The Geometry of the Gauss Map and Moduli of Abelian Varieties

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Master thesis

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For Joy

Enkele andere overwegingen

Hoe zal ik dit uitleggen, dit waarom
wat wij vinden niet is
wat wij zoeken?

Laten we de tijd laten gaan
waarheen hij wil,

en zie dan hoe weiden hun vee vinden,
wouden hun wild, luchten hun vogels,
uitzichten onze ogen

en ach, hoe eenvoud zijn raadsel vindt.

Zo andersom is alles, misschien.
Ik zal dit uitleggen.

- *Rutger Kopland*

Introduction

The purpose of this thesis is to study certain singularities of theta divisors of complex principally polarized abelian varieties using the degeneracy loci of the Gauss map.

First, abelian varieties will be defined as certain schemes of finite type over a field k, and we will state some basic results. Then, using the techniques developed in [15], we will pass to an analytic setting, where abelian varieties are defined as complex tori allowing an embedding in projective space.

Continuing along the analytic route, we define theta functions and theta divisors. At the end of chapter 1, the theta function η as defined in [7] is introduced.

In chapter 2, we return to the algebro-geometric setting to develop some of the general theory of vector bundles. In section 2.1, several well-known exact sequences of locally free sheaves are used to derive some useful identities involving the sheaf of differentials and the normal sheaf of closed immersions $Y \to X$ of smooth algebraic varieties. After a short section on determinants, we define the ramification locus of a morphism of schemes as well as the rank q degeneracy loci of a morphism of vector bundles and show that these can be related in special cases. Then we define the Gauss map of suitable closed immersions in section 2.4, and in the final section of chapter 2, we show that the theta function η from 1.5 and the Gauss map are closely related.

We use the relation between η and the Gauss map in the final chapter to obtain information on the locus $\theta_{null} \subset \mathcal{A}_g$ consisting of principally polarized abelian varieties (A, Θ) such that Θ has a singularity at a point of order 2. In particular, we prove that Θ has an ordinary double point for generic $(A, \Theta) \in \theta_{null}$.

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I would also like to thank my wife, Joy, for the breakfasts and dinners she prepared in my stead when I was busy, but most of all for always believing in me.

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Contents

1 Complex abelian varieties

This section gives a short introduction to the theory of complex abelian varieties. There is also a very rich theory of abelian varieties over arbitrary fields k, but these are not the primary focus of this paper. After giving the general definition of an abelian variety in section 1.1, we will soon restrict ourselves to studying complex abelian varieties. These turn out to be complex tori, allowing us to use the machinery of complex analysis and linear algebra.

1.1 Abelian varieties

Abelian varieties are complete varieties whose points form a group. The maps giving the group structure should be morphisms of varieties. In this section, we will use the scheme-theoretic definition. Let S be a scheme.

Definition 1.1.1. A *group scheme over S* is an S-scheme $\pi : G \to S$ equipped with S-scheme morphisms

$$\begin{array}{lcl} e_G & : & S \to G \\ m_G & : & G \times_S G \to G \\ i_G & : & G \to G \end{array}$$

such that the following diagrams commute:

(associativity)

$$\begin{array}{ccc} G \times_S G \times_S G & \xrightarrow{m_G \times \mathrm{id}_G} & G \times_S G \\ \downarrow{\scriptstyle \mathrm{id}_G \times m_G} & & \downarrow{\scriptstyle m_G} \\ G \times_S G & \xrightarrow{m_G} & G \end{array}$$

(left identity)

$$\begin{array}{ccc} S \times_S G & \xrightarrow{e_G \times \mathrm{id}_G} & G \times_S G \\ & \searrow^{\sim} \quad \swarrow_{m_G} & \\ & G & \end{array}$$

(left inverse)

$$\begin{array}{ccccc} G & \xrightarrow{\pi} & S & & \\ \downarrow{\scriptstyle \Delta_{G/S}} & & & \searrow^{e_G} & \\ G \times_S G & \xrightarrow[i_G \times \mathrm{id}_G]{} & G \times_S G & \xrightarrow[m_G]{} & G. \end{array}$$

Given a group scheme G over S and any S-scheme T, the maps e_G, m_G and i_G turn the set $G(T)$ of T-valued points of G into a group. It is also possible to view G as a representable contravariant functor

$G : \mathrm{Sch}_S \to \mathrm{Grp}$ that sends an S-scheme T to the group $G(T)$. A group scheme is called *commutative* if $m_G(p_2 \times p_1) = m_G$, where $p_i : G \times_S G \to G$ is the i-th projection map, or equivalently if $G(T)$ is an abelian group for all test schemes T.

For example, consider the affine $\mathbb{Z}$-scheme $G = \mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[x, x^{-1}]$. It can be endowed with a group scheme structure. Let $e_G : \operatorname{Spec} \mathbb{Z} \to \mathbb{G}_m$ be the morphism corresponding to the map $\mathbb{Z}[x, x^{-1}] \to \mathbb{Z}$ given by $x \mapsto 1$, let $m_G : \mathbb{G}_m \times_{\mathbb{Z}} \mathbb{G}_m \to \mathbb{G}_m$ be the morphism corresponding to $\mathbb{Z}[x, x^{-1}] \to \mathbb{Z}[x, x^{-1}, y, y^{-1}]$ given by $x \mapsto xy$ and let $i_G : \mathbb{G}_m \to \mathbb{G}_m$ be the morphisms corresponding to the map $\mathbb{Z}[x, x^{-1}] \to \mathbb{Z}[x, x^{-1}]$ given by $x \mapsto x^{-1}$. Then (G, e_G, m_G, i_G) is a group scheme. For a scheme S, the base change of $\mathbb{G}_m$ to S is denoted $\mathbb{G}_{m,S}$. As a functor, $\mathbb{G}_m$ sends a scheme T to the multiplicative group $\mathcal{O}_T(T)^*$; indeed, any morphism $T \to \mathbb{G}_m$ is given by a ring homomorphism $\operatorname{Spec} \mathbb{Z}[x, x^{-1}] \to \mathcal{O}_T(T)$, which is uniquely determined by the image of x in $\mathcal{O}_T(T)^*$.

Definition 1.1.2. Let $A \to S$ and $B \to S$ be group schemes. A *morphism of group schemes* over S is a morphism of schemes $f : A \to B$ over S such that $f \circ e_A = e_B$, where $e_A : S \to A$ and $e_B : S \to B$ are the unit sections, and such that the following diagram commutes:

$$\begin{array}{ccc} A \times_S A & \xrightarrow{(f,f)} & B \times_S B \\ \big\downarrow{\scriptstyle m_A} & & \big\downarrow{\scriptstyle m_B} \\ A & \xrightarrow{f} & B. \end{array}$$

Let k be a field. For the remainder of this paper, an *algebraic variety over k* is a scheme of finite type over $\operatorname{Spec} k$ (see example 3.2.3. on page 88 of [10]). Often, we will simply write *variety* instead of *algebraic variety.* An algebraic variety that is also a group scheme is called an *algebraic group variety.*

Definition 1.1.3. An *abelian variety* A over k is an algebraic group variety over k that is geometrically integral and proper over k.

Let A be an abelian variety over k. Note that the definition doesn't require A to be a commutative group scheme, which may inspire doubt regarding the validity of the name *abelian.* Fortunately, given an abelian variety A, we can derive that A has to be a commutative group scheme. We need some preliminary results. The following lemma is exercise 3.2.9. from [10].

Lemma 1.1.4. Let X and Y be schemes of finite type over k, with X geometrically reduced. Let $f, g : X \to Y$ be morphisms such that they induce the same map $X(\bar{k}) \to Y(\bar{k})$. Then $f = g$ as morphisms of schemes.

Proof. Note that the algebraic closure $\bar{k}$ of k is a faithfully flat k-module, so $\operatorname{Spec} \bar{k} \to \operatorname{Spec} k$ is a faithfully flat quasicompact morphism. Since these are stable under pullback, we obtain the diagram

$$\begin{array}{ccc} X_{\bar{k}} & \overset{\tilde{f}}{\underset{\tilde{g}}{\rightrightarrows}} & Y_{\bar{k}} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ X & \overset{f}{\underset{g}{\rightrightarrows}} & Y \end{array}$$

of two pullbacks packed together with vertical arrows that are faithfully flat and quasicompact. Since π_1 is faithfully flat and quasicompact, it is an epimorphism in the category of schemes, so $f\pi_1 = g\pi_1$ implies $f = g$. Hence it suffices to show that $\pi_2\tilde{f} = \pi_2\tilde{g}$, which certainly holds if $\tilde{f} = \tilde{g}$. As X is geometrically reduced, $X_{\bar{k}}$ is reduced. Therefore, we may assume that k is algebraically closed.

The map f' on k-rational points induced by f is the map $X(k) \to Y(k)$ given by $\alpha \mapsto f \circ \alpha$, where $\alpha : \operatorname{Spec} k \to X$ is a k-rational point. The map $g' : X(k) \to Y(k)$ induced by g is defined analogously. Let $x \in X$ be a closed point of X. As k is algebraically closed, x corresponds to the unique k-rational point $\alpha : \operatorname{Spec} k \to X$ with $\alpha(*) = x$, where $*$ is the point of $\operatorname{Spec} k$. By assumption $f'(\alpha) = g'(\alpha)$, which yields $f(x) = g(x)$.

Denote by X^0 the set of closed points of X. As X is of finite type over k, it holds that X^0 is dense in X (see [10], remark 2.3.49.). For each $x \in X^0$, let $V_{f(x)} \subset Y$ be an affine open containing $f(x)$. Then $U_x = f^{-1}V_{f(x)} \cap g^{-1}V_{f(x)}$ is non-empty for all $x \in X^0$ and we will show that $\mathcal{U} = \{U_x : x \in X^0\}$ is an open cover of X. Let

$$U = \bigcup_{x \in X^0} U_x$$

and suppose that $Z = X \setminus U$ is non-empty. Note that Z is closed, so that we may regard it as a closed subscheme of X by taking the reduced scheme structure - it is of finite type over k, so the subset of closed points Z^0 is dense in Z. In particular, it is non-empty, and a closed point of Z is also a closed point of X, which contradicts the fact that $X^0 \subset U$. Thus $\mathcal{U}$ is an open cover of X.

For $x \in X^0$, let $f_x = f|_{U_x}$ and $g_x = g|_{U_x}$. Let $x \in X^0$ and consider $(f_x, g_x) : U_x \to V_{f(x)} \times_k V_{f(x)}$. Let Δ_x be the image of the diagonal morphism $V_{f(x)} \to V_{f(x)} \times_k V_{f(x)}$. Since $V_{f(x)}$ is affine, Δ_x is closed, so $(f_x, g_x)^{-1}(\Delta_x)$ is closed. It also contains $X^0 \cap U_x$, so it follows that $(f_x, g_x)^{-1}(\Delta_x)$ is the closure of $X^0 \cap U_x$ in U_x, which is just U_x. This yields $f_x(y) = g_x(y)$ for all $y \in U_x$. As this holds for all $x \in X^0$, it follows that $f(x) = g(x)$ for all $x \in X$.

Let $V \subset Y$ be an affine open. If we show that $f|_U = g|_U$ for each affine open $U \subset f^{-1}V = g^{-1}V$, it follows that $f = g$, as both Y and

$f^{-1}V$ can be covered by open affines. Since X is reduced, each affine open $U \subset f^{-1}V$ is reduced. Hence we may assume that X and Y are affine, with $X = \operatorname{Spec} B$ for some reduced finitely generated k-algebra B and $Y = \operatorname{Spec} A$ for $A = k[T_1, \ldots, T_n]/I$ a finitely generated k-algebra. Then there is a closed immersion $i : Y \to \mathbb{A}^n_k$, and to show that $f = g$ it is enough to show that $if = ig$. Thus we further assume that $Y = \mathbb{A}^n_k$ and we set $A = k[T_1, \ldots, T_n]$.

Now f and g are given by ring homomorphisms $\phi : A \to B$ and $\psi : A \to B$, respectively. A point $p \in X(k)$ corresponds uniquely to a closed point of X and thus to a maximal ideal $\mathfrak{p}$ of B (cf. remark 2.1.3, [10]); in fact, p is the canonical morphism $\operatorname{Spec} B/\mathfrak{p} \to \operatorname{Spec} B$, since $B/\mathfrak{p}$ is a field extension of the algebraically closed field k and therefore itself equal to k. Then $f(p), g(p) \in \mathbb{A}^n_k(k)$ are morphisms $\operatorname{Spec} B/\mathfrak{p} \to \mathbb{A}^n_k$ given by the compositions

$$A \xrightarrow{\phi} B \longrightarrow B/\mathfrak{p} \qquad \text{and} \qquad A \xrightarrow{\psi} B \longrightarrow B/\mathfrak{p},$$

respectively. They are completely determined by the images of the T_i, and since f and g agree on closed points, it holds that $\overline{\phi(T_i)} = \overline{\psi(T_i)}$ in $B/\mathfrak{p}$ for all i. Hence $\phi(T_i) - \psi(T_i) \in \mathfrak{p}$ for all maximal ideals $\mathfrak{p}$ of B, so $\phi(T_i) - \psi(T_i)$ is in the nilradical of B by lemma 2.1.18 of [10], which is the zero ideal by assumption. It follows that $\phi = \psi$, and we are done. □

Let X, Y and Z be algebraic varieties over a field k. Assume that X is complete and geometrically integral, and assume that Y is geometrically integral. Let $f : X \times_k Y \to Z$ be a morphism. We have a commutative diagram:

$$\begin{array}{ccccc} Z & \xleftarrow[f]{} & X \times_k Y & \xrightarrow{p_1} & X \\ & & \downarrow{\scriptstyle p_2} & & \downarrow \\ & & Y & \longrightarrow & \operatorname{Spec} k. \end{array}$$

The following lemma will help in showing that abelian varieties are indeed commutative group schemes. Moreover, it will give us a classification of morphisms of schemes between abelian varieties, which turn out to be morphisms of group schemes up to translation.

Lemma 1.1.5 (Rigidity lemma)**.** If there exist $y_0 \in Y(k)$ and $z_0 \in Z(k)$ such that

$$f(X \times_k \{y_0\}) = \{z_0\}$$

then f factors through the projection map $p_2 : X \times_k Y \to Y$.

Proof. We may assume that k is algebraically closed, using the same descent technique as in the proof of lemma 1.1.4. Fix a point $x_0 \in X(k)$ and define $g : Y \to Z$ by $y \mapsto f(x_0, y)$. We have to show that $f = gp_2$. Let $U \subset Z$ be an affine open such that $z_0 \in U$. As X is universally closed over k, it holds that p_2 is a closed map. Hence $V = p_2\left(f^{-1}(Z \setminus U)\right)$ is closed in Y. Of course, $y_0 \notin V$. Let $y \notin V$ be a k-rational point. Then it holds that $f(X \times_k \{y\}) \subset U$. As $X \times_k \{y\}$ is complete and U is affine, it follows that f must be constant on $X \times_k \{y\}$. Hence $f|_{X \times_k \{y\}} = gp_2|_{X \times_k \{y\}}$ for all k-rational points $y \notin V$. As $X \times_k Y$ is the product of reduced varieties, it is reduced. Hence it follows from lemma 1.1.4 that $f = gp_2$ on the non-empty open set $X \times_k (Y \setminus V)$. As $X \times_k Y$ is the product of irreducible varieties and therefore irreducible, it follows that $X \times_k (Y \setminus V)$ is dense in $X \times_k Y$, so $f = gp_2$ on the whole of $X \times_k Y$, as was to be shown. □

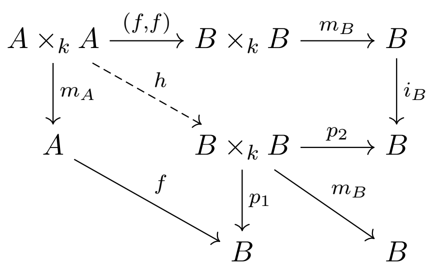
Let A be an abelian variety over k. For every $a \in A(k)$, there is a morphism of schemes $t_a : A \to A$ given by the composition

$$A \xrightarrow{\sim} A \times_k \operatorname{Spec} k \xrightarrow{\mathrm{id}_A \times a} A \times_k A \xrightarrow{m_A} A,$$

called *translation* by a. Let B be another abelian variety over k.

Corollary 1.1.6. Every morphism $f : A \to B$ of schemes over k is given by $t \circ \phi$, where $\phi : A \to B$ is a morphism of group schemes and $t : B \to B$ is a translation by $b \in B(k)$. In particular, any morphism $f : A \to B$ of schemes over k such that $f(0) = 0$ is also a morphism of group schemes over k, where $0 \in A(k)$ is the unit section.

Proof. Let $f : A \to B$ be a morphism of schemes over k. As $0 \in A(k)$, it holds that $f(0) \in B(k)$. Let $t : B \to B$ be the translation given by $b \mapsto m_B(b, i_B(f(0)))$, where m_B is the addition map on B and i_B is the inversion map on B. This map is a morphism of schemes. Therefore, we may replace f with $t \circ f$ and assume $f(0) = 0$. Consider the following commutative diagram



where m_A is the addition map on A, p_1 and p_2 are the projection maps and h is the unique map such that $p_1 \circ h = f \circ m_A$ and $p_2 \circ h = i_B \circ m_B \circ (f, f)$. Let $g = m_B \circ h : A \times_k A \to B$. It holds that $g(A \times_k \{0\}) = \{0\}$, so g

factors through the second projection $A \times A \to A$ by lemma 1.1.4. On the other hand, $g(\{0\} \times_k A) = \{0\}$, so g also factors through the first projection $A \times A \to A$. It follows that g is constant with image $\{0\}$. By uniqueness of inverse, it follows that $m_B(f, f) = fm_A$, which shows that f is a morphism of group schemes. □

Corollary 1.1.7. The abelian variety A over k is a commutative group scheme.

