Geometric constructions of the irreducible representations of $\mathbb{GL}_m(\mathbb{C})$

Nicola Sambin

Thesis advisor: Dr. Robin De Jong

Master thesis, defended on June 28, 2010

Universiteit Leiden

Mathematisch Instituut

Introduction

In presenting the contents and the spirit of his 1997 article *Geometric Methods in Representation Theory of Hecke Algebras and Quantum Groups* V. Ginzburg asserts that recent developments and discoveries “have made representation theory, to a large extent, part of algebraic geometry”.

In this brief work we are not able to describe all the deep reasons underneath this conclusion but at least we may corroborate it by showing how some representation-theoretic results follow, at times quite easily, when placed in the appropriate geometric context. In other words we give some examples of geometric constructions of the irreducible representations of the general linear group which make possible the use of certain geometrical methods to gain informations such as the dimensions, the characters of these representations, beyond obviously an explicit realization of themselves.

The choice of the constructions we exhibit has been suggested by an article of J. Kamnitzer [K]. In particular this thesis deals with a Borel-Weil type construction and Ginzburg’s work contained in his *Representation Theory and Complex Geometry*. The former dates back to the early 1950s in its original version and was extended by R. Bott in 1957; the latter is more recent and due to V. Ginzburg.
Consequently the thesis is fundamentally divided into two parts besides a short chapter in which we recall the basics of representation theory for the general linear group, mainly to fix notation used throughout these pages.

In chapter two we describe a realization of a family of irreducible representations of the general linear group on the space of global sections of certain line bundles defined on varieties of flags (of a given vector space) which in our case are not supposed to be complete, in this differing from the usual Borel-Weil construction. In fact we follow more closely the approach in Fulton’s book *Young Tableaux* which provides a constructive version of the Borel-Weil theorem. Namely in the original Borel-Weil construction we gain any irreducible representation of highest weight taken among the dominant weights on the space of global sections of a line bundle, depending on the weight, on the variety of complete flags, whereas in our case also the flag variety varies together with the weight. The pro of this choice is essentially that all the constructions become more "visible", that is for example we are able in this setting to give an explicit formula for the highest weight vector.

The last section is devoted to an important application of a result known as Atiyah-Bott (or Woods Hole) fixed point theorem (early 1960s) to this case, which permits to compute the Weyl character formula. This is an evident example of how fruitful the geometric approach can be. In a letter to Grothendieck dated 2-3 August 1964, J.P. Serre outlines it saying that "c'est d'une simplicité étonnante" and, later on, "magnifique". Indeed, once Borel-Weil theory is given, the deduction of the Weyl character formula by means of the Atiyah-Bott theorem is quite natural.

The third chapter introduces Ginzburg's construction. Using Borel-Moore homology we manufacture a convolution algebra which turns out to be a homomorphic image of the enveloping algebra $\mathcal{U}(\mathfrak{gl}_m)$. This algebra has a natural, intrinsically geometrical, action on other homology groups and in this manner we obtain irreducible representations of the general linear group.
This method permits for example to compute the dimensions of the weight spaces inside a representation by counting the irreducible components of a given variety.

As an application we work out explicitly all the simple $\mathfrak{gl}_2$-modules.

We stress throughout these pages the benefits of using geometrical methods to understand representation theory and in particular we try to do this via low-dimensional examples.

Acknowledgements

I am grateful to my advisor Dr. R. De Jong for his support in front of my everlasting insecurity and Prof. S.J. Edixhoven for his fruitful suggestions.
I would also like to thank Dr. G. Carnovale, Dr. F. Esposito and Prof. M. Garuti for their helpfulness.
I thank all the friends I left in Italy and somewhere else in Europe and the ones I shared this experience with. In particular I want to thank Arianna for her invaluable support and Francesca for all the precious time spent together.

This thesis is dedicated to my sisters, my mother and the memory of my father.

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Chapter 1

Representation theory

We give a preliminary review of some basic results in the representation theory of the general linear group $\mathbb{GL}_m(\mathbb{C})$ (later on simply denoted with $\mathbb{GL}_m$). Good references for the proofs are [FH] and [GW].

In particular we deal with representations supposed finite-dimensional and algebraic, in the sense explained below.

Definition 1.0.1. *An algebraic finite-dimensional representation of $\mathbb{GL}_m$ is a couple (π, V), where V is a finite-dimensional complex vector space and $\pi : \mathbb{GL}_m \to \mathbb{GL}(V)$ is a group homomorphism which is also required to be a morphism of algebraic varieties.*

Hence from now on, we will reserve the term representations for this particular class.

1.1 Highest weight theory

It's known that all representations of $\mathbb{GL}_m$ decompose as a direct sum of irreducible subrepresentations. This means that we can reduce our attention to the description of the irreducible representations.

These can be characterized using the highest weight theory, that is we study representations by restricting them to the maximal torus H of $\mathbb{GL}_m$ consisting of invertible diagonal matrices, hence isomorphic, as an algebraic group, to $(\mathbb{C}^\times)^m$. For a representation V of $\mathbb{GL}_m$ this restriction gives rise to a decomposition of V into the direct sum:

$$V = \oplus_{\mu \in \mathbb{Z}^m} V_\mu \tag{1.1}$$

where $t = \text{diag}(t_1, \dots, t_m) \in H$ acts on each vector $v \in V_\mu$ in the following way:

$$t.v = t_1^{\mu_1} \cdots t_m^{\mu_m} v$$

The μ's s.t. $V_\mu \neq 0$ are usually called weights of the representation V and each vector subspace V_μ weight space.
We may define a partial order on the weights, i.e. on $\mathbb{Z}^m$. For two elements $\lambda, \mu \in \mathbb{Z}^m$, we say that

$$\lambda \geq \mu \Leftrightarrow \lambda - \mu = k_1\alpha_1 + \ldots + k_{m-1}\alpha_{m-1}$$

where the k_i's are non-negative integers and $\alpha_i := (0, \dots, 1, -1, \dots, 0)$ with 1 in the i-th and -1 in the $(i+1)$-th position.

It is also convenient to introduce a subset of $\mathbb{Z}^m$.

Definition 1.1.1. *We call dominant weight a m-uple of integers belonging to the set*

$$P_m^+ := \{(\mu_1, \dots, \mu_m) \in \mathbb{Z}^m \mid \mu_i \geq \mu_{i+1} \text{ for } i = 1, \dots, m-1\}$$

The dominant weights parametrize the irreducible representations.

Theorem 1.1.2. *For each $\lambda \in P_m^+$ there exists a representation $V(\lambda)$ which is both irreducible and of highest weight λ. The set $\Pi := \{V(\lambda)\}_{\lambda \in P_m^+}$ constitutes a complete collection of the irreducible representations of $\mathbb{GL}_m$.*
Two irreducible representations $V(\lambda), V(\mu) \in \Pi$ are isomorphic if and only if $\lambda = \mu$.

Each $V(\lambda)$ can be characterized as well studying both the action of the torus H and the one of the subgroup N of $\mathbb{GL}_m$ consisting of upper triangular matrices with 1's along the diagonal.

Theorem 1.1.3. *A representation V is isomorphic to $V(\lambda)$ if and only if the vector space of the N-invariant vectors in V has dimension one and it is a weight space of weight λ.*

Example 1.1.4. The one dimensional representation defined by the m-wedge product of the standard representation $(\text{id}, \mathbb{C}^m)$ i.e.

$$\begin{array}{rccc} D: & \mathbb{GL}_m & \rightarrow & \mathbb{GL}(\wedge^m \mathbb{C}^m) = \mathbb{C}^\times \\ & g & \mapsto & \det(g) \end{array}$$

is called the determinant representation and corresponds to the irreducible representation $V(1, \dots, 1)$, whereas its dual D^* is isomorphic to $V(-1, \dots, -1)$.

Theorem 1.1.5. *Let* $\lambda \in \{\mu \in P_m^+ \mid \mu_i \geq 0$ *for* $i = 1, \ldots, m\}$*. The representation* $\pi : \mathbb{GL}_m \to \mathbb{GL}(V(\lambda))$ *is given by polynomials, i.e. after choosing a basis of* $V(\lambda)$*, so* $\mathbb{GL}(V(\lambda)) = \mathbb{GL}_N(\mathbb{C})$ *for* $N = \dim_{\mathbb{C}} V(\lambda)$*, the* N^2 *coordinate functions are polynomial functions of the* m^2 *variables.*

Consequently we call these irreducible representations polynomial representations.
By tensoring irreducible polynomial representations with a suitable power of D^* we obtain all the irreducible representations since $V(\lambda_1, \ldots, \lambda_m) \otimes (D^*)^k \cong V(\lambda_1 - k, \ldots, \lambda_m - k)$. For this reason we mainly consider thoughout these pages the polynomial representations.
This basically corresponds to studying the restriction of representations of $\mathbb{GL}_m$ to the subgroup $\mathbb{SL}_m$, which acts trivially on $\wedge^m \mathbb{C}^m$.

1.2 The action of the Lie algebra

Following Ginzburg [CG], the second realization of the irreducible representations of $\mathbb{GL}_m$ will be exhibited indirectly by describing the induced action of the Lie algebra $\mathfrak{gl}_m$.
So we need some criteria to recognize the irreducible representation $V(\lambda)$ from the Lie algebra action.

The picture is given in the following theorem, where $\mathfrak{n}$ is the subalgebra of strictly upper triangular matrices and $\mathfrak{h}$ the Cartan subalgebra of diagonal matrices in $\mathfrak{gl}_m$.

Theorem 1.2.1. *Let* (π, V) *be a representation of* $\mathbb{GL}_m$ *and* $(d\pi, V)$ *the representation of* $\mathfrak{gl}_m$ *induced by differentiation.*
The representation V *is irreducible for both the action of* $\mathbb{GL}_m$ *and* $\mathfrak{gl}_m$ *if and only if the subspace* $V^{\mathfrak{n}} := \{v \in V \mid \mathfrak{n}.v = 0\}$ *has dimension one. Moreover in this case if* $v \in V^{\mathfrak{n}}, h \in \mathfrak{h}$ *then* $h.v = \langle h, \lambda \rangle v$*, where* λ *is the highest weight for the decomposition of* V *under the action of the abelian algebra* $\mathfrak{h}$ *and* $\langle -, - \rangle$ *is the bilinear form defined by* $\langle diag(a_1, \ldots, a_m), (\lambda_1, \ldots, \lambda_m) \rangle := a_1\lambda_1 + \cdots + a_m\lambda_m$*.*
With these conditions we have $V \cong V(\lambda)$*.*

Remark 1.2.2. The algebra $\mathfrak{gl}_m$ is generated as Lie algebra by the elements $H_i := \mathrm{diag}(0, \ldots, 1, \ldots, 0)$ (with 1 in the i-th position), for $i = 1, \ldots, m$, and the so called Chevalley generators E_i, F_i for $i = 1, \ldots, m-1$, where E_i is the matrix with 1 in the $(i, i+1)$-entry and 0 elsewhere and F_i is the transpose of E_i.
So, of course, a representation of $\mathfrak{gl}_m$ can be given by simply defining consistently the action of $\mathfrak{h}$ and the Chevalley generators.

Chapter 2

Representations and line bundles

In this chapter we describe a first geometric construction of the irreducible algebraic representations of the general linear group $\mathbb{GL}_m$. Namely we will realize them on the space of global sections of certain line bundles over flag varieties. Along this first chapter we follow the approach in [F].

