-1}(\, \sigma_i(p)\,)$.*

[1] See [11, Proposition 1.6 of Chapter 3] for more details.

[2] Equations of this type are always to be understood as holding on the common domain of definition.

Example 1.1.6 *Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be two vector bundles over the same base space M with trivializations $\{\, \theta_i^E : \pi_E^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n,\ i \in I \,\}$ and $\{\, \theta_i^F : \pi_F^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^m,\ i \in I \,\}$ with respect to a cover $\{\mathcal{U}_i,\ i \in I\,\}$ of M [3]. Then any set of smooth maps $\{\, f_i : \mathcal{U}_i \to \mathcal{M}(n \times m),\ i \in I \,\}$ determines a well-defined vector bundle map $f : E \to F$, provided that $\theta_{ij}^F\, f_j = f_i\, \theta_{ij}^E$ on $\mathcal{U}_{ij}$: for $v \in E$, choose $i \in I$ with $p = \pi_E(v) \in \mathcal{U}_i$ and define $f(v) := \big(\theta_{ip}^F\big)^{-1}\Big(f_i(p)\, \theta_{ip}^E(v)\Big)$.*

Example 1.1.7 *Suppose now that the vector bundles E and F of Example 1.1.6 both have rank n and that a set of smooth maps $\{\, f_i : \mathcal{U}_i \to GL(n, \mathbb{R}),\ i \in I \,\}$ is given such that $\theta_{ij}^F\, f_j = f_i\, \theta_{ij}^E$ on $\mathcal{U}_{ij}$. Then we have also the smooth maps $\{\, f_i^{-1} : \mathcal{U}_i \to GL(n, \mathbb{R}),\ i \in I \,\}$ which satisfy $\theta_{ij}^E\, f_j^{-1} = f_i^{-1}\, \theta_{ij}^F$ on $\mathcal{U}_{ij}$. It follows that we get vector bundle maps $f : E \to F$ and $f' : F \to E$ induced by the $\{\, f_i,\ i \in I \,\}$ and $\{\, f_i^{-1},\ i \in I \,\}$ respectively. It is easy to see that f and f' are inverse of each other, so in particular E and F are isomorphic vector bundles.*

Transition functions can also be used to reconstruct the whole bundle, as explained in the following proposition.

Proposition 1.1.8 *Let M be a manifold and let $\{\mathcal{U}_i,\ i \in I\,\}$ be a cover of M. Suppose a set of smooth maps $\{\, \theta_{ij} : \mathcal{U}_{ij} \to GL(n, \mathbb{R}) \,\}$ is given satisfying the cocycle condition (1.2). Then there is a unique (up to isomorphism) vector bundle of rank n over M with the $\{\theta_{ij}\}$ as transition functions with respect to some system of local trivializations.*

Proof Define

$$E := \dot{\bigcup_{i \in I}} \mathcal{U}_i \times \mathbb{R}^n \ {}_{/\sim},$$

where $(\, i, p, x \,) \sim (\, j, q, y \,)$ by definition if $p = q \in \mathcal{U}_{ij}$ and $x = \theta_{ij}\,(p)\,(y)$ (note that the cocycle condition implies that this is a well-defined equivalence relation on the set $\bigcup \mathcal{U}_i \times \mathbb{R}^n$). Denote by $(\, i, p, x \,)_{/\sim} \in E$ the equivalence class of $(\, i, p, x \,) \in \bigcup \mathcal{U}_i \times \mathbb{R}^n$.
Define a map $\pi : E \to M$ by $(\, i, p, x \,)_{/\sim} \mapsto p$ and let $\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n$ be the bijection $(\, i, p, x \,)_{/\sim} \mapsto (\, p, x \,)$ for all $i \in I$. We can define a differentiable structure on E by requiring the θ_i's to be diffeomorphisms: this makes sense since θ_i can be obtained from a different θ_j (with $\mathcal{U}_{ij} \neq \emptyset$) by composing it with the smooth map $\mathcal{U}_{ij} \times \mathbb{R}^n \to \mathcal{U}_{ij} \times \mathbb{R}^n$, $(\, p, x \,) \mapsto \big(\, p,\ \theta_{ij}(p)\, x \,\big)$.
Then $\pi : E \to M$ becomes a vector bundle, with the $\{\theta_{ij}\}$ as transition functions.

Uniqueness follows from the fact that two vector bundles over the same manifold having the same transition functions on a given cover are isomorphic (put $\{\, f_i : \mathcal{U}_i \to GL(n, \mathbb{R}),\ p \mapsto I \,\}$ in Example 1.1.7). □

[3] Without loss of generality we can use the same cover $\{\,\mathcal{U}_i,\ i \in I\,\}$ of M for the trivializations of E and F.

It is possible to prove that every (multi-)linear operation on vector spaces (for example $V \mapsto V^*$, $(V, W) \mapsto V \otimes W$ etc) induces in a natural way a corresponding operation on vector bundles. For a proof of this general principle see [17, §3.4], [11, §5.6] or [22, §3(f)]. We will just give a few examples of this.

Example 1.1.9 *Let $\pi : E \to M$ be a vector bundle with transition functions $\{ \theta_{ij} : \mathcal{U}_{ij} \to GL(n, \mathbb{R}) \}$ with respect to a cover $\{ \mathcal{U}_i,\, i \in I \}$ of M. Define*

$$\theta_{ij}{}^* : \mathcal{U}_{ij} \to GL(n, \mathbb{R}) \ , \ p \mapsto \big(\theta_{ij}(p)^{\,t} \big)^{-1}.$$

*The $\{\theta_{ij}{}^*\}$ satisfy the cocycle condition (1.2), thus by Proposition 1.1.8 they are the transition functions of a real vector bundle E^* over M, called the **dual bundle** of E. We have $(E^*)_p \cong (E_p)^*$ in a canonical way. The isomorphism is defined as follows. For $(\, i, p, x\,)_{/\sim} \in (E^*)_p$ and $(\, i, p, y\,)_{/\sim} \in E_p$ (notation as in the proof of Proposition 1.1.8) we set*

$$\langle\, (\, i, p, x\,)_{/\sim} \,,\, (\, i, p, y\,)_{/\sim} \,\rangle \,:= x^t \cdot y$$

(note that this is well-defined). Observe that the construction of E^* does not depend on the set of transition functions defining E. Indeed, suppose that another set of transition functions $\{ \theta'_{ij} : \mathcal{U}_{ij} \to GL(n, \mathbb{R}) \}$ (without loss of generality we can assume the cover to be the same as above) determines the same vector bundle E. Consider the functions $\{ f_i : \mathcal{U}_i \to GL(n, \mathbb{R}),\ p \mapsto \theta_{ip} \circ \theta'_{ip}{}^{-1},\ i \in I \}$, where $\{ \theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n \ ,\ i \in I \}$ and $\{ \theta'_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n \ ,\ i \in I \}$ are some systems of local trivializations for E, inducing the transition functions $\{ \theta_{ij} \}$ and $\{ \theta'_{ij} \}$ respectively. Then it holds $\theta_{ij}\, f_j = f_i\, \theta'_{ij}$ and so $\theta_{ij}{}^* \big(f_j{}^t \big)^{-1} = \big(f_i{}^t \big)^{-1} \theta'_{ij}{}^*$. By Example 1.1.7, this implies that $\{ \theta_{ij}{}^* \}$ and $\{ \theta'_{ij}{}^* \}$ determine the same vector bundle E^* (up to isomorphism). A similar argument can be used to show that if E and E' are isomorphic vector bundles, then so are E^* and $E'^{\,*}$.

Example 1.1.10 *Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be vector bundles with transition functions $\{ \theta^E_{ij} : \mathcal{U}_{ij} \to GL(n, \mathbb{R}) \}$ and $\{ \theta^F_{ij} : \mathcal{U}_{ij} \to GL(m, \mathbb{R}) \}$ with respect to a common cover $\{ \mathcal{U}_i,\, i \in I \}$ of M. Define*

$$\theta_{ij} : \mathcal{U}_{ij} \to GL(\, n+m, \mathbb{R}\,) \ , \ p \mapsto \theta^E_{ij}\,(p) \oplus \theta^F_{ij}\,(p) = \left(\begin{array}{c|c} \theta^E_{ij}\,(p) & 0 \\ \hline 0 & \theta^F_{ij}\,(p) \end{array} \right).$$

*The $\{\theta_{ij}\}$ satisfy the cocycle condition (1.2), thus by Proposition 1.1.8 they are the transition functions of a real vector bundle $E \oplus F$ over M, called the **direct** (or **Whitney**) **sum** of E and F. We have $(E \oplus F)_p \cong E_p \oplus F_p$ in a canonical way. The isomorphism is given by the well-defined map $E_p \oplus F_p \to (E \oplus F)_p$,*

$$\big(\, (\, i, p, x\,)_{/\sim} \,,\, (\, i, p, y\,)_{/\sim} \big) \mapsto (\, i, p, \begin{pmatrix} x \\ y \end{pmatrix})_{/\sim}$$

(notation as in the proof of Proposition 1.1.8). Note that the construction of $E \oplus F$ does not depend on the sets of transition functions defining E and F. Moreover, if E, E' and F, F' are isomorphic vector bundles, then so are $E \oplus F$ and $E' \oplus F'$. The proof of this is similar to that in Example 1.1.9.

Example 1.1.11 *Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be vector bundles as in Example 1.1.10. Define* [4]

$$\theta_{ij} : \mathcal{U}_{ij} \to GL(nm, \mathbb{R}) \ , \ p \mapsto \theta_{ij}(p) = \theta^E_{ij}(p) \otimes \theta^F_{ij}(p) \, .$$

The $\{\theta_{ij}\}$ satisfy the cocycle condition (1.2), thus by Proposition 1.1.8 they are the transition functions of a real vector bundle $E \otimes F$ over M, called the ***tensor product*** *of E and F. We have $(E \otimes F)_p \cong E_p \otimes F_p$ in a canonical way. To see this, apply the universal factorization property of the tensor product to the bilinear (well-defined) map $E_p \times F_p \to (E \otimes F)_p$,*

$$\big((i,p,x)_{/\sim} \, , \, (i,p,y)_{/\sim} \big) \mapsto (i,p,x \otimes y)_{/\sim}$$

(notation as in the proof of Proposition 1.1.8) [5]*. Note that the construction of $E \otimes F$ does not depend on the sets of transition functions defining E and F. Moreover, if E, E' and F, F' are isomorphic vector bundles, then so are $E \otimes F$ and $E' \otimes F'$. The proof of this is similar to that in Example 1.1.9.*

Example 1.1.12 *Let $\pi : E \to M$ be a vector bundle as in Example 1.1.9 and let $r \leq n$ be a positive integer. Define* [6]

$$\bigwedge^r \theta_{ij} : \mathcal{U}_{ij} \to GL\Big(\binom{n}{r}, \mathbb{R} \Big) \ , \ p \mapsto \bigwedge^r \big(\theta_{ij}(p) \big) \, .$$

The $\{ \bigwedge^r \theta_{ij} \}$ satisfy the cocycle condition (1.2), thus by Proposition 1.1.8 they are the transition functions of a real vector bundle $\bigwedge^r E$ over M, called the ***r-th exterior power*** *of E. We have $\big(\bigwedge^r E \big)_p \cong \bigwedge^r E_p$ in a canonical way. To see*

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[4] Given two matrices $A = (a_{ij}) \in \mathcal{M}(n \times n, \mathbb{R})$ and $B = (b_{ij}) \in \mathcal{M}(m \times m, \mathbb{R})$, the matrix $A \otimes B \in \mathcal{M}(nm \times nm, \mathbb{R})$ (*Kronecker product* of A and B) is defined by

$$A \otimes B = \left(\begin{array}{c|c|c} A\, b_{11} & \dots & A\, b_{1m} \\ \hline \vdots & \ddots & \vdots \\ \hline A\, b_{m1} & \dots & A\, b_{mm} \end{array} \right) .$$

We have $(A_1 \otimes B_1)(A_2 \otimes B_2) = A_1 A_2 \otimes B_1 B_2$ (see [20, §43]).

[5] Given $x = (x_1, \dots, x_n)^t \in \mathbb{R}^n$ and $y = (y_1, \dots y_m)^t \in \mathbb{R}^m$, $x \otimes y \in \mathbb{R}^{nm}$ is defined to be $(x_1 y_1, \dots, x_n y_1, \dots, x_1 y_m, \dots, x_n y_m)^t$.

[6] Given a matrix $A = (a_{ij}) \in \mathcal{M}(n \times n, \mathbb{R})$ and a positive integer $r \leq n$, the matrix $\bigwedge^r (A)$ in $\mathcal{M}\Big(\binom{n}{r} \times \binom{n}{r}, \mathbb{R} \Big)$ (*r-adjugate* of A) is defined as follows. Denote by $a^{i_1 \dots i_r}_{j_1 \dots j_r}$ the r-rowed inner determinants of A. Then the entries of $\bigwedge^r (A)$ are the numbers $a^{i_1 \dots i_r}_{j_1 \dots j_r}$ in lexicographic order. We have $\bigwedge^r (AB) = \bigwedge^r (A) \bigwedge^r (B)$ (see [20, §45]).

this, apply the universal factorization property of the exterior power to the r-linear alternating (well-defined) map $E_p \times \ldots \times E_p \to \left(\bigwedge^r E \right)_p$,

$$\left((\, i, p, x_1 \,)_{/\sim}, \ \ldots \ , (\, i, p, x_r \,)_{/\sim} \right) \mapsto (\, i, p, \textstyle\bigwedge^r (x_1, \ldots, x_r) \,)_{/\sim}$$

(notation as in the proof of Proposition 1.1.8) [7]. *Note that the construction of $\bigwedge^r E$ does not depend on the set of transition functions defining E. Moreover, if E and E' are isomorphic vector bundles, then so are $\bigwedge^r E$ and $\bigwedge^r E'$. The proof of this is similar to that in Example 1.1.9.*

Iterating the constructions of Examples 1.1.9 - 1.1.12, we can get any combination of direct sum, tensor product and exterior power of two or more vector bundles over the same manifold M and of their duals.

Example 1.1.13 *Given vector bundles $\pi_E : E \to M$ and $\pi_F : F \to M$, we can define the vector bundle $E^* \otimes F$ over M. We have*

$$(E^* \otimes F)_p = (E_p)^* \otimes F_p = \mathrm{Hom}\,(E_p, F_p)\,.$$

This gives a correspondence between sections of $E^* \otimes F$ and vector bundles maps $E \to F$, and this correspondence is actually a vector space isomorphism

$$\Gamma\,(E^* \otimes F) \cong \mathrm{Hom}\,(E, F)\,.$$ [8]

Example 1.1.14 *Given a vector bundle $\pi : E \to M$ and an integer r as in Example 1.1.12, we can define the vector bundle $\bigwedge^r E^*$ (called the* ***bundle of r-forms*** *of E). We have*

$$\left(\textstyle\bigwedge^r E^* \right)_p = \textstyle\bigwedge^r (E_p)^*\,.$$

Thus a section of $\bigwedge^r E^*$ gives an r-linear alternating form at each fibre E_p of E, varying smoothly with p. In particular, $\Gamma\,(\, \bigwedge^r TM^* \,) = \mathcal{A}^r\,(M)$.

Example 1.1.15 *Given a vector bundle $\pi : E \to M$, we can define the vector bundle $E^* \otimes E^*$. We have*

$$(E^* \otimes E^*)_p = (E_p)^* \otimes (E_p)^* = 2\mathrm{-Lin}\,(\, E_p \times E_p, \mathbb{R}\,).$$

Thus a section of $E^* \otimes E^*$ gives a bilinear map $E_p \times E_p \to \mathbb{R}$ on each fibre of E, varying smoothly with p. A ***Riemannian metric*** *on $\pi : E \to M$ is a section h of $E^* \otimes E^*$ such that $h(p)$ is an inner product on E_p for all $p \in M$.* [9]

[7] Given $x_1 = (x_{11}, \ldots, x_{n1})^t, \ldots, x_r = (x_{1r}, \ldots, x_{nr})^t \in \mathbb{R}^n$, the vector $\bigwedge^r (x_1, \ldots, x_r)$ is defined to have as entries the r-rowed inner determinants of the matrix $(x_{ij}) \in \mathcal{M}(n \times r, \mathbb{R})$ in lexicographic order.

[8] The notation $\mathrm{Hom}\,(E, F)$ is often used in the literature to denote the vector bundle $E^* \otimes F$. In this thesis instead by $\mathrm{Hom}\,(E, F)$ we will always mean the vector space of vector bundle maps from E to F.

[9] The smoothness condition for a Riemannian metric h can equivalently be formulated as follows: given two sections $\sigma_1,\, \sigma_2 : M \to E$, the function $M \to \mathbb{R}^n$, $p \mapsto h(p) \quad \sigma_1(p), \sigma_2(p)$ should be smooth.

Finally, in the following example we show how to construct the pullback bundle f^*E of a vector bundle $\pi : E \to M$, given a smooth map $f : N \to M$.

Example 1.1.16 *Let $\pi : E \to M$ be a vector bundle and f a smooth map from a manifold N to M. Consider the set $f^*E := \{ (p, v) \in N \times E \ / \ f(p) = \pi(v) \}$ and the surjective map $\pi' : f^*E \to N$, $(p, v) \mapsto p$. Then we have a commutative diagram*

$$\begin{array}{ccc} f^*E & \xrightarrow{\bar{f}} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{f} & M \end{array}$$

where $\bar{f} : f^*E \to E$ is given by $(p, v) \mapsto v$. Let $\{ \theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n \ , \ i \in I \}$ be a system of local trivializations for $\pi : E \to M$, with respect to a cover $\{ \mathcal{U}_i, \ i \in I \}$ of M. Consider the cover $\{ f^{-1}(\mathcal{U}_i), \ i \in I \}$ of N and let

$$\theta'_i : {\pi'}^{-1}\big(f^{-1}(\mathcal{U}_i)\big) \to f^{-1}\big(\mathcal{U}_i\big) \times \mathbb{R}^n$$

be the bijection $(p, v) \mapsto \big(p, \theta_{i\, f(p)}(v)\big)$ for all $i \in I$. We can define a differentiable structure on f^*E by requiring the θ'_i's to be diffeomorphisms: this makes sense since θ'_i can be obtained from a different θ'_j (with $\mathcal{U}_{ij} \neq \emptyset$) by composing it with the smooth map $f^{-1}(\mathcal{U}_{ij}) \times \mathbb{R}^n \to f^{-1}(\mathcal{U}_{ij}) \times \mathbb{R}^n$, $(p, x) \mapsto \Big(p, \theta_{ij}\big(f(p)\big)\, x\Big)$. Then $\pi' : f^*E \to N$ becomes a vector bundle (the ***pullback bundle*** *of $\pi : E \to M$ with respect to the map $f : N \to M$) and $\bar{f} : f^*E \to E$ a vector bundle map. Note that the transition functions of f^*E with respect to the local trivializations $\{ \theta'_i, \ i \in I \}$ are given by $\theta'_{ij} = f^*\, \theta_{ij}$.*

We conclude this section with a lemma that will be needed in §3.

Lemma 1.1.17 *Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be vector bundles. Then we have:*

$$\begin{aligned} \Gamma\,(E^* \otimes F) \ &\cong \ \mathrm{Hom}\,(E, F) \cong \{ \lambda : \Gamma(E) \to \Gamma(F) \ \textit{linear over}\ \mathcal{C}^\infty(M) \} \\ &\cong \ \Gamma(E^*) \otimes_{\mathcal{C}^\infty(M)} \Gamma(F) \end{aligned}$$

where "$\cong$" means "isomorphic as $\mathcal{C}^\infty(M)$-module". In particular, $\Gamma\,(E \otimes F) \cong \Gamma(E) \otimes_{\mathcal{C}^\infty(M)} \Gamma(F)$.

To prove this we need the following lemma.

Lemma 1.1.18 *Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be vector bundles and let $\lambda : \Gamma(E) \to \Gamma(F)$ be a $\mathcal{C}^\infty(M)$-linear map. Then for all $\sigma \in \Gamma(E)$ and $p \in M$ the value of $\lambda(\sigma)$ at p depends only on $\sigma(p)$.*

Proof Let $p \in M$. We have to show that if σ, η are sections of E with $\sigma(p) = \eta(p)$ then $\lambda(\sigma)\,(p) = \lambda(\eta)\,(p)$, or equivalently that if σ is a section of E with $\sigma(p) = 0$ then $\lambda(\sigma)\,(p) = 0$. Observe first that if $\lambda : \Gamma(E) \to \Gamma(F)$ is linear over $\mathcal{C}^\infty(M)$, then λ is a *local operator*, i.e. for $\sigma \in \Gamma(E)$ the value of $\lambda(\sigma)$ at p depends only on the value of σ in a neighborhood of p, i.e. if we have $\sigma, \eta \in \Gamma(E)$ with $\sigma|_{\mathcal{U}} = \eta|_{\mathcal{U}}$ for some $\mathcal{U} \subset M$ with $p \in \mathcal{U}$, then $\lambda(\sigma)\,(p) = \lambda(\eta)\,(p)$. To see this, take a function $\psi \in \mathcal{C}^\infty(M)$ with $\operatorname{supp}(\psi) \subset \mathcal{U}$ and $\psi(p) = 1$; then $\psi\,\sigma = \psi\,\eta$ and $\psi\,\lambda(\sigma) = \lambda(\psi\,\sigma) = \lambda(\psi\,\sigma) = \psi\,\lambda(\eta)$, in particular $\lambda(\sigma)\,(p) = \lambda(\eta)\,(p)$. Consider now a section $\sigma \in \Gamma(E)$ with $\sigma(p) = 0$. Let $\mathcal{U} \subset M$ be a neighborhood of p and $u_1, \ldots, u_n$ a local frame of E on $\mathcal{U}$. Then $\sigma|_{\mathcal{U}} = \sum \sigma_i\, u_i$ where the σ_i's are functions on $\mathcal{U}$ with $\sigma_i\,(p) = 0$. Take a function $\psi \in \mathcal{C}^\infty(M)$ with $\operatorname{supp}(\psi) \subset \mathcal{U}$ and $\psi|_{\mathcal{V}} = 1$ for some open $\mathcal{V} \subset \mathcal{U}$ with $p \in \mathcal{V}$. Let $\sigma' := \sum \sigma'_i\, u'_i$, where $\sigma'_i|_{\mathcal{U}} := \psi\,\sigma_i$, $\sigma'_i|_{M\setminus\mathcal{U}} := 0$ and $u'_i|_{\mathcal{U}} := \psi\,u_i$, $u'_i|_{M\setminus\mathcal{U}} := 0$. Then $\sigma'|_{\mathcal{V}} = \sigma|_{\mathcal{V}}$ thus $\lambda(\sigma)\,(p) = \lambda(\sigma')\,(p) = \lambda\big(\sum \sigma'_i\, u'_i\big)\,(p) = \sum \sigma'_i(p)\, u'_i(p) = \sum \sigma_i(p)\, u_i(p) = 0$, as we wanted. □

Proof of Lemma 1.1.17 For $\Gamma\,(E^* \otimes F) \cong \operatorname{Hom}(E, F)$, see Example 1.1.13.

$\operatorname{Hom}(E,F) \cong \{\,\lambda : \Gamma(E) \to \Gamma(F)$ linear on $\mathcal{C}^\infty(M)\,\}$ can be proved as follows. Let $\varphi \in \operatorname{Hom}(E,F)$ and define $\lambda_\varphi : \Gamma(E) \to \Gamma(F)$ by $\sigma \mapsto (\,p \mapsto \varphi\big(\sigma(p)\big)\,)$. Then λ_φ is linear over $\mathcal{C}^\infty(M)$ and $\varphi \mapsto \lambda_\varphi$ is a homomorphism of $\mathcal{C}^\infty(M)$-modules. Conversely, let $\lambda : \Gamma(E) \to \Gamma(F)$ be linear on $\mathcal{C}^\infty(M)$ and define $\varphi_\lambda \in \operatorname{Hom}(E,F)$ by $\varphi_\lambda\,(v) = \lambda\,(\sigma_v)\,\big(\pi(v)\big)$, where σ_v is a section of E with $\sigma_v\,\big(\pi(v)\big) = v$. By Lemma 1.1.18 this is well-defined. Observe that φ_λ is smooth, so $\varphi_\lambda \in \operatorname{Hom}(E,F)$, and that $\lambda_{\varphi_\lambda} = \lambda$ and $\varphi_{\lambda_\varphi} = \varphi$. Thus $\varphi \mapsto \lambda_\varphi$ is an isomorphism of $\mathcal{C}^\infty(M)$-modules.

Finally, we can prove as follows that $\{\,\lambda : \Gamma(E) \to \Gamma(F)$ linear over $\mathcal{C}^\infty(M)\,\}$ and $\Gamma(E^*) \otimes_{\mathcal{C}^\infty(M)} \Gamma(F)$ are isomorphic. Let $\xi = \tau^* \otimes \eta \in \Gamma(E^*) \otimes_{\mathcal{C}^\infty(M)} \Gamma(F)$ and define $\lambda_\xi : \Gamma(E) \to \Gamma(F)$ by $\sigma \mapsto \big(\,p \mapsto \langle\,\tau^*(p), \sigma(p)\,\rangle\,\eta(p)\,\big)$. Then λ_ξ is linear over $\mathcal{C}^\infty(M)$ and $\xi \mapsto \lambda_\xi$ is a homomorphism of $\mathcal{C}^\infty(M)$-modules. Conversely, given a map $\lambda : \Gamma(E) \to \Gamma(F)$ which is linear over $\mathcal{C}^\infty(M)$, we can define $\xi_\lambda \in \Gamma(E^*) \otimes_{\mathcal{C}^\infty(M)} \Gamma(F)$ as follows. Let $\{\mathcal{U}_i,\ i \in I\,\}$ be a cover of M and for $i \in I$ let $u^i_1, \ldots, u^i_n$ be a local frame on $\mathcal{U}_i$. Observe that since $\lambda : \Gamma(E) \to \Gamma(F)$ is a local operator (see Lemma 1.1.16) it induces a map $\lambda_i : \Gamma(E|_{\mathcal{U}_i}) \to \Gamma(F|_{\mathcal{U}_i})$ (which is linear over $\mathcal{C}^\infty(\mathcal{U}_i)$) for all $i \in I$. Define $\xi^i_\lambda := \sum_{\alpha=1}^n (u^i_\alpha)^* \otimes \lambda\,(u^i_\alpha) \in \Gamma(E^*|_{\mathcal{U}_i}) \otimes_{\mathcal{C}^\infty(\mathcal{U}_i)} \Gamma(F|_{\mathcal{U}_i})$. Then $\lambda_{\xi^i_\lambda} = \lambda|_{\mathcal{U}_i}$ and $\xi^i_{\lambda_\zeta} = \zeta$ for all $\zeta \in \Gamma(E^*|_{\mathcal{U}_i}) \otimes_{\mathcal{C}^\infty(\mathcal{U}_i)} \Gamma(F|_{\mathcal{U}_i})$, so $\xi \mapsto \lambda_\xi$ gives an isomomorphism between $\{\,\lambda : \Gamma(E|_{\mathcal{U}_i}) \to \Gamma(F|_{\mathcal{U}_i})$ linear over $\mathcal{C}^\infty(\mathcal{U}_i)\,\}$ and $\Gamma(E^*|_{\mathcal{U}_i}) \otimes_{\mathcal{C}^\infty(\mathcal{U}_i)} \Gamma(F|_{\mathcal{U}_i})$. In particular it follows that ξ^i_λ and ξ^j_λ coincide on $\mathcal{U}_{ij}$, so if we piece the $\{\,\xi^i_\lambda,\ i \in I\,\}$ together using a partition of unity of M we get an element $\xi_\lambda \in \Gamma(E^*) \otimes_{\mathcal{C}^\infty(M)} \Gamma(F)$ such that $\xi_\lambda\,(p) = \xi^i_\lambda\,(p)$ for all $i \in I$ and $p \in \mathcal{U}_i$. Thus it holds $\lambda_{\xi_\lambda} = \lambda$ for all $\lambda : \Gamma(E) \to \Gamma(F)$ linear on $\mathcal{C}^\infty(M)$ and $\xi_{\lambda_\xi} = \xi$ for all $\xi \in \Gamma(E^*) \otimes_{\mathcal{C}^\infty(M)} \Gamma(F)$. This implies that $\xi \mapsto \lambda_\xi$ is an isomomorphism of $\mathcal{C}^\infty(M)$-modules. □

Remark 1.1.19 *Everything what we did in this paragraph for real vector bundles can also be done for* ***complex vector bundles****. All definitions, examples and results (except Examples 1.1.2 and 1.1.15) carry over to the complex case just substituting "$\mathbb{R}$" and "real" with "$\mathbb{C}$" and "complex" throughout (in particular, $\mathcal{C}^\infty(M)$ becomes $\mathcal{C}^\infty(M,\mathbb{C})$). Analogues of Examples 1.1.2 and 1.1.15 will be given in Chapter 2.*

1.2 Principal fibre bundles

Definition 1.2.1 *Let G be a Lie group. A* ***principal fibre bundle*** *with* structure group *G consists of a manifold P and an action of G on P on the right such that:*

1. *$M := {}^{P}\!/_{G}$ has a manifold structure which makes the canonical projection $\pi : P \to M$ smooth;*
2. *the condition of* local triviality *is satisfied, i.e. there exist an open cover $\{\mathcal{U}_i,\, i \in I\}$ of M and diffeomorphisms*

$$\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times G \ , \ u \mapsto \big(\pi(u),\, \varphi_i(u)\big)$$

   where $\varphi_i : \pi^{-1}(\mathcal{U}_i) \to G$ satisfies $\varphi_i\,(ug) = \varphi_i\,(u)\, g$ for all $u \in \pi^{-1}(\mathcal{U}_i)$ and $g \in G$.

In particular $\pi : P \to M$ is fibre bundle with typical fibre G. We will write $P\,(M,G,\pi)$ or $P\,(M,G)$ (or simply P) for a principal fibre bundle $\pi : P \to M$. The action $P \times G \to P$ will be denoted by $(\,u,g\,) \mapsto ug$.

Note that from 2. above it follows that the action of G on P is differentiable and free [10].

For $g \in G$, denote by $R_g : P \to P$ the map $u \mapsto ug$. Then we have $R_e = \mathrm{id}_P$ and $R_{g_1 g_2} = R_{g_2} \circ R_{g_1}$. In particular it follows that $R_g : P \to P$ is a diffeomorphism for all $g \in G$. Given $u \in P$ with $\pi(u) = p$, we have $\pi^{-1}(p) = \{\, ug,\, g \in G\,\}$ and the map $\sigma_u : g \mapsto ug$ is a diffeomorphism between G and the fibre of P over p.

A trivial bundle $M \times G$ admits a global section, for example $M \to M \times G$, $p \mapsto (\,p, e\,)$. The converse is also true: if a principal fibre bundle $P\,(M,G)$ admits a section $\sigma : M \to P$, then it is trivial. A global trivialization is given by the inverse of the smooth map $M \times G \to P$, $(\,p,g\,) \mapsto \sigma\,(p)\, g$, which is also smooth as can be seen using local trivializations. In particular, condition 2. in Definition 1.2.1 is equivalent to requiring the existence of an open cover $\{\mathcal{U}_i,\, i \in I\,\}$ of M and local sections $\{\,\sigma_i : \mathcal{U}_i \to P\,\}$. This will be used in the next example.

[10] I.e., if $ug = u$ for some $u \in P$, then $g = e$.

Example 1.2.2 *Let* $\pi_E : E \to M$ *be a vector bundle with local trivializations* $\{\, \theta_i^E : \pi_E^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n \,\}$ *with respect to a cover* $\{\mathcal{U}_i,\, i \in I\}$ *of* M*. Define* $L(E)$ *to be the set of all linear frames* [11] *on* $\pi_E : E \to M$*. We will show that* $L(E)$ *has the structure of a principal fibre bundle over* M *with structure group* $GL\,(n, \mathbb{R})$*. Define the projection* $\pi : L(E) \to M$ *to be the map sending a frame at* p *to* p*. Let* $GL\,(n, \mathbb{R})$ *act on* $L(E)$ *on the right by* $(\, u,\, A\,) \mapsto u \cdot A$*. Then we have* $L(E)\big/_{GL\,(n, \mathbb{R})} = M$*. Define sections* $\sigma_i : \mathcal{U}_i \to L(E)$ *by*

$$p \mapsto u\,(p) := \Big(\big(\, \theta_i^E\,\big)^{-1}(\,p, e_1)\,,\; \ldots\;,\; \big(\, \theta_i^E\,\big)^{-1}(\,p, e_n) \Big).$$

The $\{\, \sigma_i : \mathcal{U}_i \to P \,\}$ *induce bijections* $\{\, \theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times GL\,(n, \mathbb{R}),\, i \in I\,\}$ *and we can define a differentiable structure on* $L(E)$ *by requiring these bijections to be diffeomorphisms: this makes sense since* θ_i *can be obtained from a different* θ_j *(with* $\mathcal{U}_{ij} \neq \emptyset$*) by composing it with the smooth map* $\mathcal{U}_{ij} \times GL\,(n, \mathbb{R}) \to \mathcal{U}_{ij} \times GL\,(n, \mathbb{R})$*,* $(\,p, A\,) \mapsto (\,p, \theta_{ij}^E\,(p)\,A\,)$*. Then* $L(E)\,\big(M, GL\,(n, \mathbb{R})\big)$ *becomes a principal fibre bundle over* M*, called the* ***frame bundle*** *of the vector bundle* $\pi_E : E \to M$*. Note that* $L(E)$ *is trivial if* E *is.*

Definition 1.2.3 *A* ***homomorphism*** *of a principal fibre bundle* $Q\,(N, H)$ *into another principal fibre bundle* $P\,(M, G)$ *consists of a smooth map* $f : Q \to P$ *and a Lie group homomorphism* $f' : H \to G$ *such that* $f\,(ug) = f(u)\, f'(g)$ *for all* $u \in Q$ *and* $g \in H$*.*

Note that a homomorphism $(\, f, f'\,)$ as in Definition 1.2.3 induces a smooth [12] map $f'' : N \to M$.

Definition 1.2.4 *A homomorphism* $(\, f, f'\,) : Q\,(N, H) \to P\,(M, G)$ *is an* ***embedding*** *if the induced* $f'' : N \to M$ *is an embedding and* $f' : H \to G$ *a monomorphism (then, in particular,* $f : Q \to P$ *is an embedding). Then we call* $f(Q)\,\big(\, f(N), f(H)\,\big)$ *a* ***subbundle*** *of* $P\,(M, G)$*. If moreover* $N = M$ *and the induced* $f'' : M \to M$ *is the identity, then* $(f, f'\,) : Q\,(M, H) \to P\,(M, G)$ *is called a* ***reduction*** *(relative to* $f' : H \to G$*) of (the structure group of)* $P\,(M, G)$ *to* H *or an* ***extension*** *(relative to* $f' : H \to G$*) of (the structure group of)* $Q\,(M, H)$ *to* G*. The bundle* $Q\,(M, H)$ *is called a* ***reduced bundle*** *of* $P\,(M, G)$*. Given* $P\,(M, G)$ *and a Lie subgroup* [13] H *of* G*, we say that* $P\,(M, G)$ *is reducible to* H *if there is a*

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[11] A *linear frame* on a vector bundle $\pi_E : E \to M$ is a point p of M together with a basis of the fibre E_p.

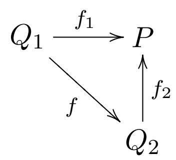
[12] Indeed, locally it is the composition $f'' = \pi_P \circ f \circ \theta^{-1} \circ s$, where $\theta : \pi_Q^{-1}(\mathcal{U}) \to \mathcal{U} \times H$ is a local trivialization for $Q\,(N, H)$ and $s : \mathcal{U} \to \mathcal{U} \times H$ is the map $p \mapsto (\,p, e\,)$.

[13] A Lie subgroup of a Lie group G consists of a Lie group H and an embedding $f : H \to G$ which is also a group homomorphism. We can (and usually will) identify H with $f(H) \leq G$, but note that the topology of $f(H)$ induced by the identification with H is in general not the relative topology with respect to G. This is the case if and only if $f(H)$ is closed in G (see [31, 3.21]). Moreover, if H is a closed subgroup of the Lie group G, then H has a unique manifold structure making it into a Lie subgroup of G (see [31, 3.42]). Note that the topology of this manifold structure must then be the relative topology.

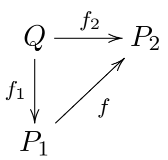
reduced bundle $Q\,(M,H)$. An ***isomorphism*** *between two principal fibre bundles $Q\,(M,G)$ and $P\,(M,G)$ is a homomorphism $(\,f,f'\,) : Q\,(M,G) \to P\,(M,G)$ such that f' and the induced f'' are the identity on G and M respectively. In particular, $f : Q \to P$ is a diffeomorphism. An isomorphism $P\,(M,G) \to P\,(M,G)$ is called an* ***automorphism*** *of $P\,(M,G)$.*

Denote by $\mathrm{Aut}(P)$ the set of all automorphisms of a principal fibre bundle $P\,(M,G)$. It is a group, with respect to composition of maps.

Definition 1.2.5 *We say that two reductions $(\,f_1,f'\,) : Q_1\,(M,H) \to P\,(M,G)$ and $(\,f_2,f'\,) : Q_2\,(M,H) \to P\,(M,G)$ are* equivalent *if there is an isomorphism $f : Q_1\,(M,H) \to Q_2\,(M,H)$ making the following diagram commutative.*



Similarly, we say that two extensions $(\,f_1,f'\,) : Q\,(M,H) \to P_1\,(M,G)$ and $(\,f_2,f'\,) : Q\,(M,H) \to P_2\,(M,G)$ are equivalent *if there is an isomorphism $f : P_1\,(M,H) \to P_2\,(M,H)$ making the following diagram commutative.*



Let $P\,(M,G)$ be a principal fibre bundle and let $\{\,\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times G,\ u \mapsto \big(\pi(u),\,\varphi_i\,(u)\,\big)\,\}$ be local trivializations with respect to a cover $\{\,\mathcal{U}_i,\ i \in I\,\}$ of M. Define maps $\theta_{ij} : \mathcal{U}_{ij} \to G$ by $p \mapsto \theta_{ij}\,(p) := \varphi_i\,(u)\,\varphi_j\,(u)^{-1}$, where $u \in \pi^{-1}(p)$. The functions $\{\,\theta_{ij} : \mathcal{U}_{ij} \to G\,\}$ are called the **transition functions** of $P\,(M,G)$ with respect to the cover $\{\,\mathcal{U}_i,\ i \in I\,\}$ and the trivializations $\{\theta_i\}$. Note that they are smooth and that they satisfy the *cocycle condition* [14]

$$\theta_{ij}\,\theta_{jk} = \theta_{ik}\,, \tag{1.3}$$

where multiplication is in G. In particular, from (1.3) it follows that $\theta_{ii} = e$ and $\theta_{ji} = \theta_{ij}^{-1}$.

Example 1.2.6 *Let $\pi_E : E \to M$ be a vector bundle and let $L(E)\,\big(M, GL\,(n,\mathbb{R})\big)$ be the frame bundle of E. Then it is easy to see that the transition functions $\{\,\theta^E_{ij} : \mathcal{U}_{ij} \to GL\,(n,\mathbb{R})\,\}$ of E with respect to a system of local trivializations $\{\,\theta^E_i : \pi_E^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n\ ,\ i \in I\,\}$ are equal to the transition functions of $L(E)$ with respect to the induced local trivializations $\{\,\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times GL\,(n,\mathbb{R})\ ,\ i \in I\,\}$.*

[14] See footnote 2.

As for vector bundles, we can use transition functions to reconstruct a whole principal fibre bundle, as explained in the following proposition.

Proposition 1.2.7 *Let M be a manifold and G a Lie group. Suppose we have a cover $\{\mathcal{U}_i,\, i \in I\}$ of M and a set of smooth maps $\{\,\theta_{ij} : \mathcal{U}_{ij} \to G\,\}$ satisfying the cocycle condition (1.3). Then there is a unique (up to isomorphism) principal fibre bundle $P\,(M, G)$ with the $\{\theta_{ij}\}$ as transition functions with respect to some system of local trivializations.*

Proof Define:

$$P := \dot{\bigcup_{i\in I}} \mathcal{U}_i \times G_{\,/\sim}$$

where $(\,i, p, g_1\,) \sim (\,j, q, g_2\,)$ by definition if $p = q \in \mathcal{U}_{ij}$ and $g_1 = \theta_{ij}\,(p)\, g_2$ (note that the cocycle condition implies that this is a well-defined equivalence relation on the set $\bigcup \mathcal{U}_i \times G$). Denote by $(\,i, p, g\,)_{/\sim} \in P$ the equivalence class of $(\,i, p, x\,) \in \bigcup \mathcal{U}_i \times G$.
Define an action of $P \times G \to P$ by $\big(\,(\,i, p, g_1\,)_{/\sim},\, g_2\big) \mapsto (\,i, p, g_1\, g_2\,)_{/\sim}$. Then ${}^{P}\!/_{G} = M$ and the canonical projection $\pi : P \to M$ is given by $(\,i, p, g\,)_{/\sim} \mapsto p$. Let $\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times G$ be the bijection $(\,i, p, g\,)_{/\sim} \mapsto (\,p, g\,)$ for all $i \in I$. We can define a differentiable structure on P by requiring the θ_i's to be diffeomorphisms: this makes sense since θ_i can be obtained from a different θ_j (with $\mathcal{U}_{ij} \neq \emptyset$) by composing it with the smooth map $\mathcal{U}_{ij} \times G \to \mathcal{U}_{ij} \times G$, $(\,p, g\,) \mapsto (\,p,\, \theta_{ij}\,(p)\, g\,)$. Then $P\,(M, G)$ becomes a principal fibre bundle, with the $\{\theta_{ij}\}$ as transition functions.

Uniqueness follows from the fact that if two principal fibre bundles $P\,(M, G)$ and $Q\,(M, G)$ have a system of local trivializations $\{\,\theta_i^P : \pi_P^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times G\,\}$ and $\{\,\theta_i^Q : \pi_Q^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times G\,\}$ with respect to the same cover $\{\mathcal{U}_i,\, i \in I\}$ of M such that the transition functions coincide, then they must be isomorphic. An isomorphism $f : P \to Q$ is defined as follows: for $u \in P$ choose $i \in I$ with $\pi_P\,(u) \in \mathcal{U}_i$ and let $f(u) := \big(\,\theta_i^Q\,\big)^{-1}\big(\,\theta_i^P\,(u)\,\big)$. □

Example 1.2.8 *Let $P\,(M, G, \pi)$ be a principal fibre bundle, N a manifold and $f : N \to M$ a smooth map. Consider the set $f^*P := \{\,(p, u) \in N \times P \;/\; f(p) = \pi(u)\,\}$ and the action $f^*P \times G \to f^*P$, $\big((p, u), g\,\big) \mapsto (p, ug)$. Then ${}^{f^*P}\!/_{G} = N$ and the canonical projection $\pi' : f^*P \to N$ is given by $(p, u) \mapsto p$. Thus we have a commutative diagram*

$$\begin{array}{ccc} f^*P & \xrightarrow{\ \bar{f}\ } & P \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ N & \xrightarrow{\ f\ } & M \end{array}$$

where $\bar{f} : f^*P \to P$ is given by $(p, u) \mapsto u$. Let $\{\,\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times G,$ $u \mapsto \big(\pi(u), \varphi_i(u)\big),\ i \in I\,\}$ be a system of local trivializations for $P\,(M, G)$, with *respect to a cover* $\{\mathcal{U}_i,\, i\in I\}$ *of* M*. Consider the cover* $\{f^{-1}(\mathcal{U}_i),\, i\in I\}$ *of* N *and let* $\theta'_i : {\pi'}^{-1}\big(f^{-1}(\mathcal{U}_i)\big) \to f^{-1}\big(\mathcal{U}_i\big)\times G$ *be the bijection* $(p,u)\mapsto \big(p,\varphi_i(u)\big)$*, for all* $i\in I$*. We can define a differentiable structure on* f^*P *by requiring the* θ'_i*'s to be diffeomorphisms: this makes sense since* θ'_i *can be obtained from a different* θ'_j *(with* $\mathcal{U}_{ij}\neq\emptyset$*) by composing it with the smooth map* $f^{-1}(\mathcal{U}_{ij})\times G\to f^{-1}(\mathcal{U}_{ij})\times G$*,* $(p,g)\mapsto \Big(p,\theta_{ij}\big(f(p)\big)\,g\Big)$*. Then* $f^*P\,(N,G)$ *becomes a principal fibre bundle (the* ***pullback bundle*** *of* $P\,(M,G)$ *with respect to the map* $f: N\to M$*) and* $(\bar f,\mathrm{id}) : f^*P\,(N,G)\to P\,(M,G)$ *a homomorphism. Note that the transition functions of* $f^*P\,(N,G)$ *with respect to the local trivializations* $\{\theta'_i,\ i\in I\}$ *are given by* $\theta'_{ij}=f^*\,\theta_{ij}$*.*

Let $P\,(M,G)$ be a principal fibre bundle and let F be a manifold on which G act differentiably on the left. There is a standard construction that associates with $P\,(M,G)$ a fibre bundle over M with fibre F; this fibre bundle will be denoted by $E=P\times_G F$ and its construction goes as follows (see [14] for more details).

Consider the following action of G on $P\times F$:

$$(\,P\times F\,)\times G\to P\times F\ ,\ \big(\,(\,u,f\,),\,g\,\big)\mapsto(\,ug,\,g^{-1}f\,).$$

Define $E := P\times_G F := {}^{(P\times F)}\!/_{G}$, and denote by $(\,u,f\,)_{/\sim}$ the equivalence class of $(\,u,f\,)\in P\times F$ in E. Define $\pi_E : E\to M$ to be the map $(\,u,f\,)_{/\sim}\mapsto\pi(u)$, where π is the projection $P\to M$. Let $\{\,\theta_i:\pi^{-1}(\mathcal{U}_i)\to\mathcal{U}_i\times G,\ u\mapsto\big(\pi(u),\,\varphi_i\,(u)\big)\,\}$ be local trivializations of $P\,(M,G)$ with respect to a cover $\{\mathcal{U}_i,\, i\in I\}$ of M and define for all $i\in I$ a bijection

$$\theta_i^E:\pi_E^{-1}(\mathcal{U}_i)\to\mathcal{U}_i\times F\ ,\ (\,u,f\,)_{/\sim}\mapsto(\,\pi(u),\,\varphi_i\,(u)\,f\,). \tag{1.4}$$

We can define a differentiable structure on E by requiring the θ_i^E's to be diffeomorphisms: this makes sense since θ_i^E can be obtained from a different θ_j^E (with $\mathcal{U}_{ij}\neq\emptyset$) by composing it with the smooth map $\mathcal{U}_{ij}\times F\to\mathcal{U}_{ij}\times F$, $(\,p,f\,)\mapsto(\,p,\,\theta_{ij}\,(p)\,f\,)$. Then $E=P\times_G F$ becomes a fibre bundle over M with fibre F.