Proof. By corollary 1.1.6, the map i_A is a morphism of group schemes. Hence A is commutative. □

There is one last result that will be useful to us in the general setting, which we state without proof. The interested reader may refer to proposition 1.5 in the unpublished book [12].

Proposition 1.1.8. The abelian variety A is smooth over k and Ω_k is free.

1.2 From algebraic varieties to manifolds

The subject of this thesis balances between algebraic geometry and analytic geometry, so it is good to show how results from one area may be transported to the other. The article [15] by Serre was instrumental in the development of the interaction between the two different areas. We exhibit the functor $h : \mathrm{Coh}(X) \to \mathrm{Coh}^{an}(X_h)$ for a projective scheme X over $\mathbb{C}$ as found in appendix B of [6], which was adapted to scheme language from [15]. First, we will give a definition of a complex analytic space.

Definition 1.2.1. A *complex analytic space* is a ringed topological space $(X, \mathcal{O})$ which admits an open cover $\mathcal{U}$ such that each $U \in \mathcal{U}$ is a locally ringed topological space $(U, \mathcal{O}_U)$ that is isomorphic to a locally ringed topological space $(Y, \mathcal{O}_Y)$ of the following form: let $D \subset \mathbb{C}^n$ be the polydisc $D = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_i| < 1 \text{ for all } 1 \leq i \leq n\}$, equipped with the standard topology. Let $f_1, \ldots, f_q$ be holomorphic functions on D and let $Y \subset D$ be the closed subset of D consisting of the common zeroes of $f_1, \ldots, f_q$. Define $\mathcal{O}_Y = \mathcal{O}_D/(f_1, \ldots, f_q)$, where $\mathcal{O}_D$ is the sheaf of holomorphic functions on D.

This definition is quite cumbersome, but we get a lot of structure in return. We can see that a complex analytic space is basically “a bunch of zero loci in polydiscs glued together”. Since we can cover $\mathbb{C}^n$ by polydiscs of radius 1, it holds that $\mathbb{C}^n$ is a complex analytic space. Any closed subspace $Y \subset \mathbb{C}^n$ that is the zero locus of holomorphic functions $f_1, \ldots, f_q$ on $\mathbb{C}^n$ therefore has a natural structure of a complex analytic subspace.

Schemes of finite type over $\mathbb{C}$ can be used to give complex analytic spaces. Let X be a scheme of finite type over $\mathbb{C}$. Let $\mathcal{X} = \{X_i : i \in I\}$ be an affine open cover of X, where $X_i = \operatorname{Spec} A_i$ for each $i \in I$. Let $i \in I$. Then A_i is isomorphic to some finitely generated $\mathbb{C}$-algebra $\mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_q)$, since X is of finite type over $\mathbb{C}$. We can view the polynomials $f_1, \dots, f_q$ as holomorphic functions on $\mathbb{C}^n$. The ideal $(f_1, \dots, f_q)$ defines a complex analytic subspace $(X_i)_h \subset \mathbb{C}^n$. Thus we get a complex analytic space $(X_i)_h$ for each $i \in I$. We can use the gluing data of the X_i to glue the $(X_i)_h$ and get a complex analytic space X_h.

Definition 1.2.2. The complex analytic space X_h is called the *complex analytic space associated to* X, or the *associated complex analytic space* of X.

Set-theoretically, it holds that $X_h = X(\mathbb{C})$, but the topology on X_h is usually much finer than the topology on $X(\mathbb{C})$ induced by X.

Next we show that associating complex analytic spaces to schemes of finite type over $\mathbb{C}$ is functorial. Let $\mathrm{SchFT}_{\mathbb{C}}$ be the category of schemes of finite type over $\mathbb{C}$ and CAS that of complex analytic spaces. Define $F : \mathrm{SchFT}_{\mathbb{C}} \to \mathrm{CAS}$ by $X \mapsto X_h$. Let $f : X \to Y$ be an arrow in $\mathrm{SchFT}_{\mathbb{C}}$. Then there is an induced arrow $F(f) : X_h \to Y_h$ in CAS, which is just the map $X(\mathbb{C}) \to Y(\mathbb{C})$ induced by f. This map is an analytic function. It clearly holds that $F(\mathrm{id}_X) = \mathrm{id}_{X_h}$ and $F(gf) = F(g)F(f)$, so F is a functor as claimed.

We mentioned the functor $h : \mathrm{Coh}(X) \to \mathrm{Coh}^{an}(X_h)$ from coherent sheaves on X to coherent analytic sheaves on X_h, the latter of which we haven't defined yet.

Definition 1.2.3. Let X_h be a complex analytic space. A *coherent analytic sheaf* on X_h is a coherent sheaf of $\mathcal{O}_{X_h}$-modules.

Given X in $\mathrm{SchFT}_{\mathbb{C}}$ and a coherent sheaf $\mathcal{F}$ on X, there is an associated coherent sheaf $\mathcal{F}_h$ on X_h. Locally, it holds that

$$\mathcal{O}_U^m \xrightarrow{\phi} \mathcal{O}_U^n \longrightarrow \mathcal{F} \longrightarrow 0,$$

but since the topology on X_h is finer than the topology on $X(\mathbb{C})$ induced by the Zariski topology, U_h is open in X_h, and $\mathcal{F}_h$ is defined locally as

$$\mathcal{O}_{U_h}^m \xrightarrow{\phi_h} \mathcal{O}_{U_h}^n \longrightarrow \mathcal{F}_h \longrightarrow 0.$$

Thus we obtain the functor $h : \mathrm{Coh}(X) \to \mathrm{Coh}^{an}(X_h)$. There are some useful facts about the relationship between a scheme X of finite type over $\mathbb{C}$ and the associated complex analytic space X_h. For example, X is smooth over $\mathbb{C}$ if and only if X_h is a complex manifold. However, the functor F is *not* an equivalence of categories and we cannot move freely from complex analytic spaces to schemes of finite type over $\mathbb{C}$. The following results, therefore, are somewhat astounding.

Theorem 1.2.4 (Serre). Let X be a projective scheme over $\mathbb{C}$. Then $h : \mathrm{Coh}(X) \to \mathrm{Coh}^{an}(X_h)$ is an equivalence of categories and for every coherent sheaf $\mathcal{F}$ on X it holds that $H^i(X, \mathcal{F}) \cong H^i(X_h, \mathcal{F}_h)$.

Serre proves this in his influential article [15] as early as 1956, although in a slightly different form. At the time of writing, the theory of schemes was still being developed and the language of category theory was not widely used, so Serre needed three theorems to state this result in his article. Nonetheless, it has been a fertile ground for many important results in algebraic geometry. He proves the following theorem of Chow as a corollary.

Theorem 1.2.5 (Chow). If X' is a compact analytic subspace of the complex manifold $\mathbb{P}^n_{\mathbb{C}}$, then there is a subscheme $X \subset \mathbb{P}^n$ such that $X_h = X'$.

Exercise 6.6 of appendix B in [6] tells us that, given projective schemes X and Y over $\mathbb{C}$ and an arrow $f' : X_h \to Y_h$, there is a unique arrow $f : X \to Y$ such that $F(f) = f'$. It follows that F restricted to projective schemes induces an equivalence of categories from the category of projective schemes over $\mathbb{C}$ to the category of projective compact analytic spaces.

Naturally, these results are useful to us as complex abelian varieties are projective (see theorem 2.25 in [12]). Their analytic counterparts are discussed in the next section.

1.3 Complex tori

Let V be a g-dimensional complex vector space.

Definition 1.3.1. A *lattice* in V is a co-compact discrete subgroup $\Lambda \subset V$.

This is not a very constructive definition. Alternatively, one may define a lattice as the free $\mathbb{Z}$-module generated by an $\mathbb{R}$-basis $(\lambda_1, \ldots, \lambda_{2g})$ of V. These two definitions are equivalent.

Definition 1.3.2. A *complex torus* A is a quotient $A = V/\Lambda$, where V is a finite dimensional complex vector space and $\Lambda \subset V$ is a lattice.

If A is an abelian variety over $\mathbb{C}$, then its associated complex analytic space A_h is a complex torus, but the converse is not true in general and depends on the existence of an ample line bundle on the complex torus.

1.4 Theta functions

Let V be a g-dimensional complex vector space, $\Lambda \subset V$ a lattice and $A = V/\Lambda$ a complex torus. Let $\pi : V \to A$ be the quotient map. Let $f : L \to A$ be a line bundle on A. Consider the pullback diagram

$$\begin{array}{ccc} \pi^*L & \longrightarrow & L \\ \big\downarrow & & \big\downarrow f \\ V & \xrightarrow{\pi} & A. \end{array}$$

Then π^*L is a line bundle on V and therefore trivial by lemma 2.1 in [1]. Let $\phi : \pi^*L \to V \times \mathbb{C}$ be an isomorphism. The natural action of Λ on V lifts to an action on the line bundle $V \times \mathbb{C}$. For $\lambda \in \Lambda$ and $(v, z) \in V \times \mathbb{C}$, it holds that

$$\lambda(v, z) = (\lambda + v, e_\lambda(v)z)\,, \tag{1.1}$$

where e_λ is a holomorphic invertible function on V. The above formula defines a group action if and only if the functions e_λ satisfy

$$e_{\lambda+\mu}(v) = e_\lambda(v + \mu)e_\mu(v), \tag{1.2}$$

the so-called *cocycle condition.* It follows that L is the quotient of $V \times \mathbb{C}$ by this action, which is defined completely by the family $(e_\lambda)_{\lambda\in\Lambda}$.

Definition 1.4.1. A *system of multipliers* for Λ is a family of invertible holomorphic functions $(e_\lambda)_{\lambda\in\Lambda}$ on V satisfying the cocycle condition.

As shown above, each system of multipliers defines a line bundle, and each line bundle corresponds to such a system. In fact, if we let S be the set of systems of multipliers for Λ, then we can equip it with a group structure and there is a surjective group homomorphism $S \to \mathrm{Pic}(A)$ (see [1], page 105). Thus the product of two systems of multipliers maps to the tensor product of the line bundles they each define.

Let $(e_\lambda)_{\lambda\in\Lambda}$ be a system of multipliers.

Definition 1.4.2. A *theta function* θ for $(e_\lambda)_{\lambda\in\Lambda}$ is a holomorphic function $\theta : V \to \mathbb{C}$ such that

$$\theta(v + \lambda) = e_\lambda(v)\theta(v)$$

for all $v \in V$ and $\lambda \in \Lambda$.

A classical result in the theory of complex abelian varieties is that each line bundle on a complex torus corresponds to a so-called *Appell-Humbert datum*, which is a pair (H, α). We need some linear algebra to set this up.

Definition 1.4.3. A *Hermitian form* H on V is a function $H : V \times V \to \mathbb{C}$ that is linear on the first coordinate and anti-linear on the second coordinate, such that $H(v, w) = \overline{H(w, v)}$ for all $v, w \in V$.

Given a Hermitian form H on V, we set E to be the imaginary part of H. Note that E is a real skew-symmetric bilinear form on V such that $E(iv, iw) = E(v, w)$ for all $v, w \in V$, where i is the imaginary unit. In fact, such skew-symmetric forms correspond bijectively to Hermitian forms. For all $v, w \in V$, it holds that

$$H(v, w) = E(iv, w) + iE(v, w).$$

Let X be the set of pairs (H, α) where H is a Hermitian form on V such that E takes values in $\mathbb{Z}$ and $\alpha : \Lambda \to U(1)$ is a map such that for all $\lambda, \mu \in \Lambda$,

$$\alpha(\lambda + \mu) = \alpha(\lambda)\alpha(\mu)(-1)^{E(\lambda,\mu)}.$$

Here, $U(1)$ is the circle group. The set X has a natural group structure, defined by $(H, \alpha) \cdot (H', \alpha') = (H + H', \alpha\alpha')$. Let $(H, \alpha) \in X$ and define

$$e_\lambda(v) = \alpha(\lambda) \exp\left(\pi \left((H(\lambda, v) + \tfrac{1}{2}H(\lambda, \lambda)\right)\right)$$

for each $\lambda \in \Lambda$. It is easy to check that this is a system of multipliers, and so $(e_\lambda)_{\lambda\in\Lambda}$ defines a line bundle on A, which we denote by $L(H, \alpha)$. Appell and Humbert proved the following theorem for two-dimensional A as early as 1891, and it was generalized to arbitrary A by Lefschetz in 1921, albeit in a more archaic form.

Theorem 1.4.4 (Appell-Humbert)**.** The map $X \to \text{Pic}(A)$ given by $(H, \alpha) \mapsto L(H, \alpha)$ is a group isomorphism.

For a proof, see theorem 2.6 of [1]. Now we want to make things more explicit, which can be done by choosing an appropriate basis of V. Suppose that there is a Hermitian form H on V such that the real part S of H is positive definite and such that the imaginary part E of H satisfies $E(\lambda, \mu) \in \mathbb{Z}$ for all $\lambda, \mu \in \Lambda$. The following result is due to Frobenius and extremely useful in this situation.

Proposition 1.4.5. Let M be a free finitely generated $\mathbb{Z}$-module of rank $2g$ and $B : M \times M \to \mathbb{Z}$ a non-degenerate skew-symmetric bilinear form. Then there exist $d_1, \ldots, d_g \in \mathbb{Z}_{>0}$ with $d_i | d_{i+1}$ for all $i = 1, \ldots, g - 1$ and a basis $(a_1, \ldots, a_g, b_1, \ldots, b_g)$ of M such that the matrix of B with respect to this basis is

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

where D is the diagonal matrix with entries $(d_1, \ldots, d_g)$.

We refer to proposition 3.1 of [1] for a proof of this. Since Λ and E satisfy the conditions of the proposition, it follows immediately that the determinant of E is the square of an integer. In the special case that $\det(E) = 1$, we say that E is *unimodular* and any basis

$$(a_1, \ldots, a_g, b_1, \ldots, b_g)$$

satisfying the conditions of proposition 1.4.5 is called *symplectic*. We have the following definition.

Definition 1.4.6. A *polarization* of A is a Hermitian form H on V such that the real part S of H is positive definite and such that the imaginary part E of H satisfies $E(\lambda, \mu) \in \mathbb{Z}$ for all $\lambda, \mu \in \Lambda$. If E is unimodular, then we call the polarization *principal*.

Let H be a principal polarization of A. Then proposition 1.4.5 gives us a symplectic basis $(\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g)$ of Λ such that the matrix of $E = \mathrm{Im}(H)$ with respect to this basis is

$$J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix},$$

where $\mathbf{1}$ is the $g \times g$ identity matrix. By lemma 3.2 of [1], it holds that $(\lambda_1, \ldots, \lambda_g)$ is a basis of V over $\mathbb{C}$. Thus, for $j \in \{1, \ldots, g\}$, we can write

$$\mu_j = \sum_{i=1}^{g} a_{ij} \lambda_i,$$

with $a_{ij} \in \mathbb{C}$. We obtain a $g \times g$-matrix $\tau = (a_{ij})$, called the *period matrix* of H. We have an identification $\Lambda = \mathbb{Z}^g \times \tau\mathbb{Z}^g$. The imaginary part of τ is positive definite, and τ itself is symmetric, see proposition 3.3 of [1] for details.

Definition 1.4.7. The *Siegel upper-half space* of degree g, denoted by $\mathbb{H}_g$, is the space of complex symmetric $g \times g$-matrices τ with positive definite imaginary part.

The Lefschetz theorem states that any complex torus A that admits a polarization can be embedded in projective space (cf. section 3.7 in [1]). By the results mentioned in section 1.2, it holds that such a complex torus is the associated complex analytic space of a complex abelian variety. In particular, any principally polarized complex torus is a complex abelian variety, which we call a *principally polarized abelian variety* over $\mathbb{C}$ or simply *ppav* over $\mathbb{C}$.

In [1] it is explained that $H^0(A, L(H, \alpha))$ can be identified with the space of theta functions for the system of multipliers corresponding to $L(H, \alpha)$, and then it is shown that $\dim H^0(A, L(H, \alpha)) = 1$ if H is a principal polarization. Moreover, for any two α, α', it holds that $L(H, \alpha')$

is the pullback of $L(H, \alpha)$ along a translation $t_a : A \to A$. Thus there is a divisor Θ of A defined up to translation by the vanishing of an element of $H^0(A, L(H, \alpha))$. The corresponding theta function is one of the main objects of interest in this paper.

Definition 1.4.8. The *Riemann theta function* of dimension g is the map $\theta : \mathbb{H}_g \times \mathbb{C}^g \to \mathbb{C}$ given by

$$(\tau, z) \mapsto \sum_{m \in \mathbb{Z}^g} e^{\pi i ({}^t m \tau m + 2 {}^t m z)}.$$

A fixed $\tau \in \mathbb{H}_g$ defines a ppav, for which the Riemann theta function is actually a theta function in the sense of definition 1.4.2, for a system of multipliers corresponding to the line bundle $L(H, \alpha)$ with $\alpha(p + \tau q) = (-1)^{{}^t pq}$. In fact, θ defines a nonzero section of that line bundle, and so its zero locus defines a specific instance of the divisor Θ in A. It is not hard to check that θ is symmetric in z, so Θ has the pleasant property of being symmetric around the origin and will therefore be called "the" theta divisor of the ppav A. Henceforth, we will *define* a ppav to be a pair (A, Θ), with Θ the theta divisor of A, since such a pair carries the same information as a pair (A, H) with H a principal polarization.