2.1 Flag varieties

We first describe flag varieties using the Plücker embedding and later we identify them with certain homogeneous spaces. This endows flag varieties with a natural action of $\mathbb{GL}_m$ and permits a more explicit description of the equivariant line bundles over them.

Definition 2.1.1. *Let $(d_i)_{i=1}^s$ be a sequence of strictly decreasing integers such that $0 \leq d_i \leq m$ for any i. The flag variety $\mathcal{F}^{d_1,\ldots,d_s}(\mathbb{C}^m)$ is the set of nested subspaces of $\mathbb{C}^m$: $\{V_\bullet := (V_1 \subset \ldots \subset V_s) \mid \dim(\mathbb{C}^m/V_i) = d_i\}$.*

Later on we will refer to the s-uple $\underline{d} := (d_1, \ldots, d_s)$ as the type of any flag $V_\bullet \in \mathcal{F}^{d_1,\ldots,d_s}(\mathbb{C}^m)$.

Such sets can be embedded inside a product of projective spaces. This requires the use of the Plücker embedding. For a finite dimensional complex vector space V we denote with $\mathbb{P}^*(V)$ the dual projective space, i.e. $\mathbb{P}(V^*)$. Then recall that, if $Gr^d(\mathbb{C}^m)$ is the Grassmannian of subspaces of $\mathbb{C}^m$ of codimension d, the Plücker embedding

$$\begin{array}{rcl} Gr^d(\mathbb{C}^m) & \longrightarrow & \mathbb{P}^*(\wedge^d \mathbb{C}^m) \\ V & \longmapsto & \ker(\wedge^d \mathbb{C}^m \to \wedge^d(\mathbb{C}^m/V)) \end{array}$$

gives a bijection from $Gr^d(\mathbb{C}^m)$ to a subvariety of $\mathbb{P}^*(\wedge^d \mathbb{C}^m)$.
The flag variety $\mathcal{F}^{d_1,\ldots,d_s}(\mathbb{C}^m)$ is a subset of $Gr^{d_1}(\mathbb{C}^m) \times \ldots \times Gr^{d_s}(\mathbb{C}^m)$ with

some incidence relationships, thus we can embed $\mathcal{F}^{d_1,\ldots,d_s}(\mathbb{C}^m) \hookrightarrow \mathbb{P}^*(\wedge^{d_1}\mathbb{C}^m) \times \ldots \times \mathbb{P}^*(\wedge^{d_s}\mathbb{C}^m)$.

Furthermore any flag variety $\mathcal{F}^{d_1,\ldots,d_s}(\mathbb{C}^m)$ can be identified with a quotient of $\mathbb{GL}_m$. Indeed let $P_{\underline{d}}$ be the subgroup of $\mathbb{GL}_m$ formed by the matrices with invertible square matrices of size $d_s, d_{s-1}-d_s, \ldots, m-d_1$ along the diagonal and arbitrary entries below.

Proposition 2.1.2. *The quotient $\mathbb{GL}_m/P_{\underline{d}}$ can be identified with the flag variety $\mathcal{F}^{d_1,\ldots,d_s}(\mathbb{C}^m)$.*

Proof. The subgroup $P_{\underline{d}}$ previously introduced is exactly the subgroup of $\mathbb{GL}_m$ fixing the flag $U_\bullet = (U_1 \subset \ldots \subset U_s)$ of subspaces of $\mathbb{C}^m$ defined by $U_i := \langle e_{d_i+1}, \ldots, e_m \rangle$.
Consequently we have a well-defined map:

$$\begin{array}{rcl} \mathbb{GL}_m/P_{\underline{d}} & \to & \mathcal{F}^{d_1,\ldots,d_s}(\mathbb{C}^m) \\ gP_{\underline{d}} & \mapsto & g \cdot U_1 \subset \ldots \subset g \cdot U_s \end{array}$$

The claim now follows from the fact that $\mathbb{GL}_m$ acts transitively on the set of all flags of fixed type. □

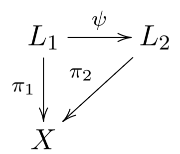
Hence the embedding of the flag variety $\mathcal{F}^{d_1,\ldots,d_s}(\mathbb{C}^m)$ previously described gives the quotient $\mathbb{GL}_m/P_{\underline{d}}$ a structure of projective variety.

2.2 Equivariant line bundles

As anticipated, the action of $\mathbb{GL}_m$ will be given on the space of global sections of certain line bundles on flag varieties. So we begin recalling a few definitions about line bundles on projective varieties.

Definition 2.2.1. *A line bundle on a projective variety X is a morphism of algebraic varieties $\pi : L \to X$ together with a one-dimensional complex vector space structure on $\pi^{-1}(x)$ for each $x \in X$, with the property that there exists an open covering $\{U_i\}_{i \in I}$ of X and isomorphisms $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ s.t. the maps $\psi_i \circ \psi_j^{-1} : (U_i \cap U_j) \times \mathbb{C} \to (U_i \cap U_j) \times \mathbb{C}$ are given by $(x, z) \mapsto (x, \phi_{ij}(z))$, with $\phi_{ij} : U_i \cap U_j \to \mathbb{GL}(\mathbb{C})$ regular.*

A morphism of line bundles $\pi_1 : L_1 \to X, \pi_2 : L_2 \to X$ is a morphism $\psi : L_1 \to L_2$ s.t. the diagram



commutes and the maps induced between the fibres $\pi_1^{-1}(x)$ and $\pi_2^{-1}(x)$ are linear. For any variety X, there is a line bundle $L = X \times \mathbb{C}$ called the trivial line bundle. Hence the condition $ii)$ in definition 2.2.1 equals to require L locally trivial.

Example 2.2.2. On any projective space $\mathbb{P}^*(V)$ there is a hyperplane line bundle $\mathcal{O}_{\mathbb{P}^*(V)}(1)$ whose fibre over a point $W \in \mathbb{P}^*(V)$ is given by the quotient V/W.
The n-th tensor power $\mathcal{O}_{\mathbb{P}^*(V)}(1)^{\otimes n}$ is a line bundle on $\mathbb{P}^*(V)$ as well and we denote it with the symbol $\mathcal{O}_{\mathbb{P}^*(V)}(n)$.
For any projective subvariety $X \subseteq \mathbb{P}^*(V)$ we call $\mathcal{O}_X(n)$ the restriction of $\mathcal{O}_{\mathbb{P}^*(V)}(n)$ to X and more generally, on a subvariety $X \subseteq \prod_{i=1}^r \mathbb{P}^*(V_i)$, we define the line bundle $\mathcal{O}_X(n_1, \ldots, n_r) := \bigotimes_{i=1}^r (\mathrm{pr}_i)^* \mathcal{O}_{\mathbb{P}^*(V_i)}(n_i)$, where pr_j is the projection $X \subseteq \prod_{i=1}^r \mathbb{P}^*(V_i) \to \mathbb{P}^*(V_j)$.

The group $\mathbb{GL}_m$ acts naturally on the flag variety $\mathcal{F}^{d_1,\ldots,d_s}(\mathbb{C}^m)$. We would like $\mathbb{GL}_m$ to act also on some line bundles over it in such a way that the projection from the line bundle onto the variety commutes with these actions. This sort of line bundles are called equivariant.

Definition 2.2.3. *An equivariant line bundle on a variety X endowed with an action of a group G is a line bundle $\pi : L \to X$, together with an action of G on L, s.t. $\pi(g.x) = g.\pi(x)$ for any $x \in L$ and $g \in G$.*

Given a line bundle L, a morphism of varieties $s : X \to L$ s.t. $\pi \circ s = \mathrm{id}$ is called a global section of L. The set $\Gamma(X, L)$ of all such sections is a vector space with the operations defined using the vector space structure on each fibre.
Furthermore, for an equivariant line bundle, $\Gamma(X, L)$ has a structure of a G-module, with the action of an element $g \in G$ given by $s \mapsto (x \mapsto g.s(g^{-1}.x))$. In fact it easy to check that $\pi(g.s(g^{-1}.x)) = g.(\pi \circ s)(g^{-1}.x) = g.\mathrm{id}(g^{-1}.x) = x$ for any $x \in X$.

Now our goal is the realization of the irreducible representation of highest weight $\lambda := (\lambda_1, \ldots, \lambda_m)$ on the space of global sections of a suitable equivariant line bundle on a flag variety.
Let $\tilde{\lambda} = (d_1^{a_1}, \ldots, d_s^{a_s})$ be the partition conjugate to λ (where a_i stands for the multiplicity of d_i).

Theorem 2.2.4. *Let L^λ be the line bundle $\mathcal{O}_{\mathbb{GL}_m/P_{\underline{d}}}(a_1, \ldots, a_s)$ for the embedding of the flag variety $\mathbb{GL}_m/P_{\underline{d}} = \mathcal{F}^{d_1,\ldots,d_s}(\mathbb{C}^m)$ inside $\mathbb{P}^*(\wedge^{d_1}\mathbb{C}^m) \times \ldots \times \mathbb{P}^*(\wedge^{d_s}\mathbb{C}^m)$. The space of sections $\Gamma(\mathbb{GL}_m/P_{\underline{d}}, L^\lambda)$ is isomorphic to the representation $V(\lambda)$.*

Proof. Postponed. □

In order to get a description as explicit as possible of this representation, we need to make beforehand a few observations.

For any equivariant line bundle $\pi : L \to \mathbb{GL}_m/P_{\underline{d}}$, $P_{\underline{d}}$ acts on the left on the fiber $(L)_{P_{\underline{d}}}$ by a character $\chi \in \hat{P}_{\underline{d}}$. Indeed for any $p \in P_{\underline{d}}$ and $x \in (L)_{P_{\underline{d}}}$ we have $\pi(p.x) = p.\pi(x) = p.P_{\underline{d}} = P_{\underline{d}}$ in $\mathbb{GL}_m/P_{\underline{d}}$ thanks to the equivariance. Hence $P_{\underline{d}}$ acts as a subgroup of $\mathrm{Aut}((L)_{P_{\underline{d}}}) \cong \mathbb{C}^\times$.
On the other hand, for any character $\chi \in \hat{P}_{\underline{d}}$, we can manufacture an equivariant line bundle $L(\chi)$ over $\mathbb{GL}_m/P_{\underline{d}}$ by defining

$$L(\chi) := (\mathbb{GL}_m \times \mathbb{C})/\sim$$

where the equivalence relation is given by $(gp, z) \sim (g, \chi(p)z)$ and taking the canonical projection $\pi_\chi : (\mathbb{GL}_m \times \mathbb{C})/\sim \to \mathbb{GL}_m/P_{\underline{d}}$. Moreover $L(\chi)$ is endowed with a natural $\mathbb{GL}_m$-action given by multiplication on the first factor, which commutes with the projection π_χ.
The following proposition will allow us to work with this kind of line bundles as well.