Lemma 1.2.9 *Let* $P\,(M,G)$ *be a principal fibre bundle,* F *a manifold with a left* G*-action and* $P\times_G F$ *the associated fibre bundle. We say that a map* $\phi: P\to F$ *is* G-equivariant *if* $\phi(ug)=g^{-1}\,\phi(u)$ *for all* $u\in P$ *and* $g\in G$*. Then we have a correspondence:*

$$\{\,\textit{sections of } P\times_G F\,\}\xleftrightarrow{\text{1-1}}\{\,G\textit{-equivariant smooth maps } P\to F\,\}.$$

Proof Given a section $\sigma: M\to P\times_G F$, define a map $\hat\sigma : P\to F$ by the relation $\sigma\big(\pi(u)\big)=\big(u,\hat\sigma(u)\big)_{/\sim}$ for $u\in P$. Then $\hat\sigma$ is G-equivariant and smooth. Smoothness can be seen as follows. Observe first that if $\{\,\theta_i:\pi^{-1}(\mathcal{U}_i)\to\mathcal{U}_i\times G$,

$i \in I\}$ are local trivializations of $P(M,G)$ with respect to a cover $\{\mathcal{U}_i,\, i \in I\}$ of M, then for $p \in \mathcal{U}_i$ we can write $\sigma(p) = \big(\theta_i^{-1}(p,e), f_i(p)\big)_{/\sim}$ for some map $f_i : \mathcal{U}_i \to F$ which is smooth because it is the composition $\theta_i^E \circ \sigma$, where θ_i^E is given by (1.4). Since $\hat{\sigma}$ is locally the composition of the map $\mathcal{U}_i \times G \to F$, $(p,g) \mapsto g^{-1} f_i(p)$ with the trivialization θ_i, it follows that it is smooth.
Conversely, given a smooth G-equivariant map $\phi : P \to F$, the map

$$\phi_0 : M \to P \times_G F \;,\; p \mapsto \big(u, \phi(u)\big)_{/\sim}$$

is well-defined and smooth, thus it is a section of $P \times_G F$.
Clearly, $\hat{\phi_0} = \phi$ for all G-equivariant maps $P \to F$ and $(\hat{\sigma})_0 = \sigma$ for all section σ of $P \times_G F$. □

Example 1.2.10 *Suppose we have a representation of the structure group G of a principal fibre bundle $P(M,G)$ on a vector space V, i.e. a Lie group homomorphism $\varrho : G \to \mathrm{Aut}(V)$. We can consider the action of G on V on the left given by $(g,v) \mapsto \varrho(g)(v)$. Then $E = P \times_G V$ is a vector bundle over M. To see this, it is enough to show that the maps*

$$V \xrightarrow{\cong} \{p\} \times V \xrightarrow{\theta_i^E \circ (\theta_j^E)^{-1}|_{\{p\}\times V}} \{p\} \times V \xrightarrow{\cong} V$$

are linear for all $p \in \mathcal{U}_{ij}$. But these maps are given by $v \mapsto \varrho\big(\theta_{ij}(p)\big)(v)$, where $\{\theta_{ij} : \mathcal{U}_{ij} \to G\}$ are the transition functions of $P(M,G)$ with respect to an open cover $\{\mathcal{U}_i,\, i \in I\}$ of M, i.e. they are the linear maps $\varrho\big(\theta_{ij}(p)\big) \in \mathrm{Aut}(V)$.
Note that to give E a vector bundle structure in the form required in Definition 1.1.1, we still have to choose a basis $(v_1, \ldots, v_n)$ of V, thus a bijection $\alpha_v : V \to \mathbb{R}^n$ (which will also induce an isomorphism $\alpha_v' : \mathrm{Aut}(V) \to GL(n,\mathbb{R})$). If we do that, then the local trivializations of E induced by (1.4) are:

$$\theta_i^E : \pi_E^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n \;,\; (u,w)_{/\sim} \mapsto \Big(\pi(u)\,,\, \alpha_v'\Big(\varrho\big(\varphi_i(u)\big)\Big)\, \alpha_v(w)\Big)$$

where as usual $\varphi_i = \mathrm{pr}_2 \circ \theta_i : \pi^{-1}(\mathcal{U}_i) \to G$, with $\{\theta_i,\, i \in I\}$ local trivializations of $P(M,G)$. In particular, it follows that the vector space structure on each fibre E_p is given by $\lambda_1 (u,w_1)_{/\sim} + \lambda_2 (u,w_2)_{/\sim} = (u, \lambda_1 w_1 + \lambda_2 w_2)_{/\sim}$, for $\lambda_1, \lambda_2 \in \mathbb{R}$, thus for every $u \in \pi^{-1}(p)$ the map $V \to E_p$, $w \mapsto (u,w)_{/\sim}$ is a vector space isomorphism. Note that $E = P \times_G F$ is trivial if P is. If $\{\theta_{ij} : \mathcal{U}_{ij} \to G\}$ are the transition functions of $P(M,G)$ with respect to the trivializations $\{\theta_i,\, i \in I\}$, then the transition functions $\{\theta_{ij}^E : \mathcal{U}_{ij} \to GL(n,\mathbb{R})\}$ of E with respect to the trivializations $\{\theta_i^E,\, i \in I\}$ are given by $\theta_{ij}^E(p) = \alpha_v'\Big(\varrho\big(\theta_{ij}(p)\big)\Big)$ for $p \in \mathcal{U}_{ij}$.
We will now construct a homomorphism from $P(M,G)$ to the frame bundle $L(E)$ of $E = P \times_G V$ and show that if the representation $\varrho : G \to \mathrm{Aut}(V)$ is faithful *(i.e. injective), then this homomorphism gives a reduction of $L(E)$ to G. Note first that for $u \in P$ with $\pi(u) = p$, the vectors $(u,v_1)_{/\sim}, \ldots, (u,v_n)_{/\sim}$ form a*

basis of the fibre E_p*, since they are sent by* θ^E_{ip} *to the vectors* $\alpha'_v\Big(\varrho\big(\varphi_i\,(u)\big)\Big)\,e_1$*,* $\ldots$*,* $\alpha'_v\Big(\varrho\big(\varphi_i\,(u)\big)\Big)\,e_n$*, which are a basis of* $\mathbb{R}^n$*. Thus we can consider the map*

$$f_v : P \to L(E)\ ,\ u \mapsto \big(\,(\,u, v_1\,)_{/\sim}, \ldots, (\,u, v_n\,)_{/\sim}\,\big).$$

Locally f_v *is the composition of local trivializations of* P *and* $L(E)$ *with the map* $\mathcal{U}_i \times G \to \mathcal{U}_i \times GL\,(n,\mathbb{R})$*,* $(\,p, g\,) \mapsto \big(p, \alpha'_v\big(\varrho(g)\big)\,\big)$*, so it is smooth. An easy calculation shows that*

$$f_v(ug) = f_v(u)\,\alpha'_v\big(\varrho(g)\big)$$

so f_v *induces a homomorphism* $(\,f_v, f'_v\,) : P\,(M,G) \to L(E)\,\big(M, GL\,(n,\mathbb{R})\big)$*, where*

$$f'_v : G \to GL\,(n,\mathbb{R})\ ,\ g \mapsto \alpha'_v\big(\varrho(g)\big).$$

If $\varrho : G \to \mathrm{Aut}(V)$ *is injective, then* $f'_v : G \hookrightarrow GL\,(n,\mathbb{R})$ *is a Lie group monomorphism; since* $f''_v : M \to M$ *is the identity, we get that in this case* $P\,(M,G)$ *is a reduced bundle of* $L(E)$*.*

Example 1.2.11 *As a particular case of Example 1.2.10, consider a principal fibre bundle* $P\,\big(M, GL\,(n,\mathbb{R})\big)$ *with structure group* $GL\,(n,\mathbb{R})$*. We can identify* P *with the frame bundle* $L(E)$ *of the associated vector bundle* $E = P \times_{GL\,(n,\mathbb{R})} \mathbb{R}^n$ *by the isomorphism:*

$$f : P \to L(E)\ ,\ u \mapsto \big(\,(\,u, e_1\,)_{/\sim}, \ldots, (\,u, e_n\,)_{/\sim}\,\big).$$

Under this identification, we can consider an element $u \in P$ *as a basis of the fibre* E_p *(where* $p = \pi(u)$*), and an element* $(\,u, x\,)_{/\sim} \in E_p$ *can be interpreted as the vector of the fibre* E_p *which has coordinates* $x \in \mathbb{R}^n$ *with respect to the basis* u*.*
Similarly, given a vector bundle $\pi_E : E \to M$ *and its frame bundle* $L(E)$*, we can identify* E *with the associated vector bundle* $L(E) \times_{GL\,(n,\mathbb{R})} \mathbb{R}^n$ *by the vector bundle isomorphism:*

$$L(E) \times_{GL\,(n,\mathbb{R})} \mathbb{R}^n \to E\ ,\ (\,u, x\,)_{/\sim} \mapsto u \cdot x.$$

Combining what was said in Example 1.2.2 and in Example 1.2.10, we see that a vector bundle is trivial if and only if its frame bundle is, or equivalently if and only if its frame bundle has a section. In particular it follows that we can identify local trivializations of a vector bundle with local sections of its frame bundle.

In the next two examples we will need the following facts about Lie groups. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Every element $g \in G$ induces an automorphism $c(g)$ of G, defined by $c(g)\,(g') = g\,g'g^{-1}$. Denote the differential of $c(g)$ by $\mathrm{Ad}(g) \in \mathrm{Aut}(\mathfrak{g})$; then the map $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is a representation of G on $\mathfrak{g}$ (see [31, 3.45]), called the *adjoint representation.*
Denote by exp the exponential map of the Lie algebra of a Lie group to the Lie group. Then for any Lie group homomorphism $f : G \to H$ we have

$$f\,\big(\exp(B)\big) = \exp\big(\,f_*(B)\big) \tag{1.5}$$

for all $B \in \mathfrak{g}$ (see [31, 3.32]). In particular, we can apply this to the homomorphism $c(g) : G \to G$ and get

$$c(g)\,\big(\exp(B)\big) = \exp\big(\mathrm{Ad}(g)\,(B)\big) \tag{1.6}$$

for all $g \in G$ and $B \in \mathfrak{g}$.

Lemma 1.2.12 *Let $f : G \to H$ be a Lie group homomorphism. Then for $g \in G$ and $B \in \mathfrak{g}$ it holds:*

$$f_*\,\big(\mathrm{Ad}(g)\,(B)\big) = \mathrm{Ad}\big(f(g)\big)\,\big(f_*(B)\big).$$

Proof We have, using (1.5) and (1.6):

$$\begin{aligned}
f_*\,\big(\mathrm{Ad}(g)\,(B)\big) &= \frac{d}{dt}\Big|_{t=0} f\big(\,g\exp(tB)\,g^{-1}\,\big) = \frac{d}{dt}\Big|_{t=0} f(g)\,f\big(\,\exp(tB)\,\big)\,f\big(g^{-1}\big) \\
&= \frac{d}{dt}\Big|_{t=0} c\big(f(g)\big)\,\Big(f\big(\,\exp(tB)\,\big)\Big) \\
&\overset{(1.5)}{=} \frac{d}{dt}\Big|_{t=0} c\big(f(g)\big)\,\Big(\exp\big(\,f_*(tB)\,\big)\Big) \\
&\overset{(1.6)}{=} \frac{d}{dt}\Big|_{t=0} \exp\Big(\mathrm{Ad}\big(f(g)\big)\,\big(\,f_*(tB)\,\big)\Big) \\
&= \frac{d}{dt}\Big|_{t=0} \exp\Big(t\,\mathrm{Ad}\big(f(g)\big)\,\big(f_*(B)\big)\Big) = \mathrm{Ad}\big(f(g)\big)\,\big(f_*(B)\big).
\end{aligned}$$

□

Consider the Lie group $\mathrm{Aut}(V)$, where V is a vector space. Its Lie algebra is $\mathrm{End}(V)$ (see [31, 3.10]) and the adjoint representation $\mathrm{Ad} : \mathrm{Aut}(V) \to \mathrm{Aut}\big(\mathrm{End}(V)\big)$ is given by:

$$\mathrm{Ad}(A)(X) = AXA^{-1} \tag{1.7}$$

for $A \in \mathrm{Aut}(V)$ and $X \in \mathrm{End}(V)$, where the operation on the right hand side is composition of maps (see [31, 3.46]).

Example 1.2.13 *Let $P\,(M, G)$ be a principal fibre bundle and let $\mathfrak{g}$ be the Lie algebra of G. Denote by $P \times_{Ad} \mathfrak{g}$ the associated fibre bundle obtained by letting G work on $\mathfrak{g}$ via the adjoint representation $Ad : G \to \mathrm{Aut}(\mathfrak{g})$. By what was said in Example 1.2.10, $P \times_{Ad} \mathfrak{g}$ is a vector bundle over M. It is called the **adjoint bundle** of $P\,(M, G)$.*
Suppose that there is a representation $\varrho : G \to \mathrm{Aut}(V)$ of G on a vector space V and let $E = P \times_G V$ be the associated vector bundle. Then we can construct a vector bundle map $\phi : P \times_{Ad} \mathfrak{g} \to E^* \otimes E$ as follows.
Observe first that the differential of ϱ gives a representation of $\mathfrak{g}$ on V, i.e. a Lie algebra homomorphism $\varrho_* : \mathfrak{g} \to \mathrm{End}(V)$.
Let $(\,u, B\,)_{/\sim} \in (P \times_{Ad} \mathfrak{g})_p$ and define $\phi\,\big((\,u, B\,)_{/\sim}\big) \in (E^* \otimes E)_p = \mathrm{Hom}(E_p, E_p)$ by $\phi\,\big(\,(\,u, B\,)_{/\sim}\big)\,\big(\,(\,u, v\,)_{/\sim}\,\big) = \big(\,u, \varrho_*(B)\,(v)\,\big)_{/\sim}$ (use (1.7) and Lemma 1.2.12

to see that this is well-defined). We have to check that ϕ is smooth. By Lemma 1.1.17, it is enough to show that, given $s \in \Gamma(P \times_{Ad} \mathfrak{g})$, the map $M \to E^* \otimes E$, $p \mapsto \phi\big(s(p)\big)$ is a section of $E^* \otimes E$, i.e. a vector bundle map $E \to E$. For this, again by Lemma 1.1.17, it is enough to show that, given $\sigma \in \Gamma(E)$, the map $M \to E$, $p \mapsto \phi\big(s(p)\big)\big(\sigma(p)\big)$ is smooth, thus a section of E. Locally on some open $\mathcal{U} \subset M$ we can write $s(p) = (\,u(p), B(p)\,)_{/\sim}$ and $\sigma(p) = (\,u(p), v(p)\,)_{/\sim}$, where u is a local section of P and consequently $B : \mathcal{U} \to \mathfrak{g}$ and $v : \mathcal{U} \to V$ are smooth maps. Then for $p \in \mathcal{U}$ we have $\phi\big(s(p)\big)\big(\sigma(p)\big) = \big(\,u(p), \varrho_*\big(B(p)\big)\,v(p)\,\big)_{/\sim}$, thus it follows that the map $p \mapsto \phi\big(s(p)\big)\big(\sigma(p)\big)$ is smooth, as we wanted to show. So $\phi : P \times_{Ad} \mathfrak{g} \to E^* \otimes E$ is a vector bundle homomorphism.
Consider now the frame bundle $L(E)\,\big(M, GL\,(n,\mathbb{R})\big)$ of a vector bundle $\pi : E \to M$ and let $L(E) \times_{Ad} \mathfrak{gl}\,(n,\mathbb{R})$ be the adjoint bundle. Then we have

$$L(E) \times_{Ad} \mathfrak{gl}\,(n,\mathbb{R}) \cong E^* \otimes E.$$

To see this, define $\phi : L(E) \times_{Ad} \mathfrak{gl}\,(n,\mathbb{R}) \to E^* \otimes E$ as above and observe that it induces a vector space isomorphism on each fibre. Conclude using Lemma 1.1.4.

Example 1.2.14 *Let $P\,(M,G)$ be a principal fibre bundle and consider the action of G on G on the left given by $(\,g_1, g_2\,) \mapsto c\,(g_1)\,(g_2)$. Denote by $P \times_{Ad} G$ the associated fibre bundle. It is a fibre bundle over M with fibre G.*
We will show that we can identify the set of sections of $P \times_{Ad} G$ with the set of automorphisms of P. Given a section $\sigma \in \Gamma(P \times_{Ad} G)$, define $f_\sigma : P \to P$ by $f_\sigma\,(u) := ug$, where $g \in G$ is determined by $\sigma\big(\pi(u)\big) = (\,u, g\,)_{/\sim}$. Then we have $f_\sigma\,(ug') = f_\sigma\,(u)\,g'$, thus $f_\sigma \in \mathrm{Aut}(P)$. Conversely, given $f \in \mathrm{Aut}(P)$, define a section σ_f of $P \times_{Ad} G$ by $\sigma_f(p) := (\,u, g\,)_{/\sim}$, where $u \in \pi^{-1}(p)$ and $g \in G$ is determined by $f(u) = ug$. Use local trivializations of the various bundles to see that f_σ and σ_f are smooth if σ and f are. Since $f_{\sigma_f} = f$ for all $f \in \mathrm{Aut}(P)$ and $\sigma_{f_\sigma} = \sigma$ for all $\sigma \in \Gamma(P \times_{Ad} G)$, it follows that we have a bijection

$$\Gamma(P \times_{Ad} G) \overset{\text{1-1}}{\longleftrightarrow} \mathrm{Aut}(P).$$

Define a group structure on $\Gamma(P \times_{Ad} G)$ as follows. For σ_1, $\sigma_2 \in \Gamma(P \times_{Ad} G)$ with $\sigma_1\,(p) = (\,u, g_1\,)_{/\sim}$ and $\sigma_2\,(p) = (\,u, g_2\,)_{/\sim}$, define $\sigma_1\,\sigma_2 \in \Gamma(P \times_{Ad} G)$ by $\sigma_1\,\sigma_2\,(p) = (\,u, g_1\,g_2\,)_{/\sim}$. Then the bijection $\Gamma(P \times_{Ad} G) \overset{\text{1-1}}{\longleftrightarrow} \mathrm{Aut}(P)$ defined above gives a group isomorphism $\Gamma(P \times_{Ad} G) \cong \mathrm{Aut}(P)$. The group $\Gamma(P \times_{Ad} G)$ (usually denoted by $\mathcal{G}$) is called the ***gauge group*** *of $P\,(M,G)$.*
Note that we have also a third description of the gauge group as the group of G-equivariant maps $P \to G$, i.e. maps $\phi : P \to G$ such that $\phi\,(ug) = g^{-1}\,\phi\,(u)\,g$ (cf. Lemma 1.2.9). The group structure is given by $\phi_1 \cdot \phi_2\,(u) := \phi_1\,(u) \cdot \phi_2\,(u)$, where the operation on the right hand side is multiplication in G.
The natural isomorphisms of this group with $\mathrm{Aut}(P)$ and $\Gamma(P \times_{Ad} G)$ are given by $\phi \mapsto f_\phi$ where $f_\phi\,(u) = u\,\phi\,(u)$, and $\phi \mapsto \sigma_\phi$ where $\sigma_\phi\,(p) = (\,u, \phi\,(u)\,)_{/\sim}$ for $u \in \pi^{-1}(p)$.

Example 1.2.15 *Let $Q\,(M,H)$ be a principal fibre bundle and $\alpha : H \to G$ a Lie group monomorphism. We will construct an extension of Q to G relative to α, i.e. a principal fibre bundle $P\,(M,G)$ and a homomorphism $(f,\alpha) : Q\,(M,H) \to P\,(M,G)$. Define $P := Q \times_H G$, where the action of H on G on the left is given by $(\,h, g\,) \mapsto \alpha(h)\,g$. P is a fibre bundle over M with fibre G. Define an action $P \times G \to P$ by $\big(\,(\,u, g_1\,)_{/\sim}, g_2\,\big) \mapsto (\,u, g_1\,g_2\,)_{/\sim}$. Then ${}^{P}\!/_{G} = M$ and the local trivializations (1.4) of P given by the general construction of the associated fibre bundle are in the form required in Definition 1.2.1. Thus $P\,(M,G)$ is a principal fibre bundle. The map $f : Q \to P$, $u \mapsto (\,u, e\,)_{/\sim}$ induces a homomorphism $(f,\alpha) : Q\,(M,H) \to P\,(M,G)$, thus makes $P\,(M,G)$ an extension of $Q\,(M,H)$ to G relative to α.*
Any other extension $(f',\alpha) : Q\,(M,H) \to P'\,(M,G)$ is equivalent to $P\,(M,G)$. Indeed, the map $\psi : P = Q \times_H G \to P'$ defined by $(\,u, g\,)_{/\sim} \mapsto f'(u)\,g$ is an isomorphism $P\,(M,G) \cong P'\,(M,G)$ and makes the diagram

$$\begin{array}{ccc} Q & \xrightarrow{\;f'\;} & P' \\ {\scriptstyle f}\downarrow & \nearrow_{\psi} & \\ P & & \end{array}$$

commutative. Observe that, using Proposition 1.2.7, we can also define an extension of $Q\,(M,H)$ to G relative to α to be the principal fibre bundle over M with structure group G and transition functions $\{\,\alpha \circ \theta_{ij} : \mathcal{U}_{ij} \to G\,\}$ with respect to the cover $\{\,\mathcal{U}_i,\, i \in I\,\}$ of M, where $\{\,\theta_{ij} : \mathcal{U}_{ij} \to H\,\}$ are the transition functions of $Q\,(M,H)$ with respect to $\{\,\mathcal{U}_i,\, i \in I\,\}$.

Let $P\,(M,G)$ be a principal fibre bundle and let H be a closed subgroup of G. From classical results about closed subgroups of a Lie group [15], we know (see for example [31, 3.58 and 3.63]):

1. the set ${}^{G}\!/_{H} = \{\,gH, g \in G\,\}$ has a unique manifold structure such that the natural projection $G \to {}^{G}\!/_{H}$, $g \mapsto gH$ is smooth and such that there exist local smooth sections of ${}^{G}\!/_{H}$ in G;
2. the action $G \times {}^{G}\!/_{H} \to {}^{G}\!/_{H}$, $(\,g_1, g_2 H\,) \mapsto g_1\,g_2\,H$ is smooth.

In particular, we can construct the fibre bundle $E = P \times_G {}^{G}\!/_{H}$. It is a fibre bundle over M with fibre ${}^{G}\!/_{H}$ and it can be identified with the set ${}^{P}\!/_{H} = \{\,uH, u \in P\,\}$ by the bijection

$$P \times_G {}^{G}\!/_{H} \to {}^{P}\!/_{H}\ ,\ (\,u, gH\,)_{/\sim} \mapsto ugH.$$

[15] See also footnote 13.

Proposition 1.2.16 *The structure group G of a principal fibre bundle $P(M,G)$ is reducible to a closed subgroup H of G if and only if the associated fibre bundle $E = P \times_G {}^{G}\!/_{H}$ has a section. In fact, we have a bijection:*

$$\{\, \textit{sections of } E = P \times_G {}^{G}\!/_{H} \equiv {}^{P}\!/_{H} \,\} \overset{\text{1-1}}{\longleftrightarrow} \{\, \textit{reductions of } P(M,G) \textit{ to } H \,\}/_{\sim}$$

where "$\sim$" is the equivalence relation of Definition 1.2.5.

Proof (sketch) Given a section $\sigma : M \to E \equiv {}^{P}\!/_{H}$, let $P' = \{\, u \in P \;/\; uH = \sigma\big(\pi(u)\big) \,\}$. Then $P'(M,H)$ is a reduced bundle of $P(M,G)$ (a proof of this uses the existence of local smooth sections of ${}^{G}\!/_{H}$ in G, see 1. above).
Conversely, suppose a reduced bundle $P'(M,H)$ of $P(M,G)$ is given, i.e. a principal fibre bundle $P'(M,H)$ and a smooth map $f : P \to P'$ such that $f(u'h) = f(u')\,h$ for $u' \in P'$ and $h \in H$ and such that the induced map $M \to M$ is the identity. Then we can define a section $\sigma : M \to E = P \times_G {}^{G}\!/_{H} \equiv {}^{P}\!/_{H}$ by $\sigma(p) := f(u')\,H$, where u' is some element in the fibre of P' at p (the proof that σ is smooth uses the fact that the map $G \to {}^{G}\!/_{H}$, $g \mapsto gH$ is smooth, see 1. above). See [14, Proposition 5.6 of Chapter I] for more details. □

Example 1.2.17 *Let $L(E)\big(M, GL(n,\mathbb{R})\big)$ be the frame bundle of a vector bundle $\pi_E : E \to M$ and consider the closed subgroup $O(n)$ of $GL(n,\mathbb{R})$. We will show that there is a correspondence:*

$$\{\, \textit{Riemannian metrics on } E \,\} \overset{\text{1-1}}{\longleftrightarrow} \{\, \textit{reductions of } L(E) \textit{ to } O(n) \,\}/_{\sim} \qquad (1.8)$$

where "$\sim$" is the equivalence relation of Definition 1.2.5. From Proposition 1.2.16, it will follow that there is also a correspondence:

$$\{\, \textit{Riemannian metrics on } E \,\} \overset{\text{1-1}}{\longleftrightarrow} \{\, \textit{sections of } {}^{L(E)}\!/_{O(n)} \,\}.$$

The correspondence (1.8) is constructed as follows.
Let h be a Riemannian metric on E and define $O_h(E) \subset L(E)$ to be the set of h-orthonormal linear frames on $\pi : E \to M$. Then $O(n)$ works on $O_h(E)$ on the right and ${}^{O_h(E)}\!/_{O(n)} = M$. Let π_Q be the restriction of $\pi : L(E) \to M$ to $O_h(E)$. Consider sections $\{\, \sigma_i : \mathcal{U}_i \to L(E),\ i \in I \,\}$, where $\{\mathcal{U}_i,\ i \in I\,\}$ is a cover of M. For $i \in I$ and for all $p \in \mathcal{U}_i$ apply the Gram-Schmidt process to $\sigma_i(p)$ and $h(p)$ to obtain an h-orthonormal basis $u_i(p)$ of the fibre E_p. The new sections $\{\, u_i : \mathcal{U}_i \to L(E),\ i \in I \,\}$ are still smooth, so they induce trivializations $\pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times GL(n,\mathbb{R})$ and these trivializations send $\pi_Q^{-1}(\mathcal{U}_i)$ to $\mathcal{U}_i \times O(n)$ and so induce a differentiable structure on $O_h(E)$ making it a reduced bundle of $L(E)$. Conversely, suppose we have a reduction of $L(E)$ to $O(n)$, i.e. a principal fibre bundle $Q\big(M, O(n)\big)$ with a smooth map $f : Q \to L(E)$ such that $f(u\,A) = f(u)\,A$ for all $A \in O(n)$ and $u \in Q$ and such that the induced map $M \to M$ is the identity. For all $p \in M$, let $h(p)$ be the metric on E_p which has matrix I with

respect to a basis $f(u)$ of E_p, where u is some element of the fibre of Q over p. Note that the definition of $h(p)$ does not depend on the choice of u and that an element of the fibre of $L(E)$ over p is an h-orthonormal basis of E_p if and only if it is the image of some element of Q. To conclude that $p \mapsto h(p)$ gives a Riemannian metric on E, we have to prove that it is smooth, i.e. that if σ_1 and σ_2 are two smooth sections $M \to E$, then the function $h(\sigma_1, \sigma_2) : M \to \mathbb{R}$, $p \mapsto h(p)\big(\sigma_1(p), \sigma_2(p)\big)$ is smooth. But if we write $\sigma_1(p) = (f(u), x_1(p))_{/\sim}$ and $\sigma_2(p) = (f(u), x_2(p))_{/\sim}$ (where we identify E with $L(E) \times_{GL(n,\mathbb{R})} \mathbb{R}^n$), then $h(\sigma_1, \sigma_2)$ is the map $p \mapsto x_1(p)^t \cdot x_2(p)$, which is smooth.

Fibre bundles associated with a principal fibre bundle $P(M,G)$ do not change under reduction or extension of the structure group of $P(M,G)$. This is explained in the following proposition.

Proposition 1.2.18 *Let $Q(M,H)$ be a reduction of the principal fibre bundle $P(M,G)$ relative to a monomorphism $\alpha : H \to G$. Suppose F is a manifold on which G acts on the left and let H act on F on the left by $(h,a) \mapsto \alpha(h)\,a$. Then the associated fibre bundles $E = P \times_G F$ and $E' = Q \times_H F$ are isomorphic.*

Proof Let $f : Q \to P$ be the map inducing the reduction and define a map λ between the total spaces of $P \times_G F$ and $Q \times_H F$ by $(u,a)_{/\sim} \mapsto (f(u),a)_{/\sim}$. It is easy to see that λ is well-defined and bijective. Let $\{\theta_i^Q : \pi_Q^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times H,\ i \in I\}$ be local trivializations of $Q(M,H)$ with respect to a cover $\{\mathcal{U}_i,\ i \in I\}$ of M and let $\{\sigma_i : \mathcal{U}_i \to \pi_Q^{-1}(\mathcal{U}_i),\ i \in I\}$ be the associated local sections of Q, i.e. $\sigma_i : p \mapsto (\theta_i^Q)^{-1}(p,e)$. Then $\{f \circ \sigma_i : \mathcal{U}_i \to \pi_P^{-1}(\mathcal{U}_i),\ i \in I\}$ are local sections of $P(M,G)$ and they induce local trivializations $\{\theta_i^P : \pi_P^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times G,\ i \in I\}$ of $P(M,G)$ such that $\theta_i^P\big(f \circ \sigma_i(p)\big) = (p,e)$. Let $\{\theta_i^E : \pi_E^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times F,\ i \in I\}$ and $\{\theta_i^{E'} : \pi_{E'}^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times F,\ i \in I\}$ be the induced local trivializations of $E = P \times_G F$ and $E' = Q \times_H F$ respectively. Then locally $\lambda = (\theta_i^P)^{-1} \circ \mathrm{id}|_{\mathcal{U}_i \times F} \circ \theta_i^Q$, thus in particular λ gives a diffeomorphism between the total spaces of $P \times_G F$ and $Q \times_H F$ and so, since the induced map on the base spaces is the identity, a fibre bundle homomorphism. □

We have proved in Example 1.2.15 that extensions of principal fibre bundles always exist and are unique up to equivalence. This is not the case for reductions: they do not always exist and, if they do, they are in general not unique. For example, a reduction of the frame bundle $L(M)$ of the tangent bundle of a manifold M to $GL(n,\mathbb{R})^+$ gives an orientation of M (and vice versa), but not all manifold are orientable and, if they do, there are two possible choices of an orientation, i.e. two different reductions of $L(M)$ to $GL(n,\mathbb{R})^+$.
The following lemma gives a description of reductions of a principal fibre bundle in terms of transition functions.

Lemma 1.2.19 *Let $P(M,G)$ be a principal fibre bundle and let H be a Lie subgroup of G. Then $P(M,G)$ is reducible to H if and only if there is a cover of M with a set of transition functions for $P(M,G)$ which take their values in H.*

For a proof of this, see [14, Proposition 5.3 of Chapter I].

In the following proposition we summarize all facts that were proved or mentioned in this section about reductions of a principal fibre bundle $P(M,G)$ to a closed Lie subgroup $H \leq G$ (cf. Definition 1.2.4, Proposition 1.2.16, Lemma 1.2.9 and Lemma 1.2.19).

Proposition 1.2.20 *Let $P(M,G)$ be a principal fibre bundle and let H be a closed Lie subgroup of G. Then a reduction of $P(M,G)$ to H can be described equivalently as:*

1. *a principal fibre bundle $Q(M,H)$ with an embedding $(f,\iota) : Q(M,H) \to P(M,G)$ inducing the identity on M, where ι is the inclusion $H \hookrightarrow G$;*
2. *a system of local trivializations of $P(M,G)$ such that the transition functions have values in H;*
3. *a section of $P \times_G {}^{G}\!/_{H}$;*
4. *a G-equivariant map $P \to {}^{G}\!/_{H}$.*

We conclude the section with a concept that plays an important role in the theory of connections on principal fibre bundles. Let $P(M,G)$ be a principal fibre bundle and let $\mathfrak{g}$ be the Lie algebra of G. We can assign to each $B \in \mathfrak{g}$ a vector field $B^* \in \Gamma(TP)$ (called the **fundamental vector field** corresponding to B) as follows (see [14] for more details). For $u \in P$, let $\sigma_u : G \to P$ be the map $g \mapsto ug$ and define $(B^*)_u := (\sigma_u)_* (B) \in T_uP$. In other words, $(B^*)_u = \frac{d}{dt}\Big|_{t=0} u \exp(tB)$. The map $\mathfrak{g} \to \Gamma(TP)$, $B \mapsto B^*$ is a Lie algebra homomorphism and, since the action of G is free, B^* never vanish on P if $B \neq 0$ (see [14, Proposition 4.1 of Chapter I]). Since the action of G sends each fibre of P into itself, we have that $(B^*)_u$ is tangent to the fibre through u for every $u \in P$. The map $\mathfrak{g} \to T_uP$, $B \mapsto (B^*)_u$ is a linear monomorphism; since the dimension of each fibre is equal to the dimension of G, it follows that $B \mapsto (B^*)_u$ is a linear isomorphism of $\mathfrak{g}$ into the tangent space at u of the fibre through u.

Lemma 1.2.21 *Let $P(M,G)$ be a principal fibre bundle and let $B^* \in \Gamma(TP)$ be the fundamental vector field corresponding to $B \in \mathfrak{g}$. Then for each $g \in G$, $(R_g)_* B^*$ is the fundamental vector field corresponding to* $\mathrm{Ad}(g^{-1})(B)$.

For a proof of this, see [14, Proposition 5.1 of Chapter I].

1.3 Connections on vector bundles

Let $\pi : E \to M$ be a vector bundle and r a positive integer. Define

$$\mathcal{A}^r(E) := \Gamma(E) \otimes_{\mathcal{C}^\infty(M)} \mathcal{A}^r(M) = \Gamma\left(E \otimes \bigwedge^r T^*M\right)$$

(cf. Lemma 1.1.17). In particular, $\mathcal{A}^0(E) = \Gamma(E)$. Elements of $\mathcal{A}^r(E)$ are called *(smooth) r-forms on M with values in E*; they can equivalently be defined as $\mathcal{C}^\infty(M)$-multilinear alternating maps

$$\Gamma(TM) \times \ldots \times \Gamma(TM) \to \Gamma(E). \text{ [16]}$$

Definition 1.3.1 *A* ***connection*** *on a vector bundle $\pi : E \to M$ is an $\mathbb{R}$-linear map $D : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ which satisfies the* Leibnitz rule*, i.e. for $f \in \mathcal{C}^\infty(M)$ and $\sigma \in \Gamma(E)$ it must hold*

$$D\,(f\sigma) = \sigma \otimes df + f\,D(\sigma). \tag{1.9}$$

Connections exist on every vector bundle, as can be proved using a partition of unity on the base space (see [22, Lemma 2 of Appendix C]).
Given a connection D_0 on $\pi : E \to M$, every other one is of the form $D_0 + \xi$, where $\xi \in \mathcal{A}^1(E^* \otimes E)$. This is explained in the next proposition.

Proposition 1.3.2 *Let $\pi : E \to M$ be a vector bundle and denote by $\mathcal{D}_E$ the set of connections on it. Then $\mathcal{D}_E$ is an affine space modeled on $\mathcal{A}^1(E^* \otimes E)$, i.e.:*

$$\mathcal{D}_E = D_0 + \mathcal{A}^1(E^* \otimes E)$$

where D_0 is a fixed connection on $\pi : E \to M$.

Proof Let D_1 and D_2 be connections on $\pi : E \to M$. For $X \in \Gamma(TM)$ and $\sigma \in \Gamma(E)$, $(D_1 - D_2)(X)\,(\sigma) := D_1(\sigma)\,(X) - D_2(\sigma)\,(X)$ is an element of $\Gamma(E)$. The map $(D_1 - D_2)(X) : \Gamma(E) \to \Gamma(E)$ is linear over $\mathcal{C}^\infty(M)$, thus by Lemma 1.1.17 we have $(D_1 - D_2)(X) \in \Gamma(E^* \otimes E)$. The map $D_1 - D_2 : \Gamma(TM) \to \Gamma\,(E^* \otimes E)$ is linear over $\mathcal{C}^\infty(M)$, thus $D_1 - D_2 \in \mathcal{A}^1(E^* \otimes E)$. In particular, $\mathcal{D}_E \subset D_0 + \mathcal{A}^1(E^* \otimes E)$.
Conversely, let $\xi \in \mathcal{A}^1(E^* \otimes E)$. We have to prove that $D_0 + \xi \in \mathcal{D}_E$. But since ξ is a $\mathcal{C}^\infty(M)$-linear map $\mathcal{A}^0(E) \to \mathcal{A}^1(E)$, it follows that $D_0 + \xi$ is a map $\mathcal{A}^0(E) \to \mathcal{A}^1(E)$ which satisfies the Leibnitz rule, thus $D_0 + \xi \in \mathcal{D}_E$. □

A connection $D : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ on a vector bundle $\pi : E \to M$ is a local operator, i.e. if two sections σ_1 and $\sigma_2 \in \mathcal{A}^0(E)$ coincide on some open $\mathcal{U} \subset M$, then $D(\sigma_1)$ and $D(\sigma_2) \in \mathcal{A}^1(E)$ also coincide on $\mathcal{U}$. To see this, let $p \in \mathcal{U}$ and take a function $f \in \mathcal{C}^\infty(M)$ with $f(p) = 1$ and $\operatorname{supp}(f) \subset \mathcal{U}$. Then $f\sigma_1 = f\sigma_2$, so $D\,(f\sigma_1) = D\,(f\sigma_2)$. Applying (1.9) and evaluating at p gives

[16] A proof of this is similar to the second part of the proof of Lemma 1.1.17. See [31, 2.18] for the proof of an analogous statement for real differential forms.

$D(\sigma_1)\,(p) = D(\sigma_2)\,(p)$. Thus a connection $D : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ induces a connection $D|_{\mathcal{U}} : \mathcal{A}^0(E|_{\mathcal{U}}) \to \mathcal{A}^1(E|_{\mathcal{U}})$ on each open $\mathcal{U} \subset M$, in such a way that it holds $D|_{\mathcal{U}}\,(\sigma|_{\mathcal{U}}) = \big(D(\sigma)\big)|_{\mathcal{U}}$ for $\sigma \in \mathcal{A}^0(E)$. The induced connection $D|_{\mathcal{U}}$ will be also denoted by D.

Let $u = (u_1, \ldots, u_n)$ be a local frame of a vector bundle $\pi : E \to M$ (i.e. a local section of the frame bundle $L(E)$) over an open $\mathcal{U} \subset M$ and let D be a connection on E. Then we can write $D(u) = u \otimes \omega_u$, i.e.

$$D\,(u_\alpha) = \sum_{\beta=1}^{n} u_\beta \otimes (\omega_u)_{\,\beta\,\alpha} \tag{1.10}$$

where $\omega_u = \big(\,(\omega_u)_{\,\alpha\beta}\,\big)$ is a matrix of 1-forms on $\mathcal{U}$ called the **connection form** of D with respect to the local frame u. If $\sigma = \sum_{\alpha=1}^{n} v_\alpha\, u_\alpha$ is a section of E over $\mathcal{U}$ (where $v_\alpha \in \mathcal{C}^\infty(\mathcal{U})$), then from (1.9) and (1.10) we get:

$$D(\sigma) = \sum_{\alpha=1}^{n} u_\alpha \otimes \Big(\, dv_\alpha + \sum_{\beta=1}^{n} (\omega_u)_{\,\alpha\beta}\; v_\beta \Big). \tag{1.11}$$

To simplify notation we can identify E with $L(E) \times_{GL\,(n,\mathbb{R})} \mathbb{R}^n$ and consider u as a local section of $L(E)$ over $\mathcal{U}$. Then we can write $\sigma = (\,u, v\,)_{/\sim}$, where $v = (v_1, \ldots, v_n)^{\,t}$, and (1.11) becomes (by abuse of notation):

$$D(\sigma) = (\,u, dv + \omega_u\, v\,)_{/\sim}.$$

Consider a system of local trivializations $\{\,\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n\;,\; i \in I\,\}$ of E with respect to a cover $\{\,\mathcal{U}_i,\, i \in I\,\}$ of M and let $\{\,u^i,\, i \in I\,\}$ be the corresponding local frames, i.e. ${u^i}_\alpha\,(p) = \theta_i^{-1}(\,p, e_\alpha\,)$ for $p \in \mathcal{U}_i$. For every $i \in I$ let ω_i be the connection form of D with respect to the local frame u^i, i.e. $D(u^i) = u^i \otimes \omega_i$. Then on $\mathcal{U}_{ij}$ it holds:

$$\omega_j = \theta_{ji}\; \omega_i\; \theta_{ij} + \theta_{ji}\; d\theta_{ij} \tag{1.12}$$

where $\{\,\theta_{ij} : \mathcal{U}_{ij} \to GL\,(n, \mathbb{R})\,\}$ are the transition functions of the trivializations $\{\,\theta_i,\, i \in I\,\}$. This can be checked using (1.9), (1.10) and the relation $u^j = u^i\,\theta_{ij}$. Conversely, we have the following lemma.

Lemma 1.3.3 *Let $\pi : E \to M$ be a vector bundle with local trivializations $\{\,\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n\;,\; i \in I\,\}$ with respect to a cover $\{\,\mathcal{U}_i,\, i \in I\,\}$ of M. Suppose we are given for each $i \in I$ an $n \times n$ matrix ω_i of 1-forms on $\mathcal{U}_i$ in such a way that (1.12) is satisfied on each $\mathcal{U}_{ij} \neq \emptyset$. Then there is a unique connection D on E with the $\{\,\omega_i,\, i \in I\,\}$ as connection forms with respect to the local frames*

$$\{\, u^i = ({u^i}_1, \ldots, {u^i}_n) : p \mapsto \big(\,\theta_i^{-1}(\,p, e_1\,),\, \ldots\, ,\theta_i^{-1}(\,p, e_n\,)\,\big),\, i \in I\,\}$$

induced by the trivializations $\{\,\theta_i,\, i \in I\,\}$.

Proof Let $\sigma \in \Gamma(E)$. On $\mathcal{U}_i$ we have $\sigma|_{\mathcal{U}_i} = \sum_{\alpha=1}^{n} \sigma_\alpha^i \, u_\alpha^i$ for some $\sigma_\alpha^i \in \mathcal{C}^\infty(\mathcal{U})$ and by (1.11) we must define

$$D\left(\sigma|_{\mathcal{U}_i}\right) = \sum_{\alpha=1}^{n} u_\alpha^i \otimes \Big(d\sigma_\alpha^i + \sum_{\beta=1}^{n} (\omega_i)_{\alpha\beta} \, \sigma_\beta^i \Big).$$

A long but straightforward calculation is needed to see that $D\left(\sigma|_{\mathcal{U}_i}\right)$ and $D\left(\sigma|_{\mathcal{U}_j}\right)$ coincide on $\mathcal{U}_{ij}$. So the local definitions of $D(\sigma)$ glue together to give a well defined element of $\mathcal{A}^1(E)$. It is easy to see that the Leibnitz rule is satisfied. □

Given a connection $D : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ on a vector bundle $\pi : E \to M$, we can extend it for every positive integer r to a $\mathbb{R}$-linear operator $D : \mathcal{A}^r(E) \to \mathcal{A}^{r+1}(E)$ by forcing the Leibnitz rule

$$D\left(\sigma \otimes \omega\right) = \sigma \otimes d\omega + D(\sigma) \wedge \omega \tag{1.13}$$

for $\sigma \in \Gamma(E)$ and $\omega \in \mathcal{A}^r(M)$ and by linear extension [17]. The exterior product $\mathcal{A}^s(E) \times \mathcal{A}^r(M) \to \mathcal{A}^{r+s}(E)$ on the right hand side is defined as follows: for $\xi = \sigma \otimes \omega \in \mathcal{A}^s(E)$ and $\omega' \in \mathcal{A}^r(M)$, $\xi \wedge \omega' := \sigma \otimes (\omega \wedge \omega')$.

Definition 1.3.4 *Given a connection $D : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ on a vector bundle $\pi : E \to M$, the **curvature** of D is the operator $R = D \circ D : \mathcal{A}^0(E) \to \mathcal{A}^2(E)$.*

It is easy to check that $R : \mathcal{A}^0(E) \to \mathcal{A}^2(E)$ is linear over $\mathcal{C}^\infty(M)$; then by Lemma 1.1.17 it follows that R is a vector bundle map $E \to E \otimes \bigwedge^2 T^*M$ i.e. $R \in \Gamma\left(E^* \otimes E \otimes \bigwedge^2 T^*M\right) = \mathcal{A}^2(E^* \otimes E)$. We can also consider $R = D \circ D$ as a map $\mathcal{A}^r(E) \to \mathcal{A}^{r+2}(E)$, for every integer $r \geq 0$. Then it is easy to check that

$$R\left(\sigma \otimes \omega\right) = R\left(\sigma\right) \wedge \omega \tag{1.14}$$

for $\sigma \in \mathcal{A}^0(E)$ and $\omega \in \mathcal{A}^r(M)$.
In general, given vector bundles $\pi_E : E \to M$ and $\pi_F : F \to M$, we can consider every form in $\mathcal{A}^r(E^* \otimes F)$ as a map $\mathcal{A}^s(E) \to \mathcal{A}^{r+s}(F)$ by defining

$$\left(\varphi \otimes \omega_1\right)\left(\sigma \otimes \omega_2\right) := \varphi(\sigma) \otimes \left(\omega_1 \wedge \omega_2\right) \tag{1.15}$$

for $\varphi \in \mathcal{A}^0(E^* \otimes F)$, $\sigma \in \mathcal{A}^0(E)$ and $\omega_1, \omega_2 \in \mathcal{A}(M)$, and linear extension. Because of (1.14), for $R \in \mathcal{A}^2(E^* \otimes E)$ this is consistent with the definition $R = D \circ D$.

Let $u = (u_1, \ldots, u_n)$ be a local frame of a vector bundle $\pi : E \to M$ over an open $\mathcal{U} \subset M$ and let D be a connection on E with curvature R. Then we can write $R(u) = u \otimes \Omega_u$, i.e.

$$R\left(u_\alpha\right) = \sum_{\beta=1}^{n} u_\beta \otimes \left(\Omega_u\right)_{\beta\,\alpha} \tag{1.16}$$

[17] Note that this is well-defined, since for $f \in \mathcal{C}^\infty(M)$ we have $D\ \ (f\sigma) \otimes \omega\ \ = D\ \ \sigma \otimes (f\omega)\ $.

where $\Omega_u = \big((\Omega_u)_{\alpha\beta} \big)$ is a matrix of 2-forms on $\mathcal{U}$ called the **curvature form** of D with respect to the local frame u. If $\sigma = \sum_{\alpha=1}^n v_\alpha \, u_\alpha$ is a section of E over $\mathcal{U}$ (where $v_\alpha \in \mathcal{C}^\infty(\mathcal{U})$), then from (1.13) and (1.16) we get:

$$R(\sigma) = \sum_{\alpha=1}^n u_\alpha \otimes \Big(\sum_{\beta=1}^n (\Omega_u)_{\alpha\beta} \, v_\beta \Big). \tag{1.17}$$

If we identify E with $L(E) \times_{GL(n,\mathbb{R})} \mathbb{R}^n$ and write $\sigma = (\, u, v \,)_{/\sim}$ as above, then (1.17) becomes (by abuse of notation):

$$R(\sigma) = (\, u, \, \Omega_u \, v \,)_{/\sim}.$$

Let $\{\, \theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n \;, \; i \in I \,\}$ be local trivializations of E with respect to a cover $\{ \mathcal{U}_i, \, i \in I \,\}$ of M and let $\{\, u^i, \, i \in I \,\}$ be the corresponding local frames. For every $i \in I$ let Ω_i be the connection form of D with respect to the local frame u^i, i.e. $R(u^i) = u^i \otimes \Omega_i$. Then on $\mathcal{U}_{ij}$ it holds:

$$\Omega_j = \theta_{ji} \, \Omega_i \, \theta_{ij} \tag{1.18}$$

where $\{\, \theta_{ij} : \mathcal{U}_{ij} \to GL(n, \mathbb{R}) \,\}$ are the transition functions of the trivializations $\{\, \theta_i, \, i \in I \,\}$. This can be checked using (1.13), (1.16) and the relation $u^j = u^i \, \theta_{ij}$.

The connection and curvature forms ω_u and Ω_u of a connection D on a vector bundle E with respect to a local frame u over $\mathcal{U} \subset M$ are related by the **structure equation**

$$\Omega_u = d\omega_u + \omega_u \wedge \omega_u \tag{1.19}$$

i.e

$$(\, \Omega_u \,)_{\alpha\beta} = d \, (\, \omega_u \,)_{\alpha\beta} + \sum_{k=1}^n (\, \omega_u \,)_{\alpha k} \wedge (\, \omega_u \,)_{k\beta}.$$

This can be proved using (1.16), (1.10) and (1.13) (see also [13]).
Exterior differentiation of (1.19) gives the **Bianchi identity**:

$$d\Omega_u = \Omega_u \wedge \omega_u - \omega_u \wedge \Omega_u. \tag{1.20}$$

Given connections on two vector bundles $\pi_E : E \to M$ and $\pi_F : F \to M$, one can define induced connections on the bundles E^*, $E \oplus F$, $E \otimes F$ and $\bigwedge^r E$ in a natural way (and get then also connections on the bundles obtained by iterating these operations). In the next examples we will describe how to do it for dual and tensor bundles. See [13, §5 of Chapter 1] for more details and for the other cases.

Example 1.3.5 *Let D be a connection on a vector bundle $\pi : E \to M$. Define a connection D^* on the dual bundle E^* by:*

$$\langle \, D^*(\eta^*), \sigma \, \rangle := d \, \langle \, \eta^*, \sigma \, \rangle - \langle \, \eta^*, D(\sigma) \, \rangle \tag{1.21}$$

for $\eta^* \in \mathcal{A}^0(E^*)$ and $\sigma \in \mathcal{A}^0(E)$. It is easy to see that the Leibnitz rule is satisfied. Let $u = (u_1, \ldots, u_n)$ be a local frame of E over some open $\mathcal{U} \subset M$ and take the dual frame $u^* = (u_1^*, \ldots, u_n^*)$ of E^* over $\mathcal{U}$. Then the connection and curvature forms of D^* with respect to u^* are given by $\big(\omega^*\big)_{u^*} = -\left(\omega_u\right)^t$ and $\big(\Omega^*\big)_{u^*} = -\left(\Omega_u\right)^t$.

Example 1.3.6 *Let D_E and D_F be connections respectively on the vector bundles $\pi_E : E \to M$ and $\pi_F : F \to M$. Define a connection $D_{E\otimes F}$ on the tensor bundle $E \otimes F$ by:*

$$D_{E\otimes F}(\sigma \otimes \eta) := D_E(\sigma) \otimes \eta + \sigma \otimes D_F(\eta) \tag{1.22}$$

for $\sigma \in \mathcal{A}^0(E)$ and $\eta \in \mathcal{A}^0(F)$ and linear extension [18]*. It is easy to see that the Leibnitz rule is satisfied. Let $u^E = (u_1^E, \ldots, u_n^E)$ and $u^F = (u_1^F, \ldots, u_m^F)$ be local frames of E and F over some open $\mathcal{U} \subset M$ and consider the frame*

$$u^E \otimes u^F = \{\, u_i^E \otimes u_j^F \,,\ i = 1, \ldots, n \,,\ j = 1, \ldots, m \,\}$$

of $E \otimes F$ over $\mathcal{U}$, where the $\{\, u_i^E \otimes u_j^F \,\}$ are ordered lexicographically. Let ω_E, ω_F, $\omega_{E\otimes F}$ and Ω_E, Ω_F, $\Omega_{E\otimes F}$ be the connection and curvature forms of D_E, D_F and $D_{E\otimes F}$ with respect to the frames u^E, u^F and $u^{E\otimes F}$ respectively. Then we have:

$$\omega_{E\otimes F} = \omega_E \otimes I_m + I_n \otimes \omega_F$$

and

$$\Omega_{E\otimes F} = \Omega_E \otimes I_m + I_n \otimes \Omega_F \,.$$ [19]

Example 1.3.7 *Let D_E and D_F be connections respectively on the vector bundles $\pi_E : E \to M$ and $\pi_F : F \to M$. Define a connection $D_{E^*\otimes F}$ on the bundle $E^* \otimes F$ by*

$$D_{E^*\otimes F}\,(\varphi) := D_F \circ \varphi - \varphi \circ D_E \tag{1.23}$$

i.e.

$$D_{E^*\otimes F}\,(\varphi)(X)\,(\sigma) := D_F\big(\varphi(\sigma)\big)(X) - \varphi\left(D_E(\sigma)(X)\right)$$

for $\varphi \in \mathcal{A}^0(E^* \otimes F)$, $X \in \Gamma(TM)$ and $\sigma \in \mathcal{A}^0(E)$, where φ and $D_{E^*\otimes F}\,(\varphi)(X)$ are regarded as $\mathcal{C}^\infty(M)$-liner maps $\mathcal{A}^0(E) \to \mathcal{A}^0(F)$ (see Lemma 1.1.17). This definition coincides with the one that is obtained by combining (1.22) and (1.21), as one can see by checking it for elements $\varphi \in \mathcal{A}^0(E^* \otimes F)$ of the form $\varphi = \tau^* \otimes \eta$, where $\tau^* \in \mathcal{A}^0(E^*)$ and $\eta \in \mathcal{A}^0(F)$.
For the extended operator $D_{E^*\otimes F} : \mathcal{A}^r(E^* \otimes F) \to \mathcal{A}^{r+1}(E^* \otimes F)$ it holds

$$D_{E^*\otimes F}\,(\xi) = D_F \circ \xi + (-1)^{r+1}\,\xi \circ D_E \tag{1.24}$$

i.e.

$$D_{E^*\otimes F}\,(\xi)\,(\sigma) = D_F\,\big(\xi(\sigma)\big) + (-1)^{r+1}\,\xi\,\big(D_E(\sigma)\big)$$

[18] Note that this is well defined since $D_{E\otimes F}\ \ (f\sigma)\otimes\eta\ \ = D_{E\otimes F}\ \ \sigma\otimes f(\eta)\ \ .$
[19] See footnote 4.

for $\xi \in \mathcal{A}^r(E^* \otimes F)$ and $\sigma \in \mathcal{A}^0(E)$, where the second term on the right hand side is defined by (1.15). Formula (1.24) can be proved by checking it for elements of the form $\xi = \varphi \otimes \omega$ and using (1.13).
In particular, let $\pi : E \to M$ be a vector bundle and let $R \in \mathcal{A}^2(E^* \otimes E)$ be the curvature of a connection D on E. Then

$$D_{E^*\otimes E}(R) = D \circ R - R \circ D = D \circ D \circ D - D \circ D \circ D = 0$$

(see the remark following (1.15)). We can interpret this in local coordinates as follows. Consider a local frame $u = (u_1, \ldots, u_n)$ of E over some open $\mathcal{U} \subset M$ and let ω_u and Ω_u be the connection and curvature forms of D with respect to u. Then on $\mathcal{U}$ we can write $R = \sum_{\alpha,\beta=1}^{n} (u_\beta^* \otimes u_\alpha) \otimes \big(\Omega_u\big)_{\alpha\beta}$ and a straightforward calculation gives

$$D_{E^*\otimes E}(R) = \sum_{\alpha,\beta=1}^{n} (u_\beta^* \otimes u_\alpha) \otimes \Big(d\Omega_u + \omega_u \wedge \Omega_u - \Omega_u \wedge \omega_u\Big)_{\alpha\beta}.$$

So we see that the formula $D_{E^*\otimes E}(R) = 0$ obtained above is a global version of the Bianchi identity (1.20).