Let $A = \mathbb{C}^g / (\tau \mathbb{Z}^g + \mathbb{Z}^g)$ be a ppav. Then its points of order 2 are obviously given by

$$\frac{\tau \epsilon + \delta}{2},$$

where $\epsilon, \delta \in \mathbb{Z}^g$ are column vectors containing ones and zeroes. Hence A has 2^{2g} points of order 2. There is a specific theta function for each point of order 2. Let $[\epsilon, \delta]$ be a pair as above.

Definition 1.4.9. The *theta function with characteristic* $[\epsilon, \delta]$ is the map $\mathbb{H}_g \times \mathbb{C}^g \to \mathbb{C}$ defined by

$$\theta \left[\begin{smallmatrix} \epsilon \\ \delta \end{smallmatrix} \right] (\tau, z) := \sum_{m \in \mathbb{Z}^g} \exp \pi i \left({}^t \Big(m + \frac{\epsilon}{2}\Big) \tau \Big(m + \frac{\epsilon}{2}\Big) + 2 {}^t \Big(m + \frac{\epsilon}{2}\Big) \Big(z + \frac{\delta}{2}\Big) \right)$$

The theta function $\theta \left[\begin{smallmatrix} \epsilon \\ \delta \end{smallmatrix} \right]$ is called *even* if the inner product $\epsilon \cdot \delta$ is even, and *odd* otherwise. It holds that

$$\theta \left(\tau, z + \frac{\tau \epsilon + \delta}{2} \right) = \exp \left(\pi i \left(-\frac{{}^t \epsilon}{2} \tau \frac{\epsilon}{2} - {}^t \epsilon \left(z + \frac{\delta}{2} \right) \right) \right) \theta \left[\begin{smallmatrix} \epsilon \\ \delta \end{smallmatrix} \right] (\tau, z),$$

which shows that $\theta \left[\begin{smallmatrix} \epsilon \\ \delta \end{smallmatrix} \right]$ is basically θ shifted by a point of order 2. An even theta function is actually even as a function of z for fixed τ, and the same goes for odd theta functions.

1.5 The theta function η

Let $\theta(\tau, z)$ be the Riemann theta function of dimension g, $d\theta$ the column vector of its first order derivatives θ_i with respect to $z_1, \ldots, z_g$ and H its Hessian, which consists of the second order derivatives θ_{ij} with respect to $z_1, \ldots, z_g$.

Definition 1.5.1. Define $\eta(\tau, z)$ as the function

$$\eta(\tau, z) = \det \begin{pmatrix} H(\tau, z) & d\theta(\tau, z) \\ {}^t d\theta(\tau, z) & 0 \end{pmatrix}.$$

It is shown in theorem 1.3 of [7] that η is a global section of the line bundle $\mathcal{O}_\Theta(\Theta)^{\otimes g+1}$ for fixed τ, and thus a theta function of order $g+1$. The definition given in [7] is slightly different from this one, namely

$$\eta(\tau, z) = {}^t d\theta H^c d\theta,$$

where H^c is the cofactor matrix of H, but it can easily be shown to be equivalent by developing the determinant to the last line and column.

$$\begin{aligned} \det \begin{pmatrix} H & d\theta \\ {}^t d\theta & 0 \end{pmatrix} &= \sum_{i=1}^{g} \theta_i \sum_{j=1}^{g} \theta_j H_{ij}^c \\ &= {}^t d\theta H^c d\theta. \end{aligned}$$

Note that η is identically zero at singular points of Θ for fixed τ, since $d\theta$ is zero at singular points and therefore the determinant becomes zero.

2 Vector bundles

In this chapter, we go back to the general framework of schemes. We develop the theory of vector bundles necessary to define the Gauss map for a closed immersion $Y \to X$ of smooth varieties over some field k, and we relate the theta function defined in section 1.5 to the Gauss map associated to the embedding $\Theta \subset A$ of the theta divisor in its principally polarized abelian variety.

2.1 Commonly occurring bundles

In algebraic geometry, it can be quite useful to associate a scheme to a sheaf in a functorial manner. This section develops some of the theory behind this. For a more extensive exposition, see section II.7 in [6] and chapter 11 in [4].

Definition 2.1.1. A ring homomorphism $\phi : A \to B$ of polynomial rings $A = R[x_1, \ldots, x_m]$ and $B = R[y_1, \ldots, y_n]$ over a ring R is called *linear* if $\phi(r) = r$ for all $r \in R$ and

$$\phi(x_j) = \sum_{i=1}^{n} a_{ij} y_i$$

for all $j = 1, \ldots, m$, where $a_{ij} \in R$. The *matrix of* ϕ is the matrix (a_{ij}).

There is a canonical injective group homomorphism $\mathrm{Mat}(n \times m, R) \to \mathrm{Hom}(A, B)$ given by

$$(a_{ij}) \mapsto \left(x_j \mapsto \sum_{i=1}^{n} a_{ij} y_i \right),$$

the image of which is the group of linear ring homomorphisms $A \to B$. If $n = m$, then we may identify A and B and we get a ring homomorphism $\mathrm{Mat}(n, R) \to \mathrm{End}(A)$. Now let X be a scheme.

Definition 2.1.2. A *geometric vector bundle* of rank n over X is a scheme $f : E \to X$ over X together with an open covering $\mathcal{U} = \{U_i : i \in I\}$ of X and isomorphisms $\phi_i : f^{-1}(U_i) \to \mathbb{A}^n_{U_i}$ such that for all i, j and $V = \operatorname{Spec} A$ an affine open contained in $U_i \cap U_j$, the automorphism $\phi_j \circ \phi_i^{-1} : \mathbb{A}^n_V \to \mathbb{A}^n_V$ corresponds to a linear automorphism of $A[x_1, \ldots, x_n]$.

For a vector bundle $f : E \to X$ and an open $U \subset X$, the restriction $f^{-1}(U) \to U$ is sometimes denoted by $f|_U : E|_U \to U$, or simply $E|_U$ if there is no possibility of confusing it with the scheme $E|_U$ without the morphism $f|_U$.

Now let $f : E \to X$ and $f' : F \to X$ be geometric vector bundles over X of ranks n and m, respectively. Let $x \in X$ and $V \subset X$ affine open such that $x \in V$ and $f^{-1}(V) \cong \mathbb{A}^n_V$. Then it holds that

$$E_x = E \times_X \operatorname{Spec} \kappa(x) \cong \mathbb{A}^n_V \times_V \operatorname{Spec} \kappa(x) \cong \mathbb{A}^n_{\kappa(x)},$$

which shows that the fibers of f are the spectra of polynomial rings over the residue fields $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$, which justifies the terminology of vector bundles.

Let $f : X \to Y$ be a morphism of schemes, $\pi : E \to X$ a vector bundle over X and $\pi' : F \to X$ a vector bundle over Y.

Definition 2.1.3. A *morphism of vector bundles* $g : E \to F$ is a morphism of schemes such that $\pi' g = f\pi$ and for some affine open cover $\mathcal{U}$ of X and each $U \in \mathcal{U}$, the map $g|_U : E|_U \to F|_U$ corresponds to a linear ring homomorphism.

Given a locally free sheaf $\mathcal{E}$ of rank n on X, we can construct an associated vector bundle $\mathbb{V}(\mathcal{E})$. Let $\operatorname{Sym}(\mathcal{E})$ be the symmetric algebra on $\mathcal{E}$. It is the quotient of the tensor algebra

$$T(\mathcal{E}) = \bigoplus_{n \geq 0} \mathcal{E}^{\otimes n},$$

where the tensor product is taken over $\mathcal{O}_X$, by the ideal $\mathcal{J}$ generated by elements of the form $a \otimes b - b \otimes a$. Let $E = \underline{\operatorname{Spec}}(\operatorname{Sym}(\mathcal{E}))$ be as in [17], tag 01LL. It comes with a morphism of schemes $f : E \to X$. If $U \subset X$ is an affine open, then $f^{-1}(U) = \operatorname{Spec} \operatorname{Sym}(\mathcal{E})(U)$ is affine.

Let $U \subset X$ be an affine open such that $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus n}$ and let $x_1, \ldots, x_n \in \mathcal{E}(U)$ be an $\mathcal{O}_U$-basis of $\mathcal{E}|_U$. It holds that $\operatorname{Sym}(\mathcal{E}|_U) = \operatorname{Sym}(\mathcal{E})|_U$ by exercise II.5.16(e) in [6]. The $\mathcal{O}_U$-module $\mathcal{E}|_U$ is free of rank n, so also quasi-coherent. Therefore it corresponds to the free $\mathcal{O}_U(U)$-module

$$\mathcal{O}_U(U)x_1 \oplus \cdots \oplus \mathcal{O}_U(U)x_n$$

of rank n, since U is affine. By proposition II.5.2 in [6], it holds that $\operatorname{Sym}(\mathcal{E}|_U)$ is the $\mathcal{O}_U$-algebra corresponding to the $\mathcal{O}_U(U)$-algebra

$$\operatorname{Sym}(\mathcal{E}(U)) \cong \mathcal{O}_U(U)[x_1, \ldots, x_n].$$

In particular, its global sections are $\operatorname{Sym}(\mathcal{E}|_U)(U) = \operatorname{Sym}(\mathcal{E}(U))$. Thus there is a natural isomorphism $g : \mathcal{O}_U(U)[x_1, \ldots, x_n] \to \operatorname{Sym}(\mathcal{E}(U))$, which corresponds to an isomorphism $\phi : f^{-1}(U) \to \mathbb{A}^n_U$. Of course, this depends on our choice of $x_1, \ldots, x_n$. If $y_1, \ldots, y_n$ is another $\mathcal{O}_U$-basis of $\mathcal{E}|_U$, then

$$y_i = \sum_{j=1}^{n} a_{ij} x_j$$

with $a_j \in \mathcal{O}_U(U)$ for all $i = 1, \ldots, n$, so the isomorphism

$$g' : \mathcal{O}_U(U)[y_1, \ldots, y_n] \longrightarrow \mathrm{Sym}(\mathcal{E}(U))$$

is such that $g' = gh$, where $h : \mathcal{O}_U(U)[y_1, \ldots, y_n] \to \mathcal{O}_U(U)[x_1, \ldots, x_n]$ is a linear automorphism given by (a_{ij}). Hence ϕ is determined up to linear automorphism.

Let $U_i, U_j \subset X$ be two affine opens on which $\mathcal{E}$ is free and let $\phi_i : f^{-1}(U_i) \to \mathbb{A}^n_{U_i}$ and $\phi_j : f^{-1}(U_j) \to \mathbb{A}^n_{U_j}$ be isomorphisms as above, with $(x_1, \ldots, x_n)$ a basis for $\mathcal{E}|_{U_i}$ and $(y_1, \ldots, y_n)$ a basis for $\mathcal{E}|_{U_j}$. Both bases restrict to a basis of $\mathcal{E}|_{U_i \cap U_j}$. Hence the isomorphism $\phi_j \circ \phi_i^{-1} : \mathbb{A}^n_{U_i \cap U_j} \to \mathbb{A}^n_{U_i \cap U_j}$ corresponds to the linear automorphism

$$\begin{aligned} \mathcal{O}_{U_j}(U_i \cap U_j)[y_1, \ldots, y_n] &\longrightarrow \mathcal{O}_{U_i}(U_i \cap U_j)[x_1, \ldots, x_n] \\ y_k &\longmapsto \sum_{l=1}^{n} a_{kl} x_l \end{aligned}$$

with $a_{kl} \in \mathcal{O}_{U_i}(U_i \cap U_j)$ for all $k = 1, \ldots, n$.

Let $\{U_i : i \in I\}$ be an open cover of X such that $\mathcal{E}|_{U_i}$ is free for all $i \in I$. For $i \in I$, let $\{V_{ij} : j \in J_i\}$ be an affine open cover of U_i. Then $\{V_{ij} : i \in I, j \in J_i\}$ is an affine open cover of X such that $\mathcal{E}|_{V_{ij}}$ is free for all $i \in I$ and $j \in J_i$.

It follows that $f : E \to X$ is a geometric vector bundle over X, which is the *geometric vector bundle associated to* $\mathcal{E}$. It is usually denoted by $\mathbb{V}(\mathcal{E})$.

We continue with a little sidebar. Let $f : X \to Y$ be any morphism of schemes and $U \subset Y$ open.

Definition 2.1.4. A *section* of f over U is a morphism $s : U \to X$ such that $f \circ s = \mathrm{id}_U$.

By definition, it holds that a section s of f over U is also a morphism $s : U \to f^{-1}(U)$. Given another open $V \subset U$, we can restrict a section s of f over U to a section over V just by restricting the map $s : U \to X$ to V. If U and V are opens with sections $s : U \to X$ and $s' : V \to X$ of f such that $s|_{U \cap V} = s'|_{U \cap V}$, we can glue them to a section $t : U \cup V \to X$ by the gluing lemma. Hence we get a sheaf $\mathcal{S}(X/Y)$ given by

$$U \mapsto \{\text{sections of } f \text{ over } U\},$$

called the *sheaf of sections of* f. It is a sheaf on Y.

Let $f : E \to X$ be a vector bundle of rank n over a scheme X, with sheaf of sections $\mathcal{S} = \mathcal{S}(E/X)$. For $x \in X$, it holds that $\mathrm{Spec}(\mathcal{S}_y \otimes_{\mathcal{O}_{X,x}} \kappa(x))$ is canonically isomorphic to the fiber E_x. Let $U \subset X$ be open and let $s, t \in \mathcal{S}(U)$ and $\tilde{a} \in \mathcal{O}_X(U)$. Let $V = \mathrm{Spec}\, A \subset U$ be an affine open with an isomorphism $\phi : f^{-1}(V) \to \mathbb{A}^n_V$, coming from the vector bundle structure of $f : E \to X$. Note that $A = \mathcal{O}_U(V)$. Then the composition

$$V \xrightarrow{s|_V} f^{-1}(V) \xrightarrow{\phi} \mathbb{A}^n_V$$

is a morphism into the affine scheme Spec $A[x_1, \ldots, x_n]$, so it corresponds to an A-algebra homomorphism $\sigma : A[x_1, \ldots, x_n] \to A$. Since the set of A-algebra homomorphisms $\sigma : A[x_1, \ldots, x_n] \to A$ has a natural $\mathcal{O}_U(V)$-module structure for each affine open $V \subset U$ such that $E|_V$ is trivial, it holds that $\mathcal{S}(U)$ has a natural $\mathcal{O}_X(U)$-module structure. This turns $\mathcal{S}$ into an $\mathcal{O}_X$-module.

Let $U \subset X$ be an affine open with an isomorphism $\phi : f^{-1}(U) \to \mathbb{A}^n_U$. We show that $\mathcal{S}|_U$ is free. A section of f over an open $V \subset U$ is a map $g : V \to E|_V$, so we also have $\phi|_V \circ g : V \to \mathbb{A}^n_V$ which corresponds to an $\mathcal{O}_U(V)$-algebra homomorphism $\sigma : \mathcal{O}_U(V)[x_1, \ldots, x_n] \to \mathcal{O}_U(V)$. Such a homomorphism is uniquely determined by the images $\sigma(x_1), \ldots, \sigma(x_n)$. Let $M(V)$ be the $\mathcal{O}_U(V)$-module of $\mathcal{O}_U(V)$-algebra homomorphisms $\sigma : \mathcal{O}_U(V)[x_1, \ldots, x_n] \to \mathcal{O}_U(V)$. There is a natural isomorphism of $\mathcal{O}_U(V)$-modules $M(V) \to \mathcal{O}_U(V)^{\oplus n}$ given by $\sigma \mapsto (\sigma(x_1), \ldots, \sigma(x_n)$. It follows that $\mathcal{S}|_U \cong \mathcal{O}_U^{\oplus n}$, so $\mathcal{S}|_U$ is free. Hence $\mathcal{S}$ is a locally free sheaf.

Now let $f : E \to X$ with $E = \mathbb{V}(\mathcal{E})$ be the geometric vector bundle over X associated to a locally free sheaf $\mathcal{E}$ of rank n, with sheaf of sections $\mathcal{S}$ as above. Given a section $s : U \to E|_U$, there is morphism $\tilde{\gamma} : f_*\mathcal{O}_{E|_U} \to \mathcal{O}_U$ of $\mathcal{O}_U$-algebras coming from $s^\# : \mathcal{O}_{E|_U} \to \mathcal{O}_U$. By lemma 24.4.6 from [17], (tag 01LQ), it holds that $f_*\mathcal{O}_{E|_U} \cong \mathrm{Sym}(\mathcal{E}|_U)$. There is a corresponding morphism $\gamma : \mathcal{E}|_U \to \mathcal{O}_U$ of $\mathcal{O}_U$-modules.

Conversely, let $U \subset X$ open and let $\gamma : \mathcal{E}|_U \to \mathcal{O}_U$ be a morphism of $\mathcal{O}_U$-modules. There is a corresponding morphism $\tilde{\gamma} : \mathrm{Sym}(\mathcal{E}|_U) \to \mathcal{O}_U$ of $\mathcal{O}_U$-algebras. If $V \subset U$ is an affine open, we get $\tilde{\gamma}(V) : \mathrm{Sym}(\mathcal{E}|_U)(V) \to \mathcal{O}_U(V)$ which corresponds to a section $s_V : V \to E|_V$. Gluing these yields a section $s : U \to E|_U$.

