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Amenability in Positivity

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Dedicated to my grandfather W. H. Roelands,
in loving memory.

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1 Introduction and overview

This thesis is about amenability. There are several contexts in which this property can be used and we will discuss what these are and how they relate to one another and, simultaneously, give an overview of what is done in this thesis. The notion of amenable groups arose in the first half of the 20th century in the context of the famous Banach-Tarski paradox:

Theorem 1.1 (Banach-Tarski) *Every closed ball in* $\mathbb{R}^3$ *is paradoxical.* ∎

In words, this means that every closed ball in $\mathbb{R}^3$ can be divided into a finite amount of disjoint subsets which can be put together again and yield two identical copies of the original ball. Reassembling these disjoint subsets is done by letting the special orthogonal group $SO(3)$ of $\mathbb{R}^3$ act on them and the key idea in the proof of this paradox lies in the observation that this group has a subgroup which is isomorphic to the free group $\mathbb{F}_2$ with two generators.

Definition: Let G be a group. Then G is **paradoxical**, or G allows a **paradoxical decomposition** if there are pairwise disjoint subsets $A_1, \dots, A_n, B_1, \dots, B_m$ in G along with $g_1, \dots, g_n, h_1, \dots, h_m \in G$ such that

$$G = \bigcup_{k=1}^{n} g_k \cdot A_k \quad \text{and} \quad G = \bigcup_{k=1}^{m} h_k \cdot B_k.$$

This free group is paradoxical when acting on itself, for if a and b are the generators of $\mathbb{F}_2$ and we define

$$W(y) := \{x \in \mathbb{F}_2 : x \text{ starts with } y\} \qquad (y = a, a^{-1}, b, b^{-1}),$$

then we can write $\mathbb{F}_2$ as the disjoint union

$$\mathbb{F}_2 = \{e_{\mathbb{F}_2}\} \cup W(a) \cup W(b) \cup W(a^{-1}) \cup W(b^{-1})$$

and if $x \in \mathbb{F}_2 \setminus W(a)$, then $a^{-1}x \in W(a^{-1})$, so $x \in aW(a^{-1})$; hence $\mathbb{F}_2 = W(a) \cup aW(a^{-1})$. Similarly, one finds that $\mathbb{F}_2 = W(b) \cup bW(b^{-1})$. Now the class of groups that do not allow these paradoxical decompositions were characterized in order to omit the behavior stated in the Banach-Tarski paradox:

Theorem 1.2 *A discrete group G is not paradoxical if and only if there exists a functional* $m \in \ell^\infty(G)^*$ *that satisfies*

i) $m(1) = \|m\| = 1$*;*

ii) $m(\delta_g * \phi) = m(\phi) \qquad (g \in G, \phi \in \ell^\infty(G))$*.* ∎

Here the function $\delta_g * \phi$ is just the translation of ϕ over g and is defined by

$$\delta_g * \phi(h) := \phi(g^{-1}h)$$

and the notation comes from a convolution of the discrete measures on $\ell^\infty(G)$. This characterization can be found in [15, Cor. 0.2.11]. The functional $m \in \ell^\infty(G)$ is called a **left invariant mean** on G and a discrete group G is said to be **amenable**, apparently as a pun, if such a mean exists on G. As an example, all finite groups are amenable, for the left invariant in question mean would be defined by

$$m(f) := \frac{1}{|G|} \sum_{g \in G} f(g).$$

Also, all compact groups are amenable as we can define the left invariant mean again in this case:

$$m(f) := \int_G f(g) dm_G(g).$$

Finally, all abelian groups are amenable as well. This is stated in [15, Ex. 1.15].

From a functional analytic point of view, we would like to extend the notion of amenability to general locally compact groups and this is done in section 2.1.4. A theorem of Johnson's characterizes the amenability of locally compact groups G in terms of the triviality of a specific cohomology group on the Banach space of integrable functions on G.

In a more general setting, a **cochain complex** is a sequence $(A_n, d_n)_{n \geq 0}$ of modules A_n together with homomorphisms

$$d_n : A_n \to A_{n+1}$$

such that $d_n \circ d_{n-1} = 0$ for all $n \geq 1$ and is denoted by

$$\{0\} \to A_0 \overset{d_0}{\to} A_1 \overset{d_1}{\to} \cdots \to A_n \overset{d_n}{\to} A_{n+1} \overset{d_{n+1}}{\to} \cdots$$

Since we have that $\mathrm{Im}(d_{n-1}) \subset \ker(d_n)$, we can consider the quotient modules

$$H(A_n) := \ker(d_n)/\mathrm{Im}(d_{n-1}) \qquad (n \geq 1)$$

which are referred to as the **cohomology groups** of A_n. These types of complexes can be constructed for Banach spaces as well, as is done in section 3.1.

Under pointwise operations, the real valued integrable functions on G allow a vector space ordering and the first goal of this thesis will be to construct a characterization of amenable locally compact groups, similar to Johnson's theorem, purely in an ordered context.

Johnson's theorem also induces the notion of an amenable Banach algebra. A remarkable result, proven in section 3.1.2, is that a Banach algebra which is amenable yields trivial cohomology groups for all $n \geq 1$. The proof uses tensor products of Banach spaces and section 3.1.1 is, partially as a general reminder, devoted to providing sufficient background knowledge about them. The second goal of this thesis is to investigate under which conditions an ordered Banach algebra yields trivial cohomology groups for all $n \geq 1$ as well.

As was mentioned above, the vector space order on the real valued integrable functions on G, in addition, turns this space into a Banach lattice algebra and the third goal of this thesis is to construct a cochain complex specifically for Banach lattice algebras such that the corresponding cohomology groups propose an alternative notion of amenability and have similar properties as in the case of general Banach algebras. In the first part of section 3.3 this construction is described. As for the triviality of the cohomology groups for all $n \geq 1$ in this respect, along the lines of the previous findings concerning Banach algebras, we shall consider tensor products of Banach lattices which will be studied, in depth, in section 3.3.1. The main result here will be that under the assumption that we have an alternatively amenable Banach lattice algebra, we obtain trivial cohomology groups for all $n \geq 1$ when using these tensor products of Banach lattices.

Just in order to clarify things, all vector spaces considered in section 2.1 until 3.2 are regarded as complex and in the remainder, the vector spaces will be real.

"A mathematician is a device for turning coffee into theorems."

- *Paul Erdős* -

2 Characterizations of amenable groups

The main goal in this chapter will be to extend the notion of amenable discrete groups to general locally compact groups and to characterize them using Johnson's theorem. Finally, we will place this result in an ordered context.

2.1 Integration theory on locally compact groups

In this section we will study the theory of integrable functions defined on locally compact groups, in terms of convolution products, dual spaces and ideals. We start with some elementary properties of topological groups and will assume that all groups here are Hausdorff.

Lemma 2.1 *For a topological group G we have:*

i) every open subgroup of G is clopen;

ii) if A and B are compact sets in G, then so is AB.

Proof: If H is an open subgroup of G, then so are all its cosets gH, because the map $x \mapsto gx$ is a homeomorphism on G. If $g \in G \setminus H$, then for all $h \in H$ we have that $gh \in G \setminus H$ and since $e \in H$, we conclude that $G \setminus H = \bigcup_{g \in G \setminus H} gH$; hence $G \setminus H$ is open. Equivalently, we find that H is closed. As for the second property, since AB is the image of the compact set $A \times B$ under the continuous map $(g, h) \mapsto gh$, we conclude that AB must be compact too. ∎

Lemma 2.2 *Let G be a locally compact group. The there exists a subgroup H of G that is clopen and σ-compact.*

Proof: Since G is locally compact, there exists a compact neighborhood N of e. By the continuity of the map $g \mapsto g^{-1}$, we find that N^{-1} is a compact neighborhood of e as well. The intersection $U := N \cap N^{-1}$ now yields a symmetric and compact neighborhood of e. For each $n \in \mathbb{N}_+$ we define $U_n := \prod_{k=1}^{n} U$ and let $H := \bigcup_{n=1}^{\infty} U_n$. It is a straightforward verification to see that H is a group. If $x \in H$, then there is a number $n \geq 1$ such that $x \in U_n$ and for an open set $V \subset U$ which contains e, we have that $x \in Vx \subset U_{n+1} \subset H$. Now V is also open since $g \mapsto gx$ is a homeomorphism; hence H is open and Lemma 2.1 now implies that H is clopen. Moreover, each U_n is also compact by Lemma 2.1 and we conclude that H is σ-compact. ∎

For a locally compact group G, we define $M(G)$ to be the space of finite and regular Borel measures on G. By the regularity of μ we mean that its variation $|\mu|$, which is defined by

$$|\mu|(E) := \sup \left\{ \sum_{k=1}^{n} |\mu(E_k)| : n \in \mathbb{N}_+,\ E_k \in \mathscr{B}(G),\ E = \biguplus_{k=1}^{n} E_k \right\} \qquad (E \in \mathscr{B}(G)),$$

is regular, where $\mathscr{B}(G)$ denotes the Borel sets of G. By the Riesz representation theorem, the space $M(G)$ can isometrically be identified with $C_0(G)^*$, which is a Banach space, with respect to the norm

$$\|\mu\| := \sup \left\{ \sum_{k=1}^{n} |\mu(E_k)| : n \in \mathbb{N}_+,\ E_k \in \mathscr{B}(G),\ G = \biguplus_{k=1}^{n} E_k \right\} \qquad (\mu \in M(G)).$$

For $\mu, \nu \in M(G)$ the convolution product $\mu * \nu$ is defined by

$$[\mu * \nu](f) := \int_G \left(\int_G f(gh) d\mu(g) \right) d\nu(h) \qquad (f \in C_0(G))$$

and is again a member of $M(G)$, since this defines an element of $C_0(G)^*$. By Fubini's theorem, the order of integration can be interchanged and this implies that $M(G)$ is an algebra. Moreover, by considering step functions, we obtain the inequality

$$\begin{aligned} |[\mu * \nu](f)| &= \left| \int_G \left(\int_G f(gh) d\mu(g) \right) d\nu(h) \right| \leq \int_G \left| \int_G f(gh) d\mu(g) \right| d|\nu|(h) \\ &\leq \int_G \left(\int_G |f(gh)| d|\mu|(g) \right) d|\nu|(h) \leq \|f\| |\mu|(G) |\nu|(G) \\ &= \|f\| \|\mu\| \|\nu\| \qquad (f \in C_0(G)), \end{aligned}$$

so $\|\mu * \nu\| \leq \|\mu\| \|\nu\|$ for all $\mu, \nu \in M(G)$. Finally, since $\mu * \delta_e = \mu$ and $\delta_e * \mu = \mu$ for all $\mu \in M(G)$, we conclude that $M(G)$ is a unital Banach algebra.

Next, we will use a fundamental measure theoretic result in order to identify $L^1(G)$ with a closed ideal of $M(G)$.

2.1.1 Integrable functions on G

Theorem 2.3 *Let G be locally compact group. Then there is a non-zero, positive regular Borel measure m_G on G, a so called **left Haar measure**, which is left invariant, that is, we have the equality $m_G(gE) = m_G(E)$ for all $g \in G$ and all Borel sets $E \subset G$. Furthermore, if $\tilde{m}_G$ is another non-zero, positive Radon measure on G that is left invariant, then there exists a unique positive constant α such that $\tilde{m}_G = \alpha\, m_G$.*

Now, having such a left Haar measure m_G, it is not generally true that m_G is σ-finite on G. For example, when $G = (\mathbb{R}, +)$ endowed with the discrete topology. Then the left Haar measure is just the counting measure, which clearly is not σ-finite, since $\mathbb{R}$ is uncountable. In order to show that we can define a convolution on $L^1(G)$, it is important to use Fubini's theorem, but this theorem requires the given measure to be σ-finite. However, as we shall see, a slight adjustment will do, so we will continue by deriving some properties of m_G that will prove useful in this respect.

Lemma 2.4 *If m_G is a left Haar measure on a locally compact group G, then $m_G(U) > 0$ for every non-empty open set U.*

Proof: Let U be a non-empty open set in G and suppose that $m_G(U) = 0$. It follows that $m_G(gU) = 0$ for all $g \in G$ and if K is a compact set in G, then $K \subset \bigcup_{g \in G} gU$ is an open cover, so there is a finite subcover $K \subset \bigcup_{k=1}^{n} g_k U$ which yields the inequality

$$m_G(K) \leq \sum_{k=1}^{n} m_G(g_k U) = 0,$$

so $m_G(K) = 0$ for all compact sets in G. But since m_G is inner regular, we find that $m_G(G) = 0$ and this contradicts the fact that m_G is non-zero; hence $m_G(U) > 0$. ∎

In the case that G is not σ-compact, by Lemma 2.2 there is a clopen subgroup H of G that is σ-finite and we let Γ be a subset of G that contains exactly one element of every coset gH, where g runs through G. We now have that G is the disjoint union of the sets γH, with $\gamma \in \Gamma$ and m_G restricted to the Borel sets of H also is a left Haar measure on H. For if A is a Borel set of H, then $A = B \cap H$ for some Borel set B of G and the regularity of this restriction follows from the fact that H is clopen and intersecting the open and compact sets with H. The left invariance of this restriction is directly inherited from m_G. Moreover, this restriction determines m_G completely, since they agree on the Borel subsets of the cosets γH for all $\gamma \in \Gamma$ by its left invariance and the remainder of this assertion is shown in the following Lemma:

Lemma 2.5 *Let G be a locally compact group with a left Haar measure m_G. If $E \subset G$ is a Borel set and $E \subset \bigcup_{k=1}^{\infty} \gamma_k H$ for some countable set $(\gamma_k)_{k\geq 1} \subset \Gamma$, then*

$$m_G(E) = \sum_{k=1}^{\infty} m_G(E \cap \gamma_k H).$$

Also, if $E \cap \gamma H \neq \emptyset$ for uncountably many $\gamma \in \Gamma$, it follows that $m_G(E) = \infty$.

Proof: The first statement follows immediately from the countable additivity of m_G. By the outer regularity of m_G, we may assume that E is open in G to verify the second statement. By Lemma 2.4 we have that $m_G(E \cap \gamma H) > 0$ whenever $E \cap \gamma H \neq \emptyset$. Define the sets Θ_n for all $n \in \mathbb{N}_+$ by

$$\Theta_n := \left\{ \gamma \in \Gamma : m_G(E \cap \gamma H) > \tfrac{1}{n} \right\}.$$

If Θ_n would be countable for all $n \geq 1$, then $\bigcup_{n=1}^{\infty} \Theta_n$ is also countable, which is impossible, so there exists a number $\varepsilon > 0$ such that $m_G(E \cap \gamma H) > \varepsilon$ for all $\gamma \in \tilde{\Gamma} \subset \Gamma$ with $\tilde{\Gamma}$ an uncountable subset. If we now take $(\gamma_k)_{k\geq 1} \subset \tilde{\Gamma}$, we find that

$$m_G(E) \geq m_G\left(\bigcup_{k=1}^{\infty} E \cap \gamma_k H \right) = \sum_{k=1}^{\infty} m_G(E \cap \gamma_k H) = \infty. \qquad \blacksquare$$

Returning to the Banach space $L^1(G)$ of measurable functions that are integrable with respect to m_G, let $f \in L^1(G)$. There exists a sequence of simple functions $(\phi_n)_{n\geq 1}$ such that $0 \leq \phi_1 \leq \phi_2 \leq \ldots |f|$ and $\phi_n \uparrow |f|$ pointwise, with the property

$$\begin{aligned} \int_G |f(g)| dm_G(g) &= \lim_{n\to\infty} \int_G \phi_n(g) dm_G(g) = \lim_{n\to\infty} \int_G \sum_{i=1}^{k_n} a_i \cdot \chi_{A_k}(g) dm_G(g) \\ &= \lim_{n\to\infty} \sum_{i=1}^{k_n} a_i \cdot m_G(A_k) \end{aligned}$$

where $m_G(A_k) < \infty$ and χ_{A_k} is the characteristic function on A_k for all k. For each $n \in \mathbb{N}_+$, define the sets

$$\Xi_n := \{g \in G : \phi_n(g) > 0\}.$$

By Lemma 2.5 we have the inclusion $\Xi_n \subset \bigcup_{j=1}^{k_n} \bigcup_{i=1}^{\infty} \gamma_i^{(j)} H = \bigcup_{i=1}^{\infty} \gamma_i'^{(n)} H$ and if $g \in G$ such that $|f(g)| > 0$, then there is a number $n \geq 1$ such that $\phi_n(g) > 0$, so we conclude that

$$\{g \in G : |f(g)| > 0\} = \bigcup_{n=1}^{\infty} \Xi_n \subset \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} \gamma_i'^{(n)} H = \bigcup_{n=1}^{\infty} \tilde{\gamma}_n H;$$

hence f vanishes outside some σ-compact set.

For $h \in G$ and $0 \leq f \in L^1(G)$ with a corresponding sequence $(\phi_n)_{n\geq 1}$ of simple functions that has the properties mentioned above, we have that $(\psi_n)_{n\geq 1}$, where ψ_n is defined by $\psi_n(g) := \phi_n(h^{-1}g)$, yields a sequence of simple functions that satisfies the inequalities $0 \leq \psi_n \leq \psi_{n+1}$ for all $n \geq 1$ and converges pointwise to $\tilde{f}(g) := f(h^{-1}g)$. By the left invariance of m_G, for a simple function ϕ_n we have the equality

$$\begin{aligned}\int_G \phi_n(g)dm_G(g) &= \int_G \sum_{k=1}^m a_k \cdot \chi_{A_k}(g)dm_G(g) = \sum_{k=1}^m a_k \cdot m_G(A_k) = \sum_{k=1}^m a_k \cdot m_G(hA_k) \\ &= \int_G \sum_{k=1}^m a_k \cdot \chi_{hA_k}(g)dm_G(g) = \int_G \sum_{k=1}^m a_k \cdot \chi_{A_k}(h^{-1}g)dm_G(g) \\ &= \int_G \psi_n(g)dm_G(g).\end{aligned}$$

The monotone convergence theorem now implies that

$$\begin{aligned}\int_G f(g)dm_G(g) &= \lim_{n\to\infty} \int_G \phi_n(g)dm_G(g) = \lim_{n\to\infty} \int_G \psi_n(g)dm_G(g) = \int_G \tilde{f}(g)dm_G(g) \\ &= \int_G f(h^{-1}g)dm_G(g). \end{aligned} \tag{1}$$

Since every $f \in L^1(G)$ can be written as

$$f = f_1^+ - f_1^- + i(f_2^+ - f_2^-) \qquad (0 \leq f_1, f_2 \in L^1_{\mathbb{R}}(G))$$

and each of these components satisfies this translation invariance property, it also holds for general $f \in L^1(G)$.

For the functions $f_1, f_2 \in L^1(G)$ define the function

$$\Psi(h, g) := f_1(h)f_2(h^{-1}g).$$

Since

$$\Omega_1 := \{g \in G : f_1(g) \neq 0\} \quad \text{and} \quad \Omega_2 := \{g \in G : f_2(g) \neq 0\}$$

are both contained in some σ-compact subset of G by our previous findings, the function Ψ vanishes outside $\Omega_1 \times \Omega_1\Omega_2$ because

$$(\Omega_1 \times \Omega_1\Omega_2)^c = [\Omega_1 \times (\Omega_1\Omega_2)^c] \cup [\Omega_1^c \times \Omega_1\Omega_2] \cup [\Omega_1^c \times (\Omega_1\Omega_2)^c],$$

so $h \notin \Omega_1$ implies that $f_1(h) = 0$ and $h \in \Omega_1, g \notin \Omega_1\Omega_2$ implies that $h^{-1}g \notin \Omega_2$; hence $f_2(h^{-1}g) = 0$. Furthermore, from the inclusions

$$\Omega_1 \subset \bigcup_{n=1}^{\infty} \gamma_n H \quad \text{and} \quad \Omega_2 \subset \bigcup_{k=1}^{\infty} \tilde{\gamma}_k H$$

it follows that

$$\Omega_1\Omega_2 \subset \bigcup_{n,k=1}^{\infty} \gamma_n H \cdot \tilde{\gamma}_k H$$

and by Lemma 2.1 we conclude that $\Omega_1\Omega_2$ is σ-compact; hence so is $\Omega_1 \times \Omega_1\Omega_2$. Fubini's theorem and the translation invariance now yield

$$\begin{aligned}\int_G\left(\int_G |f_1(h)f_2(h^{-1}g)|dm_G(h)\right)dm_G(g) &= \int_{\Omega_1\Omega_2}\left(\int_{\Omega_1} |f_1(h)f_2(h^{-1}g)|dm_G(h)\right)dm_G(g)\\ &= \int_{\Omega_1}\left(\int_{\Omega_1\Omega_2} |f_1(h)f_2(h^{-1}g)|dm_G(g)\right)dm_G(h)\\ &= \int_{\Omega_1}|f_1(h)|\left(\int_{\Omega_1\Omega_2} |f_2(h^{-1}g)|dm_G(g)\right)dm_G(h)\\ &= \int_{G}|f_1(h)|\left(\int_{G} |f_2(h^{-1}g)|dm_G(g)\right)dm_G(h)\\ &= \int_{G}|f_1(h)|\left(\int_{G} |f_2(g)|dm_G(g)\right)dm_G(h)\\ &= \|f_1\|\|f_2\|,\end{aligned}$$

which implies that the function

$$[f_1 * f_2](g) := \int_G f_1(h)f_2(h^{-1}g)dm_G(h) \qquad \text{(for almost all } g \in G)$$

is well defined and $f_1 * f_2 \in L^1(G)$, so we conclude that $L^1(G)$ is a Banach algebra.

For $f \in L^1(G)$ we can define a map $\Psi : C_0(G) \to \mathbb{C}$ by

$$\Psi(\phi) := \int_G \phi(g)f(g)dm_G(g).$$

Clearly, this defines a bounded linear functional on $C_0(G)$ and by the Riesz representation theorem there is a regular Borel measure μ on G such that

$$\int_G \phi(g)f(g)dm_G(g) = \int_G \phi(g)d\mu(g) \qquad (\phi \in C_0(G)).$$

We want to show that $\mu = fdm_G$ on $\mathscr{B}(G)$, where fdm_G is defined by

$$fdm_G(E) := \int_E f(g)dm_G(g) \qquad (E \in \mathscr{B}(G)).$$

In order to do so, we will first assume that $f \geq 0$. In this case μ is positive, so $|\mu| = \mu$, and define the set $\Omega := \{g \in G : f(g) > 0\}$. Let U be an open set in G. It is a well known fact that we have

$$\mu(U) \leq \int_G \chi_U(g)f(g)dm_G(g) = \int_U f(g)dm_G(g),$$

and, as we saw, the set Ω is contained in some countable union of compact sets in G, so we have that $\Omega = \biguplus_{k=1}^{\infty} E_k$ where $m_G(E_k) < \infty$ for all $k \geq 1$. Let $j \in \mathbb{N}_+$. For each $k \geq 1$ there exists a compact set F_k^j in G such that $F_k^j \subseteq E_k \cap U$ and

$$m_G((E_k \cap U) \setminus F_k^j) \leq 2^{-k}j^{-1}$$

by the inner regularity of m_G. It follows that for $F^j := \bigcup_{k=1}^{\infty} F_k^j$ we can write the intersection $\Omega \cap U$ as

$$\Omega \cap U = F^j \cup ((\Omega \cap U) \setminus F^j).$$

By construction, we have that $m_G((\Omega\cap U)\setminus F^j) \le j^{-1}$. Doing this for all $j \ge 1$, we obtain a countable collection $(F^j)_{j=1}^{\infty}$ of compact sets, so we may write

$$F := \bigcup_{j=1}^{\infty} F^j = \bigcup_{k=1}^{\infty} \tilde{F}_k.$$

Letting $\Xi_n := \bigcup_{k=1}^{n} \tilde{F}_k$, we find that $(\Xi_n)_{n\ge 1}$ is an increasing sequence of compacta. Furthermore, we have the inequality $(\Omega\cap U)\setminus F \subset (\Omega\cap U)\setminus F^j$ for all $j \ge 1$, so

$$m_G\left((\Omega\cap U)\setminus F\right) \le m_G\left((\Omega\cap U)\setminus F^j\right) \le j^{-1} \qquad (j \ge 1);$$

hence $m_G((\Omega\cap U)\setminus F) = 0$ and we find that we can write

$$\Omega\cap U = \left(\bigcup_{n=1}^{\infty} \Xi_n\right) \cup ((\Omega\cap U)\setminus F)$$

as the union of an increasing sequence of compacta together with a m_G- negligible set. Using Urysohn's lemma, we have functions $\phi_n \in C_c(G)$ with $0 \le \phi_n \le 1$ such that $\phi_n|_{\Xi_n} = 1$ and $\mathrm{supp}(\phi_n) \subseteq U$, where $n \in \mathbb{N}_+$. For all $g \in \Omega^c \cup U^c$ we have

$$\lim_{n\to\infty} \phi_n(g)f(g) = 0 = \chi_U(g)f(g)$$

and for all $g \in F$ we have

$$\lim_{n\to\infty} \phi_n(g)f(g) = f(g) = \chi_U(g)f(g).$$

Moreover, for $g \in (\Omega\cap U)\setminus F$ it may happen that $\phi_n(g)f(g) \not\to \chi_U(g)f(g)$, but this set is m_G- negligible, so we have the m_G- almost everywhere pointwise convergence $\phi_n f \to \chi_U f$. By the dominated convergence theorem we now have

$$\mu(U) \ge \lim_{n\to\infty} \int_G \phi_n(g)f(g)dm_G(g) = \int_{\Omega\cap U} \chi_U(g)f(g)dm_G(g) = \int_U f(g)dm_G(g),$$

so μ and $f dm_G$ coincide on all open sets in G. From the proof of [11, Thm. 14.17], we conclude that $f dm_G$ is a finite regular Borel measure, because μ is.

For $f \in L^1_{\mathbb{R}}(G)$, write $f = f^+ - f^-$ and let $\Psi : C_0(G;\mathbb{R}) \to \mathbb{R}$ be the bounded linear functional defined by

$$\Psi(\phi) := \int_G \phi(g)f(g)dm_G(g) \qquad (\phi \in C_0(G;\mathbb{R})).$$

Since $C_0(G;\mathbb{R})$ is a Banach lattice and $\mathbb{R}$ is Dedekind complete, we have that

$$\Psi^+(\phi) = \sup\left\{\int_G \varphi(g)f(g)dm_G(g) : 0 \le \varphi \le \phi,\ \varphi \in C_0(G;\mathbb{R})\right\} \qquad (0 \le \phi \in C_0(G;\mathbb{R})).$$

It is clear that we have the inequality

$$\Psi^+(\phi) \le \int_G \phi(g)f^+(g)dm_G(g)$$

and for $\Omega_+ := \{g \in G : f(g) > 0\}$, using a similar argument, we have an increasing sequence $(F_n)_{n\geq 1}$ of compacta such that

$$\Omega_+ = \left(\bigcup_{n=1}^{\infty} F_n\right) \cup N$$

with $m_G(N) = 0$. Again by Urysohn's lemma, let $\phi_n \in C_c(G;\mathbb{R})$ with $0 \leq \phi_n \leq 1$ be such that $\phi_n|_{F_n} = 1$ and $\text{supp}(\phi_n) \subseteq \overline{\Omega}_-^c$ where $n \geq 1$. It follows that we have the m_G-almost everywhere pointwise convergence $\phi_n \phi f \to \phi f^+$, so the dominated convergence theorem now implies that

$$\Psi^+(\phi) \geq \lim_{n\to\infty} \int_G \phi_n(g)\phi(g)f(g)dm_G(g) = \int_G \phi(g)f^+(g)dm_G(g).$$

Analogously, we find that

$$\Psi^-(\phi) = \int_G \phi(g)f^-(g)dm_G(g) \qquad (\phi \in C_0(G;\mathbb{R})).$$

Let μ_1 and μ_2 be the finite regular Borel measures associated with Ψ^+ and Ψ^-, respectively. Now if μ is the regular Borel measure associated with $\Psi = \Psi^+ - \Psi^-$, then it follows that $\mu = \mu_1 - \mu_2$ and our previous findings imply that $\mu = f^+ dm_G - f^- dm_G$. For any Borel set A we have

$$\begin{aligned} f^+dm_G(A) - f^-dm_G(A) &= \int_A f^+(g)dm_G(g) - \int_A f^-(g)dm_G(g) = \int_A f(g)dm_G(g) \\ &= fdm_G(A), \end{aligned}$$

so we conclude that $fdm_G = \mu$. Finally, for general $f \in L^1(G)$ we have $f = \Re(f) + i\Im(f)$ and the maps

$$\Psi_1 : \phi \mapsto \int_G \phi(g)\Re(f)dm_G(g) \qquad \text{and} \qquad \Psi_2 : \phi \mapsto \int_G \phi(g)\Im(f)dm_G(g)$$

satisfy the identity $\Psi_1 + i\Psi_2 = \Psi$, so $\Re(f)dm_G + i\Im(f)dm_G = \mu$ and analogously, we conclude that $fdm_G = \mu$.

These results allow us to consider the linear map $\Phi : L^1(G) \to M(G)$ which is defined by $f \mapsto fdm_G$. We claim that Φ is an isometry and to prove this assertion, we need a lemma.