Example 1.3.8 *Let D be a connection on a vector bundle $\pi : E \to M$. Combining (1.23) and (1.21), we get a connection $D_{E^*\otimes E^*}$ on the bundle $E^* \otimes E^*$, which can be expressed as follows. Observe first that we can consider an element h of $\mathcal{A}^0(E^* \otimes E^*)$ either as a $\mathcal{C}^\infty(M)$-linear map $\mathcal{A}^0(E) \to \mathcal{A}^0(E^*)$ or as a $\mathcal{C}^\infty(M)$-bilinear map $\mathcal{A}^0(E) \times \mathcal{A}^0(E) \to \mathcal{C}^\infty(M)$; these two interpretations are related by the formula $h(\sigma_1, \sigma_2) = \langle h(\sigma_1), \sigma_2 \rangle$ for σ_1, $\sigma_2 \in \mathcal{A}^0(E)$. We have*

$$\begin{aligned} D_{E^*\otimes E^*}(h)\,(\sigma_1, \sigma_2) &= \langle D_{E^*\otimes E^*}(h)\,(\sigma_1), \sigma_2 \rangle = \langle D^*\big(h(\sigma_1)\big) - h\big(D(\sigma_1)\big), \sigma_2 \rangle \\ &= \langle D^*\big(h(\sigma_1)\big), \sigma_2 \rangle - \langle h\big(D(\sigma_1)\big), \sigma_2 \rangle \\ &= d\,\langle h(\sigma_1), \sigma_2 \rangle - \langle h(\sigma_1), D(\sigma_2) \rangle - \langle h\big(D(\sigma_1)\big), \sigma_2 \rangle \\ &= d\,h(\sigma_1, \sigma_2) - h\,\big(\sigma_1, D(\sigma_2)\big) - h\,\big(D(\sigma_1), \sigma_2\big). \end{aligned} \tag{1.25}$$

Definition 1.3.9 *Let h be a Riemannian metric on a vector bundle $\pi : E \to M$. A connection D on E is said to be* compatible with h *(or to be an **h-connection**) if $D_{E^*\otimes E^*}(h) = 0$, i.e. if for all σ_1, $\sigma_2 \in \mathcal{A}^0(E)$ it holds:*

$$d\,h(\sigma_1, \sigma_2) = h\,\big(\sigma_1, D(\sigma_2)\big) + h\,\big(D(\sigma_1), \sigma_2\big). \tag{1.26}$$

Let D be an h-connection on a vector bundle $\pi : E \to M$ with a Riemannian metric h. Then the connection form ω_u of D with respect to an h-orthonormal local frame $u = (u_1, \ldots, u_n)$ is skew-symmetric. This can be seen by applying (1.26) to the local sections u_i, u_j for $i, j \in I$. In particular, $\omega_u(X) \in \mathfrak{o}(n)$ for all p in the domain of the local frame and $X \in T_pM$. Using the structure equation (1.19), we see that the curvature form Ω_u is also skew-symmetric. Conversely, let

D be a connection on E and suppose that each $p \in M$ has an open neighborhood with a local h-orthonormal frame $u = (u_1, \ldots, u_n)$ such that the connection form of D with respect to u is skew-symmetric. Then D is an h-connection, as can be seen by writing $\sigma_1 = \sum_{i=1}^n \sigma_1^i \, u_i$ and $\sigma_2 = \sum_{i=1}^n \sigma_2^i \, u_i$ in (1.26) and using (1.11).

Finally, in the next example we describe how a connection on a vector bundle $\pi : E \to M$ induces a connection on the pullback bundle f^*E, given a smooth map $f : N \to M$.

Example 1.3.10 *Let $\pi : E \to M$ be a vector bundle and let $f : N \to M$ be a smooth map. Observe first that every section σ of E induces a section $f^*\sigma$ of f^*E, which is defined by $f^*\sigma\,(p) := \big(\bar{f}|_{(f^*E)_p}\big)^{-1}\Big(\sigma\big(f(p)\big)\Big)$ for $p \in N$, where $\bar{f} : f^*E \to E$ is the homomorphism defined in Example 1.1.16. Consider a system $\{\, \theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^n \ , \ i \in I \,\}$ of local trivialization of E over a cover $\{\mathcal{U}_i, \, i \in I\,\}$ of M and let $\{\, u^i : \mathcal{U}_i \to L(E), \, i \in I\,\}$ be the corresponding local frames. Suppose D is a connection on E with connection forms $\{\, \omega_i, \, i \in I\,\}$ with respect to the $\{\, u^i, \, i \in I\,\}$. Then we can define f^*D to be the connection on f^*E with connection forms $\{\, f^*\omega_i, \, i \in I\,\}$ with respect to the local frames $\{\, f^*u^i, \, i \in I\,\}$. This makes sense, since the $\{\, f^*\omega_i, \, i \in I\,\}$ and the transition functions $\{\, \theta_{ij} \circ f : f^{-1}(\mathcal{U}_{ij}) \to GL\,(n, \mathbb{R})\,\}$ corresponding to the frames $\{\, f^*u^i, \, i \in I\,\}$ satisfy (1.12). Note that the diagram*

$$\begin{array}{ccc} \mathcal{A}^0(E) & \xrightarrow{f^*} & \mathcal{A}^0(f^*E) \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle f^*D} \\ \mathcal{A}^1(E) & \xrightarrow{f^*} & \mathcal{A}^1(f^*E) \end{array}$$

commutes, where the map $f^* : \mathcal{A}^1(E) \to \mathcal{A}^1(f^*E)$ is defined by $f^*(\sigma \otimes \omega) := f^*(\sigma) \otimes f^*(\omega)$ for $\sigma \in \mathcal{A}^0(E)$ and $\omega \in \mathcal{A}^1(M)$.

1.4 Connections on principal fibre bundles

Let $P\,(M, G)$ be a principal fibre bundle. For each $u \in P$ we will denote by $T_u^v P \subset T_u P$ the tangent space at u of the fibre of P through u. $T_u^v P$ is the kernel of the linear map $\pi_* : T_u P \to T_{\pi(u)} M$ (where π is the projection $P \to M$) and is called the *vertical subspace* of $T_u P$. A vector $X \in T_u P$ is called *vertical* if $X \in T_u^v P$. The map $u \mapsto T_u^v P$ is a smooth distribution on P, spanned by the fundamental vector fields corresponding to a basis of the Lie algebra of G.

Definition 1.4.1 *A **connection** A on a principal fibre bundle $P\,(M, G)$ is a smooth distribution $u \mapsto \big(T_u^h P\big)_A$ on P such that:*

1. *$T_u P = T_u^v P \oplus \big(T_u^h P\big)_A$ for all $u \in P$;*

2. $\left(T^h_{ug}P\right)_A = (R_g)_* \left(T^h_u P\right)_A$ *for all* $u \in P$ *and* $g \in G$.

$\left(T^h_u P\right)_A$ *is called the* horizontal subspace *of* $T_u P$ *with respect to the connection* A. *A vector* $X \in T_u P$ *is called* horizontal *(with respect to the connection* A*) if* $X \in \left(T^h_u P\right)_A$.

Connections exist on every principal fibre bundle, as can be proved using a partition of unity on the base space (see [14, Theorem 2.1 of Chapter II]).

By property 1. in Definition 1.4.1, every vector field $X \in \Gamma(TP)$ can be written uniquely in the form $X = \left(X^v\right)_A + \left(X^h\right)_A$, where $\left(X^v\right)_A(u) \in T^v_u P$ and $\left(X^h\right)_A(u) \in \left(T^h_u P\right)_A$ for all $u \in P$; $\left(X^v\right)_A$ and $\left(X^h\right)_A$ are called the *vertical* and *horizontal components* of X with respect to the connection A. Note that the smoothness condition for the distribution $u \mapsto \left(T^h_u P\right)_A$ is equivalent to requiring that $\left(X^v\right)_A$ and $\left(X^h\right)_A$ are smooth vector fields of P for every $X \in \Gamma(TP)$.

For each $u \in P$, the map $\pi_* : T_u P \to T_{\pi(u)} M$ has kernel $T^v_u P$. Thus, since $T_u P = T^v_u P \oplus \left(T^h_u P\right)_A$, it follows that π_* gives an isomorphism between $\left(T^h_u P\right)_A$ and $T_{\pi(u)} M$. For $Y \in T_{\pi(u)} M$ we will denote the corresponding vector of $\left(T^h_u P\right)_A$ by $\left(\hat{Y}^h_u\right)_A$ and we will call it the **horizontal lift** of Y at u. Given a vector field $Y \in \Gamma(TM)$, there exists a unique horizontal vector field $\left(\hat{Y}^h\right)_A \in \Gamma(TP)$ (called the *horizontal lift* of X) such that $\pi_* \left(\hat{Y}^h\right)_A = Y$. To see this, we have to check that the only possible definition of $\left(\hat{Y}^h\right)_A$, i.e. $\left(\hat{Y}^h\right)_A(u) := \left(\left(Y_{\pi(u)}\hat{\ }\right)^h_u\right)_A$ for $u \in P$ indeed gives a smooth vector field on P. Since smoothness is a local question, we can assume that $P(M,G)$ is trivial and take a vector field $X \in \Gamma(TP)$ such that $\pi_* X = Y$. But then it follows that $\left(\hat{Y}^h\right)_A = \left(X^h\right)_A$ is smooth. Notice that $\left(\hat{Y}^h\right)_A$ is invariant by R_g for all $g \in G$ (this follows from property 2. of Definition 1.4.1). Conversely, every horizontal vector field X on P which is invariant by R_g for all $g \in G$ is the horizontal lift of a vector field $Y \in \Gamma(TM)$. Just define $Y_p := \pi_* X_u$ for some $u \in \pi^{-1}(p)$.

Lemma 1.4.2 *Let* A *be a connection on a principal fibre bundle* $P(M,G)$. *Then for* Y, Y_1 *and* $Y_2 \in \Gamma(TM)$ *and* $f \in \mathcal{C}^\infty(M)$ *it holds:*

1. $\left(\widehat{Y_1 + Y_2}^h\right)_A = \left(\hat{Y_1}^h\right)_A + \left(\hat{Y_2}^h\right)_A$;
2. $\left(\widehat{fY}^h\right)_A = (f \circ \pi)\left(\hat{Y}^h\right)_A$;
3. $\left(\widehat{[Y_1, Y_2]}^h\right)_A = \left(\left[\left(\hat{Y_1}^h\right)_A, \left(\hat{Y_2}^h\right)_A\right]^h\right)_A$.

Proof 1. and 2. are clear. For 3., observe that since $\left(\hat{Y_1}^h\right)_A$, Y_1 and $\left(\hat{Y_2}^h\right)_A$, Y_2 are π-related, then so are $\left[\left(\hat{Y_1}^h\right)_A, \left(\hat{Y_2}^h\right)_A\right]$ and $[Y_1, Y_2]$. [20] □

[20] Given manifolds M and N and a map $f : N \to M$, two vector field $X \in \Gamma(TN)$ and $Y \in \Gamma(TM)$ are called *f-related* if $Y\ f(p) = f_*\ X(p)$ for all $p \in N$. If $X_i \in \Gamma(TN)$ and

Let A be a connection on a principal fibre bundle $P(M,G)$. We can define a 1-form ω_A on P with values in the Lie algebra $\mathfrak{g}$ of G as follows [21] . For $u \in P$, let $\theta_u : T_u^v P \to \mathfrak{g}$ be the inverse of the isomorphism $\mathfrak{g} \to T_u^v P$, $B \mapsto (B^*)_u$ and define

$$\omega_A(X) := \theta_u\big((X^v)_A \big)$$

for $X \in T_u P$. Since $T_u P = T_u^v P \oplus \big(T_u^h P\big)_A$, it follows that $\big(T_u^h P\big)_A$ is the kernel of ω_A at u. The 1-form ω_A is called the **connection form** of the connection A.

Lemma 1.4.3 *Let ω_A be the connection form of a connection A on a principal fibre bundle $P(M,G)$. Then ω_A is smooth and*

1. *$\omega_A\big((B^*)_u\big) = B$ for all $u \in P$ and $B \in \mathfrak{g}$;*
2. *${R_g}^* \omega_A = \mathrm{Ad}(g^{-1})\, \omega_A$ for all $g \in G$.*

Conversely, given a smooth $\mathfrak{g}$-valued 1-form ω on P satisfying 1. and 2., there is a unique connection A_ω on $P(M,G)$ whose connection form is ω.
Thus we have a 1-1 correspondence between the set of connections on $P(M,G)$ and the set of $\mathfrak{g}$-valued 1-forms on P satisfying 1. and 2. (such forms are called connections form *on P).*

Proof 1. follows directly from the definition of ω_A.

To prove 2., let $g \in G$, $u \in P$ and $X \in T_u P$; set $\omega_A(X) = B$, for some $B \in \mathfrak{g}$. Then we have:

$$\Big((R_g)_* X \Big)^v_A = (R_g)_* \Big((X^v)_A \Big) = (R_g)_* \Big((B^*)_u \Big) = \Big(\mathrm{Ad}(g^{-1})(B) \Big)^*_{ug}$$

where the first equality follows from 2. of definition 1.4.1 and the last from Lemma 1.2.21. Thus $\omega_A\Big((R_g)_* X\Big) = \mathrm{Ad}(g^{-1})\big(\omega_A(X)\big)$.

$Y_i \in \Gamma(TM)$ are f-related for $i = 1, 2$, then so are $[X_1, X_2]$ and $[Y_1, Y_2]$ (see [31, 1.55]).

[21] A *(smooth) r-form* on a manifold M *with values in a vector space* V is defined to be a smooth r-form on M with values in the trivial vector bundle $M \times V$ or equivalently a $\mathcal{C}^\infty(M)$-multilinear alternating map $\Gamma(TM) \times \ldots \times \Gamma(TM) \to \mathcal{C}^\infty(M, V)$. If we choose a basis $v_1, \ldots, v_n$ of V, then any $\omega \in \mathcal{A}^r(M \times V)$ can be written uniquely in the form $\omega = \sum_{i=1}^n \omega_i\, v_i$, where $\omega_i \in \mathcal{A}^r(M)$. We can define an exterior derivative $d : \mathcal{A}^r(M \times V) \to \mathcal{A}^{r+1}(M \times V)$ by

$$\omega = \sum_{i=1}^n \omega_i\, v_i \mapsto d\omega = \sum_{i=1}^n d\omega_i\, v_i.$$

Note that this definition does not depend on the choice of the basis of V.
If $\lambda : V \to W$ is a linear map and $\omega \in \mathcal{A}^r(M \times V)$, then we will denote by $\lambda(\omega) \in \mathcal{A}^r(M \times W)$ the W-valued r-form on M defined by

$$\lambda(\omega)\,(X_1, \ldots, X_r) := \lambda \quad \omega(X_1, \ldots, X_r)$$

for $X_1, \ldots, X_r \in \Gamma(TM)$.

Smoothness of ω_A will follow if we prove that for every vector field $X \in \Gamma(TP)$ the function $\omega_A(X) : P \to \mathfrak{g}$ is smooth. Since $\omega_A(X)$ is the composition

$$u \mapsto X_u \mapsto \left(X_u^v\right)_A \mapsto \theta_u\left(\left(X_u^v\right)_A\right)$$

and the map $P \to T^vP$, $u \mapsto \left(X_u^v\right)_A$ is smooth, it is enough to prove that $T^vP \to \mathfrak{g}$, $(u, X) \mapsto \theta_u(X)$ is smooth. Since $P(M, G)$ is locally trivial, we just have to show this for the bundle $\mathcal{U} \times G$, for some open $\mathcal{U} \subset M$.
Let $u = (p, g) \in \mathcal{U} \times G$ and $X \in T_u^v(\mathcal{U} \times G) = T_gG$. Then by Lemma 1.2.21 we have

$$\theta_u(X) = \mathrm{Ad}\left(g^{-1}\right)\left((R_{g^{-1}})_*(X)\right)$$

since $\theta_{(p,e)} : T^v_{(p,e)}(\mathcal{U} \times G) \cong T_eG \cong \mathfrak{g} \to \mathfrak{g}$ is the identity. So it is enough to prove that the map $TG \to \mathfrak{g}$, $(g, X) \mapsto \mathrm{Ad}\left(g^{-1}\right)\left((R_{g^{-1}})_*(X)\right)$ is smooth, or equivalently that so is the map $TG \to \mathfrak{g}$, $(g, X) \mapsto (R_{g^{-1}})_*(X)$. But this last map is the composition

$$TG \xrightarrow{\pi_G \times \mathrm{id}} G \times TG \xrightarrow{\iota \times \mathrm{id}} G \times TG \xrightarrow{\sigma \times \mathrm{id}} TG \times TG \xrightarrow{\eta_*} TG$$

where $\pi_G : TG \to G$ is the canonical projection $(g, X) \mapsto g$, $\iota : G \to G$ is the map $g \mapsto g^{-1}$, $\sigma : G \to TG$ is the section $g \mapsto (g, 0)$ and $\eta_* : TG \times TG \to TG$, $\big((g_1, X_1), (g_2, X_2)\big) \mapsto \big(g_1 g_2, (R_{g_2})_*(X_1) + (L_{g_1})_*(X_2)\big)$ is the differential of the map $\eta : G \times G \to G$, $(g_1, g_2) \mapsto g_1 g_2$. [22]

Suppose now that ω is a smooth $\mathfrak{g}$-valued 1-form on P satisfying 1. and 2. and define a connection A_ω on $P(M, G)$ by $\left(T_u^hP\right)_{A_\omega} := \{\, X \in T_uP \;/\; \omega(X) = 0 \,\}$ (note that this is the only possible definition if we want $\omega_{A_\omega} = \omega$). The map $u \mapsto \left(T_u^hP\right)_{A_\omega}$ defines a smooth distribution because it is the kernel of a smooth 1-form (see [27, Proposition 5.1 of Chapter 2]) and it is easy to see that properties 1. and 2. of Definition 1.4.1 are satisfied. □

[22] Here for all $(g_1, g_2) \in G \times G$ we identify $T_{(g_1,g_2)}\, G \times G$ with $T_{g_1}G \times T_{g_2}G$ via the isomorphism $X \mapsto \left(\mathrm{pr}_{1*}X, \mathrm{pr}_{2*}X\right)$, where $\mathrm{pr}_1 : G \times G \to G$ and $\mathrm{pr}_2 : G \times G \to G$ are the projections respectively on the first and second argument.
Note that for $X = (X_1, X_2) \in T_{(g_1,g_2)}\, G \times G \cong T_{g_1}G \times T_{g_2}G$ and $f \in \mathcal{C}^\infty(G \times G)$ we have

$$(X_1, X_2)(f) = X_1(f \circ i_1) + X_2(f \circ i_2)$$

where $i_1 : G \to G \times G$ and $i_2 : G \to G \times G$ are respectively the maps $g \mapsto (g, g_2)$ and $g \mapsto (g_1, g)$. In particular, we can apply this to calculate the differential at (g_1, g_2) of the map $\eta : G \times G \to G$, $(g, g') \mapsto g\, g'$. Let $(X_1, X_2) \in T_{(g_1,g_2)}\, G \times G$ and $\varphi \in \mathcal{C}^\infty(G)$, then

$$\begin{aligned}\eta_*(X_1, X_2)(\varphi) &= (X_1, X_2)(\varphi \circ \eta) = X_1(\varphi \circ \eta \circ i_1) + X_2(\varphi \circ \eta \circ i_2)\\ &= (R_{g_2})_*(X_1)(\varphi) + (L_{g_1})_*(X_2)(\varphi)\end{aligned}$$

thus

$$\eta_*(X_1, X_2) = (R_{g_2})_*(X_1) + (L_{g_1})_*(X_2).$$

Consider a principal fibre bundle $P\,(M,G)$ with a system of local trivializations $\{\,\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times G \ ,\ i \in I\,\}$ with respect to a cover $\{\mathcal{U}_i,\, i \in I\,\}$ of M. In the next proposition we will show that every connection form ω on $P\,(M,G)$ can be expressed as a family of local 1-forms $\{\omega_i \in \mathcal{A}^1(\mathcal{U}_i \times \mathfrak{g})\,,\, i \in I\,\}$ on M. Two different ω_i and ω_j will be related by a transformation formula that involves the *Maurer-Cartan form* on G, i.e. the smooth left-invariant $\mathfrak{g}$-valued 1-form ω_G on G defined by $\omega_G(X) := (L_{g^{-1}})_*(X)$ for $g \in G$ and $X \in T_gG$ (see also [27, §1 of Chapter 3]). We will need the following lemma.

Lemma 1.4.4 *Let $P\,(M,G)$ be a principal fibre bundle with local trivializations $\{\,\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times G \ ,\ i \in I\,\}$ with respect to a cover $\{\mathcal{U}_i,\, i \in I\,\}$ of M. Let $\omega_G \in \mathcal{A}^1(G \times \mathfrak{g})$ be the Maurer-Cartan form on G and for $i, j \in I$ with $\mathcal{U}_{ij} \neq \emptyset$ define*

$$(\,\omega_G\,)_{ij} := {\theta_{ij}}^*\,\omega_G$$

where the $\{\,\theta_{ij} : \mathcal{U}_{ij} \to G\,\}$ are the transition functions of the trivializations $\{\theta_i\,,\, i \in I\,\}$. Then we have:

$$(\,\omega_G\,)_{ji} = -\,\mathrm{Ad}\,(\theta_{ij})\,\big(\,(\,\omega_G\,)_{ij}\,\big).$$

Proof Let $p \in \mathcal{U}_{ij}$ and $Y \in T_pM$. Then:

$$\begin{aligned}(\,\omega_G\,)_{ji}\,(Y) \;&=\; \omega_G\,\big(\,{\theta_{ji}}_*(Y)\,\big) \overset{(*)}{=} \omega_G\Big(-\,(L_{\theta_{ji}(p)})_*\Big(\,(R_{\theta_{ji}(p)})_*\,\big(\,(\theta_{ij})_*(Y)\,\big)\,\Big)\Big)\\ &=\; -\,(R_{\theta_{ji}(p)})_*\,\big(\,(\theta_{ij})_*(Y)\,\big) = -\,\mathrm{Ad}\,\big(\theta_{ij}(p)\big)\,\Big(\omega_G\,\big(\,(\theta_{ij})_*(Y)\,\big)\Big)\\ &=\; -\,\mathrm{Ad}\,\big(\theta_{ij}(p)\big)\,\big(\,(\,\omega_G\,)_{ij}\,(Y)\,\big).\end{aligned}$$

The equality $(*)$ follows from the fact that $\theta_{ji} : \mathcal{U}_{ij} \to G$ is the composition $\iota \circ \theta_{ij}$, where $\iota : G \to G$ is the map $g \mapsto g^{-1}$ whose differential is given by $\iota_*\,(X) = -\,{L_{g^{-1}}}_*\big(\,{R_{g^{-1}}}_*(X)\,\big)$ for $g \in G$ and $X \in T_gG$. [23] □

Proposition 1.4.5 *Let $P\,(M,G)$ be a principal fibre bundle with local trivializations $\{\,\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times G \ ,\ i \in I\,\}$ with respect to a cover $\{\mathcal{U}_i,\, i \in I\,\}$ of M. Let $\omega \in \mathcal{A}^1(P \times \mathfrak{g})$ be a connection form on $P\,(M,G)$ and for all $i \in I$ define a form $\omega_i \in \mathcal{A}^1(\mathcal{U}_i \times \mathfrak{g})$ by*

$$\omega_i := (u_i)^*\,\omega$$

---

[23] Recall that the differential of the map $\eta : G \times G \to G$, $(g, g') \mapsto g\,g'$ is given by

$$\eta_*\,(X_1, X_2) = (R_{g_2})_*(X_1) + (L_{g_1})_*(X_2)$$

for $g_1,\, g_2 \in G$ and $X_1 \in T_{g_1}G$, $X_2 \in T_{g_2}G$. From this we can derive the differential of the map $\iota : G \to G$, $g \mapsto g^{-1}$ as follow. Observe that the composition $G \xrightarrow{\mathrm{id}\,\times\,\iota} G \times G \xrightarrow{\eta} G$ is the constant map $g \mapsto e$, so for $g \in G$ and $X \in T_gG$ we have

$$0 = \big(\,(\mathrm{id}\,\times\,\iota) \circ \eta\,\big)_*(X) = \eta_*(X, \iota_*X) = (R_{g^{-1}})_*(X) + (L_g)_*\,\big(\iota_*(X)\big)\,.$$

Thus we get $\iota_*\,(X) = -\,(L_{g^{-1}})_*\,\big((R_{g^{-1}})_*(X)\big)$.

where $u^i : \mathcal{U}_i \to P$ *is the local section* $p \mapsto \theta_i^{-1}(\,p, e\,)$*. Then on* $\mathcal{U}_{ij}$ *it holds:*

$$\omega_j = \mathrm{Ad}\,(\theta_{ji})\;(\omega_i) + (\,\omega_G\,)_{ij}, \tag{1.27}$$

where the $\{\,\theta_{ij} : \mathcal{U}_{ij} \to G\,\}$ *are the transition functions for the trivializations* $\{\,\theta_i\,,\, i \in I\,\}$ *and where* $(\,\omega_G\,)_{ij} = {\theta_{ij}}^*\,\omega_G$*.*
Conversely, suppose we have a family of 1-forms $\{\,\omega_i \in \mathcal{A}^1(\mathcal{U}_i \times \mathfrak{g})\,,\, i \in I\,\}$ *such that (1.27) is satisfied on all* $\mathcal{U}_{ij} \neq \emptyset$*. Then there exists a unique connection form* ω *on* $P\,(M, G)$ *such that* $\omega_i = {u_i}^*\,\omega$ *for all* $i \in I$*.*

Proof Let ω be a connection form on $P\,(M, G)$. We have to show that for $p \in \mathcal{U}_{ij}$ and $Y \in T_pM$ it holds:

$$\omega_j\,(Y) = \mathrm{Ad}\,\big(\theta_{ji}(p)\big)\;\big(\omega_i\,(Y)\big) + (\,\omega_G\,)_{ij}\,(Y)$$

i.e.

$$\omega\,\big({u^j}_*\,(Y)\big) = \mathrm{Ad}\,\big(\theta_{ji}(p)\big)\;\big(\,\omega\,({u^i}_*\,(Y))\,\big) + \omega_G\,\big(\,(\theta_{ij})_*\,(Y)\,\big). \tag{1.28}$$

On $\mathcal{U}_{ij}$ the map u^j is the composition $\mathcal{U}_{ij} \xrightarrow{u^i \times \theta_{ij}} \pi^{-1}(\mathcal{U}_{ij}) \times G \xrightarrow{\eta} \pi^{-1}(\mathcal{U}_{ij})$, where $\eta : \pi^{-1}(\mathcal{U}_{ij}) \times G \to \pi^{-1}(\mathcal{U}_{ij})$ is the map $(u, g) \mapsto ug$. Using the Leibnitz rule [24] we have

$$\begin{aligned} {u^j}_*\,(Y) &= \eta_*\,\big(\,{u^i}_*\,(Y), (\theta_{ij})_*(Y)\,\big) \\ &= (R_{\theta_{ij}(p)})_*\,\big({u^i}_*\,(Y)\big) + \Big((L_{\theta_{ji}(p)})_*\,\big((\theta_{ij})_*\,(Y)\big)\Big)^*_{u^j(p)}. \end{aligned} \tag{1.29}$$

Equation (1.28) follows from (1.29), applying ω to both sides and using properties 1. and 2. in Lemma 1.4.3.

Conversely, let $\{\,\omega_i \in \mathcal{A}^1(\mathcal{U}_i \times \mathfrak{g})\,,\, i \in I\,\}$ be a family of 1-forms satisfying (1.27). Suppose there is a connection form ω on $P\,(M, G)$ with $\omega_i = {u_i}^*\,\omega$ for all $i \in I$. Then for $p \in \mathcal{U}_i$ and $X \in T_{u^i(p)}P$ we have

$$\omega\,(X) = \omega_i\,\big(\pi_*(X)\big) + \theta_{u^i(p)}\,\Big(X - {u^i}_*\,\big(\pi_*(X)\big)\Big)$$

[24] Recall the *Leibnitz rule* for the differential of a map $\varphi : N_1 \times N_2 \to M$. For $(p, q) \in N_1 \times N_2$ identify $T_{(p,q)}(N_1 \times N_2)$ with $T_pN_1 \times T_qN_2$ via the isomorphism $Z \mapsto (\,{\mathrm{pr}_1}_*Z, {\mathrm{pr}_2}_*Z\,)$ where $\mathrm{pr}_1 : N_1 \times N_2 \to N_1$ and $\mathrm{pr}_2 : N_1 \times N_2 \to N_2$ are the projections respectively on the first and on the second argument. Then the differential of φ at (p, q) is given by

$$\varphi_*(X, Y) = (\varphi \circ i_1)_*(X) + (\varphi \circ i_2)_*(Y)$$

where $i_1 : N_1 \to N_1 \times N_2$ and $i_2 : N_2 \to N_1 \times N_2$ are the maps $x \mapsto (x, q)$ and $y \mapsto (p, y)$ respectively (see [14, Proposition 1.4 of Chapter I]). Notice that footnote 22 describes a particular case of it.

since $X = {u^i}_* \left(\pi_*(X)\right) + \Big(X - {u^i}_* \left(\pi_*(X)\right)\Big)$, with $X - {u^i}_* \left(\pi_*(X)\right) \in T^v_{u^i(p)}P$. Let now $u = u^i(p)\, g$ for some $g \in G$ and $X \in T_uP$. Then we have:

$$\begin{aligned}
\omega\,(X) &= \omega\,\Big(\,(R_g)_*\,\big(\,(R_{g^{-1}})_*(X)\,\big)\Big) = \mathrm{Ad}\,(g^{-1})\,\Big(\omega\,\big(\,(R_{g^{-1}})_*(X)\,\big)\Big) \\
&= \mathrm{Ad}\,(g^{-1})\,\Bigg(\omega_i\,\Big(\,\pi_*\big(\,(R_{g^{-1}})_*(X)\,\big)\Big) \\
&\qquad + \theta_{u^i(p)}\,\Big(\,(R_{g^{-1}})_*(X) - {u^i}_*\,\Big(\pi_*\,\big(\,(R_{g^{-1}})_*(X)\,\big)\Big)\Big)\Bigg) \\
&= \mathrm{Ad}\,(g^{-1})\,\Bigg(\omega_i\,\big(\pi_*(X)\big) \\
&\qquad + \theta_{u^i(p)}\,\Big(\,(R_{g^{-1}})_*(X) - {u^i}_*\,\big(\pi_*(X)\big)\,\Big)\Bigg). \qquad (1.30)
\end{aligned}$$

This proves the uniqueness part. To prove existence, define $\omega \in \mathcal{A}^1(P \times \mathfrak{g})$ by (1.30). We have only to check that this is well-defined. Then by construction it will follow that ω is a connection form and that $\omega_i = {u_i}^*\omega$ for all $i \in I$. Let $p \in \mathcal{U}_{ij}$, $u = u^i(p)\, g = u^j(p)\,\theta_{ji}(p)\, g$ and $X \in T_uP$. We have to show that

$$\begin{aligned}
&\mathrm{Ad}\,\big(\,g^{-1}\,\theta_{ij}(p)\,\big)\,\Big(\omega_j\,\big(\pi_*(X)\big) + \theta_{u^j(p)}\,\Big(\,(R_{g^{-1}\,\theta_{ij}(p)})_*(X) - {u^j}_*\,\big(\pi_*(X)\big)\,\Big)\Big) \\
&= \;\mathrm{Ad}\,(g^{-1})\,\Big(\omega_i\,\big(\pi_*(X)\big) + \theta_{u^i(p)}\,\Big(\,(R_{g^{-1}})_*(X) - {u^i}_*\,\big(\pi_*(X)\big)\,\Big)\Big)
\end{aligned}$$

thus that

$$\begin{aligned}
&\mathrm{Ad}\,\big(\,\theta_{ij}(p)\,\big)\,\Big(\omega_j\,\big(\pi_*(X)\big) + \theta_{u^j(p)}\,\Big(\,(R_{g^{-1}\,\theta_{ij}(p)})_*(X) - {u^j}_*\,\big(\pi_*(X)\big)\,\Big)\Big) \\
&\qquad = \;\omega_i\,\big(\pi_*(X)\big) + \theta_{u^i(p)}\,\Big(\,(R_{g^{-1}})_*(X) - {u^i}_*\,\big(\pi_*(X)\big)\,\Big).
\end{aligned}$$

But this follows from (1.27), (1.29) and Lemma 1.4.4. □

We will show later that there is a correspondence between connections on a vector bundle $\pi : E \to M$ and connections on its frame bundle $L(E)\,\big(M, GL\,(n,\mathbb{R})\big)$. We will then see that the forms $\{\,\omega_i \in \mathcal{A}^1(\mathcal{U}_i \times \mathfrak{gl}\,(n,\mathbb{R}))\,,\, i \in I\,\}$ associated with a connection and a system of local trivializations on $L(E)$ are just the connection forms of the corresponding connection on E, with respect to the local frames induced by the local trivializations of $L(E)$. Formula (1.27) will then appear to be a generalization of formula (1.12).

Definition 1.4.6 *Let $P\,(M,G)$ be a principal fibre bundle and suppose we have a representation $\varrho : G \to \mathrm{Aut}(V)$ of G on a vector space V. A V-valued r-form ω on P is called G-*equivariant *if ${R_g}^*\,\omega = \varrho\,(g^{-1})\,\omega$ for all $g \in G$. It is called* horizontal *if for $u \in P$ and $X_1, \ldots, X_r \in T_uP$, $\omega\,(X_1, \ldots, X_r) = 0$ when at least one of the X_i's is vertical.*

Lemma 1.4.7 *Let $P(M,G)$ be a principal fibre bundle and suppose we have a representation $\varrho : G \to \mathrm{Aut}(V)$ of G on a vector space V. Let $P \times_G V$ be the associated vector bundle. We have a 1-1 correspondence*

$$\mathcal{A}^r(P \times_G V) \overset{1\text{-}1}{\longleftrightarrow} \{\, \textit{horizontal } G\textit{-equivariant } V\textit{-valued } r\textit{-forms on } P \,\}.$$

In particular, for every principal fibre bundle $P(M,G)$ we have a 1-1 correspondence

$$\mathcal{A}^r(P \times_{Ad} \mathfrak{g}) \overset{1\text{-}1}{\longleftrightarrow} \{\, \textit{horizontal } G\textit{-equivariant } \mathfrak{g}\textit{-valued } r\textit{-forms on } P \,\}$$

where $P \times_{Ad} \mathfrak{g}$ is the adjoint bundle of $P(M,G)$ (see Example 1.2.13).

Proof Given $\xi \in \mathcal{A}^r(P \times_G V)$, define a V-valued r-form $\hat{\xi}$ on P by the relation

$$\xi\left(\pi_*(X_1), \dots, \pi_*(X_r)\right) = \left(u, \hat{\xi}\,(X_1, \dots, X_r)\right)_{/\sim}$$

for $u \in P$ and $X_1, \dots, X_r \in T_uP$. Then $\hat{\xi}$ is horizontal and G-equivariant, the last since

$$\begin{aligned} \xi\left(\pi_*\big(R_{g*}(X_1)\big), \dots, \pi_*\big(R_{g*}(X_r)\big)\right) &= \xi\left(\pi_*(X_1), \dots, \pi_*(X_r)\right) \\ = \left(u, \hat{\xi}\,(X_1, \dots, X_r)\right)_{/\sim} &= \left(ug,\ \varrho\,(g^{-1})\,\hat{\xi}\,(X_1, \dots, X_r)\right)_{/\sim}; \end{aligned}$$

thus $\hat{\xi}\left(R_{g*}(X_1), \dots, R_{g*}(X_r)\right) = \varrho\,(g^{-1})\,\hat{\xi}\,(X_1, \dots, X_r)$.
Conversely, let ω be a horizontal G-equivariant V-valued r-form on P and define $\omega_0 \in \mathcal{A}^r(P \times_G V)$ by

$$\omega_0\,(Y_1, \dots, Y_r) := \left(u, \omega\left((\hat{Y}_1)_u, \dots, (\hat{Y}_r)_u\right)\right)_{/\sim}$$

for $p \in M$ and $Y_1, \dots, Y_r \in T_pM$, where u is some element in the fibre of P over p and $(\hat{Y}_1)_u, \dots, (\hat{Y}_1)_u$ are vectors in T_uP such that $\pi_*\big((\hat{Y}_i)_u\big) = X_i$ for $i = 1, \dots, r$. Since ω is horizontal and G-equivariant, the definition of ω_0 does not depend on the choice of the vectors $(\hat{Y}_1)_u, \dots, (\hat{Y}_r)_u \in T_uP$ and of the element u in the fibre over p.
Clearly, $\hat{\omega_0} = \omega$ for all horizontal G-equivariant V-valued forms ω on P and $(\hat{\xi})_0 = \xi$ for all $\xi \in \mathcal{A}^r(P \times_G V)$. □

We will need Lemma 1.4.7 to prove the next proposition, which is an analogue of Proposition 1.3.2 and which will appear to be a generalization of it.

Proposition 1.4.8 *Let $P(M,G)$ be a principal fibre bundle and denote by $\mathcal{A}_P$ the set of connections on it. Then $\mathcal{A}_P$ is an affine space modeled on $\mathcal{A}^1(P \times_{Ad} \mathfrak{g})$, i.e.*

$$\mathcal{A}_P = A_0 + \mathcal{A}^1(P \times_{Ad} \mathfrak{g}),$$

where A_0 is a fixed connection on $P(M,G)$ and where we identify connections with the corresponding connection forms and elements of $\mathcal{A}^1(P \times_{Ad} \mathfrak{g})$ with the corresponding G-equivariant horizontal forms on P in the way described in Lemmas 1.4.3 and 1.4.7.

Proof Let A_1 and A_2 be connections on $P(M,G)$ with connection forms ω_1 and ω_2 respectively. Then ω_1 and ω_2 are G-equivariant $\mathfrak{g}$-valued 1-forms on P that take the same values on vertical vectors. Thus $\omega_1 - \omega_2$ is a G-equivariant horizontal $\mathfrak{g}$-valued 1-forms on P, so by Lemma 1.4.7 we have $\omega_1 - \omega_2 \in \mathcal{A}^1(P \times_{Ad} \mathfrak{g})$. In particular, $\mathcal{A}_P \subset A_0 + \mathcal{A}^1(P \times_{Ad} \mathfrak{g})$.
Conversely, let $\xi \in \mathcal{A}^1(P \times_{Ad} \mathfrak{g})$. We have to prove that $\omega_0 + \xi$ is a connection form on P, where ω_0 is the connection form of A_0. But this follows from the fact that, by Lemma 1.4.7, ξ is a G-equivariant horizontal $\mathfrak{g}$-valued 1-forms on P. □

Proposition 1.4.9 *Let $P(M,G)$ be a principal fibre bundle and suppose we have a representation $\varrho : G \to \mathrm{Aut}(V)$ of G on a vector space V. Let $E = P \times_G V$ be the associated vector bundle. Then every connection A on $P(M,G)$ induces a connection D_A on E, which is defined as follows. Let $Y \in \Gamma(TM)$ and $\sigma \in \mathcal{A}^0(E)$. Denote by $\hat{\sigma} : P \to V$ the G-equivariant map corresponding to σ in the sense of Lemma 1.2.9. Then $D_A(\sigma)(Y) \in \mathcal{A}^0(E)$ is defined by*

$$D_A(\sigma)(Y)(p) := \Big(u, \big(\hat{Y}^h\big)_A(u)\,(\hat{\sigma}) \Big)_{/\sim} \tag{1.31}$$

for $p \in M$, where u is some element of the fibre of P over p.

Proof Observe first that the map $M \to E$, $p \mapsto \Big(u, \big(\hat{Y}^h\big)_A(u)\,(\hat{\sigma}) \Big)_{/\sim}$ is well-defined, since

$$\begin{aligned} \big(\hat{Y}^h\big)_A(ug)\,(\hat{\sigma}) &= R_{g*}\Big(\big(\hat{Y}^h\big)_A(u)\Big)(\hat{\sigma}) = \big(\hat{Y}^h\big)_A(u)\,\big(\hat{\sigma} \circ R_g\big) \\ &= \big(\hat{Y}^h\big)_A(u)\,\big(\varrho(g^{-1}) \circ \hat{\sigma}\big) = \varrho(g^{-1})\,\Big(\big(\hat{Y}^h\big)_A(u)\,(\hat{\sigma})\Big), \end{aligned}$$

where the last equality follows from the fact that $\varrho(g^{-1}) : V \to V$ is linear.
The map $D_A(\sigma) : \Gamma(TM) \to \mathcal{A}^0(E)$ is $\mathcal{C}^\infty(M)$-linear, thus $D_A(\sigma) \in \mathcal{A}^1(E)$. To conclude that $D_A : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ is a connection, we have to check that the Leibnitz rule is satisfied. For this, let $\sigma \in \mathcal{A}^0(E)$, $f \in \mathcal{C}^\infty(M)$, $Y \in \Gamma(TM)$ and $p \in M$. Then

$$\begin{aligned} D_A(f\,\sigma)(Y)(p) &= \Big(u, \big(\hat{Y}^h\big)_A(u)\,\big((f \circ \pi)\,\hat{\sigma}\big) \Big)_{/\sim} \\ &= \Big(u, \hat{\sigma}(u)\,\big(\hat{Y}^h\big)_A(u)\,(f \circ \pi) \,+\, f(p)\,\big(\hat{Y}^h\big)_A(u)\,(\hat{\sigma}) \Big)_{/\sim} \\ &= \Big(u, \hat{\sigma}(u)\,\pi_*\big(\big(\hat{Y}^h\big)_A(u)\big)(f) \,+\, f(p)\,\big(\hat{Y}^h\big)_A(u)\,(\hat{\sigma}) \Big)_{/\sim} \\ &= \Big(u, \hat{\sigma}(u)\,Y_p(f) \Big)_{/\sim} \,+\, f(p)\,D_A(\sigma)(Y)(p) \\ &= \sigma(p)\,Y_p(f) \,+\, f(p)\,D_A(\sigma)(Y)(p), \end{aligned}$$

i.e. $D_A(f\sigma) = \sigma \otimes df + f\,D_A(\sigma)$. □

Proposition 1.4.10 *Let A be a connection on a principal fibre bundle $P\,(M,G)$. Suppose we have a representation $\varrho : G \to \mathrm{Aut}(V)$ of G on a vector space V and let $E = P \times_G V$ be the associated vector bundle. Let D_A be the connection on E induced by A, as described in Proposition 1.4.9. Then the extended operator $\mathcal{A}^r(E) \to \mathcal{A}^{r+1}(E)$ works as follows. Let $Y_1, \ldots, Y_r \in \Gamma(TM)$ and $\xi \in \mathcal{A}^r(E)$. Denote by $\hat{\xi}$ the horizontal, G-equivariant V-valued r-form on P corresponding to ξ in the sense of Lemma 1.4.7. Then $D_A(\xi)\,(Y_1, \ldots, Y_r) \in \mathcal{A}^0(E)$ is given by*

$$D_A(\xi)\,\big(Y_1, \ldots, Y_r\big)\,(p) = \left(u,\, d\hat{\xi}\,\Big(\big(\,\hat{Y}_1^h\,\big)_A(u)\,\ldots, \big(\,\hat{Y}_r^h\,\big)_A(u) \Big) \right)_{/\sim} \tag{1.32}$$

for $p \in M$, where u is some element of the fibre of P over p.

Proof It is enough to consider an element ξ of the form $\xi = \sigma \otimes \omega$, where $\sigma \in \mathcal{A}^0(E)$ and $\omega \in \mathcal{A}^r(M)$. We have

$$D_A(\xi) = \sigma \otimes d\omega + D_A(\sigma) \wedge \omega$$

thus

$$\begin{aligned} D_A(\xi)\,\big(Y_1, \ldots, Y_{r+1}\big)\,(p) &= d\omega\,\big(Y_1, \ldots, Y_{r+1}\big)\,(p)\,\sigma(p) \\ &+ \frac{1}{(r+1)!}\,\sum_\tau (-1)^\tau D_A(\sigma)(Y_{\tau(1)})\,(p)\,\omega\,\big(Y_{\tau(2)}, \ldots, Y_{\tau(r+1)}\big) \\ &= \Big(u,\, d\omega\,\big(Y_1, \ldots, Y_{r+1}\,\big)(p)\,\hat{\sigma}(u) \\ &+ \frac{1}{(r+1)!}\,\sum_\tau (-1)^\tau \big(\,\hat{Y}^h_{\tau(1)}\,\big)_A(u)\,(\hat{\sigma})\,\omega\,\big(Y_{\tau(2)}, \ldots, Y_{\tau(r+1)}\big) \Big)_{/\sim}. \end{aligned}$$

So we have to show that

$$\begin{aligned} d\hat{\xi}\,\Big(\big(\,\hat{Y}_1^h\,\big)_A(u)\,\ldots, \big(\,\hat{Y}_r^h\,\big)_A(u) \Big) &= d\omega\,\big(Y_1, \ldots, Y_{r+1}\,\big)(p)\,\hat{\sigma}(u) \\ &+ \tfrac{1}{(r+1)!}\,\textstyle\sum_\tau (-1)^\tau \big(\,\hat{Y}^h_{\tau(1)}\,\big)_A(u)\,(\hat{\sigma})\,\omega\,\big(Y_{\tau(2)}, \ldots, Y_{\tau(r+1)}\big). \end{aligned}$$

But this follows from the fact that $\hat{\xi} = \hat{\sigma}\,\pi^*\omega$, so

$$d\hat{\xi} = d\hat{\sigma} \wedge \pi^*\omega + \hat{\sigma}\,d\,(\pi^*\omega) = d\hat{\sigma} \wedge \pi^*\omega + \hat{\sigma}\,\pi^* d\omega.$$

□

The following lemma will be proved in the Appendix.

Lemma 1.4.11 *Let $P\,(M,G)$ be a principal fibre bundle and suppose we have a representation $\varrho : G \to \mathrm{Aut}(V)$ of G on a vector space V. Let $E = P \times_G V$ be the associated vector bundle. Consider a local section $\sigma : \mathcal{U} \to P$ of P over some open $\mathcal{U} \subset M$ and the induced local frame $u = f_v \circ \sigma : \mathcal{U} \to L(E)$ of E*

over $\mathcal{U}$, where $f_v : P \to L(E)$ is the map defined in Example 1.2.10, relative to a basis v of V. Let A be a connection on $P(M, G)$ with connection form ω_A and let D_A be the connection on E induced by A, as described in Proposition 1.4.9. Then the connection form ω_u of D_A with respect to the local frame u is given by $\omega_u = (\alpha'_v)_* \big(\varrho_* (\sigma^* \omega_A)\big)$ [25], where $(\alpha'_v)_*$ is the isomorphism $\mathrm{End}(V) \to \mathfrak{gl}(n, \mathbb{R})$ induced by the choice of the basis v of V.

We are now in a position to prove the correspondence between connections on a vector bundle and connections on its frame bundle.

Theorem 1.4.12 *Let $\pi : E \to M$ be a vector bundle and let $L(E)\big(M, GL(n, \mathbb{R})\big)$ be its frame bundle. Then we have a 1-1 correspondence between the spaces $\mathcal{A}_{L(E)}$ and $\mathcal{D}_E$.*

Proof Identify E with $L(E) \times_{GL(n,\mathbb{R})} \mathbb{R}^n$. Then every connection A on $L(E)$ induces a connection D_A on E, as described in Proposition 1.4.9.
Let $\{ \theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times GL(n, \mathbb{R}),\ i \in I \}$ be local trivializations of $L(E)$ and let $\{ u^i,\ i \in I \}$ be the associated local frames of E, i.e. $u^i(p) = \theta_i^{-1}(p, I)$ for $p \in \mathcal{U}_i$. Then by Lemma 1.4.11 it follows that for all $i \in I$ the connection form of D_A with respect to the local frame u^i is $(\omega_A)_i = (u^i)^* (\omega_A)$, where ω_A is the connection form of A.

Suppose now that we have a connection D on E and let $\{\omega_i,\ i \in I\}$ be the connection forms of D with respect to the local frames $\{ u^i,\ i \in I \}$. Regard the ω_i's as $\mathfrak{gl}(n, \mathbb{R})$-valued 1-forms on the $\mathcal{U}_i$'s. By (1.12) it holds

$$\omega_j = \theta_{ji}\, \omega_i\, \theta_{ij} + \theta_{ji}\, d\theta_{ij}$$

But $\theta_{ji}\, \omega_i\, \theta_{ij} = \mathrm{Ad}(\theta_{ji})\,(\omega_i)$ by (1.7) and $\theta_{ji}\, d\theta_{ij} = (\omega_{GL(n,\mathbb{R})})_{ij}$, since for $p \in \mathcal{U}_{ij}$ and $Y \in T_pM$ we have

$$\begin{aligned} (\omega_{GL(n,\mathbb{R})})_{ij}(Y) &= \omega_{GL(n,\mathbb{R})}\big((\theta_{ij})_*(Y)\big) = (L_{\theta_{ji}(p)})_*\big((\theta_{ij})_*(Y)\big) \\ &= \big(L_{\theta_{ji}(p)} \circ \theta_{ij}\big)_*(Y) = Y\big(L_{\theta_{ji}(p)} \circ \theta_{ij}\big) = \theta_{ji}(p)\, Y(\theta_{ij}). \end{aligned}$$

Thus the $\{ \omega_i,\ i \in I \}$ satisfy

$$\omega_j = \mathrm{Ad}(\theta_{ji})\,(\omega_i) + (\omega_G)_{ij}$$

so by Lemma 1.4.5 they induce a unique connection A_D on $L(E)$ such that its connection form ω_{A_D} satisfies $(\omega_{A_D})_i = \omega_i$. [26]

[25] Here we regard $\mathfrak{gl}(n, \mathbb{R})$-valued 1-forms as matrices of real 1-forms.