It follows that $\mathcal{S} \cong \mathscr{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) = \mathcal{E}^\vee$. Note that $\mathcal{E}^{\vee\vee}$ is canonically isomorphic to $\mathcal{E}$. For a morphism of vector bundles $g : E \to F$, let $\mathcal{S}(g) : \mathcal{S}(E/X) \to \mathcal{S}(F/X)$ be given by $s \mapsto g \circ s$.

Let $\gamma : \mathcal{E} \to \mathcal{F}$ be a morphism of locally free sheaves on X of ranks m and n, respectively. Let $E = \mathbb{V}(\mathcal{E}^\vee)$, $F = \mathbb{V}(\mathcal{F}^\vee)$ and let $f : E \to X$ be the usual projection map. Let $U \subset X$ be an affine open such that $\mathcal{E}|_U$ and $\mathcal{F}|_U$ are free. Suppose that

$$\mathcal{E}|_U = \bigoplus_{i=1}^{m} x_i \cdot \mathcal{O}_U \qquad \text{and} \qquad \mathcal{E}|_U = \bigoplus_{j=1}^{n} y_j \cdot \mathcal{O}_U.$$

Then $\gamma|_U : \mathcal{E}|_U \to \mathcal{F}|_U$ is given by $\gamma(x_1), \ldots, \gamma(x_m)$. Hence we have a ring homomorphism

$$\tilde{\gamma}|_U : \mathcal{O}_U(U)[x_1, \ldots, x_m] \longrightarrow \mathcal{O}_U(U)[y_1, \ldots, y_n],$$

which defines a morphism $g|_U : E|_U \to F|_U$. These morphisms glue to a morphism $\mathcal{V}(g) : E \to F$.

Let Vect(X) be the category of geometric vector bundles of fixed finite rank over X (every E in Vect(X) is of rank n over X for some $n \in \mathbb{Z}_{/geq0}$), and let LocF(X) be that of locally free sheaves of fixed finite rank on X. Then we have a functor $\mathcal{S} : \mathrm{Vect}(X) \to \mathrm{LocF}(X)$ given by $E \mapsto \mathcal{S}(E/X)$ and a functor $\mathcal{V} : \mathrm{LocF}(X) \to \mathrm{Vect}(X)$ given by $\mathcal{E} \mapsto \mathbb{V}(\mathcal{E}^\vee)$, where we dualize to make sure that the functors $\mathcal{S}$ and $\mathcal{V}$ are both covariant. This is the "right" choice, since the geometric tangent bundle of a scheme X over k is defined as $T(X) = \mathbb{V}(\Omega_{X/k}) = \mathcal{V}(\mathcal{T}_{X/k})$, where $\mathcal{T}_{X/k}$, the tangent sheaf, is the dual of $\Omega_{X/k}$. In fact, the functor $\mathcal{V}$ is an equivalence of categories.

Proposition 2.1.5. The functor $\mathcal{V} : \mathrm{LocF}(X) \to \mathrm{Vect}(X)$ is an equivalence of categories with quasi-inverse $\mathcal{S}$.

For the rest of the proof, see [4], section 11.4. All operations that exist on locally free sheaves therefore also exist on geometric vector bundles, so that we may form direct sums, tensor products, duals, exterior powers and so forth, simply by applying these operations to sheaves of sections and turning them into vector bundles using $\mathcal{V}$.

These vector bundles have affine spaces as fibers. It seems natural that we can also consider bundles with projective spaces as fibers, which gives rise to the concept of projective bundles and projectivization.

Definition 2.1.6. A *projective bundle* of rank n over X is a scheme $f : E \to X$ over X together with an open covering $\mathcal{U} = \{U_i : i \in I\}$ of X and isomorphisms $\phi_i : f^{-1}(U_i) \to \mathbb{P}^n_{U_i}$ such that for all i, j and $V = \operatorname{Spec} A$ an affine open contained in $U_i \cap U_j$, the automorphism $\phi_j \circ \phi_i^{-1} : \mathbb{P}^n_V \to \mathbb{P}^n_V$ corresponds to a linear automorphism of $A[x_0, \dots, x_n]$.

Note that a linear automorphism of $A[x_0, \dots, x_n]$ is an isomorphism of graded rings and therefore defines a map $\mathbb{P}^n_A \to \mathbb{P}^n_A$ by lemma 3.40 from [10]. Now let $f : E \to X$ be a geometric vector bundle of rank $n+1$ over X with an open cover $\{U_i : i \in I\}$ and transition maps $\psi_{ij} : \mathbb{A}^{n+1}_{U_i \cap U_j} \to \mathbb{A}^{n+1}_{U_i \cap U_j}$.

Definition 2.1.7. The *projectivization* $\mathbb{P}(E)$ of E is the projective bundle of rank n over X obtained by gluing $\mathbb{P}^n_{U_i}$ for all $i \in I$ along the transition maps ψ'_{ij}, induced by the same linear automorphism as ψ_{ij} for all $i, j \in I$.

For a locally free sheaf $\mathcal{E}$, we also have the projective bundle $\mathbb{P}(\mathcal{E})$ associated to $\mathcal{E}$, which is constructed similarly to $\mathbb{V}(\mathcal{E})$ as $\underline{\operatorname{Proj}} \operatorname{Sym} \mathcal{E}$. It holds that $\mathbb{P}(\mathcal{E} \otimes \mathcal{L}) \cong \mathbb{P}(\mathcal{E})$ for all invertible sheaves $\mathcal{L}$. A projective bundle of rank n is also called a $\mathbb{P}^n$-bundle or a projective space bundle of rank n.

It is important to note that every line bundle $\pi : E \to X$ over X can be extended to a $\mathbb{P}^1$-bundle. Given a transition map $\psi : \mathbb{A}^1_V \to \mathbb{A}^1_V$ on $V = \operatorname{Spec} A$ of the line bundle, the corresponding linear automorphism of $A[x]$ is given by $x \mapsto ax$ for some $a \in A^*$. Now we define a linear automorphism of $A[x_0, x_1]$ by $x_0 \mapsto ax_0$ and $x_1 \mapsto x_1$ (we could also choose $x_1 \mapsto ax_1$). Then we get a $\mathbb{P}^1$-bundle $\pi' : F \to X$ over X. For an affine open $U \subset X$ with $\phi : \pi'^{-1}(U) \cong \mathbb{P}^1_U$, we have that $\phi^{-1}(D_+(x_1)) = \pi^{-1}(U)$. Thus we have embedded the line bundle in the $\mathbb{P}^1$-bundle, such that locally on X it is the standard affine open $D_+(x_1)$ of $\mathbb{P}^1_U$ for open $U \subset X$. Naturally, we could have also chosen $D_+(x_0)$. Similarly, we can extend any $\mathbb{A}^n$-bundle to a $\mathbb{P}^n$-bundle.

There is also the notion of a $\mathbb{G}_m$*-bundle* over X, see section 1.1 for the definition of the affine scheme $\mathbb{G}_m$. It is defined analogously to a line bundle over X, and the transition maps correspond to maps $A[x, x^{-1}] \to A[x, x^{-1}]$ given by $x \mapsto ax$ for some $a \in A^*$, where A is the ring of some affine open V contained in the intersection of two trivializing opens $U_i, U_j \subset X$ of the bundle.

There is a one-to-one correspondence between line bundles and $\mathbb{G}_m$-bundles over X, since a linear automorphism of $A[x]$ is given by $x \mapsto ax$ for some $a \in A^*$. Taking the $\mathbb{G}_m$-bundle corresponding to a line bundle can be thought of as "throwing away the origin of the line bundle". Indeed, if in the above A is a field, then the prime ideal in $A[x]$ generated by (x) is the "origin" of $\operatorname{Spec} A[x]$, which is not a prime ideal of $A[x, x^{-1}]$. Hence we can associate to each invertible sheaf on X a $\mathbb{G}_m$-bundle in a functorial manner, and we get another equivalence of categories, this time between the category of invertible sheaves on X and that of $\mathbb{G}_m$-bundles over X. In particular, the notions of "dual" and "tensor product" carry to $\mathbb{G}_m$-bundles.

We now turn our attention to closed immmersions of varieties. The following results are proposition II.8.12 and II.8.17 from [6]. Let X be a smooth variety over k.

Proposition 2.1.8. Let Y be a closed subscheme of X with ideal sheaf $\mathcal{I}$. Then there is an exact sequence

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{X/k} \otimes \mathcal{O}_Y \longrightarrow \Omega_{Y/k} \longrightarrow 0 \tag{2.1}$$

of sheaves on Y.

Theorem 2.1.9. Let X be a smooth variety over k. Let $Y \subset X$ be an irreducible closed subscheme with sheaf of ideals $\mathcal{I}$. Then Y is smooth if and only if $\Omega_{Y/k}$ is locally free and the sequence 2.1 is exact on the left as well:

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{X/k} \otimes \mathcal{O}_Y \longrightarrow \Omega_{Y/k} \longrightarrow 0. \tag{2.2}$$

If Y is smooth, $\mathcal{I}$ is locally generated by $r = \operatorname{codim}(Y, X)$ elements and $\mathcal{I}/\mathcal{I}^2$ is locally free of rank r.

Let X be a smooth algebraic variety of dimension $n+1$ over a field k and $i : Y \to X$ a smooth closed subvariety. We have an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$$

of $\mathcal{O}_X$-modules. The sheaf $\mathcal{I}/\mathcal{I}^2$ is a locally free sheaf on Y of rank $\operatorname{codim}(Y, X)$, which is called the *conormal sheaf* of Y in X (see [6], p. 182). Its dual $\mathscr{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$ is called the *normal sheaf* of Y in X and denoted $\mathcal{N}_{Y/X}$. Dualizing (2.2) gives the exact sequence

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{O}_Y \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0, \tag{2.3}$$

where $\mathcal{T}_X = \Omega_{X/k}^{\vee}$ denotes the tangent sheaf of X, and $\mathcal{T}_Y$ that of Y. Let $\mathcal{E} = \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{O}_Y$. From now on, assume that $\operatorname{codim}(Y, X) = 1$. Then $\mathcal{N}_{Y/X}$ is a line bundle. We state proposition II.7.12 from [6] for future reference.

Proposition 2.1.10. Let X be a noetherian scheme and $\mathcal{E}$ a locally free coherent sheaf on X. Let $f : Y \to X$ be a morphism of schemes. Then a morphism $g : Y \to \mathbb{P}(\mathcal{E})$ over X is defined uniquely by an invertible sheaf $\mathcal{L}$ on Y and a surjective map $f^*\mathcal{E} \to \mathcal{L}$. For $g : Y \to \mathbb{P}(\mathcal{E})$ corresponding to $f^*\mathcal{E} \to \mathcal{L}$, it holds that $\mathcal{L} \cong g^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

By this proposition, the surjective map $\mathcal{E} \to \mathcal{N}_{Y/X}$ defines a morphism $g : Y \to \mathbb{P}(\mathcal{E})$ of schemes over Y, as $\mathcal{E}$ is here the pullback of $\mathcal{T}_X$ along i. Denote the natural projection $\mathbb{P}(\mathcal{E}) \to Y$ by π.

Let $\mathcal{U} = \{U_i \mid i \in I\}$ be an affine open cover of Y such that $\mathcal{E}|_{U_i}$ is free for all $i \in I$, with transition maps ψ_{ij}. For $i \in I$ with $U_i = \operatorname{Spec} A$, theorem II.8.13 in [6] gives us the exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n_A/U_i} \longrightarrow \mathcal{O}_{\mathbb{P}^n_A}(-1)^{\oplus n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n_A} \longrightarrow 0, \tag{2.4}$$

and note that $\mathcal{O}_{\mathbb{P}^n_A}(-1)^{\oplus n+1} \cong (\pi^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1))|_{\pi^{-1}(U_i)}$, $\Omega_{\mathbb{P}^n_A/U_i} \cong \Omega_{\mathbb{P}(\mathcal{E})/Y}|_{\pi^{-1}(U_i)}$ and $\mathcal{O}_{\mathbb{P}^n_A} \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}|_{\pi^{-1}(U_i)}$. Moreover, $\Omega_{\mathbb{P}(\mathcal{E})/Y}$ is locally free of rank n, since $\Omega_{\mathbb{P}^n_A/U_i}$ is locally free of rank n. The exact sequence agrees on overlaps $U_i \cap U_j$ using the transition maps, so it follows from [17] (tag 00AK) that we can glue the sequences (2.4) to an exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}(\mathcal{E})/Y} \longrightarrow \pi^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})} \longrightarrow 0,$$

which we then tensor with $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ to obtain

$$0 \longrightarrow \Omega_{\mathbb{P}(\mathcal{E})/Y} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \longrightarrow \pi^*\mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \longrightarrow 0.$$

By proposition II.7.11 in [6], there is a natural surjective morphism $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, which is precisely the penultimate arrow in the latter exact sequence. The surjective map $g^*\pi^*\mathcal{E} \to g^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ obtained by applying

g^* to $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is the surjective map $\mathcal{E} \to \mathcal{N}_{Y/X}$ defining g, see proposition 2.1.10. Hence applying g^* to the last exact sequence yields

$$g^*\Omega_{\mathbb{P}(\mathcal{E})/Y} \otimes \mathcal{N}_{Y/X} \longrightarrow \mathcal{E} \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0,$$

an exact sequence of locally free $\mathcal{O}_Y$-modules. By proposition 8.10 in [4], the sequence must be exact on the left as well and using the exact sequence (2.3) we see that $g^*\Omega_{\mathbb{P}(\mathcal{E})/Y} \otimes \mathcal{N}_{Y/X} \cong \mathcal{T}_Y$.

The locally free sheaves of this section correspond to geometric vector bundles, so it's a good idea to give these names. Let $\mathcal{V}$ be the functor from proposition 2.1.5. Then we let $T(X) = \mathcal{V}(\mathcal{T}_X) = \mathbb{V}(\Omega_{X/k})$, $T(X)|_Y = \mathcal{V}(\mathcal{T}_X \otimes \mathcal{O}_Y) = \mathbb{V}(\Omega_{X/k} \otimes \mathcal{O}_Y)$ and $N(Y) = \mathcal{V}(\mathcal{N}_{Y/X}) = \mathbb{V}(\mathcal{N}_{Y/X}^\vee)$ if there is no possibility of ambiguity.

2.2 Determinants

Let X be a scheme. Let $\mathcal{E}$ and $\mathcal{F}$ be locally free $\mathcal{O}_X$-modules of rank n and $\phi : \mathcal{E} \to \mathcal{F}$ a morphism of $\mathcal{O}_X$-modules.

Definition 2.2.1. The *m-th exterior power* $\bigwedge^m \mathcal{E}$ of $\mathcal{E}$ is the sheafification of the presheaf $U \mapsto \bigwedge^m_{\mathcal{O}_X(U)} \mathcal{F}(U)$.

It is easily checked that $\bigwedge^m \mathcal{E}$ is locally free of rank $\binom{n}{m}$. Hence the n-th exterior powers $\bigwedge^n \mathcal{E}$ and $\bigwedge^n \mathcal{F}$ are locally free sheaves of rank 1 and there is an induced map $\det \phi : \bigwedge^n \mathcal{E} \to \bigwedge^n \mathcal{F}$. For the map $\mathcal{V}(\phi) : \mathcal{V}(\mathcal{E}) \to \mathcal{V}(\mathcal{F})$ on vector bundles, we define $\det \mathcal{V}(\phi) = \mathcal{V}(\det \phi)$. By the equivalence of categories from proposition 2.1.5, every morphism $f : E \to F$ of vector bundles over X with $\mathrm{rk}(E) = \mathrm{rk}(F)$ has a well-defined determinant $\det f$. There is a wealth of results on determinants, see for example section 6.4 of [10].

2.3 Ramification and degeneration

We relate the concept of ramification for a morphism of schemes to degeneracy loci, as the latter are much better suited for our purposes. We also exhibit the Giambelli-Porteous formula, which will be fundamental in showing that the theta function η from definition 1.5.1 can be identified with the determinant of the tangent map of the Gauss map.

Definition 2.3.1. The *ramification locus* of a morphism of schemes $X \to Y$ is the support of the sheaf of differentials $\Omega_{X/Y}$.

Let E and F be vector bundles of rank r and s on a scheme X, respectively. Let $f : E \to F$ be a morphism of vector bundles on X. For $x \in X$, the map $f_x : E_x \to F_x$ is a linear map between finite dimensional vector spaces and thus has finite rank, the *rank of f at x*. This allows us to consider the locus $R_q(f)$ of points x of X such that f has rank at most q at x for some $q \in \mathbb{Z}_{\geq 0}$. If $q \geq \min(r, s)$, then $R_q(f)$ is obviously all of X.

We show that $R_q(f)$ can be given a natural structure of a closed subscheme of X for all $q \in \mathbb{Z}$. Let $q \in \mathbb{Z}$ and consider the map

$$f_{q+1} : \bigwedge^{q+1} E \to \bigwedge^{q+1} F$$

induced by f. For a point $x \in X$, it holds that f_x has rank at most q if and only if $f_{q+1,x}$ is the zero map; the wedge product $f_x(v_1) \wedge \cdots \wedge f_x(v_{q+1})$ is zero if and only if $f_x(v_1), \ldots, f_x(v_{q+1})$ are linearly dependent. Thus $R_q(f)$ is precisely the locus of points $x \in X$ such that $f_{q+1,x}$ is the zero map. $R_q(f)$ is empty if $q < 0$.