Proposition 2.2.5. *The category of equivariant line bundles on $\mathbb{GL}_m/P_{\underline{d}}$ with morphisms the isomorphisms of equivariant line bundles (i.e. isomorphisms of line bundles commuting with the group action) and the category of characters of $P_{\underline{d}}$ with morphisms the identity morphisms are equivalent.*

Proof. We have already described two correspondences between the objects. We hence need to verify whether they are functorial and both the compositions are naturally isomorphic to the identity functor.
To check the functoriality we simply need to prove that two isomorphic equivariant line bundles L_1, L_2 yield the same character. This follows from the equivariance, in fact, if $\phi : L_1 \to L_2$ is an isomorphism, then for any $x \in (L)_{P_{\underline{d}}}, p \in P_{\underline{d}}$ we have $\chi_1(p)\phi(x) = \phi(\chi_1(p)x) = \phi(p.x) = p.\phi(x) = \chi_2(p)\phi(x)$.
It remains to show that if χ is the character associated to the equivariant line bundle $\pi : L \to \mathbb{GL}_m/P_{\underline{d}}$, then $\pi_\chi : L(\chi) \to \mathbb{GL}_m/P_{\underline{d}}$ is isomorphic to L, whereas it is clear that the character constructed from $L(\chi)$ is χ itself. To prove the claim we first fix a non-zero element $y \in (L)_{P_{\underline{d}}}$ and define the map

$$\begin{array}{rccc} \phi: & L(\chi) & \to & L \\ & (g,z) & \mapsto & g.zy \end{array}$$

This is well defined as if $(g, z) \sim (g', z')$, that is $g' = gp^{-1}$ and $z' = \chi(p)z$, then $\phi(g', z') = g'.z'y = gp^{-1}.\chi(p)zy = g.\chi(p^{-1})\chi(p)zy = \phi(g, z)$. Besides $\pi \circ \phi = \pi_\chi$ since $\pi_\chi(g, z) = gP_{\underline{d}} = g.\pi(zy) = \pi(g.zy) = \pi(\phi(g, z))$. Finally ϕ is clearly equivariant.
We claim that ϕ admits an inverse. Let $x \in (L)_{gP_{\underline{d}}}$, then $x = g.(g^{-1}.x)$ where $g^{-1}.x \in (L)_{P_{\underline{d}}}$ and consequently it can be written as zy for a suitable $z \in \mathbb{C}$. So

we define a morphism

$$\begin{array}{rccc}\psi: & L & \to & L(\chi)\\ & x = g.(zy) & \mapsto & (g,z)\end{array}$$

which is well defined as if $g' = gp$ for some $p \in P_{\underline{d}}$, then $x = g'.(g'^{-1}.x) = gp.(p^{-1}.(g^{-1}.x)) = gp.(\chi(p^{-1})g^{-1}.x) = gp.(\chi(p^{-1})zy)$ and consequently $\psi(x) = (gp, \chi(p^{-1})z) \sim (g, \chi(p)\chi(p^{-1})z) = (g,z)$ in $L(\chi)$.
The morphism ψ is the inverse of ϕ and the proof is concluded. □

Willing to work with equivariant line bundles of the type $L(\chi)$, now we have to compute the character associated to the line bundle L^λ.
The fiber of L^λ at a point $V_\bullet := (V_1 \subset \ldots \subset V_s)$ of $\mathcal{F}^{d_1,\ldots,d_s}(\mathbb{C}^m)$ is, by definition, $(L^\lambda)_{V_\bullet} = \wedge^{d_1}(\mathbb{C}^m/V_1)^{\otimes a_1} \otimes \ldots \otimes \wedge^{d_s}(\mathbb{C}^m/V_s)^{\otimes a_s}$.
The fixed point under the action of $P_{\underline{d}}$ is given by the flag $U_\bullet := (U_1 \subset \ldots \subset U_s)$, where we recall $U_i := \langle e_{d_{i+1}}, \ldots, e_m \rangle$.
Hence, if we take a point $x := \alpha(e_1 \wedge \ldots \wedge e_{d_1})^{\otimes a_1} \otimes \ldots \otimes (e_1 \wedge \ldots \wedge e_m)^{\otimes a_s} \in (L^\lambda)_{U_\bullet}$ we have $p.x = \det(p_{d_1})^{a_1} \cdot \ldots \cdot \det(p_{d_s})^{a_s} x$, where we put $p_{d_k} = (p_{ij})_{i,j=1}^{d_k}$. This means that L^λ is isomorphic to $L(\chi_\lambda)$ for

$$\chi_\lambda(p) := \prod_{i=1}^{s} \det(p_{d_i})^{a_i}$$

Now we recall that a section of the line bundle $\pi_\chi : L(\chi_\lambda) \to \mathbb{GL}_m/P_{\underline{d}}$ is a morphism $s : \mathbb{GL}_m/P_{\underline{d}} \to (\mathbb{GL}_m \times \mathbb{C})/\sim$ s.t. $\pi_\chi \circ s = \mathrm{id}$.
More explicitly if $s \in \Gamma(\mathbb{GL}_m/P_{\underline{d}}, L(\chi_\lambda))$ we must have $s(gP_{\underline{d}}) = (g, f(g))$ where $f : \mathbb{GL}_m \to \mathbb{C}$ is a morphism s.t. $(g, f(g)) \sim (gp, f(gp)) \sim (g, \chi_\lambda(p) f(gp))$, that is f satisfies the property:

$$f(g) = \chi_\lambda(p) f(gp) \tag{2.1}$$

We are finally able to prove our main claim.

Proof of the Theorem 2.2.4. We want to prove that the map

$$\begin{array}{rcl}\mathbb{GL}_m & \longrightarrow & \mathbb{GL}(\Gamma(\mathbb{GL}_m/P_{\underline{d}}, L(\chi_\lambda))\\ g & \longmapsto & s \mapsto (h \mapsto g.s(g^{-1}.h))\end{array}$$

defines a representation which is isomorphic to the irreducible representation $V(\lambda)$.
Let s be a section and suppose $s(h) = (h, f(h))$ for a suitable function f satisfying property (2.1). Then $g.s(g^{-1}.h) = g.(g^{-1}h, f(g^{-1}.h)) = (h, f(g^{-1}.h))$, so the action translates into an action of $\mathbb{GL}_m$ on the space $S(\chi_\lambda)$ of morphisms $\mathbb{GL}_m \to \mathbb{C}$ which fulfil the requirement (2.1). More precisely, the map $(h, f(h)) \mapsto f(h)$ is

an isomorphism of representations, with the action on $S(\chi_\lambda)$ given by $g.f(h) := f(g^{-1}h)$. Consequently we can focus on this second functional space.
Now we want to know whether there is, up to scalars, a unique highest weight vector of weight λ in $S(\chi_\lambda)$.
We start by noticing that the evaluation at $1 \in \mathbb{GL}_m$ is an injective morphism between the vector space of highest weight vectors and $\mathbb{C}$.
In fact the value of a function $f \in S(\chi_\lambda)$ on any element $p \in P_{\underline{d}}$ can be computed using relation (2.1): $f(p) = \chi(p^{-1})f(1)$. Moreover any highest weight vector is by definition invariant under the action of the group N of upper triangular matrices with all 1's along the diagonal, hence $f(u.g) = u^{-1}.f(g) = f(g)$ for any $u \in N, g \in \mathbb{GL}_m$ and highest weight vector f. Now, being the subspace $N \cdot P_{\underline{d}}$ dense in $\mathbb{GL}_m$, we conclude that

$$\begin{array}{ccc} S(\chi_\lambda)^N & \to & \mathbb{C} \\ f & \mapsto & f(1) \end{array}$$

is an injective homomorphism from the vector space of N-invariant vectors to $\mathbb{C}$. Consequently the vector space $S(\chi_\lambda)^N$ is at most one dimensional. Using the character χ_λ we can exhibit a non-zero vector $F \in S(\chi_\lambda)^N$.
We claim that such a vector is given by the function $F(g) := \chi_\lambda(g^{-1}) := \prod_{i=1}^s \det((g^{-1})_{d_i})^{a_i}$.
We first observe that χ_λ extended to the whole group $\mathbb{GL}_m$ is clearly not a homomorphism, but with simple computations it seen to satisfy the relations

$$\begin{cases} \chi_\lambda(pg) = \chi_\lambda(p)\chi_\lambda(g) & \text{for any } p \in P_{\underline{d}}, g \in \mathbb{GL}_m \\ \chi_\lambda(gu) = \chi_\lambda(g) & \text{for any } g \in \mathbb{GL}_m, u \in N \end{cases}$$

The former guarantees that $F \in S(\chi_\lambda)$ since $\chi_\lambda(p)F(gp) = \chi_\lambda(p)\chi_\lambda(p^{-1}g^{-1}) = \chi_\lambda(p)\chi_\lambda(p^{-1})\chi_\lambda(g^{-1}) = \chi_\lambda(g^{-1}) = F(g)$ for any $p \in P_{\underline{d}}$, the latter that F is invariant under the action of N as $u.F(g) = F(u^{-1}g) = \chi_\lambda((u^{-1}g)^{-1}) = \chi_\lambda(g^{-1}u) = \chi_\lambda(g^{-1}) = F(g)$ for any $u \in U$.
Finally, if $t := \operatorname{diag}(t_1, \ldots, t_m)$, then

$$\begin{aligned} t.F(g) &= F(t^{-1}g) \\ &= \chi_\lambda(g^{-1}t) \\ &= \prod_{i=1}^s \det((g^{-1}t)_{d_i})^{a_i} \\ &= \prod_{i=1}^m t_i^{(\sum_{j|d_j \geq i} a_j)} \cdot \prod_{i=1}^s \det((g^{-1})_{d_i})^{a_i} \\ &= t_1^{\lambda_1} \cdot \ldots \cdot t_m^{\lambda_m} \cdot F(g) \end{aligned}$$

for any $g \in \mathbb{GL}_m$, that is to say F has exactly weight λ. Thanks to Theorem 1.1.3 this equals to say $S(\chi_\lambda) \cong V(\lambda)$. □

Example 2.2.6. We can make explicit the computation described above for the case $m = 2$.
We start from a dominant weight $\lambda = (\lambda_1, 0)$. Its conjugate is $\tilde{\lambda} = (1^{\lambda_1})$, so we know we have to look at the variety of flags of the type $(1, 0)$ inside $\mathbb{C}^2$, i.e. to $\mathbb{P}^1(\mathbb{C})$.
We then want to study the global sections of the line bundle $\mathcal{O}_{\mathbb{P}^1(\mathbb{C})}(\lambda_1)$ which is isomorphic to $L(\chi_\lambda)$ for the character $\chi_\lambda := x_{11}^{\lambda_1}$ of the subgroup

$$P_{\underline{d}} := \{g \in \mathbb{GL}_2 \mid x_{12} = 0\}$$

where x_{ij} stands for the standard coordinate function on $\mathbb{GL}_2$.
Hence the representation $V(\lambda)$ can be realized on the functional space

$$S(\chi_\lambda) := \left\{ f \in \mathbb{C}\left[\{x_{ij}\}_{i,j=1}^2, (\det)^{-1}\right] \;\middle|\; f(g) = x_{11}^{\lambda_1}(p) f(gp) \text{ for } g \in \mathbb{GL}_2, p \in P_\lambda \right\}$$

More explicitly we require

$$\begin{aligned} f\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Big) &= x_{11}^{\lambda_1}\Big(\begin{pmatrix} p & 0 \\ q & r \end{pmatrix}\Big) f\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} p & 0 \\ q & r \end{pmatrix}\Big) \\ &= p^{\lambda_1} f\Big(\begin{pmatrix} ap+bq & br \\ cp+dq & dr \end{pmatrix}\Big) \end{aligned}$$