[26] The horizontal distribution $\ T^h L(E)\ _{A_D}$ can be described as follows. For a point b of $\ L(E)\ _p$ choose a local frame $u = (u_1, \ldots, u_n)$ of E around p such that $u(p) = b$ and $D(u_\alpha)(p) = 0$ for $\alpha = 1, \ldots, n$; such frame exists. Then

$$T_b^h L(E)\ _{A_D} = u_*\ T_pM$$

where we regard u as a local section of $L(E)$. Indeed, for $Y \in T_pM$ we have $\omega_{A_D}(u_* Y) = \omega_u(Y) = 0$, where ω_u is the connection form of D with respect to u, thus (by dimensions) $u_*\ T_pM\ = \ker\ \omega_{A_D}(p)\ $.

By what we said above, the connection D_{A_D} on E induced by A_D has the $\{(\omega_{A_D})_i, i \in I\}$ as connection forms with respect to the local frames $\{u^i, i \in I\}$, thus by the uniqueness part in Lemma 1.3.3 we have $D_{A_D} = D$.
Similarly, $A_{D_A} = A$ for all $A \in \mathcal{A}_{L(E)}$, so the map $\mathcal{A}_{L(E)} \to \mathcal{D}_E$, $A \mapsto D_A$ is a bijection. [27] $\square$

Let $\pi : E \to M$ be a vector bundle and let $L(E)\big(M, GL(n, \mathbb{R})\big)$ be its frame bundle. We know that $\mathcal{D}_E$ and $\mathcal{A}_{L(E)}$ are affine spaces modeled respectively on $\mathcal{A}^1(E^* \otimes E)$ and $\mathcal{A}^1\big(L(E) \times_{Ad} \mathfrak{gl}(n, \mathbb{R})\big)$ (cf Propositions 1.3.2 and 1.4.8). In the Appendix we will show that the bijection $\mathcal{A}_{L(E)} \to \mathcal{D}_E$, $A \mapsto D_A$ of Theorem 1.4.12 is actually an affine isomorphism, whose associated linear isomorphism $\mathcal{A}^1\big(L(E) \times_{Ad} \mathfrak{gl}(n, \mathbb{R})\big) \to \mathcal{A}^1(E^* \otimes E)$ is the map induced by the isomorphism $\phi : L(E) \times_{Ad} \mathfrak{gl}(n, \mathbb{R}) \to E^* \otimes E$ described in Example 1.2.13.

Let $P(M, G)$ be a principal fibre bundle and let V be a vector space. Given a connection A on $P(M, G)$ and a V-valued r-form ω on P, define a V-valued $(r+1)$-form $d_A \omega$ on P by

$$d_A \omega\,(X_1, \ldots, X_{r+1}) := d\omega\Big(\big(X_1^h\big)_A, \ldots, \big(X_{r+1}^h\big)_A\Big)$$

for $u \in P$ and $X_1, \ldots, X_{r+1} \in T_u P$. Observe that if we have a representation $\varrho : G \to \mathrm{Aut}(V)$ of G on V and if the form ω is G-equivariant, then $d_A \omega$ is a horizontal G-equivariant V-valued form on P, thus by Lemma 1.4.7 we have in this case

$$d_A \omega \in \mathcal{A}^{r+1}(P \times_G V).$$

In particular, consider the representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ and denote by $(D^{\mathrm{Ad}})_A$ the connection on $P \times_{Ad} \mathfrak{g}$ induced by A. Then by Proposition 1.4.10 we have $d_A \xi = (D^{\mathrm{Ad}})_A(\xi)$ for all $\xi \in \mathcal{A}^r(P \times_{Ad} \mathfrak{g})$.

Let ω_A be the connection form of a connection A on a principal fibre bundle $P(M, G)$. The $\mathfrak{g}$-valued 2-form $\Omega_A = d_A \omega_A$ on P is called the **curvature form** of A. It is an element of $\mathcal{A}^2(P \times_{Ad} \mathfrak{g})$. The following lemma will be proved in the Appendix.

Lemma 1.4.13 *Let $P(M, G)$ be a principal fibre bundle and suppose we have a representation $\varrho : G \to \mathrm{Aut}(V)$ of G on a vector space V. Let A be a connection*

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[27] Another, more intuitive approach to see the correspondence between $\mathcal{D}_E$ and $\mathcal{A}_{L(E)}$ is as follows. One can prove that the concepts of connection on a vector bundle $\pi : E \to M$ and on a principal fibre bundle $P(M, G)$ are equivalent respectively to assigning a *parallel translation* $\tau_\alpha : E_{\alpha(0)} \to E_{\alpha(t)}$ $(t \in [0,1])$ along each curve $\alpha : [0,1] \to M$ and to a unique G-equivariant path lifting property on $P(M, G)$. If $P(M, G)$ is the frame bundle of $\pi : E \to M$, then these two concepts are equivalent, since parallel translation along a curve $\alpha : [0,1] \to M$ can also be thought of as carrying a frame at $\alpha(0)$ to a frame at $\alpha(t)$, for all $t \in [0,1]$. See for example [21] and [14, §1 of Chapter III].

on $P(M,G)$. Then the curvature of the connection D_A on the associated vector bundle $E = P\times_G V$ is $\phi(\Omega_A) \in \mathcal{A}^2(E^*\otimes E)$, where $\phi : \mathcal{A}^2(P\times_{Ad}\mathfrak{g}) \to \mathcal{A}^2(E^*\otimes E)$ is induced by the map $P\times_{Ad}\mathfrak{g} \to E^*\otimes E$ defined in Example 1.2.13. In particular, if A is a connection on the frame bundle $L(E)\big(M, GL(n,\mathbb{R})\big)$ of a vector bundle $\pi : E \to M$, then under the identification $L(E)\times_{GL(n,\mathbb{R})}\mathfrak{gl}(n,\mathbb{R}) \cong E^*\otimes E$ we have $\Omega_A = R_A$, where R_A is the curvature of the connection D_A on E induced by A.

Proposition 1.4.14 *Let A be a connection on a principal fibre bundle $P(M,G)$ and let ω_A and Ω_A be the connection and curvature forms of A. Then for $u \in P$ and X_1, $X_2 \in T_uP$ it holds:*

$$d\omega_A(X_1, X_2) = -\frac{1}{2}\left[\,\omega_A(X_1), \omega_A(X_2)\,\right] + \Omega_A(X_1, X_2). \tag{1.33}$$

Equation (1.33) is called the **structure equation** for a connection on a principal fibre bundle. It can be proved by checking it separately for the three cases that X_1,X_2 are both vertical, both horizontal or X_1 vertical and X_2 horizontal (see [14, Theorem 5.2 of Chapter II]. In the case when $P(M,G)$ is the frame bundle $L(E)\big(M, GL(n,\mathbb{R})\big)$ of a vector bundle $\pi : E \to M$, (1.33) reduces to the structure equation (1.19) for vector bundles. To see this, let $u : \mathcal{U} \to L(E)$ be a local frame of E on some open $\mathcal{U} \subset M$. Then by Lemma 1.4.11 and by Lemma 1.4.13, the connection and curvature forms of a connection D on E with respect to the local frame u are given by $\omega_u = u^*\,\omega_{A_D}$ and $\Omega_u = u^*\,\Omega_{A_D}$. Equation (1.33) implies

$$d\omega_u(Y_1, Y_2) = -\frac{1}{2}\left[\,\omega_u(Y_1), \omega_u(Y_2)\,\right] + \Omega_u(Y_1, Y_2)$$

for $p \in \mathcal{U}$ and Y_1, $Y_2 \in T_pM$. But this is equivalent to equation (1.19), since

$$(\omega_u \wedge \omega_u)(Y_1, Y_2) = \frac{1}{2}\big(\,\omega_u(Y_1)\,\omega_u(Y_2) - \omega_u(Y_2)\,\omega_u(Y_1)\,\big) = \frac{1}{2}\left[\,\omega_u(Y_1), \omega_u(Y_2)\,\right].$$

Proposition 1.4.15 *Let Ω_A be the curvature form of a connection A on a principal fibre bundle $P(M,G)$. Then it holds:*

$$d_A\,\Omega_A = 0. \tag{1.34}$$

Equation (1.34) is called the **Bianchi identity** for a connection on a principal fibre bundle. We refer for a proof to [14, Theorem 5.4 of Chapter II]. We will show instead that (1.34) reduces to the Bianchi identity (1.20) for vector bundles, when $P(M,G)$ is the frame bundle $L(E)\big(M, GL(n,\mathbb{R})\big)$ of a vector bundle $\pi : E \to M$. Let $(D^{\mathrm{Ad}})_A$ and D_A be the connections on $L(E)\times_{Ad}\mathfrak{gl}(n,\mathbb{R})$ and $E = L(E)\times_{GL(n,\mathbb{R})}\mathbb{R}^n$ induced by A. Regard Ω_A and $d_A\,\Omega_A$ as forms on the vector bundle $L(E)\times_{Ad}\mathfrak{gl}(n,\mathbb{R})$. Then by Proposition 1.4.10 we have $d_A\,\Omega_A = (D^{\mathrm{Ad}})_A(\Omega_A)$, as was observed above. By Lemma 1.4.13, Ω_A is equal to the curvature R_A of the connection D_A on E, under the identification $L(E)\times_{Ad}\mathfrak{gl}(n,\mathbb{R}) \cong E^*\otimes E$ described in Example 1.2.13. If we show that under this identification $(D^{\mathrm{Ad}})_A = (D_A)_{E^*\otimes E}$, then from (1.34) it will follow $(D_A)_{E^*\otimes E}\, R_A = 0$, which is the Bianchi identity for vector bundles (cf. Example 1.3.7). This is done in the following lemma.

Lemma 1.4.16 *Let A be a connection on the frame bundle $L(E)\big(M, GL\,(n,\mathbb{R})\big)$ of a vector bundle $\pi : E \to M$ and let $(D^{\mathrm{Ad}})_A$ and D_A be respectively the connections on $L(E) \times_{Ad} \mathfrak{gl}\,(n,\mathbb{R})$ and $E = L(E) \times_{GL\,(n,\mathbb{R})} \mathbb{R}^n$ induced by A, as described in Proposition 1.4.9. Identify $L(E) \times_{Ad} \mathfrak{gl}\,(n,\mathbb{R})$ with $E^* \otimes E$, as done in Example 1.2.13. Then we have:*

$$(D_A)_{E^*\otimes E} = (D^{\mathrm{Ad}})_A.$$

Proof Let $\varphi \in \mathcal{A}^0\big(L(E) \times_{Ad} \mathfrak{gl}\,(n,\mathbb{R})\big) = \mathcal{A}^0(E^* \otimes E) = \mathrm{Hom}\,(E,E)$ and $\sigma \in \mathcal{A}^0(E)$. Denote by $\hat{\varphi} : L(E) \to \mathfrak{gl}\,(n,\mathbb{R})$ and $\hat{\sigma} : L(E) \to \mathbb{R}^n$ the corresponding $GL\,(n,\mathbb{R})$-equivariant maps. Recall that for $p \in M$ we have

$$\varphi\,\big(\sigma(p)\big) = \big(\, u, \hat{\varphi}(u)\,\hat{\sigma}(u)\,\big)_{/\sim} \tag{1.35}$$

where u is some element in the fibre of $L(E)$ over p and where the operation on the right hand side is matrix multiplication $\mathfrak{gl}\,(n,\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n$ (cf. Example 1.2.13). Let $Y \in \Gamma(TM)$ and regard φ and $(D_A)_{E^*\otimes E}\,(\varphi)(Y)$ as $\mathcal{C}^\infty(M)$-linear maps $\mathcal{A}^0(E) \to \mathcal{A}^0(E)$. Then by (1.23), (1.31) and (1.35) we have

$$\begin{aligned}
(D_A)_{E^*\otimes E}\,(\varphi)(Y)(\sigma)\,(p) &= D_A\,\big(\varphi(\sigma)\big)(Y)\,(p) - \varphi\,\big(D_A(\sigma)(Y)\,(p)\big) \\
&= \big(\, u, \hat{Y}^h_A(u)\,(\widehat{\varphi(\sigma)})\,\big)_{/\sim} - \big(\, u, \hat{\varphi}(u)\,\hat{Y}^h_A(u)\,(\hat{\sigma})\,\big)_{/\sim} \\
&= \big(\, u, \hat{Y}^h_A(u)\,(\widehat{\varphi(\sigma)}) - \hat{\varphi}(u)\,\hat{Y}^h_A(u)\,\hat{\sigma}\big)_{/\sim}.
\end{aligned}$$

On the other hand, by (1.31) and (1.35) we have

$$(D^{\mathrm{Ad}})_A\,(\varphi)(Y)(\sigma)\,(p) = \big(\, u, \hat{Y}^h_A(u)\,(\hat{\varphi})\,\hat{\sigma}(u)\,\big)_{/\sim}.$$

But

$$\hat{Y}^h_A(u)\,(\widehat{\varphi(\sigma)}) - \hat{\varphi}(u)\,\hat{Y}^h_A(u)\,(\hat{\sigma}) = \hat{Y}^h_A(u)\,(\hat{\varphi})\,\hat{\sigma}(u)$$

by the Leibnitz rule, since by (1.35) we have $\widehat{\varphi(\sigma)} = \hat{\varphi}\,\hat{\sigma}$. □

Let $(f, f') : Q\,(N,H) \to P\,(M,G)$ be a homomorphism of principal fibre bundles. If the induced map $f'' : N \to M$ is a diffeomorphism, then every connection on $Q\,(N,H)$ induces a connection on $P\,(M,G)$ in a natural way. If $f' : H \to G$ is an isomorphism, then every connection on $P\,(M,G)$ induces a connection on $Q\,(N,H)$ in a natural way. This will be needed in Chapter 4 and is explained in the next two propositions.

Proposition 1.4.17 *Let $(f, f') : Q\,(N,H) \to P\,(M,G)$ be a homomorphism of principal fibre bundles such that the induced map $f'' : N \to M$ is a diffeomorphism. Let A be a connection on $Q\,(N,H)$ with connection and curvature forms ω_A and*

Ω_A. *Then there is a unique connection* $f(A)$ *on* $P(M,G)$ *such that the horizontal subspaces of* $Q(N,H)$ *with respect to* A *are mapped by* f *to horizontal subspaces of* $P(M,G)$ *with respect to* $f(A)$. *If* $\omega_{f(A)}$ *and* $\Omega_{f(A)}$ *are the connection and curvature forms of* $f(A)$, *then*

$$f^* \omega_{f(A)} = f'_* (\omega_A) \tag{1.36}$$

and

$$f^* \Omega_{f(A)} = f'_* (\Omega_A). \tag{1.37}$$

The connection $f(A)$ on $P(M,G)$ is defined as follows. For $u \in P$ with $\pi(u) = p$ choose an element u' in the fibre of Q over $f^{-1}(p)$; then $u = f(u')\, g$, for some $g \in G$. Define

$$\left(T_u^h P \right)_{f(A)} := (R_g)_* \left(f_* \left(T_{u'}^h Q \right)_A \right).$$

This definition does not depend on the choice of u' in the fibre of Q over $f^{-1}(p)$ and the distribution $u \mapsto \left(T_u^h P \right)_{f(A)}$ indeed defines a connection, whose connection and curvature forms satisfy (1.36) and (1.37). See [14, Proposition 6.1 of Chapter II] for a proof of this.

In particular, Proposition 1.4.17 holds for a reduction $Q(M,H)$ of a principal fibre bundle $P(M,G)$: every connection on $Q(M,H)$ induces a connection on $P(M,G)$ in such a way that the two connections coincide on Q, when we consider Q as a subset of P. We will denote the two connections by the same letter.

Definition 1.4.18 *Let* $P(M,G)$ *be a principal fibre bundle and let* $Q(M,H)$ *be a reduction of* $P(M,G)$. *A connection on* $P(M,G)$ *is said to be* reducible *to* $Q(M,H)$ *if it is induced by a connection in* $Q(M,H)$ *in the way described in Proposition 1.4.17.*

Note that a connection A on $P(M,G)$ is reducible to $Q(M,H)$ if and only if $\omega_A(X)$ is an element of the Lie algebra of H, for all $u \in Q$ and $X \in T_u Q$.

Proposition 1.4.19 *Let* $(f, f') : Q(N,H) \to P(M,G)$ *be a homomorphism of principal fibre bundles such that* $f' : H \to G$ *is an isomorphism. Let* A *be a connection on* $P(M,G)$ *with connection and curvature forms* ω_A *and* Ω_A. *Then there exists an unique connection* $f^*(A)$ *on* $Q(N,H)$ *such that the horizontal subspaces of* $Q(N,H)$ *with respect to* $f^*(A)$ *are mapped by* f *to horizontal subspaces of* $P(M,G)$ *with respect to* A. *If* $\omega_{f^*(A)}$ *and* $\Omega_{f^*(A)}$ *are the connection and curvature forms of* $f^*(A)$, *then*

$$f^* \omega_A = f'_* \left(\omega_{f^*(A)} \right) \tag{1.38}$$

and

$$f^* \Omega_A = f'_* \left(\Omega_{f^*(A)} \right). \tag{1.39}$$

The connection $f^*(A)$ on $Q\,(N,H)$ is defined by

$$\big(T^h_u Q\big)_{f^*(A)} := (f_*)^{-1}\left(\big(T^h_{f(u)}P\big)_A\right)$$

for $u \in Q$. Then the distribution $u \mapsto \big(T^h_u Q\big)_{f^*(A)}$ indeed defines a connection, whose connection and curvature forms satisfy (1.38) and (1.39). See [14, Proposition 6.2 of chapter II] for a proof of this.

In particular, Proposition 1.4.19 holds for the pullback bundle $f^*(P)\,(N,G)$ of a principal fibre bundle $P\,(M,G)$, where f is a smooth map $N \to M$: every connection on $P\,(M,G)$ induces a connection on $f^*(P)\,(N,G)$ whose connection and curvature forms are the pullback of the connection and curvature forms of the connection on $P\,(M,G)$.

Example 1.4.20 *Let $P\,(M,G)$ be a principal fibre bundle and $f : N \to M$ a smooth map. Suppose we have a representation $\varrho : G \to \mathrm{Aut}(V)$ of G on a vector space V. Then the vector bundle $E' = f^*(P) \times_G V$ associated with the pullback bundle $f^*(P)\,(N,G)$ is the pullback of the vector bundle $E = P \times_G V$ associated with $P\,(M,G)$. This follows from the fact that the vector bundles $f^*(P) \times_G V$ and $f^*\,(P \times_G V)$ both have transition functions*

$$\{\, \alpha'_v \circ \varrho \circ \theta_{ij} \circ f : f^{-1}(\mathcal{U}_{ij}) \to GL\,(n,\mathbb{R})\,\}$$

over the cover $\{\, f^{-1}(\mathcal{U}_i),\, i \in I\,\}$ of N, where $\alpha'_v : \mathrm{Aut}\,(V) \to GL\,(n,\mathbb{R})$ is the isomorphism induced by the choice of a basis v of V, $\{\,\theta_{ij} : \mathcal{U}_{ij} \to GL\,(n,\mathbb{R})\,\}$ are the transition functions of a system of local trivializations of $P\,(M,G)$ over a cover $\{\,\mathcal{U}_i,\, i \in I\,\}$ of M and where we consider the systems of local trivializations naturally induced on $f^*(P)\times_G V$ and $f^*\,(P\times_G V)$, as described in Examples 1.1.16, 1.2.8 and 1.2.10. By Propositions 1.4.9 and 1.4.19 and by Example 1.3.10, given a connection A on $P\,(M,G)$ we get connections D_A on E, $f^*(A)$ on $f^*(P)\,(N,G)$ and $f^*(D_A)$, $D_{f^*(A)}$ on $f^*(E) = f^*(P) \times_G V$. By Proposition 1.4.19 and Lemma 1.4.11, both connections $f^*(D_A)$ and $D_{f^*(A)}$ have connection forms $\{\,(\sigma_i \circ f)^*\,\omega_A,$ $i \in I\,\}$ with respect to the local frames on $f^*(E)$ induced by a system of local frames $\{\,\sigma_i : \mathcal{U}_i \to P,\, i \in I\,\}$ of $P\,(M,G)$. By the uniqueness part in Lemma 1.3.3, it follows that $f^*(D_A) = D_{f^*(A)}$.

Example 1.4.21 *Let $P\,(M,G)$ be a principal fibre bundle and let $Q\,(M,H)$ be a reduction of $P\,(M,G)$. Suppose that we have a representation $\varrho : G \to \mathrm{Aut}(V)$ of G on a vector space V . Then, by Proposition 1.2.18, the associated vector bundles $P \times_G V$ and $Q \times_H V$ are isomorphic. Let A be a connection on $P\,(M,G)$ which is reducible to $Q\,(M,H)$. Then it is easy to see that the connections induced by A on $P \times_G V$ and $Q \times_H V$, as described in Proposition 1.4.9, are equal.*

Example 1.4.22 *Let $\pi : E \to M$ be a vector bundle and let h be a Riemannian metric on E. Consider the reduction $O_h(E)\,\big(M, O(n)\big)$ of the frame bundle $L(E)\,\big(M, GL\,(n,\mathbb{R})\big)$ of E corresponding to h (see Example 1.2.17). We will* *show that a connection A on $L(E)$ is reducible to $O_h(E)$ if and only if the induced connection D_A on E is an h-connection. Suppose first that D_A is an h-connection. Let $u_0 \in O_h(E)$ and $X_0 \in T_{u_0}O_h(E)$; we need to show that $\omega_A(X_0) \in \mathfrak{o}(n)$ (then it will follow that the connection A is reducible to $O_h(E)$). Let $u : \mathcal{U} \to O_h(E)$ be a local h-orthonormal frame of E over some open $\mathcal{U} \subset M$ with $\pi(u_0) \in \mathcal{U}$. Let $X \in T_{u(\pi(u_0))}O_h(E)$; then $X - u_*\big(\pi_*(X)\big) \in T^v_{u(\pi(u_0))}O_h(E)$, thus $\omega_A\Big(X - u_*\big(\pi_*(X)\big)\Big) \in \mathfrak{o}(n)$. Denote by ω_u the connection form of D_A with respect to u. Then by Lemma 1.4.11 we have*

$$\omega_A\Big(X - u_*\big(\pi_*(X)\big)\Big) = \omega_A(X) - \omega_u\big(\pi_*(X)\big).$$

Since D_A is an h-connection and u is an h-orthonormal frame, it follows that $\omega_u\big(\pi_*(X)\big) \in \mathfrak{o}(n)$, so $\omega_A(X) \in \mathfrak{o}(n)$. Let $u_0 = u\big(\pi(u_0)\big)g$ for some $g \in O(n)$. Then

$$\omega_A(X_0) = \omega_A\Big(R_{g*}\big(R_g^{-1}{}_*(X_0)\big)\Big) = \mathrm{Ad}(g^{-1})\,\omega_A\big(R_g^{-1}{}_*(X_0)\big) \in \mathfrak{o}(n),$$

as we wanted. Suppose now that A is a connection on $L(E)$ reducible to $O_h(E)$. Then for all $u \in O_h(E)$ and $X \in T_u O_h(E)$ we have $\omega_A(X) \in \mathfrak{o}(n)$. Consider a local h-orthonormal frame $u : \mathcal{U} \to O_h(E)$ of E over some open $\mathcal{U} \subset M$. Denote by ω_u the connection form of D_A with respect to u. Then by Lemma 1.4.11 we have $\omega_u(X) = \omega_A\big(u_*(X)\big) \in \mathfrak{o}(n)$ for all $p \in \mathcal{U}$ and $X \in T_pM$. So ω_u is skew-symmetric and it follows that D_A is an h-connection.

Chapter 2

Complexifications

In the rest of the thesis we will consider complex vector bundles over complex manifolds and principal fibre bundles with complex structure group and complex base space. All results of Chapter 1 apply to these objects, with only minor changes. In this chapter we will make this statement more precise and we will take a closer look at some properties of connections which strictly depend on the complex structures of the base spaces and of the fibres.
Paragraph 1 contains linear-algebraic preliminaries, while in Paragraph 4 we will collect some results over complex reductive Lie groups that are needed later.

General references for this chapter are [15, Chapter IX] for 2.1 and 2.2, [32, Chapters I and III] and [2] for 2.1, 2.2 and 2.3, [16, pages 70-81] and [6, §2, §5 and §7 of Chapter 0] for 2.2 and 2.3, [13, Chapter I] for 2.3 and [25, §7 of Chapter 1 and §2 of Chapter 4], [10], [12] and [30] for 2.4.

2.1 Complexification of a real vector space with a complex structure

All results of this section will be applied in Paragraph 2.2 to tangent spaces of complex manifolds. We follow [15, §1 of Chapter IX]. Proof of all unproven statements can be found there.

Definition 2.1.1 *A* ***complex structure*** *on a real vector space V is a linear automorphism J of V such that $J^2 = -\,\mathrm{id}_V$.*

If a real vector space V has a complex structure J then it must be even-dimensional. Indeed, the automorphism J of V has no eigenvalues and thus the characteristic polynomial of J has no zero. Since the degree of the characteristic polynomial is equal to the dimension of V, it follows that V is even-dimensional. If J is a complex structure on a $2n$-dimensional real vector space V, then we can find a basis of V of the form $\{ v_1, J(v_1), \dots, v_n, J(v_n) \}$.

Let V be a real $2n$-dimensional vector space and J a complex structure on V. Then we can turn V into a complex vector space, denoted by (V, J), by setting $(a + ib)\, v := av + bJ(v)$ for $v \in V$ and $a + ib \in \mathbb{C}$. If $\{ v_1, J(v_1), \ldots, v_n, J(v_n) \}$ is a basis of V over $\mathbb{R}$, then $\{ v_1, \ldots, v_n \}$ is a basis of (V, J) over $\mathbb{C}$. In particular, $\dim_{\mathbb{C}} (V, J) = \frac{1}{2} \dim_{\mathbb{R}} V$.

Conversely, let W be a complex n-dimensional vector space. Then we can define a complex structure J on the underlying real vector space $W^{\mathbb{R}}$ by $J(w) := iw$ for $w \in W$. The map $(W^{\mathbb{R}}, J) \to W$, $w \mapsto w$ is then a $\mathbb{C}$-linear isomorphism.

The complex vector space $\overline{W} := (W^{\mathbb{R}}, -J)$ is called the *conjugate* of W. Observe that every conjugate-linear map $W \to \mathbb{C}$ can be obtained by composing a $\mathbb{C}$-linear map $\overline{W} \to \mathbb{C}$ with the conjugate-linear map $W \to \overline{W}$, $w \mapsto w$. Thus the space of conjugate-linear maps $W \to \mathbb{C}$ can be identified with $\overline{W}^*$. Similarly, the space of sesquilinear maps $W \times W \to \mathbb{C}$ can be identified with the space of bilinear maps $\overline{W} \times W \to \mathbb{C}$.

If V, V' are real vector spaces with complex structures J and J' respectively, then an $\mathbb{R}$-linear map $f : V \to V'$ induces a $\mathbb{C}$-linear map $f : (V, J) \to (V', J')$ if and only if $J' \circ f = f \circ J$.

Given a complex structure J on a real vector space V, we can define a complex structure J^* on the dual V^* by the relation

$$\langle J^*(v_1^*), v_2 \rangle := \langle v_1^*, J(v_2) \rangle$$

for $v_1^* \in V^*$ and $v_2 \in V$.

Definition 2.1.2 *Let V be an n-dimensional real vector space and consider the $2n$-dimensional real vector space $V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. Define a scalar complex multiplication on $V^{\mathbb{C}}$ by $\mu\,(v \otimes \lambda) := v \otimes \mu\lambda$ for $v \in V$ and μ, $\lambda \in \mathbb{C}$ and $\mathbb{R}$-linear extension. Then $V^{\mathbb{C}}$ becomes an n-dimensional complex vector space, called the* ***complexification*** *of V.*

Observe that if we identify V with the image of the monomorphism $V \to V^{\mathbb{C}}$, $v \mapsto v \otimes 1$, then we can write $V^{\mathbb{C}} = \{ v_1 + iv_2, v_1, v_2 \in V \}$ (briefly, $V^{\mathbb{C}} = V \oplus i\,V$); the scalar multiplication on $V^{\mathbb{C}}$ takes then the form

$$(a + ib)(v_1 + iv_2) = av_1 - bv_2 + i\,(av_2 + bv_1)$$

for $a + ib \in \mathbb{C}$ and $v_1 + iv_2 \in V^{\mathbb{C}}$.

Let V be a real vector space. We have an $\mathbb{R}$-linear map $V^{\mathbb{C}} \to V^{\mathbb{C}}$, $w \mapsto \overline{w}$ (called *conjugation*) defined by $\overline{v \otimes \lambda} := v \otimes \bar{\lambda}$ (or equivalently $\overline{v_1 + iv_2} := v_1 - iv_2$

for $v_1, v_2 \in V$). Observe that the map $V^{\mathbb{C}} \to \overline{V^{\mathbb{C}}}$, $w \mapsto \overline{w}$ is a $\mathbb{C}$-linear isomorphism.

Let V^* be the dual of a real vector space V. Then its complexification is canonically isomorphic (over $\mathbb{C}$) to the (complex) dual of the complexification of V, i.e. $(V^*)^{\mathbb{C}} = (V^{\mathbb{C}})^* = \mathrm{Hom}(V^{\mathbb{C}}, \mathbb{C})$. The dual pairing $(V^*)^{\mathbb{C}} \times V^{\mathbb{C}} \to \mathbb{C}$ is given by $\langle v_1^* \otimes \lambda, v_2 \otimes \mu \rangle := \lambda\mu \langle v_1^*, v_2 \rangle$ for λ, $\mu \in \mathbb{C}$, $v_1^* \in V^*$ and $v_2 \in V$. Note that for $w_1^* \in (V^*)^{\mathbb{C}}$ and $w_2 \in V^{\mathbb{C}}$ we have $\langle \overline{w_1^*}, w_2 \rangle = \overline{\langle w_1^*, \overline{w_2} \rangle}$.

Observe that for every real vector space V we have a canonical $\mathbb{C}$-linear isomorphism $\left(\bigwedge^r V\right)^{\mathbb{C}} \cong \bigwedge^r V^{\mathbb{C}}$ (thus in particular $\left(\bigwedge^r V^*\right)^{\mathbb{C}} = \bigwedge^r \left(V^*\right)^{\mathbb{C}} = \bigwedge^r \left(V^{\mathbb{C}}\right)^*$). It is obtained by applying the universal factorization property of the exterior power to the r-linear alternating map $V^{\mathbb{C}} \times \ldots \times V^{\mathbb{C}} \to \left(\bigwedge^r V\right)^{\mathbb{C}}$,

$$\left(v_1 \otimes \lambda_1, \ldots, v_r \otimes \lambda_r\right) \mapsto v_1 \wedge \ldots \wedge v_r \otimes \lambda_1 \ldots \lambda_r \,.$$

From now on, till the end of the section, V will be a $2n$-dimensional real vector space with a complex structure J and $V^{\mathbb{C}}$ its complexification.

We can extend J to a $\mathbb{C}$-linear automorphism of $V^{\mathbb{C}}$ by $J\,(v \otimes \lambda) := J(v) \otimes \lambda$ (equivalently, $J\,(v_1 + iv_2) := J(v_1) + iJ(v_2)$ for $v_1, v_2 \in V$). Consider the polynomial $\varphi(x) = x^2+1 = (x+i)(x-i)$. Since $\varphi(J) = 0$ and $J+i \neq 0 \neq J-i$, it follows that $\varphi(x)$ is the minimal polynomial of J, thus $+i$ and $-i$ are the eigenvalues of J. We will denote by $V^{1,0}$ (respectively $V^{0,1}$) the eigenspace of $+i$ (respectively $-i$). Then $V^{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$. Given a vector $w \in V^{\mathbb{C}}$, we will denote by $w^{1,0}$ (respectively $w^{0,1}$) the projection of w into $V^{1,0}$ (respectively $V^{0,1}$). Then we have

$$w^{1,0} = \frac{w - iJ(w)}{2} \quad \text{and} \quad w^{0,1} = \frac{w + iJ(w)}{2} \,. \tag{2.1}$$

The proof of the following lemma is immediate.

Lemma 2.1.3 *Let $V^{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ as above. Then:*

1. $V^{1,0} = \{\, v - iJ(v),\ v \in V \,\}$ *and* $V^{0,1} = \{\, v + iJ(v),\ v \in V \,\}$;
2. *the $\mathbb{R}$-linear isomorphism $V^{\mathbb{C}} \to V^{\mathbb{C}}$, $w \mapsto \overline{w}$ induces a conjugate-linear map $V^{1,0} \to V^{0,1}$;*
3. *the maps $(V, J) \to V^{1,0}$, $v \mapsto \frac{v-iJ(v)}{2}$ and $(V, -J) \to V^{0,1}$, $v \mapsto \frac{v+iJ(v)}{2}$ are $\mathbb{C}$-linear isomorphisms.* [1]

The above observations apply to the dual space V^* and the induced complex structure J^*. Thus we can write

$$\left(V^{\mathbb{C}}\right)^* = \left(V^*\right)^{\mathbb{C}} = V_{1,0} \oplus V_{0,1} \tag{2.2}$$

[1] The motivation for the factor $\frac{1}{2}$ will appear in Section 2.2.

where $V_{1,0} := \{ w^* \in (V^*)^{\mathbb{C}} \;/\; J^*(w^*) = i\, w^* \} = \{ v^* - i\, J^*(v^*),\; v^* \in V^* \}$ and $V_{0,1} := \{ w^* \in (V^*)^{\mathbb{C}} \;/\; J^*(w^*) = -i\, w^* \} = \{ v^* + i\, J^*(v^*),\; v^* \in V^* \}$. It is easy to see that $V_{1,0} = \{ w^* \in (V^*)^{\mathbb{C}} \;/\; \langle w^*, w' \rangle = 0,\; \forall w' \in V^{0,1} \}$ and $V_{0,1} = \{ w^* \in (V^*)^{\mathbb{C}} \;/\; \langle w^*, w' \rangle = 0,\; \forall w' \in V^{1,0} \}$. Thus $V_{1,0} = (V^{1,0})^*$ and $V_{0,1} = (V^{0,1})^*$ (complex duals).

Remark 2.1.4 *Observe that, using the conjugate-linear map $V^{1,0} \to V^{0,1}$ of Lemma 2.1.3, we can identify $V_{0,1}$ with the vector space of conjugate-linear maps $V^{1,0} \to \mathbb{C}$ (i.e. for $\alpha \in V_{0,1}$ we define a conjugate-linear map $\tilde{\alpha} : V^{1,0} \to \mathbb{C}$ by $\tilde{\alpha}(w) := \alpha(\overline{w})$). This identification will be always tacitly assumed in the following.*

The decomposition (2.2) induces a decomposition of the exterior algebra $\bigwedge (V^{\mathbb{C}})^*$ as follows. For integers s, $q \geq 0$ define

$$\bigwedge^{s,q} (V^{\mathbb{C}})^* := < \alpha \wedge \beta \in \bigwedge^{s+q} (V^{\mathbb{C}})^* \;/\; \alpha \in \bigwedge^s V_{1,0}\;,\; \beta \in \bigwedge^q V_{0,1} >_{\mathbb{C}}$$

where $< \quad >_{\mathbb{C}}$ denotes the $\mathbb{C}$-linear span. Then for every integer $r \geq 0$ we have

$$\bigwedge^r (V^{\mathbb{C}})^* = \bigoplus_{s+q=r} \bigwedge^{s,q} (V^{\mathbb{C}})^*. \tag{2.3}$$

Complex conjugation in $(V^{\mathbb{C}})^* = (V^*)^{\mathbb{C}}$ can be extended to $\bigwedge (V^{\mathbb{C}})^*$ in a natural way. In particular we obtain an $\mathbb{R}$-linear isomorphism between $\bigwedge^{s,q} (V^{\mathbb{C}})^*$ and $\bigwedge^{q,s} (V^{\mathbb{C}})^*$. Note that if $\alpha \in \bigwedge^r (V^{\mathbb{C}})^*$ and $w_1, \ldots, w_r \in V^{\mathbb{C}}$, then

$$\overline{\alpha}\,(w_1, \ldots, w_r) = \overline{\alpha\,(\overline{w_1}, \ldots, \overline{w_r})}\,.$$

The proof of the following proposition is straightforward.

Proposition 2.1.5 *Let h be an Hermitian inner product on $V^{1,0}$ and denote by $\underline{h}$ the Hermitian inner product on (V, J) induced by h, via the isomorphism $(V, J) \to V^{1,0}$, $v \mapsto \frac{v - iJ(v)}{2}$. Then:*

1. *The map $\tilde{\underline{h}} := \mathrm{Re}\,(\underline{h}) : V \times V \to \mathbb{R}$ is an inner product on V such that $\tilde{\underline{h}}\,\big(J(v_1), J(v_2)\big) = \tilde{\underline{h}}\,(v_1, v_2)$ for all v_1, $v_2 \in V$. It holds*

$$\underline{h}\,(v_1, v_2) = \tilde{\underline{h}}\,(v_1, v_2) + i\, \tilde{\underline{h}}\,\big(v_1, J(v_2)\big).$$

2. *The $\mathbb{C}$-linear extension of $\tilde{\underline{h}} : V \times V \to \mathbb{R}$ to $V^{\mathbb{C}}$ is a $\mathbb{C}$-bilinear symmetric non-degenerate map $\tilde{h} : V^{\mathbb{C}} \times V^{\mathbb{C}} \to \mathbb{C}$ such that for w, w_1 and $w_2 \in V^{\mathbb{C}}$ it holds*

   - $\tilde{h}\,(\overline{w_1}, \overline{w_1}) = \overline{\tilde{h}\,(w_1, w_2)}$;
   - $\tilde{h}\,(w, \overline{w}) > 0$ *if* $w \neq 0$;
   - $\tilde{h}\,(w_1, w_2) = 0$ *if* w_1, $w_2 \in V^{1,0}$ *or* w_1, $w_2 \in V^{0,1}$;
   - $h\,(w_1, w_2) = 2\,\tilde{h}\,(w_1, \overline{w_2})$ *for* w_1, $w_2 \in V^{1,0}$.

3. For every positive integer r, $\tilde{h} : V^{\mathbb{C}} \times V^{\mathbb{C}} \to \mathbb{C}$ extends to a $\mathbb{C}$-bilinear symmetric non-degenerate map $\bigwedge^r \tilde{h} : \bigwedge^r(V^{\mathbb{C}}) \times \bigwedge^r(V^{\mathbb{C}}) \to \mathbb{C}$ by defining

$$\bigwedge^r \tilde{h}\,(\, w_1 \wedge \ldots \wedge w_r, w_1' \wedge \ldots \wedge w_r' \,) := \det \Big(\big(\, \tilde{h}\,(w_i, w_j')\,\big)_{ij} \Big)$$

for $w_1, \ldots, w_r, w_1', \ldots, w_r' \in V^{\mathbb{C}}$ and linear extension. Then we have

$$\bigwedge^r \tilde{h}\,(\bar{\alpha}, \bar{\beta}) = \overline{\bigwedge^r \tilde{h}\,(\alpha, \beta)}$$

for $\alpha,\ \beta \in \bigwedge^r(V^{\mathbb{C}})$.

4. The map $-\frac{1}{2}\operatorname{Im}(\underline{h}) : V \times V \to \mathbb{R}$ is an element of $\bigwedge^2 V^*$ and its $\mathbb{C}$-linear extension k to $V^{\mathbb{C}}$ is an element of $\bigwedge^{1,1}\big(V^{\mathbb{C}}\big)^*$. It holds

$$k\,(w_1, w_2) = -\frac{1}{2}\,\tilde{h}\,\big(w_1, J(w_2)\big)$$

for $w_1,\ w_2 \in V^{\mathbb{C}}$.

2.2 Complex and almost complex manifolds

Definition 2.2.1 *A **complex** n-dimensional **manifold** consists of a topological manifold M together with a family $\{\,(\mathcal{U}_i, \varphi_i),\ i \in I\,\}$ such that:*

- *$\{\,\mathcal{U}_i,\ i \in I\,\}$ is a cover of M;*
- *for every $i \in I$, φ_i is a homeomorphism between $\mathcal{U}_i$ and an open subset of $\mathbb{C}^n$;*
- *for every $i,\ j \in I$ with $\mathcal{U}_{ij} \neq \emptyset$, the* transition function *$\varphi_i \circ \varphi_j^{-1} : \varphi_j\,(\mathcal{U}_{ij}) \to \varphi_i\,(\mathcal{U}_{ij})$ is holomorphic.*

A chart *for the complex manifold M consists of an open subset $\mathcal{U}$ of M and a homeomorphism φ between $\mathcal{U}$ and an open subset of $\mathbb{C}^n$ such that for every $i \in I$ with $\mathcal{U} \cap \mathcal{U}_i \neq \emptyset$ the map $\varphi_i \circ \varphi^{-1} : \varphi\,(\mathcal{U} \cap \mathcal{U}_i) \to \varphi_i\,(\mathcal{U} \cap \mathcal{U}_i)$ is holomorphic. A continuous map f between two complex manifolds N and M is* holomorphic *if for every $p \in N$ and some (and hence all) charts $(\mathcal{U}, \varphi)$ of N around p and $(\mathcal{V}, \psi)$ of M around $f(p)$ the map $\psi \circ f \circ \varphi^{-1} : \varphi\big(\, f^{-1}(\mathcal{V}) \cap \mathcal{U}\,\big) \to \psi\big(\,\mathcal{V} \cap f(\mathcal{U})\,\big)$ is holomorphic.*

Note that every complex n-dimensional manifold has the structure of a real $2n$-dimensional manifold and that a holomorphic map between complex manifolds is in particular a smooth map between the underlying real manifolds.

Let M be a complex manifold and $p \in M$. We will denote by $T_p^{\mathbb{C}}M$ the *holomorphic tangent space* of M at p, i.e. the complex vector space of $\mathbb{C}$-linear derivations of germs of holomorphic functions on M, and by $T_p^{\mathbb{R}}M$ the tangent space at p of the underlying smooth manifold, thus the real vector space of $\mathbb{R}$-linear derivations of germs of smooth functions on M. If $\varphi = (z_1, \ldots, z_n) : \mathcal{U} \to \mathbb{C}^n$ is a chart of M around p then $\{ \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n} \}$ [2] is a basis of $T_p^{\mathbb{C}} M$ over $\mathbb{C}$ (see [7, p. 152]) and $\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n} \}$ a basis of $T_p^{\mathbb{R}} M$ over $\mathbb{R}$, where $z_j = x_j + iy_j$ for $j = 1, \ldots, n$. Note that we have a canonical $\mathbb{R}$-linear isomorphism $\phi : T_p^{\mathbb{R}} M \to \big(T_p^{\mathbb{C}} M\big)^{\mathbb{R}}$, defined by $\phi(X)\,(f) := X\big(\mathrm{Re}(f)\big) + i\, X\big(\mathrm{Im}(f)\big)$ for $X \in T_p^{\mathbb{R}} M$ and for a holomorphic germ f on M. In local coordinates we have, using the Cauchy-Riemann equations, $\phi\,(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial z_j}$ and $\phi\,(\frac{\partial}{\partial y_j}) = i \frac{\partial}{\partial z_j}$. The complex structure of $T_p^{\mathbb{C}} M$ induces a complex structure on $\big(T_p^{\mathbb{C}} M\big)^{\mathbb{R}}$ as explained in §1 and thus, via the $\mathbb{R}$-linear isomorphism $\phi : T_p^{\mathbb{R}} M \to \big(T_p^{\mathbb{C}} M\big)^{\mathbb{R}}$, a complex structure J on $T_p^{\mathbb{R}} M$ (i.e., we define J to be the unique complex structure on $T_p^{\mathbb{R}} M$ turning $\phi : (T_p^{\mathbb{R}} M, J) \to T_p^{\mathbb{C}} M$ into a $\mathbb{C}$-linear isomorphism). Locally we have $J\,(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j}$ for $j = 1, \ldots, n$.

All results of Section 2.1 apply now to the real vector space $T_p^{\mathbb{R}} M$ with the complex structure J. We will denote by $T_p M$ the complexification of $T_p^{\mathbb{R}} M$ (and call it the *tangent space* of the complex manifold M) and by $T'_p M$ and $T''_p M$ respectively the eigenspaces of $+i$ and $-i$ for the endomorphism J of $T_p M$. Then $T_p M = T'_p M \oplus T''_p M$. Note that we can identify $T_p M$ with the complex vector space of $\mathbb{C}$-linear derivations of germs of smooth $\mathbb{C}$-valued functions on M, by defining $(X \otimes \lambda)\,(f) := \lambda \Big(X\big(\mathrm{Re}(f)\big) + i\, X\big(\mathrm{Im}(f)\big) \Big)$ for $X \otimes \lambda \in T_p M$ and for a smooth germ f.

Consider a chart $\varphi = (z_1, \ldots, z_n) : \mathcal{U} \to \mathbb{C}^n$ of M around p, and let $z_j = x_j + iy_j$ for $j = 1, \ldots, n$. Define elements $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \bar{z}_j}$ of $T_p M$ by

$$\frac{\partial}{\partial z_j} := \frac{1}{2} \Big(\frac{\partial}{\partial x_j} \otimes 1 - \frac{\partial}{\partial y_j} \otimes i \Big) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2} \Big(\frac{\partial}{\partial x_j} \otimes 1 + \frac{\partial}{\partial y_j} \otimes i \Big). \tag{2.4}$$

Then, by Lemma 2.1.3, $\{ \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n} \}$ and $\{ \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n} \}$ are bases over $\mathbb{C}$ of $T'_p M$ and $T''_p M$ respectively. Note that the $\frac{\partial}{\partial z_j} \in T_p M$ just defined is an extension of $\frac{\partial}{\partial z_j} \in T_p^{\mathbb{C}} M$ since, because of the Cauchy-Riemann equations, the two operators coincide on holomorphic functions. Observe that when we identify $\frac{\partial}{\partial z_j} \in T_p^{\mathbb{C}} M$ with $\frac{\partial}{\partial z_j} \in T'_p M$ we are actually applying the inverse of the isomorphism $\phi : (T_p^{\mathbb{R}} M, J) \to T_p^{\mathbb{C}} M$ defined above composed with the isomorphism $(T_p^{\mathbb{R}} M, J) \to T'_p M$ of Lemma 2.1.3. From now on, the identification $T_p^{\mathbb{C}} M \equiv T'_p M$

[2] By abuse of notation, we write $\frac{\partial}{\partial z_j}$ for the element of $T_p^{\mathbb{C}} M$ defined by

$$\frac{\partial}{\partial z_j}\,(f) := \frac{\partial}{\partial z_j}\,(f \circ \varphi^{-1})$$

for $f : \mathcal{U} \to \mathbb{C}$ holomorphic. Similarly for $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial y_j}$.

will be always tacitly assumed.

Let $\{ dz_1, \ldots, dz_n \}$ and $\{ d\overline{z}_1, \ldots, d\overline{z}_n \}$ be the dual bases for $T'_P M^*$ and $T''_P M^*$ of $\{ \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n} \}$ and $\{ \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n} \}$ respectively. Then we have

$$dz_j = dx_j + i\, dy_j \quad \text{and} \quad d\overline{z}_j = dx_j - i\, dy_j \tag{2.5}$$

for $j = 1, \ldots, n$, where $\{ dx_1, dy_1, \ldots, dx_n, dy_n \}$ is the basis of T_pM^* dual to $\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n} \}$.

Given a complex manifold M, we will denote by $T^{\mathbb{R}}M$ the tangent bundle of the real manifold underlying M and by $TM := \bigcup_{p\in M} T_pM$ the tangent bundle of M. The last is a complex vector bundle and its construction is analogous to that of the tangent bundle of a real manifold. Similarly, we obtain complex vector bundles $T^{\mathbb{C}}M := \bigcup_{p\in M} T^{\mathbb{C}}_p M$ and $(T^{\mathbb{R}}M, J) := \bigcup_{p\in M}(T^{\mathbb{R}}_p M, J)$. Note that the $\mathbb{C}$-linear isomorphisms $\phi : (T^{\mathbb{R}}_p M, J) \to T^{\mathbb{C}}_p M$ $(p \in M)$ defined above induce a vector bundle isomorphism ϕ between $(T^{\mathbb{R}}M, J)$ and $T^{\mathbb{C}}M$.

We can now apply Example 1.1.14 to TM (see also Remark 1.1.19) and construct the complex vector bundle $\bigwedge^r TM^*$. We define $\mathcal{A}^r(M) := \Gamma\left(\bigwedge^r TM^*\right)$. Elements of $\mathcal{A}(M) := \bigoplus_{r\geq 0} \mathcal{A}^r(M)$ are called *complex differential forms* on M. It is easy to see that the decomposition (2.3), applied to the tangent spaces of M, yields a decomposition

$$\mathcal{A}^r(M) = \bigoplus_{s+q=r} \mathcal{A}^{s,q}(M)$$

where $\mathcal{A}^{s,q}(M) := \{ \omega \in \mathcal{A}^r(M) \;/\; \omega(p) \in \bigwedge^{s,q}(T_pM)^*, \forall p \in M \}$. We will denote by $\pi^{s,q}$ the projection $\mathcal{A}^r(M) \to \mathcal{A}^{s,q}(M)$. On the domain of local holomorphic coordinates $(z_1, \ldots, z_n)$ of M, every differential form $\omega \in \mathcal{A}^{s,q}(M)$ can be written in the form

$$\omega = \sum_{|I|=s,\, |J|=q} f_{I,J}\, dz_I \wedge d\overline{z}_J \tag{2.6}$$

where the $f_{I,J}$ are smooth $\mathbb{C}$-valued functions.

Conjugation on $\Gamma(TM)$ and $\mathcal{A}(M)$ is defined pointwise as described in Section 2.1 (note in particular that in local coordinates we have $\overline{\frac{\partial}{\partial z_j}} = \frac{\partial}{\partial \overline{z}_j}$ and $\overline{dz_j} = d\overline{z}_j$ for $j = i, \ldots, n$). If $\omega \in \mathcal{A}^{s,q}(M)$, then $\overline{\omega} \in \mathcal{A}^{q,s}(M)$ and

$$\overline{\omega}\,(X_1, \ldots, X_r) = \overline{\omega\,(\overline{X}_1, \ldots, \overline{X}_r)}$$

for $X_1, \ldots, X_r \in \Gamma(TM)$. In local coordinates, if ω is given by (2.6) then

$$\overline{\omega} = \sum_{|I|=s,\, |J|=q} \overline{f_{I,J}}\, d\overline{z}_I \wedge dz_J\,. \tag{2.7}$$

Note that conjugation commutes with the wedge product, i.e. for $\omega_1, \omega_2 \in \mathcal{A}(M)$ we have $\overline{\omega_1 \wedge \omega_2} = \overline{\omega_1} \wedge \overline{\omega_2}$.