Lemma 2.6 *Let μ be a positive regular Borel measure on G. Then $C_c(G)$ is dense in $L^1(G,\mu)$ with respect to the L^1-norm.*

Proof: Let $E \subset \mathscr{B}(G)$ with $\mu(E) < \infty$ and consider the function $\chi_E \in L^1(G,\mu)$. Let $\varepsilon > 0$. By the regularity of μ there exists an open set U and a compact set F such that $F \subset A \subset U$ with $\mu(U \setminus F) < \varepsilon$. Urysohn's Lemma implies that there exists a function $\phi \in C_c(G)$ with $0 \leq \phi \leq 1$ such that $\phi|_F = 1$ and $\text{supp}(\phi) \subset U$. We now find that

$$\int_G |(\chi_E - \phi)(g)|d\mu(g) = \int_{U\setminus F} |(\chi_E - \phi)(g)|d\mu(g) \leq \mu(U \setminus F) < \varepsilon,$$

so all characteristic functions can be approximated in $L^1(G,\mu)$ by a function in $C_c(G)$. If we now consider a simple function $\sum_{k=1}^m a_k \cdot \chi_{A_k}$, then there exists $\phi_k \in C_c(G)$ such that $\phi_k|_{F_k} = 1$, $\mathrm{supp}(\phi_k) \subset U_k$ and $F_k \subset A_k \subset U_k$ with

$$\mu(U_k \setminus F_k) < \frac{\varepsilon}{m \cdot \max_{1\le k\le m} |a_k|}$$

for all $1 \le k \le m$. This implies that

$$\begin{aligned}\int_G \left| \sum_{k=1}^m a_k \cdot \chi_{A_k}(g) - \sum_{k=1}^m a_k \cdot \phi_k(g) \right| d\mu(g) &\le \max_{1\le k\le m} |a_k| \sum_{k=1}^m \int_{U_k \setminus F_k} |(\chi_{A_k} - \phi_k)(g)| d\mu(g) \\ &< \varepsilon,\end{aligned}$$

so $C_c(G)$ is dense in the set of all simple functions in $L^1(G,\mu)$. Since for every $f \in L^1(G,\mu)$ there exists a sequence of simple functions $(\phi_n)_{n\ge1}$ such that

$$\lim_{n\to\infty} \int_G |(f - \phi_n)(g)| d\mu(g) = 0,$$

the set of all such simple functions is dense in $L^1(G,\mu)$ and we conclude that $C_c(G)$ must also be dense in $L^1(G,\mu)$. ∎

Now for $0 \le \phi \in C_0(G)$ we have the inequality

$$|\Psi|(\phi) \le \int_G \phi(g)|f(g)| dm_G(g)$$

and if we define the function $\tilde{f} \in L^1(G, |f| dm_G)$ by

$$\tilde{f}(g) := \begin{cases} \frac{\overline{f(g)}}{|f(g)|} & \text{if } f(g) \neq 0, \\ 0 & \text{if } f(g) = 0, \end{cases}$$

then Lemma 2.6 implies that there exists a sequence $(\phi_n)_{n\ge1}$ in $C_c(G)$ such that

$$\int_G |\phi_n(g) - \tilde{f}(g)||f(g)| dm_G(g) \to 0.$$

Let $\psi_n := \min\{\|\phi_n\|^{-1}, 1\}\phi_n$ for all $n \ge 1$. If $\tilde{f}(g) = 0$, then

$$|\psi_n(g) - \tilde{f}(g)| = |\psi_n(g)| \le |\phi_n(g)| = |\psi_n(g) - \tilde{f}(g)|$$

and if $|\tilde{f}(g)| = 1$, then for $\lambda \in \mathbb{T}$ such that $\lambda \tilde{f}(g) = 1$ we find

$$\begin{aligned}|\psi_n(g) - \tilde{f}(g)|^2 = |\lambda\psi_n(g) - 1|^2 &= |\psi_n(g)|^2 - 2\Re(\lambda\psi_n(g)) - 1 \\ &\le |\phi_n(g)|^2 - 2\Re(\lambda\phi_n(g)) - 1 \\ &= |\phi_n(g) - \tilde{f}(g)|^2,\end{aligned}$$

so $|\psi_n(g) - \tilde{f}(g)| \le |\phi_n(g) - \tilde{f}(g)|$ for all $g \in G$ and it follows that

$$\int_G |\psi_n(g) - \tilde{f}(g)||f(g)| dm_G(g) \to 0.$$

This implies that we have the convergence

$$\left| \int_G \phi(g)\psi_n(g)f(g)dm_G(g) - \int_G \phi(g)\tilde{f}(g)f(g)dm_G(g) \right| \to 0;$$

hence

$$|\Psi|(\phi) \geq \lim_{n\to\infty} \left| \int_G \phi(g)\psi_n(g)f(g)dm_G(g) \right| = \int_G \phi(g)|f(g)|dm_G(g),$$

from which we conclude that

$$|\Psi|(\phi) = \int_G \phi(g)|f(g)|dm_G(g) \qquad (\phi \in C_0(G)),$$

so $|fdm_G| = |f|dm_G$ by our previous findings. Since $\|fdm_G\| = |fdm_G|(G)$, we conclude that $\|fdm_G\| = \|f\|$ and it follows that Φ is an isometry.

If $N \in \mathscr{B}(G)$ is a m_G-null set, then we clearly have that

$$\int_N \phi(g)dm_G(g) = 0$$

for all simple functions ϕ, so $fdm_G(N) = 0$ and it follows that the range of Φ lies in the set of all measures in $M(G)$ that are absolutely continuous with respect to m_G. To show that the range of Φ is precisely this set, we need a classical measure theoretical result.

Theorem 2.7 (Radon-Nikodym) *If (X, Σ, ν) is a σ-finite measure space and μ is a measure on X which is absolutely continuous with respect to ν, then there is a unique function $f \in L^1(X, \nu)$ such that*

$$\mu(E) = \int_E f(x)d\nu(x)$$

for every $E \in \Sigma$.

Since G is not necessarily σ-finite, we need to do some more work. Let $\mu \in M(G)$ be absolutely continuous with respect to m_G. As we have shown above, by the regularity and finiteness of μ there are compact sets $(F_n)_{n\geq 1}$ such that $G = (\bigcup_{n=1}^{\infty} F_n) \cup N$ with $\mu(N) = 0$. Define the measure $\tilde{m}_G$ by

$$\tilde{m}_G(E) := m_G\left(E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)\right)$$

for all $E \in \mathscr{B}(G)$. Because $m_G(F_n) < \infty$ for all $n \geq 1$, we find that $\tilde{m}_G$ is σ-finite. If E is a Borel set such that $\tilde{m}_G(E) = 0$, then we clearly have that $\mu(E) = 0$, since $\mu(N \cap E) = 0$, so μ is absolutely continuous with respect to $\tilde{m}_G$. Theorem 2.7 now yields a function $f \in L^1(G, \tilde{m}_G)$ such that

$$\mu(E) = \int_E f(g)d\tilde{m}_G(g) \qquad (E \in \mathscr{B}(G)).$$

If we define $\tilde{f} : G \to \mathbb{C}$ by

$$\tilde{f}(g) := \begin{cases} f(g) & \text{if } \ g \in \bigcup_{n=1}^{\infty} F_n, \\ 0 & \text{if } g \in N, \end{cases}$$

then $\tilde{f} \in L^1(G)$ and, if we write $F := \bigcup_{n=1}^{\infty} F_n$, we have

$$\mu(E) = \int_E f(g) d\tilde{m}_G(g) = \int_{E\cap F} f(g) d\tilde{m}_G(g) = \int_{E\cap F} \tilde{f}(g) dm_G(g) = \int_E \tilde{f}(g) dm_G(g)$$

for all Borel sets E in G; hence $\Phi(\tilde{f}) = \mu$ and we conclude that the range of Φ equals the set of measures in $M(G)$ that are absolutely continuous with respect to m_G. Moreover, this set is closed, since Φ is an isometry. If $f_1, f_2 \in L^1(G)$, let $\Psi \in C_0(G)^*$ be associated with $f_1 * f_2 dm_G$ and $\Psi_1, \Psi_2 \in C_0(G)^*$ be associated with $f_1 dm_G$ and $f_2 dm_G$, respectively. For $\phi \in C_0(G)$ we conclude from our findings concerning the convolution on $L^1(G)$ that

$$\int_G \phi(hg) f_2(g) dm_G(g) = \int_G \phi(g) f_2(h^{-1}g) dm_G(g) \qquad (h \in G), \tag{2}$$

by considering a sequence of simple functions $(\phi_n)_{n\geq 1}$ with $\phi_n(g) \to \xi(g) := \phi(hg) f_2(g)$ for m_G-almost all $g \in G$ which exists, since $\xi \in L^1(G)$, and then changing to the simple functions $(\psi_n)_{n\geq 1}$ defined by $\psi_n(g) := \phi_n(h^{-1}g)$ for which we have the convergence $\psi_n(g) \to \xi'(g) := \phi(g) f_2(h^{-1}g)$ for m_G-almost all $g \in G$. Since integrating ϕ_n and ψ_n over G results in the same values for all $n \geq 1$, the dominated converge theorem yields the desired equality. From (2) and the modification of Fubini's theorem it follows that

$$\begin{aligned}
[\Psi_1 * \Psi_2](\phi) &= \int_G \left(\int_G \phi(gh) f_1(g) f_2(h) dm_G(g) \right) dm_G(h) \\
&= \int_G f_1(g) \left(\int_G \phi(gh) f_2(h) dm_G(h) \right) dm_G(g) \\
&= \int_G f_1(g) \left(\int_G \phi(h) f_2(g^{-1}h) dm_G(h) \right) dm_G(g) \\
&= \int_G \phi(h) \left(\int_G f_1(g) f_2(g^{-1}h) dm_G(g) \right) dm_G(h) \\
&= \int_G \phi(h) [f_1 * f_2](h) dm_G(h) = \Psi(\phi),
\end{aligned}$$

so $\Phi(f_1 * f_2) = \Phi(f_1) * \Phi(f_2)$ and we conclude that $L^1(G)$ is isometrically isomorphic, as a Banach algebra, to the set of measures in $M(G)$ that are absolutely continuous with respect to m_G.

2.1.2 $L^1(G)$ as an ideal of $M(G)$

Next, we want to show is that this set also forms an ideal in $M(G)$. To that end, we will need a lemma and we shall investigate to what extent m_G fails to be right invariant.

Lemma 2.8 *For $\mu \in M(G)$ the following statements are equivalent:*

i) If F is a compact set in G such that $m_G(F) = 0$, then $\mu(F) = 0$.

ii) μ is absolutely continuous with respect to m_G.

Proof: *i)* $\Rightarrow$ *ii)*: Let $E \in \mathscr{B}(G)$ and $\varepsilon > 0$. By the inner regularity of $|\mu|$, there is a compact set F in G with $F \subset E$ such that $|\mu|(E \setminus F) < \varepsilon$. If $m_G(E) = 0$, then $m_G(F) = 0$, so $|\mu|(E) < \varepsilon$ and since $\varepsilon > 0$ was arbitrary, we conclude that $|\mu|(E) = 0$ which implies that $\mu(E) = 0$; hence μ is absolutely continuous with respect to m_G.

ii) $\Rightarrow$ *i)*: This implication obviously holds. ∎

For $g \in G$ we can define $\mu_g(E) := m_G(Eg)$ for all $E \in \mathscr{B}(G)$. It is a straightforward verification that shows μ_g is a non-zero Borel measure. As for inner regularity, If $E \in \mathscr{B}(G)$ and F is a compact set with $F \subset E$, then $Kg \subset E_g$ and Fg is compact by Lemma 2.1. Conversely, if $F \subset Eg$ is compact, then analogously, we have that Fg^{-1} is compact and $Fg^{-1} \subset E$; hence

$$\begin{aligned}\mu_g(E) = m_G(Eg) &= \sup\{m_G(F) : F \subset Eg,\ F \text{ is compact}\}\\ &= \sup\{m_G(Fg) : F \subset E,\ F \text{ is compact}\}\\ &= \sup\{\mu_g(F) : F \subset E,\ F \text{ is compact}\}.\end{aligned}$$

In a similar way one can show that μ_g is also outer regular and by the associative law that holds in G, we find μ_g is left invariant. By Theorem 2.3 there exists a unique positive constant α such that $\mu_g = \alpha\, m_G$ and we can define a function $\Delta : G \to (0, \infty)$ by $g \mapsto \alpha$. The function Δ is called the **modular function** of G. Since we are concerned with measuring left and right actions on Borel sets in G, it comes naturally to examine the integration of left and right translates of the functions in $L^1(G)$. For $f \in L^1(G)$ we define the left translate $L_h f$ and the right translate $R_h f$ by $L_h f(g) := f(hg)$ and $R_h f(g) := f(gh)$ for all $h \in G$.

Lemma 2.9 *If $f \in C_c(G)$, then f is left and right uniformly continuous.*

Proof: Let $f \in C_c(G)$, $\varepsilon > 0$ and define $F := \text{supp}(f)$. By the continuity of f, for each $g \in F$ there is an open neighborhood U_g of e such that $|f(gh) - f(g)| < \frac{1}{2}\varepsilon$ whenever $h \in U_g$. Since the map $(g, h) \mapsto gh$ is continuous, there exist open neighborhoods V_1 and V_2 of e such that $V_1 V_2 \subset U_g$ and define

$$V_g := V_1 \cap V_2 \cap V_1^{-1} \cap V_2^{-1}.$$

This set V_g is an open symmetric neighborhood of e. Now, the collection $(gV_g)_{g\in F}$ forms an open cover for F, so there are $g_1, \ldots, g_n \in F$ such that $F \subset \bigcup_{k=1}^n g_k V_{g_k}$. Let $V := \bigcap_{k=1}^n V_{g_k}$ and choose $h \in V$. If $g \in F$, then there is a number $1 \leq k \leq n$ such that $g_k^{-1} g \in V_{g_k}$, so $gh = g_k(g_k^{-1} g)h \in g_k U_{g_k}$; hence

$$|f(gh) - f(g)| \leq |f(gh) - f(g_k)| + |f(g_k) - f(g)| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

If $gh \in F$, then, analogously, there is a number $1 \leq k \leq n$ such that $gh \in g_k V_{g_k}$, so $g \in g_k U_{g_k}$; hence $|f(gh) - f(g)| < \frac{1}{2}\varepsilon$. Finally, if neither g nor gh are elements of F, then $f(gh) = f(g) = 0$, so we conclude that $\sup_{g\in G} |(R_h f - f)(g)| < \varepsilon$ whenever $h \in V$. In a similar way, we can show that f is left absolutely continuous by considering the multiplications on the right hand side. ∎

Lemma 2.10 *The modular function Δ is a continuous homomorphism from G into $(\mathbb{R}_+, \times)$. Moreover, for every $f \in L^1(G)$ we have*

$$\int_G R_h f(g) dm_G(g) = \Delta(h^{-1}) \int_G f(g) dm_G(g).$$

Proof: Let $g, h \in G$ and $E \in \mathscr{B}(G)$ with $0 < m_G(E) < \infty$. Since we have

$$\Delta(gh) m_G(E) = m_G(E(gh)) = m_G((Eg)h) = \Delta(h) m_G(Eg) = \Delta(h)\Delta(g) m_G(E)$$

by the associative law that holds in G, it follows that Δ is a homomorphism from G into $(\mathbb{R}_+, \times)$. If we consider the characteristic function χ_E, we find that

$$\int_G \chi_E(gh)dm_G(g) = \int_G \chi_{Eh^{-1}}(g)dm_G(g) = m_G(Eh^{-1}) = \Delta(h^{-1})m_G(E)$$
$$= \Delta(h^{-1}) \int_G \chi_E(g)dm_G(g),$$

so the desired equality holds for all simple functions on G. Now, for $f \in L^1(G)$, the equality follows from the dominated convergence theorem and the appropriate changing of the corresponding simple functions, as we have shown previously. As for the continuity, by Lemma 2.9 the function $h \mapsto \int_G R_h f$ is continuous for all $f \in C_c(G)$, so if $(h_\alpha)_\alpha$ is a net in G such that $h_\alpha \to h$ for some $h \in G$, then

$$\lim_\alpha \Delta(h_\alpha^{-1}) \int_G f(g)dm_G(g) = \lim_\alpha \int_G R_{h_\alpha} f(g)dm_G(g) = \int_G R_h f(g)dm_G(g)$$
$$= \Delta(h^{-1}) \int_G f(g)dm_G(g)$$

and since the map $h \mapsto h^{-1}$ is continuous, we conclude that $\Delta(h_\alpha) \to \Delta(h)$. ∎

Now, let $\mu, \nu \in M(G)$ and suppose that μ is absolutely continuous with respect to m_G. Let F be a compact set in G with $m_G(F) = 0$. Since $m_G(Fg) = \Delta(g)m_G(F) = 0$ for all $g \in G$, it follows that $\mu(Fg) = 0$ for all $g \in G$. In the same spirit, we now find that

$$[\mu * \nu](F) = \int_G \left(\int_G \chi_F(gh)d\mu(g) \right) d\nu(h) = \int_G \mu(Fh^{-1})d\nu(h) = 0;$$

hence Lemma 2.8 now implies that $\mu * \nu$ is absolutely continuous with respect to m_G. Furthermore, we have that $m_G(gF) = m_G(F) = 0$ for all $g \in G$, so the modification of Fubini's theorem yields

$$[\nu * \mu](F) = \int_G \left(\int_G \chi_F(gh)d\nu(g) \right) d\mu(h) = \int_G \left(\int_G \chi_F(gh)d\mu(h) \right) d\nu(g)$$
$$= \int_G \mu(g^{-1}F)d\nu(g) = 0$$

and analogously, we find that $\nu * \mu$ is absolutely continuous with respect to m_G and this shows that the set of all measures in $M(G)$ that are absolutely continuous with respect to m_G is an ideal in $M(G)$.

Furthermore, we want to show is that $L^1(G)$ has a bounded approximate identity and in order to do so, we need the continuity of left and right translates of functions in $L^1(G)$.

Lemma 2.11 *Let $(g_\alpha)_\alpha$ be a net in G such that $g_\alpha \to e$. If $f \in L^1(G)$, then*

$$\lim_\alpha \int_G |(L_{g_\alpha} f - f)(g)|\, dm_G(g) = 0 \quad \textit{and} \quad \lim_\alpha \int_G |(R_{g_\alpha} f - f)(g)|\, dm_G(g) = 0.$$

Proof: Let V be a compact neighborhood of e. For $f \in C_c(G)$, let

$$F := \text{supp}(f)V \cap V\text{supp}(f).$$

Then F is compact by Lemma 2.1 and $\mathrm{supp}(L_h f) \subset F$, $\mathrm{supp}(R_h f) \subset F$ for all $h \in V$. This yields the inequality

$$\int_G |(L_h f - f)(g)| dm_G(g) = \int_F |(L_h - f)(g)| dm_G(g) \leq m_G(F) \sup_{g \in G} |(L_h f - f)(g)|.$$

Let $\varepsilon > 0$. By Lemma 2.9 there is an open neighborhood $\tilde{V}$ of e such that

$$\sup_{g \in G} |(L_h f - f)(g)| < m_G(F)^{-1} \varepsilon$$

whenever $h \in \tilde{V}$. Also we have an α such that $g_\beta \in V \cap \tilde{V}$ whenever $\beta \geq \alpha$, so

$$\int_G |(L_{g_\beta} f - f)(g)| dm_G(g) < \varepsilon$$

for every $\beta \geq \alpha$ and we conclude that

$$\lim_\alpha \int_G |(L_{g_\alpha} f - f)(g)| \, dm_G(g) = 0.$$

Analogously, it follows that we also have this convergence for the right translates $R_{g_\alpha} f$. Now, let $f \in L^1(G)$. It follows from (1) that $\|L_h f\| = \|f\|$ and by Lemma 2.10 there exists a constant $K > 0$ such that $\|R_h f\| = \Delta(h^{-1}) \|f\| \leq K \|f\|$ for all $h \in V$, since the continuous image of a compact set is compact. Again, let $\varepsilon > 0$. By Lemma 2.6 there is a function $\phi \in C_c(G)$ such that $\|f - \phi\| < \varepsilon$ and we find that

$$\begin{aligned}\int_G |(R_{g_\alpha} f - f)(g)| dm_G(g) &\leq \int_G |(R_{g_\alpha}(f - \phi))(g)| dm_G(g) + \int_G |(R_{g_\alpha} \phi - \phi)(g)| + \|f - \phi\| \\ &\leq (K+1)\varepsilon + \int_G |(R_{g_\alpha} \phi - \phi)(g)| dm_G(g) \to (K+1)\varepsilon,\end{aligned}$$

and since $\varepsilon > 0$ was arbitrary, we conclude that

$$\lim_\alpha \int_G |(R_{g_\alpha} f - f)(g)| \, dm_G(g) = 0.$$

In a similar way, the limit concerning $L_{g_\alpha} f$ is shown. ∎

Theorem 2.12 *$L^1(G)$ has a bounded approximate identity which is positive and bounded by 1.*

Proof: Let $\mathcal{F}$ be the collection of symmetric compact neighborhoods of e. Then $\mathcal{F}$ can be partially ordered by inclusion and for each $F \in \mathcal{F}$ define the functions $\psi_F := m_G(F)^{-1} \chi_F$. Clearly, we have $\psi_F \in L^1(G)$ and $\|\psi_F\| = 1$ for all $F \in \mathcal{F}$, by Lemma 2.4 and the fact that m_G is a regular Borel measure. Let $f \in L^1(G)$. Since $\psi_F(g) = \psi_F(g^{-1})$ for all $g \in G$, it follows from (1) that for all $h \in G$ we have

$$\begin{aligned}[f * \psi_F](h) - f(h) &= \int_G f(g) \psi_F(g^{-1}h) dm_G(g) - f(h) \int_G \psi_F(g) dm_G(g) \\ &= \int_G f(hg) \psi_F(g^{-1}) dm_G(g) - f(h) \int_G \psi_F(g) dm_G(g) \\ &= \int_G (R_g f - f)(h) \psi_F(g) dm_G(g)\end{aligned}$$

and this identity yields the inequality

$$\begin{aligned}\int_G |(f * \psi_F - f)(h)|dm_G(h) &\leq \int_G \left(\int_G |(R_g f - f)(h)\psi_F(g)|dm_G(h)\right) dm_G(g) \\ &= \int_G |\psi_F(g)|\|R_g f - f\|dm_G(g) \\ &\leq \sup_{g\in F} \|R_g f - f\|\end{aligned}$$

For every $F \in \mathcal{F}$ choose an element $g_F \in F$. Then $(g_F)_F$ is a net in G with $g_F \to e$. Let $\varepsilon > 0$. By Lemma 2.11 there is an $\tilde{F} \in \mathcal{F}$ such that $\|R_{g_F} f - f\| < \varepsilon$ whenever $F \subset \tilde{F}$. Since $g_F \in F$ was arbitrary, it follows that $\sup_{g\in F} \|R_g f - f\| \leq \varepsilon$ for all $F \subset \tilde{F}$; hence

$$\lim_F \|f * \psi_F - f\| = 0.$$

In a similar way, we can show that for all $h \in G$ we have

$$\begin{aligned}[\psi_F * f](h) - f(h) &= \int_G \psi_F(g)f(g^{-1}h)dm_G(g) - f(h)\int_G \psi_F(g)dm_G(g) \\ &= \int_G \psi_F(g^{-1})f(g^{-1}h)dm_G(g) - f(h)\int_G \psi_F(g)dm_G(g) \\ &= \int_G \psi_F(g)f(gh)dm_G(g) - f(h)\int_G \psi_F(g)dm_G(g) \\ &= \int_G (L_g f - f)(h)\psi_F(g)dm_G(g)\end{aligned}$$

and analogously, it follows that we have the limit

$$\lim_F \|\psi_F * f - f\| = 0;$$

hence $(\psi_F)_F$ is a bounded approximate identity in $L^1(G)$. ∎

As for the convolution of a measure $\mu \in M(G)$ with a function $f \in L^1(G)$, let $\phi \in C_0(G)$. By applying the modification of Fubini's theorem and (1), it follows that

$$\begin{aligned}[\mu * f dm_G](\phi) &= \int_G \left(\int_G \phi(gh)f(h)d\mu(g)\right) dm_G(h) = \int_G \left(\int_G \phi(gh)f(h)dm_G(h)\right) d\mu(g) \\ &= \int_G \left(\int_G \phi(h)f(g^{-1}h)dm_G(h)\right) d\mu(g) \\ &= \int_G \phi(h)\left(\int_G f(g^{-1}h)d\mu(g)\right) dm_G(h)\end{aligned}$$

and therefore, if we consider the function ξ on G defined by

$$h \mapsto \int_G f(g^{-1}h)d\mu(g),$$

the same properties imply that $\xi \in L^1(G)$, since $\|\xi\| \leq \|f\|\|\mu\|$. We find that $\mu * f dm_G$ can be identified with ξdm_G and this validates the identity

$$[\mu * f](g) = \int_G f(h^{-1}g)d\mu(h) \qquad (m_G\text{—almost everywhere}). \tag{3}$$

Similarly, if we use Lemma 2.10, we get

$$\begin{aligned}
[f dm_G * \mu](\phi) &= \int_G \left(\int_G R_h(\phi R_{h^{-1}} f)(g) dm_G(g) \right) d\mu(h) \\
&= \int_G \left(\int_G \phi(g) f(gh^{-1}) \Delta(h^{-1}) dm_G(g) \right) d\mu(h) \\
&= \int_G \phi(g) \left(\int_G f(gh^{-1}) \Delta(h^{-1}) d\mu(h) \right) dm_G(g)
\end{aligned}$$

and the function η on G defined by

$$g \mapsto \int_G f(gh^{-1}) \Delta(h^{-1}) d\mu(h)$$

also lies in $L^1(G)$, since we have the integral inequality

$$\begin{aligned}
\int_G \left(\int_G |f(gh^{-1}) \Delta(h^{-1})| dm_G(g) \right) d|\mu|(h) &= \int_G \Delta(h^{-1}) \left(\int_G |f(g) \Delta(h)| dm_G(g) \right) d|\mu|(h) \\
&\leq \|f\| \|\mu\|
\end{aligned}$$

and the modification of Fubini's theorem now implies that

$$\int_G \left(\int_G |f(gh^{-1}) \Delta(h^{-1})| d|\mu|(h) \right) dm_G(g) \leq \|f\| \|\mu\|.$$

Analogously, we can now validate the identity

$$[f * \mu](g) = \int_G f(gh^{-1}) \Delta(h^{-1}) d\mu(h) \qquad (m_G\text{−almost everywhere}). \tag{4}$$

Summarizing our findings, we obtain the following result:

Theorem 2.13 *Let G be a locally compact group. Then the set of all measures in $M(G)$ which are absolutely continuous with respect to m_G is a closed ideal of $M(G)$ and, via the map $f \mapsto f dm_G$, is isometrically isomorphic to $L^1(G)$ as a Banach algebra.*

2.1.3 The dual space of $L^1(G)$

The last we wish to investigate regarding $L^1(G)$ is its dual space. It is a well known fact that if μ is a σ-finite measure on a space (X, Σ, μ), we have $L^1(X, \mu)^* = L^\infty(X, \mu)$; for a proof, see [5, Thm. 4.5.1]. However, if we have a measure μ which is not σ-finite, it is not true in general that we can identify $L^\infty(G, \mu)$ with $L^1(G, \mu)^*$ via the canonical isometry that associates $\phi \in L^\infty(G, \mu)$ with the functional $\Psi_\phi \in L^1(G, \mu)^*$ where

$$\Psi_\phi(f) := \int_G f(g) \phi(g) d\mu(g) \qquad (f \in L^1(G, \mu)).$$

For example, consider the measure space $(\mathbb{R}, \Sigma, \mu)$ where Σ is the σ-algebra consisting of those subsets $E \subset \mathbb{R}$ such that either E or E^c is countable and let μ be the counting measure. Then the functions in $L^1(\mathbb{R}, \mu)$ are those that vanish outside a countable set and satisfy

$$\|f\| = \sum_{x \in \Theta_f} |f(x)| < \infty$$

where $\Theta_f := \{x \in \mathbb{R} : f(x) \neq 0\}$. Define the functional ψ on $L^1(\mathbb{R}, \mu)$ by

$$\psi(f) := \sum_{x \in \Theta_f^+} f(x) \qquad (f \in L^1(\mathbb{R}, \mu))$$

where $\Theta_f^+ := \{x \in \mathbb{R}_+ : f(x) \neq 0\}$. Clearly, we have that ψ is linear and bounded. If ϕ is a function that satisfies

$$\psi(f) = \int_{\mathbb{R}} f(x)\phi(x)d\mu(x) \qquad (f \in L^1(\mathbb{R}, \mu)),$$

then we must have that $\phi = 1$ on $\mathbb{R}_+$, but $\phi^{-1}(\{1\}) = (0, \infty) \notin \Sigma$, so ϕ is not Σ-measurable and we conclude that no function in $L^\infty(\mathbb{R}, \mu)$ corresponds with ψ.