Now we show that $R_q(f)$ has a natural closed subscheme structure. Let $\operatorname{Spec} A = U \subset X$ be an affine open such that $E|_U$ and $F|_U$ are trivial and choose bases. Then $\bigwedge^{q+1} E|_U$ and $\bigwedge^{q+1} F|_U$ are trivial of rank r_{q+1} and s_{q+1} respectively, and $f_{q+1}|_U$ corresponds to an $r_{q+1} \times s_{q+1}$-matrix M with entries in A. Let $I \subset A$ be the ideal generated by the entries of M. Then $\overline{M}$ is the zero matrix in A/I, so f_q is zero everywhere on the closed subscheme $\operatorname{Spec} A/I \subset U$, and nowhere else on U. Let $\operatorname{Spec} B = V \subset X$ be another affine open such that $E|_V$ and $F|_V$ are trivial, and define the $r_{q+1} \times s_{q+1}$-matrix N with entries in B similarly to M. Denote by J the ideal generated by the entries of M. The following easy lemma will help us along.

Lemma 2.3.2. Let M be an $l \times m$-matrix and M' an $m \times n$-matrix with entries in a commutative ring A. Let I, I' and J be the ideals generated by the entries of M, M' and MM', respectively. Then it holds that $J \subset I \cdot I'$.

Proof. Note that $(MM')_{ij}$ is given by

$$(MM')_{ij} = \sum_{k=1}^{m} M_{ik} M'_{kj}$$

and it follows that $(MM')_{ij} \in I \cdot I'$ for all $1 \leq i \leq l$ and $1 \leq j \leq n$. Hence $J \subset I \cdot I'$. □

Let M and N be as before the lemma, restricted to an affine open $\operatorname{Spec} C = W \subset U \cap V$. The transition maps $\mathbb{A}^r_W \to \mathbb{A}^r_W$ and $\mathbb{A}^s_W \to \mathbb{A}^s_W$ are given by invertible matrices P and Q with entries in C, and it holds that $M = P^{-1} N Q$. By lemma 2.3.2, it holds that $I \subset J$, and writing $N = PMQ^{-1}$, we also see that $J \subset I$. Hence $I = J$ on the overlap $U \cap V$ and it follows that $R_q(f)$ is a closed subscheme given by the ideal sheaf $\mathcal{I}$ locally generated by the entries of matrices defining f. We have the following definition.

Definition 2.3.3. The *degeneracy locus of rank* q of f is the closed subscheme $R_q(f)$ of points $x \in X$ where f_x has rank at most q.

Suppose that $r = s$. In this case, the degeneracy locus $R_{r-1}(f)$ is rather special. On affine opens, it corresponds to the vanishing locus of

$$\det f : \bigwedge^r E \to \bigwedge^r F,$$

the determinant of f.

Now let X be a smooth quasi-projective variety over k of dimension n, $0 \le q < \min(r, s)$ and $f : E \to F$ as before. The closed immersion $R_q(f) \subset X$ defines a group homomorphism $A_i(R_q(f)) \to A_i(X)$ on Chow groups for all i (see [6], appendix A and [17], tag 02RV). Let $\mathbf{d}(q) = (r - q)(s - q)$.

Theorem 2.3.4 (Giambelli-Thom-Porteous formula)**.** There is a class $\mathscr{R}_q(f)$ in $A_{n-\mathbf{d}(q)}(R_q(f))$, such that the image of $\mathscr{R}_q(f)$ in $A_{n-\mathbf{d}(q)}(X)$ is

$$\det\left(c_{s-q-i+j}(F - E)\right)_{1 \le i,j \le r-q} \cap [X],$$

where c denotes the Chern class.

Fulton greatly elaborates on this theme in [3], which is the culmination of much important work on determinants and degeneracy loci. In fact, this theorem is one of the simplest cases of Fulton's theorem 10.1. By theorem 8.2 from [3], if $R_q(f)$ is of codimension $\mathbf{d}(q)$ in X (which is the case for suitably generic f), then $\mathscr{R}_q = [R_q]$. We understand $c_i(F - E)$ to be the Chern class

$$c_i(F - E) = c_i(F) - c_i(E) = c_i(\mathcal{F}) - c_i(\mathcal{E})$$

by taking sheaves of sections and using the equivalence of categories of proposition 2.1.5.

Assume that $r = s$, $q = r - 1$ and $\mathscr{R}_q(f) = [R_q(f)]$, which will be an interesting case for the tangent map of the Gauss map. According to the theorem, the image of $[R_q(f)]$ in $A_{r-1}(X)$ is

$$(c_1(F) - c_1(E)) \cap [X].$$

Let $f : X \to Y$ be a morphism of smooth varieties over k of dimension n. By II.8.11 from [6], there is the exact sequence

$$f^*\Omega_{Y/k} \xrightarrow{\phi} \Omega_{X/k} \longrightarrow \Omega_{X/Y} \longrightarrow 0.$$

Thus the ramification locus of f is the set of points $x \in X$ where the natural map $f^*\Omega_{Y/k,x} \to \Omega_{X/k,x}$ fails to be surjective. Since $f^*\Omega_{Y/k}$ and

$\Omega_{X/k}$ are locally free of rank n, there is a morphism $\mathcal{V}(\phi) : \mathcal{V}(f^*\Omega_{Y/k}) \to \mathcal{V}(\Omega_{X/k})$ of vector bundles over X of rank n. For $x \in X$, $\mathcal{V}(\phi)_x$ corresponds to

$$f^*\Omega_{Y/k,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x) \xrightarrow{\phi_x} \Omega_{X/k,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x),$$

which is surjective if and only if ϕ_x is surjective, as $\Omega_{X/k,x}$ and $f^*\Omega_{Y/k,x}$ are flat over $\mathcal{O}_{X,x}$. Hence $\mathcal{V}(\phi)_x$ has maximal rank if and only if x is not in the ramification locus of f. Therefore, the ramification locus of f is precisely the underlying closed set of the degeneracy locus $R_{n-1}(\phi) = R_{n-1}(\mathcal{V}(\phi))$.

2.4 The Gauss map

The Gauss map is a useful tool in algebraic geometry, because it gives us an insight into the nature of certain embeddings; particularly so in the case of projective varieties. In order to define the Gauss map, we need Grassmannians. A Grassmannian is, loosely speaking, a scheme representing d-dimensional subspaces of an n-dimensional vector space. Define a functor $\mathbf{GR}_{d,n} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$ by

$$S \mapsto \left\{ \mathcal{F} \subset \mathcal{O}_S^{\oplus n} \mid \mathcal{O}_S^{\oplus n}/\mathcal{F} \text{ is locally free of rank } n-d \right\},$$

see [4], chapter 8.4 for a motivation and an elaboration on this definition. At least note that $\mathbf{GR}_{d,n}(\operatorname{Spec} k)$ can be identified with the set $\{U \subset k^n \mid k^n/U \text{ is a } k\text{-vector space of dimension } n-d\}$, which is the set of d-dimensional subspaces of k^n. Proposition 8.14 of [4] tells us that $\mathbf{GR}_{d,n}$ is representable for all $d, n \in \mathbb{Z}_{\geq 1}$, which justifies the following definition.

Definition 2.4.1. For $d, n \in \mathbb{Z}_{\geq 1}$, the *Grassmannian* $\mathbf{GR}_{d,n}$ is the scheme representing the functor $\mathbf{GR}_{d,n}$.

For more on representable functors, see [11], section III.2 on the Yoneda lemma. Analogously to the definition of a classical projective space $\mathbb{P}(V)$ with V a vector space as being the one-dimensional subspaces of V, one would hope that there is a canonical isomorphism $\mathbf{GR}_{1,n+1} \cong \mathbb{P}^n_{\mathbb{Z}}$, which turns out to be the case (cf. [4] section 8.5). Corollary 8.15 of [4] tells us that there is a finite open covering of $\mathbf{GR}_{d,n}$ by schemes isomorphic to $\mathbb{A}^{d(n-d)}$ and that $\mathbf{GR}_{d,n}$ is smooth of relative dimension $d(n-d)$ over $\operatorname{Spec} \mathbb{Z}$.

It is a good thing to have these Grassmannians, because they give us the notion of "linear subspaces" of the affine space $\mathbb{A}^n_S$ over an arbitrary scheme S. Moreover, they allow us to define dual projective spaces in a natural way. If we set $\mathbb{P}^n_{\mathbb{Z}} = \mathbf{GR}_{1,n+1}$, then $(\mathbb{P}^n_{\mathbb{Z}})^\vee = \mathbf{GR}_{n,n+1}$ is its dual. For an arbitrary scheme S, we set $\mathbf{GR}_{d,n,S} = \mathbf{GR}_{d,n} \times_{\mathbb{Z}} S$. In case $S = \operatorname{Spec} k$ for some field k, we also write $\mathbf{GR}_{d,n,k}$.

We can further generalize Grassmannians. Let S be a scheme, $\mathcal{E}$ a quasi-coherent $\mathcal{O}_S$-module and $m \in \mathbb{Z}_{\geq 0}$. Define a functor $\mathbf{GR}^m(\mathcal{E}) : \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Set}$ by

$$(h : T \to S) \mapsto \{\mathcal{F} \subset h^*(\mathcal{E}) \mid h^*\mathcal{E}/\mathcal{F} \text{ is locally free of rank } m\}.$$

By proposition 8.17 in [4], $\mathbf{GR}^m(\mathcal{E})$ is representable by an S-scheme.

Definition 2.4.2. For $m \in \mathbb{Z}_{\geq 0}$, the *Grassmannian* $\mathbf{GR}^m(\mathcal{E})$ *of quotients of* $\mathcal{E}$ *of rank* m is the S-scheme representing the functor $\mathbf{GR}^m(\mathcal{E})$.

Let $\mathbb{P}(\mathcal{E})$ be as in section 2.1. Then for each S-scheme $h : T \to S$, an S-morphism $T \to \mathbb{P}(\mathcal{E})$ is given by an invertible sheaf $\mathcal{L}$ together with a surjective map $h^*\mathcal{E} \to \mathcal{L}$ (proposition II.7.12 of [6]), the kernel of which is an inclusion $\mathcal{F} \subset h^*\mathcal{E}$ such that $h^*\mathcal{E}/\mathcal{F} \cong \mathcal{L}$. Thus we see that $\mathbf{GR}^1(\mathcal{E})(T) \cong \mathbb{P}(\mathcal{E})(T)$ for all S-schemes T, and it follows that $\mathbf{GR}^1(\mathcal{E})$ can be identified with $\mathbb{P}(\mathcal{E})$.

Now let $f : Y \to X$ be a closed immersion of smooth varieties over k, with $\dim X = n$ and $\dim Y = d$. The idea of the Gauss map is to make a morphism $Y \to \mathbf{GR}_{d,n,k}$ by sending a point x to the d-dimensional linear subspace T_xY of T_xX and identifying all the tangent spaces T_xX. We have the sequence 2.3 from section 2.1

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{O}_Y \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0,$$

yielding a morphism $Y \to \mathbf{GR}^{n-d}(\mathcal{T}_X \otimes \mathcal{O}_Y)$ of schemes over Y. Regrettably, there is no canonical way to define a morphism $\mathbf{GR}^{n-d}(\mathcal{T}_X \otimes \mathcal{O}_Y) \to \mathbf{GR}_{d,n,k}$ involving $\mathcal{T}_Y$, unless $\mathcal{T}_X$ (or equivalently Ω_X) is free of rank n so that $\mathcal{T}_X \otimes \mathcal{O}_Y \cong \mathcal{O}_Y^{\oplus n}$. It holds that

$$\mathbf{GR}^{n-d}(\mathcal{T}_X \otimes \mathcal{O}_Y) \cong \mathbf{GR}^{n-d}(\mathcal{O}_Y^{\oplus n}) \cong \mathbf{GR}_{d,n,Y} = \mathbf{GR}_{d,n,k} \times_k Y,$$

which comes equipped with a natural projection map $\mathbf{GR}^{n-d}(\mathcal{T}_X \otimes \mathcal{O}_Y) \to \mathbf{GR}_{d,n,k}$ (cf. [4], section 8.7) upon fixing an isomorphism $\mathcal{T}_X \otimes \mathcal{O}_Y \to \mathcal{O}_Y^{\oplus n}$. We end up with the following definition.

Definition 2.4.3. Let $f : Y \to X$ be a closed immersion of smooth varieties over k with $\dim X = n$ and $\dim Y = d$, such that $\Omega_{X/k}$ is free of rank n. Fix an isomorphism $\mathcal{T}_X \otimes \mathcal{O}_Y \to \mathcal{O}_Y^{\oplus n}$. Then the *Gauss map* of f is the morphism $\Gamma : Y \to \mathbf{GR}_{d,n,k}$ which is the composition

$$Y \longrightarrow \mathbf{GR}^{n-d}(\mathcal{T}_X \otimes \mathcal{O}_Y) \longrightarrow \mathbf{GR}_{d,n,k},$$

where the first map corresponds to the canonical inclusion $\mathcal{T}_Y \to \mathcal{T}_X \otimes \mathcal{O}_Y$ and the second map is the projection map.

There are two special cases, the first being when $d = n-1$ so that Y is of codimension 1 in X and the Gauss map is a map into $(\mathbb{P}^n_k)^\vee$, and the second being when $d = 1$ so that Y is a curve and the Gauss map is a map into $\mathbb{P}^n_k$. The first case will be of most interest to us, since the theta divisor of a principally polarized abelian variety is a subscheme of codimension 1. Sadly, the theta divisor Θ of a ppav is not always smooth, in which case we only get a morphism from the smooth locus Θ^s of Θ to dual projective space.

Suppose that we are given a closed immersion $f : Y \to X$ of smooth varieties over k with $\dim Y = d$, $\dim X = n$ and $\Omega_{X/k}$ free, with a Gauss map $\Gamma : Y \to \mathbf{GR}_{d,n,k}$. It is good to check that Γ does what it is supposed to do at k-valued points, namely embed a tangent space of Y at x in the tangent space of X at x, which gives a point in $\mathbf{GR}_{d,n,Y}(k)$ and thus in $\mathbf{GR}_{d,n,k}(k)$. Let $x \in Y(k)$. Then $\mathcal{T}_{Y,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x) = T_xY$ and $(\mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{O}_Y)_x \otimes_{\mathcal{O}_{X,x}} \kappa(x) = \mathcal{T}_{X,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x) = T_xX$ are k-vector spaces of dimension d and n, respectively. Therefore the injective map $\mathcal{T}_Y \to \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ of locally free sheaves gives an injective map $T_xY \to T_xX$ of k-vector spaces and we might just as well think of it as an embedding $T_xY \subset T_xX$. This embedding defines a point (x, p) of $\mathbf{GR}_{d,n,Y}(k)$, the projection of which is just p, so $\Gamma(x) = p$.

The current definition of the Gauss map comes at a fairly steep price; even most smooth varieties have sheaves of differentials that are not free, projective space being a prime example. Luckily the above definition can be fixed to include the case $X = \mathbb{P}^n_k$, such that the Gauss map exists for all smooth projective varieties.

Set $X = \operatorname{Proj} k[T_0, \ldots, T_n]$ and let $L = \mathbb{V}(\mathcal{O}_X(1))$ be the tautological line bundle over X (cf. [10], example 1.19). Its trivializations are over the standard opens $U_i = D_+(T_i) \subset X$, for which we have

$$L|_{U_i} = \operatorname{Spec} k\left[\tfrac{T_0}{T_i}, \ldots, \tfrac{T_n}{T_i}\right][T_i],$$

with obvious isomorphisms $\phi_i : L|_{U_i} \to \mathbb{A}^1_{U_i}$. For $i, j \in \{0, \ldots, n\}$ with $i \neq j$, let $U_{ij} = U_i \cap U_j$ and $\mathbb{A}^1_{U_i} = \operatorname{Spec} \mathcal{O}_X(U_i)[T]$. The transition maps $\phi_{ij} : \mathbb{A}^1_{U_{ij}} \to \mathbb{A}^1_{U_{ij}}$ are given by the linear automorphism $T \to \frac{T_j}{T_i}T$.