In particular for $p = r = 1$ this implies

$$f\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Big) = f\Big(\begin{pmatrix} a+bq & b \\ c+dq & d \end{pmatrix}\Big)$$

which means that $f \in \mathbb{C}\left[x_{12}, x_{22}, (\det)^{-1}\right]$.
Whereas for $q = 0, r = 1$ we have

$$f\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Big) = p^{\lambda_1} f\Big(\begin{pmatrix} ap & b \\ cp & d \end{pmatrix}\Big)$$

which, together with the previous observation, forces $f = \det^{-\lambda_1} h$, with $h \in \mathbb{C}[x_{12}, x_{22}]$.
Finally, for $p = 1, q = 0$ we compute

$$\begin{aligned} (ad-bc)^{-\lambda_1} h(b,d) &= f\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Big) \\ &= f\Big(\begin{pmatrix} a & br \\ c & dr \end{pmatrix}\Big) \\ &= (ad-bc)^{-\lambda_1} r^{-\lambda_1} h(br, qr) \end{aligned}$$

from which we may infer that $S(\chi_\lambda)$ is composed of all functions f of the form $(\det)^{-\lambda_1}h$ for $h \in \mathbb{C}[x_{12}, x_{22}]_{\lambda_1}$ homogeneous polynomial of degree λ_1 in x_{12}, x_{22}. Thus $\dim V(\lambda) = \dim S(\chi_\lambda) = \lambda_1 + 1$. Moreover we have the weight decomposition:

$$S(\chi_\lambda) = \bigoplus_{k=0}^{\lambda_1} \left\langle (\det)^{-\lambda_1} x_{12}^k x_{22}^{\lambda_1 - k} \right\rangle$$

since

$$\begin{aligned} \operatorname{diag}(t_1, t_2).(\det)^{-\lambda_1} x_{12}^k x_{22}^{\lambda_1-k}(g) &= (\det)^{-\lambda_1} x_{12}^k x_{22}^{\lambda_1-k}(\operatorname{diag}(t_1^{-1}, t_2^{-1})g) \\ &= (t_1 t_2)^{\lambda_1} \cdot t_1^{-k} t_2^{-\lambda_1+k} (\det)^{-\lambda_1} x_{12}^{\lambda_1-k} x_{22}^k(g) \\ &= t_1^{\lambda_1-k} t_2^k (\det)^{-\lambda_1} x_{12}^k x_{22}^{\lambda_1-k}(g) \end{aligned}$$

and hence $(\det)^{-\lambda_1} x_{12}^k x_{22}^{\lambda_1-k}$ is a weight vector of weight $(\lambda_1 - k, k)$.

2.3 Weyl character formula

We realized the irreducible representation $V(\lambda)$ on the space $\Gamma(\mathbb{GL}_m/P_{\underline{d}}, L^\lambda)$ of global sections of the line bundle $\pi_\lambda : \mathcal{O}_{\mathbb{GL}_m/P_{\underline{d}}}(a_1, \ldots, a_s) \to \mathbb{GL}_m/P_{\underline{d}}$, with $(d_1^{a_1}, \ldots, d_s^{a_s})$ the partition conjugate to λ.
The space we called $\Gamma(\mathbb{GL}_m/P_{\underline{d}}, L^\lambda)$ is actually the 0-cohomology group of a certain sheaf. Indeed the assignment

$$\begin{aligned} \mathcal{L}_\lambda : \quad \mathrm{Op}(\mathbb{GL}_m/P_{\underline{d}}) &\to \quad \mathrm{Vect} \\ U \quad &\mapsto \quad \{s : U \to L \mid \pi_\lambda \circ s = \mathrm{id}_U\} \end{aligned}$$

with the obvious restriction morphisms defines a sheaf of vector spaces on $\mathbb{GL}_m/P_{\underline{d}}$ and the vector space of global sections is $H^0(\mathbb{GL}_m/P_{\underline{d}}, \mathcal{L}_\lambda) \cong \Gamma(\mathbb{GL}_m/P_{\underline{d}}, L^\lambda)$.
In particular we showed that $H^0(\mathbb{GL}_m/P_{\underline{d}}, \mathcal{L}_\lambda) \neq 0$. This is proved to be the only non-vanishing cohomology group of the sheaf $\mathcal{L}_\lambda$.

Proposition 2.3.1. *In the notation used above, for any dominant weight λ we have*

$$H^i(\mathbb{GL}_m/P_{\underline{d}}, \mathcal{L}_\lambda) = 0$$

for any $i \neq 0$.

Proof. See e.g. [D]. □

Any element $g \in \mathbb{GL}_m$ acts as an endomorphism of $\Gamma(\mathbb{GL}_m/P_{\underline{d}}, L^\lambda)$. Now we want to compute the trace of this endomorphism using a deep result known as Atiyah-Bott fixed point theorem.
Consider a projective non-singular variety X endowed with a vector bundle $\pi : F \to X$ and a morphism $f : X \to X$.

By means of the morphism f we can define the pullback vector bundle $\pi' : f^*F \to X$ as $\{(x,z) \mid f(x) = \pi(z)\}$, i.e. with $(f^*F)_x = (F)_{f(x)}$. Furthermore f induces a map between the spaces of sections of the sheaves associated to these vector bundles:

$$\begin{array}{rccc} \Gamma_f: & \Gamma(X, \mathcal{L}_F) & \to & \Gamma(X, \mathcal{L}_{f^*F}) \\ & s & \mapsto & s \circ f \end{array}$$

Now, if there is a vector bundle morphism $\phi : f^*F \to F$, the composition $\phi \circ \Gamma_f$ gives an endomorphism, denoted with (f, ϕ), of $\Gamma(X, \mathcal{L}_F)$ which induces endomorphisms $H^k(f, \phi)$ of all cohomology groups $H^k(X, \mathcal{L}_F)$.
Any vector bundle map $\phi : f^*F \to F$ is called a lifting of f to F. With this terminology M. Atiyah and R. Bott established the following result.

Theorem 2.3.2 (Atiyah-Bott). *Let X be a non-singular projective variety, F a vector bundle on X and $f : X \to X$ a morphism whose graph is transversal to the diagonal Δ in $X \times X$. Also let ϕ be a lifting of f to F. Then*

$$Tr^{\bullet}((f, \phi)) := \sum_{q=0}^{\infty} (-1)^q \, Tr(H^k(f, \phi)) = \sum_{y \in S} \frac{Tr(\phi_{|y})}{\det(1 - (df)_y)}$$

where S is the set of fixed points of f and $(df)_y$ is the application tangent to f at the point y.

Proof. See [AB]. □

We can apply the previous result in case $X = \mathcal{F}^{d_1, \ldots, d_s}(\mathbb{C}^m)$, $F = L^\lambda$ and f is the map $l_{g^{-1}} : \mathbb{GL}_m/P_{\underline{d}} \to \mathbb{GL}_m/P_{\underline{d}}$ given by left multiplication by the element $g^{-1} \in \mathbb{GL}_m$.
As a lifting of the map $l_{g^{-1}}$ we can take the map $\Lambda_g : (l_{g^{-1}})^* L^\lambda \to L^\lambda$ defined on the fibres by

$$\begin{array}{rcl} ((l_{g^{-1}})^* L^\lambda)_x & \to & (L^\lambda)_x \\ y & \mapsto & g.y \end{array}$$

which induces the map

$$\begin{array}{rcl} \Gamma(\mathbb{GL}_m/P_{\underline{d}}, (l_{g^{-1}})^* L^\lambda) & \to & \Gamma(\mathbb{GL}_m/P_{\underline{d}}, L^\lambda) \\ s & \mapsto & g \circ s \end{array}$$

Thus the map $(l_{g^{-1}}, \Lambda_g)$ corresponds exactly to the endomorphism induced by the action of g on $\Gamma(\mathbb{GL}_m/P_{\underline{d}}, L^\lambda)$ as defined in the previous section. Consequently if we denote with $\mathrm{Tr}(g)$ the trace of this endomorphism, thanks to Theorem 2.3.2 and Proposition 2.3.1, we have

$$\mathrm{Tr}(g) = \sum_{y \in S} \frac{\mathrm{Tr}(\Lambda_{g|y})}{\det(1 - (dl_{g^{-1}})_y)} \tag{2.2}$$

We restrict the computation to the elements $g \in H$, in fact all semisimple elements in $\mathbb{GL}_m$ are conjugate to an element of H and the trace is known to be a class function.
Moreover suppose for simplicity $\lambda_1 > \lambda_2 > \ldots > \lambda_m > 0$, so that we work with the complete flag variety $\mathcal{F}^{m,\ldots,1}(\mathbb{C}^m)$ and the bundle $\mathcal{O}_{\mathbb{GL}_m/P_{\underline{d}}}(\lambda_m, \ldots, \lambda_1 - \lambda_2)$. Consider an element $t = \mathrm{diag}(t_1, \ldots, t_m) \in H$, with the t_i's distinct. The fixed points of $l_{t^{-1}}$ are exactly the flags of the form

$$V_\bullet^\sigma := (0 \subseteq \langle e_{\sigma(m)} \rangle \subseteq \langle e_{\sigma(m)}, e_{\sigma(m-1)} \rangle \subseteq \ldots \subseteq \langle e_{\sigma(m)}, \ldots, e_{\sigma(1)} \rangle)$$

where $(e_i)_{i=1}^m$ is the canonical basis of $\mathbb{C}^m$ and $\sigma \in \mathcal{S}_m$ a permutation.
Both the fibre of the line bundle $(l_{t^{-1}})^* L^\lambda$ and of L^λ above $V_\bullet^\sigma$ equal

$$\wedge^m (\mathbb{C}^m)^{\otimes \lambda_m} \otimes \ldots \otimes (\mathbb{C}^m / \langle e_{\sigma(m)}, \ldots, e_{\sigma(2)} \rangle)^{\lambda_1 - \lambda_2}$$

hence

$$\begin{aligned} \mathrm{Tr}(\Lambda_{t|V_\bullet^\sigma}) &= (t_{\sigma^{-1}(1)} \cdot \ldots \cdot t_{\sigma^{-1}(m)})^{\lambda_m} \cdot \ldots \cdot (t_{\sigma^{-1}(1)})^{\lambda_1 - \lambda_2} \\ &= \prod_{i=1}^m (t_{\sigma^{-1}(i)})^{\lambda_i} \end{aligned}$$

It is left to compute the denominators in the expression 2.2.
The tangent space at the origin $V_\bullet^{\mathrm{id}}$ is $T_{V_\bullet^{\mathrm{id}}}(\mathbb{GL}_m/P_{\underline{d}}) = \mathfrak{gl}_m/\mathfrak{p}_{\underline{d}}$ where $\mathfrak{p}_{\underline{d}} := \mathrm{Lie}(P_{\underline{d}})$, hence a basis is given by the set of matrices $\{E_{ij}\}_{1 \leq i < j \leq m}$ (E_{ij} being the matrix with one in the (i,j)-entry and zero elsewhere).
We claim that the differential $dl_{t^{-1}}$ acts on a vector $Y \in \mathfrak{gl}_m/\mathfrak{p}_{\underline{d}}$ as the adjoint map $\mathrm{Ad}(t^{-1})$. In fact, if we denote with $C_{t^{-1}}$ the conjugation action of t^{-1}, we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{GL}_m & \xrightarrow{C_{t^{-1}}} & \mathbb{GL}_m \\ \downarrow & & \downarrow \\ \mathbb{GL}_m/P_{\underline{d}} & \xrightarrow[l_{t^{-1}}]{} & \mathbb{GL}_m/P_{\underline{d}} \end{array}$$