Since m_G need not be σ-finite, we will introduce a slight modification in the definition of $L^\infty(G)$. Suppose that m_G is not σ-finite. We define a set $E \subset G$ to be **locally m_G−Borel** if $E \cap F$ is a Borel set in G for all Borel sets F with $m_G(F) < \infty$. Accordingly, a set $E \subset G$ is said to be **locally m_G−null** if $m_G(E \cap F) = 0$ for all Borel sets F with $m_G(F) < \infty$. For a statement about elements of G to be true **locally m_G−almost everywhere** we mean that it holds except on a set contained in a locally m_G−null set. Finally, a function $f : G \to \mathbb{C}$ is said to be **locally m_G−measurable** is $f^{-1}(E)$ is locally m_G−Borel for every Borel set $E \subset \mathbb{C}$. Now, for the sets

$$\mathcal{N} := \{f : G \to \mathbb{C} : f = 0 \text{ locally } m_G\text{−everywhere}\}$$

and

$$\mathcal{L}^\infty(G) := \{f : G \to \mathbb{C} : f \text{ is measurable and bounded locally } m_G\text{− almost everywhere}\},$$

we redefine $L^\infty(G)$ to be the quotient space $\mathcal{L}^\infty(G)/\mathcal{N}$. Let $f, g \in L^\infty(G)$ and M_f and M_g be the sets in G on which f and g are bounded. Then $M_f \cap M_g$ is the set on which $f + g$ is bounded. Suppose that $M_f \cap M_g = \emptyset$. Then $G = M_f^c \cup M_g^c$ and both M_f^c and M_g^c are locally m_G−null sets; hence $M_f^c \cup M_g^c$ is a locally m_G−null set. Lemma 2.2 now implies that $m_G(\gamma H) = 0$ for all $\gamma \in \Gamma$; hence $m_G(G) = 0$ which contradicts the fact that m_G is not σ-finite. We find that $f + g \in L^\infty(G)$, so $L^\infty(G)$ is a linear space, since its clearly closed under scalar multiplication. For $f \in L^\infty(G)$ define

$$\|f\| := \inf\{\alpha : |f(g)| \leq \alpha \text{ locally } m_G\text{−almost everywhere}\}.$$

It is a straightforward verification to show that this defines a norm on $L^\infty(G)$. Now, let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $L^\infty(G)$. For the corresponding M_{f_n} we must have that $M := \bigcap_{n=1}^\infty M_{f_n} \neq \emptyset$ by the same argument used to show that $L^\infty(G)$ is a linear space and M^c is a locally m_G−null set. For $g \in M$ we have that $(f_n(g))_{n\geq 1}$ is a Cauchy sequence, so we may define a function f on G by $f(g) := \lim_{n\to\infty} f_n(g)$. Also, since $(\|f_n\|)_{n\geq 1}$ is a Cauchy sequence, it is bounded by say $K > 0$, so $f|_M$ is bounded by K; hence $f \in L^\infty(G)$. To show that $f_n \to f$ on M is just routine. Let $\varepsilon > 0$ and pick $g \in M$. There exists a number $N \in \mathbb{N}_+$ such that $|f(g) - f_n(g)| < \frac{1}{2}\varepsilon$ whenever $n \geq N$. Also, there is a number $N' \in \mathbb{N}_+$ such that $\|f_n - f_m\| < \frac{1}{2}\varepsilon$ if $n, m \in N'$. If $\tilde{N} := \max\{N, N'\}$, then for $m \geq \tilde{N}$ we have

$$|f(g) - f_m(g)| \leq |f(g) - f_n(g)| + |f_n(g) - f_m(g)| < \varepsilon$$

and since $g \in M$ was arbitrary, we conclude that $\|f - f_m\| \leq \varepsilon$ whenever $m \geq \tilde{N}$; hence $f_n \to f$ on M and we conclude that $L^\infty(G)$ is a Banach space with this norm.

Now, if $\phi \in L^\infty(G)$ and $f \in L^1(G)$, then $f\phi$ is measurable, since the set

$$\Omega_f := \{g \in G : f(g) \neq 0\}$$

is σ-finite and $M_\phi^c \cap \Omega_f$ is a m_G-null set. Furthermore, we find that

$$|\Psi_\phi(f)| \leq \int_G |f(g)\phi(g)|dm_G(g) = \int_{\Omega_f \cap M_\phi} |f(g)\phi(g)|dm_G(g) \leq \|\phi\|\|f\|,$$

so $\|\Psi_\phi\| \leq \|\phi\|$. To prove that the other inequality holds, consider the set

$$E := \{g \in G : |\phi(g)| > \|\phi\| - \varepsilon\}.$$

Suppose that E is locally m_G-null. Then we can define a function $\tilde{\phi}$ on G by $\tilde{\phi} := \phi$ on $M_\phi \setminus E$ and $\tilde{\phi} := 0$ on $M_\phi^c \cup E$. By definition of $L^\infty(G)$ we must have that $\tilde{\phi} \sim \phi$, so

$$\|\phi\| = \|\tilde{\phi}\| \leq \|\phi\| - \varepsilon$$

which is absurd; hence E is not locally m_G-null. This implies that there exists a Borel set A in G with $m_G(A) < \infty$ such that $0 < m_G(A \cap E) < \infty$. Now for $B := A \cap E$ we can define the function f on G by $f(g) := \xi(g) \cdot \chi_B(g)$ where

$$\xi(g) := \begin{cases} \frac{\overline{\phi(g)}}{|\phi(g)|} & \text{if } \phi(g) \neq 0 \text{ and } g \in M_\phi, \\ 0 & \text{if } \phi(g) = 0 \text{ or } g \in M_\phi^c. \end{cases}$$

It follows that $f \in L^1(G)$ and

$$\|f\| = \int_G |\xi(g) \cdot \chi_B(g)|dm_G(g) \leq \int_G \chi_B(g)dm_G(g) = m_G(B),$$

so these findings yield the inequality

$$\Psi_\phi(f) = \int_G \xi(g) \cdot \chi_B(g)\phi(g)dm_G(g) = \int_B |\phi(g)|dm_G(g) \geq (\|\phi\| - \varepsilon)m_G(B).$$

Since $|\Psi_\phi(f)| \leq \|\Psi_\phi\|\|f\| \leq \|\Psi_\phi\|m_G(B)$, we find that $\|\Psi_\phi\| \geq \|\phi\| - \varepsilon$; hence $\|\Psi_\phi\| \geq \|\phi\|$ because $\varepsilon > 0$ was arbitrary. We conclude that $L^\infty(G)$ can be mapped isometrically into $L^1(G)^*$ this way. Conversely, let $\Phi \in L^1(G)^*$. In the spirit of Lemma 2.2 the restriction of Φ to $L^1(\gamma H)$ corresponds with an essentially bounded measurable function ϕ_γ on γH such that

$$\Phi|_{L^1(\gamma H)}(f) = \int_{\gamma H} f(g)\phi_\gamma(g)dm_G(g) \qquad (f \in L^1(\gamma H))$$

for every $\gamma \in \Gamma$, since γH is σ-compact. Define the function ϕ on G by $\phi := \phi_\gamma$ on γH. Let $E \subset \mathbb{C}$ be a Borel set. Then $\phi^{-1}(E) \cap \gamma H = \phi_\gamma^{-1}(E)$ is a Borel set in G for all $\gamma \in \Gamma$. Now, if $F \subset G$ is a Borel set with $m_G(F) < \infty$, then F intersects only countably many γH, so $\phi^{-1}(E) \cap F$ intersects only countably many γH and we conclude that it is a Borel set in G; hence ϕ is locally m_G-measurable. Let M_γ be the m_G-null set on which ϕ_γ is not bounded for all $\gamma \in \Gamma$ and define $M := \bigcup_{\gamma \in \Gamma} M_\gamma$. Using a similar argument, we find that M is locally m_G-null. Also, we have

$$\|\phi\| = \sup_{\gamma \in \Gamma} \|\phi_\gamma\| = \sup_{\gamma \in \Gamma} \|\Phi|_{L^1(\gamma H)}\| \leq \|\Phi\|,$$

so $\phi \in L^\infty(G)$. Let $f \in L^1(G)$. We have shown that f vanishes outside some σ-compact set $\Xi := \biguplus_{k=1}^\infty \gamma_k H$. Let $\varepsilon > 0$. By Lemma 2.6 there is a function $\phi \in C_c(G)$ such that $\|f - \phi\| < \varepsilon$. Write $F := \mathrm{supp}(\phi)$. Then the elementary properties of a measure now imply that

$$\begin{aligned}
\int_G \left| f(g) - \sum_{k=1}^N f(g)\cdot \chi_{\gamma_k H}(g) \right| dm_G(g) &= \int_{\Xi \setminus (\bigcup_{k=1}^N \gamma_k H)} |f(g)| dm_G(g) \\
&\leq \int_{\Xi \setminus (\bigcup_{k=1}^N \gamma_k H)} |f(g) - \phi(g)| dm_G(g) + \int_{\Xi \setminus (\bigcup_{k=1}^N \gamma_k H)} |\phi(g)| dm_G(g) \\
&< \varepsilon + \sup_{g\in F} |\phi(g)| \cdot m_G\left((\Xi \cap F) \setminus \bigcup_{k=1}^N \gamma_k H \right) \\
&\to \varepsilon
\end{aligned}$$

as $N \to \infty$ and since $\varepsilon > 0$ was arbitrary, we now find that

$$\begin{aligned}
\Phi(f) = \Phi\left(\sum_{k=1}^\infty f \cdot \chi_{\gamma_k H} \right) &= \sum_{k=1}^\infty \Phi(f \cdot \chi_{\gamma_k H}) = \sum_{k=1}^\infty \int_{\gamma_k H} f(g)\phi(g) dm_G(g) \\
&= \int_\Xi f(g)\phi(g) dm_G(g) = \int_G f(g)\phi(g) dm_G(g).
\end{aligned}$$

This implies the inequality $\|\Phi\| \leq \|\phi\|$ and the identity $\Phi = \Psi_\phi$. In what follows, by $L^\infty(G)$ we will mean the space defined above. Note that if m_G is σ-finite and f is a locally m_G-measurable function on $G = \bigcup_{k=1}^\infty G_k$ that is bounded, except on a locally m_G-null set, then for a Borel set E in $\mathbb{C}$ we have that $f^{-1}(E) \cap G_k$ is Borel for all $k \geq 1$; hence $f^{-1}(E)$ is Borel and we conclude that f is measurable. Also, if N is a locally m_G-null set, then $m_G(N \cap G_k) = 0$ for all $k \geq 1$, so N is a null set and we conclude that in this case our modification of $L^\infty(G)$ agrees with the original one. This yields the following result:

Theorem 2.14 *Let G be a locally compact group. Then the dual space of $L^1(G)$ equals $L^\infty(G) := \mathcal{L}^\infty(G)/\mathcal{N}$.*

As for the left and right convolutions of $M(G)$ on $L^\infty(G)$, these are defined by

$$(\mu * \phi)(g) := \int_G \phi(h^{-1}g) d\mu(h) \qquad \text{and} \qquad (\phi * \mu)(g) := \int_G \phi(gh^{-1}) d\mu(g)$$

for all $\mu \in M(G)$ and $\phi \in L^\infty(G)$ locally m_G-almost everywhere.

2.1.4 Amenable groups

We have acquired enough results to be able to define amenability when considering a general locally compact group.

Definition: Let G be a locally compact group, and let E be a subspace of $L^\infty(G)$ that contains the constant functions. A functional $m \in E^*$ is a **mean** if $m(1_G) = \|m\| = 1$.

If we also require E to be closed under complex conjugation, means on E have the important property that they are positive, as the following lemma shows:

Lemma 2.15 *Let G be a locally compact group and E be a subspace of $L^\infty(G)$ that contains the constant functions and is closed under complex conjugation. Then a linear functional $m : E \to \mathbb{C}$ with $m(1_G) = 1$ is a mean on E if and only if m is positive.*

Proof: Suppose that m is a mean on E. Let $\phi \in E$ be such that $\phi(G) \subset \mathbb{R}$. Without loss of generality, we may assume that $\|\phi\| \leq 1$. Let $\alpha, \beta \in \mathbb{R}$ be such that $m(\phi) = \alpha + i\beta$. Then for all $t \in \mathbb{R}$ we have that

$$(\beta + t)^2 \leq |\alpha + (\beta + t)i|^2 = |m(\phi + it1_G)|^2 \leq \|\phi + it1_G\|^2 = \sup_{g\in G} |\phi(g) + it|^2 \leq 1 + t^2,$$

so $2\beta t \leq 1 - \beta^2$ for all $t \in \mathbb{R}$; hence $\beta = 0$ and we conclude that $m(\phi) \in \mathbb{R}$ whenever we have $\phi(G) \subset \mathbb{R}$. Now, let $\phi \in E$ be such that $\phi \geq 0$. Again, we may assume that $\|\phi\| \leq 1$. Define $\psi := 2\phi - 1_G$. We have that $\psi(G) \subset \mathbb{R}$ and $\|\psi\| \leq 1$. Using that m is a mean on E, it follows that

$$m(\phi) = \tfrac{1}{2}m(\psi + 1_G) = \tfrac{1}{2}(m(\psi) + 1) \geq \tfrac{1}{2}(1 - |m(\psi)|) \geq \tfrac{1}{2}(1 - \|\psi\|) \geq 0,$$

and we find that m is positive.

Conversely, Suppose that $\phi \in E$ is such that $\phi(G) \subset \mathbb{R}$. If we define $\psi := \|\phi\|1_G - \phi$, then clearly $\psi \geq 0$, so $m(\psi) \geq 0$. This implies that $m(\phi) \in \mathbb{R}$ and that $m(\phi) \leq \|\phi\|$. If we replace ϕ with $-\phi$, we find that $-m(\phi) \leq \|\phi\|$ in a similar way, so $|m(\phi)| \leq \|\phi\|$. Now, for an arbitrary $\phi \in E$, let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ be such that $m(\lambda\phi) = |m(\phi)|$. Since E is closed under complex conjugation, we have that $\Re(\lambda\phi), \Im(\lambda\phi) \in E$ and since

$$m(\Re(\lambda\phi)) + im(\Im(\lambda\phi)) = m(\lambda\phi) \geq 0,$$

it follows that $m(\Im(\lambda\phi)) = 0$; hence $|m(\phi)| = m(\Re(\lambda\phi)) \leq \|\Re(\lambda\phi)\| \leq \|\lambda\phi\| = \|\phi\|$. We conclude that $\|m\| = 1 = m(1_G)$, so m is a mean on E. ∎

Definition: Let G be a locally compact group, and let E be a subspace of $L^\infty(G)$ which contains the constant functions and is closed under complex conjugation.

i) E is called **left invariant** if $\delta_g * \phi \in E$ for all $\phi \in E$ and $g \in G$.

ii) If E is left invariant, then a mean m on E is called **left invariant** if

$$m(\delta_g * \phi) = m(\phi) \qquad (g \in G,\ \phi \in E).$$

Definition: A locally compact group G is called **amenable** if there is a left invariant mean on $L^\infty(G)$.

2.2 Modules over a Banach algebra

An element $e \in \mathfrak{A}$ in a Banach algebra with the property $ae = ea = a$ for all $a \in \mathfrak{A}$ is called an **identity**. Not all Banach algebras have an identity, for example, consider $C_0(\mathbb{R})$ under pointwise multiplication, but the following lemma will provide us with a way to construct a unital Banach algebra in which the original one can be embedded isometrically.

Lemma 2.16 *Let $\mathfrak{A}$ be a Banach algebra without an identity. Let $\mathfrak{A}^\# := \mathfrak{A} \times \mathbb{F}$, where $\mathbb{F}$ is either $\mathbb{C}$ or $\mathbb{R}$, and define the algebraic operations on $\mathfrak{A}^\#$ by* *i)* $(a, \alpha) + (b, \beta) := (a + b, \alpha + \beta)$;

ii) $\beta(a, \alpha) := (\beta a, \beta\alpha)$;

iii) $(a, \alpha)(b, \beta) := (ab + \alpha b + \beta a, \alpha\beta)$.

If we define the norm $\|(a, \alpha)\| := \|a\| + |\alpha|$ *on* $\mathfrak{A}^\#$*, then* $\mathfrak{A}^\#$ *is a Banach algebra with respect to this norm, it has the identity* $(0, 1)$ *and the map* $a \mapsto (a, 0)$ *is an isometry.*

Proof: It is clear that $\mathfrak{A}^\#$ is a Banach space under this norm and that the map $a \mapsto (a, 0)$ is an isometry. Let $(a, \alpha), (b, \beta) \in \mathfrak{A}^\#$. Then the inequality

$$\begin{aligned}\|(a, \alpha)(b, \beta)\| &= \|(ab + \alpha b + \beta a, \alpha\beta)\| = \|ab + \alpha b + \beta a\| + |\alpha\beta| \\ &\leq \|a\|\|b\| + |\beta|\|a\| + |\alpha|\|b\| + |\alpha||\beta| \\ &= \|(a, \alpha)\|\|(b, \beta)\|\end{aligned}$$

implies that $\mathfrak{A}^\#$ is indeed a Banach algebra. ∎

The Banach algebra $\mathfrak{A}^\#$ is called the **unitization** of $\mathfrak{A}$.

Definition: Let $\mathfrak{A}$ be a Banach algebra. A Banach space E which is also a left $\mathfrak{A}$-module is said to be a **left Banach $\mathfrak{A}$-module** if there is a constant $k > 0$ such that

$$\|a \cdot x\| \leq k\|a\|\|x\| \qquad (a \in \mathfrak{A},\ x \in E).$$

Note that we do not require here that $e \cdot x = x$ for all $x \in E$ where $e \in \mathfrak{A}$ is an identity. In a similar way, we have the notion of a **right Banach $\mathfrak{A}$-module** and a Banach space E which is both a left and a right Banach $\mathfrak{A}$-module such that the actions commute, is called a **Banach $\mathfrak{A}$-bimodule**. Since E^* is a Banach space too, and a left action of $\mathfrak{A}$ on E can be viewed as a multiplicative bounded linear map $\varphi : \mathfrak{A} \to B(E)$ where $a \cdot x := \varphi(a)(x)$, we can consider a right action of $\mathfrak{A}$ on E^* by taking the adjoint operator $\varphi(a)'$, which is anti-multiplicative operation, and putting $f \cdot a := \varphi(a)'(f)$ where $f \cdot a(x) = f(\varphi(a)(x))$. In this way E becomes a right Banach $\mathfrak{A}$-module. Similarly, for a right action of $\mathfrak{A}$ on E, which can be viewed as a multiplicative bounded linear map $\varphi : \mathfrak{A}^{\text{opp}} \to B(E)$, where $\mathfrak{A}^{\text{opp}}$ is the **opposite Banach algebra** of $\mathfrak{A}$ in which the order of multiplication is reversed, that is, we have $a \star b := ba$ for all $a, b \in \mathfrak{A}^{\text{opp}}$, we can define a left action of $\mathfrak{A}$ on E^* by putting $a \cdot f := \varphi(a)'(f)$ where $a \cdot f(x) = f(\varphi(a)(x))$. It follows that E^* becomes a Banach $\mathfrak{A}$-bimodule if E is.

For a Banach algebra $\mathfrak{A}$ and a Banach $\mathfrak{A}$-bimodule E a bounded linear map $D : \mathfrak{A} \to E$ that satisfies

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in \mathfrak{A})$$

is called a **derivation**. Let $x \in E$. We define the map $\text{ad}_x : \mathfrak{A} \to E$ by $a \mapsto a \cdot x - x \cdot a$. It follows from

$$a \cdot \text{ad}_x(b) + \text{ad}_x(a) \cdot b = a \cdot (b \cdot x - x \cdot b) + (a \cdot x - x \cdot a) \cdot b = (a \cdot b) \cdot x - x \cdot (a \cdot b) = \text{ad}_x(ab)$$

that ad_x is a derivation from $\mathfrak{A}$ to E for all $x \in E$. Derivations of this form are called **inner derivations**. The set of derivations from $\mathfrak{A}$ to E is denoted by $\mathcal{Z}^1(\mathfrak{A}, E)$ and the set of inner derivations from $\mathfrak{A}$ to E is denoted by $\mathcal{B}^1(\mathfrak{A}, E)$. It is a straightforward verification to show that $\mathcal{Z}^1(\mathfrak{A}, E)$ is a linear subspace of the bounded linear operators from $\mathfrak{A}$ to E, which will be denoted by $\mathcal{L}(\mathfrak{A}, E)$, and $\mathcal{B}^1(\mathfrak{A}, E)$ is a linear subspace of $\mathcal{Z}^1(\mathfrak{A}, E)$.

Moreover, if $(D_n)_{n\geq 1}$ is a sequence in $\mathcal{Z}^1(\mathfrak{A}, E)$ such that $\|D_n - T\| \to 0$ as $n \to \infty$ for some $T \in \mathcal{L}(\mathfrak{A}, E)$, then for $a, b \in \mathfrak{A}$ we have

$$\begin{aligned}\|T(ab) - (a \cdot T(b) - T(a) \cdot b)\| & \\ &\leq \|T(ab) - D_n(ab)\| + \|a \cdot D_n(b) + D_n(a) \cdot b - (a \cdot T(b) + T(a) \cdot b)\| \\ &\leq \|ab\|\|T - D_n\| + k\|a\|\|D_n(b) - T(b)\| + k\|b\|\|D_n(a) - T(a)\| \\ &\leq (1 + 2k)\|a\|\|b\|\|D_n - T\| \to 0\end{aligned}$$

as $n \to \infty$, so T is a derivation and we conclude that $\mathcal{Z}^1(\mathfrak{A}, E)$ is closed in $\mathcal{L}(\mathfrak{A}, E)$. However, it is not necessarily the case that $\mathcal{B}^1(\mathfrak{A}, E)$ is closed in $\mathcal{Z}^1(\mathfrak{A}, E)$. This can be found in [15, 2.1, pp. 38].

Definition: Let $\mathfrak{A}$ be a Banach algebra and E a Banach $\mathfrak{A}$-bimodule. Then

$$\mathcal{H}^1(\mathfrak{A}, E) := \mathcal{Z}^1(\mathfrak{A}, E)/\mathcal{B}^1(\mathfrak{A}, E)$$

is called the **first order Hochschild cohomology group** of $\mathfrak{A}$ with coefficients in E.

2.2.1 Cohen's factorization theorem

Before we start to investigate the properties of $\mathcal{H}^1(\mathfrak{A}, E)$, some important results are needed with respect to the representations of elements in the Banach $\mathfrak{A}$-bimodule, in particular cases.

Lemma 2.17 *Let $\mathfrak{A}$ be a Banach algebra and E a left Banach $\mathfrak{A}$-module. Let $C \in \mathbb{R}$ with $C > 1$, $\gamma := (4C)^{-1}$ and $a \in \mathfrak{A}$ with $\|a\| \leq C$. Then $1 - \gamma + \gamma a \in \mathrm{Inv}(\mathfrak{A}^{\#})$ where*

$$\mathrm{Inv}(\mathfrak{A}^{\#}) := \{x \in \mathfrak{A} : \exists y \in \mathfrak{A},\ xy = yx = e\},$$

and $f := (1 - \gamma + \gamma a)^{-1}$ satisfies:

i) $\|f\| \leq 2$;

ii) given $\varepsilon > 0$, there exists a number $\eta > 0$ such that $\|f \cdot x - x\| \leq \varepsilon\|x\|$ whenever $x \in E$ and $\|a \cdot x - x\| \leq \eta\|x\|$.

Proof: Let $\kappa > 0$ be such that $\|a \cdot x\| \leq \kappa\|a\|\|x\|$ for all $a \in \mathfrak{A}$ and all $x \in E$. We have the inequalities $0 < \gamma < \frac{1}{4}$, $1 - \gamma > \frac{3}{4}$ and $\gamma C/(1 - \gamma) < \frac{1}{3}$, hence $\|\gamma(1 - \gamma)^{-1}a\| < \frac{1}{3}$ and it follows that $1 + \gamma(1 - \gamma)^{-1}a \in \mathrm{Inv}(\mathfrak{A}^{\#})$ using the von Neumann series. Moreover, we find that

$$\|f\| = \|(1 - \gamma + \gamma a)^{-1}\| \leq (1 - \gamma)^{-1}\left(1 + \sum_{k=1}^{\infty}\|\gamma(1 - \gamma)^{-1}a\|^k\right) < \tfrac{3}{2}(1 - \gamma)^{-1} \leq 2.$$

Let $\varepsilon > 0$ and choose a number $N \in \mathbb{N}_+$ such that

$$(1 - \gamma)^{-1}\sum_{k=N+1}^{\infty}\gamma^k(1 - \gamma)^{-k}(\kappa C^k + 1) < \varepsilon/2. \tag{5}$$

Choose $\eta > 0$ such that

$$\eta(1-\gamma)^{-1}\sum_{k=1}^{N}\gamma^k(1-\gamma)^{-k}\kappa(1+C+\cdots+C^{k-1}) < \varepsilon/2. \tag{6}$$

Since

$$f = (1-\gamma)^{-1}\left(1+\sum_{k=1}^{\infty}\gamma^k(\gamma-1)^{-k}a^k\right) \quad \text{and} \quad 1 = (1-\gamma)^{-1}\left(1+\sum_{k=1}^{\infty}\gamma^k(\gamma-1)^{-k}\right),$$

we have that

$$f\cdot x - x = (1-\gamma)^{-1}\sum_{k=1}^{\infty}\gamma^k(\gamma-1)^{-k}(a^k\cdot x - x) \qquad (x\in E).$$

Using (5) we obtain the inequality

$$\begin{aligned}(1-\gamma)^{-1}\sum_{k=N+1}^{\infty}\gamma^k(1-\gamma)^{-k}\|a^k\cdot x - x\| &\le (1-\gamma)^{-1}\sum_{k=N+1}^{\infty}\gamma^k(1-\gamma)^{-k}(\kappa C^k\|x\|+\|x\|)\\ &\le (1-\gamma)^{-1}\sum_{k=N+1}^{\infty}\gamma^k(1-\gamma)^{-k}\cdot(\kappa C^k+1)\|x\|\\ &\le \frac{\varepsilon}{2}\|x\|;\end{aligned}$$

hence

$$\|f\cdot x - x\| \le (1-\gamma)^{-1}\sum_{k=1}^{N}\gamma^k(1-\gamma)^{-k}\|a^k\cdot x - x\| + \frac{\varepsilon}{2}\|x\|. \tag{7}$$

Finally, since we have

$$\|a^k\cdot x - x\| = \|(1+a+\cdots+a^{k-1})\cdot(a\cdot x - x)\| \le \kappa(1+C+\cdots+C^{k-1})\|a\cdot x - x\|,$$

it follows from (6) and (7) that $\|f\cdot x - x\| \le \varepsilon\|x\|$ whenever $\|a\cdot x - x\| \le \eta\|x\|$. ∎

For a left, respectively right, Banach $\mathfrak{A}$-bimodule E a net $(e_\lambda)_{\lambda\in\Lambda}$ in $\mathfrak{A}$ is said to be a **bounded approximate identity for** E if $\sup_\lambda\|e_\lambda\| < \infty$ and if $\|e_\lambda\cdot x - x\| \to 0$, respectively $\|x\cdot e_\lambda - x\| \to 0$, for all $x\in E$. Similarly, a Banach $\mathfrak{A}$-bimodule E has a bounded approximate identity for E if we have both $\|e_\lambda\cdot x - x\|\to 0$ and $\|x\cdot e_\lambda - x\|\to 0$ for all $x\in E$.

In such situations we have an important result concerning the representation of the elements in E, which will play an important role when we derive some useful properties of $\mathcal{H}^1(\mathfrak{A}, E)$.