The exterior derivative d on real differential forms is extended to $\mathcal{A}(M)$ by complex linearity. Then for a differential form $\omega \in \mathcal{A}^{s,q}(M)$ which is locally given by (2.6) we have, using (2.4) and (2.5):

$$\begin{aligned} d\omega &= \sum \Big(\frac{\partial f_{I,J}}{\partial x_j}\, dx_j + \frac{\partial f_{I,J}}{\partial y_j}\, dy_j \Big) \wedge dz_I \wedge d\overline{z}_J \\ &= \sum \frac{\partial f_{I,J}}{\partial z_j}\, dz_j \wedge dz_I \wedge d\overline{z}_J + \frac{\partial f_{I,J}}{\partial \overline{z}_j}\, d\overline{z}_j \wedge dz_I \wedge d\overline{z}_J\,. \end{aligned} \tag{2.8}$$

It follows that

$$d\big(\mathcal{A}^{s,q}(M)\big) \subset \mathcal{A}^{s+1,q}(M) \oplus \mathcal{A}^{s,q+1}(M).$$

We will denote by $\partial : \mathcal{A}^{s,q}(M) \to \mathcal{A}^{s+1,q}(M)$ the operator $\pi^{s+1,q} \circ d$ and by $\overline{\partial} : \mathcal{A}^{s,q}(M) \to \mathcal{A}^{s,q+1}(M)$ the operator $\pi^{s,q+1} \circ d$. Then $d = \partial + \overline{\partial}$.
Note that the exterior derivative commutes with conjugation, i.e. for all $\omega \in \mathcal{A}(M)$ we have $d\overline{\omega} = \overline{d\omega}$. This is easy to see, using (2.7) and (2.8).

Let h be an Hermitian metric on M, i.e. a $\mathcal{C}^\infty$ field of Hermitian inner products in the fibres of $T^{\mathbb{C}}M$. Via the vector bundle isomorphism $\phi : (T^{\mathbb{R}}M, J) \to T^{\mathbb{C}}M$, we obtain then an Hermitian metric $\underline{h}$ on $(T^{\mathbb{R}}M, J)$. By Proposition 2.1.5 it follows that $\tilde{h} := \mathrm{Re}\,(\underline{h})$ is a Riemannian metric on the real manifold underlying M and that the $\mathbb{C}$-linear extension k of $-\frac{1}{2}\,\mathrm{Im}\,(\underline{h})$ is an element of $\mathcal{A}^{1,1}(M)$ (the *associated* $(1,1)$*-form* of the metric).

Definition 2.2.2 *Let h be an Hermitian metric on a complex manifold. Then the associated $(1,1)$-form k is called the* Kähler form *of h. The metric h is said to be a* Kähler metric *if its Kähler form is closed, i.e. if $dk = 0$. A complex manifold admitting a Kähler metric is called a* Kähler manifold.

Let $\varphi = (z_1, \ldots, z_n) : \mathcal{U} \to \mathbb{C}^n$ be a chart for M. Then an Hermitian metric h can be written on $\mathcal{U}$ in the form

$$h = \sum_{\alpha,\beta} h_{\alpha\beta}\, dz_\alpha \otimes d\overline{z}_\beta$$

where $\big(h_{\alpha\beta}(p)\big)_{\alpha\beta} := \big(h(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta})(p)\big)_{\alpha\beta}$ is a positive-definite Hermitian matrix for every $p \in \mathcal{U}$ and where we regard the $d\overline{z}_\beta$'s as conjugate-linear maps $T_p^{\mathbb{C}}M \to \mathbb{C}$ (see Remark 2.1.4). [3]

[3] It can be shown that h is a Kähler metric if and only if it approximates the Euclidean metric up to order 2 at each point, i.e. if and only if we can find around each $p \in M$ a chart $\varphi = (z_1, \ldots, z_n)$ with $\varphi(p) = 0$ and such that $h_{\alpha\beta}(p) = I$ and $dh_{\alpha\beta}(p) = 0$ for all α, β. See [6, p. 107].

Let $k = \sum_{\alpha,\beta} 2\, k(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta})\, dz_\alpha \wedge d\bar{z}_\beta \in \mathcal{A}^{1,1}(M)$ be the Kähler form of h. Using the relations in Proposition 2.1.5, it is easy to see that $k(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}) = \frac{i}{4}\, h(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta})$. Thus we have

$$k = \frac{i}{2} \sum_{\alpha,\beta} h_{\alpha\beta}\, dz_\alpha \wedge d\bar{z}_\beta. \tag{2.9}$$

Definition 2.2.3 *An* almost complex structure *on a real manifold M is a tensor field $J \in \Gamma\,(TM^* \otimes TM)$ such that $J^2 = -\mathrm{id}$. An* ***almost complex manifold*** *is a real manifold with an almost complex structure.*

If a real manifold admits an almost complex structure, then it must be even-dimensional and orientable (for the last, see [15, Proposition 2.1 of Chapter IX]).

Given a complex manifold M, we can define a tensor field $J \in \Gamma\,(T^{\mathbb{R}}M^* \otimes T^{\mathbb{R}}M)$ pointwise as described in the beginning of the paragraph. This gives an almost complex structure on the underlying real manifold.

Definition 2.2.4 *Let (M, J) and (M', J') be almost complex manifolds. A smooth map $f : M \to M'$ is said to be* almost holomorphic *if $f_* \circ J = J' \circ f_*$.*

Proposition 2.2.5 *A smooth map $f : M \to M'$ between complex manifolds is holomorphic if and only if it is almost holomorphic with respect to the induced almost complex structures on M and M'.*

This can be proved locally, using the Cauchy-Riemann equations (see [15, Propositions 2.2 and 2.3 of Chapter IX]).

Definition 2.2.6 *Let J be an almost complex structure on a real manifold M. The* torsion *of J is the tensor field $N_J \in \Gamma\,(TM^* \otimes TM^* \otimes TM)$ given by*

$$N_J\,(X,Y) := 2\Big(\big[J(X), J(Y)\big] - [X,Y] - J\big([X, J(Y)]\big) - J\big([J(X), Y]\big)\Big)$$

for X, $Y \in \Gamma(TM)$. The almost complex structure J is said to be integrable *if $N_J = 0$.*

The induced almost complex structure on a complex manifold is integrable. For a proof of this, see [15, Theorem 2.5 of Chapter IX]). The converse is a classical result due to Newlander and Nirenberg (see [24]).

Theorem 2.2.7 *Let J be an integrable almost complex structure on a real manifold M. Then M has a unique complex structure which induces the almost complex structure J.*

Observe that some of the definitions in this section also make sense when M is a real manifold. For every $p \in M$ we can consider the complexification $\big(T_pM\big)^{\mathbb{C}}$ of the tangent space and interpret it as the space of $\mathbb{C}$-linear derivations of germs of smooth $\mathbb{C}$-valued functions on M. We can construct the complex vector bundles $TM^{\mathbb{C}} := \bigcup_{p\in M} \big(T_pM\big)^{\mathbb{C}}$ and $\bigwedge^r \big(TM^{\mathbb{C}}\big)^*$. We define $\mathcal{A}^r(M)^{\mathbb{C}} := \Gamma\Big(\bigwedge^r \big(TM^{\mathbb{C}}\big)^*\Big)$ and call elements of $\mathcal{A}(M)^{\mathbb{C}} := \bigoplus_{r\geq 0} \mathcal{A}^r(M)^{\mathbb{C}}$ complex differential forms on M. Conjugation on $TM^{\mathbb{C}}$ and $\mathcal{A}(M)^{\mathbb{C}}$ and the exterior derivative

$$d : \mathcal{A}^r(M)^{\mathbb{C}} \to \mathcal{A}^{r+1}(M)^{\mathbb{C}}$$

are defined as above. Finally, if $Y_1 \otimes \lambda_1,\ Y_2 \otimes \lambda_2 \in TM^{\mathbb{C}}$, we can define the Lie bracket $[Y_1 \otimes \lambda_1, Y_2 \otimes \lambda_2] \in TM^{\mathbb{C}}$ as usual by

$$[Y_1 \otimes \lambda_1, Y_2 \otimes \lambda_2]\,(f) := (Y_1 \otimes \lambda_1)\big((Y_2 \otimes \lambda_2)\,(f)\big) - (Y_2 \otimes \lambda_2)\big((Y_1 \otimes \lambda_1)\,(f)\big)$$

for $f \in \mathcal{C}^\infty(M, \mathbb{C})$. Then $[Y_1 \otimes \lambda_1, Y_2 \otimes \lambda_2] = [Y_1, Y_2] \otimes \lambda_1\lambda_2$.

2.3 Connections on complex vector bundles

Let $\pi : E \to M$ be a complex vector bundle over a real (respectively complex) manifold. A (smooth) r-form on M with values in E is an element of $\mathcal{A}^r(E) := \Gamma(E) \otimes_{\mathcal{C}^\infty(M,\mathbb{C})} \mathcal{A}^r(M)^{\mathbb{C}}$ (respectively $\mathcal{A}^r(E) := \Gamma(E) \otimes_{\mathcal{C}^\infty(M,\mathbb{C})} \mathcal{A}^r(M)$). A connection on E is a $\mathbb{C}$-linear map $D : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ which satisfies the Leibnitz rule (for $\mathbb{C}$-valued smooth functions on M). Observe that all results of Paragraph 1.3, with the exception of Example 1.3.8 and Definition 1.3.9, also apply to connections on complex vector bundles. In this section we will first show how those two points have to be changed in the complex case, and then we will describe some specific properties of connections on complex and holomorphic vector bundles over complex manifolds.

Example 2.3.1 *Let $\pi : E \to M$ be a complex vector bundle with transition functions $\{\, \theta_{ij} : \mathcal{U}_{ij} \to GL(n, \mathbb{C}) \,\}$ with respect to a cover $\{\, \mathcal{U}_i,\ i \in I \,\}$ of M. The functions $\{\, \overline{\theta_{ij}} : \mathcal{U}_{ij} \to GL(n, \mathbb{C}) \,\}$ satisfy the cocycle condition (1.2), thus by Proposition 1.1.8 (see Remark 1.1.19) they are the transition functions of a complex vector bundle $\bar{E}$ over M, called the* ***conjugate bundle*** *of E. We have $\bar{E}_p \cong \overline{E_p}$ in a canonical way. The isomorphism is given by the well-defined $\mathbb{C}$-linear map $(\, i, p, x \,)_{/\sim} \mapsto (\, i, p, \bar{x} \,)_{/\sim}$ (notation as in the proof of Proposition 1.1.8). Note that the construction of $\bar{E}$ does not depend on the set of transition functions defining E. Moreover, if E and E' are isomorphic vector bundles, then so are $\bar{E}$ and $\bar{E}'$. The proof of this is similar to that in Example 1.1.9.*

Example 2.3.2 *Let $\pi : E \to M$ be a complex vector bundle. Combining Examples 2.3.1, 1.1.9 and 1.1.11, we can define the complex vector bundle $E^* \otimes \bar{E}^*$. Since $(E^* \otimes \bar{E}^*)_p = (E_p)^* \otimes \overline{(E_p)}^*$ for all $p \in M$, we see that a section of $E^* \otimes \bar{E}^*$ gives* *a sesquilinear map $E_p \times E_p \to \mathbb{C}$ on each fibre of E, varying smoothly with p. An* ***Hermitian metric*** *on $\pi : E \to M$ is a section h of $E^* \otimes \bar{E}^*$ such that $h(p)$ is an Hermitian inner product on E_p, for all $p \in M$.*

Example 2.3.3 *Let $\pi : E \to M$ be a complex vector bundle and let $\bar{E}$ be the conjugate bundle. Observe first that sections of E can also be regarded as sections of $\bar{E}$ (and vice versa) because for every $p \in M$ the sets underlying the vector spaces E_p and $\bar{E}_p$ are equal (under the canonical isomorphism $\bar{E}_p \cong \overline{E_p}$ described in Example 2.3.1) We define a map $Q : \mathcal{A}^r(E) \to \mathcal{A}^r(\bar{E})$ by*

$$Q\,(\sigma \otimes \varphi) := \sigma \otimes \bar{\varphi}$$

for $\sigma \in \mathcal{A}^0(E)$, $\varphi \in \mathcal{A}^r(M)$ and linear extension (in particular, note that for $f \in \mathcal{C}^\infty(M, \mathbb{C})$ we have $Q\,(f\sigma) = \bar{f} \cdot \sigma$, where the operation on the right hand side is pointwise scalar multiplication in the fibres of $\bar{E}$). Given a connection D on E, define a connection $\bar{D}$ on $\bar{E}$ by

$$\bar{D}(\sigma) := Q\,\big(D(\sigma)\big) \tag{2.10}$$

for $\sigma \in \mathcal{A}^0(\bar{E})$. If ω_u and Ω_u are the connection and curvature forms of D with respect to a local frame u of E, then $\overline{\omega_u}$ and $\overline{\Omega_u}$ are the connection and curvature forms of $\bar{D}$ with respect to the same u, considered as a frame of $\bar{E}$.

Example 2.3.4 *Let $\pi : E \to M$ be a complex vector bundle and let D be a connection on E. Combining (1.23), (1.21) and (2.10), we get a connection $D_{E^* \otimes \bar{E}^*}$ on the bundle $E^* \otimes \bar{E}^*$. With a calculation similar to that in Example 1.3.8 we get, for $h \in \mathcal{A}^0(E^* \otimes \bar{E}^*)$ and σ_1, $\sigma_2 \in \mathcal{A}^0(E)$*

$$D_{E^* \otimes \bar{E}^*}(h)\,(\sigma_1, \sigma_2) = d\,h(\sigma_1, \sigma_2) - h\,\big(\sigma_1, D(\sigma_2)\big) - h\,\big(D(\sigma_1), \sigma_2\big)$$

where h is considered as a $\mathcal{C}^\infty(M, \mathbb{C})$-sesquilinear map $\mathcal{A}^0(E) \times \mathcal{A}^0(E) \to \mathcal{C}^\infty(M, \mathbb{C})$ and where the extensions $h : \mathcal{A}^0(E) \times \mathcal{A}^1(E) \to \mathcal{A}^1(M)$ and $h : \mathcal{A}^1(E) \times \mathcal{A}^0(E) \to \mathcal{A}^1(M)$ on the right hand side are defined by $h(\sigma_1, \sigma_2 \otimes \varphi) := h(\sigma_1, \sigma_2)\,\overline{\varphi}$ and $h(\sigma_1 \otimes \varphi, \sigma_2) := h(\sigma_1, \sigma_2)\,\varphi$ for σ_1, $\sigma_2 \in \mathcal{A}^0(E)$ and $\varphi \in \mathcal{A}^1(M)$ and by linear extension.

Definition 2.3.5 *Let $\pi : E \to M$ be a complex vector bundle with an Hermitian metric h. A connection D on E is said to be* compatible with h *(or to be an* ***h-connection****) if $D_{E^* \otimes \bar{E}^*}(h) = 0$, i.e. if for all σ_1, $\sigma_2 \in \mathcal{A}^0(E)$ it holds*

$$d\,h(\sigma_1, \sigma_2) = h\,\big(\sigma_1, D(\sigma_2)\big) + h\,\big(D(\sigma_1), \sigma_2\big). \tag{2.11}$$

If D is an h-connection, then its connection and curvature forms ω_u and Ω_u with respect to an h-orthonormal frame u are skew-Hermitian [4] (in particular, we have $\omega_u(X) \in \mathfrak{u}(n)$ for every real vector X). Conversely, if around each point of M

[4] I.e. $\omega_{\alpha\beta} = -\overline{\omega_{\beta\alpha}}$ and $\Omega_{\alpha\beta} = -\overline{\Omega_{\beta\alpha}}$ for all α, β.

there is a local h-orthonormal frame u such that the connection form ω_u of a connection D with respect to it is skew-Hermitian (equivalently, if $\omega_u(X) \in \mathfrak{u}(n)$ for every real vector X), then D is an h-connection. This all can be seen similarly to the real case.

Let $\pi : E \to M$ be a complex vector bundle over a complex manifold. For integers s, $q \geq 0$, define

$$\mathcal{A}^{s,q}(E) := \Gamma(E) \otimes_{\mathcal{C}^\infty(M,\mathbb{C})} \mathcal{A}^{s,q}(M).$$

Then for every integer $r \geq 0$ we have $\mathcal{A}^r(E) := \bigoplus_{s+q=r} \mathcal{A}^{s,q}(E)$. We will denote by $\pi^{s,q}$ the projection $\mathcal{A}^r(E) \to \mathcal{A}^{s,q}(E)$.

Let $D : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ be a connection on E. We can write

$$D = D' + D'' \tag{2.12}$$

where $D' = \pi^{1,0} \circ D : \mathcal{A}^0(E) \to \mathcal{A}^{1,0}(E)$ and $D'' = \pi^{0,1} \circ D : \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E)$. Extend D' and D'' to operators $\mathcal{A}^r(E) \to \mathcal{A}^{r+1}(E)$ by defining

$$D'(\sigma \otimes \omega) = \sigma \otimes \partial\omega + D'(\sigma) \wedge \omega$$

and

$$D''(\sigma \otimes \omega) = \sigma \otimes \overline{\partial}\omega + D''(\sigma) \wedge \omega$$

for $\sigma \in \mathcal{A}^0(E)$, $\omega \in \mathcal{A}^r(M)$ and linear extension. Then we have

$$D'\big(\mathcal{A}^{s,q}(E)\big) \subset \mathcal{A}^{s+1,q}(E) \quad \text{and} \quad D''\big(\mathcal{A}^{s,q}(E)\big) \subset \mathcal{A}^{s,q+1}(E).$$

Note that (2.12) holds also for these extended operators. Let $R \in \mathcal{A}^2(E^* \otimes E)$ be the curvature of D. Using (2.12) we get

$$R = D' \circ D' + (D' \circ D'' + D'' \circ D') + D'' \circ D'' \tag{2.13}$$

where $D' \circ D' \in \mathcal{A}^{2,0}(E^* \otimes E)$, $D' \circ D'' + D'' \circ D' \in \mathcal{A}^{1,1}(E^* \otimes E)$ and $D'' \circ D'' \in \mathcal{A}^{0,2}(E^* \otimes E)$.

Definition 2.3.6 *A* ***holomorphic vector bundle*** *is a complex vector bundle over a complex manifold which has a system of local trivializations with holomorphic transition functions.*

Example 2.3.7 *The complex tangent bundle $T^{\mathbb{C}}M$ of a complex manifold M is a holomorphic vector bundle.*

Note that we can define a complex manifold structure on the total space of a holomorphic vector bundle $\pi : E \to M$ by requiring the local trivializations in Definition 2.3.6 to be biholomorphic. Then the projection $\pi : E \to M$ becomes a holomorphic map. A local frame for $\pi : E \to M$ over an open $\mathcal{U} \subset M$ is said to be holomorphic if so is the induced map $\pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{C}^n$ with respect to the complex structure on E just defined. Let $u = (u_1, \cdots, u_n)$ be a local holomorphic frame for E over an open $\mathcal{U} \subset M$. Then a section $\sigma = \sum_{\alpha=1}^n v_\alpha u_\alpha$ of E over $\mathcal{U}$ is holomorphic if and only if so are the functions $v_1, \ldots, v_n : \mathcal{U} \to \mathbb{C}$. If u' is another local holomorphic frame for E over $\mathcal{U}$, then we can write $u' = ua$, where $a : \mathcal{U} \to GL(n, \mathbb{C})$ is a holomorphic map.

Let $\pi : E \to M$ be a holomorphic vector bundle. We can define an exterior derivative $\overline{\partial}$ on the space of differential forms with values in E as follows. Every $\xi \in \mathcal{A}^r(E)$ can be written locally in the form $\xi = \sum_{\alpha=1}^n u_\alpha \otimes \omega_\alpha$, where $u = (u_1, \cdots, u_n)$ is a local holomorphic frame for E and where the ω_α's are local r-forms on M. We define $\overline{\partial}\xi := \sum_{\alpha=1}^n u_\alpha \otimes \overline{\partial}\omega_\alpha$. Note that this definition does not depend on the choice of the local holomorphic frame. Indeed, given another holomorphic frame $u' = ua$, where a is a $GL(n, \mathbb{C})$-valued holomorphic map, we have $\xi = \sum_{\alpha,\beta=1}^n u'_\beta \otimes a_{\beta\alpha}\, \omega_\alpha$ and

$$\sum_{\alpha,\beta=1}^n u'_\beta \otimes \overline{\partial}\,(a_{\beta\alpha}\,\omega_\alpha) = \sum_{\alpha,\beta=1}^n u'_\beta \otimes a_{\beta\alpha}\,\overline{\partial}\omega_\alpha = \sum_{\alpha=1}^n u_\alpha \otimes \overline{\partial}\omega_\alpha\,,$$

since $\overline{\partial} a_{\beta\alpha} = 0$. In particular, the local definitions of $\overline{\partial}\xi$ glue together to give a well-defined differential form. Note that $\overline{\partial}$ satisfies the Leibnitz rule

$$\overline{\partial}\,(f\sigma) = \sigma \otimes \overline{\partial} f + f\,\overline{\partial}\sigma \tag{2.14}$$

for $f \in \mathcal{C}^\infty(M, \mathbb{C})$ and $\sigma \in \mathcal{A}^0(E)$, that $\overline{\partial}^{\,2} = 0$ and that $\overline{\partial}\left(\mathcal{A}^{s,q}(E)\right) \subset \mathcal{A}^{s,q+1}(E)$. Observe also that a section σ of E is holomorphic if and only if $\overline{\partial}\sigma = 0$, thus $\overline{\partial}$ determines the holomorphic structure of $\pi : E \to M$.

Definition 2.3.8 *A* ***semiconnection*** *on a complex vector bundle $\pi : E \to M$ over a complex manifold is an operator $\delta : \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E)$ which satisfies the Leibnitz rule (2.14).*

We will denote by $\overline{\mathcal{D}}(E)$ the space of semiconnections on $\pi : E \to M$. Note that $\overline{\mathcal{D}}(E)$ is an affine space modeled on $\mathcal{A}^{0,1}(E^* \otimes E)$ (the proof of this is similar to that of Proposition 1.3.2), and that every semiconnection $\delta \in \overline{\mathcal{D}}(E)$ can be extended to operators $\delta : \mathcal{A}^{s,q}(E) \to \mathcal{A}^{s,q+1}(E)$ by forcing the Leibnitz rule, just in the same way we have done for connections (see page 34).

Definition 2.3.9 *A semiconnection δ on a complex vector bundle over a complex manifold is said to be* integrable *if $\delta \circ \delta = 0$.*

Let $\pi : E \to M$ be a complex vector bundle over a complex manifold. Note that the operator $\overline{\partial}$ associated to a holomorphic structure on $\pi : E \to M$ is an integrable semiconnection. Conversely, we have the following theorem (see for a proof [1, Theorem 5.1]).

Theorem 2.3.10 *Suppose we have an integrable semiconnection δ on a complex vector bundle $\pi : E \to M$ over a complex manifold. Then there exists a unique structure of a holomorphic vector bundle on E such that $\delta = \overline{\partial}$.*

From Theorem 2.3.10 it follows that we can identify integrable semiconnections on a complex vector bundle over a complex manifold with holomorphic structures on it.

We conclude this section by proving the *Chern correspondence* in the vector bundle case.

Proposition 2.3.11 *Let $\pi : E \to M$ be a complex vector bundle over a complex manifold and let h be an Hermitian metric on it. Then the map $D \mapsto D''$ gives a bijection between the space of h-connections on $\pi : E \to M$ and $\overline{\mathcal{D}}(E)$. Under this bijection, integrable semiconnections on $\pi : E \to M$ correspond precisely to h-connections with curvature of type $(1,1)$.*

Proof Observe first that if D is an h-connection then by (2.11) we have

$$\partial h(\sigma_1, \sigma_2) = h\left(\sigma_1, D''(\sigma_2)\right) + h\left(D'(\sigma_1), \sigma_2\right) \tag{2.15}$$

for σ_1,$\sigma_2 \in \mathcal{A}^0(E)$. This shows that an h-connection D is uniquely determined by D'', and thus that the map $D \to D''$ is injective.
To show surjectivity, let D'' be a semiconnection on $\pi : E \to M$ and define D' by (2.15). Then $D := D' + D''$ is easily seen to be an h-connection.
For the last statement, recall that $D'' \circ D''$ is the $(0,2)$-component of the curvature R_D of a connection D (see (2.13)). From this it follows immediately that if R_D has type $(1,1)$ then D'' is integrable. Conversely, suppose that D'' is integrable. Then R_D has no $(0,2)$-component. Applying ∂ to both sides of (2.15) we get

$$0 = h\left(\sigma_1, D'' \circ D''(\sigma_2)\right) + h\left(D' \circ D'(\sigma_1), \sigma_2\right)$$

for all σ_1,$\sigma_2 \in \mathcal{A}^0(E)$, thus we see that the $(2,0)$-component $D' \circ D'$ of R_D also vanishes, so R_D is a $(1,1)$-form. □

Note that by Theorem 2.3.10 the bijection in Proposition 2.3.11 induces a 1-1 correspondence between holomorphic structures on $\pi : E \to M$ and h-connections with curvature of type $(1,1)$.

Definition 2.3.12 *Given an Hermitian metric h on a complex vector bundle $\pi : E \to M$ over a complex manifold, the h-connection corresponding to a semiconnection δ is called the* ***Chern connection*** *on $\pi : E \to M$ with respect to the metric h and the semiconnection δ, and is denoted by $D_{h,\delta}$.*

We will see in Chapter 3 how the 1-1 correspondences in Proposition 2.3.11 generalize to principal fibre bundles.

2.4 Complexification of Lie algebras and Lie groups

Let $\mathfrak{g}$ be a complex Lie algebra, i.e. a complex vector space with a Lie bracket operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is bilinear over $\mathbb{C}$, antisymmetric and which satisfies the Jacobi identity. Then multiplication by i induces on the underlying real vector space a complex structure J such that

$$\big[J(v_1), v_2\big] = J\big([v_1, v_2]\big) \tag{2.16}$$

for all v_1, v_2. Conversely, let $\mathfrak{g}$ be a real Lie algebra with a complex structure J such that (2.16) holds for all v_1, v_2. Then the complex vector space $(\mathfrak{g}, J)$ with the induced Lie bracket becomes a complex Lie algebra.

A complex Lie group G is by definition a complex manifold with a group structure such that the group operations are holomorphic. Since in particular for all $g \in G$ the left translation $L_g : G \to G$ is holomorphic, it follows that if $X \in \Gamma(T^{\mathbb{R}}G)$ is a left invariant vector field, then so is $J(X)$, where J is the natural almost complex structure on the real manifold underlying G. Thus J induces a complex structure on the Lie algebra $\mathfrak{g}^{\mathbb{R}}$ of the real Lie group underlying G. We will now show that condition (2.16) is satisfied, and thus that the complex vector space $(\,\mathfrak{g}^{\mathbb{R}}, J\,)$ has the structure of a complex Lie algebra. We know that for all $g \in G$ the automorphism $c(g) : G \to G$ is holomorphic, thus $\mathrm{Ad}(g)$ is a $\mathbb{C}$-linear automorphism of $(\,\mathfrak{g}^{\mathbb{R}}, J\,)$. From this it follows that $\mathrm{ad}(X)$ is a $\mathbb{C}$-linear endomorphism of $(\,\mathfrak{g}^{\mathbb{R}}, J\,)$ for all $X \in \mathfrak{g}^{\mathbb{R}}$, where $\mathrm{ad} : \mathfrak{g}^{\mathbb{R}} \to \mathrm{End}(\mathfrak{g}^{\mathbb{R}})$ denotes the differential of $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g}^{\mathbb{R}})$ at the identity. Since $\mathrm{ad}(X)\,(Y) = [X, Y]$ (see [31, 3.47]), we get that $\big[X, J(Y)\big] = J\big([X, Y]\big)$ for all $X, Y \in \mathfrak{g}^{\mathbb{R}}$, which is equivalent to (2.16). The complex Lie algebra $(\,\mathfrak{g}^{\mathbb{R}}, J\,)$ will be denoted by $\mathfrak{g}$ and called the Lie algebra of the complex Lie group G. We will often identify it with the holomorphic tangent space of G at the identity, with induced Lie bracket.
We have seen that $\mathrm{ad}\big(J(X)\big)(Y) = \big[J(X), Y\big] = J\big([X, Y]\big) = J\big(\mathrm{ad}(X)(Y)\big)$ for all $X, Y \in \mathfrak{g}^{\mathbb{R}}$. From this it follows that $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is $\mathbb{C}$-linear, and thus that the representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is holomorphic. It can be shown that the exponential map $\exp : \mathfrak{g} \to G$ is also holomorphic, for example by checking that its differential (see [30, 2.14]) is $\mathbb{C}$-linear.

Let $\mathfrak{g}$ be a real Lie algebra. If we define a Lie bracket on the complex vector space $\mathfrak{g}^{\mathbb{C}}$ by $[v_1 \otimes \lambda_1, v_2 \otimes \lambda_2] := [v_1, v_2] \otimes \lambda_1\lambda_2$, then $\mathfrak{g}^{\mathbb{C}}$ becomes a complex Lie algebra, called the *complexification* of $\mathfrak{g}$. Let now $\mathfrak{g}$ be a complex Lie algebra and denote by $\mathfrak{g}^{\mathbb{R}}$ the underlying real Lie algebra. A *real form* of $\mathfrak{g}$ is a Lie subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}^{\mathbb{R}}$ such that ${\mathfrak{g}_0}^{\mathbb{C}} = \mathfrak{g}$, in the sense that the inclusion $\mathfrak{g}_0 \hookrightarrow \mathfrak{g}$ induces a $\mathbb{C}$-linear isomorphism ${\mathfrak{g}_0}^{\mathbb{C}} \cong \mathfrak{g}$. Observe that the conjugation on $\mathfrak{g}$ with respect to a real form $\mathfrak{g}_0$ (i.e. the conjugation on $\mathfrak{g} = {\mathfrak{g}_0}^{\mathbb{C}}$ defined in Section 2.1) is an $\mathbb{R}$-linear automorphism σ of $\mathfrak{g}$ such that $\sigma^2 = \mathrm{id}_{\mathfrak{g}}$, $\sigma(\lambda v) = \bar{\lambda}\,\sigma(v)$ for $\lambda \in \mathbb{C}$, $v \in \mathfrak{g}$ and $\sigma\big([v_1, v_2]\big) = \big[\sigma(v_1), \sigma(v_2)\big]$ for v_1, $v_2 \in \mathfrak{g}$. Conversely, given an $\mathbb{R}$-linear automorphism σ of $\mathfrak{g}$ which satisfies the three conditions above, the set

$\mathfrak{g}_0 := \{ v \in \mathfrak{g} \ / \ \sigma(v) = v \}$ is a real form of $\mathfrak{g}$ and σ is the conjugation with respect to $\mathfrak{g}_0$. Note that if we drop the conditions involving the Lie algebra structure in the discussion above, we get a 1-1 correspondence between real forms of a complex vector space W (i.e. real subspaces V such that $V^{\mathbb{C}} = W$) and conjugations on W (i.e. automorphisms σ of W with $\sigma^2 = \mathrm{id}_W$ and $\sigma(\lambda w) = \bar{\lambda}\,\sigma(w)$ for $\lambda \in \mathbb{C}$, $w \in W$).

Example 2.4.1 *$X \mapsto \bar{X}$ and $X \mapsto -\bar{X}^t$ are conjugations on the complex Lie algebra $\mathfrak{gl}(n,\mathbb{C})$ and the corresponding real forms are respectively $\mathfrak{gl}(n,\mathbb{R})$ and $\mathfrak{u}(n)$.*

From now on, all Lie groups are assumed to be connected.

Definition 2.4.2 *A **real form** of a complex Lie group G is a real Lie subgroup of G whose Lie algebra is a real form of the Lie algebra of G. A **complex reductive Lie group** is a complex Lie group which has a compact real form.*

Example 2.4.3 *From Example 2.4.1 we see that $GL(n,\mathbb{R})$ and $U(n)$ are real forms of $GL(n,\mathbb{C})$; since $U(n)$ is compact (see [3, Theorem 1, §1 of Chapter 1]), it follows that $GL(n,\mathbb{C})$ is a complex reductive Lie group.*

Theorem 2.4.4 *Let K be a compact real Lie group. Then there exists a unique complex reductive Lie group G (called the **complexification** of K) such that K is a real form of G. Moreover, K is a maximal compact Lie subgroup of G and any other real form K' of G is of the form $K' = gKg^{-1}$, for some $g \in G$.*

For a proof of this, see [25, Theorem 2.7 of Chapter 4] and [10, Theorem 5.1 of Chapter XVII].

From now on, till the end of the section, G will be a complex reductive Lie group, K a compact real form of G , $\mathfrak{k}$ and $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ the Lie algebras of K and G.

Note that in particular K is a closed Lie subgroup of G [5], thus ${G}/{K}$ has a natural manifold structure making the projection $G \to {G}/{K}$ smooth (see p. 28). It turns out that the map $i\mathfrak{k} \to {G}/{K}$, $B \mapsto \exp(B)\,K$ is a diffeomorphism: it follows easily form the next theorem, which is proved in [8, 2.1].

Theorem 2.4.5 *Let $P := \exp(i\mathfrak{k})$. Then P is a closed submanifold of G and the maps $\exp : i\mathfrak{k} \to P$ and $K \times P \to G$, $(k,g) \mapsto kg$ are diffeomorphisms.*

Example 2.4.6 *Consider the complex reductive Lie group $GL(n,\mathbb{C})$ with the compact real form $U(n)$. Then $i\mathfrak{u}(u)$ and $\exp\big(i\mathfrak{u}(u)\big)$ are respectively the spaces of Hermitian matrices and of positive definite Hermitian matrices. Every $A \in GL(n,\mathbb{C})$*

[5] Continuous images of compact spaces are compact and compact subspaces of Hausdorff spaces are closed.

*can be written uniquely in the form $A = XY$, with $X \in U(n)$ and $Y \in \exp\big(i\mathfrak{u}(u)\big)$ (*polar decomposition of $GL(n,\mathbb{C})$*)* [6]*. The maps* $\exp : i\mathfrak{u}(u) \to \exp\big(i\mathfrak{u}(u)\big)$ *and $U(n) \times \exp\big(i\mathfrak{u}(u)\big) \to GL(n,\mathbb{C})$, $(X,Y) \mapsto XY$ are diffeomorphisms. This all can be also proved directly, by concrete calculations with matrices (see [3, §IV and § V of Chapter 1]).*

We want to define a conjugation, which will be used in the proof of Theorem 3.2.5, on the holomorphic tangent spaces of G at points of $P = \exp(i\mathfrak{k})$. We will first define a natural conjugation on the holomorphic tangent bundle of $GL(n,\mathbb{C})$, and then generalize it to G.

Consider first the conjugation σ on the complex vector space $\mathfrak{gl}\,(n,\mathbb{C})$ given by $X \mapsto \bar{X}^t$. The corresponding real form is the real subspace $i\mathfrak{u}(n)$ of Hermitian matrices [7]. Note that $\sigma(XY) = \sigma(Y)\,\sigma(X)$.
Since $GL(n,\mathbb{C})$ is an open submanifold of $\mathfrak{gl}\,(n,\mathbb{C})$, for every $A \in GL(n,\mathbb{C})$ we can identify $T^{\mathbb{C}}_A\, GL(n,\mathbb{C})$ with $T^{\mathbb{C}}_A\, \mathfrak{gl}\,(n,\mathbb{C}) = \mathfrak{gl}\,(n,\mathbb{C})$ and get thus a conjugation σ on the holomorphic tangent bundle of $GL(n,\mathbb{C})$. Then it holds

$$\sigma\left((L_{A^{-1}})_*\,(X)\right) = (R_{A^{-1}})_*\big(\sigma(X)\big)$$

for $A \in \exp\big(i\mathfrak{u}(n)\big)$ and $X \in T^{\mathbb{C}}_A\, GL(n,\mathbb{C})$. To see this, let $t \mapsto A_t$ be a curve in $GL(n,\mathbb{C})$ with $A_0 = A$ and $\frac{d}{dt}\Big|_{t=0} A_t = X$; then

$$(L_{A^{-1}})_*\,(X) = \frac{d}{dt}\Big|_{t=0} A^{-1}A_t = A^{-1}\,\frac{d}{dt}\Big|_{t=0} A_t = A^{-1}X. \tag{2.17}$$

Similarly, $(R_{A^{-1}})_*\big(\sigma(X)\big) = \sigma(X)A^{-1}$. Since $A^{-1} \in \exp\big(i\mathfrak{u}(n)\big)$ is Hermitian (see Example 2.4.6), we have

$$\sigma\left((L_{A^{-1}})_*\,(X)\right) = \sigma\left(A^{-1}X\right) = \sigma(X)\,\sigma(A^{-1}) = \sigma(X)\,A^{-1} = (R_{A^{-1}})_*\big(\sigma(X)\big).$$

Consider now the complex reductive Lie group G. Define a conjugation σ on the complex vector space $\mathfrak{g} = T_eG$ to be the reflexion about the real form $i\mathfrak{k}$ and on T_gG, for $g \in P = \exp(i\mathfrak{k})$, to be the map $X \mapsto (R_g)_*\Big(\sigma\left((L_{g^{-1}})_*(X)\right)\Big)$.

Lemma 2.4.7 *For $g \in P = \exp(i\mathfrak{k})$, the conjugation σ on T_gG defined above is the reflexion about the tangent space of P at g.*

To prove this, we will need the following theorem and lemma.

[6] Note that for $n = 1$ this reduces to the representation of a complex number as the product of its absolute value and a unitary number $e^{i\varphi}$.

[7] Note that it is a natural choice to consider $i\mathfrak{u}(n)$ instead of $\mathfrak{u}(n)$ as real form of the *vector space* $\mathfrak{gl}\,(n,\mathbb{C})$, since for every Hermitian matrix $X \in i\mathfrak{u}(n)$ there exists a unitary matrix $A \in U(n)$ such that AXA^{-1} is diagonal with real eigenvalues. Note also that $\mathfrak{gl}(1,\mathbb{C}) = \mathbb{C}$ and $i\mathfrak{u}(1) = \mathbb{R}$. Observe, on the other hand, that $i\mathfrak{u}(n)$ is not a real form of the *Lie algebra* $\mathfrak{gl}\,(n,\mathbb{C})$.

Theorem 2.4.8 *Every complex reductive Lie group G with compact real form K has a holomorphic faithful representation $\varrho : G \hookrightarrow GL(n, \mathbb{C})$ with $\varrho(K) \subset U(n)$.*

Proof It is a classical result that a compact Lie group K can be embedded in some $U(n)$ by a faithful representation $\varrho_0 : K \hookrightarrow U(n)$ (see [26, Theorem 6.1.1]) and it can be shown that ϱ_0 can be uniquely extended to a holomorphic faithful representation $\varrho : G \hookrightarrow GL(n, \mathbb{C})$ (see [12, Remark after Proposition 7.12] and [10, Theorem 5.2 of Chapter XVII]). $\square$

Lemma 2.4.9 *Let $\varrho : G \hookrightarrow GL(n, \mathbb{C})$ be a representation as in Theorem 2.4.8 and let σ be the conjugation on the holomorphic tangent bundle of $GL(n, \mathbb{C})$ and on T_gG (for $g \in P = \exp(i\mathfrak{k})$) defined above. Then for $X \in T_gG$ we have*

$$\varrho_* \big(\sigma(X)\big) = \sigma\big(\varrho_*(X)\big).$$

Proof Consider first a vector $Y \in T_eG = \mathfrak{g}$; write $Y = Y_1 + i\,Y_2$, with Y_1, $Y_2 \in i\mathfrak{k}$. Then, since ϱ is holomorphic, we have

$$\varrho_* \big(\sigma(Y)\big) = \varrho_* (Y_1 - i\,Y_2) = \varrho_* (Y_1) - i\varrho_*(Y_2) = \sigma\big(\varrho_*(Y_1) + i\varrho_*(Y_2)\big) = \sigma\big(\varrho_*(X)\big).$$

Using this and (2.17) we get, for $X \in T_gG$

$$\begin{aligned}
\varrho_* \big(\sigma(X)\big) &= \varrho_* \Big((R_g)_* \Big(\sigma\big((L_{g^{-1}})_*(X)\big)\Big)\Big) = (R_{\varrho(g)})_* \Big(\varrho_* \Big(\sigma\big((L_{g^{-1}})_*(X)\big)\Big)\Big) \\
&= (R_{\varrho(g)})_* \Big(\sigma\Big(\varrho_*\big((L_{g^{-1}})_*(X)\big)\Big)\Big) \\
&= (R_{\varrho(g)})_* \Big(\sigma\Big((L_{\varrho(g^{-1})})_*\big(\varrho_*(X)\big)\Big)\Big) = \sigma\big(\varrho(g^{-1})\varrho_*(X)\big)\,\varrho(g) \\
&= \sigma\big(\varrho_*(X)\big)\,\sigma\big(\varrho(g^{-1})\big)\,\varrho(g) = \sigma\big(\varrho_*(X)\big)
\end{aligned}$$

where the last equality follows from the fact that $\varrho(g^{-1}) \in \exp\big(i\mathfrak{u}(n)\big)$ is an Hermitian matrix. $\square$

Proof of Lemma 2.4.7 Let $g = \exp(X)$, for some $X \in i\mathfrak{k}$. Since $\exp : i\mathfrak{k} \to P$ is a diffeomorphism (see Theorem 2.4.5), we have a vector space isomorphism $\exp_*$ between $T_X i\mathfrak{k}$ and T_gP. Thus we have to show that $\sigma\big(\exp_*(Y)\big) = \exp_*(Y)$ for all $Y \in T_X i\mathfrak{k}$.
Consider first the conjugation on the holomorphic tangent bundle of $GL(n, \mathbb{C})$. Let $A \in i\mathfrak{u}(n)$, $B \in i\mathfrak{u}(n) = T_A\big(i\mathfrak{u}(n)\big)$ and write $B = \frac{d}{dt}\Big|_{t=0}(A + tB)$. Then $\exp_*(B) = \frac{d}{dt}\Big|_{t=0}\exp(A + tB) = \lim_{t\to 0} \frac{\exp(A+tB)-\exp(A)}{t}$. Since $\frac{\exp(A+tB)-\exp(A)}{t}$ is an Hermitian matrix for all t, we have $\sigma\big(\exp_*(B)\big) = \exp_*(B)$.
Let now $Y \in T_X(i\mathfrak{k})$ and consider a representation $\varrho : G \hookrightarrow GL(n, \mathbb{C})$ as in Theorem 2.4.8. Then $\varrho_*\big(\exp_*(Y)\big) \in \exp_*\big(i\mathfrak{u}(n)\big)$, thus $\sigma\Big(\varrho_*\big(\exp_*(Y)\big)\Big) = \varrho_*\big(\exp_*(Y)\big)$. But $\sigma\Big(\varrho_*\big(\exp_*(Y)\big)\Big) = \varrho_*\Big(\sigma\big(\exp_*(Y)\big)\Big)$ by Lemma 2.4.9, thus $\sigma\big(\exp_*(Y)\big) = \exp_*(Y)$, as we wanted. $\square$

2.5 Complex principal fibre bundles

A principal fibre bundle is said to be a **complex principal fibre bundle** if the base space is a complex manifold and the structure group a complex Lie group. When we apply results of Paragraph 1.2 to complex principal fibre bundles, we always mean that we apply them to the underlying real principal fibre bundles. For example, by a reduction of a complex principal fibre bundle $P\,(M,G)$ we mean a principal fibre bundle $Q\,(M,H)$, where H is a Lie subgroup of the real Lie group underlying G, with a smooth map $f : P \to Q$ and a Lie group monomorphism $f' : H \to G$, as required in Definition 1.2.4.
All examples in Paragraph 1.2 which describe relations between real vector bundles and real principal fibre bundles can be modified in order to obtain relations between complex vector bundles over complex manifolds and complex principal fibre bundles. So, by substituting $\mathbb{R}$ with $\mathbb{C}$ in Examples 1.2.2, 1.2.10 and 1.2.11, we see that there is a 1-1 correspondence between complex vector bundles (over complex manifolds) and (complex) principal fibre bundles with structure group $GL(n,\mathbb{C})$ (in Example 1.2.10, V has to be a complex vector space and $\mathrm{Aut}(V)$ the group of $\mathbb{C}$-linear automorphisms of V).
Given a complex principal fibre bundle $P\,(M,G)$, we can define the adjoint bundle $P \times_{Ad} \mathfrak{g}^{\mathbb{R}}$, as done in Example 1.2.13. If we consider the Lie algebra of G as a complex vector space, then $P \times_{Ad} \mathfrak{g}$ becomes a complex vector bundle. Let $\varrho : G \to \mathrm{Aut}(V)$ be a holomorphic representation of G on a complex vector space V. Then $\varrho_* : \mathfrak{g} \to \mathrm{End}(V)$ is a $\mathbb{C}$-linear homomorphism and the map $\phi : P \times_{Ad} \mathfrak{g} \to E^* \otimes E$ defined in Example 1.2.13, where $E = P \times_G V$, is a homomorphism of complex vector bundles. In particular, for every complex vector bundle $\pi : E \to M$ we have

$$L(E) \times_{Ad} \mathfrak{gl}\,(n,\mathbb{C}) \cong E^* \otimes E.$$

In Example 1.2.17, we substitute $O(n)$ and $GL\,(n,\mathbb{R})$ with $U(n)$ and $GL(n,\mathbb{C})$ and "Riemannian metric" with "Hermitian metric". Then for every complex vector bundle $\pi : E \to M$ we get 1-1 correspondences

$$\{\,\text{Hermitian metrics on } E\,\} \;\overset{1\text{-}1}{\longleftrightarrow}\; \{\,\text{reductions of } L(E) \text{ to } U(n)\,\}/\sim$$
$$\overset{1\text{-}1}{\longleftrightarrow}\; \{\,\text{sections of } L(E)/U(n)\,\}. \tag{2.18}$$

We will denote by $U_h(E)$ the bundle of h-orthonormal frames of a complex vector bundle $\pi : E \to M$ with an Hermitian metric h.

Let $P\,(M,G)$ be a complex principal fibre bundle and let A be a connection on the underlying real principal fibre bundle. For every $p \in M$ and $u \in \pi^{-1}(p)$, we can extend the horizontal lift $T_p^{\mathbb{R}}M \to T_uP$, $Y \mapsto \big(\hat{Y}_u^h\big)_A$ to the complexifications T_pM and $\big(T_uP\big)^{\mathbb{C}}$ by defining

$$\big(\,(\widehat{Y \otimes \lambda})_u^h\,\big)_A := \big(\hat{Y}_u^h\big)_A \otimes \lambda$$

for $Y \in T_p^{\mathbb{R}}M$ and $\lambda \in \mathbb{C}$ and linear extension. Similarly, we can do so for the horizontal and vertical projections $T_uP \to T_uP$, $X \mapsto \big(X_u^h\big)_A$ and $X \mapsto \big(X_u^v\big)_A$. Observe that Lemma 1.4.2 also holds for these extended maps.
Consider the connection form ω_A of A. It is a 1-form on P with values in the real vector space $\mathfrak{g}^{\mathbb{R}}$ and it extends to a complex $\mathfrak{g}$-valued 1-form on P by complex linearity (i.e. for $X \otimes \lambda \in TP^{\mathbb{C}}$ we define $\omega_A(X \otimes \lambda) := \lambda\, \omega_A(X)$). Note that properties 1. and 2. in Lemma 1.4.3 also hold for the complex form ω_A. In 2., ${R_g}^*$ has to be interpreted as the $\mathbb{C}$-linear extension to the space of complex differential forms (and the property holds because $\mathrm{Ad}(g^{-1})$ is a $\mathbb{C}$-linear automorphism of $\mathfrak{g}$). The formulation of 1. remains the same as in the real case (indeed, $(B^*)_u$ is a real vector). Conversely, given a complex $\mathfrak{g}$-valued 1-form ω on P satisfying properties 1. and 2., we can restrict it to a real $\mathfrak{g}^{\mathbb{R}}$-valued 1-form which still satisfies properties 1. and 2. and thus determines a connection A_ω on $P\,(M,G)$. Then the complex connection form of A_ω is the initial ω. Thus we see that there is a 1-1 correspondence between the set of connections on a complex principal fibre bundle $P\,(M,G)$ and the set of complex $\mathfrak{g}$-valued 1-forms on P satisfying properties 1. and 2. of Lemma 1.4.3. In the following, by connection form of a connection on a complex principal fibre bundle we will always mean the complex form. Observe that Lemma 1.4.4 and Proposition 1.4.5 also hold in the complex case (after extending the Maurer-Cartan form to a complex 1-form on G). In particular, given a system of local trivializations $\{\,\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times G\ ,\ i \in I\,\}$, we can express every connection form ω on $P\,(M,G)$ as a family of local complex $\mathfrak{g}$-valued 1-forms $\{\,\omega_i,\, i \in I\,\}$ on M.
In Definition 1.4.6 and Lemma 1.4.7, we can consider a complex vector space V and the complex vector bundle $P \times_G V$. Then we get a 1-1 correspondence between $\mathcal{A}^r(P \times_G V)$ and the set of horizontal G-equivariant V-valued complex r-forms on $P\,(M,G)$. Using this, we can extend Proposition 1.4.8 to the complex case. Similarly, all other results of §1.4 are easily seen to hold also for complex principal fibre bundles (but note that for all facts proved in the Appendix we need the representation $\varrho : G \to \mathrm{Aut}(V)$ to be holomorphic). In Example 1.4.22 we consider a complex vector bundle $\pi : E \to M$ with an Hermitian metric h and substitute $O(n)$, $GL\,(n,\mathbb{R})$ and $O_h(E)$ with $U(n)$, $GL(n,\mathbb{C})$ and $U_h(E)$.

The following definitions and results are specific of the complex case.

Definition 2.5.1 *An* ***almost holomorphic structure*** *on a complex principal fibre bundle $P\,(M,G)$ is an almost complex structure J in the total space P such that the projection $\pi : P \to M$ and the action $\lambda : P \times G \to P$ are almost holomorphic, i.e.*

1. $\pi_* \circ J = J_M \circ \pi_*$
2. $J\big(\lambda_*(Y,B)\big) = \lambda_*\big(J(Y), J_G(B)\big)$ *for $u \in P$, $Y \in T_uP$ and $g \in G$, $B \in T_gG$*

where J_M and J_G are the natural almost complex structures of the complex manifolds M and G.

Applying the Leibnitz rule to the map $\lambda : P \times G \to P$ on both sides of 2., we obtain

$$J\big((R_g)_*(Y)\big) + J\Big(\big(\omega_G(B)^*\big)_u\Big) = (R_g)_*\big(J(Y)\big) + \Big(\omega_G\big(J_G(B)\big)^*\Big)_u \tag{2.19}$$

for $u \in P$, $Y \in T_uP$ and $g \in G$, $B \in T_gG$, where ω_G is the Maurer-Cartan form on G. In particular, for $B = 0 \in T_gG$ this gives

$$J\big((R_g)_*(Y)\big) = (R_g)_*\big(J(Y)\big) \tag{2.20}$$

for $u \in P$, $Y \in T_uP$, thus the almost holomorphic structure J is G-invariant. Moreover, from formulas (2.19) and (2.20) we see that condition 2. is equivalent to

$$J\big((B^*)_u\big) = \big(J_G(B)^*\big)_u \tag{2.21}$$

for all $B \in \mathfrak{g}$ and $u \in P$, i.e. to requiring the restriction of J to the vertical subspaces to coincide with the almost complex structure induced on them by the identification of their tangent spaces with $\mathfrak{g}$.