Looking at the differential we get

$$\begin{array}{ccc} \mathfrak{gl}_m & \xrightarrow{Ad(t^{-1})} & \mathfrak{gl}_m \\ \downarrow & & \downarrow \\ \mathfrak{gl}_m/\mathfrak{p}_{\underline{d}} & \xrightarrow[dl_{t^{-1}}]{} & \mathfrak{gl}_m/\mathfrak{p}_{\underline{d}} \end{array}$$

and taking the quotients we obtain the desired result.
Thus in particular then we have $(dl_{t^{-1}})_{V_\bullet^{\mathrm{id}}}(E_{ij}) = t_i^{-1}t_jE_{ij}$, so

$$\det(1-(dl_{t^{-1}})_{V_\bullet^{\mathrm{id}}}) = \prod_{i<j}(1-t_i^{-1}t_j) = \left(t_1^{m-1}\cdot\ldots\cdot t_{m-1}\right)^{-1}\cdot\prod_{i<j}(t_i-t_j)$$

Similarly, if with an abuse of notation we denote with σ the permutation matrix defined by $\sigma(e_i) = e_{\sigma(i)}$, then we have $T_{V_\bullet^\sigma}(\mathbb{GL}_m/P_{\underline{d}}) = \sigma\cdot\mathfrak{gl}_m/\mathfrak{p}_{\underline{d}}\cdot\sigma^{-1}$. Consequently a basis for the tangent space at the flag $V_\bullet^\sigma$ is given by $E_{\sigma^{-1}(i),\sigma^{-1}(j)}$, with $i<j$, and we obtain

$$\det(1-(dl_{t^{-1}})_{V_\bullet^\sigma}) = \left((t_{\sigma^{-1}(1)})^{m-1}\cdot\ldots\cdot t_{\sigma^{-1}(m-1)}\right)^{-1}\cdot\mathrm{sgn}(\sigma)\prod_{i<j}(t_i-t_j)$$

Finally the equation 2.2 can now be rewritten as

$$\begin{aligned}\mathrm{Tr}(t) &= \sum_\sigma \frac{\prod_{i=1}^m (t_{\sigma^{-1}(i)})^{\lambda_i}}{\left((t_{\sigma^{-1}(1)})^{m-1}\cdot\ldots\cdot t_{\sigma^{-1}(m-1)}\right)^{-1}\cdot\mathrm{sgn}(\sigma)\prod_{i<j}(t_i-t_j)}\\ &= \frac{1}{\prod_{i<j}(t_i-t_j)}\cdot\Big(\sum_\sigma \mathrm{sgn}(\sigma)\prod_{i=1}^m (t_{\sigma^{-1}(i)})^{\lambda_i+m-i}\Big)\\ &= \frac{\det(t_j^{\lambda_i+m-i})}{\det(t_j^{m-i})}\end{aligned}$$

The last term gives the character of the representation $V(\lambda)$.

Theorem 2.3.3 (Weyl). *The character of the irreducible representation $\pi:\mathbb{GL}_m\to V(\lambda)$, for $\lambda=(\lambda_1,\ldots,\lambda_m)$, is the function $H\to\mathbb{C}$ given by:*

$$Tr(\pi(diag(t_1,\ldots,t_m))) := \frac{\det(t_j^{\lambda_i+m-i})}{\det(t_j^{m-i})}$$

Proof. See e.g. [FH], §24.1. □

Example 2.3.4. The formula due to Weyl provides informations on the dimension of each weight space of a given representation. Consider for example the representation $V(2,1,0)$ of $\mathbb{GL}_3$. The trace of an element $t=\mathrm{diag}(t_1,t_2,t_3)$ is given by

$$\begin{aligned}
\mathrm{Tr}(t) &= \frac{\det\begin{pmatrix} t_1^4 & t_2^4 & t_3^4 \\ t_1^2 & t_2^2 & t_3^2 \\ 1 & 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} t_1^2 & t_2^2 & t_3^2 \\ t_1 & t_2 & t_3 \\ 1 & 1 & 1 \end{pmatrix}} \\
&= \frac{(t_1^2 - t_2^2)(t_1^2 - t_3^2)(t_2^2 - t_3^2)}{(t_1 - t_2)(t_1 - t_3)(t_2 - t_3)} \\
&= t_1^2 t_2 + t_1^2 t_3 + t_1 t_2^2 + t_1 t_3^2 + 2t_1 t_2 t_3 + t_2^2 t_3 + t_2 t_3^2
\end{aligned}$$

The coefficient of the monomial $t_1^{\mu_1} t_2^{\mu_2} t_3^{\mu_3}$ is precisely the dimension of the weight space relative to $\mu = (\mu_1, \mu_2, \mu_3)$. Hence in this case we can deduce that there are 6 one-dimensional weight spaces, whereas the one relative to the weight $(1, 1, 1)$ has dimension 2.

Chapter 3

Ginzburg construction

In this chapter we will provide another geometric construction of the irreducible representations of $\mathbb{GL}_m$ due to Ginzburg [CG]. This time vector spaces involved are certain homology groups of conormal bundles of flag varieties.
Besides another geometric interpretation of these representations, we will gain a method to compute the dimensions of weight spaces in each $V(\lambda)$ by counting the number of irreducible components of certain varieties.
Finally we will provide some concrete examples of this construction, recovering in particular the description of the simple $\mathfrak{gl}_2$-modules.

3.1 Borel-Moore homology

Borel-Moore homology will be the functor used to construct representations of $\mathbb{GL}_m$. For a detailed treatment of the properties of BM homology we refer to [B].
Let Y be a locally compact topological space that has the homotopy type of a finite CW-complex. Furthermore we assume that Y admits a closed embedding into a C^∞-manifold M and there exists an open neighborhood $U \supset Y$ in M s.t. Y is a homotopy retract of U. For such space Y, BM homology is the homology of the chain complex of locally finite chains in Y, i.e. we consider chains of type $\sum_{i=0}^{\infty} a_i\sigma_i$, where σ_i is a singular simplex, $a_i \in \mathbb{C}$, s.t. for any compact set $C \subseteq Y$ there are only finitely many indexes i with the property that $C \cap \operatorname{supp}(\sigma_i) \neq \emptyset$ and $a_i \neq 0$.
From now on we will reserve the notation H_k for the k-th BM homology group.

We give a description of some properties of BM homology we are going to exploit in the Ginzburg construction.

i) If Y is an algebraic variety of complex dimension n, possibly non compact, then we have a well-defined fundamental class $[Y] \in H_{2n}(Y)$. Moreover every subvariety gives a fundamental class.

If $Y_1, \dots, Y_k$ are the n-dimensional irreducible components of Y then the fundamental classes $[Y_1], \dots, [Y_k]$ form a basis for the vector space $H_{2n}(Y)$, also denoted with $H_{top}(Y)$.

ii) BM homology is a covariant functor with respect to proper maps, i.e. if $f : X \to Y$ is a map s.t. $f^{-1}(C)$ is compact for any compact set $C \subseteq Y$, then we may define the pushforward

$$f_* : H_\bullet(X) \to H_\bullet(Y)$$

iii) If $f : X \to Y$ is a locally trivial fibre bundle with equidimensional fibre of dimension d then we can define a pullback morphism

$$f^* : H_\bullet(Y) \to H_{\bullet+d}(X)$$

In case of a trivial fibration $f : Y \times F \to Y$ this morphism is defined by the assignment $c \mapsto c \boxtimes [F]$, where $[F]$ stands for the fundamental class of F and $\boxtimes$ for Künneth isomorphism $H_\bullet(Y) \otimes H_\bullet(F) \cong H_\bullet(Y \times F)$.

iv) If M is a smooth manifold of real dimension n and Z, Z' are two closed subsets, each of which is supposed to be a homotopy retract of an open set in M, we can define a bilinear pairing

$$\cap : H_i(Z) \times H_j(Z') \to H_{i+j-n}(Z \cap Z')$$

3.2 Convolution in Borel-Moore homology

The use of properties listed above permits the construction of a convolution-type product in the BM homology of a given variety.
We describe it in its most general setting. Later on we will apply the same machinery to a precise choice of varieties.

3.2.1 Convolution product

Let M_1, M_2, M_3 be three smooth manifolds of real dimension, respectively, d_1, d_2, d_3 and let

$$Z_{12} \subseteq M_1 \times M_2, \quad Z_{23} \subseteq M_2 \times M_3$$

be closed subsets in the sense explained in iv).
We define the set-theoretic composition $Z_{12} \circ Z_{23}$ as

$$\{(m_1, m_3) \in M_1 \times M_3 \mid \exists m_2 \in M_2 \text{ s.t. } (m_1, m_2) \in Z_{12}, (m_2, m_3) \in Z_{23}\}$$

This can be viewed as a generalization of composition of functions as one sees in case both Z_{12} and Z_{23} are graphs of functions.

Now denote with p_{ij} the projection $M_1 \times M_2 \times M_3 \to M_i \times M_j$ and assume that the map

$$p_{13} : p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to M_1 \times M_3$$

is proper. In particular this implies that its image

$$p_{13}(p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23})) = p_{13}((Z_{12} \times M_3) \cap (M_1 \times Z_{23})) = Z_{12} \circ Z_{23}$$

is a closed subset in $M_1 \times M_3$.

We are now able to define a bilinear map

$$\begin{array}{rcl} H_i(Z_{12}) \times H_j(Z_{23}) & \to & H_{i+j-d_2}(Z_{12} \circ Z_{23}) \\ (c_{12}, c_{23}) & \mapsto & c_{12} * c_{23} \end{array} \tag{3.1}$$

in the following way.
First we consider the pullbacks

$$\begin{array}{ccc} H_i(Z_{12}) & \to & H_{i+d_3}(Z_{12} \times M_3) \\ c_{12} & \mapsto & c_{12} \boxtimes [M_3] \end{array} \qquad \begin{array}{ccc} H_j(Z_{23}) & \to & H_{j+d_1}(M_1 \times Z_{23}) \\ c_{23} & \mapsto & [M_1] \boxtimes c_{23} \end{array}$$

and then we perform the intersection pairing

$$H_{i+d_3}(Z_{12} \times M_3) \times H_{j+d_1}(Z_{23} \times M_1) \quad \to \quad H_{i+j-d_2}((Z_{12} \times M_3) \cap (M_1 \times Z_{23}))$$

Finally we compose with the pushforward

$$(p_{13})_* : \quad H_{i+j-d_2}((Z_{12} \times M_3) \cap (M_1 \times Z_{23})) \quad \to \quad H_{i+j-d_2}(Z_{12} \circ Z_{23})$$

which is well-defined under the assumption on the map p_{13}.
Hence explictly we have

$$c_{12} * c_{23} := (p_{13})_*\big((c_{12} \boxtimes [M_3]) \cap ([M_1] \boxtimes c_{23})\big)$$

The convolution just introduced is associative.