Theorem 2.18 (Cohen's factorization theorem) *Let $\mathfrak{A}$ be a Banach algebra with a bounded approximate identity for a left Banach $\mathfrak{A}$-module E. Let $z\in E$ and $\delta > 0$. Then there exists an $a\in\mathfrak{A}$ and a $y\in E$ such that $z = a\cdot y$ and $\|z-y\| < \delta$.*

Proof: Let $\kappa > 0$ be such that $\|a \cdot x\| \leq \kappa \|a\| \|x\|$ for all $a \in \mathfrak{A}$ and all $x \in E$. There is nothing to prove when $z = 0$, so fix $z \in E \setminus \{0\}$ and $\delta > 0$. Let $(e_\lambda)_{\lambda \in \Lambda}$ be a bounded approximate identity in $\mathfrak{A}$ for E and let $C > 1$ be such that $\sup_{\lambda \in \Lambda} \|e_\lambda\| \leq C$. Define $\gamma := (4C)^{-1}$ and $f_\lambda := (1 - \gamma + \gamma e_\lambda)^{-1}$, which lies in the unitization $\mathfrak{A}^\#$. and is well defined by Lemma 2.17. We will construct a sequence $(\lambda_n)_{n \geq 1} \subset \Lambda$, and define $e_n := e_{\lambda_n}$, such that

$$b_n := (1-\gamma)^n + \sum_{k=1}^{n} \gamma(1-\gamma)^{k-1} e_k \in \mathrm{Inv}(\mathfrak{A}^\#), \;\; \|t_n z - t_{n-1} z\| \leq \delta 2^{-n} \;\; (n = 1, 2, \ldots), \quad (8)$$

where $t_0 := 1$ and $t_n := b_n^{-1}$ $(n = 1, 2, \ldots)$. The existence of λ_1, satisfying (8), follows from Lemma 2.17 when we take $x = z$ and $\varepsilon = \delta(2\|z\|)^{-1}$. Suppose that $\lambda_1, \ldots, \lambda_m$ have been chosen to satisfy (8) for $n = 1, \ldots, m$. Now, if we define

$$u_\lambda := (1-\gamma)^m + \sum_{k=1}^{m} \gamma(1-\gamma)^{k-1} f_\lambda e_k,$$

then it follows that

$$u_\lambda - b_m = \sum_{k=1}^{m} \gamma(1-\gamma)^{k-1} (f_\lambda e_k - e_k),$$

so we can make $\|u_\lambda - b_m\|$ arbitrarily small, provided that

$$\|e_\lambda e_k - e_k\| \leq \xi \;\; (k = 1, \ldots, m), \quad (9)$$

with $\xi > 0$ sufficiently small, by using Lemma 2.17 repeatedly and minimizing the obtained ξ_j $(j = 1, \ldots, m)$. Since $\mathrm{Inv}(\mathfrak{A}^\#)$ is open and the mapping $x \mapsto x^{-1}$ is continuous on $\mathrm{Inv}(\mathfrak{A}^\#)$, it follows that $u_\lambda \in \mathrm{Inv}(\mathfrak{A}^\#)$ and $\|u_\lambda^{-1} - t_m\|$ is arbitrarily small when (9) holds, with $\xi > 0$ sufficiently small. Now choose λ_{m+1} so that $\|e_{m+1} z - z\| \leq \xi$ with $\xi > 0$ so small that $u_{\lambda_{m+1}} \in \mathrm{Inv}(\mathfrak{A}^\#)$ and

$$2\kappa \left\| u_{\lambda_{m+1}}^{-1} - t_m \right\| \|z\| + \kappa \|t_m\| \left\| f_{\lambda_{m+1}} z - z \right\| \leq \delta 2^{-(m+1)}. \quad (10)$$

Since $(1 - \gamma + \gamma e_\lambda) f_\lambda = 1$, we have that

$$\begin{aligned}(1 - \gamma + \gamma e_{m+1}) u_{\lambda_{m+1}} &= (1-\gamma)^{m+1} + \gamma(1-\gamma)^m e_{m+1} + \sum_{k=1}^{m} \gamma(1-\gamma)^{k-1} e_k \\ &= (1-\gamma)^{m+1} + \sum_{k=1}^{m+1} \gamma(1-\gamma)^{k-1} e_k \\ &= b_{m+1}.\end{aligned}$$

This implies that $b_{m+1} \in \mathrm{Inv}(\mathfrak{A}^\#)$ and its inverse t_{m+1} is given by $t_{m+1} = u_{\lambda_{m+1}}^{-1} \cdot f_{\lambda_{m+1}}$. Therefore, using (10) and the fact that $\|f_{\lambda_{m+1}}\| \leq 2$, by Lemma 2.17, we find

$$\begin{aligned}\|t_{m+1} z - t_m z\| &= \left\| u_{\lambda_{m+1}}^{-1} \cdot f_{\lambda_{m+1}} z - t_m z \right\| = \left\| u_{\lambda_{m+1}}^{-1} \cdot f_{\lambda_{m+1}} z + t_m f_{\lambda_{m+1}} z - t_m f_{\lambda_{m+1}} z - t_m z \right\| \\ &\leq \kappa \left\| u_{\lambda_{m+1}}^{-1} - t_m \right\| \left\| f_{\lambda_{m+1}} z \right\| + \kappa \|t_m\| \left\| f_{\lambda_{m+1}} z - z \right\| \\ &\leq 2\kappa \left\| u_{\lambda_{m+1}}^{-1} - t_m \right\| \|z\| + \kappa \|t_m\| \left\| f_{\lambda_{m+1}} z - z \right\| \\ &\leq \delta 2^{-(m+1)};\end{aligned}$$

hence λ_{m+1} satisfies (8) and we have a recursive construction for $(\lambda_n)_{n\geq 1}$. Define $y_n := t_n z$. Then $z = b_n y_n$ and $(y_n)_{n\geq 1}$ is a Cauchy sequence in E which must converge to some $y \in E$ and has the property

$$\|z-y\| = \lim_{n\to\infty} \|z - t_n z\| = \lim_{n\to\infty} \|t_0 z - t_n z\| \leq \lim_{n\to\infty} \sum_{k=1}^{n} \|t_{k-1} z - t_k z\| \leq \delta \lim_{n\to\infty} \sum_{k=1}^{n} 2^{-k} = \delta.$$

Also, since $0 < 1-\gamma < 1$, the series in (8) converges to, say

$$a := \sum_{k=1}^{\infty} \gamma(1-\gamma)^{k-1} e_k \in \mathfrak{A},$$

so $\lim_{n\to\infty} b_n = a$ and we conclude that $z = \lim_{n\to\infty} b_n \cdot y_n = a \cdot y$. ∎

It is completely analogous to obtain the statement of Cohen's factorization theorem and the preceding lemma for a right action, that is, when E is a right Banach $\mathfrak{A}$-module. One simply considers the opposite Banach algebra $\mathfrak{A}^{\mathrm{opp}}$ of $\mathfrak{A}$. It follows that E now is a left Banach $\mathfrak{A}^{\mathrm{opp}}$-module that yields the desired properties.

2.2.2 Neo-unital modules and extensions of derivations to larger modules

Lemma 2.19 *Let $\mathfrak{A}$ be a Banach algebra with a bounded right approximate identity, and let E be a Banach $\mathfrak{A}$-bimodule such that $\mathfrak{A} \cdot E = \{0\}$. Then $\mathcal{H}^1(\mathfrak{A}, E^*) = \{0\}$.*

Proof: Since E^* is also a Banach $\mathfrak{A}$-bimodule we have for $\phi \in E^*$, $a \in \mathfrak{A}$ and $x \in E$ that

$$(\phi \cdot a)(x) := \phi(a \cdot x) = \phi(0) = 0;$$

hence $\phi \cdot a = 0$ and so $E^* \cdot \mathfrak{A} = \{0\}$. Let $D \in \mathcal{Z}^1(\mathfrak{A}, E^*)$. Then for $a, b \in \mathfrak{A}$ we have

$$D(ab) = a \cdot D(b) + D(a) \cdot b = a \cdot D(b).$$

Let $(e_\alpha)_\alpha$ be a bounded right approximate identity for $\mathfrak{A}$, and let $\phi \in E^*$ be a w^*-accumulation point of $(D(e_\alpha))_\alpha$, which exists, since this net is bounded and must have a convergent subnet by the Banach-Alaoglu theorem. Without loss of generality, we may assume that $\phi = \lim_\alpha D(e_\alpha)$. It follows that

$$\begin{aligned} D(a) &= w^* - \lim_\alpha D(a \cdot e_\alpha) = w^* - \lim_\alpha a \cdot D(e_\alpha) = a \cdot w^* - \lim_\alpha D(e_\alpha) \\ &= a \cdot \phi = a \cdot \phi - \phi \cdot a \qquad (a \in \mathfrak{A}); \end{aligned}$$

hence $D = \mathrm{ad}_\phi$. ∎

A similar result can be obtained when it is assumed that $\mathfrak{A}$ has a bounded left approximate identity and $E \cdot \mathfrak{A} = \{0\}$.

Definition: Let $\mathfrak{A}$ be a Banach algebra. A Banach $\mathfrak{A}$-bimodule E is called **neo-unital** if $E = \{a \cdot x \cdot b : a, b \in \mathfrak{A}\ x \in E\}$.

Lemma 2.20 *For a Banach algebra $\mathfrak{A}$ with bounded approximate identity the following are equivalent:*

i) $\mathcal{H}^1(\mathfrak{A}, E^*) = \{0\}$ *for each Banach* $\mathfrak{A}$*-bimodule* E.

ii) $\mathcal{H}^1(\mathfrak{A}, E^*) = \{0\}$ *for each neo-unital Banach* $\mathfrak{A}$*-bimodule* E.

Proof: The implication *i)* $\Rightarrow$ *ii)* obviously holds.

ii) $\Rightarrow$ *i)*: Let E be a Banach $\mathfrak{A}$-bimodule and define

$$E_0 := \{a \cdot x \cdot b : a, b \in \mathfrak{A},\ x \in E\} \quad \text{and} \quad E_1 := \{x \cdot b : b \in \mathfrak{A},\ x \in E\}.$$

We will show, by using Theorem 2.18, that E_0 and E_1 are closed submodules of E over $\mathfrak{A}$. Let $B := \mathrm{Sp}\{a \cdot x \cdot b : a, b \in \mathfrak{A},\ x \in E\}$ and consider its closure $\overline{B}$. It is easily checked that $\overline{B}$ is a Banach $\mathfrak{A}$-bimodule and that the bounded approximate identity $(e_\alpha)_\alpha$ in $\mathfrak{A}$, with $\sup_\alpha \|e_\alpha\| = M < \infty$, is a bounded approximate identity for B. Now for $x \in \overline{B}$ and a sequence $(x_n)_{n\geq 1}$ in B such that $\|x - x_n\| \to 0$, we find that

$$\begin{aligned}\|e_\alpha \cdot x - x\| &\leq \|e_\alpha \cdot x - e_\alpha \cdot x_n\| + \|e_\alpha \cdot x_n - x_n\| + \|x_n - x\| \\ &\leq (\kappa\|e_\alpha\| - 1)\|x - x_n\| + \|e_\alpha \cdot x_n - x_n\| \\ &\leq (\kappa M + 1)\|x - x_n\| + \|e_\alpha \cdot x_n - x_n\| \\ &\to (\kappa M + 1)\|x - x_n\|,\end{aligned}$$

so $\|e_\alpha \cdot x - x\|$ can be made arbitrarily small; hence $(e_\alpha)_\alpha$ is a left bounded approximate identity for $\overline{B}$. Similarly, one shows that this also is a right bounded approximate identity for $\overline{B}$. By Theorem 2.18 there are elements $a \in \mathfrak{A}$ and $y_1 \in \overline{B}$ such that $x = a \cdot y_1$. Repeating this, we have $y_1 = y_2 \cdot b$ for some $b \in \mathfrak{A}$ and $y_2 \in \overline{B}$, so $x = a \cdot y_2 \cdot b$. This yields the inclusions

$$E_0 \subset \overline{B} \subset \{a \cdot x \cdot b : a, b \in \mathfrak{A},\ x \in \overline{B}\} \subset E_0;$$

hence $E_0 = \overline{B}$. Analogously, we can show that the closure of $C := \mathrm{Sp}\{x \cdot b : b \in \mathfrak{A},\ x \in E\}$ equals E_1, so E_0 and E_1 are closed submodules of E. Furthermore, we have also shown that E_0 is neo-unital.

Let $D \in \mathcal{Z}^1(\mathfrak{A}, E_1^*)$ and $\pi_0 : E_1^* \to E_0^*$ be the restriction map. Since E_0 is neo-unital and $\pi_0 \circ D \in \mathcal{Z}^1(\mathfrak{A}, E_0^*) = \mathcal{B}^1(\mathfrak{A}, E_0^*)$ by our assumption, there exists a functional $\phi_0 \in E_0^*$ such that $\pi_0 \circ D = \mathrm{ad}_{\phi_0}$. By the Hahn-Banach theorem we have an extension ϕ_1 of ϕ_0 to E_1. Since

$$[D(a) - \mathrm{ad}_{\phi_1}(a)](x) = [\pi_0 \circ D(a) - \mathrm{ad}_{\phi_0}(a)](x) = 0$$

for all $a \in \mathfrak{A}$ and $x \in E_0$, it follows that $D - \mathrm{ad}_{\phi_1} \in \mathcal{Z}^1(\mathfrak{A}, E_1^* \cap E_0^\perp)$. The mapping $\eta : E_1^* \cap E_0^\perp \to (E_1/E_0)^*$ with $\eta(f)(\overline{x}) := f(x)$ is well defined and induces a bounded linear bijection which also is a module homomorphism, so $E_1^* \cap E_0^\perp \cong (E_1/E_0)^*$ as Banach $\mathfrak{A}$-bimodules. Because we have $\mathfrak{A} \cdot (E_1/E_0) = \{0\}$, it follows from Lemma 2.19 that $\mathcal{H}^1(\mathfrak{A}, E_1^* \cap E_0^\perp) = \{0\}$ by using the isomorphism η. Hence, there exists a functional $\psi_1 \in E_1^* \cap E_0^\perp$ such that $D - \mathrm{ad}_{\phi_1} = \mathrm{ad}_{\psi_1}$, that is,

$$D = \mathrm{ad}_{\phi_1} + \mathrm{ad}_{\psi_1} = \mathrm{ad}_{\phi_1 + \psi_1}.$$

Since $D \in \mathcal{Z}^1(\mathfrak{A}, E_1^*)$ was arbitrary, we conclude that $\mathcal{H}^1(\mathfrak{A}, E_1^*) = \{0\}$.

Now, let $D \in \mathcal{Z}^1(\mathfrak{A}, E^*)$ and $\pi_1 : E^* \to E_1^*$ be the restriction map. Then we have $\pi_1 \circ D \in \mathcal{Z}^1(\mathfrak{A}, E_1^*)$ and by the foregoing, there is a $\phi_1 \in E_1^*$ such that $\pi_1 \circ D = \mathrm{ad}_{\phi_1}$. Using the Hahn-Banach theorem again, we have an extension $\phi \in E^*$ of ϕ_1, so that $D - \mathrm{ad}_\phi \in \mathcal{Z}^1(\mathfrak{A}, E_1^\perp)$. Analogously, we have that $E_1^\perp \cong (E/E_1)^*$ as Banach $\mathfrak{A}$-bimodules and $(E/E_1) \cdot \mathfrak{A} = \{0\}$ and Lemma 2.19 yields $\mathcal{H}^1(\mathfrak{A}, (E/E_1)^*) = \{0\}$, so $\mathcal{H}^1(\mathfrak{A}, E_1^\perp) = \{0\}$; hence there is a $\psi \in E_1^\perp$ such that $D - \mathrm{ad}_\phi = \mathrm{ad}_\psi$ or equivalently, $D = \mathrm{ad}_{\phi+\psi}$. Since $D \in \mathcal{Z}^1(\mathfrak{A}, E^*)$ was arbitrary, it finally follows that $\mathcal{H}^1(\mathfrak{A}, E^*) = \{0\}$. ∎

In what follows, we wish to extend a derivation to a larger algebra, that is, if $\mathfrak{A}$ is a Banach algebra which is contained as a closed ideal in another Banach algebra $\mathfrak{B}$, it should extend to $\mathfrak{B}$. We can define a topology, the so called **strict topology**, on $\mathfrak{B}$ with respect to $\mathfrak{A}$ through the collection of seminorms $(p_a)_{a\in\mathfrak{A}}$, where

$$p_a(b) := \|ba\| + \|ab\| \qquad (b \in \mathfrak{B}).$$

Using this topology we are able to prove the following:

Theorem 2.21 *Let $\mathfrak{A}$ be a Banach algebra with bounded approximate identity which is contained as a closed ideal in a Banach algebra $\mathfrak{B}$, let E be a neo-unital Banach $\mathfrak{A}$-bimodule, and let $D \in \mathcal{Z}^1(\mathfrak{A}, E^*)$. Then E is a Banach $\mathfrak{B}$-bimodule in a canonical fashion, and there is a $\tilde{D} \in \mathcal{Z}^1(\mathfrak{B}, E^*)$ such that:*

i) $\tilde{D}|_{\mathfrak{A}} = D$;

ii) $\tilde{D}$ is continuous with respect to the strict topology on $\mathfrak{B}$ and the w^*-topology on E^*.

Proof: First, we will show that E is a Banach $\mathfrak{B}$-bimodule. For $x \in E$, let $a \in \mathfrak{A}$ and $y \in E$ be such that $x = a \cdot y$, since E is neo-unital. Now, for $b \in \mathfrak{B}$, define $b \cdot x := (ba) \cdot y$. We claim that this action is well defined, that is, independent of the choices for a and y. Let $a' \in \mathfrak{A}$ and $y' \in E$ be such that $x = a' \cdot y'$ and let $(e_\alpha)_\alpha$ be a bounded approximate identity for $\mathfrak{A}$. Then

$$(ba) \cdot x = \lim_\alpha (be_\alpha)a \cdot y = \lim_\alpha (be_\alpha)a' \cdot y' = (ba') \cdot y'$$

for all $b \in \mathfrak{B}$. The fact that this defines an additive and bounded action, can be found in [4, Thm. 3.1], so we conclude that it turns E into a left Banach $\mathfrak{B}$-module. Similarly, we can define a right Banach $\mathfrak{B}$-module structure on E, so that E becomes a Banach $\mathfrak{B}$-bimodule.

In order to extend D, define $\tilde{D} : \mathfrak{B} \to E^*$ by

$$\tilde{D}(b) := w^* - \lim_\alpha (D(be_\alpha) - b \cdot D(e_\alpha)). \tag{11}$$

We will prove that $\tilde{D}$ is well defined by showing that the limit in (11) exists. Let $x \in E$ and, again using that E is neo-unital, let $a \in \mathfrak{A}$ and $y \in E$ be such that $x = y \cdot a$. Then, when using the action of $\mathfrak{A}$ on E^*, we obtain

$$\begin{aligned}
[D(be_\alpha) - b \cdot D(e_\alpha)](x) &= [D(be_\alpha) - b \cdot D(e_\alpha)](y \cdot a) \\
&= [a \cdot D(be_\alpha) - ab \cdot D(e_\alpha)](y) \\
&= [D(abe_\alpha) - D(a) \cdot be_\alpha - D(abe_\alpha) + D(ab)e_\alpha](y) \\
&= [D(ab) \cdot e_\alpha - D(a) \cdot be_\alpha](y) \\
&= D(ab)(e_\alpha \cdot y) - D(a)(be_\alpha \cdot y) \\
&\to D(ab)(y) - D(a)(b \cdot y)
\end{aligned}$$

for all $b \in \mathfrak{B}$, by using the fact that E is neo-unital again for x, so the limit in (11) exists. If $M := \sup_\alpha \|e_\alpha\| < \infty$, then for $b \in \mathfrak{B}$ it follows that

$$\|\tilde{D}(b)\| \leq \liminf_\alpha \|D(be_\alpha) - b \cdot D(e_\alpha)\| \leq \sup_\alpha \|D(be_\alpha) - b \cdot D(e_\alpha)\| \leq M(\kappa + 1)\|D\|\|b\|,$$

where $\kappa > 0$ is such that $\|b \cdot x\| \leq \kappa \|b\| \|x\|$ for all $b \in \mathfrak{B}$ and all $x \in E$. This implies that $\tilde{D}$ is bounded. Moreover, for $a \in \mathfrak{A}$ we have that

$$\tilde{D}(a) = w^* - \lim_\alpha (D(ae_\alpha) - a \cdot D(e_\alpha)) = w^* - \lim_\alpha D(a) \cdot e_\alpha = D(a);$$

hence $\tilde{D}$ extends D. Furthermore, for $b \in \mathfrak{B}$ and $a \in \mathfrak{A}$ we have

$$\begin{aligned} \tilde{D}(b) \cdot a &= w^* - \lim_\alpha (D(be_\alpha) \cdot a - b \cdot D(e_\alpha) \cdot a) \\ &= w^* - \lim_\alpha (D(be_\alpha a) - be_\alpha \cdot D(a) - b \cdot D(e_\alpha a) + be_\alpha \cdot D(a)) \\ &= D(ba) - b \cdot D(a). \end{aligned} \tag{12}$$

To prove that $\tilde{D}$ is continuous with respect to the strict topology on $\mathfrak{B}$ and the w^*-topology on E^*, let $(b_\alpha)_\alpha$ be a net in $\mathfrak{B}$ converging to b with respect to the strict topology on $\mathfrak{B}$, that is, $p_a(b_\alpha - b) \to 0$ for all $a \in \mathfrak{A}$. Let $x \in E$. Since E is neo-unital, there exist elements $a, c \in \mathfrak{A}$ such that $x = a \cdot y \cdot c$ for some $y \in E$. Using (12) we now find that

$$\begin{aligned} |\tilde{D}(b_\alpha - b)(x)| &= |\tilde{D}(b_\alpha - b)(a \cdot y \cdot c)| = |[\tilde{D}(b_\alpha - b) \cdot a](y \cdot c)| \\ &= |[D(a(b_\alpha - b)) - (b_\alpha - b) \cdot D(a)](y \cdot c)| \\ &= |[c \cdot D(a(b_\alpha - b)) - c(b_\alpha - b) \cdot D(a)](y)| \\ &\leq \|D\| \|y\| (\|c\| p_a(b_\alpha - b) + \|a\| p_c(b_\alpha - b)) \\ &\to 0; \end{aligned}$$

hence $w^* - \lim_\alpha \tilde{D}(b_\alpha) = \tilde{D}(b)$.

It remains to be shown that $\tilde{D}$ is a derivation. It follows from the definition of the strict topology on $\mathfrak{B}$ that $be_\alpha \to b$ for all $b \in \mathfrak{B}$. Let $b, c \in \mathfrak{B}$. Then

$$\begin{aligned} \tilde{D}(bc) &= w^* - \lim_\alpha \left(w^* - \lim_\beta D((be_\alpha)(ce_\beta)) \right) \\ &= w^* - \lim_\alpha \left(w^* - \lim_\beta ((be_\alpha) D(ce_\beta) + D(be_\alpha) \cdot ce_\beta) \right) \\ &= w^* - \lim_\alpha \left(be_\alpha \cdot \tilde{D}(c) + D(be_\alpha) \cdot c \right) \\ &= b \cdot \tilde{D}(c) + \tilde{D}(b) \cdot c; \end{aligned}$$

hence $\tilde{D} \in \mathcal{Z}^1(\mathfrak{B}, E^*)$. ∎

2.3 Johnson's theorem

Enough preparations have been made to state a fundamental theorem for our investigations:

Theorem 2.22 *Let E be a neo-unital Banach $L^1(G)$-bimodule and $D \in \mathcal{Z}^1(L^1(G), E^*)$. Then E is a Banach $M(G)$-bimodule in a canonical fashion, and there exists a derivation $\tilde{D} \in \mathcal{Z}^1(M(G), E^*)$ that extends D and is continuous with respect to the strict topology on $M(G)$ and the w^*-topology on E^*. In particular, $\tilde{D}$ is uniquely determined by its values on $\{\delta_g : g \in G\}$.*

Proof: All the properties follow immediately from Theorem 2.21 and our previous findings, except the last. In this respect, we will show that the subspace of discrete measures, which equals the closure of $\mathrm{Sp}\{\delta_g : g \in G\}$, is strictly dense in $M(G)$. First, notice that $M(G)$ can be isometrically embedded in $C_b(G)^*$ via the map

$$\Psi(\mu)(f) \mapsto \int_G f(g)d\mu(g) \qquad (\mu \in M(G),\ f \in C_b(G)),$$

so we view $M(G)$ as a subspace of $C_b(G)^*$. Consider the sequence space

$$\ell^1(G) := \left\{ f : G \to \mathbb{R} : \mathrm{supp}(f) \text{ is countable and } \sum_{g\in G} |f(g)| < \infty \right\}$$

and the map $\Phi : \ell^1(G) \to M(G)$ which is defined by

$$f \mapsto \sum_{g\in G} f(g)\delta_g \qquad (f \in \ell^1(G)).$$

The sequence space $\ell^1(G)$ is a Banach space and under the convolution

$$[f_1 * f_2](g) := \sum_{h\in G} f_1(gh)f_2(h^{-1})$$

it becomes a Banach algebra with unit $f := \chi_{\{e\}}$. Clearly, the map Φ is linear and similar as in the case of $L^1(G)$, it follows that Φ is a unital, isometric homomorphism between Banach algebras. Moreover, the range of Φ equals the closure of $\mathrm{Sp}\{\delta_g : g \in G\}$.

Let $\xi \in C_b(G)$ and suppose that $\Phi(f)(\xi) = 0$ for all $f \in \ell^1(G)$. Then we must have that

$$\delta_g(\xi) = \int_G \xi(h)d\delta_g(h) = \xi(g) = 0 \qquad (g \in G),$$

so $\xi = 0$; hence $\ell^1(G)$ is w^*-dense in $M(G)$.

Now, the weak strict topology on $M(G)$ with respect to $L^1(G)$ is defined by the seminorms $p_{f,\phi}$, where

$$p_{f,\phi}(\mu) := \left| \int_G [\mu * f](g)\phi(g)dm_G(g) \right| + \left| \int_G [f * \mu](g)\phi(g)dm_G(g) \right|$$

for $f \in L^1(G),\ \phi \in L^\infty(G),\ \mu \in M(G)$. By using the modification of Fubini's theorem and (3), for the first integral, we obtain the equality

$$\begin{aligned}\int_G [\mu * f](g)dm_G(g) &= \int_G \left(\int_G f(h^{-1}g)d\mu(h) \right) \phi(g)dm_G(g) \\ &= \int_G \left(\int_G f(h^{-1}g)\phi(g)dm_G(g) \right) d\mu(h).\end{aligned}$$

Define the function ξ on G by

$$\xi(h) := \int_G f(h^{-1}g)\phi(g)dm_G(g).$$

Let $(h_\alpha)_\alpha$ be a net in G such that $h_\alpha \to h$ for some $h \in G$. Then by the continuity of the inversion and composition on G, we also have the convergence $hh_\alpha^{-1} \to e$. By Lemma 2.11 and (1) we have that

$$\begin{aligned}\left|\int_G (f(h_\alpha^{-1}g) - f(h^{-1}g))\phi(g)dm_G(g)\right| &\leq \|\phi\| \int_G |f(h_\alpha^{-1}g) - f(h^{-1}g)|dm_G(g) \\ &= \|\phi\| \int_G |L_h(L_{h_\alpha^{-1}}f - L_{h^{-1}}f)(g)|dm_G(g) \\ &= \|\phi\|\|L_{hh_\alpha^{-1}}f - f\| \to 0;\end{aligned}$$

hence ξ is continuous. Furthermore, we have the inequality

$$\sup_{h\in G} |\xi(h)| \leq \sup_{h\in G} \int_G |f(h^{-1}g)\phi(g)|dm_G(g) \leq \|\phi\|\|f\|,$$

so it follows that $\xi \in C_b(G)$. Similarly, for the second integral, using (4), we have the identity

$$\begin{aligned}\int_G [f * \mu](g)dm_G(g) &= \int_G \left(\int_G f(gh^{-1})\Delta(h^{-1})\phi(g)d\mu(h)\right) dm_G(g) \\ &= \int_G \left(\int_G f(gh^{-1})\Delta(h^{-1})\phi(g)dm_G(g)\right) d\mu(h).\end{aligned}$$

Now, define the function η on G by

$$\eta(h) := \int_G f(gh^{-1})\Delta(h^{-1})\phi(g)dm_G(g).$$

Analogously, it follows from Lemma 2.11 and Lemma 2.10 that

$$|\Delta(h_\alpha^{-1}) - \Delta(h^{-1})|\|\phi\| \left(\int_G |f(gh_\alpha^{-1}) - f(gh^{-1})|dm_G(g) + \int_G f(gh^{-1})dm_G(g)\right) \to 0$$

and also that

$$|\Delta(h^{-1})|\|\phi\| \int_G |f(gh_\alpha^{-1}) - f(gh^{-1})|dm_G(g) \to 0;$$

hence η is continuous. Finally, Lemma 2.10 yields the inequality

$$\sup_{h\in G} |\eta(h)| \leq \sup_{h\in G} \|\phi\| \int_G |R_{h^{-1}}f(g)\Delta(h^{-1})|dm_G(g) = \|\phi\|\|f\|,$$

so $\eta \in C_b(G)$. Our findings now imply that if $(\mu_\alpha)_\alpha$ is a net in $M(G)$ that converges in the w^*-topology to μ in $M(G)$, then we also have the convergence $p_{f,\phi}(\mu_\alpha - \mu) \to 0$ for all $f \in L^1(G)$ and $\phi \in L^\infty(G)$; hence the identity operator from $M(G)$ with the w^*-topology onto $M(G)$ with the weakly strict topology is continuous. We conclude from this property that $\ell^1(G)$ must be weakly strictly dense in $M(G)$ also.

It follows from [15, Ex. A.2.4] that $(M(G), \text{strict})^* = (M(G), w - \text{strict})^*$ and since $\ell^1(G)$ is a convex subspace of $M(G)$, we have

$$\overline{\ell^1(G)}^{\text{strict}} = \overline{\ell^1(G)}^{w-\text{strict}} = M(G);$$

hence $\ell^1(G)$ is strictly dense in $M(G)$. ∎

Since $C_b(G) \subset L^\infty(G)$ is a subspace containing the constant functions and is also closed under complex conjugation, we immediately have the following result:

Lemma 2.23 *Let G be a locally compact group. If G is amenable, then there is a left invariant mean on $C_b(G)$.*

Before we continue, we need an important fixed point theorem:

Theorem 2.24 (Day's fixed point theorem) *For a locally compact group G the following are equivalent:*

i) G is amenable.

ii) If G acts affinely on a compact, convex subset K of a locally convex vector space E, that is,

$$g \cdot (tx + (1-t)y) = t(g \cdot x) + (1-t)(g \cdot y) \qquad (g \in G,\ x, y \in K,\ t \in [0,1]),$$

such that $\phi : G \times K \to K$, $(g, x) \mapsto g \cdot x$ is separately continuous, then there is a fixed point $\xi \in K$ with respect to this map, that is, $g \cdot \xi = \xi$ for all $g \in G$.