We will denote the set of almost holomorphic structures on a complex principal fibre bundle $P\,(M, G)$ by $\overline{\mathcal{C}}(P)$.

Definition 2.5.2 *A **holomorphic structure** on a complex principal fibre bundle is an integrable almost holomorphic structure on it.*

Using Theorem 2.2.7 and Proposition 2.2.5, we see that a holomorphic structure on a complex principal fibre bundle $P\,(M, G)$ can be equivalently defined as a complex manifold structure on the total space P such that the projection $\pi : P \to M$ and the action $P \times G \to P$ are holomorphic. A third equivalent description is given by the following proposition.

Proposition 2.5.3 *A complex principal fibre bundle admits a holomorphic structure if and only if it has a system of local trivializations with holomorphic transition functions.*

Proof Suppose that $P\,(M, G)$ is a complex principal fibre bundle which has a system of local trivializations with holomorphic transition functions. We can define a complex manifold structure on P by requiring these local trivializations to be biholomorphic. Then the projection $\pi : P \to M$ and the action $P \times G \to P$ become holomorphic maps, thus we get a holomorphic structure on $P\,(M, G)$.
Conversely, suppose that the total space P of a complex principal fibre bundle is a complex manifold and that the projection $\pi : P \to M$ and the action $P \times G \to P$ are holomorphic. Let $\dim_{\mathbb{C}} M = n$ and $\dim_{\mathbb{C}} G = r$. Since $\pi : P \to M$ is a holomorphic map of constant rank n, for every point u_0 of P we can find complex charts $(\mathcal{V}, \varphi)$ for P around u_0 and $(\mathcal{U}, \psi)$ for M around $p_0 := \pi(u_0)$ with $\varphi(u_0) = 0$, $\psi(p_0) = 0$, $\pi(\mathcal{V}) \subset \mathcal{U}$ and such that $\psi \circ \pi \circ \varphi^{-1} : \mathbb{C}^{n+r} \to \mathbb{C}^n$ is the projection

$(x, y) \mapsto (x, 0)$ [8]. Define a local section of P around p_0 by $p \mapsto \varphi^{-1}\Big((\psi(p), 0)\Big)$. Since this can be done around each point of M, we get a system of local holomorphic sections of $P\,(M, G)$, thus a system of local trivializations with holomorphic transition functions. $\square$

We will denote the set of holomorphic structures on a complex principal fibre bundle $P\,(M, G)$ by $\mathcal{C}(P)$.

Example 2.5.4 *Let $\pi : E \to M$ be a complex vector bundle over a complex manifold and let $L(E)$ be its frame bundle. Since every system of local trivializations of E induces a system of local trivializations of $L(E)$ with the same transition functions and vice versa, we see that $L(E)$ has a holomorphic structure if and only if $\pi : E \to M$ is a holomorphic vector bundle. More generally, let $P\,(M, G)$ be a complex principal fibre bundle and assume we have a holomorphic representation $\varrho : G \to \mathrm{Aut}(V)$ of G on a complex vector space V. Suppose that $P\,(M, G)$ has a holomorphic structure and let $\{\, \theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times G \;,\; i \in I \,\}$ be a system of local trivializations with holomorphic transition functions $\{\, \theta_{ij} : \mathcal{U}_{ij} \to G \,\}$. Then the transition functions $\{\, \theta^E_{ij} = \alpha'_v \circ \varrho \circ \theta_{ij} : \mathcal{U}_{ij} \to GL(n, \mathbb{C}) \,\}$ of the induced local trivializations on the associated complex vector bundle $E = P \times_G V$ (with respect to a basis v of V) are also holomorphic, thus $E = P \times_G V$ is a holomorphic vector bundle.*

The previous example can be generalized even more, as explained in the next proposition.

Proposition 2.5.5 *Let $P\,(M, G)$ be a complex principal fibre bundle and suppose that we have a holomorphic representation $\varrho : G \to \mathrm{Aut}(V)$ of G on a complex vector space V. Then every almost holomorphic structure J on $P\,(M, G)$ induces a semiconnection δ_J on the associated vector bundle $E = P \times_G V$, which is defined as follows. Let $Y \in \Gamma(TM)$ and $\sigma \in \mathcal{A}^0(E)$. Denote by $\hat{\sigma} : P \to V$ the G-equivariant map corresponding to σ in the sense of Lemma 1.2.9. Then $\delta_J(\sigma)\,(Y) \in \mathcal{A}^0(E)$ is defined by*

$$\delta_J(\sigma)\,(Y)\,(p) := \Big(u, \widehat{Y}_u^{\;0,1}\,(\hat{\sigma}) \Big)_{/\sim}$$

for $p \in M$, where u is some element of the fibre of P over p and $\widehat{Y}_u$ is any vector of T_uP with $\pi_*(\widehat{Y}_u) = Y_p$.

Proof We will first show that the map $M \to E$, $p \mapsto \Big(u, \widehat{Y}_u^{\;0,1}\,(\hat{\sigma}) \Big)_{/\sim}$ is well-defined. Two vectors of T_uP that both project to Y_p differ by a vertical vector

[8] A proof of an analogous statement for real manifolds and smooth maps can be found for example in [27, Theorem 1.31 of Chapter 1]. The complex case can be proved similarly, using the Inverse Function Theorem for holomorphic functions between open subsets of $\mathbb{C}^n$ (see [6, p.18]).

$(B^*)_u$, where B is some element of $\mathfrak{g}$. But by (2.1), (2.21) and Lemma A.1, and since ϱ is holomorphic, we have

$$\begin{aligned}
{(B^*)_u}^{0,1}(\hat\sigma) &= \left(\frac{(B^*)_u + i\,J\big((B^*)_u\big)}{2}\right)(\hat\sigma) = \left(\frac{(B^*)_u + i\left(J_G(B)^*\right)_u}{2}\right)(\hat\sigma) \\
&= \frac{-\,\varrho_*(B)\left(\hat\sigma(u)\right) - i\,\varrho_*\left(J_G(B)\right)\left(\hat\sigma(u)\right)}{2} \\
&= \frac{-\,\varrho_*(B)\left(\hat\sigma(u)\right) + \varrho_*(B)\left(\hat\sigma(u)\right)}{2} = 0\,.
\end{aligned}$$

This shows that the definition of $p \mapsto \left(u, {\widehat{Y}_u}^{0,1}(\hat\sigma)\right)_{/\sim}$ does not depend on the choice of $\widehat{Y}_u \in T_uP$. It also does not depend on the choice of u in the fibre of P over p, since for $g \in G$ we can take ${\widehat{Y}_{ug}}^{0,1} = (R_g)_*(\widehat{Y}_u)$, and so by (2.20) we have

$$\begin{aligned}
{\widehat{Y}_{ug}}^{0,1}(\hat\sigma) &= \big((R_g)_*(\widehat{Y}_u)\big)^{0,1}(\hat\sigma) = (R_g)_*\left({\widehat{Y}_u}^{0,1}\right)(\hat\sigma) \\
&= {\widehat{Y}_u}^{0,1}(\hat\sigma\circ R_g) = {\widehat{Y}_u}^{0,1}\left(\varrho\,(g^{-1})\circ\hat\sigma\right) = \varrho\,(g^{-1})\left({\widehat{Y}_u}^{0,1}(\hat\sigma)\right)
\end{aligned}$$

where the last equality follows from the fact that $\varrho\,(g^{-1}) : V \to V$ is linear.

The map $\delta_J(\sigma) : \Gamma(TM) \to \mathcal{A}^0(E)$ is $\mathcal{C}^\infty(M,\mathbb{C})$-linear, thus $\delta_J(\sigma) \in \mathcal{A}^1(E)$. In fact, we have $\delta_J(\sigma) \in \mathcal{A}^{0,1}(E)$ because if $Y \in T'_pM$ then by Lemma 2.1.3 we have $Y = Z - i\,J_M(Z)$, for some $Z \in T^{\mathbb{R}}_pM$ and thus, since

$$\pi_*\left(\hat Z_u - i\,J(\hat Z_u)\right) = Z - \pi_*\left(i\,J(\hat Z_u)\right) = Z - i\,J_M(Z) = Y$$

by condition 1. of Definition 2.5.1, we have

$$\delta_J(\sigma)\,(Y) = \left(u, \left(\hat Z_u - i\,J(\hat Z_u)\right)^{0,1}(\hat\sigma)\right)_{/\sim} = 0.$$

To conclude that $\delta_J : \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E)$ is a semiconnection, we have to check that the Leibnitz rule is satisfied. For this, let $\sigma \in \mathcal{A}^0(E)$, $f \in \mathcal{C}^\infty(M,\mathbb{C})$, $Y \in \Gamma(TM)$ and $p \in M$. Then

$$\begin{aligned}
\delta_J\,(f\,\sigma)\,(Y)\,(p) &= \left(u, {\widehat{Y}_u}^{0,1}\left((f\circ\pi)\,\hat\sigma\right)\right)_{/\sim} \\
&= \left(u,\, \hat\sigma(u)\,{\widehat{Y}_u}^{0,1}(f\circ\pi) \;+\; f(p)\,{\widehat{Y}_u}^{0,1}(\hat\sigma)\right)_{/\sim} \\
&= \left(u,\, \hat\sigma(u)\,{Y_p}^{0,1}(f) \;+\; f(p)\,{\widehat{Y}_u}^{0,1}(\hat\sigma)\right)_{/\sim} \\
&= \sigma(p)\,\overline{\partial}f(Y_p) \;+\; f(p)\,\delta_J(\sigma)\,(Y)\,(p)\,,
\end{aligned}$$

i.e. $\delta_J\,(f\sigma) = \sigma\otimes\overline{\partial}f + f\,\delta_J(\sigma)$. □

Note that if $J \in \mathcal{C}(P)$ then Proposition 2.5.5 reduces to the case described in Example 2.5.4. Indeed, we will now show that if $J \in \mathcal{C}(P)$ then $\delta_J = \overline{\partial}$, where $\overline{\partial}$

denotes the operator associated to the holomorphic structure induced by J on E, in the sense of Example 2.5.4. Let $\sigma \in \mathcal{A}^0(E)$ and write it on some $\mathcal{U} \subset M$ as $\sigma = (\, u, x\,)_{/\sim}$, where $u : \mathcal{U} \to P$ is a local holomorphic section of $P\,(M,G)$ and x a function $\mathcal{U} \to V$. We have to prove that $\overline{\partial}\sigma\,(Y) = \delta_J(\sigma)\,(Y)$ for $p \in \mathcal{U}$ and $Y \in T_pM$. Let $(\, f_v, \alpha'_v \circ \varrho\,) : P\,(M,G) \hookrightarrow L(E)\big(M, GL(n,\mathbb{C})\big)$ be the reduction described in Example 1.2.10, with respect to a basis $(v_1, \dots, v_n)$ of V; note that the map $f_v : P \to L(E)$ is holomorphic, with respect to the holomorphic structure induced by J on $\pi : E \to M$ and on $L(E)$, as described in Example 2.5.4. If we identify E with $L(E) \times_{GL(n,\mathbb{C})} \mathbb{C}^n$, then we can write $\sigma = \big(\, f_v(u), \alpha_v(x)\,\big)_{/\sim}$ (where now "$\sim$" denotes the equivalence relation on the set $L(E) \times \mathbb{C}^n$). This is clear from the way the map $f_v : P \to L(E)$ is defined. Observe that $f(u)$ is a local holomorphic frame of $\pi : E \to M$, with respect to the holomorphic structure induced by J. Thus we have

$$\begin{aligned} \overline{\partial}\sigma\,(Y) &= \Big(\, f_v\big(u(p)\big)\,,\, \overline{\partial}\,\alpha_v(x)\,(Y)\,\Big)_{/\sim} \\ &= \Big(\, f_v\big(u(p)\big)\,,\, \alpha_v\big(Y^{0,1}(x)\big)\,\Big)_{/\sim} = \big(\, u(p)\,,\, Y^{0,1}(x)\,\big)_{/\sim} \end{aligned}$$

and

$$\delta_J(\sigma)\,(Y) = \Big(\, u(p)\,,\, u_*(Y)^{\,0,1}\,(\hat{\sigma})\,\Big)_{/\sim}\,.$$

But $Y^{0,1}(x) = Y^{0,1}\,(\hat{\sigma} \circ u) = u_*(Y^{0,1})\,(\hat{\sigma}) = u_*(Y)^{\,0,1}\,(\hat{\sigma})$, because u is holomorphic.

Let $P\,(M,G)$ be a complex principal fibre bundle and let $\mathcal{G} = \mathrm{Aut}(P)$ be its gauge group (see Example 1.2.14). We define an action of $\mathcal{G}$ on $\overline{\mathcal{C}}(P)$ on the right by $(J, \varphi) \mapsto \varphi(J)$, where

$$\varphi(J)\,(X) := {\varphi_*}^{-1}\Big(J\,\big(\varphi_*(X)\big)\Big)$$

for $u \in P$ and $X \in T_uP$.

Proposition 2.5.6 *Let $\mathcal{G}$ be the gauge group of a complex principal fibre bundle $P\,(M,G)$. Then the action $\overline{\mathcal{C}}(P) \times \mathcal{G} \to \overline{\mathcal{C}}(P)$ defined above leaves $\mathcal{C}(P)$ invariant.*

Proof Let $J \in \mathcal{C}(P)$ and $\varphi \in \mathcal{G}$. We have to show that $\varphi(J) \in \mathcal{C}(P)$. Let X_1, $X_2 \in \Gamma(TP)$. A straightforward calculation shows that

$$N_{\varphi(J)}\,(X_1, X_2) = {\varphi_*}^{-1}\Big(N_J\,\big(\,\varphi_*(X_1), \varphi_*(X_2)\,\big)\Big)$$

where N_J and $N_{\varphi(J)}$ are the torsions of J and $\varphi(J)$ respectively (see Definition 2.2.6). Since $N_J = 0$, it follows that $N_{\varphi(J)} = 0$ and thus $\varphi(J) \in \mathcal{C}(P)$. □

Definition 2.5.7 *Two (almost) holomorphic structures J_1 and J_2 on a complex principal fibre bundle $P\,(M,G)$ are* isomorphic *if they are in the same $\mathcal{G}$-orbit with respect to the action $\overline{\mathcal{C}}(P) \times \mathcal{G} \to \overline{\mathcal{C}}(P)$ defined above, i.e. if there exists $\varphi \in \mathcal{G}$ such that $\varphi(J_1) = J_2$ (equivalently, such that $J_1 \circ \varphi_* = \varphi_* \circ J_2$).*

Chapter 3

The Chern correspondence

Throughout this chapter, G will be a complex reductive Lie group with a compact real form K and $P(M,G)$ a complex principal fibre bundle.

In Paragraph 3.1 we will consider a fixed K-reduction $Q(M,K)$ of $P(M,G)$ and we will show that there is a 1-1 correspondence between the set of almost holomorphic structures on $P(M,G)$ and the space of connections on $Q(M,K)$. We will prove that this is an extension of the 1-1 correspondence, described in Proposition 2.3.11, between semiconnections and h-connections on a complex vector bundle with an Hermitian metric h. Using the second part of Proposition 2.3.11 and Theorem 2.3.10 we will then derive that connections on $Q(M,K)$ with curvature form of type $(1,1)$ correspond precisely to the holomorphic structures of $P(M,G)$.

In Paragraph 3.2 we will consider a fixed almost holomorphic structure J on $P(M,G)$ and the corresponding connection, regarded as a connection on $P(M,G)$, with respect to a K-reduction. We will describe the way this connection changes when we vary the reduction.

In this chapter we will always regard K as a subset of G. Note that K will have then the relative topology, because it is a closed subgroup [1]. The Lie algebras of G and K will be denoted respectively by $\mathfrak{g}$ and $\mathfrak{k}$. We will identify $\mathfrak{g}$ with $\mathfrak{k} \oplus i\mathfrak{k}$ and, in particular, denote the almost complex structure J_G of G simply as multiplication by i. Given a reduction $Q(M,K)$ of $P(M,G)$, we will consider the total space Q as a subset (with the relative topology) of P.

3.1 The Chern correspondence

Throughout this section, $Q(M,K)$ will be a fixed K-reduction of $P(M,G)$.

[1] See footnotes 13 of Chapter I and 5 of Chapter II.

Theorem 3.1.1 *Let J be an almost holomorphic structure on $P\,(M,G)$. Then the distribution $u \in Q \mapsto T_uQ \cap J\big(T_uQ\big)$ defines a connection A_J on $Q\,(M,K)$. The map $J \mapsto A_J$ gives a 1-1 correspondence (the* ***Chern correspondence****) between the set $\overline{\mathcal{C}}(P)$ of almost holomorphic structures on $P\,(M,G)$ and the space $\mathcal{A}(Q)$ of connections on $Q\,(M,K)$.*

Proof Let $J \in \overline{\mathcal{C}}(P)$ and define $\big(T_u^hQ\big)_{A_J} := T_uQ \cap J\big(T_uQ\big)$ for $u \in Q$. We have to show that the distribution $u \mapsto \big(T_u^hQ\big)_{A_J}$ is smooth and satisfies conditions 1. and 2. of Definition 1.4.1. Observe first that $\big(T_u^hQ\big)_{A_J} \cap T_u^vQ = (0)$. Indeed, suppose that we have an $X \in \big(T_u^hQ\big)_{A_J} \cap T_u^vQ$ and write $X = (B^*)_u$, for some $B \in \mathfrak{k}$. Then by (2.21) we have $J(X) = (\,iB^*\,)_u$. But $J(X) \in T_uQ$ because $X \in J\big(T_uQ\big)$, thus it must be $iB \in \mathfrak{k}$; so $B \in \mathfrak{k} \cap i\mathfrak{k} = (0)$ and $X = 0$. To conclude that $\big(T_u^hQ\big)_{A_J} \oplus T_u^vQ = T_uQ$, observe that

$$\begin{aligned}
\dim\Big(\big(T_u^hQ\big)_{A_J}\Big) &= \dim\Big(T_uQ \cap J\big(T_uQ\big)\Big) \\
&= \dim\big(T_uQ\big) + \dim\Big(J\big(T_uQ\big)\Big) - \dim\Big(T_uQ + J\big(T_uQ\big)\Big) \\
&\geq \dim_{\mathbb{R}} M
\end{aligned}$$

thus $\big(T_u^hQ\big)_{A_J} \oplus T_u^vQ$ has the required dimension. Property 2. of Definition 1.4.1 can be easily proved using (2.20), thus it remains to show that $u \mapsto \big(T_u^hQ\big)_{A_J}$ is a smooth distribution. For this, let $(X_1, \ldots, X_{m+r})$ be a local frame of the tangent bundle of Q over some open $\mathcal{U} \subset Q$, where $m = \dim_{\mathbb{R}} M$ and $r = \dim K$. Consider $X_1, \ldots, X_{m+r}$ and $J(X_1), \ldots, J(X_{m+r})$ as smooth maps $\mathcal{U} \to TP$ and let $u_0 \in \mathcal{U}$; since $X_1(u_0), \ldots, X_{m+r}(u_0)$ span $T_{u_0}Q$ and $J(X_1)(u_0), \ldots, J(X_{m+r})(u_0)$ span $J\big(T_{u_0}Q\big)$ and since $T_{u_0}Q + J\big(T_{u_0}Q\big) = T_{u_0}P$, there must be r elements of the set $\{\, J(X_1)(u_0), \ldots, J(X_{m+r})(u_0)\,\}$, say $J(X_{m+1})(u_0), \ldots, J(X_{m+r})(u_0)$, such that $X_1(u_0), \ldots, X_{m+r}(u_0), J(X_{m+1})(u_0), \ldots, J(X_{m+r})(u_0)$ is a basis of $T_{u_0}P$; the same then holds in a small neighborhood $\mathcal{V}$ of u_0. Thus $J(X_1), \ldots, J(X_m)$ can be written in $\mathcal{V}$ as combinations of $X_1, \ldots, X_{m+r}, J(X_{m+1}), \ldots, J(X_{m+r})$. Let $J(X_i) = \sum_{j=1}^{m+r} a_{ij}X_j + \sum_{j=1}^{r} b_{ij}J(X_{m+j})$ on $\mathcal{V}$, for $i = 1, \ldots, m$. Then $\{\sum_{j=1}^{m+r} a_{ij}X_j = J(X_i) - \sum_{j=1}^{r} b_{ij}J(X_{m+j}),\ i = 1, \ldots, m\,\}$ span the distribution $u \mapsto \big(T_u^hQ\big)_{A_J}$, showing that the distribution is smooth.

We will now construct the inverse of the map $J \mapsto A_J$. Let A be a connection on $Q\,(M,K)$ and define an almost holomorphic structure J_A on $P\,(M,G)$ as follows. Extend first A to a connection on $P\,(M,G)$ (see the remark after Proposition 1.4.17). Then, for $u \in P$ and $X \in T_uP$, with $(X^v)_A = (B^*)_u$ for some $B \in \mathfrak{g}$, define

$$J_A\,(X) := (\,iB^*\,)_u + \Big(J_M\,\widehat{\big(\pi_*(X)\big)}_u^{\,h}\Big)_A. \tag{3.1}$$

Note that J_A is indeed an almost holomorphic structure on $P\,(M,G)$, since it satisfies (2.21) and condition 1. of Definition 1.4.1.

The two constructions are inverse of each other. Consider first a connection A on $Q(M,K)$; we need to show that $A = A_{J_A}$, i.e. that for $u \in Q$ we have $\left(T^h_u Q\right)_A = T_u Q \cap J_A\big(T_u Q\big)$. Let $X \in \left(T^h_u Q\right)_A$. Then using (3.1) we get

$$X = J_A\left(\left(-J_M\widehat{\left(\pi_*(X)\right)}^h_u\right)_A\right)$$

thus $X \in T_u Q \cap J_A\big(T_u Q\big)$. Since $\left(T^h_u Q\right)_A$ and $T_u Q \cap J_A\big(T_u Q\big)$ have the same dimension, it follows that they are equal.
Consider now an almost holomorphic structure J on $P(M,G)$; we have to show that $J = J_{A_J}$. It is enough to show that $J(X) = J_{A_J}(X)$ for $u \in Q$ and $X \in \left(T^h_u Q\right)_{A_J}$, since both J and J_{A_J} are G-invariant and since they coincide on vertical vectors. By (3.1) we have

$$J_{A_J}(X) = \left(J_M\widehat{\left(\pi_*(X)\right)}^h_u\right)_{A_J}.$$

But we have also

$$J(X) = \left(J_M\widehat{\left(\pi_*(X)\right)}^h_u\right)_{A_J}.$$

Indeed, $J(X) \in \left(T^h_u Q\right)_{A_J} = T_u Q \cap J\big(T_u Q\big)$ since X does and, by condition 1. of Definition 1.4.1, $\pi_*\big(J(X)\big) = J_M\big(\pi_*(X)\big)$. □

Example 3.1.2 *Let $\varrho : G \hookrightarrow GL(n,\mathbb{C})$ be a holomorphic faithful representation of G with $\varrho(K) \subset U(n)$ (see Theorem 2.4.8). Suppose that we have an almost holomorphic structure J on $P(M,G)$. Then J induces a semiconnection δ_J on the associated vector bundle $E = P \times_G \mathbb{C}^n$, as described in Proposition 2.5.5. Let $(f,\varrho) : P(M,G) \hookrightarrow L(E)\big(M, GL(n,\mathbb{C})\big)$ be the reduction described in Example 1.2.10. The reduction $Q(M,K) \hookrightarrow P(M,G)$ induces a metric h on E, which is defined as follows. For all $p \in M$, $h(p)$ is the metric on E_p which has matrix I with respect to a basis $f(u)$ of E_p, where u is some element of the fibre of Q over p (see also Example 1.2.17). Let A_J be the connection on $Q(M,K)$ corresponding to the almost holomorphic structure J of $P(M,G)$ under the Chern correspondence and extend it to a connection on $P(M,G)$, as explained in the remark after Proposition 1.4.17.*

We will show that the induced connection D_{A_J} is the Chern connection of E with respect to the metric h and the induced semiconnection δ_J.

A_J can be extended to a connection on $L(E)\big(M, GL(n,\mathbb{C})\big)$, which is then reducible to $U_h(E)\big(M, U(n)\big)$; so, using Examples 1.4.21 and 1.4.22, we see that D_{A_J} is an h-connection. Thus it remains to show that ${D_{A_J}}'' = \delta_J$. Let $\sigma : M \to E$ be a section of $E = P \times_G \mathbb{C}^n$. It is enough to prove that $\delta_J(\sigma)(Y) = {D_{A_J}}''(\sigma)(Y)$ for

$p \in M$ and $Y \in T_p^{\mathbb{R}} M$. By Proposition 2.5.5 and Proposition 1.4.9 we have

$$\delta_J(\sigma)\,(Y) = \big(\, u\,,\, \widehat{Y}_u{}^{0,1}(\hat{\sigma})\,\big)_{/\sim}$$

and

$$D_{A_J}{}''(\sigma)\,(Y) = D_{A_J}(\sigma)\,\big(Y^{0,1}\big) = \Big(\, u\,,\, \big(\,\widehat{Y^{0,1}}^{\,h}\,\big)_{A_J}(u)\,(\hat{\sigma})\,\Big)_{/\sim}\,,$$

where $\hat{\sigma}$ is the G-equivariant function on P corresponding to σ, u some element in the fibre of P over p and $\widehat{Y}_u$ some vector of T_uP with $\pi_*(\widehat{Y}_u) = Y$. So we have to show that $\widehat{Y}_u{}^{0,1} = \big(\,\widehat{Y^{0,1}}^{\,h}\,\big)_{A_J}(u)$. We know that

$$\pi_*\left(\widehat{Y}_u{}^{0,1}\right) = Y^{0,1} = \pi_*\left(\big(\,\widehat{Y^{0,1}}^{\,h}\,\big)_{A_J}(u)\right),$$

thus it is enough to prove that $\widehat{Y}_u{}^{0,1} \in \big(T_u^h P\big)_{A_J}$. Using (2.1) and complex linearity of ω_{A_J} (see page 74), we have

$$\begin{aligned} \omega_{A_J}\left(\widehat{Y}_u{}^{0,1}\right) &= \omega_{A_J}\left(\frac{\widehat{Y}_u + i\,J\big(\widehat{Y}_u\big)}{2}\right) \\ &= \frac{\omega_{A_J}\big(\widehat{Y}_u\big) + i\,\omega_{A_J}\Big(J\big(\widehat{Y}_u\big)\Big)}{2}\,. \end{aligned} \tag{3.2}$$

Set $\big(\,\widehat{Y}_u^{\,v}\,\big)_{A_J} = (B^*)_u$, for some $B \in \mathfrak{g}$. Since

$$J\Big(\big(T_u^h P\big)_{A_J}\Big) = \big(T_u^h P\big)_{A_J}$$

we have, using (2.21)

$$\Big(\,J\big(\widehat{Y}_u\big)^{\,v}\,\Big)_{A_J} = J\Big(\big(\,\widehat{Y}_u^{\,v}\,\big)_{A_J}\Big) = J\,\big(\,(B^*)_u\,\big) = (\,iB^*\,)_u\,.$$

Thus $\omega_{A_J}\big(\widehat{Y}_u\big) = B$ and $\omega_{A_J}\Big(J\big(\widehat{Y}_u\big)\Big) = iB$, so from (3.2) we get

$$\omega_{A_J}\left(\widehat{Y}_u{}^{0,1}\right) = 0\,,$$

i.e. $\widehat{Y}_u{}^{0,1} \in \big(T_u^h P\big)_{A_J}$, as we wanted.

Note that to prove $D_{A_J}{}'' = \delta_J$ in the previous example we have only used the fact that $J\Big(\big(T_u^h P\big)_{A_J}\Big) = \big(T_u^h P\big)_{A_J}$ (in particular, we didn't need the injectivity of the reduction $\varrho : G \hookrightarrow GL(n,\mathbb{C})$). Thus exactly the same argument can be used to prove the statement of the next example.

Example 3.1.3 *Suppose we have a holomorphic representation $\varrho : G \to \mathrm{Aut}(V)$ of G on a vector space V and consider the associated vector bundle $E = P \times_G V$. Let J be an almost holomorphic structure on $P(M,G)$ and A a connection satisfying $J\Big(\big(T_u^h P\big)_A\Big) = \big(T_u^h P\big)_A$, for all $u \in P$. Then we have $D_{A_J}{}'' = \delta_J$, where D_A and δ_J are the connection and the semiconnection on E induced by A and J respectively. In particular, if J is a holomorphic structure on $P(M,G)$ then we have $D_{A_J}{}'' = \overline{\partial}$, where $\overline{\partial}$ is the operator associated to the induced holomorphic structure on E.*

Example 3.1.4 *Let $\pi : E \to M$ be a complex vector bundle over a complex manifold and let h be an Hermitian metric on it. Consider the frame bundle $L(E)\big(M, GL(n,\mathbb{C})\big)$ and the $U(n)$-reduction $U_h(E)\big(M, U(n)\big)$ associated to h, under the correspondence (2.18). By Theorem 3.1.1, Theorem 1.4.12, Example 1.4.22 and Proposition 2.3.11 we have 1-1 correspondences*

$$\overline{\mathcal{C}}\big(L(E)\big) \stackrel{\text{1-1}}{\longleftrightarrow} \mathcal{A}\big(U_h(E)\big) \stackrel{\text{1-1}}{\longleftrightarrow} \{\, h\text{-connections on } \pi : E \to M \,\} \stackrel{\text{1-1}}{\longleftrightarrow} \overline{\mathcal{D}}(E).$$

By Example 3.1.2 the composition of these correspondences sends an element J of $\overline{\mathcal{C}}\big(L(E)\big)$ to the induced semiconnection δ_J on $E = L(E) \times_{GL(n,\mathbb{C})} \mathbb{C}^n$. In particular we see that the map $J \mapsto \delta_J$ described in Proposition 2.5.5 gives a 1-1 correspondence between semiconnections on a complex vector bundle over a complex manifold and almost complex structures on its frame bundle.

We will use Example 3.1.2 to prove the next proposition, which is a generalization of the second part of Proposition 2.3.11.

Proposition 3.1.5 *Under the Chern correspondence $\overline{\mathcal{C}}(P) \stackrel{\text{1-1}}{\longleftrightarrow} \mathcal{A}(Q)$, the elements of $\mathcal{A}^{1,1}(Q) := \{\, A \in \mathcal{A}(Q) \ / \ \Omega_A \in \mathcal{A}^{1,1}(P \times_{Ad} \mathfrak{g}) \,\}$ correspond precisely to the integrable almost holomorphic structures of $P(M,G)$, i.e. we have a 1-1 correspondence $\mathcal{C}(P) \stackrel{\text{1-1}}{\longleftrightarrow} \mathcal{A}^{1,1}(Q)$.*

Proof Let $\varrho : G \hookrightarrow GL(n,\mathbb{C})$ be a holomorphic faithful representation of G with $\varrho(K) \subset U(n)$. Consider the associated vector bundle $E = P \times_G \mathbb{C}^n = Q \times_K \mathbb{C}^n$ and the metric h on E defined in Example 3.1.2. Recall that we have a reduction $(f, \varrho) : P(M,G) \hookrightarrow L(E)\big(M, GL(n,\mathbb{C})\big)$.

Let J be a holomorphic structure on $P(M,G)$ and let A_J be the corresponding connection of $Q(M,K)$. By Example 3.1.2 we know that the connection D_{A_J} on E induced by A_J is the Chern connection of E with respect to the metric h and the holomorphic structure on E induced by J. Then, by Proposition 2.3.11, the curvature R_{A_J} of D_{A_J} is an element of $\mathcal{A}^{1,1}(E^* \otimes E)$. Since, by Lemma 1.4.13, $R_{A_J} = \phi(\Omega_{A_J})$, where $\phi : \mathcal{A}^2(P \times_{Ad} \mathfrak{g}) \to \mathcal{A}^2(E^* \otimes E)$ is induced by the injective map $P \times_{Ad} \mathfrak{g} \to E^* \otimes E$ defined in Example 1.2.13, it follows that $\Omega_{A_J} \in \mathcal{A}^{1,1}(P \times_{Ad} \mathfrak{g})$, thus $A_J \in \mathcal{A}^{1,1}(Q)$.

Conversely, let $A \in \mathcal{A}^{1,1}(Q)$ and extend it to a connection on $L(E)\,\big(M, GL(n,\mathbb{C})\big)$, which is then reducible to $U_h(E)\,\big(M, U(n)\big)$. By Examples 1.4.21 and 1.4.22, the connection D_A on E induced by A is an h-connection and by hypothesis its curvature $R_A = \phi\,(\Omega_A)$ is of type $(1,1)$. Consider the holomorphic structure on E corresponding to D_A, in the sense of Proposition 2.3.11 and Theorem 2.3.10. This holomorphic structure on E induces a holomorphic structure J on $L(E)$, as explained in Example 2.5.4. Then (the extension of) A is the connection on $U_h(E)$ corresponding to J under the Chern correspondence.

Let $u \in P$ and $X \in T_uP$. Write $X = (B^*)_u + \big(X^h\big)_A$, for some $B \in \mathfrak{g}$. Then, since $f_*\Big(\big(T_u^hP\big)_A\Big) = \big(T_{f(u)}^hL(E)\big)_A$, we have $f_*(X) = \big(\varrho_*(B)^*\big)_u + \big(f_*(X)^h\big)_A$ and by (3.1)

$$
\begin{aligned}
J\big(f_*(X)\big) &= \big(i\varrho_*(B)^*\big)_{f(u)} + \left(J_M\left(\widehat{\pi_*\big(f_*(X)\big)}\right)^h_{f(u)}\right)_A \\
&= \big(\varrho_*(iB)^*\big)_{f(u)} + f_*\left(\left(J_M\left(\widehat{\pi_{P*}(X)}\right)^h_u\right)_A\right) \\
&= f_*\left(\big(iB^*\big)_u + \left(J_M\left(\widehat{\pi_{P*}(X)}\right)^h_u\right)_A\right) \\
&= f_*\big(J_A(X)\big) \qquad (3.3)
\end{aligned}
$$

where π denotes the projection $L(E) \to M$ and π_P the projection $P \to M$. Let N_{J_A} and N_J be the torsions of the almost holomorphic structures J_A and J on P and $L(E)$ respectively (see Definition 2.2.6). Then for X_1,$X_2 \in \Gamma(TP)$ we have, using (3.3) and the fact that J is integrable,

$$
f_*\big(N_{J_A}(X_1,X_2)\big) = N_J\big(f_*(X_1), f_*(X_2)\big) = 0\,.
$$

Since $f : P \to L(E)$ is an immersion, it follows that $N_{J_A}(X_1,X_2) = 0$, thus J_A is integrable. □

Let $\mathcal{G}^P = \Gamma(P \times_{Ad} G)$ and $\mathcal{G}^Q = \Gamma(Q \times_{Ad} K)$ be the gauge groups of $P\,(M,G)$ and $Q\,(M,K)$ respectively. By Proposition 1.2.18, we know that $P \times_{Ad} G = Q \times_{Ad} G$. Since the natural inclusion $Q \times_{Ad} K \subset Q \times_{Ad} G$ induces a group monomorphism $\Gamma(Q \times_{Ad} K) \hookrightarrow \Gamma(Q \times_{Ad} G) = \Gamma(P \times_{Ad} G)$, we can regard $\mathcal{G}^Q$ as a subgroup of $\mathcal{G}^P$. Equivalently, the inclusion $\mathcal{G}^Q \subset \mathcal{G}^P$ can be described as follows. For $\varphi \in \mathcal{G}^Q = \mathrm{Aut}(Q)$ and $u \in P$ with $u = u'g$, for some $u' \in Q$ and $g \in G$, we define $\varphi(u) := \varphi(u')\,g$.

We have a natural action of $\mathcal{G}^Q$ on $\mathcal{A}(Q)$ on the right, given by $(A,\varphi) \mapsto \varphi^*(A)$, where $\varphi^*(A)$ is defined by

$$
\big(T_u^hQ\big)_{\varphi^*(A)} := (\varphi_*)^{-1}\Big(\big(T_{\varphi(u)}^hQ\big)_A\Big)
$$

for $u \in Q$, as in Proposition 1.4.19. Note that $\mathcal{A}^{1,1}(Q)$ is invariant under this action. This can be seen as follows. Let $A \in \mathcal{A}^{1,1}(Q)$ and $\varphi \in \mathcal{G}^Q$. Let Ω_A, $\Omega_{\varphi^*(A)} \in \mathcal{A}^2(P \times_{Ad} \mathfrak{g})$ be the curvature forms of A and $\varphi^*(A)$ respectively and denote the corresponding G-equivariant horizontal forms on P by $\widehat{\Omega_A}$ and $\widehat{\Omega_{\varphi^*(A)}}$. By Proposition 1.4.19 we know that $\widehat{\Omega_{\varphi^*(A)}} = \varphi^*(\widehat{\Omega_A})$. Let $p \in M$ and $Y_1, Y_2 \in T_pM$. Then $\Omega_{\varphi^*(A)}(Y_1, Y_2) = \big(\varphi^{-1}(u), \widehat{\Omega_A}(\hat{Y}_1, \hat{Y}_2)\big)_{/\sim}$, where u is some element in the fibre of p over P and $\hat{Y}_1$, $\hat{Y}_2$ are vectors of T_uP with $\pi_*(\hat{Y}_i) = Y_i$ $(i = 1, 2)$. Suppose that $Y_1, Y_2 \in T'_pM$ (or $Y_1, Y_2 \in T''_pM$). Then, since $\Omega_A \in \mathcal{A}^{1,1}(P \times_{Ad} \mathfrak{g})$, we have $0 = \Omega_A(Y_1, Y_2) = \big(u, \widehat{\Omega_A}(\hat{Y}_1, \hat{Y}_2)\big)_{/\sim}$, thus $\widehat{\Omega_A}(\hat{Y}_1, \hat{Y}_2) = 0$ and $\Omega_{\varphi^*(A)}(Y_1, Y_2) = 0$. This proves that $\Omega_{\varphi^*(A)} \in \mathcal{A}^{1,1}(P \times_{Ad} G)$ and thus $\varphi^*(A) \in \mathcal{A}^{1,1}(Q)$, as we wanted.

We have seen at the end of Section 2.5 that $\mathcal{G}^P$ works on the right on $\overline{\mathcal{C}}(P)$ by $(J, \varphi) \mapsto \varphi(J) := {\varphi_*}^{-1} \circ J \circ \varphi$ and that this action leaves $\mathcal{C}(P)$ invariant. Using the Chern correspondence $\overline{\mathcal{C}}(P) \overset{1\text{-}1}{\longleftrightarrow} \mathcal{A}(Q)$ (and the restriction $\mathcal{C}(P) \overset{1\text{-}1}{\longleftrightarrow} \mathcal{A}^{1,1}(Q)$) we get an action

$$\mathcal{A}(Q) \times \mathcal{G}^P \to \mathcal{A}(Q)\ ,\ (A, \varphi) \mapsto A_{\varphi(J_A)}$$

which can be restricted to $\mathcal{A}^{1,1}(Q) \times \mathcal{G}^P \to \mathcal{A}^{1,1}(Q)$.

Proposition 3.1.6 *The action of $\mathcal{G}^P$ on $\mathcal{A}(Q)$ defined above extends the natural $\mathcal{G}^Q$-action on $\mathcal{A}(Q)$.*

Proof Let $A \in \mathcal{A}(Q)$ and $\varphi \in \mathcal{G}^Q$. Then for $u \in Q$ we have

$$\begin{aligned} \left(T^h_uQ\right)_{A_{\varphi(J_A)}} &= T_uQ \cap \varphi(J_A)\left(T_uQ\right) = T_uQ \cap {\varphi_*}^{-1}\Big(J_A\left(\varphi_*(T_uQ)\right)\Big) \\ &= {\varphi_*}^{-1}\Big(T_{\varphi(u)}Q \cap J_A\left(T_{\varphi(u)}Q\right)\Big) = {\varphi_*}^{-1}\Big(\left(T^h_{\varphi(u)}Q\right)_A\Big) \\ &= \left(T^h_uQ\right)_{\varphi^*(A)}, \end{aligned}$$

thus $\varphi^*(A) = A_{\varphi(J_A)}$. □

Recall that two (almost) holomorphic structures on $P(M, G)$ are said to be isomorphic if they are in the same $\mathcal{G}^P$-orbit. From what was said above, we conclude that we can identify the moduli space of isomorphism classes of (almost) holomorphic structures on $P(M, G)$ with the quotient ${\mathcal{A}^{1,1}(Q)}/{\mathcal{G}^P}$ (respectively ${\mathcal{A}(Q)}/{\mathcal{G}^P}$).

3.2 Chern connections

Definition 3.2.1 *Given an almost holomorphic structure J on $P(M, G)$ and a reduction $Q(M, K) \hookrightarrow P(M, G)$, the connection on $Q(M, K)$ corresponding to J under the Chern correspondence, extended to a connection on $P(M, G)$, is denoted* *by $A_{J,Q}$ and is called the* ***Chern connection*** *of $P(M,G)$ with respect to the reduction $Q(M,K)$ and the almost holomorphic structure J.*

In this section we want to study the dependence of $A_{J,Q}$ on the reduction $Q(M,K)$ of $P(M,G)$, when J is fixed.

Given a reduction $Q(M,K)$ of $P(M,G)$, we will denote by $T_uQ^{\perp}$, for $u \in Q$, the linear subspace $\{ (B^*)_u \,/\, B \in i\mathfrak{k} \}$ of T_uP. Then $T_uP = T_uQ \oplus T_uQ^{\perp}$.

Lemma 3.2.2 *Let $A_{J,Q}$ be the Chern connection of $P(M,G)$ with respect to a reduction $Q(M,K)$ and an almost holomorphic structure J. Then for $u \in Q$ and $X \in T_uQ$ we have*

$$\left(X^v\right)_{A_{J,Q}} = -\,J\left(\mathrm{pr}_{T_uQ^{\perp}}\big(J(X)\big)\right).$$

Proof We have to prove that $J\left(\mathrm{pr}_{T_uQ^{\perp}}\big(J(X)\big)\right) \in T_u^vP$ and

$$X + J\left(\mathrm{pr}_{T_uQ^{\perp}}\big(J(X)\big)\right) \in \left(T_u^hP\right)_{A_{J,Q}} = T_uQ \cap J\big(T_uQ\big).$$

Let $\mathrm{pr}_{T_uQ^{\perp}}\big(J(X)\big) = (B^*)_u$, for some $B \in i\mathfrak{k}$. Then by (2.21) we have

$$J\left(\mathrm{pr}_{T_uQ^{\perp}}\big(J(X)\big)\right) = J\left((B^*)_u\right) = \left(iB^*\right)_u \in T_u^vQ.$$

Thus $J\left(\mathrm{pr}_{T_uQ^{\perp}}\big(J(X)\big)\right) \in T_u^vP$ and $X + J\left(\mathrm{pr}_{T_uQ^{\perp}}\big(J(X)\big)\right) \in T_uQ$. It remains to show that $X + J\left(\mathrm{pr}_{T_uQ^{\perp}}\big(J(X)\big)\right) \in J\big(T_uQ\big)$. But this holds, because it is equivalent to $J(X) - \mathrm{pr}_{T_uQ^{\perp}}\big(J(X)\big) \in T_uQ$. □

Consider the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$. Since $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ is $\mathbb{C}$-linear for all $g \in G$ and since $\mathrm{Ad}(K) \subset \mathrm{Aut}(\mathfrak{k})$, we see that $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ induces an action $K \times i\mathfrak{k} \to i\mathfrak{k}$, $(k, iB) \mapsto \mathrm{Ad}(k)(iB) = i\,\mathrm{Ad}(k)(B)$. If we have a reduction $Q(M,K)$ of $P(M,G)$, we can then consider the associated vector bundle $Q \times_{Ad} i\mathfrak{k}$. Given a section $s \in \Gamma(Q \times_{Ad} i\mathfrak{k})$, we will denote as usual by $\hat{s}$ the corresponding K-equivariant map $Q \to i\mathfrak{k}$. Let $e^{\hat{s}}$ be the composition $\exp \circ \hat{s} : Q \to G$. Using (1.6) we get, for $u \in Q$ and $k \in K$

$$e^{\hat{s}}(uk) = e^{\hat{s}(uk)} = e^{\mathrm{Ad}(k^{-1})\left(\hat{s}(u)\right)} = c(k^{-1})\left(e^{\hat{s}(u)}\right) = c(k^{-1})\left(e^{\hat{s}}(u)\right). \tag{3.4}$$

Thus $e^{\hat{s}} : Q \to G$ is K-equivariant with respect to the action $K \times G \to K$, $(k,g) \mapsto c(k)(g)$ and so it induces a section e^s of $Q \times_{Ad} G = P \times_{Ad} G$, thus an element $e^s \in \mathcal{G}^P$. Consider now the map $\mathrm{pr}(e^{\hat{s}}) : Q \to {G}/{K}$, $u \mapsto e^{\hat{s}}(u)K$. For $u \in Q$ and $k \in K$ we have

$$\begin{aligned} \mathrm{pr}(e^{\hat{s}})(uk) &= e^{\hat{s}}(uk)\,K = c(k^{-1})\left(e^{\hat{s}}(u)\right)K = k^{-1}\,e^{\hat{s}}(u)\,K \\ &= k^{-1}\mathrm{pr}(e^{\hat{s}})(u)\,. \end{aligned} \tag{3.5}$$

Thus $\mathrm{pr}\,(e^{\hat{s}})$ is K-equivariant, with respect to the action $K \times {}^{G}\!/_{K}$, $(k, gK) \mapsto kgK$ and so it induces a section $\mathrm{pr}\,(e^{s})$ of the fibre bundle $Q \times_K {}^{G}\!/_{K}$. This will be used in the proof of the next proposition.

Proposition 3.2.3 *Let $Q_0\,(M, K)$ be a fixed K-reduction of $P\,(M, G)$. Then for every $s \in \Gamma\,(Q_0 \times_{Ad}\, i\mathfrak{k})$ the set $Q_s := e^s\,(Q_0) := \{\, u\, e^{\hat{s}(u)}\,,\, u \in Q_0\,\}$ defines a K-reduction of $P\,(M, G)$. Moreover, every K-reduction of $P\,(M, G)$ can be written in this form.*

Proof We will show that there is a canonical bijection between $\Gamma\,(Q_0 \times_{Ad}\, i\mathfrak{k})$ and the set of K-reductions of $P\,(M, G)$, and that under this bijection a section s of $Q_0 \times_{Ad}\, i\mathfrak{k}$ corresponds to the reduction of $P\,(M, G)$ with total space Q_s.
By Propositions 1.2.16 and 1.2.18 we already know that there are natural 1-1 correspondences

$$\{\,\text{reductions of } P\,(M,G) \text{ to } K\,\} \overset{\text{1-1}}{\longleftrightarrow} \Gamma\left(P \times_G {}^{G}\!/_{K}\right) \overset{\text{1-1}}{\longleftrightarrow} \Gamma\left(Q_0 \times_K {}^{G}\!/_{K}\right) .$$

We claim that the natural map $\Gamma\,(Q_0 \times_{Ad}\, i\mathfrak{k}) \to \Gamma\,(Q_0 \times_K {}^{G}\!/_{K})$, $s \mapsto \mathrm{pr}\,(e^s)$ defined above is the 1-1 correspondence that we need to conclude the proof.
We will first show that composition with $\mathrm{pr} \circ \exp : i\mathfrak{k} \to {}^{G}\!/_{K}$ gives a bijection $\phi \mapsto \mathrm{pr}\,(e^{\phi})$ between the set of K-equivariant maps $Q_0 \to i\mathfrak{k}$ and of K-equivariant maps $Q_0 \to {}^{G}\!/_{K}$, and thus induces a 1-1 correspondence between $\Gamma\,(Q_0 \times_{Ad}\, i\mathfrak{k})$ and $\Gamma\,(Q_0 \times_K {}^{G}\!/_{K})$. Since we know that $\mathrm{pr} \circ \exp : i\mathfrak{k} \to {}^{G}\!/_{K}$ is a bijection (see the remark before Theorem 2.4.5), we only have to check that a map $\phi : Q_0 \to i\mathfrak{k}$ is K-equivariant if and only if so is $\mathrm{pr}\,(e^{\phi}) : Q_0 \to {}^{G}\!/_{K}$. Suppose first that ϕ is K-equivariant. Then by (3.4), with ϕ replacing $\hat{s}$, we see that $\mathrm{pr}\,(e^{\phi})$ is K-equivariant. Conversely, suppose that $\mathrm{pr}\,(e^{\phi})$ is K-equivariant, then for $u \in Q_0$ and $k \in K$ we have, reversing the calculations of (3.4) and (3.5), with ϕ in the place of $\hat{s}$

$$e^{\phi(uk)}\,K = \mathrm{pr}\,(e^{\phi})\,(uk) = k^{-1}\mathrm{pr}\,(e^{\phi})\,(u) = c\,(k^{-1})\big(e^{\phi}(u)\big)\,K = e^{\,\mathrm{Ad}\,(k^{-1})\,\phi(u)}\,K\ .$$

Since $\mathrm{pr} \circ \exp : i\mathfrak{k} \to {}^{G}\!/_{K}$ is bijective, it follows that $\phi(uk) = \mathrm{Ad}(k^{-1})\,\phi(u)$, so $\phi : Q_0 \to i\mathfrak{k}$ is K-equivariant. Thus $\Gamma\,(Q_0 \times_{Ad}\, i\mathfrak{k}) \to \Gamma\,(Q_0 \times_K {}^{G}\!/_{K})$, $s \mapsto \mathrm{pr}\,(e^s)$ is a bijection.
Let $s \in \Gamma\,(Q_0 \times_{Ad}\, i\mathfrak{k})$ and $p \in M$. Then the fibre over p of the K-reduction of $P\,(M, G)$ corresponding to s under the canonical bijections described above can be written as $u_0\, \mathrm{pr}\,(e^{\hat{s}})(u_0) = u_0\, e^{\hat{s}(u_0)}\,K$, for some u_0 in the fibre of $Q_0\,(M, K)$ over p. But $u_0\, e^{\hat{s}(u_0)}\,K = \{\, u\, e^{\hat{s}(u)}\ /\ u \in Q_0 \text{ with } \pi(u) = p\,\}$, since for $k \in K$ we have $u_0\, e^{\hat{s}(u_0)}\,k = (u_0 k)\, e^{\hat{s}(u_0 k)}$. Thus indeed the total space of this reduction is $Q_s = \{\, u\, e^{\hat{s}(u)}\,,\, u \in Q_0\,\}$, as we wanted. □

Example 3.2.4 *Let h_0 be a fixed Hermitian metric on a complex vector bundle $\pi : E \to M$ and consider the $U(n)$-reduction $U_{h_0}(E)$ of the frame bundle*

$L(E)\left(M, GL(n,\mathbb{C})\right)$, corresponding to h_0 under (2.18). Under the identification of Example 1.2.13, given $s \in \Gamma\left(U_{h_0}(E) \times_{Ad} i\,\mathfrak{u}(n)\right)$ we can consider

$$e^s \in \mathcal{G}^{L(E)} = \Gamma\left(L(E) \times_{Ad} GL(n,\mathbb{C})\right) \subset \Gamma\left(L(E) \times_{Ad} \mathfrak{gl}\,(n,\mathbb{C})\right)$$

(see also (1.7)) as a section of $E^* \otimes E$, i.e. an endomorphism of E. Consider the $U(n)$-reduction

$$U_h(E) := e^s\left(U_{h_0}(E)\right) = \{\, u\, e^{\hat{s}(u)}\,,\, u \in U_{h_0}(E)\,\}$$

of $L(E)$, corresponding to a metric h on $\pi : E \to M$. Then, for every $p \in M$, $h(p)$ has matrix I with respect to the basis $u_0\, e^{\hat{s}(u_0)}$ of E_p, where u_0 is some h_0-orthonormal basis of E_p. In other words, the automorphism $e^s(p)$ of E_p sends an h_0-orthonormal basis u_0 to an h-orthonormal basis $u_0\, e^{\hat{s}(u_0)}$, thus it is an isometry between (E_p, h_0) and (E_p, h). It is easy to see that h can be equivalently defined by

$$h\left(e^s(\sigma_1), \sigma_2\right) = h_0\left(\sigma_1, \sigma_2\right)$$

for σ_1, $\sigma_2 \in \mathcal{A}^0(E)$.