Proposition 3.2.1. *Consider four manifolds M_i, with $i = 1, \dots, 4$ and three closed subsets $Z_{i,i+1} \subseteq M_i \times M_{i+1}$, with $i = 1, \dots, 3$. Let p_{ij}^{rst} (later simply p_{ij}) be the projection $M_r \times M_s \times M_t \to M_i \times M_j$ and suppose we can define for such maps the pushforwards as in the general construction. Then the following equation holds:*

$$(c_{12} * c_{23}) * c_{34} = c_{12} * (c_{23} * c_{34})$$

for $c_{12} \in H_\bullet(Z_{12}), c_{23} \in H_\bullet(Z_{23}), c_{34} \in H_\bullet(Z_{34})$.

Proof. By definition we have $(c_{12} * c_{23}) * c_{34} =$

$$\begin{aligned}
&= (p_{13})_*((c_{12} \boxtimes [M_3]) \cap ([M_1] \boxtimes c_{23})) * c_{34} \\
&= (p_{14})_*\Big(((p_{13})_*((c_{12} \boxtimes [M_3]) \cap ([M_1] \boxtimes c_{23})) \boxtimes [M_4]) \cap ([M_1] \boxtimes c_{34})\Big) \\
&= (p_{14})_*\Big((c_{12} \boxtimes [M_3 \boxtimes M_4]) \cap ([M_1] \boxtimes c_{23} \boxtimes [M_4]) \cap ([M_1 \boxtimes M_2] \boxtimes c_{34})\Big) \\
&= (p_{14})_*\Big((c_{12} \boxtimes [M_4]) \cap ([M_1] \boxtimes (p_{24})_*((c_{23} \boxtimes [M_4]) \cap ([M_2] \boxtimes c_{34})))\Big) \\
&= c_{12} * (c_{23} * c_{34})
\end{aligned}$$

as claimed. □

3.2.2 Convolution product for varieties and their conormal bundles

We now describe a relationship between the convolution product for varieties and the convolution product for their conormal bundles, which will reveal to be worthy for the following sections.
So let M_1, M_2, M_3 be three complex manifolds, $Z_{12} \subseteq M_1 \times M_2, Z_{23} \subseteq M_2 \times M_3$ complex submanifolds and Z_{13} their set-theoretic composition $Z_{12} \circ Z_{23}$. Moreover denote with Y_{ij} the conormal bundle $T^*_{Z_{ij}}(M_i \times M_j)$ and, with abuse of notation, with p_{ij} both the projection $M_1 \times M_2 \times M_3 \to M_i \times M_j$ and the one $T^*(M_1 \times M_2 \times M_3) \to T^*(M_i \times M_j)$.
The results are collected in the following theorem.

Theorem 3.2.2. *Assume that Z_{12} and Z_{23} satisfy two conditions:*

a) the intersection of $p_{12}^{-1}(Z_{12})$ and $p_{23}^{-1}(Z_{23})$ is transverse;

b) the map $p_{13} : p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to Z_{13}$ is a smooth locally trivial fibration with compact fibre L.

Then the following holds:

i) we have the set-theoretic equality $Y_{12} \circ Y_{23} = Y_{13}$;

ii) the map $p_{13} : p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23}) \to Y_{13}$ is a smooth locally trivial fibration with fibre L;

iii) in $H_\bullet(Y_{13})$ we have the identity: $[Y_{12}] * [Y_{23}] = \chi(L)\,[Y_{13}]$, where $\chi(L)$ is the Euler characteristic of L.

Proof. [CG], 2.7.26. □

3.2.3 The convolution algebra

We consider the special case in which $M_1 = M_2 = M_3 = M$.
Let N be a variety and $\pi : M \to N$ be a proper map.
Put $Z_{12} = Z_{23} = Z := \{(m_1, m_2) \in M \times M \mid \pi(m_1) = \pi(m_2)\}$.
It's straightforward to verify that $Z \circ Z = Z$, hence we have a convolution map

$$H_\bullet(Z) \times H_\bullet(Z) \to H_\bullet(Z)$$

Corollary 3.2.3. *The vector space $H_\bullet(Z)$ endowed with the convolution product is an associative algebra. The unit is given by the fundamental class of $M_\Delta := \{(m, m) \mid m \in M\} \subseteq Z$.*

The homology group of each fibre of the proper map $\pi : M \to N$ has a natural structure of $H_\bullet(Z)$-module.
Indeed choose $x \in N$ and set $M^x := \pi^{-1}(x)$. We can apply the convolution construction for $M_1 = M_2 = M$ and $M_3 = \{\text{pt}\}$. Now let $Z_{12} = Z$ and $Z_{23} = M^x \times \{\text{pt}\}$, which can be identified with M^x. We see that $Z \circ M^x = M^x$ and this turns $H_\bullet(M^x)$ into a left $H_\bullet(Z)$-module with the action given by convolution:

$$\begin{array}{ccc} H_\bullet(Z) \times H_\bullet(M^x) & \to & H_\bullet(M^x) \\ (c, c') & \mapsto & c.c' := c * c' \end{array}$$

If $n := \dim_{\mathbb{R}} M$, the subspace $H_n(Z)$ is a subalgebra of $H_\bullet(Z)$ as can be deduced from the following general observation.

Remark 3.2.4. Let M_1, M_2, M_3 be smooth varieties of real dimensions m_1, m_2, m_3 respectively. Let $Z_{12} \subseteq M_1 \times M_2, Z_{23} \subseteq M_2 \times M_3$ be closed subsets and let $p = \frac{1}{2}(m_1 + m_2), q = \frac{1}{2}(m_2 + m_3), r = \frac{1}{2}(m_1 + m_3)$. Then, assuming p, q, r are integers, we immediately deduce from (3.1) that the convolution induces a map:

$$H_p(Z_{12}) \times H_q(Z_{23}) \to H_r(Z_{12} \circ Z_{23})$$

The action of the subalgebra $H_n(Z)$ on $H_\bullet(M^x)$ is degree preserving, that is, we have $H_n(Z).H_j(M^x) \subseteq H_j(M^x)$ for any $j \in \mathbb{Z}_{\geq 0}$.

3.3 Main construction

In this section we apply the convolution construction to flag varieties to provide in this way a geometrical construction of $\mathfrak{gl}_m$-irreducible modules.
This approach is due to Ginzburg [CG]. He established a link between the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_m(\mathbb{C}))$ and the geometry of flag varieties, but his construction actually holds for $\mathfrak{gl}_m$ as well.

We begin fixing the notation we will use throughout the section[1].
For a fixed integer $d \geq 1$ we denote with $\mathcal{F}(\mathbb{C}^d)$ the smooth compact manifold of m-step partial flags in $\mathbb{C}^d$. This space is in general disconnected with connected components parametrized by m-uples $\mu = (\mu_1, \ldots, \mu_m)$ of positive integers s.t. $\sum_i \mu_i = d$, the connected component correponding to μ being

$$\mathcal{F}_\mu(\mathbb{C}^d) := \Big\{ V_\bullet = (0 = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_{m-1} \subseteq V_m = \mathbb{C}^d) \mid \dim(V_i/V_{i-1}) = \mu_i \Big\}$$

Now denote with the symbol $\mathcal{N}_m$ the variety

$$\Big\{ X \in \mathrm{End}(\mathbb{C}^d) \mid X^m = \mathbb{O} \Big\}$$

and consider

$$M_m := \Big\{ (X, V_\bullet) \in \mathcal{N}_m \times \mathcal{F}(\mathbb{C}^d) \mid X(V_i) \subseteq V_{i-1}, \text{ for } 1 \leq i \leq m \Big\}$$

This variety, with the canonical projection $\pi : M_m \to \mathcal{F}(\mathbb{C}^d)$, is a vector bundle on $\mathcal{F}(\mathbb{C}^d)$ which is isomorphic to the cotangent bundle $T^*\mathcal{F}(\mathbb{C}^d)$.

The decomposition of $\mathcal{F}(\mathbb{C}^d)$ into connected components gives rise to a decomposition of M_m:

$$M_m = \sqcup_\mu M_\mu$$

where $M_\mu := T^*(\mathcal{F}_\mu(\mathbb{C}^d))$.

The fibres of the projection $M_m \to \mathcal{N}_m$ are called m-step Springer fibres and the fibre above $X \in \mathcal{N}_m$, denoted with $\mathcal{F}(\mathbb{C}^d)^X$, can also be viewed as a subvariety of $\mathcal{F}(\mathbb{C}^d)$.
Clearly also each of these fibres inherits the decomposition

$$\mathcal{F}(\mathbb{C}^d)^X = \sqcup_\mu \mathcal{F}_\mu(\mathbb{C}^d)^X$$

where $\mathcal{F}_\mu(\mathbb{C}^d)^X := \mathcal{F}_\mu(\mathbb{C}^d) \cap \mathcal{F}(\mathbb{C}^d)^X$.

Now define[2]

$$Z := \big\{ ((X, V_\bullet), (X', V'_\bullet)) \in M_m \times M_m \mid X = X' \big\} \subseteq M_m \times M_m$$

[1]We will use a different notation from the one of the second chapter to denote flag varieties. The reason is that we want to label the different connected components of the flag varieties in a way which stresses their link with the weight spaces of the representation we construct.

[2]We will implicitly use the convention that under the isomorphism

$$T^*\mathcal{F}(\mathbb{C}^d) \times T^*\mathcal{F}(\mathbb{C}^d) \cong T^*(\mathcal{F}(\mathbb{C}^d) \times \mathcal{F}(\mathbb{C}^d))$$

the standard symplectic form on the right hand side corresponds to $\omega_1 - \omega_2$ where ω_1 and ω_2 are the symplectic forms on the first and second factors of the left hand side respectively.

Proposition 3.3.1. *The variety Z is the union of the conormal bundles to the $\mathbb{GL}_d$-orbits in $\mathcal{F}(\mathbb{C}^d) \times \mathcal{F}(\mathbb{C}^d)$. The closures of these conormal bundles are precisely the irreducible components of Z.*

Proof. See Proposition 3.3.4 in [CG]. □

According to the results of previous sections, $H_\bullet(Z)$ is an associative algebra and $H_\bullet(\mathcal{F}(\mathbb{C}^d)^X)$ is a left $H_\bullet(Z)$-module for any $X \in \mathcal{N}_m$.
We will focus on a certain subalgebra of $H_\bullet(Z)$ and a submodule of $H_\bullet(\mathcal{F}(\mathbb{C}^d)^X)$. We will then exhibit a surjective homomorphism of $\mathcal{U}(\mathfrak{gl}_m(\mathbb{C}))$ onto the former which will make the latter into an irreducible $\mathfrak{gl}_m$-module. In particular the highest weight will be proved to be linked to the Jordan type of the matrix X.

So we introduce the vector space $H(Z)$ defined as the vector subspace of $H_\bullet(Z)$ spanned by the fundamental classes of the irreducible components of Z. This vector space is actually a subalgebra of $H_\bullet(Z)$ as follows from remark 3.2.4 using a result proved by Spaltenstein [Sp] which states that all the irreducible components of Z contained in $M_\mu \times M_\eta$ have dimension equal to $\frac{1}{2}\dim(M_\mu \times M_\eta)$. Similarly we define $H(\mathcal{F}(\mathbb{C}^d)^X)$ as the submodule of $H_\bullet(\mathcal{F}(\mathbb{C}^d)^X)$ spanned by the fundamental classes of the irreducible components of $\mathcal{F}(\mathbb{C}^d)^X$.

Next theorem is the fundamental tool to give a $\mathfrak{gl}_m$-module structure to the $H(Z)$-modules just introduced.