Proof: *i)* $\Rightarrow$ *ii)*: Fix $x_0 \in K$ and let $A(K)$ denote the set of all continuous affine functions on K. For $f \in E^*$, $x, y \in K$ and $0 < t < 1$, we have that $tx + (1-t)y \in K$ and

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y),$$

so $f|_K$ is an affine function and we conclude that $\{f|_K : f \in E^*\} \subset A(K)$. For every $f \in A(K)$ we define the function $\varphi_f : G \to \mathbb{C}$ by $\varphi_f(g) := f(g \cdot x_0)$. Since ϕ is continuous in the first variable, it follows that φ_f is continuous. Moreover, we have that

$$\sup_{g \in G} |\varphi_f(g)| = \sup_{g \in G} |f(g \cdot x_0)| \leq \sup_{x \in K} |f(x)| = \|f\|,$$

so $\varphi_f \in C_b(G)$. Since G is amenable, it follows from Lemma 2.23 that there exists a left invariant mean m on $C_b(G)$ and we claim that there is a $\xi \in K$ such that $m(\varphi_f) = f(\xi)$ for all $f \in A(K)$ and that ξ is the desired fixed point. Consider the set

$$\Omega := \left\{ \sum_{k=1}^{n} t_k \delta_{g_k} : n \in \mathbb{N}_+,\ g_1, \ldots, g_n \in G,\ t_k \geq 0,\ t_1 + \cdots + t_n = 1 \right\}.$$

When we view $M(G)$ as a subspace of $C_b(G)^*$ again, it follows that

$$\int_G 1_G(g) d\left(\sum_{k=1}^{n} t_k \delta_{g_k}\right)(g) = \sum_{k=1}^{n} t_k = 1 = \sum_{k=1}^{n} t_k \delta_{g_k}(G) = \left\| \sum_{k=1}^{n} t_k \delta_{g_k} \right\|,$$

so Ω is a set of means for $C_b(G)$. Suppose there exists a mean $\tilde{m}$ on $C_b(G)$ that is not in the w^*-closure of Ω. The w^*-closure of Ω is convex and the spaces $(C_b(G)^*, w^*)^*$ and $C_b(G)$ coincide, so by the Hahn-Banach theorem there exists a function $f \in C_b(G)$ such that $\Re(\psi_f(\tilde{m})) = \Re(\tilde{m}(f)) < \alpha$ and $\Re(\psi_f(\omega)) \geq \alpha$ for all ω in the w^*-closure of Ω and some $\alpha \in \mathbb{R}$. In particular, this means that $\Re(\psi_f(\delta_g)) = \Re(f(g)) \geq \alpha$ for all $g \in G$, so $\Re(f) \geq \alpha$, but now by Lemma 2.15 we have $\Re(\tilde{m}(f)) \geq \Re(\tilde{m}(\alpha)) = \alpha$, which is absurd, so Ω is w^*-dense in $M(G)$. Let $(m_\alpha)_\alpha$ be a net in Ω such that $w^* - \lim_\alpha m_\alpha = m$. Now, for every m_α we have

$$m_\alpha(\varphi_f) = \sum_{k=1}^{n_\alpha} t_k^\alpha f(g_k^\alpha \cdot x_0) = f\left(\sum_{k=1}^{n_\alpha} t_k^\alpha (g_k^\alpha \cdot x_0)\right)$$

and $x_\alpha := \sum_{k=1}^{n_\alpha} t_k^\alpha (g_k^\alpha \cdot x_0) \in K$, since $f \in A(K)$ and K being convex, so without loss of generality, we may assume that $x_\alpha \to \xi$ for some $\xi \in K$, because K is also compact. It follows that for all $f \in A(K)$ we now have

$$f(\xi) = \lim_\alpha f(x_\alpha) = \lim_\alpha m_\alpha(\varphi_f) = m(\varphi_f).$$

For $g \in G$ and $\vartheta \in E^*$, the function $f_{g,\vartheta} : K \to \mathbb{C}$ defined by $f_{g,\vartheta}(x) := \vartheta(g \cdot x)$ is continuous, since ϕ is continuous in the second variable and because G acts affinely on K, we find that $f_{g,\vartheta} \in A(K)$. We have the identity

$$\varphi_{f_{g,\vartheta}}(h) = f_{g,\vartheta}(h \cdot x_0) = \vartheta(gh \cdot x_0) = f_{e,\vartheta}(gh \cdot x_0) = \varphi_{f_{e,\vartheta}}(gh) = (\delta_{g^{-1}} * \varphi_{f_{e,\vartheta}})(h)$$

for all $g, h \in G$ and $\vartheta \in E^*$, and this yields the equality

$$\vartheta(g \cdot \xi) = f_{g,\vartheta}(\xi) = m(\varphi_{f_{g,\vartheta}}) = m((\delta_{g^{-1}} * \varphi_{f_{e,\vartheta}})) = m(\varphi_{f_{e,\vartheta}}) = f_{e,\vartheta}(\xi) = \vartheta(\xi)$$

for all $g \in G$ and $\vartheta \in E^*$, but E^* separates the points of E, so it follows that $g \cdot \xi = \xi$ for all $g \in G$ and ξ is our desired fixed point.

ii) $\Rightarrow$ *i)*: For $g \in G$ and $f \in C_b(G)$, we have that $\delta_g * f \in C_b(G)$ by definition of the convolution of $M(G)$ on $L^\infty(G)$ and define the set

$$L_G := \{f \in C_b(G) : g \mapsto \delta_g * f \text{ is continuous}\}.$$

By [15, Prop. A.2.3] this defines a closed invariant subspace of $L^\infty(G)$, which allows us to consider the set $\mathcal{M}$ of means on L_G. Define a group action on $\mathcal{M}$ by $g \cdot m(f) := m(\delta_g * f)$ for all $f \in L_G$ and $m \in \mathcal{M}$. By [15, Prop. A.2.5] this action is well defined and clearly, we have $\|g \cdot m\| \leq 1$ for all $g \in G$. Because $\delta_g * 1_G = 1_G$ for all $g \in G$, the identity $g \cdot m(1_G) = m(1_G) = 1$ implies that $g \cdot m \in \mathcal{M}$ for all $g \in G$. Clearly, we have that $\mathcal{M}$ is a convex subset of the locally convex space L_G^* equipped with the w^*-topology and by the Banach-Alaoglu theorem, it follows that $\mathcal{M}$ is w^*-compact. For $g \in G$, $m_1, m_2 \in \mathcal{M}$ and $t \in [0,1]$ we have

$$\begin{aligned} g \cdot (tm_1 + (1-t)m_2)(f) &= (tm_1 + (1-t)m_2)(\delta_g * f) = t(g \cdot m_1)(f) + (1-t)(g \cdot m_2)(f) \\ &= (t(g \cdot m_1) + (1-t)(g \cdot m_2))(f) \qquad (f \in L_G), \end{aligned}$$

so G acts affinely on $\mathcal{M}$. Now, consider the function $\phi : G \times \mathcal{M} \to \mathcal{M}$ defined by $(g, m) \mapsto g \cdot m$. By definition of L_G and the fact that all $m \in \mathcal{M}$ are continuous, for a net $(g_\alpha)_\alpha$ in G with $g_\alpha \to g$ for some $g \in G$ we have

$$g_\alpha \cdot m(f) = m(\delta_{g_\alpha} * f) \to m(\delta_g * f) = g \cdot m(f) \qquad (f \in L_G),$$

so ϕ is continuous in the first variable. On the other hand, if $(m_\alpha)_\alpha$ is a w^*-convergent net in $\mathcal{M}$ with $m_\alpha \to m$ for some $m \in \mathcal{M}$, then clearly, we have

$$g \cdot m_\alpha(f) = m_\alpha(\delta_g * f) \to m(\delta_g * f) = g \cdot m(f) \qquad (f \in L^\infty(G)),$$

so ϕ is continuous in the second variable as well. By the hypothesis, there is a mean $\bar{m} \in \mathcal{M}$ such that $g \cdot \bar{m} = \bar{m}$ for all $g \in G$ and this implies that $\bar{m}(\delta_g * f) = \bar{m}(f)$ for all $f \in L_G$ and $g \in G$; hence $\bar{m}$ is a left invariant mean on L_G. Finally, by [15, Thm. 1.1.9] we have that G must be amenable. ∎

Theorem 2.25 (Johnson's theorem) *Let G be a locally compact group. Then the following statements are equivalent:*

i) *G is amenable.*

ii) *$\mathcal{H}^1(L^1(G),(L^\infty(G)/\mathbb{C}1_G)^*)=\{0\}$ for the two representations of $L^1(G)$ defined by*

$$f\cdot\phi:=f*\phi \quad \text{and} \quad \phi\cdot f:=\left(\int_G f(g)dm_G(g)\right)\phi \quad (f\in L^1(G),\ \phi\in L^\infty(G)).$$

iii) *$\mathcal{H}^1(L^1(G),E^*)=\{0\}$ for every Banach $L^1(G)$-bimodule E.*

Proof: *i)* $\Rightarrow$ *iii)*: Let E be a Banach $L^1(G)$-bimodule. It follows from Lemma 2.20 that we may assume, without any loss of generality, that E is neo-unital. Consider $D\in\mathcal{Z}^1(L^1(G),E^*)$ and let $\tilde{D}\in\mathcal{Z}^1(M(G),E^*)$ be the extension of D according to Theorem 2.22. We will show that $\tilde{D}\in\mathcal{B}^1(M(G),E^*)$ by using Theorem 2.24. Define the set

$$K:=\{\tilde{D}(\delta_g)\cdot\delta_{g^{-1}}:g\in G\}$$

and let Ω be the w^*-closure of the convex hull $\mathrm{co}(K)$. Since we have the equality $\|\delta_g\|=1$ for all $g\in G$, it follows that for the constant $\kappa>0$ corresponding to the module action of $M(G)$ on E^* we have $\|\psi\|\leq\kappa\|\tilde{D}\|$ for all $\psi\in\Omega$ and the Banach-Alaoglu theorem implies that Ω is w^*-compact. Define an action of G on E^* by

$$g\cdot\phi:=\delta_g\cdot\phi\cdot\delta_{g^{-1}}+\tilde{D}(\delta_g)\cdot\delta_{g^{-1}} \quad (g\in G,\ \phi\in E^*). \tag{13}$$

We claim that (13) induces a group action on E^*. Let $g,h\in G$ and $\phi\in E^*$. Then the derivation property of $\tilde{D}$ yields

$$\begin{aligned}(gh)\cdot\phi&=\delta_{gh}\cdot\phi\cdot\delta_{(gh)^{-1}}+\tilde{D}(\delta_{gh})\cdot\delta_{(gh)^{-1}}\\&=\delta_g\cdot(\delta_h\cdot\phi\cdot\delta_{h^{-1}})\cdot\delta_{g^{-1}}+\Big(\delta_g\cdot\tilde{D}(\delta_h)+\tilde{D}(\delta_g)\cdot\delta_h\Big)\cdot\delta_{h^{-1}}*\delta_{g^{-1}}\\&=\delta_g\cdot\Big(\delta_h\cdot\phi\cdot\delta_{h^{-1}}+\tilde{D}(\delta_h)\cdot\delta_{h^{-1}}\Big)\cdot\delta_{g^{-1}}+\tilde{D}(\delta_g)\cdot\delta_{g^{-1}}\\&=g\cdot(h\cdot\phi)\end{aligned}$$

and since $\delta_e\cdot\phi=\phi\cdot\delta_e=\phi$ for all $\phi\in E^*$, it follows that

$$\tilde{D}(\delta_e)=\tilde{D}(\delta_e*\delta_e)=\delta_e\cdot\tilde{D}(\delta_e)+\tilde{D}(\delta_e)\cdot\delta_e=\tilde{D}(\delta_e)+\tilde{D}(\delta_e),$$

so $\tilde{D}(\delta_e)=0$ and we conclude that

$$e\cdot\phi=\delta_e\cdot\phi\cdot\delta_e+\tilde{D}(\delta_e)\cdot\delta_e=\phi,$$

which proves our claim.

Furthermore, for $g\in G$, $\phi_1,\phi_2\in E^*$ and $t\in[0,1]$, the linearity of $\tilde{D}$ and the fact that E^* is a Banach $M(G)$-bimodule imply that

$$g\cdot(t\phi_1+(1-t)\phi_2)=t(g\cdot\phi_1)+(1-t)(g\cdot\phi_2);$$

hence G acts affinely on E^*. Let $(g_\alpha)_\alpha$ be a net G with $g_\alpha\to g$. By Lemma 2.11 the nets $(\delta_{g_\alpha})_\alpha$ and $(\delta_{g_\alpha^{-1}})_\alpha$ converge, in the strict topology, to δ_g, respectively $\delta_{g^{-1}}$; hence

$\tilde{D}(\delta_{g_\alpha}) \to \tilde{D}(\delta_g)$ in the w^*-topology and $g_\alpha \cdot \phi \to g \cdot \phi$, since E is a Banach $M(G)$-bimodule. On the other hand, if $(\phi_\alpha)_\alpha$ is a net in E^* that converges in the w^*-topology to $\phi \in E^*$, then $\delta_g \cdot \phi_\alpha \cdot \delta_{g^{-1}} \to \delta_g \cdot \phi \cdot \delta_{g^{-1}}$ in the w^*-topology, also because E is a Banach $M(G)$-bimodule. We conclude that the group action is continuous in both variables with respect to the strict topology on $M(G)$ and the w^*-topology on E^*. Let $\phi \in K$ be such that $\phi = \tilde{D}(\delta_h) \cdot \delta_{h^{-1}}$ for some $h \in G$ and let $g \in G$. Then

$$\begin{aligned} g \cdot \phi = g \cdot \left(\tilde{D}(\delta_h) \cdot \delta_{h^{-1}}\right) &= \delta_g \cdot \left(\tilde{D}(\delta_h) \cdot \delta_{h^{-1}}\right) \cdot \delta_{g^{-1}} + \tilde{D}(\delta_g) \cdot \delta_{g^{-1}} \\ &= \left(\delta_g \cdot \tilde{D}(\delta_h)\right) \cdot \delta_{(gh)^{-1}} + \left(\tilde{D}(\delta_g) \cdot \delta_h\right) \cdot \delta_{(gh)^{-1}} \\ &= \left(\tilde{D}(\delta_g * \delta_h) - \tilde{D}(\delta_g) \cdot \delta_h\right) \cdot \delta_{(gh)^{-1}} + \left(\tilde{D}(\delta_g) \cdot \delta_h\right) \cdot \delta_{(gh)^{-1}} \\ &= \tilde{D}(\delta_{gh}) \cdot \delta_{(gh)^{-1}} \in K. \end{aligned}$$

Moreover, as G acts affinely on E^*, we have that $\mathrm{co}(K)$ is invariant under the group action and since the group action is also continuous in the second variable, we find that for $g \in G$ and $\phi \in \Omega$ with a net $(\phi_\alpha)_\alpha$ in $\mathrm{co}(K)$ such that $\phi_\alpha \to \phi$ in w^*-topology, we also have $g \cdot \phi_\alpha \to g \cdot \phi$; hence Ω is invariant under the group action. By Theorem 2.24 there exists a functional $\psi \in \Omega$ such that

$$g \cdot \psi = \delta_g \cdot \psi \cdot \delta_{g^{-1}} + \tilde{D}(\delta_g) \cdot \delta_{g^{-1}} = \psi \quad (g \in G),$$

so $\tilde{D}(\delta_g) = \psi \cdot \delta_g - \delta_g \cdot \psi$ for all $g \in G$; hence $\tilde{D} = \mathrm{ad}_{-\psi}$ by Theorem 2.22.

ii) $\Rightarrow$ *i)*: The modification of Fubini's theorem and (1) show that the right action of $L^1(G)$ on $L^\infty(G)$ is well defined, since

$$\int_G [f_1 * f_2](g) dm_G(g) = \int_G f_1(g) dm_G(g) \int_G f_2(g) dm_G(g).$$

Clearly, we have that $\mathbb{C}1_G$ is a Banach $L^1(G)$-submodule of $L^\infty(G)$, hence $L^\infty(G)/\mathbb{C}1_G$ is a Banach $L^1(G)$-bimodule. Let $n \in L^\infty(G)^*$ be such that $n(1_G) = 1$ and consider the inner derivation $\mathrm{ad}_n : L^1(G) \to L^\infty(G)^*$. Since

$$\mathrm{ad}_n(f)(1_G) = n \cdot f(1_G) - f \cdot n(1_G) = \left(\int_G f(g) dm_G(g)\right)(n(1_G) - n(1_G)) = 0$$

for all $f \in L^1(G)$, this defines a derivation mapping into $(\mathbb{C}1_G)^\perp \cong (L^\infty(G)/\mathbb{C}1_G)^*$, so there is a functional $\tilde{n} \in (\mathbb{C}1_G)^\perp$ such that $\mathrm{ad}_n = \mathrm{ad}_{\tilde{n}}$. Let $\tilde{m} := n - \tilde{n}$. We have that $\tilde{m}(1_G) = 1$ and for $\phi \in L^\infty(G)$ and $0 \leq f \in L^1(G)$ with $\|f\| = 1$, it follows that

$$\begin{aligned} \tilde{m}(f * \phi) &= (n - \tilde{n})(f * \phi) = (n \cdot f - \tilde{n} \cdot f)(\phi) = (f \cdot n - \tilde{n} \cdot f - \mathrm{ad}_n(f))(\phi) \\ &= (f \cdot n - \tilde{n} \cdot f - \mathrm{ad}_{\tilde{n}}(f))(\phi) = (f \cdot n - f \cdot \tilde{n})(\phi) = (n - \tilde{n})(\phi \cdot f) \\ &= \tilde{m}(\phi \cdot f) = \tilde{m}(\phi). \end{aligned}$$

For $g \in G$ we have that $f * \delta_g = \Delta(g^{-1}) R_{g^{-1}} f$ by (4), so $0 \leq f * \delta_g \in L^1(G)$ and $\|f * \delta_g\| = 1$ by Lemma 2.10. Also, if we view ϕ as an element of $L^1(G)^*$, then it follows that

$$\begin{aligned} [\delta_g * \psi_\phi](f) = \psi_\phi(f * \delta_g) &= \Delta(g^{-1}) \int_G R_{g^{-1}} f(h) \phi(h) dm_G(h) \\ &= \Delta(g^{-1}) \int_G R_{g^{-1}}(f R_g \phi)(h) dm_G(h) \\ &= \int_G f(h) R_g \phi(h) dm_G(h) = \psi_{R_g \phi}(f) \qquad (f \in L^1(G)), \end{aligned}$$

so $\delta_g * \phi = R_g\phi \in L^\infty(G)$. In view of these results, we now have the identity

$$\tilde{m}(\delta_g * \phi) = \tilde{m}(f * (\delta_g * \phi)) = \tilde{m}((f * \delta_g) * \phi)) = \tilde{m}(\phi) \qquad (g \in G, \phi \in L^\infty(G))$$

and since $L^\infty(G)$ is a Banach lattice, for $0 \leq \phi \in L^\infty(G)$ we have the identity

$$|\tilde{m}|(\phi) = \sup\{\tilde{m}(\varphi) : |\varphi| \leq \phi\},$$

from which we can conclude that

$$\begin{aligned}|\tilde{m}|(\delta_g * \phi) &= \sup\{\tilde{m}(\varphi) : |\varphi| \leq R_g\phi\} = \sup\{\tilde{m}(\varphi) : |R_{g^{-1}}\varphi| \leq \phi\} \\ &= \sup\{\tilde{m}(R_{g^{-1}}\varphi) : |R_{g^{-1}}\varphi| \leq \phi\} \\ &= \sup\{\tilde{m}(\varphi) : |\varphi| \leq \phi\} = |\tilde{m}|(\phi);\end{aligned}$$

hence $|\tilde{m}|$ is positive and left invariant with $|\tilde{m}|(1_G) \geq 1$, so $m := (|\tilde{m}|(1_G))^{-1}|\tilde{m}|$ is a left invariant mean on $L^\infty(G)$ by Lemma 2.15 and it follows that G is amenable.

iii) $\Rightarrow$ *ii)*: This implication is obvious. ∎

2.3.1 Johnson's theorem in an ordered context

In order to state the following concluding corollary of this chapter, we need the notion of positivity when considering left and right actions of $L^1(G)$ on a Banach $L^1(G)$-module E which is an ordered Banach space. We say that an ordered Banach space E is a **positive Banach $L^1(G)$-bimodule** if all left actions $\varphi : L^1(G) \to B(E)$ are positive, that is, $\varphi(f)(x) \geq 0$ whenever $0 \leq x \in E$ and $0 \leq f \in L^1(G)$ and similarly, all right actions are positive as well. If E is a Banach lattice we refer to E as a **Banach lattice $L^1(G)$-bimodule**.

Corollary 2.26 (Johnson's theorem for Banach lattices) *Let G be a locally compact group. Then the following are equivalent:*

- *i) G is amenable.*
- *ii) $\mathcal{H}^1(L^1(G), \{e\}^\perp) = \{0\}$ for all unital positive Banach lattice $L^1(G)$-bimodules $E \ni e$ for which $\mathbb{C}e$ is invariant.*

Proof: *i)* $\Rightarrow$ *ii)*: This follows immediately form Theorem 2.25, because $\{e\}^\perp \cong (E/\mathbb{C}e)^*$.

ii) $\Rightarrow$ *i)*: Since $L^\infty(G)$ is a unital Banach lattice and the actions

$$f \cdot \phi := f * \phi \quad \text{and} \quad \phi \cdot f := \left(\int_G f(g)dm_G(g)\right)\phi \quad (f \in L^1(G),\ \phi \in L^\infty(G))$$

are positive, we find that $L^\infty(G)$ is a unital positive Banach lattice $L^1(G)$-bimodule. As we saw in the proof of Theorem 2.25, these actions turn the quotient space $L^\infty(G)/\mathbb{C}1_G$ into a Banach $L^1(G)$-bimodule, so the amenability of G follows analogously. ∎

3 Hochschild cohomology groups of Banach algebras

In this chapter we will define what amenability is in the context of general Banach algebras and will characterize them in terms of cohomology groups, the so called Hochschild cohomology groups. Having this, we turn to general ordered Banach algebras and investigate if we have a similar characterization of amenability in this ordered context. Finally, we consider Banach lattice algebras and construct alternative Hochschild cohomology groups specifically for this situation in order to obtain a new notion of amenability in this case and deduce an analogy of the properties derived for amenable Banach algebras.

Taking Johnson's theorem into account, it would be practical to characterize a Banach algebra $\mathfrak{A}$ as amenable in the following way:

Definition: A Banach algebra $\mathfrak{A}$ is said to be **amenable** if $\mathcal{H}^1(\mathfrak{A}, E^*) = \{0\}$ for all Banach $\mathfrak{A}$-bimodules E.

3.1 Constructing Hochschild cohomology groups of order $n \in \mathbb{N}_+$

In the previous chapter we defined $\mathcal{H}^1(\mathfrak{A}, E) := \mathcal{Z}^1(\mathfrak{A}, E)/\mathcal{B}^1(\mathfrak{A}, E)$ to be the first order Hochschild cohomology group with coefficients in E for a Banach algebra $\mathfrak{A}$ and a Banach $\mathfrak{A}$-bimodule E, which indicates the possibility of defining cohomology groups for higher orders $n \in \mathbb{N}_+$. As we shall see, this is indeed the case.

Definition: Let $\mathfrak{A}$ be a Banach algebra and E a Banach $\mathfrak{A}$-bimodule.

i) Let $\mathcal{L}^0(\mathfrak{A}, E) := E$ and for $n \in \mathbb{N}_+$, let

$$\mathcal{L}^n(\mathfrak{A}, E) := \{T : \mathfrak{A}^n \to E : \ T \text{ is bounded and } n\text{-linear}\}.$$

The elements of $\mathcal{L}^n(\mathfrak{A}, E)$ are called n-**cochains**.

ii) For $n \in \mathbb{N}$, define the maps $\delta^n : \mathcal{L}^n(\mathfrak{A}, E) \to \mathcal{L}^{n+1}(\mathfrak{A}, E)$ by

$$\begin{aligned}\delta^n(T)(a_1, \ldots, a_{n+1}) := a_1 \cdot T(a_2, \ldots, a_{n+1}) \\ + \sum_{k=1}^{n} (-1)^k T(a_1, \ldots, a_k a_{k+1}, \ldots, a_{n+1}) \\ + (-1)^{n+1} T(a_1, \ldots, a_n) \cdot a_{n+1}.\end{aligned}$$

The mapping δ^n is called the n-**coboundary operator**.

iii) Let $\mathcal{B}^0(\mathfrak{A}, E) := \{0\}$ and for $n \in \mathbb{N}_+$, define $\mathcal{B}^n(\mathfrak{A}, E) := \operatorname{ran}(\delta^{n-1})$. The elements of $\mathcal{B}^n(\mathfrak{A}, E)$ are called n-**coboundaries**.

iv) For $n \in \mathbb{N}$, define $\mathcal{Z}^n(\mathfrak{A}, E) := \ker(\delta^n)$. The elements of $\mathcal{Z}^n(\mathfrak{A}, E)$ are called n-**cocycles**.

v) The sequence

$$\{0\} \to E \xrightarrow{\delta^0} \mathcal{L}(\mathfrak{A}, E) \xrightarrow{\delta^1} \mathcal{L}^2(\mathfrak{A}, E) \xrightarrow{\delta^2} \cdots \xrightarrow{\delta^{n-1}} \mathcal{L}^n(\mathfrak{A}, E) \xrightarrow{\delta^n} \mathcal{L}^{n+1}(\mathfrak{A}, E) \xrightarrow{\delta^{n+1}} \cdots \quad (14)$$

is called the **Hochschild cochain complex**.

In the case where $n = 0$, the coboundary operator δ^0 is defined by $\delta^0(x)(a) := a \cdot x - x \cdot a$, so we can identify $\mathcal{Z}^0(\mathfrak{A}, E)$ with the subspace

$$\{x \in E : a \cdot x = x \cdot a \text{ for all } a \in \mathfrak{A}\}.$$

Furthermore, according to this definition, the set of 1-coboundaries $\mathcal{B}^1(\mathfrak{A}, E) = \operatorname{ran}(\delta^0)$ consists of all inner derivations from $\mathfrak{A}$ into E and for the coboundary operator δ^1 we have that

$$\delta^1(T)(a_1, a_2) = a_1 \cdot T(a_2) - T(a_1 a_2) + T(a_1) \cdot a_2 \qquad (T \in \mathcal{L}(\mathfrak{A}, E));$$

hence $\ker(\delta^1)$ consists of all bounded linear operators mapping $\mathfrak{A}$ into E with the property $T(a_1 a_2) = a_1 \cdot T(a_2) + T(a_1) \cdot a_2$ for all $a_1, a_2 \in \mathfrak{A}$, that is, all derivations form $\mathfrak{A}$ into E. We conclude that the definition above and the definitions of $\mathcal{B}^1(\mathfrak{A}, E)$ and $\mathcal{Z}^1(\mathfrak{A}, E)$ stated in the previous chapter coincide.

In order to define higher order Hochschild cohomology groups, it is essential to have a well defined quotient, so we need to show that $\mathcal{B}^n(\mathfrak{A}, E) \subset \mathcal{Z}^n(\mathfrak{A}, E)$ for all $n \in \mathbb{N}_+$.

Lemma 3.1 *Let $\mathfrak{A}$ be a Banach algebra and E a Banach $\mathfrak{A}$-bimodule. Then we have the inclusion $\mathcal{B}^n(\mathfrak{A}, E) \subset \mathcal{Z}^n(\mathfrak{A}, E)$ for all $n \in \mathbb{N}$.*

Proof: We will prove the statement by using an induction argument on n. The case where $n = 0$ and $n = 1$ is clear, since $\{0\} \subset \mathcal{Z}^0(\mathfrak{A}, E)$ and all inner derivations obviously are derivations. Now suppose that $n \geq 1$ and that $\mathcal{B}^k(\mathfrak{A}, E) \subset \mathcal{Z}^k(\mathfrak{A}, E)$ for all $k \leq n$ and let $T \in \mathcal{L}^{n+1}(\mathfrak{A}, E)$ be such that $T \in \operatorname{ran}(\delta^n)$. We want to show that $\delta^{n+1}(T) = 0$. Before we do, for the space $\mathcal{L}(\mathfrak{A}, E)$, consider the actions

$$(a \cdot T)(b) := a \cdot T(b) \quad \text{and} \quad (T \cdot a)(b) := T(ab) - T(a) \cdot b \quad (a, b \in \mathfrak{A},\ T \in \mathcal{L}(\mathfrak{A}, E)). \tag{15}$$

It is a straightforward verification to show that $a \cdot T$ and $T \cdot a$ are linear maps from $\mathfrak{A}$ to E and that these definitions yield a bimodule structure on $\mathcal{L}(\mathfrak{A}, E)$. Moreover, if $\kappa > 0$ is such that $\|a \cdot x\| \leq \kappa \|a\| \|x\|$ and $\|x \cdot a\| \leq \kappa \|x\| \|a\|$ for all $a \in \mathfrak{A}$ and $x \in E$, the inequalities

$$\|(a \cdot T)(b)\| \leq \kappa \|T\| \|a\| \|b\| \quad \text{and} \quad \|(T \cdot a)(b)\| \leq (1+\kappa) \|T\| \|a\| \|b\| \quad (a, b \in \mathfrak{A},\ T \in \mathcal{L}(\mathfrak{A}, E))$$

imply that $\mathcal{L}(\mathfrak{A}, E)$ is a Banach $\mathfrak{A}$-bimodule with respect to these actions. Now, we can identify every $T \in \mathcal{L}^{n+1}(\mathfrak{A}, E)$ with a map $\overline{T} \in \mathcal{L}^n(\mathfrak{A}, \mathcal{L}(\mathfrak{A}, E))$ by putting

$$\overline{T}(a_1, \ldots, a_n)(a_{n+1}) := T(a_1, \ldots, a_{n+1}).$$

By definition of the coboundary operator δ^{n+1} and (15), we now have

$$\begin{aligned}
\delta^{n+1}(T)(a_1, \ldots, a_{n+2}) &= a_1 \cdot T(a_2, \ldots, a_{n+2}) + \sum_{k=1}^{n+1} (-1)^k T(a_1, \ldots, a_k a_{k+1}, \ldots, a_{n+2}) \\
&\quad + (-1)^{n+2} T(a_1, \ldots, a_{n+1}) \cdot a_{n+2} \\
&= (a_1 \cdot \overline{T}(a_2, \ldots, a_{n+1}))(a_{n+2}) + \sum_{k=1}^{n} (-1)^k \overline{T}(a_1, \ldots, a_k a_{k+1}, \ldots, a_{n+1})(a_{n+2}) \\
&\quad + (-1)^{n+1} (\overline{T}(a_1, \ldots, a_n) \cdot a_{n+1})(a_{n+2}),
\end{aligned}$$

so the equality $\delta^{n+1}(T)(a_1,\dots,a_{n+2}) = \delta^n(\overline{T})(a_1,\dots,a_{n+1})(a_{n+2})$ now yields the identity

$$\overline{\delta^{n+1}(T)}(a_1,\dots,a_{n+1})(a_{n+2}) = \delta^{n+1}(T)(a_1,\dots,a_{n+2}) = \delta^n(\overline{T})(a_1,\dots,a_{n+1})(a_{n+2});$$

hence $\overline{\delta^{n+1}(T)} = \delta^n(\overline{T})$. By our assumption, we have $T = \delta^n(F)$ for some $F \in \mathcal{L}^n(\mathfrak{A}, E)$ and analogously we have the identity $\overline{T} = \overline{\delta^n(F)} = \delta^{n-1}(\overline{F})$; hence the induction hypothesis now implies that $\overline{\delta^{n+1}(T)} = 0$. So, for $a_1,\dots,a_{n+2} \in \mathfrak{A}$ we have that

$$\overline{\delta^{n+1}(T)}(a_1,\dots,a_{n+1}) = 0$$

and therefore, it follows that $\delta^{n+1}(T)(a_1,\dots,a_{n+2}) = 0$, thus $\delta^{n+1}(T) = 0$ and we conclude that $\mathcal{B}^{n+1}(\mathfrak{A}, E) = \operatorname{ran}(\delta^n) \subset \ker(\delta^{n+1}) = \mathcal{Z}^{n+1}(\mathfrak{A}, E)$; hence the desired inclusion holds for all $n \in \mathbb{N}_+$. ▮

Lemma 3.1 now guarantees that the following definition makes sense:

Definition: Let $\mathfrak{A}$ be a Banach algebra and E be a Banach $\mathfrak{A}$-bimodule. For $n \in \mathbb{N}_+$, the quotient

$$\mathcal{H}^n(\mathfrak{A}, E) := \mathcal{Z}^n(\mathfrak{A}, E)/\mathcal{B}^n(\mathfrak{A}, E)$$

is said to be the **n-th Hochschild cohomology group** of $\mathfrak{A}$ with coefficients in E.