Let M be the $\mathbb{G}_{m,k}$-bundle associated to L, so that

$$M|_{U_i} = \operatorname{Spec} k\left[\tfrac{T_0}{T_i}, \ldots, \tfrac{T_n}{T_i}\right][T_i, T_i^{-1}].$$

It will now be shown that M is actually isomorphic to affine $(n+1)$-space over k with the origin removed. Let

$$X' = \mathbb{A}^{n+1}_k \setminus \{0\} = \bigcup_{i=0}^{n} D(T_i)$$

be this scheme, where the $D(T_i) = \operatorname{Spec} k[T_0, \ldots, T_n, T_i^{-1}] \subset \mathbb{A}_k^{n+1}$ are basis opens. We get isomorphisms $\psi_i : M|_{U_i} \to D(T_i)$, where

$$D(T_i) = \operatorname{Spec} k[T_0, \ldots, T_n, T_i^{-1}] \subset X',$$

given by $T_l \mapsto \frac{T_l}{T_i}$ for $l \neq i$ and $T_i \mapsto T_i$. Restricting ψ_i to U_{ij}, we see that $T_j \mapsto \frac{T_j}{T_i}$ becomes invertible, so $\psi_i|_{U_{ij}}$ maps into $D_+(T_i) \cap D_+(T_j) = D_+(T_iT_j)$. Moreover, it is easy to see that $\psi_i|_{U_{ij}} = \psi_j|_{U_{ij}}\phi_{ij}$. By the gluing lemma (cf. [6], exercise II.2.12), we get an isomorphism $M \to X'$. This shows that the projection $M \to X$ can be identified with the canonical surjection $\pi : X' \to X$ from remark 3.15 in [4]. This allows us to think of points in $\mathbb{P}_k^n$ as one-dimensional linear subspaces in $\mathbb{A}_k^{n+1}$ with the origin removed, just as in the classical construction of projective space. Let $Y' = Y \times_X X'$ so that we get a pullback diagram

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & X' \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{i} & X. \end{array}$$

Since closed immersion are stable under pullback and the inclusion $i : Y \to X$ is a closed immersion, $i' : Y' \to X'$ is also a closed immersion. Furthermore, X, X' and Y are smooth over k, so Y' is also smooth over k by proposition 6.15 of [4]. Note that $\dim Y' = d + 1$ and

$$\mathcal{T}_{X'} = \mathcal{T}_{\mathbb{A}_k^{n+1}}\Big|_{X'} \cong \mathcal{O}_{X'}^{\oplus n+1},$$

since $X' \subset \mathbb{A}_k^{n+1}$ is an open subscheme and the tangent sheaf of $\mathbb{A}_k^{n+1}$ is free of rank $n+1$. Hence the inclusion $i' : Y' \to X'$ meets the requirements of definition 2.4.3 and yields the Gauss map $\Gamma' : Y' \to \mathbf{GR}_{d+1,n+1,k}$. It remains to show that Γ' factors through π_Y. Let $x, y \in Y'$ such that $\pi_Y(x) = \pi_Y(y)$. As Y' is a $\mathbb{G}_{m,k}$-bundle over Y, we can choose an affine open neighbourhood $U \subset Y$ of $\pi_Y(x)$ such that $Y'|_U \cong U \times_k \mathbb{G}_{m,k}$, and the map $U \to \mathbb{G}_{m,k}$ given by $u \mapsto 1$ defines a section $s|_U : U \to Y'|_U$. We get a commutative diagram

$$\begin{array}{ccc} Y'|_U & \xrightarrow{\gamma} & \mathbf{GR}_{d+1,n+1,k} \times_k (U \times_k \mathbb{G}_{m,k}) \\ {\scriptstyle s|_U}\uparrow\downarrow{\scriptstyle \pi_Y|_U} & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{p\gamma s|_U} & \mathbf{GR}_{d+1,n+1,k} \times_k U \\ & & \downarrow \\ & & \mathbf{GR}_{d+1,n+1,k}, \end{array}$$

where the square (without $s|_U$) is cartesian and any composition of arrows $Y'|_U \to \mathbf{GR}_{d+1,n+1,k}$ is $\Gamma'|_U$. Thus we have $\Gamma'(x) = \Gamma'(y)$. Hence Γ' factors through π_Y, yielding $\Gamma : Y \to \mathbf{GR}_{d+1,n+1,k}$. Intuitively, Γ assigns to each k-valued point $x \in Y(k)$ the d-dimensional projective tangent space of that point in $\mathbb{P}^n_k(k)$.

Definition 2.4.4. The *Gauss map* of a smooth closed subvariety Y of $\mathbb{P}^n_k$ of dimension d is the map $\Gamma : Y \to \mathbf{GR}_{d+1,n+1,k}$ constructed above.

Let $f : Y \to X$ be another closed immersion of smooth varieties over k with $\dim Y = d$, $\dim X = n$ and $\Omega_{X/k}$ free. Then f has a Gauss map $\Gamma : Y \to \mathbf{GR}_{d,n,k}$. Let $\mathcal{E} = \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{O}_Y$. We get an exact sequence

$$\Gamma^*\Omega_{\mathbf{GR}_{d,n,k}/k} \longrightarrow \Omega_{Y/k} \longrightarrow \Omega_{Y/X} \longrightarrow 0,$$

where the first arrow defines a map $d\Gamma : T(Y) \to \Gamma^*T(\mathbf{GR}_{d,n,k})$ of vector bundles over Y, which is subject to the Giambelli-Porteous formula. This will be useful, provided that we can figure out more about $\Gamma^*T(\mathbf{GR}_{d,n,k})$. The Gauss map Γ is the factorization

$$Y \xrightarrow{g} \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbf{GR}_{d,n,k},$$

where the first map is the morphism $g : Y \to \mathbb{P}(\mathcal{E})$ given by the surjection $\mathcal{E} \to \mathcal{N}_{Y/X}$ (see section 2.1), and the second map is the projection map $\mathbf{GR}_{d,n,Y} \to \mathbf{GR}_{d,n,k}$ after identifying $\mathbb{P}(\mathcal{E})$ with $\mathbf{GR}_{d,n,Y}$, using the fact that $\mathcal{E}$ is free. The diagram

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}) & \xrightarrow{\pi} & \mathbf{GR}_{d,n,k} \\ \downarrow\uparrow g & & \downarrow \\ Y & \longrightarrow & \operatorname{Spec} k \end{array}$$

is a pullback diagram (not considering g), so in particular it holds that $\pi^*\Omega_{\mathbf{GR}_{d,n,k}/k} = \Omega_{\mathbb{P}(\mathcal{E})/Y}$. At the end of section 2.1, we showed that $g^*\Omega_{\mathbb{P}(\mathcal{E})/Y} \cong \mathcal{T}_Y \otimes \mathcal{N}^\vee_{Y/X}$, so it also holds that

$$\Gamma^*\Omega_{\mathbf{GR}_{d,n,k}/k} = g^*\pi^*\Omega_{\mathbf{GR}_{d,n,k}/k} = g^*\Omega_{\mathbb{P}(\mathcal{E})/Y} \cong \mathcal{T}_{Y/k} \otimes \mathcal{N}^\vee_{Y/X}.$$

It follows that $\Gamma^*T(\mathbf{GR}_{d,n,k}) \cong \mathbb{V}(\mathcal{T}_{Y/k} \otimes \mathcal{N}^\vee_{Y/X}) = T(Y)^\vee \otimes N(Y)$.

Lemma 2.4.5. If $\dim Y = n-1$, then $d\Gamma$ determines a global section σ of the invertible sheaf $(\mathcal{O}_X(Y)|_Y)^{\otimes n+1}$.

Proof. Assume that $\dim Y = n-1$. Then it holds that $d\Gamma : T(Y) \to T(Y)^\vee \otimes N(Y)$ is a morphism of vector bundles of rank $n-1$, corresponding to the morphism

$$\phi : \mathcal{T}_{Y/k} \otimes \mathcal{N}^\vee_{Y/X} \longrightarrow \Omega_{Y/k}$$

of locally free sheaves of rank $n-1$, so it has a determinant $\det\phi$. We denote $\det\Omega_{Y/k} = \omega_{Y/k}$. Then

$$\det\left(\mathcal{T}_{Y/k}\otimes\mathcal{N}^{\vee}_{Y/X}\right) = \det\mathcal{T}_{Y/k}\otimes\left(\mathcal{N}^{\vee}_{Y/X}\right)^{\otimes n-1} = \omega^{\vee}_{Y/k}\otimes\left(\mathcal{N}^{\vee}_{Y/X}\right)^{\otimes n-1},$$

so, after tensoring with the appropriate invertible sheaves, $\det\phi$ can be seen to define a map

$$\mathcal{O}_X \longrightarrow \omega^{\otimes 2}_{Y/k}\otimes\mathcal{N}^{\otimes n-1}_{Y/X},$$

and we may identify $\det\phi$ with the image of 1 in $\omega^{\otimes 2}_{Y/k}\otimes\mathcal{N}^{\otimes n-1}_{Y/X}$, which is a global section. Applying corollary 6.4.2 in [10] to the exact sequence 2.2

$$0\longrightarrow\mathcal{I}/\mathcal{I}^2\longrightarrow\Omega_{X/k}\otimes\mathcal{O}_Y\longrightarrow\Omega_{Y/k}\longrightarrow 0$$

gives

$$\det\left(\mathcal{I}/\mathcal{I}^2\right)\otimes\det\Omega_{Y/k}\cong\det\left(\Omega_{X/k}\otimes\mathcal{O}_Y\right).$$

As $\Omega_{X/k}$ is free, it follows that $\Omega_{X/k}\otimes\mathcal{O}_Y$ is free and $\det\left(\Omega_{X/k}\otimes\mathcal{O}_Y\right) = \mathcal{O}_Y$. Furthermore, $\det\left(\mathcal{I}/\mathcal{I}^2\right) = \mathcal{I}/\mathcal{I}^2 = \mathcal{N}^{\vee}_{Y/X}$. Hence

$$\mathcal{N}^{\vee}_{Y/X}\otimes\omega_{Y/k}\cong\mathcal{O}_Y.$$

It follows that $\omega_{Y/k}\cong\mathcal{N}_{Y/X}$. Since $f: Y\to X$ defines a smooth divisor, it holds that $\mathcal{O}(Y)$ is an invertible sheaf on X and its dual $\mathcal{I} = \mathcal{O}(-Y)$ is the ideal sheaf that gives the exact sequence

$$0\longrightarrow\mathcal{I}\longrightarrow\mathcal{O}_X\longrightarrow f_*\mathcal{O}_Y\longrightarrow 0.$$

It holds that $\mathcal{N}^{\vee}_{Y/X} = \mathcal{I}/\mathcal{I}^2\cong\mathcal{I}\otimes_{\mathcal{O}_X}\mathcal{O}_Y = \mathcal{O}(-Y)\otimes_{\mathcal{O}_X}\mathcal{O}_Y$, so $\mathcal{N}_{Y/X}\cong(\mathcal{O}_X(Y)|_Y)$. Hence

$$\omega^{\otimes 2}_{Y/k}\otimes\mathcal{N}^{\otimes n-1}_{Y/X}\cong\mathcal{N}^{\otimes n+1}_{Y/X}\cong(\mathcal{O}_X(Y)|_Y)^{\otimes n+1}$$

and thus $\det\phi$ determines a global section σ of the invertible sheaf $(\mathcal{O}_X(Y)|_Y)^{\otimes n+1}$, the vanishing locus of which consists precisely of the points $y\in Y$ such that $\det\phi_y$ is the zero map. □

Corollary 2.4.6. The vanishing locus of σ is the ramification locus of the Gauss map.

Proof. It follows from section 2.3 that the vanishing locus of σ is actually the degeneracy locus R_{n-2} of $d\Gamma$, which in turn is the ramification locus of Γ. □

The Giambelli-Thom-Porteous formula 2.3.4 gives that the image of $[R_{n-2}]$ in $A_{n-1}(Y)$ is

$$\big(c_1(T(Y)^\vee \otimes N(Y)) - c_1(T(Y))\big) \cap [Y].$$

Using the properties listed in section 3 of appendix A in [6], it holds that $c_1(T(Y)) = c_1(T(X)|_Y) - c_1(N(Y))$, so

$$\begin{aligned} c_1(T(Y)^\vee \otimes N(Y)) &= c_1(T(Y)^\vee) + (n-1)c_1(N(Y)) \\ &= c_1(N(Y)) - c_1(T(X)|_Y) + (n-1)c_1(N(Y)) \\ &= nc_1(N(Y)) - c_1(T(X)|_Y). \end{aligned}$$

As $T(X)|_Y$ is a trivial bundle, it holds that $c_1(T(X)|_Y) = 0$. It follows that the image of $[R_{n-2}]$ in $A_{n-1}(Y)$ is

$$\begin{aligned} \big(c_1(T(Y)^\vee \otimes N(Y)) - c_1(T(Y))\big) &= nc_1(N(Y)) - c_1(T(Y)) \\ &= (n+1)c_1(N(Y)). \end{aligned}$$

Returning to locally free sheaves, we have $c_1(N(Y)) = c_1(\mathcal{N}_{Y/X})$. Hence the image of $[R_{n-2}]$ in $A_{n-1}(Y)$ is

$$(n+1)c_1(\mathcal{N}_{Y/X}) = c_1\left(\mathcal{N}_{Y/X}^{\otimes n+1}\right) = c_1\left((\mathcal{O}_X(Y)|_Y)^{\otimes n+1}\right).$$

If $f : Y \to X$ is a closed immersion such that $\Omega_{X/k}$ is free but Y is singular, not all is lost. The smooth locus Y^s is open and dense in Y by corollary II.8.16 of [6]. Let $S = Y - Y^s$ be the singular locus of Y and $X' = X - S$. Note that X' is open and dense in X, so $\Omega_{X'/k} = \Omega_{X/k}|_{X'}$ is free. Then

$$\begin{array}{ccc} Y^s & \longrightarrow & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \longrightarrow & X \end{array}$$

is a pullback diagram, so f' is a closed immersion and has a Gauss map $\Gamma : Y^s \to \mathbf{GR}_{d,n,k}$, which defines a rational map $\Gamma : Y \to \mathbf{GR}_{d,n,k}$. In this case, lemma 2.4.5 gives a rational section σ of $(\mathcal{O}_X(Y)|_Y)^{\otimes n+1}$, defined at least at Y^s.

2.5 On η and the Gauss map

Let (A, Θ) be a principally polarized abelian variety of dimension g over $\mathbb{C}$. The embedding $i : \Theta \to A$ is a closed immersion, as Θ is an effective Cartier divisor. The smooth locus Θ^s is open and dense in Θ and the singular locus $S = \Theta \setminus \Theta^s$ is closed in Θ and hence in A. Thus we have that

$$i' : \Theta^s \longrightarrow A \setminus S,$$

the pullback of i along $A \setminus S \to A$, is a closed immersion. Moreover, the tangent sheaf $\mathcal{T}_A$ of A is trivial by proposition 1.1.8, so i has a Gauss map $\Gamma : \Theta^s \to (\mathbb{P}^{g-1}_{\mathbb{C}})^\vee$ on the smooth locus Θ^s of Θ. From the previous section 2.4 we have $\Gamma^* T(X) \cong T(\Theta^s)^\vee \otimes N(\Theta^s)$ and a global section σ of the invertible sheaf $(\mathcal{O}_A(\Theta^s)|_{\Theta^s})^{\otimes g+1}$ coming from $\det d\Gamma$.

The theta function η from definition 1.5.1 is only defined on the manifold A_h of $\mathbb{C}$-valued points. We will now show that it corresponds exactly to the determinant of $d\Gamma$ restricted to $\mathbb{C}$-valued points, at least on the smooth locus of Θ. From now on, we consider only the complex analytic version of all geometric objects, and we will just write A and Θ instead of A_h and Θ_h to ease the notation. Thus we have $A = \mathbb{C}^g/(\mathbb{Z}^g \oplus \tau\mathbb{Z}^g)$ for some $\tau \in \mathbb{H}_g$, and Θ is the zero locus of the Riemann theta function $\theta(\tau, z)$ for the fixed τ. Furthermore, we have natural isomorphisms $T_x A \cong \mathbb{C}^g$ for all $x \in A$. For improved readability, we shall write θ_i instead of $\frac{\partial \theta}{\partial z_i}(\tau)$. Choosing standard coordinates $(z_1, \ldots, z_g)$ for $\mathbb{A}^g$, we get an identification $(\mathbb{P}^{g-1})^\vee \to \mathbb{P}^{g-1}$ sending a hyperplane in $\mathbb{A}^g$ with equation $\sum a_i z_i = 0$ to the point $(a_1 : \cdots : a_g)$. We now think of Γ as a map $\Theta^s \to \mathbb{P}^{g-1}$ using this identification; we saw in section 2.4 that Γ sends a point $x \in \Theta^s$ to the embedding $T_x\Theta^s \subset T_x A$, which is the hyperplane in $\mathbb{C}^g$ defined by the equation

$$\sum_{i=1}^{g} \theta_i(x) z_i = 0,$$

hence $\Gamma(x) = (\theta_1(x) : \cdots : \theta_g(x))$. Note that we've chosen local coordinates for A around x, so there is an open $U \subset A$ and a map $\phi : U \to \mathbb{A}^g$ that is just the identity on U, which follows from the fact that $\mathbb{A}^g$ is the universal covering space of A. The normal bundle $N(\Theta^s)$ becomes trivial, because we have chosen the local defining equation θ for Θ. Hence we now have $\Gamma^* T(X) \cong T(\Theta^s)^\vee$. Let $x \in \Theta^s$. Then $d\Gamma_x : T_x\Theta^s \to T_{\Gamma(x)}(\mathbb{P}^{g-1})$ is a map between $(g-1)$-dimensional $\mathbb{C}$-vector spaces. Define a map $\tilde{\Gamma} : \mathbb{A}^g \to \mathbb{A}^g$ by

$$y \mapsto (\theta_1(y), \ldots, \theta_g(y)),$$

and let $\pi : \mathbb{A}^g \setminus \{0\} \to \mathbb{P}^{g-1}$ be the usual canonical projection. Note that $\tilde{\Gamma}\phi$ restricted to $U \cap \Theta^s$ is actually a map into $\mathbb{A}^g \setminus \{0\}$. Hence Γ factors as $\pi\tilde{\Gamma}\phi$ on $U \cap \Theta^s$. It holds that the tangent map of ϕ is the identity on all tangent spaces. The following lemma is a well-known result from differential geometry.