Theorem 3.3.2. *There is a natural surjective algebra homomorphism*

$$\mathcal{U}(\mathfrak{gl}_m) \to H(Z)$$

Description of the homomorphism. We assign the images of the Chevalley generators and the elements of the Cartan subalgebra of $\mathfrak{gl}_m$. These elements are known to generate the whole Lie algebra $\mathfrak{gl}_m$. All that is needed then is to verify the surjectivity and whether the assignment is consistent; but for this we refer to [CG].
First for each partition μ, whenever $\mu + \alpha_i$, respectively $\mu - \alpha_i$ (the α_i's are as in Chapter 1), is a partition as well, we define the subsets

$$\begin{aligned} Z_{\mu+\alpha_i,\mu} &:= \left\{ (V_\bullet, V'_\bullet) \in \mathcal{F}_{\mu+\alpha_i}(\mathbb{C}^d) \times \mathcal{F}_\mu(\mathbb{C}^d) \mid \begin{array}{c} V_j = V'_j \text{ for } j \neq i, \\ V_i \supseteq V'_i, \dim(V_i/V'_i) = 1 \end{array} \right\} \\ Z_{\mu-\alpha_i,\mu} &:= \left\{ (V_\bullet, V'_\bullet) \in \mathcal{F}_{\mu-\alpha_i}(\mathbb{C}^d) \times \mathcal{F}_\mu(\mathbb{C}^d) \mid \begin{array}{c} V_j = V'_j \text{ for } j \neq i, \\ V_i \subseteq V'_i, \dim(V'_i/V_i) = 1 \end{array} \right\} \end{aligned}$$

for $i = 1, \ldots, m-1$.
Note that each $Z_{\mu\pm\alpha_i,\mu}$ is a $\mathbb{GL}_d$-orbit in $\mathcal{F}_{\mu\pm\alpha_i}(\mathbb{C}^d) \times \mathcal{F}_\mu(\mathbb{C}^d)$.

Then we can give the following assignments:

$$E_i \;\mapsto\; \sum_\mu \left[T^*_{Z_{\mu+\alpha_i,\mu}}(\mathcal{F}_{\mu+\alpha_i}(\mathbb{C}^d)\times\mathcal{F}_\mu(\mathbb{C}^d))\right]$$

$$F_i \;\mapsto\; \sum_\mu \left[T^*_{Z_{\mu-\alpha_i,\mu}}(\mathcal{F}_{\mu-\alpha_i}(\mathbb{C}^d)\times\mathcal{F}_\mu(\mathbb{C}^d))\right]$$

where the summations are taken over subsets of partitions of d with the prescriptions mentioned above. Finally for the elements of the Cartan subalgebra we define:

$$H_i \;\mapsto\; \sum_\mu \mu_i \left[T^*_\Delta(\mathcal{F}_\mu(\mathbb{C}^d)\times\mathcal{F}_\mu(\mathbb{C}^d))\right]$$

where Δ is the diagonal subvariety. □

The previous theorem permits to give each $H(\mathcal{F}(\mathbb{C}^d)^X)$, for $X \in \mathcal{N}_m$, a $\mathfrak{gl}_m$-module structure.
In this manner we will be able to obtain irreducible representations of the general linear group. In particular we will study the way the highest weight of a representation is linked to the Jordan type of X.

Any $d \times d$-matrix is similar to a Jordan form matrix; in particular for a nilpotent matrix $X \in \mathcal{N}_m$ we can find in its conjugacy class a matrix with at most d nilpotent blocks along the diagonal. Furthermore we can suppose that these blocks are arranged in a weakly decreasing order with respect to their sizes. So we can establish a bijection between the conjugacy classes of $\mathcal{N}_m$ and the weakly decreasing sequences of d positive integers $\nu := (\nu_1, \dots, \nu_d)$ with the property that $\nu_1 + \ldots + \nu_d = d$ and that $\nu_i \leq m$ for any i, being $X^m = \mathbb{O}$. We will refer to the sequence relative to a matrix X as its Jordan type.
Now let $\lambda = (\lambda_1, \dots, \lambda_s)$ be the conjugate of ν, i.e. with $\lambda_i := \#\{j \mid \nu_j \geq i\}$. We notice that the condition $\nu_i \leq m$ for any i forces $\lambda_k = 0$ for $k > m$, hence the m-uple $\lambda = (\lambda_1, \dots, \lambda_m)$ can be thought of as an element of P_m^+.

Theorem 3.3.3. *Let $X \in \mathcal{N}_m$ be a matrix of Jordan type ν and let λ be the conjugate of ν. Then the vector space $H(\mathcal{F}(\mathbb{C}^d)^X)$ is the irreducible representation of $\mathfrak{gl}_m$ of heighest weight λ.*

Proof. What remains to show is that $H(\mathcal{F}(\mathbb{C}^d)^X)$ is irreducible and its highest weight vector has precisely weight λ. We give a proof of the second part, referring to [CG] for a proof of the first.
It follows from the definition of the action of the H_i's that the weight spaces for the representation $H(\mathcal{F}(\mathbb{C}^d)^X)$ are the fundamental classes of the irreducible components of $\mathcal{F}(\mathbb{C}^d)^X$. In particular the fundamental classes of the irreducible components inside

$$\mathcal{F}_\mu(\mathbb{C}^d)^X := \left\{V_\bullet = (0 = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_{m-1} \subseteq V_m = \mathbb{C}^d) \mid \dim(V_i/V_{i-1}) = \mu_i\right\}$$

are weight vectors of weight μ. Hence we prove our claim simply showing that $\mathcal{F}_\lambda(\mathbb{C}^d)^X \neq 0$ and, whenever $\mathcal{F}_\mu(\mathbb{C}^d)^X$ is non-empty, then $\lambda \geq \mu$.
We start proving that $\dim(\ker X^k) = \lambda_1 + \ldots + \lambda_k$, for $k > 0$. We can perform induction on k. Indeed $\dim(\ker X) = \#\{j \mid \nu_j \geq 1\} = \lambda_1$ by definition. In general

$$\begin{aligned} \dim(\ker X^k) &= \dim(\ker X^{k-1}) + \#\{j \mid \nu_j \geq k\} \\ &= (\lambda_1 + \ldots + \lambda_{k-1}) + \lambda_k \end{aligned}$$

where in the second passage we used the inductive hypothesis.
This actually proves that $\mathcal{F}_\lambda(\mathbb{C}^d)^X \neq 0$.
Then we want to show that $\mu_1 + \ldots + \mu_k \leq \lambda_1 + \ldots + \lambda_k$ for any k. By definition we have:

$$\begin{aligned} \mu_1 + \ldots + \mu_k &= \dim(V_1) + \ldots + \dim(V_k/V_{k-1}) \\ &= \dim(V_k) \\ &\leq \dim(\ker X^k) \\ &\leq \lambda_1 + \ldots + \lambda_k \end{aligned}$$

Consequently we obtain $\lambda - \mu = k_1\alpha_1 + \ldots + k_{m-1}\alpha_{m-1}$, with $k_i = \sum_{j=1}^i \lambda_j - \sum_{j=1}^i \mu_j \geq 0$ for any i. □

The theorem provides immediately a method to compute the dimensions of the weight spaces of irreducible representations.

Corollary 3.3.4. *The dimension of the weight space relative to the weight μ inside the irreducible representation of heighest weight λ equals the number of irreducible components of the subvariety $\mathcal{F}_\mu(\mathbb{C}^d)^X \subseteq \mathcal{F}(\mathbb{C}^d)^X$, for X a nilpotent matrix of Jordan type ν conjugate of λ.*

Example 3.3.5. We work out part of a first concrete example, namely we want to apply the Ginzburg construction to the realization of the irreducible representation of $\mathfrak{gl}_3$ of highest weight $\lambda = (2, 1, 0)$.
The conjugate of λ is λ itself. This means that we have to study BM homology of the fibre $\mathcal{F}(\mathbb{C}^3)^X$, where $\mathcal{F}(\mathbb{C}^3)$ is the variety of 3-step flags in $\mathbb{C}^3$ and $X = E_{12}$ (recall that the choice of a matrix conjugate to X leads to an isomorphic representation).
The connected components of $\mathcal{F}_3(\mathbb{C}^3)^X$ are the single points:

- $\mathcal{F}_{(2,1,0)}(\mathbb{C}^3)^X = \{0 \subseteq \langle e_1, e_3\rangle \subseteq \mathbb{C}^3 \subseteq \mathbb{C}^3\}$
- $\mathcal{F}_{(2,0,1)}(\mathbb{C}^3)^X = \{0 \subseteq \langle e_1, e_3\rangle \subseteq \langle e_1, e_3\rangle \subseteq \mathbb{C}^3\}$
- $\mathcal{F}_{(1,2,0)}(\mathbb{C}^3)^X = \{0 \subseteq \langle e_1\rangle \subseteq \mathbb{C}^3 \subseteq \mathbb{C}^3\}$

- $\mathcal{F}_{(1,0,2)}(\mathbb{C}^3)^X = \{0 \subseteq \langle e_1 \rangle \subseteq \langle e_1 \rangle \subseteq \mathbb{C}^3\}$
- $\mathcal{F}_{(0,1,2)}(\mathbb{C}^3)^X = \{0 \subseteq 0 \subseteq \langle e_1 \rangle \subseteq \mathbb{C}^3\}$
- $\mathcal{F}_{(0,2,1)}(\mathbb{C}^3)^X = \{0 \subseteq 0 \subseteq \langle e_1, e_3 \rangle \subseteq \mathbb{C}^3\}$

and the component

$$\mathcal{F}_{(1,1,1)}(\mathbb{C}^3)^X = \left\{ V_\bullet = (0 \subseteq V_1 \subseteq V_2 \subseteq \mathbb{C}^3) \;\middle|\; \begin{array}{c} \dim V_1 = 1, \dim V_2 = 2 \\ X.\mathbb{C}^3 \subseteq V_2, X.V_2 \subseteq V_1, X.V_1 = 0 \end{array} \right\}$$

For a flag $V_\bullet \in \mathcal{F}_{(1,1,1)}(\mathbb{C}^3)^X$, V_1 can be chosen among all lines contained in $\ker X$, i.e. $V_1 = \langle a_1e_1 + a_3e_3 \rangle$ for $(a_1, a_3) \in \mathbb{C}^2 \smallsetminus \{(0,0)\}$. Moreover we have $V_2 \supseteq X.\mathbb{C}^3 = \langle e_1 \rangle$, hence $V_2 = \langle e_1, v \rangle$ with $v = v_1e_1 + v_2e_2 + v_3e_3$ s.t. $X.V_2 \subseteq \langle a_1e_1 + a_3e_3 \rangle$.
This last condition implies $X.v = v_2e_1 = \gamma(a_1e_1 + a_3e_3)$ for some $\gamma \in \mathbb{C}$. Now we have two possibilities:

i) $v_2 = 0$, so $V_2 = \langle e_1, e_3 \rangle, V_1 = \langle a_1e_1 + a_3e_3 \rangle$;

ii) $a_3 = 0$, so $V_2 = \langle e_1, v_1e_1 + v_2e_2 + v_3e_3 \rangle, V_1 = \langle e_1 \rangle$.