It would be useful, in order to study these higher order Hochschild cohomology groups in more detail, to device a way which enables us to compute $\mathcal{H}^n(\mathfrak{A}, E)$ explicitly. A key tool in this light is the fact that the spaces $\mathcal{L}^n(\mathfrak{A}, E)$ can be made into Banach $\mathfrak{A}$-bimodules.

Lemma 3.2 *Let $\mathfrak{A}$ be a Banach algebra and E be a Banach $\mathfrak{A}$-bimodule. For all $n \in \mathbb{N}_+$, the space $\mathcal{L}^n(\mathfrak{A}, E)$ becomes a Banach $\mathfrak{A}$-bimodule for the actions*

$$(a \cdot T)(a_1,\dots,a_n) := a \cdot T(a_1,\dots,a_n)$$

and

$$\begin{aligned}(T \cdot a)(a_1,\dots,a_n) := T(aa_1,\dots,a_n) + \sum_{k=1}^{n-1}(-1)^k T(a, a_1,\dots,a_k a_{k+1},\dots,a_n) \\ + (-1)^n T(a, a_1,\dots,a_{n-1}) \cdot a_n\end{aligned}$$

for all $a, a_1,\dots,a_n \in \mathfrak{A}, T \in \mathcal{L}^n(\mathfrak{A}, E)$.

Proof: Clearly, we have that $\mathcal{L}(\mathfrak{A}, E)$ is a Banach space. Now let $n \geq 2$. Suppose $(T_k)_{k\geq 1}$ is a Cauchy sequence in $\mathcal{L}^n(\mathfrak{A}, E)$. From the inequality

$$\|(T_i - T_j)(a_1,\dots,a_n)\| \leq \|T_i - T_j\|\|a_1\|\cdots\|a_n\|$$

we conclude that $(T_k(a_1,\dots,a_n))_{k\geq 1}$ is a Cauchy sequence in E for all $a_1,\dots,a_n \in \mathfrak{A}$. This allows us to define a map $T : \mathfrak{A}^n \to E$ by

$$T(a_1,\dots,a_n) := \lim_{k\to\infty} T_k(a_1,\dots,a_n).$$

It is easy to see that T is n-linear and since we have $|\|T_i\| - \|T_j\|| \leq \|T_i - T_j\|$ for all $i, j \geq 1$ it follows that $(\|T_k\|)_{k\geq 1}$ is a Cauchy sequence and is therefore bounded. This implies that $\|T\|$ is finite and we conclude that $T \in \mathcal{L}^n(\mathfrak{A}, E)$. Let $a_1,\dots,a_n \in \mathfrak{A}$ be such that $\|a_i\| \leq 1$ for $1 \leq i \leq n$. Then there exists a number $M_1 \in \mathbb{N}_+$ such that $\|(T - T_k)(a_1, \ldots, a_n)\| < \frac{1}{2}\varepsilon$ whenever $k \geq M_1$. Also, there is a number $M_2 \in \mathbb{N}_+$ such that $\|T_i - T_j\| \leq \frac{1}{2}\varepsilon$ whenever $i, j \geq M_2$. It follows that

$$\begin{aligned}\|(T - T_k)(a_1, \ldots, a_n)\| &\leq \|(T - T_M)(a_1, \ldots, a_n)\| + \|(T_M - T_k)(a_1, \ldots, a_n)\| \\ &\leq \|T - T_M\| + \|T_M - T_k\| < \varepsilon\end{aligned}$$

where $M := \max\{M_1, M_2\}$ and whenever $k \geq M$, so $\|T - T_k\| < \varepsilon$; hence $\lim_{k\to\infty} T_k = T$ and we conclude that $\mathcal{L}^n(\mathfrak{A}, E)$ is a Banach space.

We now turn to the actions defined in the statement. If $n = 0$, there is nothing to prove and note that in the case where $n = 1$ the module actions coincide with the ones mentioned in (15). So, suppose that $n \geq 2$. It can also be shown that these actions define a bimodule structure on $\mathcal{L}^n(\mathfrak{A}, E)$. Finally, If $\kappa > 0$ is such that $\|a \cdot x\| \leq \kappa\|a\|\|x\|$ and $\|x \cdot a\| \leq \kappa\|x\|\|a\|$ for all $a \in \mathfrak{A}$ and $x \in E$, we find that for $a_1, \ldots, a_n \in \mathfrak{A}^n$ with $\|a_i\| \leq 1$ for $1 \leq i \leq n$, we have

$$\|(a \cdot T)(a_1, \ldots, a_n)\| \leq \kappa\|a\|\|T(a_1, \ldots, a_n)\| \leq \kappa\|a\|\|T\| \qquad (a \in \mathfrak{A},\ T \in \mathcal{L}^n(\mathfrak{A}, E)),$$

so $\|a \cdot T\| \leq \kappa\|a\|\|T\|$ and if we fix $a \in \mathfrak{A}$ with $\|a\| = 1$, it follows from the inequality

$$\begin{aligned}\|(T \cdot a)(a_1, \ldots, a_n)\| \leq \|T(aa_1, \ldots, a_n)\| + \sum_{k=1}^{n-1} \|T(a, a_1, \ldots, a_k a_{k+1}, \ldots, a_n)\| \\ + \kappa\|T(a, a_1, \ldots, a_{n-1})\|\end{aligned}$$

that $\|T \cdot a\| \leq (n + \kappa)\|T\|$. By definition of the right action on $\mathcal{L}^n(\mathfrak{A}, E)$, it follows that $T \cdot (\lambda a) = \lambda \cdot (T \cdot a)$ for all $\lambda > 0$, so if $0 \neq a \in \mathfrak{A}$, then

$$\|a\|^{-1}\|T \cdot a\| = \|T \cdot (\|a\|^{-1}a)\| \leq (n + \kappa)\|T\|;$$

hence $\|T \cdot a\| \leq (n + \kappa)\|T\|\|a\|$. Clearly, if $a = 0$ we have that $T \cdot a = 0$ and therefore, we conclude that $\mathcal{L}^n(\mathfrak{A}, E)$ is a Banach $\mathfrak{A}$-bimodule with respect to these actions for all $n \geq 2$. ∎

Now that we can view the spaces $\mathcal{L}^k(\mathfrak{A}, E)$ as Banach $\mathfrak{A}$-bimodules for all $k \in \mathbb{N}$, the corresponding n-coboundary operators will be denoted by

$$\delta_k^n : \mathcal{L}^n(\mathfrak{A}, \mathcal{L}^k(\mathfrak{A}, E)) \to \mathcal{L}^{n+1}(\mathfrak{A}, \mathcal{L}^k(\mathfrak{A}, E)) \qquad (n \in \mathbb{N}).$$

With these operators we can now establish a useful relation.

Lemma 3.3 *Let $\mathfrak{A}$ be a Banach algebra and E be a Banach $\mathfrak{A}$-bimodule. Let $k \in \mathbb{N}_+$. Then, for $n \in \mathbb{N}$, the map*

$$\tau^n : \mathcal{L}^{n+k}(\mathfrak{A}, E) \to \mathcal{L}^n(\mathfrak{A}, \mathcal{L}^k(\mathfrak{A}, E))$$

defined by

$$(\tau^n(T)(a_1, \ldots, a_n))(a_{n+1}, \ldots, a_{n+k}) := T(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+k}),$$

where $T \in \mathcal{L}^{n+k}(\mathfrak{A}, E)$ and $a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+k} \in \mathfrak{A}$, is an isometric isomorphism such that $\delta_k^n \circ \tau^n = \tau^{n+1} \circ \delta^{n+k}$.

Proof: Note that for $k = 1$ we have the identity mentioned in proof of Lemma 3.1. First, we deal with proving that τ^n is an isometric isomorphism for all $n \in \mathbb{N}$. In the case where $n = 0$, we find that τ^0 is just the identity operator on $\mathcal{L}^k(\mathfrak{A}, E)$, which is an isometric isomorphism. Suppose that $n \in \mathbb{N}_+$ and let $T \in \mathcal{L}^{n+k}(\mathfrak{A}, E)$. Then, by definition, we have that $\tau^n(T)$ is n-linear and $\tau^n(T)(a_1, \dots, a_n)$ is k-linear for all $a_1, \dots, a_n \in \mathfrak{A}$. From the inequality

$$\|\tau^n(T)(a_1, \dots, a_n)(a_{n+1}, \dots, a_{n+k})\| \leq \|T\|\|a_1\| \cdots \|a_{n+k}\|$$

we find that $\tau^n(T)(a_1, \dots, a_n)$ and $\tau^n(T)$ are well defined and $\|\tau^n(T)\| \leq \|T\|$. It is a straightforward verification to show that τ^n is linear and if $a_1, \dots, a_n, a_{n+1}, \dots, a_{n+k} \in A$ are such that $\|a_i\| \leq 1$ for $1 \leq i \leq n+k$, then

$$\begin{aligned}
\|T(a_1, \dots, a_n, a_{n+1}, \dots, a_{n+k})\| &= \|(\tau^n(T)(a_1, \dots, a_n))(a_{n+1}, \dots, a_{n+k})\| \\
&\leq \|\tau^n(T)(a_1, \dots, a_n)\| \\
&\leq \|\tau^n(T)\|,
\end{aligned}$$

so $\|T\| = \|\tau^n(T)\|$ and it follows that τ^n is an isometry. Let $\tilde{T} \in \mathcal{L}^n(\mathfrak{A}, \mathcal{L}^k(\mathfrak{A}, E))$ and define the map $T : \mathfrak{A}^{n+k} \to E$ by

$$T(a_1, \dots, a_n, a_{n+1}, \dots, a_{n+k}) := (\tilde{T}(a_1, \dots, a_n))(a_{n+1}, \dots, a_{n+k}).$$

Clearly, the map T is n-linear and analogously we find that $\|T\| = \|\tilde{T}\|$, so $T \in \mathcal{L}^{n+k}(\mathfrak{A}, E)$. By definition of τ^n we find that $\tau^n(T)(a_1, \dots, a_n) = \tilde{T}(a_1, \dots, a_n)$, so $\tau^n(T) = \tilde{T}$ and we conclude that τ^n is an isometric isomorphism.

For the identity concerning the coboundary operators, first suppose that $n = 0$. Then $\delta_k^0 \circ \mathrm{id} : \mathcal{L}^k(\mathfrak{A}, E) \to \mathcal{L}(\mathfrak{A}, \mathcal{L}^k(\mathfrak{A}, E))$ and if $T \in \mathcal{L}^k(\mathfrak{A}, E)$ and $a_1, \dots, a_{k+1} \in \mathfrak{A}$, the module actions defined on $\mathcal{L}^k(\mathfrak{A}, E)$ by Lemma 3.2 yield

$$\begin{aligned}
(\delta_k^0 \circ \mathrm{id}(T)(a_1))(a_2, \dots, a_{k+1}) &= (a_1 \cdot T - T \cdot a_1)(a_2, \dots, a_{k+1}) \\
&= a_1 \cdot T(a_2, \dots, a_{k+1}) + \sum_{i=1}^{k} (-1)^i T(a_1, \dots, a_i a_{i+1}, \dots, a_{k+1}) \\
&\quad + (-1)^{k+1} T(a_1, \dots, a_k) \cdot a_{k+1},
\end{aligned}$$

so by definition of τ, we find that

$$(\delta_k^0 \circ \mathrm{id}(T)(a_1))(a_2, \dots, a_{k+1}) = \delta^k(T)(a_1, \dots, a_{k+1}) = (\tau \circ \delta^k(T)(a_1))(a_2, \dots, a_{k+1}),$$

from which we conclude that $\delta_k^0 \circ \mathrm{id}(T) = \tau \circ \delta^k(T)$; hence $\delta_k^0 \circ \tau^0 = \tau \circ \delta^k$. Now let $n \in \mathbb{N}_+$ and consider the map $\delta_k^n \circ \tau^n : \mathcal{L}^{n+k}(\mathfrak{A}, E) \to \mathcal{L}^{n+1}(\mathfrak{A}, \mathcal{L}^k(\mathfrak{A}, E))$. If $T \in \mathcal{L}^{n+k}(\mathfrak{A}, E)$ and $a_1, \dots, a_{n+k+1} \in \mathfrak{A}$, by definition of τ^n and the left module action on $\mathcal{L}^k(\mathfrak{A}, E)$, we get

$$\begin{aligned}
(\delta_k^n \circ \tau^n(T)(a_1, \dots, &a_{n+1}))(a_{n+2}, \dots, a_{n+k+1}) \\
&= (a_1 \cdot \tau^n(T)(a_2, \dots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i \tau^n(T)(a_1, \dots, a_i a_{i+1}, \dots, a_{n+1}) \\
&\quad + (-1)^{n+1} \tau^n(T)(a_1, \dots, a_n) \cdot a_{n+1})(a_{n+2}, \dots, a_{n+k+1}) \\
&= a_1 \cdot T(a_2, \dots, a_{n+k+1}) + \sum_{i=1}^{n} (-1)^i T(a_1, \dots, a_i a_{i+1}, \dots, a_{n+k+1}) \\
&\quad + (-1)^{n+1} (\tau^n(T)(a_1, \dots, a_n) \cdot a_{n+1})(a_{n+2}, \dots, a_{n+k+1})
\end{aligned}$$

and from the right action defined on $\mathcal{L}^k(\mathfrak{A}, E)$ it now follows that

$$\begin{aligned}
&(\tau^n(T)(a_1, \dots, a_n) \cdot a_{n+1})(a_{n+2}, \dots, a_{n+k+1}) \\
&\qquad = T(a_1, \dots, a_{n+1}a_{n+2}, \dots, a_{n+k+1}) + \sum_{i=1}^{k-1}(-1)^i T(a_1, \dots, a_{n+1+i}a_{n+2+i}, \dots, a_{n+k+1}) \\
&\qquad\quad + (-1)^k T(a_1, \dots, a_{n+k}) \cdot a_{n+k+1}.
\end{aligned}$$

Multiplying this identity by $(-1)^{n+1}$ and substituting now yields

$$\begin{aligned}
&(\delta_k^n \circ \tau^n(T)(a_1, \dots, a_{n+1}))(a_{n+2}, \dots, a_{n+k+1}) \\
&\qquad = a_1 \cdot T(a_2, \dots, a_{n+k+1}) + \sum_{i=1}^{n+k}(-1)^i T(a_1, \dots, a_i a_{i+1}, \dots, a_{n+k+1}) \\
&\qquad\quad + (-1)^{n+k+1} T(a_1, \dots, a_{n+k}) \cdot a_{n+k+1};
\end{aligned}$$

hence

$$\begin{aligned}
&(\delta_k^n \circ \tau^n(T)(a_1, \dots, a_{n+1}))(a_{n+2}, \dots, a_{n+k+1}) \\
&\qquad = \delta^{n+k}(T)(a_1, \dots, a_{n+k+1}) = (\tau^{n+1} \circ \delta^{n+k}(T)(a_1, \dots, a_{n+1}))(a_{n+2}, \dots, a_{n+k+1}),
\end{aligned}$$

from which we conclude that $\delta_k^n \circ \tau^n(T) = \tau^{n+1} \circ \delta^{n+k}(T)$, so for all $n \in \mathbb{N}_+$ we have the identity $\delta_k^n \circ \tau^n = \tau^{n+1} \circ \delta^{n+k}$. ∎

The results obtained in Lemma 3.3 now allow us to reduce the order of $\mathcal{H}^n(\mathfrak{A}, E)$ in the following way:

Theorem 3.4 *Let $\mathfrak{A}$ be a Banach algebra and E a Banach $\mathfrak{A}$-bimodule. If $k \in \mathbb{N}$, as linear spaces, we have the isomorphy*

$$\mathcal{H}^{n+k}(\mathfrak{A}, E) \cong \mathcal{H}^n(\mathfrak{A}, \mathcal{L}^k(\mathfrak{A}, E)) \qquad (n \in \mathbb{N}_+).$$

Proof: If $k = 0$, there is nothing to prove, so, let $k \in \mathbb{N}_+$. For $n \in \mathbb{N}_+$, The map $\tau^n : \mathcal{L}^{n+k}(\mathfrak{A}, E) \to \mathcal{L}^n(\mathfrak{A}, \mathcal{L}^k(\mathfrak{A}, E))$ in Lemma 3.3 yields

$$\begin{aligned}
T \in \mathcal{Z}^{n+k}(\mathfrak{A}, E) &\iff \delta^{n+k}(T) = 0 \iff \tau^{n+1} \circ \delta^{n+k}(T) = 0 \iff \delta_k^n \circ \tau^n(T) = 0 \\
&\iff \tau^n(T) \in \mathcal{Z}^n(\mathfrak{A}, \mathcal{L}^k(\mathfrak{A}, E))
\end{aligned}$$

and if $T \in \mathcal{B}^{n+k}(\mathfrak{A}, E)$, then there is a map $S \in \mathcal{L}^{(n-1)+k}(\mathfrak{A}, E)$ such that $\delta^{(n-1)+k}(S) = T$. Again, by Lemma 3.3 we find that

$$\tau^n(T) = \tau^n \circ \delta^{(n-1)+k}(S) = \delta_k^{n-1} \circ \tau^{n-1}(S) \in \mathcal{B}^n(\mathfrak{A}, \mathcal{L}^k(\mathfrak{A}, E)).$$

On the other hand, if $T \in \mathcal{L}^{n+k}(\mathfrak{A}, E)$ is such that $\tau^n(T) \in \mathcal{B}^n(\mathfrak{A}, \mathcal{L}^k(\mathfrak{A}, E))$, then there is a map $S \in \mathcal{L}^{n-1}(\mathfrak{A}, \mathcal{L}^k(\mathfrak{A}, E))$ such that $\tau^n(T) = \delta_k^{n-1}(S)$. Since τ^n is an isomorphism, there also exists a map $\tilde{S} \in \mathcal{L}^{(n-1)-k}(\mathfrak{A}, E)$ such that $\tau^{n-1}(\tilde{S}) = S$. It now follows that we have

$$\tau^n(T) = \delta_k^{n-1} \circ \tau^{n-1}(\tilde{S}) = \tau^n \circ \delta^{(n-1)+k}(\tilde{S}) = \tau^n(\delta^{(n-1)+k}(\tilde{S}))$$

and because τ^n is injective, this implies that $T = \delta^{(n-1)+k}(\tilde{S}) \in \mathcal{B}^{n+k}(\mathfrak{A}, E)$. The map $\psi : \mathcal{H}^{n+k}(\mathfrak{A}, E) \to \mathcal{H}^n(\mathfrak{A}, \mathcal{L}^k(\mathfrak{A}, E))$ defined by $\psi(\overline{T}) := \overline{\tau^n(T)}$ will be the desired isomorphism. By our previous findings and the fact that τ^n is an isomorphism, the map ψ is well defined and bijective, and it is clearly linear. We conclude that the isomorphy holds for all $n \in \mathbb{N}_+$. ∎

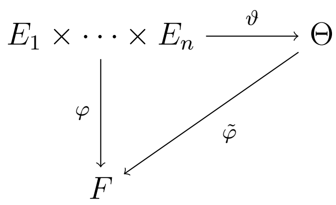
The preceding theorem basically states that every Hochschild cohomology group can be derived from a first order Hochschild cohomology group. However, in exchange for the reduction of the order, we have to transform the possibly simple coefficient module E into the more complicated one $\mathcal{L}^k(\mathfrak{A}, E)$.

The main theorem we will be dealing with in this chapter characterizes an amenable Banach algebra $\mathfrak{A}$ in terms of $\mathcal{H}^n(\mathfrak{A}, E^*)$ for all $n \in \mathbb{N}_+$. The Banach spaces $\mathcal{L}^n(\mathfrak{A}, E)$ consisting of n-linear maps motivate us to consider tensor products of the Banach spaces $\mathfrak{A}$ and E and, as we shall see, will be used to prove the characterization. As a general reminder, we will devote the next section to them.

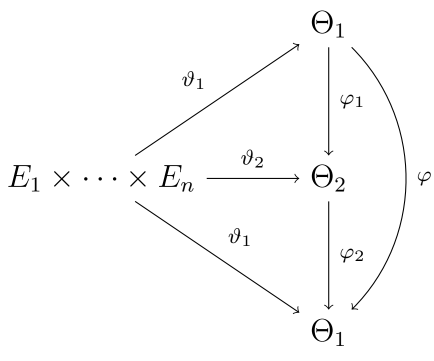
3.1.1 Tensor products of Banach spaces

For the linear spaces $E_1, \ldots, E_n$ a tensor product is defined through the universal property of n-linear maps that map the Cartesian product $E_1 \times \cdots \times E_n$ into a linear space Θ in the following way:

Definition: Let $E_1, \ldots, E_n$ be linear spaces. A **tensor product** of $E_1, \ldots, E_n$ is a pair (Θ, ϑ), where Θ is a linear space and $\vartheta : E_1 \times \cdots \times E_n \to \Theta$ is an n-linear map that satisfies the universal property for n-linear maps, that is, if F is a linear space and we have $\varphi : E_1 \times \cdots \times E_n \to F$ which is an n-linear map, then there is a unique linear map $\tilde{\varphi} : \Theta \to F$ such that we have the commutative diagram



It is a well known fact that tensor products exist and that they are unique up to an isomorphism. For if (Θ_1, ϑ_1) and (Θ_2, ϑ_2) are two tensor products of $E_1, \ldots, E_n$, then the universal property implies that the linear map φ in the diagram



must be the identity map on Θ_1, so $\varphi_1 \circ \varphi_2 = \mathrm{id}_{\Theta_1}$ and similarly, we find that $\varphi_2 \circ \varphi_2 = \mathrm{id}_{\Theta_2}$; hence φ_1 and $\varphi_2 = \varphi_1^{-1}$ are isomorphisms. It follows that we may refer to *the* tensor product of $E_1, \ldots, E_n$ and instead of (Θ, ϑ) we write $E_1 \otimes \cdots \otimes E_n$ for Θ.

The elements of $E_1 \otimes \cdots \otimes E_n$ are called **tensors** and elements of the form

$$x_1 \otimes \cdots \otimes x_n := \vartheta(x_1, \ldots, x_n) \qquad (x_1 \in E_1, \ldots, x_n \in E_n)$$

are called **elementary tensors**. In general, the collection of elementary tensors do not form a linear space, so the tensor product necessarily must contain all finite linear combinations of these elementary tensors. Let

$$F := \left\{ \sum_{k=1}^{m} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)} : \ x_i^{(1)}, \ldots, x_i^{(m)} \in E_i \text{ for all } 1 \leq i \leq n \right\}$$

and define the map $\varphi : E_1 \times \cdots \times E_n \to F$ by $\varphi(x_1, \ldots, x_n) := x_1 \otimes \cdots \otimes x_n$. Clearly, we have that F is a linear space and φ is n-linear. The universal property of $E_1 \otimes \cdots \otimes E_n$ now implies that there is a unique linear map $\tilde{\varphi} : E_1 \otimes \cdots \otimes E_n \to F$ such that

$$\tilde{\varphi}(x_1 \otimes \cdots \otimes x_n) = \varphi(x_1, \ldots, x_n) = x_1 \otimes \cdots \otimes x_n,$$

so $\tilde{\varphi}$ is the identity map on F. Now let L be any linear space and $\tau : E_1 \times \ldots \times E_n \to L$ be an n-linear map. If $\phi_1, \phi_2 : F \to L$ are linear maps with the property that

$$\tau(x_1, \ldots, x_n) = \phi_k(x_1 \otimes \cdots \otimes x_n) \qquad (k = 1, 2, \ x_1 \in E_1, \ldots, x_n \in E_n),$$

then by definition of F we must have that $\phi_1 = \phi_2$, so F satisfies the universal property and we conclude that $\tilde{\varphi}$ is the identity map on $E_1 \otimes \cdots \otimes E_n$; hence all tensors are finite linear combinations of elementary tensors.

Now that we have established a better understanding on what tensor products of linear spaces look like, the next step would be to consider Banach spaces $E_1, \ldots, E_n$. A priori, it is not clear that the existing tensor product $E_1 \otimes \cdots \otimes E_n$ is a Banach space, or even, have a suitable norm. If we manage to find one, then completing $E_1 \otimes \cdots \otimes E_n$ with respect to this norm would do the trick. In order to use the Banach space structure of the spaces $E_1, \ldots, E_n$, a useful property of such a norm would be that we have

$$\|x_1 \otimes \cdots \otimes x_n\| = \|x_1\| \cdots \|x_n\| \qquad (x_1 \in E_1, \ldots, x_n \in E_n).$$

Norms on $E_1 \otimes \cdots \otimes E_n$ that have this property are called **cross norms**. Along these lines, define for $\mathbf{x} \in E_1 \otimes \cdots \otimes E_n$

$$\|\mathbf{x}\|_\pi := \inf \left\{ \sum_{k=1}^{m} \|x_1^{(k)}\| \cdots \|x_n^{(k)}\| : \ \mathbf{x} = \sum_{k=1}^{m} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)} \right\}.$$

First, we claim that $\| \cdot \|_\pi$ is a norm on $E_1 \otimes \cdots \otimes E_n$ and additionally, satisfies the inequality

$$\|\mathbf{x}\| \leq \|\mathbf{x}\|_\pi \qquad (\mathbf{x} \in E_1 \otimes \cdots \otimes E_n)$$

for any cross norm $\| \cdot \|$ on $E_1 \otimes \cdots \otimes E_n$. It is clear that $\| \cdot \|_\pi$ is positive and if

$$\mathbf{x} = \sum_{i=1}^{m} x_1^{(i)} \otimes \cdots \otimes x_n^{(i)} \quad \text{and} \quad \mathbf{y} = \sum_{i=1}^{k} y_1^{(i)} \otimes \cdots \otimes y_n^{(i)},$$

then we have the inequality

$$\|\mathbf{x} + \mathbf{y}\|_\pi \leq \sum_{i=1}^{m} \|x_1^{(i)}\| \cdots \|x_n^{(i)}\| + \sum_{i=1}^{k} \|y_1^{(i)}\| \cdots \|y_n^{(i)}\|$$

and by taking the infimum over all the representations for $\mathbf{x}$ and $\mathbf{y}$ we obtain the desired identity $\|\mathbf{x}+\mathbf{y}\|_\pi \leq \|\mathbf{x}\|_\pi + \|\mathbf{y}\|_\pi$. In a similar way, we get $\|\lambda\mathbf{x}\|_\pi \leq |\lambda|\|\mathbf{x}\|_\pi$ for all $\lambda \neq 0$ and the inequality $|\lambda|\|\mathbf{x}\|_\pi = |\lambda|\|\frac{1}{\lambda}\lambda\mathbf{x}\|_\pi \leq \|\lambda\mathbf{x}\|$ implies that $\|\cdot\|_\pi$ is a seminorm, since $\|\lambda\mathbf{x}\|_\pi = |\lambda|\|\mathbf{x}\|_\pi$ clearly also holds for $\lambda = 0$. Now suppose that $\mathbf{x} \neq 0$. For $\phi_i \in E_i^*$ we have an n-linear map $\tau : E_1 \times \ldots \times E_n \to \mathbb{C} \otimes \cdots \otimes \mathbb{C}$ defined by

$$\tau(x_1, \ldots, x_n) := \phi_1(x_1) \otimes \cdots \otimes \phi_n(x_n),$$

so by the universal property of $E_1 \otimes \cdots \otimes E_n$ there is a unique linear map

$$\phi : E_1 \otimes \cdots \otimes E_n \to \mathbb{C} \otimes \cdots \otimes \mathbb{C}$$

such that $\phi(x_1 \otimes \cdots \otimes x_n) = \phi_1(x_1) \otimes \cdots \otimes \phi_n(x_n)$. Moreover, the map $\vartheta : \mathbb{C} \times \cdots \times \mathbb{C} \to \mathbb{C}$ defined by $\vartheta(\lambda_1, \ldots, \lambda_n) := \prod_{k=1}^n \lambda_k$ is n-linear and satisfies the universal property, so $\mathbb{C} \otimes \cdots \otimes \mathbb{C} \cong \mathbb{C}$; hence we a unique linear map $\varphi : E_1 \otimes \cdots \otimes E_n \to \mathbb{C}$ with the property that $\varphi(x_1 \otimes \cdots \otimes x_n) := \phi_1(x_1) \cdots \phi_n(x_n)$ and we shall denote this map by $\phi_1 \otimes \cdots \otimes \phi_n$. Returning to the matter at hand, it follows from [15, Prop. B.2.8] that $\|\mathbf{x}\|_\pi = 0$ implies that $\mathbf{x} = 0$, so we conclude that $\|\cdot\|_\pi$ is a norm on $E_1 \otimes \cdots \otimes E_n$. Furthermore, if $\|\cdot\|$ is any cross norm on $E_1 \otimes \cdots \otimes E_n$, then for $\mathbf{x} \in E_1 \otimes \cdots \otimes E_n$ and any representation for $\mathbf{x}$, we have

$$\|\mathbf{x}\| \leq \sum_{k=1}^{m} \|x_1^{(k)}\| \cdots \|x_n^{(k)}\|,$$

so by taking the infimum over all such representations, we conclude that $\|\mathbf{x}\| \leq \|\mathbf{x}\|_\pi$ for all $\mathbf{x} \in E_1 \otimes \cdots \otimes E_n$.