Let J be an almost holomorphic structure on $P\,(M,G)$ and let $A_{J,Q}$ be the Chern connection of $P\,(M,G)$ with respect to a reduction $Q\,(M,K)$. Given an element $\phi \in \mathcal{G}^P$ (regarded as a G-equivariant map $P \to G$), we define a $\mathfrak{g}$-valued complex 1-form $d_Q\,\phi$ on P by

$$d_Q\,\phi\,(X) := \omega_G\left(\phi_*\left(\left(X^h\right)_{A_{J,Q}}\right)\right)$$

for $u \in P$ and $X \in (T_uP)^{\mathbb{C}}$, where ω_G is the Maurer-Cartan form on G, extended to the complexified tangent bundle of G. Then $d_Q\,\phi$ is horizontal and G-equivariant, the last since

$$\begin{aligned} d_Q\,\phi\left((R_g)_*(X)\right) &= \omega_G\left(\phi_*\left(\left((R_g)_*(X)^h\right)_{A_{J,Q}}\right)\right) \\ &= \omega_G\left(\phi_*\left((R_g)_*\left(\left(X^h\right)_{A_{J,Q}}\right)\right)\right) \\ &= \omega_G\left((L_{g^{-1}})_*\,(R_g)_*\left(\phi_*\left(\left(X^h\right)_{A_{J,Q}}\right)\right)\right) \\ &= \mathrm{Ad}\,(g^{-1})\,\omega_G\left(\phi_*\left(\left(X^h\right)_{A_{J,Q}}\right)\right) = \mathrm{Ad}\,(g^{-1})\left(d_Q\,\phi\,(X)\right) \end{aligned}$$

for $g \in G$. Thus by Lemma 1.4.7 we have $d_Q\,\phi \in \mathcal{A}^1(P \times_{Ad} \mathfrak{g})$. We define $\partial_Q\,\phi \in \mathcal{A}^{1,0}(P \times_{Ad} \mathfrak{g})$ and $\overline{\partial}_Q\,\phi \in \mathcal{A}^{0,1}(P \times_{Ad} \mathfrak{g})$ to be the $(1,0)$ and $(0,1)$ components of $d_Q\,\phi$. Note that $\partial_Q\,\phi$ and $\overline{\partial}_Q\,\phi$ remains of type $(1,0)$ and $(0,1)$

also when considered as forms on P. Indeed, suppose we have $X \in T''_u P$; then $J(X) = -iX$ and, by property 1. of Definition 2.5.1,

$$J_M\left(\pi_*(X)\right) = \pi_*\left(J(X)\right) = \pi_*(-iX) = -i\pi_*(X),$$

thus $\pi_*(X) \in T''_{\pi(u)}M$. Similarly, if $X \in T'_u P$ then $\pi_*(X) \in T'_{\pi(u)}M$. Thus we have $\partial_Q \phi(X) = 0$ for all $X \in T''_u P$ and $\overline{\partial}_Q \phi(X) = 0$ for all $X \in T'_u P$ and this means that $\partial_Q \phi$ and $\overline{\partial}_Q \phi$ are respectively of type $(1,0)$ and $(0,1)$.

We are now in a position to formulate and prove the following theorem, which is the main result of this section.

Theorem 3.2.5 *Let $Q_0(M,K)$ and $Q(M,K)$ be reductions of $P(M,G)$ and write $Q = e^s(Q_0)$, for some $s \in \Gamma(Q_0 \times_{Ad} i\mathfrak{k})$, as described in Proposition 3.2.3. Let J be an almost holomorphic structure on $P(M,G)$ and consider the associated Chern connections $A := A_{J,Q}$ and $A_0 := A_{J,Q_0}$ on $P(M,G)$. Then the connection forms ω and ω_0 of A and A_0 are related by the formula*

$$\omega - \omega_0 = \partial_{Q_0} e^{-2s}$$

(recall that $e^{-2s} \in \mathcal{G}^P$, see page 86).

Proof Since $\omega - \omega_0$ and $\partial_{Q_0} e^{-2s}$ are both horizontal G-equivariant forms on P, it is enough to prove

$$(\omega - \omega_0)(X_0) = \partial_{Q_0} e^{-2s}(X_0)$$

for $u_0 \in Q_0$ and $X_0 \in \left(T_{u_0}^{\,h} Q_0\right)_{A_0} = \left(T_{u_0}^{\,h} P\right)_{A_0}$.

Let $e^{\hat{s}}$ be the K-equivariant map $Q_0 \to G$ corresponding to e^s, and f_s the map $Q_0 \to Q$, $u \mapsto u\, e^{\hat{s}}(u)$. Denote by $\mu : P \times G \to P$ the action $(u,g) \mapsto ug$. Then f_s is the composition $\mu \circ (\mathrm{id} \times e^{\hat{s}})$. By the Leibniz rule [2], the differential of f_s at u_0 is given by

$$f_{s\,*}(X) = \mu_*\left(X, (e^{\hat{s}})_*(X)\right) = (R_{e^{\hat{s}}(u_0)})_*(X) + \Big(\omega_G\left((e^{\hat{s}})_*(X)\right)\Big)^*_{f_s(u_0)}$$

for $X \in T_{u_0}Q$. For $X_0 \in \left(T_{u_0}^{\,h} Q_0\right)_{A_0}$ this becomes

$$f_{s\,*}(X_0) = (R_{e^{\hat{s}}(u_0)})_*(X_0) + \Big(d_{Q_0} e^s(X_0)\Big)^*_{f_s(u_0)}. \tag{3.6}$$

Since $J(X_0) \in \left(T_{u_0}^h Q_0\right)_{A_0}$ too, we also have

$$f_{s\,*}\left(J(X_0)\right) = (R_{e^{\hat{s}}(u_0)})_*\left(J(X_0)\right) + \Big(d_{Q_0} e^s\left(J(X_0)\right)\Big)^*_{f_s(u_0)}. \tag{3.7}$$

[2] See footnote 24 of Chapter I.

By (3.6) and since $(\,R_{e^{\hat{s}(u_0)}}\,)_*(X_0)$ is A_0-horizontal, we have

$$(\omega - \omega_0)\big(f_{s\,*}(X_0)\big) = \omega\left(\,(\,R_{e^{\hat{s}(u_0)}}\,)_*(X_0) + \left(d_{Q_0}\,e^s\,(X_0)\right)^*_{f_s(u_0)}\right) - d_{Q_0}\,e^s\,(X_0)\,. \tag{3.8}$$

Using Lemma 3.2.2, (2.20), (2.21) and (3.7) we get

$$\begin{aligned}
&\left(\left(\,(\,R_{e^{\hat{s}(u_0)}}\,)_*(X_0) + \left(d_{Q_0}\,e^s\,(X_0)\right)^*_{f_s(u_0)}\right)^v\right)_A \\
&= -\,J\left(\mathrm{pr}_{T_{u_0}Q^\perp}\left(J\left(\,(\,R_{e^{\hat{s}(u_0)}}\,)_*(X_0) + \left(d_{Q_0}\,e^s\,(X_0)\right)^*_{f_s(u_0)}\right)\right)\right) \\
&= -\,J\left(\mathrm{pr}_{T_{u_0}Q^\perp}\left(\,(\,R_{e^{\hat{s}(u_0)}}\,)_*\big(J(X_0)\big) + \left(i\,d_{Q_0}\,e^s\,(X_0)\right)^*_{f_s(u_0)}\right)\right) \\
&= -\,J\left(\mathrm{pr}_{T_{u_0}Q^\perp}\left(\,-\left(d_{Q_0}\,e^s\,\big(J(X_0)\big)\right)^*_{f_s(u_0)} + \left(i\,d_{Q_0}\,e^s\,(X_0)\right)^*_{f_s(u_0)}\right)\right) \\
&= -\,J\left(\mathrm{pr}_{T_{u_0}Q^\perp}\left(\left(-\,d_{Q_0}\,e^s\,\big(J(X_0)\big)\,+\,i\,d_{Q_0}\,e^s\,(X_0)\right)^*_{f_s(u_0)}\right)\right) \\
&= -\,J\left(\left(\mathrm{pr}_{i\mathfrak{k}}\left(-\,d_{Q_0}\,e^s\,\big(J(X_0)\big)\,+\,i\,d_{Q_0}\,e^s\,(X_0)\right)\right)^*_{f_s(u_0)}\right) \\
&= \left(-\,i\,\mathrm{pr}_{i\mathfrak{k}}\left(-\,d_{Q_0}\,e^s\,\big(J(X_0)\big)\,+\,i\,d_{Q_0}\,e^s\,(X_0)\right)\right)^*_{f_s(u_0)}\,.
\end{aligned}$$

Thus

$$\begin{aligned}
&\omega\left(\,(\,R_{e^{\hat{s}(u_0)}}\,)_*(X_0) + \left(d_{Q_0}\,e^s\,(X_0)\right)^*_{f_s(u_0)}\right) \\
&= -\,i\,\mathrm{pr}_{i\mathfrak{k}}\left(-\,d_{Q_0}\,e^s\,\big(J(X_0)\big)\,+\,i\,d_{Q_0}\,e^s\,(X_0)\right) \\
&= -\,i\,\mathrm{pr}_{i\mathfrak{k}}\left(2i\,\tfrac{d_{Q_0}\,e^s\,(X_0)+i\,d_{Q_0}\,e^s\,\left(J(X_0)\right)}{2}\right) = -\,i\,\mathrm{pr}_{i\mathfrak{k}}\left(2i\,\overline{\partial}_{Q_0}\,e^s\,(X_0)\right)\,,
\end{aligned}$$

since

$$\begin{aligned}
\frac{d_{Q_0}\,e^s\,(X_0) + i\,d_{Q_0}\,e^s\,\big(J(X_0)\big)}{2} &= \frac{d_{Q_0}\,e^s\,(X_0) +\,d_{Q_0}\,e^s\,\big(i\,J(X_0)\big)}{2} \\
&= d_{Q_0}\,e^s\,\big(\frac{X_0 + i\,J(X_0)}{2}\big) = \overline{\partial}_{Q_0}\,e^s\,(X_0)\,.
\end{aligned}$$

We claim that

$$\mathrm{pr}_{i\mathfrak{k}}\left(2i\,\overline{\partial}_{Q_0}\,e^s\,(X_0)\right) = i\,\overline{\partial}_{Q_0}\,e^s\,(X_0) - i\,\mathrm{Ad}\left(e^{\hat{s}}(u_0)\right)\left(\,\partial_{Q_0}\,e^s\,(X_0)\,\right) \tag{3.9}$$

(we will prove this later). Using this and (3.8) we get

$$
\begin{aligned}
&(\omega-\omega_0)\left(f_{s\,*}(X_0)\right)\\
&= -i\Big(\, i\,\overline{\partial}_{Q_0}\, e^s\,(X_0) - i\,\mathrm{Ad}\left(e^{\hat{s}}(u_0)\right)\left(\,\partial_{Q_0}\, e^s\,(X_0)\,\right)\Big) - d_{Q_0} e^s(X_0)\\
&= -\,\partial_{Q_0}\, e^s\,(X_0) -\, \mathrm{Ad}\left(e^{\hat{s}}(u_0)\right)\left(\,\partial_{Q_0}\, e^s\,(X_0)\,\right)\,. \qquad (3.10)
\end{aligned}
$$

So, using (3.6) and the fact that $\omega-\omega_0$ is a G-equivariant horizontal form, we have

$$
\begin{aligned}
(\omega-\omega_0)\,(X_0) &= \mathrm{Ad}\left(e^{\hat{s}}(u_0)\right)\Big(\,(\omega-\omega_0)\left(\,(R_{e^{\hat{s}}(u_0)})_*(X_0)\,\right)\Big)\\
&= \mathrm{Ad}\left(e^{\hat{s}}(u_0)\right)\Big(\,(\omega-\omega_0)\left(\,f_{s\,*}(X_0)\,\right)\Big)\\
&= \mathrm{Ad}\left(e^{\hat{s}}(u_0)\right)\Big(\, -\,\partial_{Q_0}\, e^s\,(X_0) -\, \mathrm{Ad}\left(e^{\hat{s}}(u_0)\right)\left(\,\partial_{Q_0}\, e^s\,(X_0)\,\right)\Big)\\
&= -\,\mathrm{Ad}\left(e^{\hat{s}}(u_0)\right)\left(\,\partial_{Q_0}\, e^s\,(X_0)\,\right) -\, \mathrm{Ad}\left(\,e^{\hat{s}}(u_0)^2\,\right)\left(\,\partial_{Q_0}\, e^s\,(X_0)\,\right)\,.
\end{aligned}
$$

What we need to show is thus

$$
\partial_{Q_0}\, e^{-2s}\,(X_0) = -\,\mathrm{Ad}\left(e^{\hat{s}}(u_0)\right)\left(\,\partial_{Q_0}\, e^s\,(X_0)\,\right) -\, \mathrm{Ad}\left(\,e^{\hat{s}}(u_0)^2\,\right)\left(\,\partial_{Q_0}\, e^s\,(X_0)\,\right)\,.
$$

Since $\partial_{Q_0}\, e^{-2s}\,(X_0) = d_{Q_0}\, e^{-2s}\left(\frac{X_0 - i\,J(X_0)}{2}\right)$ and since X_0 and $J(X_0)$ are A_0-horizontal vectors, it is enough to prove

$$
d_{Q_0}\, e^{-2s}\,(X) = -\,\mathrm{Ad}\left(e^{\hat{s}}(u_0)\right)\left(\,d_{Q_0}\, e^s\,(X)\,\right) -\, \mathrm{Ad}\left(\,e^{\hat{s}}(u_0)^2\,\right)\left(\,d_{Q_0}\, e^s\,(X)\,\right)
$$

for all $X \in \left(T_{u_0}{}^{h}Q_0\right)_{A_0}$.

For this, observe first that $\widehat{e^{-2s}} : Q_0 \to G$ is the composition

$$
Q_0 \xrightarrow{e^{\hat{s}}} G \xrightarrow{g \mapsto g^{-1}} G \xrightarrow{g \mapsto g^2} G
$$

and thus its differential $T_{u_0}Q_0 \to T_{e^{\hat{s}}(u_0)^{-2}}\, G$ is given by the composition [3]

$$
\begin{aligned}
Y \;\longmapsto\;& e^{\hat{s}}{}_*(Y) \longmapsto -\,(L_{e^{\hat{s}}(u_0)^{-1}})_*\Big(\,(R_{e^{\hat{s}}(u_0)^{-1}})_*\left(e^{\hat{s}}{}_*(Y)\,\right)\Big)\\
\longmapsto\;& (R_{e^{\hat{s}}(u_0)^{-1}})_*\Big(\, -\,(L_{e^{\hat{s}}(u_0)^{-1}})_*\Big(\,(R_{e^{\hat{s}}(u_0)^{-1}})_*\left(e^{\hat{s}}{}_*(Y)\,\right)\Big)\Big)\\
&+(L_{e^{\hat{s}}(u_0)^{-1}})_*\Big(\, -\,(L_{e^{\hat{s}}(u_0)^{-1}})_*\Big(\,(R_{e^{\hat{s}}(u_0)^{-1}})_*\left(e^{\hat{s}}{}_*(Y)\,\right)\Big)\Big)\,.
\end{aligned}
$$

[3] See footnotes 22 and 23 of Chapter I.

Thus for $X \in \left(T_{u_0}{}^h Q_0\right)_{A_0}$ we have

$$
\begin{aligned}
d_{Q_0}\, e^{-2s}\,(X) &= \omega_G \Big((\widehat{e^{-2s}})_*\,(X) \Big) \\
&= \omega_G \Bigg((R_{e^{\hat{s}}(u_0)^{-1}})_* \Bigg(-\, (L_{e^{\hat{s}}(u_0)^{-1}})_* \Big((R_{e^{\hat{s}}(u_0)^{-1}})_* \big(e^{\hat{s}}{}_*(X) \big) \Big) \Bigg) \\
&\quad + (L_{e^{\hat{s}}(u_0)^{-1}})_* \Bigg(-\, (L_{e^{\hat{s}}(u_0)^{-1}})_* \Big((R_{e^{\hat{s}}(u_0)^{-1}})_* \big(e^{\hat{s}}{}_*(X) \big) \Big) \Bigg) \Bigg) \\
&= -\operatorname{Ad}\left(e^{\hat{s}}(u_0)^2 \right) \Big(\omega_G\big(e^{\hat{s}}{}_*(X) \big) \Big) - \operatorname{Ad}\left(e^{\hat{s}}(u_0) \right) \Big(\omega_G\big(e^{\hat{s}}{}_*(X) \big) \Big) \\
&= -\operatorname{Ad}\left(e^{\hat{s}}(u_0)^2 \right) \left(d_{Q_0}\, e^{s}\,(X) \right) - \operatorname{Ad}\left(e^{\hat{s}}(u_0) \right) \left(d_{Q_0}\, e^{s}\,(X) \right) ,
\end{aligned}
$$

as we wanted.

It remains to prove formula (3.9),

$$
\operatorname{pr}_{i\mathfrak{k}} \Big(2i\, \overline{\partial}_{Q_0}\, e^{s}\,(X_0) \Big) = i\, \overline{\partial}_{Q_0}\, e^{s}\,(X_0) - i \operatorname{Ad}\left(e^{\hat{s}}(u_0) \right) \left(\partial_{Q_0}\, e^{s}\,(X_0) \right) .
$$

For this, we will use the conjugation σ on the holomorphic tangent spaces of G at points of $\exp(i\mathfrak{k})$ which was defined in Section 2.4. Recall that on T_gG, for $g \in \exp(i\mathfrak{k})$, σ is the reflexion about the tangent space of $\exp(i\mathfrak{k})$ at g (in particular, on $\mathfrak{g} = T_eG$ the reflexion about $i\mathfrak{k}$) and that for $X \in T_gG$ it holds

$$
\sigma(X) = (R_g)_* \Big(\sigma \left((L_{g^{-1}})_*\,(X) \right) \Big) . \tag{3.11}
$$

Thus we have

$$
\begin{aligned}
\operatorname{pr}_{i\mathfrak{k}} \Big(2i\, \overline{\partial}_{Q_0}\, e^{s}\,(X_0) \Big) &= \frac{2i\, \overline{\partial}_{Q_0}\, e^{s}\,(X_0) + \sigma\left(2i\, \overline{\partial}_{Q_0}\, e^{s}\,(X_0) \right)}{2} \\
&= i\, \overline{\partial}_{Q_0}\, e^{s}\,(X_0) - i\, \sigma\left(\overline{\partial}_{Q_0}\, e^{s}\,(X_0) \right),
\end{aligned}
$$

so we need to show that $\sigma\left(\overline{\partial}_{Q_0}\, e^{s}\,(X_0) \right) = \operatorname{Ad}\left(e^{\hat{s}}(u_0) \right) \left(\partial_{Q_0}\, e^{s}\,(X_0) \right)$. But, using (3.11), we have

$$
\begin{aligned}
\sigma\left(\overline{\partial}_{Q_0}\, e^{s}\,(X_0) \right) &= \sigma\Big(d_{Q_0}\, e^{s}\left(\tfrac{X_0 + i\, J(X_0)}{2} \right) \Big) = \sigma\left(\frac{d_{Q_0}\, e^{s}\,(X_0) + i\, d_{Q_0}\, e^{s}\big(J(X_0) \big)}{2} \right) \\
&= \frac{\sigma\left(d_{Q_0}\, e^{s}\,(X_0) \right) - i\, \sigma\Big(d_{Q_0}\, e^{s}\big(J(X_0) \big) \Big)}{2} \\
&= \frac{\sigma\Big((L_{e^{\hat{s}}(u_0)^{-1}})_* \left(e^{\hat{s}}{}_*(X_0) \right) \Big) - i\, \sigma\Big((L_{e^{\hat{s}}(u_0)^{-1}})_* \Big(e^{\hat{s}}{}_* \big(J(X_0) \big) \Big) \Big)}{2}
\end{aligned}
$$

$$= \frac{(R_{e^{\hat{s}}(u_0)^{-1}})_* \Big(\sigma \big(e^{\hat{s}}{}_*(X_0) \big) \Big) - i\, (R_{e^{\hat{s}}(u_0)^{-1}})_* \bigg(\sigma \Big(e^{\hat{s}}{}_* \big(J(X_0) \big) \Big) \bigg)}{2}$$

$$= \frac{(R_{e^{\hat{s}}(u_0)^{-1}})_* \big(e^{\hat{s}}{}_*(X_0) \big) - i\, (R_{e^{\hat{s}}(u_0)^{-1}})_* \Big(e^{\hat{s}}{}_* \big(J(X_0) \big) \Big)}{2}$$

$$= (R_{e^{\hat{s}}(u_0)^{-1}})_* \Big(e^{\hat{s}}{}_* \left(\tfrac{X_0 - i\, J(X_0)}{2} \right) \Big) = \mathrm{Ad} \left(e^{\hat{s}}(u_0) \right) \Big(d_{Q_0}\, e^{s} \left(\tfrac{X_0 - i\, J(X_0)}{2} \right) \Big)$$

$$= \mathrm{Ad} \left(e^{\hat{s}}(u_0) \right) \left(\partial_{Q_0}\, e^{s} \left(X_0 \right) \right),$$

as we wanted. □

Corollary 3.2.6 *Suppose that J in Theorem 3.2.5 is a holomorphic structure on $P\,(M, G)$. Then the curvature forms Ω and Ω_0 of A and A_0 are related by the formula*

$$\Omega - \Omega_0 = \overline{\partial} \left(\partial_{Q_0}\, e^{-2s} \right)$$

where $\overline{\partial}$ is the operator associated with the holomorphic structure induced by J on the vector bundle $P \times_{Ad} \mathfrak{g}$.

Proof We have $\Omega = d_A\, \omega$ and $\Omega_0 = d_{A_0}\, \omega_0$, thus

$$\Omega - \Omega_0 = d_A\, (\omega - \omega_0) + (d_A - d_{A_0})\, (\omega_0). \tag{3.12}$$

Denote by $(D^{\mathrm{Ad}})_A$ the connection induced by A on $P \times_{Ad} \mathfrak{g}$. Then we know that $(D^{\mathrm{Ad}})_A = d_A$ (see page 49). We will now calculate the second term of the right hand side of (3.12). Let $X_1,\ X_2 \in \Gamma\,(TP)$. Then, using the structure equation (1.33) and the fact that Ω_0 is a horizontal form, we get

$$\begin{aligned}(d_A - d_{A_0})\,(\omega_0)\,(X_1, X_2) &= d\omega_0 \Big(\left(X_1^h\right)_A, \left(X_2^h\right)_A \Big) - d\omega_0 \Big(\left(X_1^h\right)_{A_0}, \left(X_2^h\right)_{A_0} \Big) \\ &= -\tfrac{1}{2} \Big[\omega_0 \Big(\left(X_1^h\right)_A \Big),\, \omega_0 \Big(\left(X_2^h\right)_A \Big) \Big] + \Omega_0 \Big(\left(X_1^h\right)_A, \left(X_2^h\right)_A \Big) \\ &\quad - \Omega_0 \Big(\left(X_1^h\right)_{A_0}, \left(X_2^h\right)_{A_0} \Big) \\ &= -\tfrac{1}{2} \Big[\omega_0 \Big(\left(X_1^h\right)_A \Big),\, \omega_0 \Big(\left(X_2^h\right)_A \Big) \Big]. \end{aligned} \tag{3.13}$$

Note that for $X \in TP$ we have

$$\begin{aligned} \left(X^h\right)_A - \left(X^h\right)_{A_0} &= X - \left(X^v\right)_A - \left(X^h\right)_{A_0} = \left(X^v\right)_{A_0} - \left(X^v\right)_A \\ &= \omega_0\,(X)^* - \omega\,(X)^* = (\omega_0 - \omega)\,(X)^*\,, \end{aligned}$$

thus

$$\omega_0 \Big(\left(X^h\right)_A \Big) = \omega_0 \Big(\left(X^h\right)_{A_0} + (\omega_0 - \omega)\,(X)^* \Big) = (\omega_0 - \omega)\,(X).$$

Using this, (3.13) becomes

$$(d_A - d_{A_0})\,(\omega_0)\,(X_1, X_2) = -\frac{1}{2} \left[(\omega - \omega_0)\,(X_1)\,,\, (\omega - \omega_0)\,(X_2) \right],$$

briefly

$$(d_A - d_{A_0})\,(\omega_0) = -\frac{1}{2}\,\big[\,\omega - \omega_0\,,\,\omega - \omega_0\,\big].$$

Thus we have

$$\Omega - \Omega_0 = (D^{\mathrm{Ad}})_A\,(\omega - \omega_0) - \frac{1}{2}\,\big[\,\omega - \omega_0\,,\,\omega - \omega_0\,\big].$$

We know that $\omega - \omega_0 = \partial_{Q_0}\,e^{-2s} \in \mathcal{A}^{1,0}(P \times_{Ad} \mathfrak{g})$ and, by Proposition 3.1.5, that $\Omega_0 - \Omega \in \mathcal{A}^{1,1}(P \times_{Ad} \mathfrak{g})$. Thus we get

$$\Omega - \Omega_0 = (D^{\mathrm{Ad}})_A\,''\,(\omega - \omega_0).$$

But $(D^{\mathrm{Ad}})_A\,'' = \overline{\partial}$, as explained in Example 3.1.3, thus we conclude

$$\Omega - \Omega_0 = \overline{\partial}\,(\omega - \omega_0) = \overline{\partial}\,\big(\,\partial_{Q_0}\,e^{-2s}\,\big).$$

□

Proposition 3.2.7 *Let $Q_0\,(M,K)$ and $Q\,(M,K)$ be reductions of $P\,(M,G)$ and write $Q = e^s\,(Q_0)$, for some $s \in \Gamma\,(Q_0 \times_{Ad} i\mathfrak{k})$. Let $A := A_{J,Q}$ and $A_0 := A_{J,Q_0}$ be the Chern connections with respect to an almost holomorphic structure J on $P\,(M,G)$. Denote by f_s the map $Q_0 \to Q$, $u \mapsto u\,e^{\hat{s}(u)}$ and consider the connection $f_s^*(A)$ on $Q_0\,(M,K)$, as defined in Proposition 1.4.19. Then the connection and curvature forms $\omega_{f_s^*(A)}$ and $\Omega_{f_s^*(A)}$ of $f_s^*(A)$ are given by*

$$\omega_{f_s^*(A)} - \omega_0 = \overline{\partial}_{Q_0}\,e^s - \mathrm{Ad}\,(e^{\hat{s}})\,\big(\partial_{Q_0}\,e^s\,\big)$$

where ω_0 is the connection form of A_0, and

$$\Omega_{f_s^*(A)} = \mathrm{Ad}\,(e^{-\hat{s}})\,(\Omega)$$

where Ω is the curvature form of A. In particular, if J is integrable then

$$\Omega_{f_s^*(A)} = \mathrm{Ad}\,(e^{-\hat{s}})\,\Big(\,\Omega_0 + \overline{\partial}\,\big(\,\partial_{Q_0}\,e^{-2s}\,\big)\,\Big).$$

Proof For the first equality, it is enough to prove that

$$(\omega_{f_s^*(A)} - \omega_0)\,(X) = \overline{\partial}_{Q_0}\,e^s\,(X) - \mathrm{Ad}\,\big(e^{\hat{s}(u)}\big)\,\big(\partial_{Q_0}\,e^s\,(X)\,\big)$$

for $u \in Q_0$ and $X \in \big(\,T_u^h Q_0\,\big)_{A_0}$. We have seen in the proof of Theorem 3.2.5 that

$$\omega\,\big(f_{s*}(X)\big) = \overline{\partial}_{Q_0}\,e^s\,(X) - \mathrm{Ad}\,\big(e^{\hat{s}(u)}\big)\,\big(\,\partial_{Q_0}\,e^s\,(X)\,\big),$$

where ω is the connection form of A (see formula (3.10)). Since $\omega_{f_s^*(A)} = f_s^*(\omega)$ (see Proposition 1.4.19), we get

$$\begin{aligned}(\omega_{f_s^*(A)} - \omega_0)\,(X) &= \omega\,\big(f_{s*}(X)\big) - \omega_0\,(X) = \omega\,\big(f_{s*}(X)\big) \\ &= \overline{\partial}_{Q_0}\,e^s\,(X) - \mathrm{Ad}\,\big(e^{\hat{s}(u)}\big)\,\big(\,\partial_{Q_0}\,e^s\,(X)\,\big),\end{aligned}$$

as we wanted.
Let now $X_1,\, X_2 \in T_u Q_0$. Then, since Ω is G-equivariant and horizontal, we have, using (3.6)

$$\begin{aligned} \mathrm{Ad}\left(e^{-\hat{s}(u)}\right)\left(\Omega(X_1, X_2)\right) &= \Omega\Big(\,(R_{e^{\hat{s}(u)}})_*(X_1), (R_{e^{\hat{s}(u)}})_*(X_2)\,\Big) \\ &= \Omega\left(\,f_{s*}(X_1), f_{s*}(X_2)\,\right) = f_s^*\,\Omega\,(X_1, X_2) \\ &= \Omega_{f_s^*(A)}\,(X_1, X_2)\;, \end{aligned}$$

where the last equality follows from Proposition 1.4.19.
The last statement is then a consequence of this and Corollary 3.2.6. □

Chapter 4

The Hermite-Einstein equation

The aim of this chapter is to discuss some aspects of an important application of the Chern correspondence and in particular of Theorem 3.2.5 and its corollary.

Let G be a complex reductive Lie group with a compact real form K and let $P\,(M, G)$ be a complex principal fibre bundle over a compact base space. One is interested in finding a K-reduction $Q\,(M, K)$ of $P\,(M, G)$ such that the curvature form $\Omega_{A_{J,Q}}$ of the corresponding Chern connection with respect to a fixed holomorphic structure J on $P\,(M, G)$ satisfies the *Hermite-Einstein equation*

$$\Lambda_g\left(\Omega_{A_{J,Q}}\right) = C \tag{4.1}$$

for some constant C in the center of $\mathfrak{g}$. Here Λ_g is the *contraction operator* associated to an Hermitian metric g on M, which maps (1,1)-forms to 0-forms.

In [19] it is proved that under certain conditions on J and for certain values of C the Hermite-Einstein equation has a solution. The strategy of the proof is to fix a reduction $Q_0\,(M, K)$ and to write an arbitrary reduction $Q\,(M, K)$ as $Q = e^s\,(Q_0)$, for some $s \in \Gamma(Q_0 \times_{Ad} i\mathfrak{k})$. Then, using the formula we obtained in Corollary 3.2.6, (4.1) becomes a differential equation for s, which is solved. In this chapter we will not go into this, but we will only give a necessary condition for C in order to have a solution. In the vector bundle case, we will show that this condition determines C. We will only consider the case when g is a Kähler metric on M, but we will mention that actually only a much weaker assumption on g is needed. We refer to [18] and [19] for a discussion of how this fact can be used to generalize the results of this chapter to the Hermitian case.

The condition on C is obtained in the last section of this chapter, using the results of Paragraphs 4.2 and 4.3. Section 4.1 contains some algebraic preliminaries. General references for the first three sections of this chapter are [15, Chapter XII], [2, Appendix on Geometry of Characteristic Classes], [4, Chapter 4] and, only for 4.3, [22, Appendix C], [6, §3 of Chapter 3], [32, §3 of Chapter III], [13, §2 of Chapter II] and [11, Chapter 19].

4.1 Invariant polynomials

Throughout this section, V will be a complex n-dimensional vector space [1].

Let k be a positive integer. A linear map $\underbrace{V \otimes \ldots \otimes V}_{k} \to \mathbb{C}$ is said to be *symmetric* if $f\left(v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(k)}\right) = f\left(v_1 \otimes \ldots \otimes v_k\right)$, for all permutations σ of $(1, \ldots, k)$. The vector space of symmetric linear maps $f : \underbrace{V \otimes \ldots \otimes V}_{k} \to \mathbb{C}$ will be denoted by $S^k(V^*)$. We set $S^0(V^*) := \mathbb{C}$ and

$$S^*(V^*) := \bigoplus_{k=0}^{\infty} S^k(V^*).$$

We define a product in $S^*(V^*)$ as follows. For $f_1 \in S^k(V^*)$ and $f_2 \in S^l(V^*)$, $f_1 \cdot f_2$ is an element of $S^{k+l}(V^*)$, given by

$$f_1 \cdot f_2\left(v_1 \otimes \ldots \otimes v_{k+l}\right) := \frac{1}{(k+l)!} \sum_{\sigma} f_1\left(v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(k)}\right) f_2\left(v_{\sigma(k+1)} \otimes \ldots \otimes v_{\sigma(k+l)}\right)$$

for $v_1, \ldots, v_{k+l} \in V$, where the summation is taken over all permutations σ of $(1, \ldots, k+l)$. The product is extended to $S^*(V^*)$ by distributivity. This gives $S^*(V^*)$ the structure of a $\mathbb{C}$-algebra.

Let $v_1^*, \ldots, v_n^*$ be a basis of V^*. A *(homogeneous) polynomial function* on V is a function $f : V \to \mathbb{C}$ that can be expressed as a (homogeneous) polynomial of $v_1^*, \ldots, v_n^*$. This concept does not depend on the choice of the basis of V^*. Observe that the set of polynomial function on V has a natural $\mathbb{C}$-algebra structure.

Proposition 4.1.1 *Given a map $f \in S^k(V^*)$, define a polynomial function $\tilde{f}$ on V by $\tilde{f}(v) := f\left(v \otimes \ldots \otimes v\right)$, for $v \in V$. Then $f \mapsto \tilde{f}$ is a vector space isomorphism between $S^k(V^*)$ and the vector space of homogeneous polynomial functions of degree k on V. The linear extension of this map gives an algebra isomorphism between $S^*(V^*)$ and the algebra of polynomial functions on V.*

Proof It is clear that $f \mapsto \tilde{f}$ is linear and that $\widetilde{f_1 \cdot f_2} = \tilde{f}_1 \cdot \tilde{f}_2$. To complete the proof we will exhibit the inverse of $f \mapsto \tilde{f}$. Let P be a homogeneous polynomial function of degree k on V and write it as $P = \sum_{1 \leq i_1, \ldots, i_k \leq n} a_{i_1, \ldots, i_k} v_{i_1}^* \ldots v_{i_k}^*$, where $a_{i_1, \ldots, i_k} \in \mathbb{C}$. Define $P_0 \in S^k(V^*)$ by

$$P_0\left(v_1 \otimes \ldots \otimes v_k\right) := \frac{1}{k!} \sum_{1 \leq i_1, \ldots, i_k \leq n} a_{i_1, \ldots, i_k} \sum_{\sigma} v_{i_1}^*(v_{\sigma(1)}) \ldots v_{i_k}^*(v_{\sigma(k)})$$

[1] Of course, all result of the first three paragraphs of this chapter (except Proposition 4.3.3 and Remark 4.3.4) also hold in the real case: Paragraph 4.1 for real vector spaces, Paragraph 4.2 for real principal fibre bundles and Paragraph 4.3 for complex vector bundles over real manifolds.

where the second summation is taken over all permutations σ of $(1,\ldots,k)$. Then it is easy to see that $\tilde{P}_0 = P$ for all homogeneous polynomial functions P of degree k on V, and $(\tilde{f})_0 = f$ for all $f \in S^k(V^*)$, thus $P \mapsto P_0$ is the inverse of $f \mapsto \tilde{f}$. □

Let G be a Lie group and suppose we have a representation $\varrho : G \to \mathrm{Aut}(V)$ of G on V. Then a map $f \in S^k(V^*)$ is said to be *G-invariant* if it holds

$$f\left(\varrho(g)(v_1) \otimes \ldots \otimes \varrho(g)(v_k)\right) = f\left(v_1 \otimes \ldots \otimes v_k\right)$$

for all $g \in G$ and $v_1,\ldots,v_k \in V$. A polynomial function P on V is said to be *G-invariant* if $P\left(\varrho(g)(v)\right) = P(v)$, for all $g \in G$ and $v \in V$.
Note that the G-invariant forms are a subalgebra of $S^*(V^*)$ and that, under the isomorphism of Proposition 4.1.1, they correspond to G-invariant polynomial functions of V.

Example 4.1.2 *Let G be a complex Lie group with Lie algebra $\mathfrak{g}$ and consider the representation* $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$. *We will denote by $I^k(G)$, for a non-negative integer k, the vector space of G-invariant forms in $S^k(\mathfrak{g}^*)$, and by $I(G)$ the algebra of G-invariant forms in $S^*(\mathfrak{g}^*)$. Elements of $I(G)$ are called invariant polynomial functions (or simply* invariant polynomials*) on G.*

Example 4.1.3 *The maps* $\det : \mathfrak{gl}(n,\mathbb{C}) \to \mathbb{C}$ *and* $\mathrm{trace} : \mathfrak{gl}(n,\mathbb{C}) \to \mathbb{C}$ *are invariant polynomials on $GL(n,\mathbb{C})$.*

4.2 The Weil homomorphism

Let $P(M,G)$ be a complex principal fibre bundle and let A be a connection on it, with curvature form Ω_A. Regard Ω_A as a complex $\mathfrak{g}$-valued 2-form on P and consider the $\underbrace{\mathfrak{g}\otimes\ldots\otimes\mathfrak{g}}_{k}$-valued complex form ${\Omega_A}^k := \underbrace{\Omega_A \wedge \ldots \wedge \Omega_A}_{k}$ on P [2]. Given an invariant polynomial $f \in I^k(G)$, we get then a form $f\left({\Omega_A}^k\right) \in \mathcal{A}^{2k}(P)^{\mathbb{C}}$; note that $f\left({\Omega_A}^k\right)$ projects to a unique $2k$-form $f\left({\Omega_A}^k\right)_0$ on M, which is given by

$$f\left({\Omega_A}^k\right)_0(Y_1,\ldots,Y_{2k}) := f\left({\Omega_A}^k\right)(X_1,\ldots,X_{2k})$$

[2] The wedge product for vector-valued differential forms on a manifold N is defined as follows. Let V and W be two vector spaces. Given a V-valued k-form ω_1 and a W-valued l-form ω_2 on N, $\omega_1 \wedge \omega_2$ is a $V \otimes W$-valued $(k+l)$-form on N, defined by

$$(\omega_1 \wedge \omega_2)(X_1,\ldots,X_{k+l}) := \frac{1}{(k+l)!}\sum_\sigma (-1)^\sigma \omega_1\ X_{\sigma(1)},\ldots,X_{\sigma(k)}\ \otimes \omega_2\ X_{\sigma(k+1)},\ldots,X_{\sigma(k+l)}$$

for $X_1,\ldots,X_{k+l} \in \Gamma(TN)$, where the summation is taken over all permutations σ of $(1,\ldots,k+l)$. It is easy to see that the wedge product is associative and that it holds

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$$

where d is the exterior derivative defined in footnote 21 of Chapter I.
See for example [27, §5 of Chapter I] for more details over differential forms with values in a vector space.

for $p \in M$ and $Y_1, \ldots, Y_{2k} \in T_pM$, where $X_1, \ldots, X_{2k}$ are vectors in T_uP, for some u in the fibre of P over p, with $\pi_*(X_i) = Y_i$, for $i = 1, \ldots, 2k$. Observe that this is well-defined, because $f\big({\Omega_A}^k\big)$ is horizontal (since so is Ω_A) and because for $g \in G$ it holds ${R_g}^* f\big({\Omega_A}^k\big) = f\big({\Omega_A}^k\big)$ (since Ω_A is G-equivariant and f is G-invariant). In the following, $f\big({\Omega_A}^k\big)_0 \in \mathcal{A}^{2k}(M)$ will be also denoted by $f\big({\Omega_A}^k\big)$. It will be clear from the context when $f\big({\Omega_A}^k\big)$ will be regarded as a form on P and when as a form on M.

Theorem 4.2.1 *Let $P\,(M,G)$ be a complex principal fibre bundle and $I(G)$ the algebra of invariant polynomials on G. Then:*

1. *for each $f \in I^k(G)$ and each connection A on $P\,(M,G)$, the $2k$-form $f\big({\Omega_A}^k\big)$ on M defined above is closed;*
2. *for each $f \in I^k(G)$, the element $\big[\, f\big({\Omega_A}^k\big)\,\big]$ of the De Rham cohomology group $H^{2k}(M,\mathbb{C})$, where A is any connection on $P\,(M,G)$, is well-defined, i.e. it does not depend on the choice of A;*
3. *the map $I(G) \to H^*(M,\mathbb{C})$, $f \mapsto \big[\, f\big({\Omega_A}^k\big)\,\big]$ (where A is any connection on $P\,(M,G)$ and where $k = \deg(f)$) is an algebra homomorphism (the **Weil homomorphism**).*

Proof We follow [15] for 1. and [22] for 2. See these references for more details.

Let $f \in I^k(G)$ and let A be a connection on $P\,(M,G)$, with curvature form Ω_A. We will show that $d\,f\big({\Omega_A}^k\big) = 0$, where $f\big({\Omega_A}^k\big)$ is regarded as a form on P. Then the same will be true, also when we regard $f\big({\Omega_A}^k\big)$ as a form on M.
Observe first that if $\tilde{\varphi} \in \mathcal{A}^r(M)$ and $\varphi = \pi^*\,\tilde{\varphi}$, where π is the projection $P \to M$, then $d\varphi = d_A\varphi$. Indeed, for $X_1, \ldots, X_r \in \Gamma(TP)$ we have

$$\begin{aligned} d\varphi\,(X_1,\ldots,X_r) &= d\,(\pi^*\,\tilde{\varphi})\,(X_1,\ldots,X_r) = \pi^*\,d\varphi\,(X_1,\ldots,X_r) \\ &= \pi^*\,d\varphi\,\Big(\,\big(X_1^h\big)_A,\ldots,\big(X_r^h\big)_A\Big) \\ &= d\,(\pi^*\,\tilde{\varphi})\,\Big(\,\big(X_1^h\big)_A,\ldots,\big(X_r^h\big)_A\Big) \\ &= d\varphi\,\Big(\,\big(X_1^h\big)_A,\ldots,\big(X_r^h\big)_A\Big) = d_A\varphi\,(X_1,\ldots,X_r). \end{aligned}$$

Thus we have to show that $d_A\,f\big({\Omega_A}^k\big) = 0$. But, using the the Bianchi identity $d_A\,\Omega_A = 0$ (see Proposition 1.4.15), we have

$$d_A\,f\big({\Omega_A}^k\big) = f\,\big(\,d_A\,{\Omega_A}^k\,\big) = k\,f\,\big(\,d_A\,\Omega_A \wedge {\Omega_A}^{k-1}\,\big) = 0.$$

Let now A_0 and A_1 be connections on $P\,(M,G)$, with connection and curvature forms ω_0, ω_1 and Ω_0, Ω_1 respectively. We have to show that for $f \in I^k(G)$ it holds $\big[\, f\big({\Omega_0}^k\big)\,\big] = \big[\, f\big({\Omega_1}^k\big)\,\big]$. Consider the map $\mathrm{pr}_1 : M \times \mathbb{R} \to M$, $(p,t) \mapsto p$ and the pullback bundle $\mathrm{pr}_1{}^*(P)\,(M \times \mathbb{R}, G, \pi')$ (see Example 1.2.8). Let A_0' and A_1' be the connections induced on $\mathrm{pr}_1{}^*(P)\,(M \times \mathbb{R}, G)$ by A_0 and A_1 respectively (see Proposition 1.4.19) and let ω_0' and ω_1' be their connection forms. Then we have $\omega_0' = \overline{\mathrm{pr}}_1{}^*\,\omega_0$ and $\omega_1' = \overline{\mathrm{pr}}_1{}^*\,\omega_1$ where $\overline{\mathrm{pr}} : \mathrm{pr}_1{}^*(P) \to P$ is the map between total spaces inducing the homomorphism $\mathrm{pr}_1{}^*(P)\,(M \times \mathbb{R}, G) \to P\,(M, G)$, as described in Example 1.2.8. Define a connection A on $\mathrm{pr}_1{}^*(P)\,(M \times \mathbb{R}, G)$ with connection form ω_A by

$$\omega_A := \omega_0' + (\mathrm{pr}_2 \circ \pi')\,(\omega_1' - \omega_0')$$

where pr_2 is the projection $M \times \mathbb{R} \to \mathbb{R}$. Note that, by Proposition 1.3.2, ω_A is indeed a connection form. Consider the maps $i_0 : M \to M \times \mathbb{R}$, $p \mapsto (p, 0)$ and $i_1 : M \to M \times \mathbb{R}$, $p \mapsto (p, 1)$. Then $P\,(M, G)$ is the pullback bundle of $\mathrm{pr}_1{}^*(P)\,(M \times \mathbb{R}, G)$ with respect to i_0 and with respect to i_1 and the connections on $P\,(M, G)$ induced by A are A_0 and A_1 respectively; indeed, an easy calculation shows that $\omega_0 = \overline{i_0}^*\,\omega_A$ and $\omega_1 = \overline{i_1}^*\,\omega_A$, where $\overline{i_0}$ and $\overline{i_1}$ are the maps $P \to \mathrm{pr}_1{}^*(P)$ corresponding to i_0 and i_1 respectively. In particular, we have $\Omega_0 = \overline{i_0}^*\,\Omega_A$ and $\Omega_1 = \overline{i_1}^*\,\Omega_A$, where Ω_A is the curvature form of A, and so $f\big(\Omega_0{}^k\big) = i_0{}^*\Big(f\big(\Omega_A{}^k\big)\Big)$ and $f\big(\Omega_1{}^k\big) = i_1{}^*\Big(f\big(\Omega_A{}^k\big)\Big)$. Since i_0 and $i_1 : M \to M \times \mathbb{R}$ are homotopic, it follows that $\big[\,f\big(\Omega_0{}^k\big)\,\big] = \big[\,f\big(\Omega_1{}^k\big)\,\big]$.

3. can be seen as follows. Given a permutation σ of $(1, \ldots, k+l)$, denote by T_σ the automorphism of $\underbrace{\mathfrak{g} \otimes \ldots \otimes \mathfrak{g}}_{k+l}$ defined by $T_\sigma\,(B_1 \otimes \ldots \otimes B_{k+l}\,) = B_{\sigma(1)} \otimes \ldots B_{\sigma(k+l)}$. Then, since Ω_A has degree 2, we have $\Omega_A{}^{k+l} = T_\sigma \circ \Omega_A{}^{k+l}$. Thus for $f_1 \in I^k(G)$ and $f_2 \in I^l(G)$ it holds

$$\begin{aligned}
f_1 \cdot f_2\,\big(\Omega_A{}^{k+l}\big) &= \frac{1}{(k+l)!}\sum_\sigma (f_1 \otimes f_2) \circ T_\sigma \circ \Omega_A{}^{k+l} \\
&= \frac{1}{(k+l)!}\sum_\sigma (f_1 \otimes f_2) \circ \Omega_A{}^{k+l} = (f_1 \otimes f_2)\,\Omega_A{}^k \wedge \Omega_A{}^l \\
&= f_1\big(\Omega_A{}^k\big) \wedge f_2\big(\Omega_A{}^l\big).
\end{aligned}$$

□

4.3 Chern classes of a complex vector bundle

Let $\pi : E \to M$ be a complex vector bundle of rank n over a complex manifold and let D be a connection on it, with curvature R_D. Consider a system of local trivializations $\{\,\theta_i : \pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{C}^n,\ i \in I\,\}$ of E with respect to a cover $\{\mathcal{U}_i,\ i \in I\,\}$ of M and let $\{\,u^i,\ i \in I\,\}$ be the corresponding local frames, i.e. $u^i{}_\alpha(p) = \theta_i^{-1}(p, e_\alpha)$ for $p \in \mathcal{U}$ and $\alpha = 1, \ldots, n$. For every $i \in I$, let $(\Omega_D)_i$ be the curvature form of D with respect to the local frame u^i. Given an invariant polynomial $f \in I^k\big(GL(n, \mathbb{C})\big)$, we can consider the local forms $\{\,f\big(\,(\Omega_D)_i{}^k\,\big) \in \mathcal{A}^{2k}(\mathcal{U}_i)$,

$i \in I\}$. By (1.18) we have $(\Omega_D)_j = \theta_{ji}\,(\Omega_D)_i\,\theta_{ij}$ on $\mathcal{U}_{ij}$ and so, since f is $GL(n,\mathbb{C})$-invariant, it follows that $f\big(\,(\Omega_D)_j^{\,k}\,\big) = f\big(\,(\Omega_D)_i^{\,k}\,\big)$ on $\mathcal{U}_{ij}$ (recall that the adjoint representation of $GL(n,\mathbb{C})$ is given by $\mathrm{Ad}(A)(X) = AXA^{-1}$ for $A \in GL(n,\mathbb{C})$ and $X \in \mathfrak{gl}\,(n,\mathbb{C})$, see (1.7)). Thus the $\{\,f\big(\,(\Omega_D)_i^{\,k}\big),\ i \in I\,\}$ glue together to give a well-defined global $2k$-form on M, denoted by $f\big(\,{R_D}^k\big)$.

The following theorem is an immediate consequence of Theorem 4.2.1, Theorem 1.4.12 and Lemma 1.4.13. A direct proof can be found for example in [22, p. 296-298] or [13, p. 37-38].