Lemma 2.5.1. Let $f : \mathbb{C}^g \to \mathbb{C}^g$ be given by $y \mapsto (f_1(y), \ldots, f_n(y))$ with the $f_i : \mathbb{C}^g \to \mathbb{C}$ holomorphic for all i. Then the tangent map $df_y : \mathbb{C}^g \to \mathbb{C}^g$ is given by the Jacobi-matrix

$$\left(\frac{\partial f_i}{\partial z_j}\right)_{i,j},$$

evaluated at y.

For $y \in \mathbb{A}^g$, lemma 2.5.1 gives us that $d\tilde{\Gamma}_y : \mathbb{C}^g \to \mathbb{C}^g$ is given by the Jacobi-matrix of $(\theta_1, \ldots, \theta_g)$ evaluated in y, which is actually the Hessian

$$H(y) = \left(\frac{\partial^2 \theta}{\partial z_i \partial z_j}(\tau, y) \right)_{i,j}$$

of $\theta(\tau, y)$. Now all that remains to understand $d\Gamma_x$ is the tangent map $d\pi$. Let $y \in \mathbb{A}^g \setminus \{0\}$. Considering y as a point in $\mathbb{C}^g \setminus \{0\}$, we get a linear subspace $\mathbb{C}y \subset \mathbb{C}^g$, which defines the point $\pi(y)$. The tangent space $T_{\pi(y)}\mathbb{P}^{g-1}$ can then be identified with $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}y, \mathbb{C}^g/\mathbb{C}y)$ according to [4], section (8.9). Hence $d\pi_y : \mathbb{C}^g \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}y, \mathbb{C}^g/\mathbb{C}y)$ is the canonical map $v \mapsto (y \mapsto \bar{v})$, or equivalently $d\pi_y : \mathbb{C}^g \to \mathbb{C}^g/\mathbb{C}y$ given by $v \mapsto \bar{v}$.

Now let $v \in T_xA$ be a point that is contained in the subspace $T_x\Theta^s \subset T_xA$. Then $d\phi_x(v) = v$, $d\tilde{\Gamma}_x(v) = H(x)v$ and $d\pi_{\tilde{\Gamma}(x)}(H(x)v) = \overline{H(x)v}$, which ultimately shows that

$$d\Gamma_x(v) = \overline{H(x)v} \in \mathbb{C}^g/\mathbb{C}\tilde{\Gamma}(x).$$

There is an obvious isomorphism $\mathbb{C}^g/\mathbb{C}\tilde{\Gamma}(x) \to (T_x\Theta^s)^\vee$ given by $\bar{v} \mapsto (w \mapsto {}^t v w)$, which is well-defined since ${}^t\tilde{\Gamma}(x)w = 0$, viewing w as vector in $\mathbb{C}^g$. Hence we have not only a linear map $d\Gamma_x : T_x\Theta^s \to (T_x\Theta^s)^\vee$, but even a symmetric bilinear form $T_x\Theta^s \times T_x\Theta^s \to \mathbb{C}$, given by $(v, w) \mapsto {}^t v H(x) w$. Note that $\det d\Gamma_x = 0$ if and only if this symmetric bilinear form is degenerate. The following lemma gives a geometric interpretation of this last statement. Let $\hat{\ } : \mathbb{C}^g \setminus \{0\} \to \mathbb{P}^{g-1}$ denote the usual projection map.

Lemma 2.5.2. Let $a \in \mathbb{C}^g \setminus \{0\}$ and $H \in \mathrm{Mat}(g, \mathbb{C})$ a symmetric matrix with associated quadratic form q on $\mathbb{C}^g$ and associated symmetric bilinear form $B : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}$. Let $L = \{x \in \mathbb{C}^g \mid {}^t a x = 0\}$. Then $\hat{L}$ is a hyperplane in $\mathbb{P}^{g-1}$. Let $Q = \{\hat{x} \in \mathbb{P}^{g-1} \mid q(x) = {}^t x H x = 0\}$ be a quadric in $\mathbb{P}^{g-1}$. Then the following are equivalent:

1. The hyperplane $\hat{L}$ is tangent to Q.
2. It holds that
$$\det \begin{pmatrix} H & a \\ {}^t a & 0 \end{pmatrix} = 0.$$
3. The quadric $\hat{L} \cap Q$ in $\hat{L}$ is singular.
4. The symmetric bilinear form $B|_{L \times L}$ is degenerate.

Proof. Note that L and Q are tangent if and only if there exists a nonzero $p \in \mathbb{C}^g$ such that $\hat{p} \in Q$ and $L = \{\hat{q} \in \mathbb{P}^{g-1} \mid {}^t q H p = 0\}$. This is equivalent to p satisfying (i) $Hp = \lambda a$ for some $\lambda \in \mathbb{C}$, (ii) ${}^t a p = 0$ and (iii) ${}^t p H p = 0$, but (i) and (ii) together imply (iii), so we can drop (iii). Then (i) and (ii) can be stated together as

$$\begin{pmatrix} H & a \\ {}^t a & 0 \end{pmatrix} \cdot \begin{pmatrix} p \\ -\lambda \end{pmatrix} = 0,$$

which means that the matrix has a nontrivial kernel, or equivalenty zero determinant. Hence 1. and 2. are equivalent.

Statements 3. and. 4. are equivalent by definition. The symmetric bilinear form $B|_{L\times L}$ is degenerate if and only if there exists a $\hat{p} \in L$ such that ${}^t qHp = 0$ for all $\hat{q} \in L$, but this is equivalent to 1. The result follows. $\square$

Even though there is no obvious extension of Γ to U, we do have the map $\psi = \tilde{\Gamma}\phi$ given by $x \mapsto (\theta_1(x), \ldots, \theta_g(x))$, with the corresponding tangent map $T_x A \to T_{\psi(x)}\mathbb{A}^g$ given by $v \mapsto H(x)v$ upon identifying $T_x A$ and $T_{\psi(x)}\mathbb{A}^g$ with $\mathbb{C}^g$. For $x \in U$, we get a bilinear form $B_x : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}^g$, given by $(v, w) \mapsto {}^t vHw$.

Write $d\theta$ for the column vector with entries θ_i. We have the hyperplane $L = \{x \in \mathbb{P}^{g-1} \mid {}^t d\theta x = 0\} = \mathbb{P}(T_x\Theta)$ and the quadric $Q = \{(x \in \mathbb{P}^{g-1} \mid {}^t xHx = 0\}$ in $\mathbb{P}(T_x A) = \mathbb{P}^{g-1}$. Lemma 2.5.2 yields that for $x \in \Theta^s$,

$$\eta(x) = \det \begin{pmatrix} H(x) & d\theta(x) \\ {}^t d\theta(x) & 0 \end{pmatrix} = 0$$

if and only if $B_x|_{T_x\Theta^s \times T_x\Theta^s}$ is degenerate, the latter being the case if and only if $\det d\Gamma_x = 0$. It follows that $\eta(x) = 0$ if and only if $\sigma(x) = 0$, where σ is the section from lemma 2.4.5. It follows that there is an invertible global section α of $\mathcal{O}_{\Theta^s}$ such that $\eta = \alpha\sigma$. We can't expect the global sections of $\mathcal{O}_{\Theta^s}$ to be precisely those of $\mathcal{O}_\Theta$, so it's good to know that a generic principally polarized abelian variety has a smooth theta divisor. See section 3.1 for details. Henceforth, assume that $\Theta^s = \Theta$. Then $\Theta \to A \to \operatorname{Spec}\mathbb{C}$ is a composition of proper morphisms and therefore proper, so it follows that $\Gamma(\Theta, \mathcal{O}_\Theta) = \mathbb{C}$ by corollary 3.21 from [10]. Hence $\alpha \in \mathbb{C}^*$ and η and σ differ by a nonzero scalar. Thus σ also defines a theta function on Θ. This is pleasing, as σ is defined more intrinsically than η as the determinant of the tangent map of the Gauss map, which exists for all closed immersions $Y \to X$ over k with $\operatorname{codim}(Y, X) = 1$ and $\Omega_{X/k}$ free.

3 Singularities of theta divisors

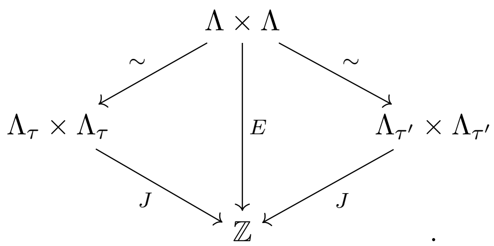
In this chapter, we further develop the framework of complex abelian varieties by defining and studying the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties. This moduli space is of great interest in many different areas of mathematics, and it is worthwhile to learn more of its geometry. We will turn our attention to the locus $\theta_{null} \subset \mathcal{A}_g$ of ppav's (A, Θ) such that Θ is singular at a point of order 2. We prove the main results of this thesis in the final section.

3.1 The moduli space $\mathcal{A}_g$

Principally polarized abelian varieties are essentially parametrized by the Siegel upper-half space $\mathbb{H}_g$, as observed in section 1.4. Given $\tau \in \mathbb{H}_g$, we get a g-dimensional principally polarized abelian variety $A = \mathbb{C}^g/(\mathbb{Z}^g \oplus \tau\mathbb{Z}^g)$ together with a symplectic basis of the lattice $\Lambda_\tau = \mathbb{Z}^g \oplus \tau\mathbb{Z}^g$. Since we would like to describe principally polarized abelian varieties without being stuck with a choice of basis, we have to pass to an appropriate quotient of $\mathbb{H}_g$. Remember that J is the matrix

$$J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}.$$

Let $A = V/\Lambda$ be an abelian variety with principal polarization H. Suppose that $\tau, \tau' \in \mathbb{H}_g$ are such that E is given by the matrix J with respect to both associated symplectic bases. We have the following commutative diagram



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Suppose that we have an automorphism $f : \mathbb{C}^g \to \mathbb{C}^g$ that induces a $\mathbb{Z}$-linear isomorphism $\Lambda_\tau \to \Lambda_{\tau'}$ given by $A \in \mathrm{Mat}(2g, \mathbb{Z})$. Then the isomorphism given by A commutes with the diagram if and only if

$$J(Av, Aw) = {}^t(Av)J(Aw) = {}^tv{}^tAJAw = {}^tvJw = J(v, w)$$

for all $v, w \in \Lambda_\tau$, that is, $J = {}^tAJA$. The $2g \times 2g$-matrices with this property form a multiplicative group.

Definition 3.1.1. The *symplectic group* $\mathrm{Sp}(2g, \mathbb{Z})$ is the multiplicative group consisting of $2g \times 2g$-matrices $A \in \mathrm{Mat}(2g, \mathbb{Z})$ such that ${}^tAJA = J$.

Let f be an automorphism of $\mathbb{C}^g$ that induces an isomorphism $\Lambda_\tau \to \Lambda_{\tau'}$ given by a matrix $A \in \mathrm{Sp}(2g, \mathbb{Z})$. For all $p, q \in \mathbb{Z}^g$, it holds that

$$f(p + \tau q) = A \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix} = p' + \tau' q',$$

where $p', q' \in \mathbb{Z}^g$. Thus it holds that

$$\begin{pmatrix} \mathbf{1} & \tau' \end{pmatrix} = f \begin{pmatrix} \mathbf{1} & \tau \end{pmatrix} A^{-1}.$$

Upon transposing, this becomes

$$\begin{pmatrix} \mathbf{1} \\ \tau' \end{pmatrix} = {}^t A^{-1} \begin{pmatrix} \mathbf{1} \\ \tau \end{pmatrix} {}^t f.$$

We have the identity $-J = ({}^t AJA)^{-1} = -A^{-1} J {}^t A^{-1}$. Letting $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$, this yields

$${}^t A^{-1} = -JAJ = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix},$$

and so we have

$$\begin{pmatrix} \mathbf{1} \\ \tau' \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ \tau \end{pmatrix} {}^t f = \begin{pmatrix} (d - c\tau)^t f \\ (-b + a\tau)^t f \end{pmatrix}.$$

Hence ${}^t f = (d - b\tau)^{-1}$ and $\tau' = (-b + a\tau)(d - c\tau)^{-1}$, which defines an action $(A, \tau) \mapsto (-b + a\tau)(d - c\tau)^{-1}$ of $\mathrm{Sp}(2g, \mathbb{Z})$ on $\mathbb{H}_g$. Two matrices τ, τ' define the same principally polarized abelian variety if and only if they are conjugate under this action. To polish things up a bit, note that the map $A \mapsto tAt$ with

$$t = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

defines an automorphism of $\mathrm{Sp}(2g, \mathbb{Z})$. Composing the action with this automorphism yields $(A, \tau) \mapsto (a\tau + b)(c\tau + d)^{-1}$, which is the action we will use to define $\mathcal{A}_g$ (see proposition 7.1 in [2]).

Definition 3.1.2. The *coarse moduli space of ppav's* is the quotient $\mathcal{A}_g := \mathbb{H}_g / \mathrm{Sp}(2g, \mathbb{Z})$ of the action of $\mathrm{Sp}(2g, \mathbb{Z})$ on $\mathbb{H}_g$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = (a\tau + b)(c\tau + d)^{-1}.$$

The described action of $\mathrm{Sp}(2g, \mathbb{Z})$ on $\mathbb{H}_g$ is properly discontinuous according to proposition 7.3 in [2], which makes $\mathcal{A}_g$ into a complex analytic space by theorem 7.2 from [2]. Unfortunately, it is not compact and therefore the techniques of section 1.2 do not apply. There is, however, an algebraic version of $\mathcal{A}_g$ that corresponds to the one just defined.

Definition 3.1.3. Let F : Sch $\to$ Set be a contravariant functor. A *coarse moduli space* for F is an object X of Sch together with a natural transformation $\mu : F \to h_X$, where h_X is the functor of points, such that

1. for all schemes Y, any natural transformation $\nu : F \to h_Y$ factors through μ (the pair (X, μ) is initial among such pairs) and
2. if P is a one-point scheme, then the map

$$\mu(P) : F(P) \to h_X(P) = X(P)$$

is a bijection.

Usually, F assigns a set of geometric objects to a scheme S, as is the case for the moduli space of ppav's. The notion required is that of *principally polarized abelian schemes*, which we will not define rigorously here; a principally polarized abelian scheme over S may be thought of as an S-scheme A such that the fibers are ppav's. Let F : Sch $\to$ Set be the functor sending a scheme S to the set of principally polarized abelian schemes over S (without going into details, it holds that abelian schemes are stable under pullback, so that F can be seen to be contravariant). Then F has a coarse moduli space, denoted by $\mathcal{A}_g$. It is, in some sense, the best scheme to represent F, and $\mathcal{A}_g(\mathbb{C})$ is the analytic version of $\mathcal{A}_g$ from definition 3.1.2.

For a generic ppav $(A, \Theta) \in \mathcal{A}_g$, the theta divisor Θ is smooth. The locus $N_0 \subset \mathcal{A}_g$ consisting of pairs (A, Θ) with a singular theta divisor defines a divisor of $\mathcal{A}_g$, which has two irreducible components θ_{null} and N_0' (cf. [14], page 21). It is important to get a better understanding of N_0 in order to fully understand the space $\mathcal{A}_g$. We will do this by restricting our attention to the component θ_{null}.

Definition 3.1.4. A *theta constant* is a function $\mathbb{H}_g \to \mathbb{C}$ which is the restriction of the theta function $\theta\left[\begin{smallmatrix}\epsilon\\ \delta\end{smallmatrix}\right] : \mathbb{H}_g \times \mathbb{C}^g \to \mathbb{C}$ to $\mathbb{H}^g \times \{0\}$.

An *even* (respectively *odd*) theta constant is the theta constant of an even (respectively odd) theta function. Because an odd theta function is odd as a function of z, it vanishes at all points $(\tau, 0)$. We now want to consider the zero locus θ_{null} of the product of all even theta constants, which lives on $\mathbb{H}_g$. For $\tau, \tau' \in \mathbb{H}_g$ that are conjugate under the action of $\mathrm{Sp}(2g, \mathbb{Z})$, it turns out that $\tau \in \theta_{null}$ if and only if $\tau' \in \theta_{null}$, so θ_{null} is well-defined on $\mathcal{A}_g$ (cf. [5], pages 5-6).

Definition 3.1.5. The *theta-null divisor* θ_{null} is the image in $\mathcal{A}_g$ of the zero locus of the product of all even theta constants.

For $(A, \Theta) \in \theta_{null}$, it holds that Θ contains a point x of order two. It is easy to check that x is singular in Θ, simply by showing that the partial derivatives of θ are zero at x. Let $H\left[\begin{smallmatrix}\epsilon\\\delta\end{smallmatrix}\right](\tau,z)$ be the *Hessian* of $\theta\left[\begin{smallmatrix}\epsilon\\\delta\end{smallmatrix}\right](\tau,z)$, the $g\times g$-matrix whose entries are the second order partial derivatives of $\theta\left[\begin{smallmatrix}\epsilon\\\delta\end{smallmatrix}\right]$ with respect to z_j, if we let $z=(z_1,\ldots,z_g)$. For $0\le h\le g$, define

$$\theta^h_{null}=\left\{\tau\in\mathbb{H}_g \mid \exists\left[\begin{smallmatrix}\epsilon\\\delta\end{smallmatrix}\right]\text{ even}:\theta\left[\begin{smallmatrix}\epsilon\\\delta\end{smallmatrix}\right](\tau,0)=0,\ \mathrm{rk}\left(H\left[\begin{smallmatrix}\epsilon\\\delta\end{smallmatrix}\right](\tau,0)\right)\le h\right\}.$$

Naturally, it holds that

$$\theta^0_{null}\subset\theta^1_{null}\subset\cdots\subset\theta^{g-1}_{null}\subset\theta^g_{null}=\theta_{null},$$

and it is shown in [5] that the θ^h_{null} are well-defined on $\mathcal{A}_g$, which is where they will live from now on.