The second claim we make is that $\|\cdot\|_\pi$ defines a cross norm on $E_1 \otimes \cdots \otimes E_n$. By definition of $\|\cdot\|_\pi$ we have the inequality

$$\|x_1 \otimes \cdots \otimes x_n\|_\pi \leq \|x_1\| \cdots \|x_n\| \qquad (x_1 \in E_1, \ldots, x_n \in E_n)$$

and for the other inequality we may assume, without loss of generality, that $0 \neq x_i \in E_i$ for all $1 \leq i \leq n$. The Hahn-Banach theorem implies that we may choose linear functionals $\phi_i \in E_i^*$ with $\|\phi_i\| = 1$ that satisfy $\phi_i(x_i) = \|x_i\|$ for all $1 \leq i \leq n$ and it follows that

$$\|x_1\| \cdots \|x_n\| = \phi_1 \otimes \cdots \otimes \phi_n(x_1 \otimes \cdots \otimes x_n).$$

Analogously, for any representation

$$x_1 \otimes \cdots \otimes x_n = \sum_{k=1}^{m} y_1^{(k)} \otimes \cdots \otimes y_n^{(k)}$$

we find that

$$\begin{aligned} \|x_1\| \cdots \|x_n\| &= \phi_1 \otimes \cdots \otimes \phi_n \left(\sum_{k=1}^{m} y_1^{(k)} \otimes \cdots \otimes y_n^{(k)} \right) = \sum_{k=1}^{m} \phi_1(y_1^{(k)}) \cdots \phi_n(y_n^{(k)}) \\ &\leq \sum_{k=1}^{m} |\phi_1(y_1^{(k)})| \cdots |\phi_n(y_n^{(k)})| \leq \|\phi_1\| \cdots \|\phi_n\| \sum_{k=1}^{m} \|y_1^{(k)}\| \cdots \|y_n^{(k)}\| \\ &= \sum_{k=1}^{m} \|y_1^{(k)}\| \cdots \|y_n^{(k)}\|, \end{aligned}$$

so by taking the infimum over all such representations we obtain the inequality

$$\|x_1\| \cdots \|x_n\| \leq \|x_1 \otimes \cdots \otimes x_n\|_\pi;$$

hence $\| \cdot \|_\pi$ is a cross norm on $E_1 \otimes \cdots \otimes E_n$.

The norm $\| \cdot \|_\pi$ is called the **projective norm** and the completion of $E_1 \otimes \cdots \otimes E_n$ with respect to the projective norm is said to be the **projective tensor product** of $E_1, \ldots, E_n$ and is denoted by $E_1 \hat{\otimes} \cdots \hat{\otimes} E_n$. The projective tensor product also satisfies a more specific universal property:

Theorem 3.5 *Let $E_1, \ldots, E_n$ be Banach spaces. Then for every Banach space F and every bounded n-linear map $\varphi : E_1 \times \cdots \times E_n \to F$, there is a unique bounded linear map $\hat{\varphi} : E_1 \hat{\otimes} \cdots \hat{\otimes} E_n \to F$ such that we have the commutative diagram*

$$\begin{array}{ccc} E_1 \times \cdots \times E_n & \xrightarrow{\vartheta} & E_1 \hat{\otimes} \cdots \hat{\otimes} E_n \\ {\scriptstyle \varphi}\downarrow & \swarrow {\scriptstyle \hat{\varphi}} & \\ F & & \end{array}$$

Moreover, this correspondence is norm preserving, that is, $\|\hat{\varphi}\| = \|\varphi\|$; hence

$$\mathcal{L}^n(E_1, \ldots, E_n; F) \cong \mathcal{L}(E_1 \hat{\otimes} \cdots \hat{\otimes} E_n; F).$$

Proof: Let $\varphi : E_1 \times \cdots \times E_n \to F$ be a bounded n-linear map. By the universal property of $E_1 \otimes \cdots \otimes E_n$, there exists a unique linear map $\phi : E_1 \otimes \cdots \otimes E_n \to F$ such that

$$\varphi(x_1, \ldots, x_n) = \phi(x_1 \otimes \cdots \otimes x_n), \qquad (x_1 \in E_1, \ldots, x_n \in E_n).$$

Let $\mathbf{x} \in E_1 \otimes \cdots \otimes E_n$ and

$$\mathbf{x} = \sum_{k=1}^{m} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)}$$

be a representation. We have the inequality

$$\begin{aligned} \|\phi(\mathbf{x})\| &= \left\| \sum_{k=1}^{m} \phi(x_1^{(k)} \otimes \cdots \otimes x_n^{(k)}) \right\| \leq \sum_{k=1}^{m} \|\phi(x_1^{(k)} \otimes \cdots \otimes x_n^{(k)})\| = \sum_{k=1}^{m} \|\varphi(x_1^{(k)}, \ldots, x_n^{(k)})\| \\ &\leq \|\varphi\| \sum_{k=1}^{m} \|x_1^{(k)}\| \cdots \|x_n^{(k)}\| \end{aligned}$$

and taking the infimum over such representations, we find that $\|\phi(\mathbf{x})\| \leq \|\varphi\| \|\mathbf{x}\|_\pi$, so ϕ is bounded on $E_1 \otimes \cdots \otimes E_n$ with $\|\phi\| \leq \|\varphi\|$. Furthermore, if $x_i \in E_i$ with $\|x_i\| \leq 1$ for $1 \leq i \leq n$, then $\|x_1 \otimes \cdots \otimes x_n\|_\pi = \|x_1\| \cdots \|x_n\| \leq 1$ and

$$\|\varphi(x_1, \ldots, x_n)\| = \|\phi(x_1 \otimes \cdots \otimes x_n)\| \leq \|\phi\|,$$

so $\|\varphi\| \leq \|\phi\|$; hence $\|\phi\| = \|\varphi\|$. It is a straightforward verification to show that ϕ is unique. ∎

When considering the ordinary tensor product $E_1 \otimes \cdots \otimes E_n$, we have shown that every tensor is a finite linear combination of elementary tensors. A corresponding representation for elements in the projective tensor product $E_1 \hat{\otimes} \cdots \hat{\otimes} E_n$ also exists:

Theorem 3.6 *Let $E_1, \ldots, E_n$ be Banach spaces and $\mathbf{x} \in E_1 \hat{\otimes} \cdots \hat{\otimes} E_n$. Then there are sequences $(x_i^{(k)})_{k \geq 1}$ in E_i for $1 \leq i \leq n$ such that*

$$\sum_{k=1}^{\infty} \|x_1^{(k)}\| \cdots \|x_n^{(k)}\| < \infty \quad \text{and} \quad \mathbf{x} = \sum_{k=1}^{\infty} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)}. \tag{16}$$

Moreover, the value $\|\mathbf{x}\|_\pi$ is obtained by taking the infimum over all the series such that (16) is satisfied.

Proof: Let F be the subspace of $E_1 \hat{\otimes} \cdots \hat{\otimes} E_n$ consisting of all elements that satisfy (16) and let $\|\cdot\|_{\tilde{\pi}} : F \to \mathbb{R}_{\geq 0}$ be defined by

$$\|\mathbf{x}\|_{\tilde{\pi}} := \inf \left\{ \sum_{k=1}^{\infty} \|x_1^{(k)}\| \cdots \|x_n^{(k)}\| : \ \mathbf{x} = \sum_{k=1}^{\infty} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)} \in F \right\}.$$

Clearly, this defines a seminorm on F. Suppose $\mathbf{x} \in F$ and let

$$\mathbf{x} = \sum_{k=1}^{\infty} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)} \in F$$

be any representation. Since we have the inequality

$$\|\mathbf{x}\|_\pi \leq \sum_{k=1}^{\infty} \|x_1^{(k)} \otimes \cdots \otimes x_n^{(k)}\|_\pi = \sum_{k=1}^{\infty} \|x_1^{(k)}\| \cdots \|x_n^{(k)}\|,$$

taking the infimum over all such representations yields the inequality $\|\mathbf{x}\|_\pi \leq \|\mathbf{x}\|_{\tilde{\pi}}$; hence $\|\cdot\|_{\tilde{\pi}}$ is a norm on F. We claim that $(F, \|\cdot\|_{\tilde{\pi}})$ is a Banach space. Accordingly, let $(\mathbf{x}_i)_{i \geq 1}$ be a sequence in F such that

$$\sum_{i=1}^{\infty} \|\mathbf{x}_i\|_{\tilde{\pi}} < \infty.$$

By definition of $\|\cdot\|_{\tilde{\pi}}$, for every i there is a representation

$$\mathbf{x}_i = \sum_{k=1}^{\infty} x_1^{(k,i)} \otimes \cdots \otimes x_n^{(k,i)} \quad \text{such that} \quad \|\mathbf{x}_i\|_{\tilde{\pi}} \geq \sum_{k=1}^{\infty} \|x_1^{(k,i)}\| \cdots \|x_n^{(k,i)}\| - 2^{-i},$$

so it follows that we have the inequality

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \|x_1^{(k,i)}\| \cdots \|x_n^{(k,i)}\| \leq \sum_{i=1}^{\infty} \|\mathbf{x}_i\|_{\tilde{\pi}} + 1 < \infty;$$

hence

$$\sum_{i=1}^{\infty} \mathbf{x}_i = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} x_1^{(k,i)} \otimes \cdots \otimes x_n^{(k,i)} \in F$$

and clearly,

$$\sum_{i=1}^{\infty} \mathbf{x}_i = \lim_{N\to\infty} \sum_{i=1}^{N} \mathbf{x}_i.$$

Since every absolutely convergent series with respect to $\|\cdot\|_{\tilde{\pi}}$ converges in F, we conclude that $(F, \|\cdot\|_{\tilde{\pi}})$ is a Banach space. Because $\|\cdot\|_{\pi}$ and $\|\cdot\|_{\tilde{\pi}}$ coincide on $E_1 \otimes \cdots \otimes E_n$, we now have

$$(F, \|\cdot\|_{\tilde{\pi}}) = \overline{(E_1 \otimes \cdots \otimes E_n, \|\cdot\|_{\tilde{\pi}})} = \overline{(E_1 \otimes \cdots \otimes E_n, \|\cdot\|_{\pi})} = E_1 \hat{\otimes} \cdots \hat{\otimes} E_n,$$

so $E_1 \hat{\otimes} \cdots \hat{\otimes} E_n = F$. ∎

3.1.2 The amenability of $\mathfrak{A}$ in terms of $\mathcal{H}^n(\mathfrak{A}, E^*)$

We have acquired enough background knowledge about tensor products on Banach spaces to continue with the characterization of amenable Banach algebras $\mathfrak{A}$. However, before we state the main theorem, we need a duality property of the bounded n-linear maps $\mathcal{L}^n(\mathfrak{A}, E)$ in order to prove this result.

Lemma 3.7 *Let $E_1, \ldots, E_n, F$ be Banach spaces. Then for $\psi \in (E_1 \hat{\otimes} \cdots \hat{\otimes} E_n \hat{\otimes} F)^*$ we can define $\tilde{\psi} \in \mathcal{L}^n(E_1, \ldots, E_n; F^*)$ with*

$$\tilde{\psi}(x_1, \ldots, x_n)(y) := \psi(x_1 \otimes \cdots \otimes x_n \otimes y).$$

This assignment induces an isometric isomorphism

$$\Phi : (E_1 \hat{\otimes} \cdots \hat{\otimes} E_n \hat{\otimes} F)^* \to \mathcal{L}^n(E_1, \ldots, E_n; F^*).$$

Proof: Let $\Phi : (E_1 \hat{\otimes} \cdots \hat{\otimes} E_n \hat{\otimes} F)^* \to \mathcal{L}^n(E_1, \ldots, E_n; F^*)$ be defined by $\Phi(\psi) := \tilde{\psi}$. If we fix $y \in F$, it is easily verified that $\Phi(\psi)$ is n-linear on $E_1 \times \cdots \times E_n$ with

$$(\Phi(\psi)(x_1, \ldots, x_n))(y) \in \mathbb{C} \qquad (x_1 \in E_1, \ldots, x_n \in E_n).$$

Moreover, the map $\Phi(\psi)(x_1, \ldots, x_n)$ is linear on F. If $\|y\| \leq 1$, the inequality

$$|(\Phi(\psi)(x_1, \ldots, x_n))(y)| = |\psi(x_1 \otimes \cdots \otimes x_n \otimes y)| \leq \|\psi\| \|x_1\| \cdots \|x_n\|$$

implies that $\|\Phi(\psi)(x_1, \ldots, x_n)\| \leq \|\psi\| \|x_1\| \cdots \|x_n\|$, so $\Phi(\psi)(x_1, \ldots, x_n) \in F^*$; hence Φ is well defined. Clearly, the map Φ is linear with

$$\|\Phi(\psi)\| = \sup_{\|x_i\| \leq 1} \|\Phi(\psi)(x_1, \ldots, x_n)\| \leq \|\psi\|.$$

Let $\mathbf{x} \in E_1 \hat{\otimes} \cdots \hat{\otimes} E_n \hat{\otimes} F$ and

$$\mathbf{x} = \sum_{k=1}^{\infty} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)} \otimes y_k$$

be any representation, which exists by Theorem 3.6. Again, the continuity of ψ implies that

$$\begin{aligned} |\psi(\mathbf{x})| &= \left| \psi\left(\sum_{k=1}^{\infty} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)} \otimes y_k \right) \right| = \left| \sum_{k=1}^{\infty} \psi(x_1^{(k)} \otimes \cdots \otimes x_n^{(k)} \otimes y_k) \right| \\ &\leq \sum_{k=1}^{\infty} \left| (\Phi(\psi)(x_1^{(k)}, \ldots, x_n^{(k)}))(y_k) \right| \leq \sum_{k=1}^{\infty} \left\| \Phi(\psi)(x_1^{(k)}, \ldots, x_n^{(k)}) \right\| \|y_k\| \\ &\leq \|\Phi(\psi)\| \sum_{k=1}^{\infty} \|x_1^{(k)}\| \cdots \|x_n^{(k)}\| \|y_k\|, \end{aligned}$$

so taking the infimum over all such representations yields $|\psi(\mathbf{x})| \leq \|\Phi(\psi)\|\|\mathbf{x}\|_{\tilde{\pi}}$; hence $\|\psi\| \leq \|\Phi(\psi)\|$ and we conclude that Φ is an isometry. In conclusion, choose a map $\varphi \in \mathcal{L}^n(E_1, \ldots, E_n; F^*)$ and define $\psi : E_1 \times \cdots \times E_n \times F \to \mathbb{C}$ by

$$\psi(x_1, \ldots, x_n, y) := \varphi(x_1, \ldots, x_n)(y).$$

Clearly, the map ψ is n-linear, so by Theorem 3.5 there is a unique bounded linear map $\hat{\psi} : E_1 \hat{\otimes} \cdots \hat{\otimes} E_n \hat{\otimes} F \to \mathbb{C}$ with $\|\hat{\psi}\| = \|\psi\|$ such that

$$\hat{\psi}(x_1 \otimes \cdots \otimes x_n \otimes y) = \varphi(x_1, \ldots, x_n)(y);$$

hence $\Phi(\hat{\psi}) = \varphi$ and we conclude that Φ is an isometric isomorphism. ∎

Theorem 3.8 *For a Banach algebra $\mathfrak{A}$ the following are equivalent:*

i) $\mathfrak{A}$ *is amenable.*

ii) $\mathcal{H}^n(\mathfrak{A}, E^*) = \{0\}$ *for every Banach $\mathfrak{A}$-bimodule E and for all $n \in \mathbb{N}_+$.*

Proof: *i)* $\Rightarrow$ *ii)*: We already have $\mathcal{H}^1(\mathfrak{A}, E^*) = \{0\}$ for every Banach $\mathfrak{A}$-bimodule E, so suppose that $n \in \mathbb{N}_+$ with $n > 1$. Let E be a Banach $\mathfrak{A}$-bimodule and define

$$F := \overbrace{\mathfrak{A} \hat{\otimes} \cdots \hat{\otimes} \mathfrak{A}}^{n-1} \hat{\otimes} E$$

and a bimodule action of $\mathfrak{A}$ on F through

$$(a_1 \otimes \cdots \otimes a_{n-1} \otimes x) \cdot a := a_1 \otimes \cdots \otimes a_{n-1} \otimes x \cdot a$$

and

$$\begin{aligned} a \cdot (a_1 \otimes \cdots \otimes a_{n-1} \otimes x) := \; & aa_1 \otimes \cdots \otimes a_{n-1} \otimes x \\ & + \sum_{k=1}^{n-2} (-1)^k a \otimes a_1 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_{n-1} \otimes x \\ & + (-1)^{n-1} a \otimes a_1 \otimes \cdots \otimes a_{n-2} \otimes a_{n-1} \cdot x. \end{aligned}$$

It can be shown that these actions define a bimodule structure on F and if $\kappa > 0$ is such that $\|a \cdot x\| \leq \kappa \|a\|\|x\|$ and $\|x \cdot a\| \leq \kappa \|x\|\|a\|$ for all $a \in \mathfrak{A}$ and $x \in E$, then, for fixed $a \in \mathfrak{A}$, the map

$$\varphi_a : \overbrace{\mathfrak{A} \times \cdots \times \mathfrak{A}}^{n-1} \times E \to F$$

defined by $\varphi_a(a_1, \ldots, a_{n-1}, x) := (a_1 \otimes \cdots \otimes a_{n-1} \otimes x) \cdot a$ has norm $\|\varphi_a\| \leq \kappa \|a\|$ and by Theorem 3.5 the right action generalizes to F with $\|\mathbf{x} \cdot a\|_{\tilde{\pi}} \leq \kappa \|a\|\|\mathbf{x}\|_{\tilde{\pi}}$ for all $\mathbf{x} \in F$. Similarly, for fixed $a \in \mathfrak{A}$, the map

$$\psi_a : \overbrace{\mathfrak{A} \times \cdots \times \mathfrak{A}}^{n-1} \times E \to F$$

defined by $\psi_a(a_1, \ldots, a_{n-1}, x) := a \cdot (a_1 \otimes \cdots \otimes a_{n-1} \otimes x)$ has norm $\|\psi_a\| \leq (n + \kappa)\|a\|$ and generalizes the left action to F with $\|a \cdot \mathbf{x}\|_{\tilde{\pi}} \leq (n + \kappa)\|a\|\|\mathbf{x}\|_{\tilde{\pi}}$ for all $\mathbf{x} \in F$, so the bimodule actions of $\mathfrak{A}$ on F are well defined and F is a Banach $\mathfrak{A}$-bimodule. It follows from Lemma 3.7 that $\mathcal{L}^{n-1}(\mathfrak{A}, E^*) \cong F^*$ and if $a \in \mathfrak{A}$, then the bimodule actions on F^* are defined by

$$(a \cdot \mathbf{f})(\mathbf{x}) := \mathbf{f}(\mathbf{x} \cdot a) \quad \text{and} \quad (\mathbf{f} \cdot a)(\mathbf{x}) := \mathbf{f}(a \cdot \mathbf{x}) \qquad (\mathbf{f} \in F^*,\ \mathbf{x} \in F).$$

We claim that these coincide with the module actions defined on $\mathcal{L}^{n-1}(\mathfrak{A}, E^*)$. For the isometric isomorphism Φ defined in Lemma 3.7, the elements $a, a_1, \ldots, a_{n-1} \in \mathfrak{A}$, $\mathbf{f} \in F^*$ and $x \in E$ we have that

$$\begin{aligned}(\Phi(a \cdot \mathbf{f})(a_1, \ldots, a_{n-1}))(x) &= (a \cdot \mathbf{f})(a_1 \otimes \cdots \otimes a_{n-1} \otimes x) = \mathbf{f}((a_1 \otimes \cdots \otimes a_{n-1} \otimes x) \cdot a) \\ &= \mathbf{f}(a_1 \otimes \cdots \otimes a_{n-1} \otimes x \cdot a) = (\Phi(\mathbf{f})(a_1, \ldots, a_{n-1}))(x \cdot a) \\ &= (a \cdot \Phi(\mathbf{f}))(a_1, \ldots, a_{n-1})(x),\end{aligned}$$

which implies that $\Phi(a \cdot \mathbf{f}) = a \cdot \Phi(\mathbf{f})$ and the equalities

$$\begin{aligned}&(\Phi(\mathbf{f} \cdot a)(a_1, \ldots, a_{n-1}))(x) = (\mathbf{f} \cdot a)(a_1 \otimes \cdots \otimes a_{n-1} \otimes x) = \mathbf{f}(a \cdot (a_1 \otimes \cdots \otimes a_{n-1} \otimes x)) \\ &\quad = \mathbf{f}(aa_1 \otimes \cdots \otimes a_{n-1} \otimes x) + \sum_{k=1}^{n-2} (-1)^k \mathbf{f}(a \otimes a_1 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes x) \\ &\qquad + (-1)^{n-1} \mathbf{f}(a \otimes a_1 \otimes \cdots \otimes a_{n-1} \cdot x) \\ &\quad = (\Phi(\mathbf{f})(aa_1, \ldots, a_{n-1}))(x) + \sum_{k=1}^{n-2} (-1)^k (\Phi(\mathbf{f})(a, a_1, \ldots, a_k a_{k+1}, \ldots a_{n-2}))(x) \\ &\qquad + (-1)^{n-1} (\Phi(\mathbf{f})(a, a_1, \ldots, a_{n-2}) \cdot a_{n-1})(x) \\ &\quad = \left((\Phi(\mathbf{f}) \cdot a) \left(a_1, \ldots, a_{n-1} \right) \right) (x)\end{aligned}$$

imply that $\Phi(\mathbf{f} \cdot a) = \Phi(\mathbf{f}) \cdot a$, which proves our claim; hence $\mathcal{L}^{n-1}(\mathfrak{A}, E^*) \cong F^*$ as Banach $\mathfrak{A}$-bimodules. Finally, by Theorem 3.4 we have

$$\mathcal{H}^n(\mathfrak{A}, E^*) \cong \mathcal{H}^1(\mathfrak{A}, \mathcal{L}^{n-1}(\mathfrak{A}, E^*)) \cong \mathcal{H}^1(\mathfrak{A}, F^*) = \{0\}.$$

ii) $\Rightarrow$ *i)*: This implication is true by definition of an amenable Banach algebra. ∎

We wish to have a similar result, that is, under what circumstances does $\mathcal{H}^1(\mathfrak{A}, E^*) = \{0\}$ imply that $\mathcal{H}^n(\mathfrak{A}, E^*) = \{0\}$ for all $n \in \mathbb{N}_+$, when we assume that $\mathfrak{A}$ is an ordered Banach algebra with generating cone $\mathfrak{A}^+$ and the module actions on the Banach $\mathfrak{A}$-bimodules E, which also have a vector space order with generating cone E^+, behave accordingly. In the next section we will first examine these assumptions in an elementary form.

3.2 The triviality of $\mathcal{H}^n(\mathfrak{A}, E^*)$ for ordered Banach algebras and regular Banach $\mathfrak{A}$-bimodules E

A Banach algebra $\mathfrak{A}$ with a vector space order is said to be an **ordered Banach algebra** if the multiplication on $\mathfrak{A}$ is positive, that is, for all $a, b \in \mathfrak{A}$ with $a, b \geq 0$ we have that $ab \geq 0$. If we assume that the Banach algebra $\mathfrak{A}$ has a vector space order with generating cone $\mathfrak{A}^+$, then for an ordered Banach $\mathfrak{A}$-bimodule E with a generating cone E^+, it would be a reasonable assumption to have that the actions are regular, that is, for

$$\varphi : \mathfrak{A} \to B(E)$$

describing a left module action on E we would have $\varphi = \varphi_1 - \varphi_2$ where φ_1 and φ_2 are both positive bounded operators mapping $\mathfrak{A}$ into $B(E)$, and similarly, for

$$\psi : \mathfrak{A}^{\mathrm{opp}} \to B(E)$$

describing the right module action on E. Note that this implies that for all $a \in \mathfrak{A}$ we have $a_1, a_2 \geq 0$ such that $a = a_1 - a_2$, so $\varphi(a) = (\varphi_1(a_1) + \varphi_2(a_2)) - (\varphi_1(a_2) + \varphi_2(a_1))$ and $\psi(a) = (\psi_1(a_1) + \psi_2(a_2)) - (\psi_1(a_2) + \psi_2(a_1))$, thus $\varphi(a)$ and $\psi(a)$ are regular as well for all $a \in \mathfrak{A}$. We shall refer to such an ordered Banach $\mathfrak{A}$-bimodule E as a **regular Banach $\mathfrak{A}$-bimodule**. In order to construct an analogue of Theorem 3.8 for an ordered Banach space and regular Banach $\mathfrak{A}$-bimodules E we need to introduce an order on the n-fold projective tensor product

$$\mathfrak{A}\hat{\otimes} \cdots \hat{\otimes}\mathfrak{A}\hat{\otimes}E$$

where $n \in \mathbb{N}_+$. We would like this Banach space to have a proper generating cone, so $\mathfrak{A}$ and E need some additional properties in order to guarantee this. The following theorem states which additional assumptions are sufficient in a more general setting:

Theorem 3.9 *Let $X_1, \ldots, X_n$ be ordered Banach spaces. Then the properties*

i) for every $x \in X_i$ there exist $x_1, x_2 \in X_i^+$ such that $x = x_1 - x_2$ for all $1 \leq i \leq n$ and there is a constant $K_i > 0$ such that $\|x_1\| \leq K_i\|x\|$ and $\|x_2\| \leq K_i\|x\|$ for all $x \in X_i$ for all $1 \leq i \leq n$;

ii) $(X_i^*)^+$ acts faithfully on X_i^+ for all $1 \leq i \leq n$;

imply that the n-fold projective tensor product $X_1\hat{\otimes} \cdots \hat{\otimes}X_n$ has a proper generating cone

$$(X_1\hat{\otimes} \cdots \hat{\otimes}X_n)^+ := \left\{ \sum_{k=1}^{\infty} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)} : \ \sum_{k=1}^{\infty} \|x_1^{(k)}\| \cdots \|x_n^{(k)}\| < \infty \ \textit{and} \ x_i^{(k)} \in X_i^+ \right\}$$

which induces a vector space order on $X_1\hat{\otimes} \cdots \hat{\otimes}X_n$. Moreover, the analogue of property i) holds in $X_1\hat{\otimes} \cdots \hat{\otimes}X_n$ and $[(X_1\hat{\otimes} \cdots \hat{\otimes}X_n)^*]^+$ acts faithfully on $(X_1\hat{\otimes} \cdots \hat{\otimes}X_n)^+$.

Proof: Clearly, this set is closed under addition and the scalar multiplication of $\mathbb{R}_{\geq 0}$. Let $\mathbf{x} \in X_1\hat{\otimes} \cdots \hat{\otimes}X_n$. By Theorem 3.6 we can write

$$\mathbf{x} = \sum_{k=1}^{\infty} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)} \qquad \text{with} \qquad \sum_{k=1}^{\infty} \|x_1^{(k)}\| \cdots \|x_n^{(k)}\| < \infty.$$

Property $i)$ implies that for all $k \geq 1$ there are $\xi_i^{(k)}, \zeta_i^{(k)} \in X_i^+$ such that

$$x_i^{(k)} = \xi_i^{(k)} - \zeta_i^{(k)}$$

for $1 \leq i \leq n$, so $\mathbf{x}$ can be written as the alternating sum of 2^n series that lie in $(X_1\hat{\otimes} \cdots \hat{\otimes}X_n)^+$ and since property $i)$ implies that each of these series is bounded by

$$\left(\prod_{i=1}^{n} K_i \right) \sum_{k=1}^{\infty} \|x_1^{(k)}\| \cdots \|x_n^{(k)}\| < \infty,$$

the element $\mathbf{x}$ can be written as the difference of two elements in $(X_1\hat{\otimes}\cdots\hat{\otimes}X_n)^+$ and it follows that the defined cone is generating. Suppose that

$$\mathbf{x} \in (X_1\hat{\otimes}\cdots\hat{\otimes}X_n)^+ \cap -(X_1\hat{\otimes}\cdots\hat{\otimes}X_n)^+$$

and let $\varphi_i \in (X_i^*)^+$ for $1 \le i \le n$. We can now write

$$\mathbf{x} = \sum_{k=1}^{\infty} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)} = -\sum_{k=1}^{\infty} y_1^{(k)} \otimes \cdots \otimes y_n^{(k)} \qquad (x_i^{(k)}, y_i^{(k)} \ge 0,\ 1 \le i \le n)$$

and since all φ_i are continuous, the representations of $\mathbf{x}$ and Theorem 3.5 now yield

$$\begin{aligned}\varphi_1 \otimes \cdots \otimes \varphi_n(\mathbf{x}) &= \sum_{k=1}^{\infty} \varphi_1(x_1^{(k)}) \cdots \varphi_n(x_n^{(k)}) \ge 0 \ge -\sum_{k=1}^{\infty} \varphi_1(y_1^{(k)}) \cdots \varphi_n(y_n^{(k)}) \\ &= \varphi_1 \otimes \cdots \otimes \varphi_n(\mathbf{x}),\end{aligned}$$

so for all k we have $\varphi_1(x_1^{(k)}) \cdots \varphi_n(x_n^{(k)}) = 0$ for all $\varphi_i \in (X_i^*)^+$. By property ii), we must have, for all $k \ge 1$, that $x_i^{(k)} = 0$ for some $1 \le i \le n$; hence $\mathbf{x} = 0$ and we conclude that the cone $(X_1\hat{\otimes}\cdots\hat{\otimes}X_n)^+$ is proper. ∎

Under these assumptions we can now formulate an analogue of Theorem 3.8 for ordered Banach algebras.