This proves that $\mathcal{F}_{(1,1,1)}(\mathbb{C}^3)^X$ has two irreducible components, each of which is naturally isomorphic to $\mathbb{P}^1(\mathbb{C})$, glued together at the point $U_\bullet := (0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_3 \rangle \subseteq \mathbb{C}^3)$. Hence the dimension of the weight space relative to the weight $(1,1,1)$ in the irreducible representation $V(2,1,0)$ equals 2, whereas $\dim V(2,1,0) = 2 + 1 \cdot 6 = 8$.
To show how the procedure works to compute the action of the generators of $\mathfrak{gl}_3$, we study e.g. the induced map

$$E_1 : H_{top}(\mathcal{F}_{(1,1,1)}(\mathbb{C}^3)^X) \to H_{top}(\mathcal{F}_{(2,0,1)}(\mathbb{C}^3)^X)$$

If we define

$$Y_1 := \{V_\bullet = (0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, v_1e_1 + v_2e_2 + v_3e_3 \rangle \subseteq \mathbb{C}^3)\},$$
$$Y_2 := \{V_\bullet = (0 \subseteq \langle a_1e_1 + a_3e_3 \rangle \subseteq \langle e_1, e_3 \rangle \subseteq \mathbb{C}^3)\},$$

then a basis of $H_{top}(\mathcal{F}_{(1,1,1)}(\mathbb{C}^3)^X)$ is given by the fundamental classes $[Y_1], [Y_2]$. Instead, as seen before, $\mathcal{F}_{(2,0,1)}(\mathbb{C}^3)^X$ is just a single point which we denote with Y_3.
The action on E_1 on the first basis vector is given by

$$E_1.[Y_1] = \left[T^*_{Z_{(2,0,1),(1,1,1)}}(\mathcal{F}_{(2,0,1)}(\mathbb{C}^3) \times \mathcal{F}_{(1,1,1)}(\mathbb{C}^3))\right] * [Y_1]$$

The conormal bundle $T^*_{Z_{(2,0,1),(1,1,1)}}(\mathcal{F}_{(2,0,1)}(\mathbb{C}^d) \times \mathcal{F}_{(1,1,1)}(\mathbb{C}^d))$ can be identified with the subvariety of Z given by

$$\{((Y, V_\bullet), (Y, V'_\bullet)) \in M_{(2,0,1)} \times M_{(1,1,1)} \mid (V_\bullet, V'_\bullet) \in Z_{(2,0,1),(1,1,1)}\}$$

Call p_{ij} the projection from $M_{(2,0,1)} \times M_{(1,1,1)} \times \{pt\}$ to the product of the i-th and the j-th factor. Then we have

$$p_{12}^{-1}(T^*_{Z_{(2,0,1),(1,1,1)}}(\mathcal{F}_{(2,0,1)}(\mathbb{C}^3) \times \mathcal{F}_{(1,1,1)}(\mathbb{C}^3))) \cap p_{23}^{-1}(\{(X, Y_1)\} \times \{pt\}) =$$

$$= \{(X, Y_3)\} \times \{(X, U_\bullet)\} \times \{pt\}$$

consequently $E_1. [Y_1] = [Y_3]$.
Similarly, for the action on the other basis vector we compute

$$p_{12}^{-1}(T^*_{Z_{(2,0,1),(1,1,1)}}(\mathcal{F}_{(2,0,1)}(\mathbb{C}^3) \times \mathcal{F}_{(1,1,1)}(\mathbb{C}^3))) \cap p_{23}^{-1}(\{(X, Y_2)\} \times \{pt\}) =$$

$$= \{(X, Y_3)\} \times \{(X, Y_2)\} \times \{pt\}$$

Hence $E_1. [Y_2] = \chi(Y_2)\, [Y_3] = 2 \cdot [Y_3]$ and E_1, with respect to the bases previously introduced, is the linear operator $(1\ ,\ 2)$.

3.3.1 Irreducible $\mathfrak{gl}_2$-modules

Our goal now is to illustrate the case $m = 2$. The irreducible $\mathfrak{gl}_2$-modules are parametrized by couples of integers (λ_1, λ_2) with the condition $\lambda_1 \geq \lambda_2$. By tensoring with a suitable power of the determinant representation we know that we can suppose $\lambda_1 = \lambda \geq 0$ and $\lambda_2 = 0$.
The conjugate of $(\lambda, 0)$ is $\nu := (1, \ldots, 1) \in \mathbb{Z}^\lambda$, hence for the representation with highest weight λ we look at the 2-step flag variety

$$\mathcal{F}(\mathbb{C}^\lambda)^{\mathbb{O}} = \mathcal{F}(\mathbb{C}^\lambda) := \Big\{ V_\bullet := (0 \subseteq V \subseteq \mathbb{C}^\lambda) \Big\}$$

This decomposes as

$$\mathcal{F}(\mathbb{C}^\lambda) = \sqcup_{0 \leq \mu \leq \lambda} Gr^\lambda_\mu$$

where $Gr^\lambda_\mu := \big\{ V_\bullet := (0 \subseteq V \subseteq \mathbb{C}^\lambda) \mid \dim(V) = \mu \big\}$ is the Grassmannian variety of μ-subspaces inside $\mathbb{C}^\lambda$.
Since Grassmannian varieties are irreducible, we can immediately deduce that all weight spaces in $V(\lambda)$ are one-dimensional and that $\dim V(\lambda, 0) = \lambda + 1$.

Now we proceed computing explicitly the action of the elements $H_1, H_2, E := E_1, F := F_1$ of $\mathfrak{gl}_2$ on $\mathcal{F}(\mathbb{C}^\lambda)$ trying to recover the formulas for the $\mathfrak{gl}_2$-action on the module $V(\lambda)$.

So now we follow the construction in Theorem 3.3.2.
Simplifying the notation for this low-dimensional case, we introduce the varieties

$$\begin{aligned} Z^+_\mu &:= \big\{ (V_\bullet, V'_\bullet) \in Gr^\lambda_{\mu+1} \times Gr^\lambda_\mu \mid V \supseteq V' \big\} \text{ for } 0 \leq \mu < \lambda; \\ Z^-_\mu &:= \big\{ (V_\bullet, V'_\bullet) \in Gr^\lambda_{\mu-1} \times Gr^\lambda_\mu \mid V \subseteq V' \big\} \text{ for } 0 < \mu \leq \lambda \end{aligned}$$

Then we define the images of the generators of $\mathfrak{gl}_2$:

$$\begin{array}{rcl} H_1 & \mapsto & \sum_{\mu=0}^{\lambda} \mu \left[T^*_{\Delta}(Gr^\lambda_\mu \times Gr^\lambda_\mu)\right] \\ H_2 & \mapsto & \sum_{\mu=0}^{\lambda} (\lambda-\mu) \left[T^*_{\Delta}(Gr^\lambda_\mu \times Gr^\lambda_\mu)\right] \\ E & \mapsto & \sum_{\mu=0}^{\lambda-1} \left[T^*_{Z^+_\mu}(Gr^\lambda_{\mu+1} \times Gr^\lambda_\mu)\right]; \\ F & \mapsto & \sum_{\mu=1}^{\lambda} \left[T^*_{Z^+_\mu}(Gr^\lambda_{\mu-1} \times Gr^\lambda_\mu)\right] \end{array}$$

We first compute $H_1.\left[Gr^\lambda_\mu\right]$ using the procedure explained in Theorem 3.2.2.
In this case we have $M_1 := Gr^\lambda_\mu, M_2 := Gr^\lambda_\mu, M_3 := \{pt\}$. Moreover we consider the subvarieties $\Delta \subseteq Gr^\lambda_\mu \times Gr^\lambda_\mu$ and $Gr^\lambda_\mu \cong Gr^\lambda_\mu \times \{pt\}$.
By definition their set-theoretic composition is $\Delta \circ Gr^\lambda_\mu = Gr^\lambda_\mu$, hence

$$\begin{array}{rcl} H_1.\left[Gr^\lambda_\mu\right] & = & \mu \left[T^*_{\Delta}(Gr^\lambda_\mu \times Gr^\lambda_\mu)\right] * \left[Gr^\lambda_\mu\right] \\ & = & \mu \cdot \chi(L) \left[Gr^\lambda_\mu\right] \end{array}$$

where $\chi(L)$ is the Euler characteristic of the fibre L of the map

$$\Delta = \Delta \cap (Gr^\lambda_\mu \times Gr^\lambda_\mu) \to Gr^\lambda_\mu$$

over a generic point, thus is clearly one.
The computation for H_2 proceeds exactly in the same fashion, giving

$$H_2.\left[Gr^\lambda_\mu\right] = (\lambda-\mu)\left[Gr^\lambda_\mu\right]$$

The action of E on $\left[Gr^\lambda_\mu\right]$ instead is given by the convolution

$$\left[T^*_{Z^+_\mu}(Gr^\lambda_{\mu+1} \times Gr^\lambda_\mu)\right] * \left[Gr^\lambda_\mu\right]$$

so we need to apply the same argument used above but now for the case $M_1 := Gr^\lambda_{\mu+1}, M_2 := Gr^\lambda_\mu, M_3 := \{pt\}$. Since $Z^+_\mu \circ Gr^\lambda_\mu = Gr^\lambda_{\mu+1}$, we infer that

$$E.\left[Gr^\lambda_\mu\right] = \chi(L')\left[Gr^\lambda_{\mu+1}\right]$$

where L' is a generic fibre of the map $Z^+_\mu = Z^+_\mu \cap (Gr^\lambda_{\mu+1} \times Gr^\lambda_\mu) \to Gr^\lambda_{\mu+1}$. The fibre above a generic point $V_\bullet \in Gr^\lambda_{\mu+1}$ is the subvariety

$$\left\{(V_\bullet, V'_\bullet) \in Gr^\lambda_{\mu+1} \times Gr^\lambda_\mu \mid V' \subseteq V\right\} \cong \mathbb{P}^*(V)$$

whose Euler characteristic is $\chi(\mathbb{P}^\mu) = \mu + 1$.
Hence we finally obtain

$$E.\left[Gr^\lambda_\mu\right] = (\mu+1)\left[Gr^\lambda_\mu\right]$$

The action of F can be computed in a similar way, but now with $M_1 := Gr^{\lambda}_{\mu-1}, M_2 := Gr^{\lambda}_{\mu}, M_3 := \{pt\}$.
The set theoretic composition $Z^{-}_{\mu} \circ Gr^{\lambda}_{\mu}$ gives $Gr^{\lambda}_{\mu-1}$ whereas the fibre of the map $Z^{-}_{\mu} = Z^{-}_{\mu} \cap (Gr^{\lambda}_{\mu-1} \times Gr^{\lambda}_{\mu}) \to Gr^{\lambda}_{\mu-1}$ over a generic point $V_{\bullet} \in Gr^{\lambda}_{\mu-1}$ is the subvariety

$$\left\{(V_{\bullet}, V'_{\bullet}) \in Gr^{\lambda}_{\mu-1} \times Gr^{\lambda}_{\mu} \mid V \subseteq V'\right\} \cong \mathbb{P}(\mathbb{C}^{\lambda}/V)$$

Being $\chi(\mathbb{P}^{\lambda-\mu}) = \lambda - \mu + 1$, we conclude that

$$F.\left[Gr^{\lambda}_{\mu}\right] = (\lambda - \mu + 1)\left[Gr^{\lambda}_{\mu-1}\right]$$

The formulas deduced are precisely the ones defining the action of $\mathfrak{gl}_2$ on the standard basis of the irreducible representation $V(\lambda)$.

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