3.2 Partial toroidal compactification

In the previous section we have defined the coarse moduli space $\mathcal{A}_g$ of principally polarized abelian varieties, but this space is not compact. Hence whenever we study a "continuous" family of principally polarized abelian varieties of dimension g (e.g. a curve in $\mathcal{A}_g$), we are bound to approach an object which is *not* a g-dimensional abelian variety, but rather a *deformation* of an abelian variety. Including these in the moduli space will make life a lot easier, which is precisely what we will do in this section: we give a partial compactification of $\mathcal{A}_g$, which then allows us to study θ_{null} at the boundary $\partial\mathcal{A}_g$ of this compactification. This will lead to some interesting conclusions in the next section.

A very important compactification of $\mathcal{A}_g$ is the *Satake compactification* $\mathcal{A}^*_g$, which set-theoretically consists of

$$\mathcal{A}^*_g=\coprod_{i=0}^{g}\mathcal{A}_i.$$

The closure of each $\mathcal{A}_i\subset\mathcal{A}^*_g$ is homeomorphic to $\mathcal{A}^*_i$, and $\mathcal{A}^*_g$ itself is a projective variety, containing $\mathcal{A}_g$ as an open subset. However, we will not be studying the Satake compactification, but merely the *partial toroidal compactification* $\overline{\mathcal{A}}^1_g$, which is not actually compact, but "compact enough" for our purposes. It is a blow-up of $\mathcal{A}_g\sqcup\mathcal{A}_{g-1}$ along $\mathcal{A}_{g-1}$, which set-theoretically amounts to

$$\overline{\mathcal{A}}^1_g=\mathcal{A}_g\sqcup\partial\mathcal{A}_g,$$

and there is a natural surjective morphism $p:\overline{\mathcal{A}}^1_g\to\mathcal{A}_g\sqcup\mathcal{A}_{g-1}$ which is the identity on $\mathcal{A}_g$. The boundary $\partial\mathcal{A}_g$ consists of so-called *rank* 1 *degenerations* of g-dimensional ppav's. Let's make this precise, following the steps on pages 3 and 4 of [14].

(1) Let $(B,\Xi)\in\mathcal{A}_{g-1}$ be a ppav with theta divisor Ξ.

(2) Let G be an algebraic group that fits in an exact sequence

$$0 \longrightarrow \mathbb{G}_m \longrightarrow G \longrightarrow B \longrightarrow 0$$

of algebraic groups that splits locally.

(3) Consider G as a $\mathbb{G}_m$-bundle over B and take its associated $\mathbb{P}^1$-bundle $\pi : \tilde{G} \to B$ as in section 2.1. Now there are two canonical sections of π, one sends every point $x \in B$ to 0 in $\pi^{-1}(x) \cong \mathbb{P}^1_k$ and the other sends $x \in B$ to $\infty \in \pi^{-1}(x)$. Let $\tilde{G}_0$ and $\tilde{G}_\infty$ be the images of these sections, respectively. Then $G = \tilde{G} \setminus (\tilde{G}_0 \sqcup \tilde{G}_\infty)$.

(4) We obtain a variety $\overline{G}$ by gluing $\tilde{G}_0$ and $\tilde{G}_\infty$ with a translation by a point $b \in B$. This variety $\overline{G}$ is then a *rank* 1 *degeneration* of a g-dimensional ppav, and it comes equipped with a divisor D. Hence we have defined what a pair $(\overline{G}, D) \in \partial\mathcal{A}_g$ looks like, to some extent. For the algebraic details of this, see [14].

The important thing to note in the above construction is that rank 1 degenerations $(\overline{G}, D)$ are defined by exact sequences as in step (2). It holds that equivalence classes of such sequences yield the dual abelian variety B^t of B by theorem 8.9 of [12]. However, two different sequences may give rise to the same $\tilde{G}$; this turns out to be the case precisely when the two sequences differ by an automorphism of (B, Ξ). Hence the projection map $p : \partial\mathcal{A}_g \to \mathcal{A}_{g-1}$ has $B/\operatorname{Aut}(B, \Xi)$ as fiber over (B, Ξ), since B is principally polarized and therefore $B \cong B^t$. The automorphism group of a generic $(B, \Xi) \in \mathcal{A}_{g-1}$ is $\{\pm 1\}$, so its fiber $\pi^{-1}(B, \Xi)$ is $B/\{\pm 1\}$, which is called the *Kummer variety* of B. Moreover, proposition VII.8 from [9] tells us that $\operatorname{Aut}(B, \Xi)$ is always finite, so each fiber has dimension $g - 1$. It holds that $\dim \mathcal{A}_{g-1} = (g-1)g/2$, so

$$\dim \partial\mathcal{A}_g = \frac{(g-1)g}{2} + g - 1 = \frac{(g+2)(g-1)}{2} = \dim \mathcal{A}_g - 1,$$

as we would expect. Note that each fiber $B/\operatorname{Aut}(B, \Xi)$ also inherits a divisor, $\Xi/\operatorname{Aut}(B, \Xi)$.

A pair $(\overline{G}, D) \in \partial\mathcal{A}_g$ can also be described analytically as a degenerating family of abelian varieties $(A(t), \Theta(t))$ with period matrix $\tau(t)$ such that

$$\tau(t) \longrightarrow \begin{pmatrix} \tau' & \omega \\ {}^t\omega & i \cdot \infty \end{pmatrix}$$

as $t \to 0$, where τ' is the period matrix of $p(\overline{G}, D) = (B, \Xi)$ and $\omega \in \mathbb{C}^{g-1}$. Hence $\overline{G}$ is given by the pair (τ', ω). The divisor D is given by the zeroes of the function

$$\tilde{\theta}((\tau', \omega), (z', z)) = \theta(\tau', z') + e^{2\pi i z}\theta(\tau', z' + \omega),$$

where $\theta(\tau', z')$ is the Riemann theta function of genus $g-1$, $z = z_g$ defines the algebraic coordinate $e^{2\pi iz}$ of $\mathbb{G}_m$ and $z' = (z_1, \ldots, z_{g-1})$ is the analytic coordinate on B for a local trivialization $G = U \times \mathbb{G}_m$ (cf. [14]). On the algebraic group G, there is an involution ρ given by $(z', z) \mapsto (-z'-\omega, -z)$, indeed

$$(-z' - \omega, -z) \mapsto (-(-z' - \omega) - \omega, z) = (z', z).$$

Then $D \cap G$ is symmetric with respect to ρ, since

$$\begin{aligned}\tilde{\theta}((\tau', \omega), (-z' - \omega, -z)) &= \theta(\tau', -z' - \omega) + e^{-2\pi iz}\theta(\tau', -z') \\ &= \theta(\tau, z' + \omega) + e^{-2\pi iz}\theta(\tau, z')\end{aligned}$$

and

$$\begin{aligned}\theta(\tau, z' + \omega) + e^{-2\pi iz}\theta(\tau, z') = 0 &\iff \theta(\tau, z') = -e^{2\pi iz}\theta(\tau', z' + \omega) \\ &\iff \tilde{\theta}(z', z) = 0.\end{aligned}$$

Note that ρ has a typical fixed point $(-\omega/2, 0)$. The partial derivatives of $\tilde{\theta}$ are

$$\begin{aligned}\frac{\partial\tilde{\theta}(z', z)}{\partial z_j} &= \frac{\partial\theta(\tau', z')}{\partial z_j} + e^{2\pi iz}\frac{\partial\theta(\tau', z' + \omega)}{\partial z_j} \qquad 1 \le j \le g-1 \\ \frac{\partial\tilde{\theta}(z', z)}{\partial e^{2\pi iz}} &= \theta(\tau', z' + \omega).\end{aligned}$$

Let b be the image of ω in B. It follows that $G \cap D$ is singular at (z', a) with $a \in \mathbb{C}$ and $p = z'$ in B if and only if $p, p + b \in \Xi$ and either $T_p\Xi = T_{p+b}\Xi$ or both p and $p+b$ are singular points of Ξ. Thus the fixed point $(-\omega/2, 0)$ of ρ is a singular point of D if and only if

$$\tilde{\theta}(-\omega/2, 0) = \theta(\tau', -\omega/2) + \theta(\tau', \omega/2) = 2\theta(\tau', \omega/2) = 0$$

and either $T_{-b/2}\Xi = T_{b/2}\Xi$ or Ξ is singular at $-b/2$ and $b/2$, but either of the latter conditions always holds because Ξ is symmetric. Hence D is singular at $(-\omega/2, 0)$ if and only if $b/2 \in \Xi$.

Therefore, it makes sense to define a locus $2_B(\Xi) = \{(\tau', 2x) \mid x \in \Xi\}$ of rank 1 degenerations of abelian varieties in the fiber $p^{-1}(B, \Xi)$. In [14], it is shown that

$$\overline{\theta_{null}} \cap \partial\mathcal{A}_g = \left(\bigcup_{(B,\Xi)\in\mathcal{A}_{g-1}} 2_B(\Xi)\right) \cup p^{-1}(\theta_{null,g-1}),$$

which describes the closure of θ_{null} completely in terms of abelian varieties of rank $g-1$, the usefulness of which is immediately apparent. We will use it to show the existence of a point in $(\overline{\theta_{null}} \setminus \overline{\theta_{null}^{g-1}}) \cap \partial\mathcal{A}_g$. From this it follows that $\theta_{null}^{g-1} \subsetneq \theta_{null}$, since if $\theta_{null}^{g-1} = \theta_{null}$ in $\mathcal{A}_g$, then surely their closures in the partial toroidal compactification $\mathcal{A}_g \cup \partial\mathcal{A}_g$ would coincide. This is a clear example of the use of a good compactification.

3.3 Theorem (Grushevsky-Salvati Manni)

In this final section, we combine the properties of the boundary $\partial\mathcal{A}_g$ from section 3.2 and the theta function η from sections 1.5 and 2.5 in the form of a proposition and a theorem, which is theorem 3 of [5].

Proposition 3.3.1. The codimension of the locus $\overline{\theta^h_{null}}$ in $\overline{\theta_{null}} \cap \partial\mathcal{A}_g$ is at most $(g-h)^2$ if it is nonempty.

Proof. Let $\theta(\tau', z)$ be the Riemann theta function of dimension $g-1$. Let

$$X = \left(\bigcup_{\substack{(B,\Xi)\in\mathcal{A}_{g-1},\\ \Xi \text{ smooth}}} 2_B(\Xi) \right)$$

and consider $\overline{\theta^h_{null}} \cap X$. As $2_B(\Xi)$ has codimension 1 in the fiber over (B, Ξ), and the theta divisor of a generic abelian variety in $\mathcal{A}_{g-1}$ is smooth, it follows that $\dim X = \dim \partial\mathcal{A}_g - 1 = \dim(\overline{\theta_{null}} \cap \partial\mathcal{A}_g)$, so it suffices to prove

$$\operatorname{codim}\left(\overline{\theta^h_{null}} \cap X, X\right) \leq (g-h)^2.$$

Let $H(\tau', z')$ be the Hessian of $\theta(\tau', z')$ and $d\theta(\tau', z')$ the column vector consisting of its first derivatives. Then the Hessian of $\tilde{\theta}((\tau', \omega), (z', z))$ is given by

$$e^{2\pi i z} \begin{pmatrix} H(\tau', z'+\omega) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} H(\tau', z') & d\theta(\tau', z'+\omega) \\ {}^t d\theta(\tau', z'+\omega) & 0 \end{pmatrix},$$

which evaluated in $(-\omega/2, 0)$ becomes

$$\begin{pmatrix} \mathbf{2} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} H(\tau', \omega/2) & d\theta(\tau', \omega/2) \\ {}^t d\theta(\tau', \omega/2) & 0 \end{pmatrix},$$

using the fact that the Hessians are equal at $-\omega/2$ and $\omega/2$ as Ξ is symmetric. Since we are only interested in the rank of this matrix and the first term of the product is obviously invertible, we define

$$M(\tau', \omega) = \begin{pmatrix} H(\tau', \omega/2) & d\theta(\tau', \omega/2) \\ {}^t d\theta(\tau', \omega/2) & 0 \end{pmatrix}.$$

Hence $(\overline{G}, D)$ given by $(\tau', \omega) \in 2_B(\Xi) \subset X$ is in the closure of θ^h_{null} if and only if the rank of $M(\tau', \omega)$ is is at most h. Thus far, we did not use the fact that Ξ is smooth. The Hessian $H(\tau', \omega/2)$ defines the tangent map $d\Gamma_x : T_x\Xi \to (T_x\Xi)^\vee$ of the Gauss map $\Gamma : \Xi \to \mathbb{P}^{g-2}$, where x is the image of $\omega/2$ in Ξ. The ranks of $M = M(\tau', \omega/2)$ and $d\Gamma_x$ are related.

Let $H = H(\tau', \omega/2)$ and $d\theta = d\theta(\tau', \omega/2)$. Furthermore, let $(z', z) \in \mathbb{C}^g$ with $z' \in \mathbb{C}^{g-1}$ and $z \in \mathbb{C}$. Then it holds that

$$\begin{aligned}(z, z') \in \ker M &\iff Hz' = -d\theta z \text{ and } {}^t d\theta z' = 0\\ &\iff z' \in T_x\Xi \text{ and } Hz' \in \mathbb{C} \cdot d\theta\\ &\iff z' \in \ker d\Gamma_x.\end{aligned}$$

It follows that $\dim \ker M = \dim \ker d\Gamma_x$, and so $\operatorname{rk}(M) \leq h$ if and only if $\operatorname{rk}(d\Gamma_x) \leq h - 1$, or equivalently if and only if x is in the degeneracy locus $R_{h-1}(d\Gamma)$ as in definition 2.3.3. Thus we have

$$\overline{\theta^h_{null}} \cap 2_B(\Xi) = 2_B(R_{h-1}(d\Gamma)),$$

seen as a subscheme of $\Xi/\operatorname{Aut}(B, \Xi)$ in the same way as $2_B(\Xi)$. According to theorem 8.2 of [3], each component of $R_{h-1}(d\Gamma)$ has codimension at most $(g-h)^2$ in Ξ. Hence it follows that

$$\operatorname{codim}\left(\overline{\theta^h_{null}} \cap 2_B(\Xi), \Xi/\operatorname{Aut}(B, \Xi)\right) \leq (g-h)^2.$$

This ultimately yields that $\overline{\theta^h_{null}} \cap X$ has codimension at most $(g-h)^2$ in X, as claimed. □

We have an easy corollary. Let X be as in the proof of proposition 3.3.1 and $2_B(\operatorname{Sing}(\Xi)) = \{(\tau', 2x) \mid x \in \operatorname{Sing}(\Xi)\}$, where $\operatorname{Sing}(\Xi)$ is the singular locus of the theta divisor Ξ of the ppav B with period matrix τ'. Note that $\operatorname{codim}(2_B(\operatorname{Sing}(\Xi)), 2_B(\Xi)) = \operatorname{codim}(\operatorname{Sing}(\Xi), \Xi)$.

Corollary 3.3.2. Assume that $g \geq 2$. Then

$$\overline{\theta^1_{null}} \cap \partial\mathcal{A}_g \subset \left(\bigcup_{\substack{(B,\Xi)\in\mathcal{A}_{g-1},\\ \Xi \text{ singular}}} 2_B(\operatorname{Sing}(\Xi)) \right).$$

Proof. Let $(\tau', \omega) \in 2_B(\Xi)$, such that Ξ is smooth at the image $b/2$ of $\omega/2$. Then it holds that $d\theta(\tau', \omega/2) \in \mathbb{C}^{g-1}$ is a nonzero vector. It follows from elementary linear algebra that $\operatorname{rk}(M(\tau', \omega)) \geq 2$. Hence

$$(\tau', \omega) \notin \overline{\theta^1_{null}}.$$

Thus, letting $(\tau', \omega) \in \overline{\theta^1_{null}} \cap \partial\mathcal{A}_g$, it follows that Ξ is singular at the image $b/2$ of $\omega/2$. Hence $(\tau', \omega) \in 2_B(\operatorname{Sing}(\Xi))$, as was to be shown. □

While not particularly thrilling for large g, this corollary might be interesting for small g. The main result we want to prove is the following.

Theorem 3.3.3 (Grushevsky-Salvati Manni)**.**

$$\theta_{null}^{g-1} \subsetneq \theta_{null}.$$

Proof. From the proof of proposition 3.3.1, we have

$$\overline{\theta_{null}^{h}} \cap 2_B(\Xi) = R_{h-1}(d\Gamma)$$

for generic $(B, \Xi) \in \mathcal{A}_{g-1}$, where $R_{h-1}(d\Gamma)$ is the rank $h-1$ degeneracy subscheme of $d\Gamma$, with $\Gamma : \Xi \to \mathbb{P}^{g-2}$ the Gauss map. For $h = g-1$, we get $R_{g-2}(d\Gamma)$, which is the ramification locus of Γ. By corollary 9.11 in [8], the Gauss map is generically finite and dominant, so in particular it does not ramify everywhere. Hence

$$\overline{\theta_{null}^{g-1}} \cap X \subsetneq X \qquad \text{and} \qquad \overline{\theta_{null}^{g-1}} \subsetneq \overline{\theta_{null}}.$$

The desired result follows from the observations at the end of section 3.2. □

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