Theorem 4.3.1 *Let $\pi : E \to M$ be a complex vector bundle of rank n over a complex manifold and $I\big(GL(n,\mathbb{C})\big)$ the algebra of invariant polynomials on $GL(n,\mathbb{C})$. Then:*

1. *for each $f \in I^k\big(GL(n,\mathbb{C})\big)$ and each connection D on $\pi : E \to M$, the $2k$-form $f\big({R_D}^k\big)$ on M defined above is closed;*
2. *for each $f \in I^k\big(GL(n,\mathbb{C})\big)$, the element $\big[\,f\big({R_D}^k\big)\,\big]$ of the De Rham cohomology group $H^{2k}(M,\mathbb{C})$, where D is any connection on $\pi : E \to M$, is well-defined, i.e. it does not depend on the choice of D;*
3. *the map $I\big(GL(n,\mathbb{C})\big) \to H^*(M,\mathbb{C})$, $f \mapsto \big[\,f\big({R_D}^k\big)\,\big]$ (where D is any connection on $\pi : E \to M$ and where $k = \deg(f)$) is an algebra homomorphism.*

For $k = 1,\dots,n$, define homogeneous polynomial functions f_k of degree k on $\mathfrak{gl}\,(n,\mathbb{C})$ by

$$\det\,(I_n + \frac{i}{2\pi}\,X) = 1 + f_1(X) + f_2(X) + \cdots + f_n(X)$$

for $X \in \mathfrak{gl}\,(n,\mathbb{C})$. Since

$$\det\,(I_n + \frac{i}{2\pi}\,AXA^{-1}) = \det\big(\,A\,(I_n + \frac{i}{2\pi}\,X)\,A^{-1}\,\big) = \det\,(I_n + \frac{i}{2\pi}\,X)$$

for all $A \in GL(n,\mathbb{C})$ and $X \in \mathfrak{gl}\,(n,\mathbb{C})$, it follows that the f_k's are $GL(n,\mathbb{C})$-invariant, thus $f_k \in I^k\big(GL(n,\mathbb{C})\big)$, for $k = 1,\dots,n$. For example, note that $f_1 : \mathfrak{gl}\,(n,\mathbb{C}) \to \mathbb{C}$ is given by $X \mapsto \frac{i}{2\pi}\,\mathrm{tr}\,(X)$.

Definition 4.3.2 *Let $\pi : E \to M$ be a complex vector bundle of rank n over a complex manifold and for $k = 1,\dots,n$ let $f_k \in I^k\big(GL(n,\mathbb{C})\big)$ be the invariant polynomial defined above. The k-th Chern form of $\pi : E \to M$ with respect to a connection D is the closed $2k$-form $c_k(E,D) := f_k\big({R_D}^k\big)$ on M. The k-**th Chern class** of $\pi : E \to M$ is $c_k(E) := \big[\,f_k\big({R_D}^k\big)\,\big] \in H^{2k}(M,\mathbb{C})$.*

Proposition 4.3.3 *Let M be a compact m-dimensional Kähler manifold with Kähler metric g and denote by $k_g \in \mathcal{A}^{1,1}(M)$ the associated Kähler form (see* *page 62). Let $\pi : E \to M$ be a complex vector bundle over M and consider the first Chern form $c_1(E, D)$ of E with respect to a connection D. Then*

$$\deg(E) := \int_M c_1(E, D) \wedge {k_g}^{m-1}$$

does not depend on the connection D. It is called the ***degree*** *of the vector bundle $\pi : E \to M$.*

Proof Let D' be another connection on $\pi : E \to M$. Then by Theorem 4.3.1 we have $c_1(E, D') = c_1(E, D) + d\alpha$, for some $\alpha \in \mathcal{A}^1(M)$. Thus

$$\begin{aligned}\int_M c_1(E, D') \wedge {k_g}^{m-1} &= \int_M \big(c_1(E, D) + d\alpha \big) \wedge {k_g}^{m-1} \\ &= \int_M c_1(E, D) \wedge {k_g}^{m-1} + \int_M d\alpha \wedge {k_g}^{m-1},\end{aligned}$$

so we need to show that $\int_M d\alpha \wedge {k_g}^{m-1} = 0$. But this follows from the Theorem of Stokes, since, using the fact that k_g is closed, we have

$$\int_M d\alpha \wedge {k_g}^{m-1} = \int_M d\,(\,\alpha \wedge {k_g}^{m-1}\,).$$

□

Remark 4.3.4 *Let g be an Hermitian metric on a complex m-dimensional manifold M and let $k_g \in \mathcal{A}^{1,1}(M)$ be its Kähler form. Then g is called a* Gauduchon metric *if $\partial\overline{\partial}\, {k_g}^{m-1} = 0$. Note that if g is a Kähler metric, then in particular it is Gauduchon, because from $dk_g = 0$ it follows that $\partial k_g = 0$ and $\partial\overline{\partial}\, {k_g}^{m-1} = 0$.*

In Proposition 4.3.3 we have defined the degree of a complex vector bundle over a compact Kähler manifold. We will now show that if we have a fixed holomorphic structure on a complex vector bundle over a compact manifold then, in order to define its degree, we only need the metric on the base space to be Gauduchon.

Let $\pi : E \to M$ be a holomorphic vector bundle over a compact complex manifold and suppose that we have a Gauduchon metric g on M, with Kähler form k_g. Let h be an Hermitian metric on $\pi : E \to M$ and consider the Chern connection D_h. Define $\deg(E) := \int_M c_1(E, D_h) \wedge {k_g}^{m-1}$. We claim that this definition does not depend on the choice of the Hermitian metric h on $\pi : E \to M$. Indeed, let $D_{h'}$ be the Chern connection on $\pi : E \to M$ with respect to another Hermitian metric h'. Then it can be shown that $c_1(E, D_{h'}) = c_1(E, D_h) + \overline{\partial}\partial\beta$, for some $\beta \in \mathcal{C}^\infty(M)$ (see [18, Lemma 1.1.18]). Thus

$$\begin{aligned}\int_M c_1(E, D_{h'}) \wedge {k_g}^{m-1} &= \int_M \big(c_1(E, D_h) + \overline{\partial}\partial\beta \big) \wedge {k_g}^{m-1} \\ &= \int_M c_1(E, D_h) \wedge {k_g}^{m-1} + \int_M \overline{\partial}\partial\beta \wedge {k_g}^{m-1},\end{aligned}$$

so all we need to show is that $\int_M \overline{\partial}\partial\beta \wedge {k_g}^{m-1} = 0$. *We have*

$$\begin{aligned}
\overline{\partial}\partial\beta \wedge {k_g}^{m-1} &= \overline{\partial}\,(\,\partial\beta \wedge {k_g}^{m-1}\,) + \partial\beta \wedge \overline{\partial}\,{k_g}^{m-1} \\
&= \overline{\partial}\,(\,\partial\beta \wedge {k_g}^{m-1}\,) + \partial\,(\,\beta \wedge \overline{\partial}\,{k_g}^{m-1}\,) - \beta \wedge \partial\overline{\partial}\,{k_g}^{m-1} \\
&= \overline{\partial}\,(\,\partial\beta \wedge {k_g}^{m-1}\,) + \partial\,(\,\beta \wedge \overline{\partial}\,{k_g}^{m-1}\,)
\end{aligned}$$

where in the last equality we use $\partial\overline{\partial}\,{k_g}^{m-1} = 0$. *Since* $\partial\beta \wedge {k_g}^{m-1}$ *and* $\beta \wedge \overline{\partial}\,{k_g}^{m-1}$ *are respectively of type* $(m, m-1)$ *and* $(m-1, m)$ *and by the Theorem of Stokes we have*

$$\int_M \overline{\partial}\,(\,\partial\beta \wedge {k_g}^{m-1}\,) = \int_M d\,(\,\partial\beta \wedge {k_g}^{m-1}\,) = 0$$

and

$$\int_M \partial\,(\,\beta \wedge \overline{\partial}\,{k_g}^{m-1}\,) = \int_M d\,(\,\beta \wedge \overline{\partial}\,{k_g}^{m-1}\,) = 0.$$

Thus $\int_M \overline{\partial}\partial\beta \wedge {k_g}^{m-1} = 0$, *as we wanted.*

It can be shown (see [5, p.502]) that if M *is compact then for every Hermitian metric* g *there exists a positive function* $\varphi \in \mathcal{C}^\infty(M)$ *such that* $g_0 := \varphi \cdot g$ *is Gauduchon. If* M *is connected and* $m \geq 2$, *then* g_0 *is unique up to a positive constant. This fact can be used to define the degree of holomorphic vector bundles on arbitrary compact Hermitian manifolds. See [18, Remark 1.3.17] for a discussion of this.*

4.4 The contraction operator Λ

Let g be an Hermitian metric on a complex manifold M (i.e. an Hermitian metric on the vector bundle $T^{\mathbb{C}}M = T'M$). Denote by $\underline{g}$ the Hermitian metric on $(T^{\mathbb{R}}M, J)$ induced by g via the vector bundle isomorphism $\phi : (T^{\mathbb{R}}M, J) \to T^{\mathbb{C}}M$ described in Section 2.2, and by $\underline{g}^*$ the dual metric on $(T^{\mathbb{R}}M^*, J^*)$, which is defined pointwise by requiring the dual of a $\underline{g}$-orthonormal frame on $(T^{\mathbb{R}}M, J)$ to be $\underline{g}^*$-orthonormal. By Proposition 2.1.5, $\underline{g}^*$ induces a $\mathbb{C}$-bilinear symmetric non-degenerate map $\tilde{g}^* : TM^* \times TM^* \to \mathbb{C}$, and this can be extended to a $\mathcal{C}^\infty(M, \mathbb{C})$-bilinear symmetric non-degenerate map $\bigwedge \tilde{g}^* : \mathcal{A}^r(M) \times \mathcal{A}^r(M) \to \mathcal{C}^\infty(M, \mathbb{C})$, for every positive integer r. Recall that we have $\overline{\bigwedge \tilde{g}^*\,(\alpha, \beta)} = \bigwedge \tilde{g}^*\,(\bar{\alpha}, \bar{\beta})$, for all α, $\beta \in \mathcal{A}^r(M)$ and note that if $\alpha \in \mathcal{A}^{s,q}(M)$, $\beta \in \mathcal{A}^{s+q}(M)$ and $\bigwedge \tilde{g}^*\,(\alpha, \beta) \neq 0$, then $\beta \in \mathcal{A}^{q,s}(M)$.

Definition 4.4.1 *Let* M *be a complex manifold and* g *an Hermitian metric on it, with Kähler form* $k_g \in \mathcal{A}^{1,1}(M)$. *The* Lefschetz operator *associated to* g *is the map* $L_g : \mathcal{A}(M) \to \mathcal{A}(M)$, $\alpha \mapsto \alpha \wedge k_g$. *The* contraction operator *associated to* g *is the map* $\Lambda_g : \mathcal{A}^r(M) \to \mathcal{A}^{r-2}(M)$ *defined by*

$$\bigwedge \tilde{g}^*\,(\,\Lambda_g(\alpha), \beta\,) = \bigwedge \tilde{g}^*\,(\,\alpha, L_g(\beta)\,)$$

for $r > 2$, $\alpha \in \mathcal{A}^r(M)$ *and* $\beta \in \mathcal{A}^{r-2}(M)$ *and by*

$$\Lambda_g(\alpha) = \bigwedge \tilde{g}^*\,(\,\alpha, k_g\,)$$

for $\alpha \in \mathcal{A}^2(M)$.

Lemma 4.4.2 *Let* M *be an* m*-dimensional complex manifold with an Hermitian metric* g. *Let* $k_g \in \mathcal{A}^{1,1}(M)$ *be the Kähler form,* L_g *the Lefschetz operator and* Λ_g *the contraction operator associated to* g. *Then:*

1. Λ_g *maps* $\mathcal{A}^{s,q}(M)$ *to* $\mathcal{A}^{s-1,q-1}(M)$*;*

2. L_g *and* Λ_g *are* real operators, *i.e.* $L_g(\bar{\alpha}) = \overline{L_g(\alpha)}$ *and* $\Lambda_g(\bar{\alpha}) = \overline{\Lambda_g(\alpha)}$ *for all* $\alpha \in \mathcal{A}(M)$*;*

3. *for every* $\alpha \in \mathcal{A}^2(M)$ *we have*

$$L_g{}^{m-1}(\alpha) = \frac{1}{m}\,\Lambda_g(\alpha)\,k_g{}^m.$$

Proof Let $\alpha \in \mathcal{A}^{s,q}(M)$, for $s+q > 2$; then either $\Lambda_g(\alpha) = 0 \in \mathcal{A}^{s-1,q-1}(M)$, or we can find a form $\beta \in \mathcal{A}^{s+q-2}(M)$ such that

$$\bigwedge \tilde{g}^*\big(\,\alpha, L_g(\beta)\,\big) = \bigwedge \tilde{g}^*\big(\,\Lambda_g(\alpha), \beta\,\big) \neq 0.$$

In this last case, by what was observed before Definition 4.4.1, we must have $L_g(\beta) \in \mathcal{A}^{q,s}(M)$ and thus $\beta \in \mathcal{A}^{q-1,s-1}(M)$ and again $\Lambda_g(\alpha) \in \mathcal{A}^{s-1,q-1}(M)$. Let now $\alpha \in \mathcal{A}^{s,q}(M)$, for $s+q = 2$; then either $\Lambda_g(\alpha) = 0 \in \mathcal{A}^{s-1,q-1}(M)$, or $\bigwedge \tilde{g}^*\,(\,\alpha, k_g\,) = \Lambda_g(\alpha) \neq 0$. In this last case we must have then $\alpha \in \mathcal{A}^{1,1}(M)$. Since $\Lambda_g(\alpha) \in \mathcal{A}^0(M)$, 1. is proved also in this case.

Let $\alpha \in \mathcal{A}^r(M)$. Then $L_g(\bar{\alpha}) = \overline{L_g(\alpha)}$, since $\overline{k_g} = k_g$ (as can be seen using the local expression of k_g given in formula (2.9)). The second equality in 2. follows if $r > 2$ from the fact that

$$\begin{aligned}\bigwedge \tilde{g}^*\big(\,\Lambda_g(\bar{\alpha}), \beta\,\big) &= \bigwedge \tilde{g}^*\big(\,\bar{\alpha}, L_g(\beta)\,\big) = \overline{\bigwedge \tilde{g}^*\big(\,\alpha, \overline{L_g(\beta)}\,\big)} \\ &= \overline{\bigwedge \tilde{g}^*\big(\,\alpha, L_g(\bar{\beta})\,\big)} = \overline{\bigwedge \tilde{g}^*\big(\,\Lambda_g(\alpha), \bar{\beta}\,\big)} = \bigwedge \tilde{g}^*\big(\,\overline{\Lambda_g(\alpha)}, \beta\,\big)\end{aligned}$$

for all $\beta \in \mathcal{A}^{r-2}(M)$, and if $r = 2$ from

$$\Lambda_g(\bar{\alpha}) = \bigwedge \tilde{g}^*\,(\bar{\alpha}, k_g) = \overline{\bigwedge \tilde{g}^*\,(\alpha, k_g)} = \overline{\Lambda_g(\alpha)}.$$

Let $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_m}$ be a local orthonormal frame of M. With respect to this frame the local expression of k_g given in (2.9) becomes

$$k_g = \frac{i}{2}\sum_{k=1}^{m} dz_k \wedge d\overline{z_k}.$$

We will prove 3. for a form $\alpha = dz_k \wedge d\overline{z_k}$, for $k \in \{1, \ldots, m\}$. This is the only case we need to consider, because for forms of the type $dz_k \wedge dz_l$ and $d\overline{z_k} \wedge d\overline{z_l}$, for $k, l \in \{1, \ldots, m\}$ or $dz_k \wedge d\overline{z_l}$, for $k, l \in \{1, \ldots, m\}$ with $k \neq l$, both sides of the equality vanish. We have

$$L_g{}^{m-1}(\alpha) = \frac{i^{m-1}}{2^{m-1}}\,(m-1)!\; dz_1 \wedge d\overline{z_1} \wedge \ldots \wedge dz_m \wedge d\overline{z_m}$$

and

$$\frac{1}{m}\,\Lambda_g(\alpha)\,k_g{}^m = \frac{1}{m}\,\Lambda_g(\alpha)\,m!\,\frac{i^m}{2^m}\; dz_1 \wedge d\overline{z_1} \wedge \ldots \wedge dz_m \wedge d\overline{z_m}\,,$$

thus we need to show that $\Lambda_g(\alpha) = -2\,i$. But

$$\begin{aligned}\Lambda_g(\alpha) &= \textstyle\bigwedge \tilde{g}^*\,(\,dz_k \wedge d\overline{z_k}\,,\, k_g\,) = \frac{i}{2}\sum_{l=1}^m \bigwedge \tilde{g}^*\,(\,dz_k \wedge d\overline{z_k}\,,\, dz_l \wedge d\overline{z_l}\,) \\ &= \frac{i}{2}\sum_{l=1}^m \det\begin{pmatrix} \tilde{g}^*(dz_k, dz_l) & \tilde{g}^*(dz_k, d\overline{z_l}) \\ \tilde{g}^*(d\overline{z_k}, dz_l) & \tilde{g}^*(d\overline{z_k}, d\overline{z_l})\end{pmatrix} = \frac{i}{2}\sum_{l=1}^m -4\,\delta_{kl} = -2\,i.\end{aligned}$$

□

Let $\pi : E \to M$ be a complex vector bundle over a complex manifold and let g be an Hermitian metric on M. Then the associated contraction operator Λ_g on M can be extended in a natural way to the space of differential forms on M with values in E. Let $\xi \in \mathcal{A}^r(E)$ and write it locally as $\xi = \sum_{i=1}^n u_i \otimes \omega_i$, where $(u_1, \ldots, u_n)$ is a local frame of $\pi : E \to M$ and where every ω_i is a local r-form on M. Then $\Lambda_g(\xi)$ is defined locally by $\Lambda_g(\xi) := \sum_{i=1}^n u_i \otimes \Lambda_g(\omega_i)$. This definition does not depend on the choice of the local frame, because Λ_g is $\mathcal{C}^\infty(M, \mathbb{C})$-linear.

4.5 The Hermite-Einstein equation

Let G be a complex reductive Lie group with a compact real form K and let $P\,(M, G)$ be a complex principal fibre bundle. Suppose that M is compact and Kähler and let k_g and Λ_g be the Kähler form and the contraction operator associated to a Kähler metric g. Let J be a fixed holomorphic structure on $P\,(M, G)$ and consider the Chern connection $A_{J,Q}$ on $P\,(M, G)$ corresponding to a K-reduction $Q\,(M, K)$. By Proposition 3.1.5 we know that the curvature form $\Omega_{A_{J,Q}}$ is an element of $\mathcal{A}^{1,1}(P \times_{Ad} \mathfrak{g})$; using point 1 of Lemma 4.4.2, we see thus that $\Lambda_g\big(\,\Omega_{A_{J,Q}}\,\big) \in \mathcal{A}^0(P \times_{Ad} \mathfrak{g})$.

Lemma 4.5.1 *Let* $\mathfrak{z}(\mathfrak{g}) := \{\, C \in \mathfrak{g} \;/\; [C, X] = 0,\ \forall\ X \in \mathfrak{g}\,\}$ *be the* center *of* $\mathfrak{g}$. *Then we have* $\mathfrak{z}(\mathfrak{g}) = \{\, C \in \mathfrak{g} \;/\; \mathrm{Ad}(g)\,(C) = C,\ \forall\ g \in G\,\}$. *In particular, every* $C \in \mathfrak{z}(\mathfrak{g})$ *can be regarded as an element of* $\mathcal{A}^0(P \times_{Ad} \mathfrak{g})$.

Proof Let $C \in \mathfrak{g}$ with $\mathrm{Ad}(g)\,(C) = C$, for all $g \in G$. Then for all $X \in \mathfrak{g}$ we have

$$[C, X] = [-X, C] = \mathrm{ad}(-X)\,(C) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}\,\big(\exp(-tX)\big)\,(C) = 0,$$

thus $C \in \mathfrak{z}(\mathfrak{g})$.
To prove the other inclusion we need the following fact (see [31, 3.50(a)]). Let $Z(G) := \{\, g \in G \;/\; c(g')\,(g) = g,\, \forall\, g' \in G \,\}$ be the *center* of G. Then $Z(G)$ is a closed subgroup of G and its Lie algebra is $\mathfrak{z}(\mathfrak{g})$.
Let $C \in \mathfrak{z}(\mathfrak{g})$. Then $\exp(tC) \in Z(G)$ for all $t \in \mathbb{R}$, and so using (1.6) we have

$$\exp(tC) = c(g)\left(\exp(tC)\right) = \exp\left(\mathrm{Ad}(g)(tC)\right) = \exp\left(t\,\mathrm{Ad}(g)(C)\right)$$

for all $g \in G$. Since this hold for all $t \in \mathbb{R}$, it follows that $C = \mathrm{Ad}(g)\,(C)$, for all $g \in G$.
For the last statement, observe that , given $C \in \mathfrak{z}(\mathfrak{g})$, the function $P \to \mathfrak{g}$, $u \mapsto C$ is G-equivariant and thus induces a section of $P \times_{Ad} \mathfrak{g}$. □

The **Hermite-Einstein equation** is the equation

$$\Lambda_g\left(\Omega_{A_{J,Q}}\right) = C \tag{4.2}$$

where C is some element of $\mathfrak{z}(\mathfrak{g})$ and where $\Omega_{A_{J,Q}}$ is the curvature of the Chern connection corresponding to a variable K-reduction $Q\,(M,K)$ of $P\,(M,G)$, with respect to the fixed holomorphic structure J. Using the results of Section 4.2, we will give now a necessary condition for C in order to have a solution of this equation.

Suppose that (4.2) holds, for some K-reduction $Q\,(M,K)$ of $P\,(M,G)$. Then for every $f \in I^1(G)$ we have, using point 3. of Lemma 4.4.2,

$$\begin{aligned}\int_M f(C)\,{k_g}^m &= \int_M f\left(\Lambda_g\left(\Omega_{A_{J,Q}}\right)\right){k_g}^m = \int_M \Lambda_g\left(f\left(\Omega_{A_{J,Q}}\right)\right){k_g}^m \\ &= m\int_M f\left(\Omega_{A_{J,Q}}\right) \wedge {k_g}^{m-1}\end{aligned}$$

where m is the dimension of M. We claim that $\delta_f(P) := \int_M f\left(\Omega_{A_{J,Q}}\right) \wedge {k_g}^{m-1}$ does not depend on the reduction $Q\,(M,G)$. Indeed, let A and A' be two connections on $P\,(M,G)$, with curvature forms Ω and Ω'. Then by Theorem 4.2.1 we have $f(\Omega') = f(\Omega) + d\alpha$, for some $\alpha \in \mathcal{A}^1(M)$, and this allows us to conclude, as in the proof of Proposition 4.3.3, that $\int_M f\,(\Omega') \wedge {k_g}^{m-1} = \int_M f\,(\Omega) \wedge {k_g}^{m-1}$.

Thus we see that (4.2) can have a solution only if

$$f(C) = \frac{m\,\delta_f(P)}{\int_M {k_g}^m} \tag{4.3}$$

for all $f \in I^1(G)$.

Example 4.5.2 *Let $\pi : E \to M$ be a holomorphic vector bundle over a compact Kähler manifold and let $L(E)\left(M, GL(n,\mathbb{C})\right)$ be its frame bundle, with induced holomorphic structure J. To every Hermitian metric h on $\pi : E \to M$*

there corresponds a $U(n)$-reduction $U_h(E)\big(M,U(n)\big)$ of $L(E)\big(M,GL(n,\mathbb{C})\big)$ (see page 73). Consider the Chern connections D_h on $\pi : E \to M$ and $A_{J,U_h(E)}$ on $L(E)\big(M,GL(n,\mathbb{C})\big)$. By Example 3.1.4 we know that D_h is the connection on $\pi : E \to M$ associated to $A_{J,U_h(E)}$, in the sense of Theorem 1.4.12. Thus, by Lemma 1.4.13, the curvature R_h of D_h is equal to the curvature form $\Omega_{A_{J,U_h(E)}}$ of $A_{J,U_h(E)}$. The Hermite-Einstein equation becomes in this case

$$\Lambda_g\,(R_h) = C \tag{4.4}$$

where C is some element of $\mathfrak{z}\big(\mathfrak{gl}\,(n,\mathbb{C})\big)$ and where the Hermitian metric h plays the role of the K-reduction $Q\,(M,G)$ of $P\,(M,G)$ in (4.2). Observe that

$$\mathfrak{z}\,\big(\mathfrak{gl}\,(n,\mathbb{C})\big) = \{\, X \in \mathfrak{gl}\,(n,\mathbb{C}) \;/\; XY = YX,\; \forall Y \in \mathfrak{gl}\,(n,\mathbb{C}) \,\} = \{\, \lambda\, I_n \;/\; \lambda \in \mathbb{C} \,\}.$$

Since $\Lambda_g\,(R_h)$ is a map $L(E) \to \mathfrak{gl}\,(n,\mathbb{C})$ that on $U_h(E)$ takes values in $\mathfrak{u}(n)$ (by the remark after Definition 2.3.5 and by point 2. of Lemma 4.4.2), we must have $\lambda \in i\,\mathbb{R}$. Equation (4.4) thus becomes

$$\Lambda_g\,(R_h) = i\,c\,I_n\,, \tag{4.5}$$

for some $c \in \mathbb{R}$. Consider the invariant polynomial $\operatorname{tr} \in I^1\big(\mathfrak{gl}\,(n,\mathbb{C})\big)$*. Condition (4.3) gives in this case*

$$c = -\,\frac{m}{n}\,\frac{2\,\pi\,\deg(E)}{\int_M {k_g}^m}\,,$$

where m is the dimension of M.

Note that, by Remark 4.3.4, in the previous example we only need the metric g to be Gauduchon. The same is true in the general case of principal fibre bundles. This can be seen as follows.

Let G be a complex reductive Lie group with a compact real form K and $P\,(M,G)$ a complex principal fibre bundle. Consider two K-reductions $Q_0\,(M,K)$ and $Q\,(M,K)$ of $P\,(M,G)$, with $Q = e^s(Q_0)$ for some $s \in \Gamma\,(Q_0 \times_{Ad} i\mathfrak{k})$. Then, by Corollary 3.2.6, the curvature forms $\Omega_{A_{J,\,Q_0}}$ and $\Omega_{A_{J,\,Q}}$ of the Chern connections $A_{J,\,Q_0}$ and $A_{J,\,Q}$ with respect to a fixed holomorphic structure J on $P\,(M,G)$ are related by the formula $\Omega_{A_{J,\,Q}} = \Omega_{A_{J,\,Q_0}} + \overline{\partial}\,\big(\,\partial_{Q_0}\,e^{-2s}\,\big)$. Thus, for every invariant polynomial $f \in I^1(G)$ we have $f\,\big(\,\Omega_{A_{J,\,Q}}\,\big) = f\,\big(\,\Omega_{A_{J,\,Q_0}}\,\big) + \overline{\partial}\,\Big(f\,\big(\,\partial_{Q_0}\,e^{-2s}\,\big)\Big)$. It can be shown that $f\,\big(\,\partial_{Q_0}\,e^{-2s}\,\big) = \partial\,\big(f(-2s)\big)$ (see [19, p. 28]), where $f(-2s)$ is regarded as a function on M. So we have $f\,\big(\,\Omega_{A_{J,\,Q}}\,\big) = f\,\big(\,\Omega_{A_{J,\,Q_0}}\,\big) + \overline{\partial}\partial\,\big(f(-2s)\big)$ and this allows us to conclude, as in Remark 4.3.4, that

$$\int_M f\,\big(\,\Omega_{A_{J,\,Q}}\,\big) \wedge {k_g}^{m-1} = \int_M f\,\big(\,\Omega_{A_{J,\,Q_0}}\,\big) \wedge {k_g}^{m-1}$$

if g is a Gauduchon metric. Thus also in this case we see that the number $\delta_f(P) := \int_M f\,\big(\,\Omega_{A_{J,\,Q}}\,\big) \wedge {k_g}^{m-1}$ does not depend on the choice of the K-reduction

$Q(M, K)$ used to define it.

This fact is particularly important, since, as mentioned in Remark 4.3.4, every compact complex manifold admits a Gauduchon metric (while there are topological obstructions for the existence of a Kähler metric on a compact complex manifold, see for example [32, p.191]).

Appendix A

Let $P(M,G)$ be a principal fibre bundle, $\varrho : G \to \mathrm{Aut}(V)$ a representation of G on a vector space V and $E = P \times_G V$ the associated vector bundle. We will prove here some results that are stated in Paragraph 1.4. Throughout this appendix we will need the following lemma.

Lemma A.1 *Let $\phi : P \to V$ be a G-equivariant map, i.e. a map such that $\phi(ug) = \varrho(g^{-1})\big(\phi(u)\big)$ for all $u \in P$ and $g \in G$. Then for $B \in \mathfrak{g}$ and $u \in P$ we have*

$$(B^*)_u(\phi) = -\,\varrho_*(B)\big(\phi(u)\big)$$

where $B^* \in \Gamma(TP)$ is the fundamental vector field corresponding to B and where $\varrho_* : \mathfrak{g} \to \mathrm{End}(V)$ is the differential of ϱ.

Proof Recall that $(B^*)_u = \frac{d}{dt}\Big|_{t=0} u \exp(tB)$. We have

$$\begin{aligned}(B^*)_u(\phi) &= \frac{d}{dt}\Big|_{t=0} \phi\big(u \exp(tB)\big) = \frac{d}{dt}\Big|_{t=0} \varrho\big(\exp(tB)^{-1}\big)\big(\phi(u)\big) \\ &= \frac{d}{dt}\Big|_{t=0} \varrho\big(\exp(-tB)\big)\big(\phi(u)\big) = -\,\varrho_*(B)\big(\phi(u)\big).\end{aligned}$$

□

We proved in Paragraph 1.4 that every connection A on P induces a connection D_A on E. This is defined by

$$D_A(\sigma)(Y)(p) := \Big(u, \big(\hat{Y}^h\big)_A(u)(\hat{\sigma})\Big)_{/\sim} \tag{A.1}$$

for $\sigma \in \mathcal{A}^0(E)$, $Y \in \Gamma(TM)$ and $p \in M$, where u is some element of the fibre of P over p and where $\hat{\sigma}$ denotes the G-equivariant map $P \to V$ corresponding to σ (see Lemma 1.2.9).

Recall that if we choose a basis $v = (v_1, \ldots, v_n)$ of V, then the map f_v from P to the total space of the frame bundle of E defined by

$$f_v : P \to L(E)\,,\ u \mapsto \big((u, v_1)_{/\sim}, \ldots, (u, v_n)_{/\sim}\big)$$

induces a homomorphism of principal fibre bundles (see Example 1.2.10).

Proposition A.2 *Consider a local section $\sigma : \mathcal{U} \to P$ of P over some open $\mathcal{U} \subset M$ and the induced local frame $u = f_v \circ \sigma : \mathcal{U} \to L(E)$ of E over $\mathcal{U}$, relative to a basis v of V. Let A be a connection on P with connection form ω_A and let D_A be the connection on E induced by A. Then the connection form ω_u of D_A with respect to the local frame u is given by $\omega_u = (\alpha'_v)_* \big(\varrho_* \left(\sigma^* \omega_A\right)\big)$, where $(\alpha'_v)_*$ is the isomorphism $\mathrm{End}(V) \to \mathfrak{gl}\,(n, \mathbb{R})$ induced by the basis v of V (see Example 1.2.10).*

Proof Observe first that if we write $u = (u_1, \ldots, u_n)$ then for all $p \in \mathcal{U}$ and $\alpha = 1, \ldots, n$ we have $u_\alpha(p) = \big(\sigma(p), v_\alpha\big)_{/\sim}$, in other words

$$\hat{u_\alpha}\left(\sigma(p)\right) = v_\alpha\,. \tag{A.2}$$

We have to show that

$$D_A\left(u_\alpha\right) = \sum_{\beta=1}^{n} u_\beta \otimes \Big((\alpha'_v)_* \big(\varrho_*\left(\sigma^* \omega_A\right)\big)\Big)_{\beta\,\alpha}\,,$$

thus that for $Y \in \Gamma(TM)$ and $p \in \mathcal{U}$ we have

$$D_A\left(u_\alpha\right)(Y)\,(p) = \sum_{\beta=1}^{n} u_\beta(p) \Bigg((\alpha'_v)_* \bigg(\varrho_*\Big(\omega_A\left(\sigma_*(Y_p)\right)\Big)\bigg)\Bigg)_{\beta\,\alpha}.$$

By (A.1), Lemma A.1 and (A.2) we have

$$\begin{aligned}
D_A\left(u_\alpha\right)(Y)\,(p) &= \Big(\sigma(p), \big(\hat{Y}^h\big)_A \left(\sigma(p)\right)(\hat{u_\alpha})\Big)_{/\sim} \\
&= \bigg(\sigma(p), \Big(\sigma_*(Y_p) - \Big(\omega_A\left(\sigma_*(Y_p)\right)\Big)^*_{\sigma(p)}\Big)(\hat{u_\alpha})\bigg)_{/\sim} \\
&= \bigg(\sigma(p),\, Y_p\left(\hat{u_\alpha} \circ \sigma\right) + \varrho_*\Big(\omega_A\left(\sigma_*(Y_p)\right)\Big)\Big(\hat{u_\alpha}\left(\sigma(p)\right)\Big)\bigg)_{/\sim} \\
&= \bigg(\sigma(p),\, \varrho_*\Big(\omega_A\left(\sigma_*(Y_p)\right)\Big)(v_\alpha)\bigg)_{/\sim}.
\end{aligned}$$

On the other hand, since $u_\beta(p) = \big(\sigma(p), v_\beta\big)_{/\sim}$ we have

$$\begin{aligned}
&\textstyle\sum_{\beta=1}^{n} u_\beta(p) \Bigg((\alpha'_v)_* \bigg(\varrho_*\Big(\omega_A\left(\sigma_*(Y_p)\right)\Big)\bigg)\Bigg)_{\beta\,\alpha} \\
&= \Bigg(\sigma(p),\, \textstyle\sum_{\beta=1}^{n} v_\beta \Bigg((\alpha'_v)_* \bigg(\varrho_*\Big(\omega_A\left(\sigma_*(Y_p)\right)\Big)\bigg)\Bigg)_{\beta\,\alpha}\Bigg)_{/\sim} \\
&= \bigg(\sigma(p),\, \varrho_*\Big(\omega_A\left(\sigma_*(Y_p)\right)\Big)(v_\alpha)\bigg)_{/\sim}.
\end{aligned}$$

□

Recall that the space $\mathcal{D}_E$ of connections on E and the space $\mathcal{A}_P$ of connections on P are affine spaces modeled on $\mathcal{A}^1(E^* \otimes E)$ and $\mathcal{A}^1(P \times_{Ad} \mathfrak{g})$ respectively (see Propositions 1.3.2 and 1.4.8). We have a vector bundle map

$$\phi : P \times_{Ad} \mathfrak{g} \to E^* \otimes E$$

which is defined by

$$\phi\big(\,(\,u, B\,)_{/\sim}\big)\big(\,(\,u, v\,)_{/\sim}\big) = \big(\,u, \varrho_*(B)\,(v)\,\big)_{/\sim} \tag{A.3}$$

for $p \in M$, $(\,u, B\,)_{/\sim} \in (P \times_{Ad} \mathfrak{g})_p$ and $(\,u, v\,)_{/\sim} \in E_p$ (see Example 1.2.13). We will denote the induced maps $\mathcal{A}^r(P \times_{Ad} \mathfrak{g}) \to \mathcal{A}^r(E^* \otimes E)$ also by ϕ.

Proposition A.3 *The map $\mathcal{A}_P \to \mathcal{D}_E$, $A \mapsto D_A$ is an affine homomorphism with associated linear map $\phi : \mathcal{A}^1(P \times_{Ad} \mathfrak{g}) \to \mathcal{A}^1(E^* \otimes E)$.*

Proof Let $\xi \in \mathcal{A}^1(P \times_{Ad} \mathfrak{g})$ and denote by $\hat{\xi}$ the corresponding G-equivariant horizontal $\mathfrak{g}$-valued 1-form on P (see Lemma 1.4.7). Recall that for $Y \in \Gamma(TM)$ and $p \in M$ we have

$$\xi\,(Y_p) = \big(\,u,\, \hat{\xi}\,(\hat{Y}_u)\,\big)_{/\sim} \tag{A.4}$$

where u is some element in the fibre of P over p and $\hat{Y}$ is a vector field of P such that $\pi_*(\hat{Y}) = Y$. Let A be a connection on P. We have to show that

$$D_{A+\xi} = D_A + \phi(\xi).$$

Observe first that for $Y \in \Gamma(TM)$ we have

$$\big(\,\hat{Y}^h\,\big)_{A+\xi} = \big(\,\hat{Y}^h\,\big)_A - \big(\,\hat{\xi}(\hat{Y})\,\big)^* \tag{A.5}$$

since

$$\begin{aligned}
\big(\,\hat{Y}^h\,\big)_{A+\xi} &= \hat{Y} - \big(\,\hat{Y}^v\,\big)_{A+\xi} = \hat{Y} - \big(\,\omega_{A+\xi}\,(\hat{Y})\,\big)^* \\
&= \hat{Y} - \big(\,\omega_A\,(\hat{Y})\,\big)^* + \big(\,\omega_A\,(\hat{Y}) - \omega_{A+\xi}\,(\hat{Y})\,\big)^* \\
&= \big(\,\hat{Y}^h\,\big)_A - \big(\,\hat{\xi}(\hat{Y})\,\big)^*.
\end{aligned}$$

The last equality follows from the fact that by definition $A + \xi$ is the connection on P with connection form $\omega_{A+\xi} = \omega_A + \hat{\xi}$. Let now $\sigma \in \mathcal{A}^0(E)$ and $p \in M$. Then by (A.1), (A.5), Lemma A.1, (A.3) and (A.4) we have:

$$\begin{aligned}
D_{A+\xi}\,(\sigma)(Y)\,(p) &= \Big(u, \big(\,\hat{Y}^h\,\big)_{A+\xi}\,(u)\,(\hat{\sigma})\Big)_{/\sim} \\
&= \Big(u, \big(\,\hat{Y}^h\,\big)_A(u)\,(\hat{\sigma}) - \big(\,\hat{\xi}(\hat{Y}_u)\,\big)^*_u\,(\hat{\sigma})\Big)_{/\sim} \\
&= D_A(\sigma)(Y)\,(p) + \Big(\,u,\, \varrho_*\big(\,\hat{\xi}(\hat{Y}_u)\,\big)(\hat{\sigma}(u))\,\Big)_{/\sim} \\
&= D_A(\sigma)(Y)\,(p) + \phi\,\Big(\big(\,u,\, \hat{\xi}\,(\hat{Y}_u)\,\big)_{/\sim}\Big)\Big(\big(\,u,\, \hat{\sigma}(u)\,\big)_{/\sim}\Big) \\
&= D_A(\sigma)(Y)\,(p) + \phi\,\big(\xi(Y_p)\big)\big(\sigma(p)\big)\,.
\end{aligned}$$

□

Recall that the curvature form of a connection on $P\,(M,G)$ is an element of $\mathcal{A}^2(P \times_{Ad}\, \mathfrak{g})$ and that the curvature of a connection on E is an element of $\mathcal{A}^2(E^* \otimes E)$.

Proposition A.4 *Let A be a connection on P with curvature form Ω_A. Denote by R_A the curvature of the connection D_A on E. Then we have*

$$R_A = \phi\,(\Omega_A).$$

Proposition A.4 is an immediate consequence of the next two lemmas.

Lemma A.5 *Let A be a connection on P with curvature form Ω_A and let D_A be the connection on E induced by A. Then for $Y_1,\, Y_2 \in \Gamma(TM)$ and $\sigma \in \mathcal{A}^0(E)$ we have*

$$\begin{aligned}&\phi\,(\Omega_A)\,(Y_1, Y_2)(\sigma)\\ &= \tfrac{1}{2}\Big(D_A\,\big(\,D_A\,(\sigma)(Y_2)\,\big)\,(Y_1) - D_A\,\big(\,D_A\,(\sigma)(Y_1)\,\big)\,(Y_2) - D_A\,(\sigma)\big(\,[Y_1, Y_2]\,\big)\Big).\end{aligned}$$

Lemma A.6 *Let $\pi : F \to M$ be a vector bundle and let R be the curvature of a connection D on F. Then for $Y_1,\, Y_2 \in \Gamma(TM)$ and $\sigma \in \mathcal{A}^0(F)$ it holds:*

$$R\,(Y_1, Y_2)(\sigma) = \frac{1}{2}\Big(D\big(\,D(\sigma)(Y_2)\,\big)\,(Y_1) - D\big(\,D(\sigma)(Y_1)\,\big)\,(Y_2) - D(\sigma)\big(\,[Y_1, Y_2]\,\big)\Big).$$

Proof of Lemma A.5 Let $p \in M$ and let u be some element in the fibre of P over p. By (A.1) we have $\widehat{D_A\,(\sigma)(Y_2)} = \big(\hat{Y}_2^h\big)_A\,(\hat{\sigma})$ and

$$D_A\,\big(\,D_A\,(\sigma)(Y_2)\,\big)\,(Y_1)\,(p) = \Big(\,u, \big(\,\hat{Y_1}^h\,\big)_A(u)\,\Big(\big(\,\hat{Y_2}^h\,\big)_A\,(\hat{\sigma})\Big)\,\Big)_{/\sim}.$$

Similarly

$$D_A\,\big(\,D_A\,(\sigma)(Y_1)\,\big)\,(Y_2)\,(p) = \Big(\,u, \big(\,\hat{Y_2}^h\,\big)_A(u)\,\Big(\big(\,\hat{Y_1}^h\,\big)_A\,(\hat{\sigma})\Big)\,\Big)_{/\sim},$$

thus

$$\begin{aligned}&D_A\,\big(\,D_A\,(\sigma)(Y_2)\,\big)\,(Y_1)\,(p) - D_A\,\big(\,D_A\,(\sigma)(Y_1)\,\big)\,(Y_2)\,(p)\\ &= \Big(\,u,\ \Big[\,\big(\,\hat{Y_1}^h\,\big)_A, \big(\,\hat{Y_2}^h\,\big)_A\Big](u)\,(\hat{\sigma})\Big)_{/\sim}.\end{aligned}$$

Recall that $\left(\widehat{[Y_1,Y_2]^h}\right)_A = \left(\left[\left(\hat{Y_1}^h\right)_A, \left(\hat{Y_2}^h\right)_A\right]^h\right)_A$ (see Lemma 1.4.2). Using this, Lemma A.1 and (A.3) we get:

$$\begin{aligned}
&\Big(D_A\left(D_A(\sigma)(Y_2)\right)(Y_1) - D_A\left(D_A(\sigma)(Y_1)\right)(Y_2) - D_A(\sigma)\left([Y_1,Y_2]\right)\Big)(p) \\
&= \left(u,\ \left[\left(\hat{Y_1}^h\right)_A, \left(\hat{Y_2}^h\right)_A\right](u)\,(\hat{\sigma})\right)_{/\sim} - \left(u,\ \left(\widehat{[Y_1,Y_2]^h}\right)_A(u)\,(\hat{\sigma})\right)_{/\sim} \\
&= \left(u,\ \left(\left[\left(\hat{Y_1}^h\right)_A, \left(\hat{Y_2}^h\right)_A\right]^v\right)_A(u)\,(\hat{\sigma})\right)_{/\sim} \\
&= \left(u,\ \left(\omega_A\left(\left[\left(\hat{Y_1}^h\right)_A, \left(\hat{Y_2}^h\right)_A\right](u)\right)\right)^*_u(\hat{\sigma})\right)_{/\sim} \\
&= \left(u,\ -\varrho_*\left(\omega_A\left(\left[\left(\hat{Y_1}^h\right)_A, \left(\hat{Y_2}^h\right)_A\right](u)\right)\right)\left(\hat{\sigma}(u)\right)\right)_{/\sim} \\
&= -\,\phi\left(\left(u,\ \omega_A\left(\left[\left(\hat{Y_1}^h\right)_A, \left(\hat{Y_2}^h\right)_A\right](u)\right)\right)_{/\sim}\right)\left(\sigma(p)\right).
\end{aligned}$$

Thus we have to show that

$$\Omega_A(Y_1,Y_2)(p) = \left(u,\ -\frac{1}{2}\,\omega_A\left(\left[\left(\hat{Y_1}^h\right)_A, \left(\hat{Y_2}^h\right)_A\right](u)\right)\right)_{/\sim}.$$

But this follows from

$$\begin{aligned}
\Omega_A(Y_1,Y_2)(p) &= \left(u,\ d\omega_A\left(\left(\hat{Y_1}^h\right)_A(u), \left(\hat{Y_2}^h\right)_A(u)\right)\right)_{/\sim} \\
= \Bigg(u,\ \tfrac{1}{2}\Bigg(&\left(\hat{Y_1}^h\right)_A(u)\left(\omega_A\left(\left(\hat{Y_2}^h\right)_A\right)\right) - \left(\hat{Y_2}^h\right)_A(u)\left(\omega_A\left(\left(\hat{Y_1}^h\right)_A\right)\right) \\
&-\omega_A\left(\left[\left(\hat{Y_1}^h\right)_A, \left(\hat{Y_2}^h\right)_A\right](u)\right)\Bigg)\Bigg)_{/\sim}
\end{aligned}$$

since $\omega_A\left(\left(\hat{Y_2}^h\right)_A\right) = \omega_A\left(\left(\hat{Y_1}^h\right)_A\right) = 0.$ [1] □

Proof of Lemma A.6 Let $p \in M$ and let $u : \mathcal{U} \to L(F)$ be a local frame of F over an open $\mathcal{U} \subset M$ with $p \in \mathcal{U}$. Identify F with $L(F) \times_{GL(n,\mathbb{R})} \mathbb{R}^n$ and write $\sigma = \left(u, v\right)_{/\sim}$ on $\mathcal{U}$, where v is a smooth function $\mathcal{U} \to \mathbb{R}^n$. Let ω_u and Ω_u be the connection and curvature forms of D with respect to u. Recall the relations

$$D(\sigma) = \left(u, dv + \omega_u\, v\right)_{/\sim} \tag{A.6}$$

[1] For a 1-form ω on a manifold M with values in a vector space V and for $Y_1, Y_2 \in \Gamma(TM)$ it holds:

$$d\omega(Y_1,Y_2) = \frac{1}{2}\ \ Y_1\ \ \omega(Y_2)\ \ - Y_2\ \ \omega(Y_1)\ \ - \omega\ \ [Y_1,Y_2]$$

(see [14, Proposition 3.11 of Chapter I] or [27, Lemma 5.15 of Chapter 1]).

$$R(\sigma) = (\, u, \Omega_u \, v\,)_{/\sim} \tag{A.7}$$

and the structure equation

$$\Omega_u = d\omega_u + \omega_u \wedge \omega_u\,. \tag{A.8}$$

By (A.6) we have $D(\sigma)(Y_2) = \big(\, u, Y_2(v) + \omega_u(Y_2)\, v\,\big)_{/\sim}$ on $\mathcal{U}$ and

$$\begin{aligned} D\big(\, D(\sigma)(Y_2)\,\big)\,(Y_1)\,(p) \;&=\; \Big(\, u(p),\, Y_1(p)\,\big(\, Y_2(v) + \omega_u(Y_2)\, v\,\big) \\ &\quad + \omega_u\,\big(Y_1(p)\big)\Big(\, Y_2(p)\,(v) + \omega_u\,\big(Y_2(p)\big)\, v(p)\,\Big)\,\Big)_{/\sim} \\ &=\; \Big(\, u(p),\, Y_1(p)\,\big(Y_2(v)\big) + Y_1(p)\,\big(\omega_u(Y_2)\big)\, v(p) \\ &\quad + \omega_u\,\big(Y_2(p)\big)\, Y_1(p)(v) + \omega_u\,\big(Y_1(p)\big)\, Y_2(p)\,(v) \\ &\quad + \omega_u\,\big(Y_1(p)\big)\;\omega_u\,\big(Y_2(p)\big)\, v(p)\,\Big)_{/\sim}. \end{aligned}$$

Similarly

$$\begin{aligned} D\big(\, D(\sigma)(Y_1)\,\big)\,(Y_2)\,(p) \;&=\; \Big(\, u(p),\, Y_2(p)\,\big(Y_1(v)\big) + Y_2(p)\,\big(\omega_u(Y_1)\big)\, v(p) \\ &\quad + \omega_u\,\big(Y_1(p)\big)\, Y_2(p)(v) + \omega_u\,\big(Y_2(p)\big)\, Y_1(p)\,(v) \\ &\quad + \omega_u\,\big(Y_2(p)\big)\;\omega_u\,\big(Y_1(p)\big)\, v(p)\,\Big)_{/\sim} \end{aligned}$$

and

$$D(\sigma)\big(\, [Y_1, Y_2]\,\big)(p) = \Big(\, u(p),\, [Y_1, Y_2]\,(p)(v) + \omega_u\big(\, [Y_1, Y_2](p)\,\big)\, v(p)\,\Big)_{/\sim}.$$

Thus the right hand side in the statement is equal to

$$\begin{aligned} &\Big(\, u(p), \tfrac{1}{2}\,\Big(Y_1(p)\,\big(\,\omega_u(Y_2)\,\big) - Y_2(p)\,\big(\,\omega_u(Y_1)\,\big) \\ &+ \Big[\,\omega_u\,\big(Y_1(p)\big), \omega_u\,\big(Y_2(p)\big)\,\Big] - \omega_u\big(\, [Y_1, Y_2](p)\big)\Big)\, v(p)\Big)_{/\sim}. \end{aligned}$$

On the other hand, using (A.7) and (A.8) we get:

$$\begin{aligned} R\,(Y_1, Y_2)(\sigma)\,(p) &= \Big(\, u(p),\, \Omega_u\,\big(\, Y_1(p), Y_2(p)\,\big)\, v(p)\,\Big)_{/\sim} \\ &= \Big(\, u(p), \Big(d\omega_u\,\big(\, Y_1(p), Y_2(p)\,\big) + \,\omega_u \wedge \omega_u\,\big(\, Y_1(p), Y_2(p)\,\big)\Big)\, v(p)\,\Big)_{/\sim} \\ &= \Big(\, u(p),\, \tfrac{1}{2}\,\Big(Y_1(p)\,\big(\omega_u(Y_2)\big) - Y_2(p)\,\big(\omega_u(Y_1)\big) \\ &\quad + \Big[\,\omega_u\,\big(Y_1(p)\big), \omega_u\,\big(Y_2(p)\big)\,\Big] - \,\omega_u\big(\, [Y_1, Y_2](p)\big)\Big)\, v(p)\Big)_{/\sim} \end{aligned}$$

where the last equality follows from

$$d\omega_u\left(\,Y_1(p),Y_2(p)\,\right)=\frac{1}{2}\Big(Y_1(p)\left(\omega_u(Y_2)\right)-Y_2(p)\left(\omega(Y_1)\right)-\omega_u\left([Y_1,Y_2](p)\right)\Big)$$ [2]

and

$$\begin{aligned}\omega_u\wedge\omega_u\left(\,Y_1(p),Y_2(p)\,\right)&=\tfrac{1}{2}\Big(\omega_u\left(Y_1(p)\right)\omega_u\left(Y_2(p)\right)-\omega_u\left(Y_2(p)\right)\omega_u\left(Y_1(p)\right)\Big)\\&=\tfrac{1}{2}\Big[\omega_u\left(Y_1(p)\right),\omega_u\left(Y_2(p)\right)\Big]\,.\end{aligned}$$

□

[2] See footnote 1.

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