Theorem 3.10 *Let $\mathfrak{A}$ be an ordered Banach algebra that satisfies the properties of Theorem 3.9. If $\mathcal{H}^1(\mathfrak{A}, E^*) = \{0\}$ for every regular Banach $\mathfrak{A}$-bimodule E satisfying the properties of Theorem 3.9, then $\mathcal{H}^n(\mathfrak{A}, E^*) = \{0\}$ for all $n \in \mathbb{N}_+$ for all regular Banach $\mathfrak{A}$-bimodules E satisfying the properties of Theorem 3.9.*

Proof: Suppose that $n \in \mathbb{N}_+$ with $n > 1$. Let E be a regular Banach $\mathfrak{A}$-bimodule satisfying the properties of Theorem 3.9. Similar to the proof we gave for Theorem 3.8, consider the Banach $\mathfrak{A}$-bimodule

$$F := \overbrace{\mathfrak{A}\hat{\otimes}\cdots\hat{\otimes}\mathfrak{A}}^{n-1}\hat{\otimes}E$$

with the corresponding two-sided actions of $\mathfrak{A}$. If $\varphi : \mathfrak{A} \to B(F)$ represents the left action of $\mathfrak{A}$ on F and $\xi : \mathfrak{A} \to B(E)$ the left action of $\mathfrak{A}$ on E, then, since ξ is regular, we can write $\xi = \xi_1 - \xi_2$ with $\xi_1, \xi_2 \ge 0$ and so

$$\varphi = \varphi_0 + \sum_{k=1}^{n-2} (-1)^k \varphi_k + (-1)^{n-1}\varphi_{n-1}$$

where $\varphi_0(a)(a_1 \otimes \cdots \otimes a_{n-1} \otimes x) := aa_1 \otimes \cdots \otimes a_{n-1} \otimes x$ and

$$\varphi_k(a)(a_1 \otimes \cdots \otimes a_{n-1} \otimes x) := a \otimes a_1 \otimes \cdots \otimes a_k a_{k+1} \otimes a_{n-2} \otimes x \quad (1 \le k \le n-2)$$

and

$$\varphi_{n-1}(a)(a_1 \otimes \cdots \otimes a_{n-1} \otimes x) := a \otimes a_1 \otimes \cdots \otimes a_{n-2} \otimes \xi(a_{n-1})(x).$$

We can identify φ_{n-1} with $\hat{\xi_1} - \hat{\xi_2}$ where

$$\hat{\xi_1}(a)(a_1 \otimes \cdots \otimes a_{n-1} \otimes x) := a \otimes a_1 \otimes \cdots \otimes a_{n-2} \otimes \xi_1(a_{n-1})(x)$$

and

$$\hat{\xi_2}(a)(a_1 \otimes \cdots \otimes a_{n-1} \otimes x) := a \otimes a_1 \otimes \cdots \otimes a_{n-2} \otimes \xi_2(a_{n-1})(x),$$

so by Theorem 3.5 and Theorem 3.9 we find that φ_i is positive for $0 \leq i < n-1$ and φ_{n-1} is regular; hence φ can be written as the difference of two positive operators and we conclude that the left action of $\mathfrak{A}$ on F is regular. On the other hand, if $\psi : \mathfrak{A}^{\text{opp}} \to B(F)$ represents the right action of $\mathfrak{A}$ on F and $\phi : \mathfrak{A}^{\text{opp}} \to B(E)$ represents the right action of $\mathfrak{A}$ on E, then, since ϕ is regular, we can write $\phi = \phi_1 - \phi_2$ with $\phi_1, \phi_2 \geq 0$ and identify ψ with $\hat{\phi_1} - \hat{\phi_2}$ where

$$\hat{\phi_1}(a)(a_1 \otimes \cdots \otimes a_{n-1} \otimes x) := a_1 \otimes \cdots \otimes a_{n-1} \otimes \phi_1(a)(x)$$

and

$$\hat{\phi_2}(a)(a_1 \otimes \cdots \otimes a_{n-1} \otimes x) := a_1 \otimes \cdots \otimes a_{n-1} \otimes \phi_2(a)(x).$$

Analogously, we conclude that ψ is regular, so F is a regular Banach $\mathfrak{A}$-bimodule. By Theorem 3.9, it follows that F has a generating cone and if $\mathbf{x} \in F^+$ is such that $\mathbf{f}(\mathbf{x}) = 0$ for all $\mathbf{f} \in (F^*)^+$, then by writing

$$\mathbf{x} = \sum_{k=1}^{\infty} a_1^{(k)} \otimes \cdots \otimes a_{n-1}^{(k)} \otimes x_k \qquad (a_1^{(k)}, \ldots, a_{n-1}^{(k)} \in \mathfrak{A}^+,\ x_k \in E^+,\ k \geq 1),$$

and choosing $\sigma \in (\mathfrak{A}^*)^+$ and $\tau \in (E^*)^+$ arbitrarily, it follows that $\sigma \otimes \cdots \otimes \sigma \otimes \tau$ is a positive functional on F for which we have

$$\sum_{k=1}^{\infty} \sigma(a_1^{(k)}) \cdots \sigma(a_{n-1}^{(k)}) \tau(x_k) = 0.$$

This implies that $\sigma(a_1^{(k)}) \cdots \sigma(a_{n-1}^{(k)}) \tau(x_k) = 0$ for all $k \geq 1$, so $\sigma(a_i^{(k)}) = 0$ for some $1 \leq i \leq n-1$ or $\tau(x_k) = 0$. Since $(\mathfrak{A}^*)^+$ and $(E^*)^+$ act faithfully on $\mathfrak{A}^+$ and E^+ respectively, we must have that $a_i^{(k)} = 0$ or $x_k = 0$. Either way, it follows that

$$a_1^{(k)} \otimes \cdots \otimes a_{n-1}^{(k)} \otimes x_k = 0 \qquad (k \geq 1);$$

hence $\mathbf{x} = 0$. The hypothesis now yields $\mathcal{H}^1(\mathfrak{A}, F^*) = \{0\}$ and the verifications in the proof for Theorem 3.8 concerning $\mathcal{L}^{n-1}(\mathfrak{A}, E^*)$ and Lemma 3.7, in conclusion, imply that we have the isomorphy $\mathcal{H}^n(\mathfrak{A}, E^*) \cong \mathcal{H}^1(\mathfrak{A}, F^*) = \{0\}$. ∎

If we want to reinforce the structure of the ordered Banach algebra $\mathfrak{A}$, a logical next step would be to assume that $\mathfrak{A}$ is a Riesz space with order norm, that is, for all $a, b \in \mathfrak{A}$ with $|a| \leq |b|$ it follows that $\|a\| \leq \|b\|$. Since this norm is already complete, the assumption turns $\mathfrak{A}$ into a Banach lattice algebra. The bimodules E over $\mathfrak{A}$ that are considered here are assumed to be Banach lattices as well, in order to respect this specific structure of $\mathfrak{A}$. In doing so, we will refer to such a bimodule E as a **Banach lattice $\mathfrak{A}$-bimodule**. Again, when the actions of $\mathfrak{A}$ on a Banach lattice $\mathfrak{A}$-bimodule are regular, we say that E is a **regular Banach lattice $\mathfrak{A}$-bimodule**. Our goal in the next section will be to investigate under which circumstances we can construct an analogue of Theorem 3.8 with respect to this added lattice structure.

3.3 Hochschild cohomology groups for Banach lattice algebras

If $\mathfrak{A}$ is a Banach lattice algebra and E is a regular Banach lattice $\mathfrak{A}$-bimodule, then the Hochschild cochain complex, defined in Section 3.1, for ordinary Banach algebras needs to be furnished so that we have a suitable order related alternative when taking into account both Lemma 3.1 and Theorem 3.4. It is not generally the case, for instance, that the spaces $\mathcal{L}^n(\mathfrak{A}, E)$ are Banach lattices, let alone, a Banach lattice $\mathfrak{A}$-bimodule. A counterexample can be found in the concluding remarks. However, when we consider regular n-linear operators $\varphi : \mathfrak{A}^n \to E$, that is, φ can be written as the difference of two n-positive operators, which are operators satisfying $\phi(a_1, \ldots, a_n) \geq 0$ whenever $a_i \geq 0$, we can do the following:

Definition: Let $\mathfrak{A}$ be a Banach lattice algebra and E a regular Banach lattice $\mathfrak{A}$-bimodule.

i) Let $\mathcal{L}_r^0(\mathfrak{A}, E) := E$ and for $n \in \mathbb{N}_+$, let

$$\mathcal{L}_r^n(\mathfrak{A}, E) := \{\varphi : \mathfrak{A}^n \to E : \ \varphi \text{ is regular and } n\text{-linear}\}.$$

The elements of $\mathcal{L}_r^n(\mathfrak{A}, E)$ will be called **regular n-cochains**.

ii) For $n \in \mathbb{N}$, define the maps $\eta^n : \mathcal{L}_r^n(\mathfrak{A}, E) \to \mathcal{L}_r^{n+1}(\mathfrak{A}, E)$ by

$$\begin{aligned}\eta^n(\varphi)(a_1, \ldots, a_{n+1}) := a_1 \cdot \varphi(a_2, \ldots, a_{n+1}) \\ + \sum_{k=1}^{n} (-1)^k \varphi(a_1, \ldots, a_k a_{k+1}, \ldots, a_{n+1}) \\ + (-1)^{n+1} \varphi(a_1, \ldots, a_n) \cdot a_{n+1}.\end{aligned}$$

The mapping η^n will be called the **regular n-coboundary operator**.

iii) Let $\mathcal{B}_r^0(\mathfrak{A}, E) := \{0\}$ and for $n \in \mathbb{N}_+$, define $\mathcal{B}_r^n(\mathfrak{A}, E) := \operatorname{ran}(\eta^{n-1})$. The elements of $\mathcal{B}_r^n(\mathfrak{A}, E)$ will be called **regular n-coboundaries**.

iv) For $n \in \mathbb{N}$, define $\mathcal{Z}_r^n(\mathfrak{A}, E) := \ker(\eta^n)$. The elements of $\mathcal{Z}_r^n(\mathfrak{A}, E)$ will be called **regular n-cocycles**.

v) The sequence

$$\{0\} \to E \xrightarrow{\eta^0} \mathcal{L}_r(\mathfrak{A}, E) \xrightarrow{\eta^1} \mathcal{L}_r^2(\mathfrak{A}, E) \xrightarrow{\eta^2} \cdots \xrightarrow{\eta^{n-1}} \mathcal{L}_r^n(\mathfrak{A}, E) \xrightarrow{\eta^n} \mathcal{L}_r^{n+1}(\mathfrak{A}, E) \xrightarrow{\eta^{n+1}} \cdots$$

will be called the **regular Hochschild cochain complex**.

For $\varphi \in \mathcal{L}_r^n(\mathfrak{A}, E)$ we can write $\varphi = \varphi_1 - \varphi_2$ with $\varphi_1, \varphi_2 \geq 0$. Let $\xi : \mathfrak{A} \to B(E)$ and $\chi : \mathfrak{A}^{\text{opp}} \to B(E)$ be the operators describing the left action, respectively, the right action of $\mathfrak{A}$ on E. Now, since $\xi = \xi_1 - \xi_2$ with $\xi_1, \xi_2 \geq 0$ and $\chi = \chi_1 - \chi_2$ with $\chi_1, \chi_2 \geq 0$, it follows that the maps

$$(a_1, \ldots, a_{n+1}) \mapsto \xi_1(a_1)(\varphi_1(a_2, \ldots, a_{n+1})) + \xi_2(a_1)(\varphi_2(a_2, \ldots, a_{n+1}))$$

and

$$(a_1, \ldots, a_{n+1}) \mapsto \xi_2(a_1)(\varphi_2(a_2, \ldots, a_{n+1})) + \xi_1(a_1)(\varphi_1(a_2, \ldots, a_{n+1}))$$

are positive, so $(a_1, \ldots, a_{n+1}) \mapsto a_1 \cdot \varphi(a_2, \ldots, a_{n+1})$ is regular. Similarly, the maps

$$(a_1, \ldots, a_{n+1}) \mapsto \chi_1(a_{n+1})(\varphi_1(a_1, \ldots, a_n)) + \chi_2(a_{n+1})(\varphi_2(a_1, \ldots, a_n))$$

and

$$(a_1, \ldots, a_{n+1}) \mapsto \chi_2(a_{n+1})(\varphi_2(a_1, \ldots, a_n)) + \chi_1(a_{n+1})(\varphi_1(a_1, \ldots, a_n))$$

are positive, so $(a_1, \ldots, a_{n+1}) \mapsto \varphi(a_1, \ldots, a_n) \cdot a_{n+1}$ is regular. Clearly, the alternating sum occurring in the definition of the regular n-coboundary operator is regular; hence η^n is well defined for all $n \geq 1$. For η^0, it follows from the decomposition $x = x^+ - x^-$ that $\eta^0(x)$ is regular for all $x \in E$. As for the desired inclusion

$$\mathcal{B}_r^n(\mathfrak{A}, E) \subset \mathcal{Z}_r^n(\mathfrak{A}, E)$$

for all $n \in \mathbb{N}_+$, along the lines of the proof given for Lemma 3.1, we desire the space $\mathcal{L}_r(\mathfrak{A}, E)$ to be a regular Banach lattice $\mathfrak{A}$-bimodule. The assumption that E is Dedekind complete will prove to be sufficient and in fact, this also holds for $\mathcal{L}_r^n(\mathfrak{A}, E)$. Before we prove this, we need a characterizing property for Banach lattices, an appropriate norm and the fact that these regular operators are norm bounded.

Lemma 3.11 *Every regular operator mapping a Banach lattice into a normed Riesz space is norm bounded.*

Proof: Let E be a Banach lattice and F a normed Riesz space. Consider a regular operator $\varphi : E \to F$ and assume that φ is not norm bounded. Then there exists a sequence $(x_n)_{n\geq 1}^{\infty}$ in E such that $\|x_n\| = 1$ and $\|\varphi(x_n)\| \geq n^3$. Since we have $\||x_n|\| = \|x_n\|$, it follows from

$$\sum_{n=1}^{\infty} \frac{\||x_n|\|}{n^2} < \infty$$

and the fact that E is a Banach lattice that

$$x := \sum_{n=1}^{\infty} \frac{|x_n|}{n^2} \in E.$$

Furthermore, we see that $0 \leq n^{-2}|x_n| \leq x$ for all $n \geq 1$, so writing $\varphi = \varphi_1 - \varphi_2$ with $\varphi_1, \varphi_2 \geq 0$, we have the inequality $(\varphi_1 + \varphi_2)(n^{-2}|x_n|) \leq (\varphi_1 + \varphi_2)(x)$ for all $n \geq 1$. Now from the inequality $|\varphi(x)| \leq (\varphi_1 + \varphi_2)(|x|)$ for all $x \in E$, we find that

$$n \leq \left\|\varphi\left(n^{-2}x_n\right)\right\| = \left\|\left|\varphi\left(n^{-2}x_n\right)\right|\right\| \leq \left\|(\varphi_1 + \varphi_2)\left(n^{-2}|x_n|\right)\right\| \leq \|(\varphi_1 + \varphi_2)(x)\|$$

for all $n \geq 1$, which leads to a contradiction; hence φ is norm bounded. ∎

We wish to add that this property also generalizes to n-linear regular operators when considering a sequence $((x_1^k, \ldots, x_n^{(k)}))_{k\geq 1}$ with $\|x_i^{(k)}\| = 1$ for all $1 \leq i \leq n$ and $k \geq 1$ such that $\|\varphi(x_1^{(k)}, \ldots, x_n^{(k)})\| \geq n^3$.

Lemma 3.12 *Let E be a normed Riesz space. Then E is a Banach lattice if and only if every increasing norm Cauchy sequence of E^+ is norm convergent.*

Proof: Suppose that $(x_n)_{n\geq 1}$ is a norm Cauchy sequence in a normed Riesz space E satisfying the given property. For each $m \geq 1$ there exists a number $N_m \in \mathbb{N}_+$ such that $\|x_n - x_m\| < 2^{-m}$ whenever $n, m \geq N_m$ and $N_m < N_{m+1}$ for all $m \geq 1$. By defining $\xi_m := x_{N_m}$, we construct a subsequence $(\xi_m)_{m\geq 1}$ such that we have $\|\xi_{m+1} - \xi_m\| < 2^{-m}$ for all $m \geq 1$. Now let

$$y_m := \sum_{i=1}^{m} (\xi_{i+1} - \xi_i)^+ \qquad \text{and} \qquad z_m := \sum_{i=1}^{m} (\xi_{i+1} - \xi_i)^-$$

for all $m \geq 1$. Clearly, the sequences $(y_m)_{m\geq 1}$ and $(z_m)_{m\geq 1}$ are increasing and if $k > l \geq 1$, it follows that

$$\begin{aligned}\|y_k - y_l\| &= \left\| \sum_{i=l+1}^{k} (\xi_{i+1} - \xi_i)^+ \right\| \leq \sum_{i=l+1}^{k} \|(\xi_{i+1} - \xi_i)^+\| \leq \sum_{i=l+1}^{k} \||\xi_{i+1} - \xi_i|\| \\ &= \sum_{i=l+1}^{k} \|\xi_{i+1} - \xi_i\| < \sum_{i=l+1}^{k} 2^{-i} = 2^{-l} - 2^{-k} < 2^{-l},\end{aligned}$$

so $(y_m)_{m\geq 1}$ is an increasing norm Cauchy sequence in E^+. Analogously, we can also show that $(z_m)_{m\geq 1}$ is an increasing norm Cauchy sequence in E^+. So, there exist vectors $y, z \in E$ such that $y_m \to y$ and $z_m \to z$; hence $y_m - z_m \to y - z$. Since we have $y_m - z_m = \xi_{m+1} - \xi_1$, we conclude that

$$\xi_{m+1} = y_m - z_m + \xi_1 \to y - z + \xi_1.$$

The original Cauchy sequence must have the same limit, so E is a Banach lattice because this sequence was arbitrary. ∎

Lemma 3.13 *Let E and F be Banach lattices. Then the map $\|\cdot\|_r : \mathcal{L}_r^n(E, F) \to \mathbb{R}_+$ defined by*

$$\|\varphi\|_r := \inf\{\|\psi\| : \pm\varphi \leq \psi\}$$

is a norm.

Proof: The case where $n = 0$ is simply $\|x\|_r = \||x|\| = \|x\|$ and when $n = 1$, we have $\|\varphi\|_r = \||\varphi|\|$, which corresponds to the regular operator norm and our statement holds by [1, Thm. 1.32]. Suppose that $n \geq 2$. Using Lemma 3.11 and the fact that for every $\varphi \in \mathcal{L}_r^n(E, F)$ we have $\varphi = \varphi_1 - \varphi_2$ with φ_1 and φ_2 both n-positive satisfying $\pm\varphi \leq \varphi_1 + \varphi_2$, we find that $\|\varphi\|_r$ is well defined. Clearly, we have that $\|\cdot\|_r$ defines a seminorm on $\mathcal{L}_r^n(E, F)$. Now suppose that $\|\varphi\|_r = 0$. Then for all $k \geq 1$ there are $\xi_k \in \mathcal{L}_r^n(E, F)$ with $\pm\varphi \leq \xi_k$ such that $\|\xi_k\| \leq 2^{-k}$. Let $x_i \in E^+$ for all $1 \leq i \leq n$. Then we have $\pm\varphi(x_1, \dots, x_n) \leq \xi_k(x_1, \dots, x_n)$ and because F is a Banach lattice, it follows that $|\varphi(x_1, \dots, x_n)| \leq \xi_k(x_1, \dots, x_n)$, so

$$\|\varphi(x_1, \dots, x_n)\| \leq \|\xi_k(x_1, \dots, x_n)\| \leq \|\xi_k\|\|x_1\| \cdots \|x_n\| \to 0.$$

Since every $x_i \in E$ can be written as $x_i = x_i^+ - x_i^-$ for all $1 \leq i \leq n$ and φ is n-linear, we conclude that $\varphi = 0$; hence $\|\cdot\|_r$ is a norm. ∎

The norm $\|\cdot\|_r$ will also be called the **regular operator norm** here and it allows us to state the following theorem, which we will prove in section 3.4:

Theorem 3.14 *Let E and F be Banach lattices with F Dedekind complete. Then, with respect to the regular operator norm, we have that $\mathcal{L}_r^n(E, F)$ is a Dedekind complete Banach lattice.*

Now for a Banach lattice algebra $\mathfrak{A}$ and a regular Banach lattice $\mathfrak{A}$-bimodule E which is Dedekind complete, on $\mathcal{L}_r(\mathfrak{A}, E)$, we can again consider the actions

$$(a \cdot \varphi)(b) := a \cdot \varphi(b) \quad \text{and} \quad (\varphi \cdot a)(b) := \varphi(ab) - \varphi(a) \cdot b \quad (a, b \in \mathfrak{A},\ \varphi \in \mathcal{L}_r(\mathfrak{A}, E)).$$

Let $\xi = \xi_1 - \xi_2$ describe the left action of $\mathfrak{A}$ on E with $\xi_1, \xi_2 \geq 0$ and define

$$\chi(a)(\varphi)(b) := \xi(a)(\varphi(b)) \qquad (a, b \in \mathfrak{A},\ \varphi \in \mathcal{L}_r(\mathfrak{A}, E)).$$

Then $\chi(a)(\varphi) = \xi(a) \circ \varphi$, so $a \cdot \varphi \in \mathcal{L}_r(\mathfrak{A}, E)$. We find that $\chi : \mathfrak{A} \to \mathcal{L}(\mathcal{L}_r(\mathfrak{A}, E))$. Let $\chi_1(a)(\varphi)(b) := \xi_1(a)(\varphi(b))$ and $\chi_2(a)(\varphi)(b) := \xi_2(a)(\varphi(b))$. It follows that we have $\chi_1(a)(\varphi) = \xi_1(a) \circ \varphi$ and $\chi_2(a)(\varphi) = \xi_2(a) \circ \varphi$. Clearly, the equality $\chi_1 - \chi_2 = \chi$ holds and if $a \in \mathfrak{A}^+$, then $\chi_1(a)(\varphi) \geq 0$ and $\chi_2(a)(\varphi) \geq 0$ whenever $\varphi \geq 0$, so $\chi_1, \chi_2 \geq 0$ and we conclude that the left action on $\mathcal{L}_r(\mathfrak{A}, E)$ is regular. Similarly, one shows that the right action on $\mathcal{L}_r(\mathfrak{A}, E)$ is regular. If $\kappa > 0$ is such that $\|a \cdot x\| \leq \kappa \|a\| \|x\|$ and $\|x \cdot a\| \leq \kappa \|x\| \|a\|$ for all $a \in \mathfrak{A}$ and $x \in E$, then it follows from the inequality $\|\varphi\| \leq \|\varphi\|_r$ that $\mathcal{L}_r(\mathfrak{A}, E)$ is a regular Banach lattice $\mathfrak{A}$-bimodule. For a $\varphi \in \mathcal{L}_r^{n+1}(\mathfrak{A}, E)$ we can define the function $\overline{\varphi} : \mathfrak{A} \times \cdots \times \mathfrak{A} \to \mathcal{L}_r(\mathfrak{A}, E)$ by

$$\overline{\varphi}(a_1, \ldots, a_n)(a_{n+1}) := \varphi(a_1, \ldots, a_{n+1}).$$

By writing $a_i = a_i^+ - a_i^-$ for all $1 \leq i \leq n$ and using the fact that φ is n-linear yields the difference of two positive operators which is regular in the last coordinate, so $\overline{\varphi}$ is well defined and $\overline{\varphi} \in \mathcal{L}_r^n(\mathfrak{A}, E)$. It now follows from the proof used in Lemma 3.1 that we have the inclusion $\mathcal{B}_r^n(\mathfrak{A}, E) \subset \mathcal{Z}_r^n(\mathfrak{A}, E)$ for all $n \in \mathbb{N}_+$.

Definition: Let $\mathfrak{A}$ be a Banach lattice algebra and E a regular Banach lattice $\mathfrak{A}$-bimodule which is Dedekind complete. Then for $n \in \mathbb{N}_+$, the quotient

$$\mathcal{H}_r^n(\mathfrak{A}, E) := \mathcal{Z}_r^n(\mathfrak{A}, E) / \mathcal{B}_r^n(\mathfrak{A}, E)$$

will be referred to as the **regular n-th Hochschild cohomology group** of $\mathfrak{A}$ with coefficients in E.

For the actions of $\mathfrak{A}$ on $\mathcal{L}_r^n(\mathfrak{A}, E)$, which are defined by

$$(a \cdot \varphi)(a_1, \ldots, a_n) := a \cdot \varphi(a_1, \ldots, a_n)$$

and

$$(\varphi \cdot a)(a_1, \ldots, a_n) := \varphi(aa_1, \ldots, a_n) + \sum_{k=1}^{n-1} (-1)^k \varphi(a, a_1, \ldots, a_k a_{k+1}, \ldots, a_n) + (-1)^n \varphi(a, a_1, \ldots, a_{n-1}) \cdot a_n,$$

it is similarly shown that they are regular and again, since $\|\varphi\| \leq \|\varphi\|_r$, it follows from Lemma 3.2 that $\mathcal{L}_r^n(\mathfrak{A}, E)$ are regular Banach lattice $\mathfrak{A}$-bimodules for all $n \geq 1$.

Also, when considering the map $\tau^n : \mathcal{L}_r^{n+k}(\mathfrak{A}, E) \to \mathcal{L}_r^n(\mathfrak{A}, \mathcal{L}_r^k(\mathfrak{A}, E))$ defined by

$$(\tau^n(\varphi)(a_1, \ldots, a_n))(a_{n+1}, \ldots, a_{n+k}) := \varphi(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+k}),$$

we have already argued that this map is well defined. This can be seen by splitting the first n coordinates into positive and negative parts and then using the fact that the map in question is $(n+k)$-linear. Analogous to the proof of Lemma 3.3, we find that τ^n is an isomorphism of linear spaces. Since this is the only property of τ^n used in the proof of Theorem 3.4, we conclude that

$$\mathcal{H}_r^{n+k}(\mathfrak{A}, E) \cong \mathcal{H}_r^n(\mathfrak{A}, \mathcal{L}_r^k(\mathfrak{A}, E)) \qquad (n \in \mathbb{N}_+,\ k \in \mathbb{N}). \tag{17}$$

Since the norm dual of a Banach lattice is a Dedekind complete Banach lattice, this allows us, upholding the tradition, to give the following definition:

Definition: A Banach lattice algebra $\mathfrak{A}$ is said to be **regularly amenable** if we have that $\mathcal{H}_r^1(\mathfrak{A}, E^*) = \{0\}$ for all regular Banach lattice $\mathfrak{A}$-bimodules E.

In order to justify this definition, let G be an amenable group and consider the Banach lattice algebra $L^1(G)$. By Theorem 2.25 we have that $\mathcal{H}^1(L^1(G), E^*) = \{0\}$ for all Banach $L^1(G)$-bimodules, so if E is a regular Banach lattice $L^1(G)$-bimodule and $\varphi \in \ker(\eta^1)$, then, by Lemma 3.11, we have that $\varphi : \mathfrak{A} \to \mathcal{L}_r(\mathfrak{A}, E^*) \subset \mathcal{L}(\mathfrak{A}, E^*)$ is a derivation; hence there is a functional $f \in E^*$ such that $\varphi = \mathrm{ad}_f$. This is a regular operator and lies in $\mathrm{ran}(\eta^0)$, so $\mathcal{H}_r^1(\mathfrak{A}, E^*) = \{0\}$ and we conclude that $L^1(G)$ is regularly amenable for all amenable groups G. Moreover, if a Banach lattice algebra $\mathfrak{A}$ is amenable, then it must be regularly amenable as well.

Just as in the previous cases, we want to characterize a regularly amenable Banach lattice algebra $\mathfrak{A}$ in terms of $\mathcal{H}_r^n(\mathfrak{A}, E^*)$ for all $n \in \mathbb{N}_+$ and accordingly, we will investigate tensor products of Banach lattices first in the next section.

3.3.1 Tensor products of Banach lattices

We will start with considering Archimedean Riesz spaces $E_1, \ldots, E_n$ and herewith, along the lines of [9] and [10], present a general construction of such tensor products. A key tool here is Kakutani's representation theorem, which will allow us to embed the algebraic tensor product $E_1 \otimes \cdots \otimes E_n$ in an Archimedean Riesz space, in fact, a Banach lattice, if $E_1, \ldots, E_n$ all have order units. In the case where we have order units, it is possible to create a norm on the given Riesz space:

Lemma 3.15 *Let E be an Archimedean Riesz space with order unit e. Define*

$$\|x\|_e := \inf\{\lambda \in [0, \infty) : \; -\lambda e \leq x \leq \lambda e\} \qquad (x \in E).$$

Then $\|\cdot\|_e$ defines a norm on E.

Proof: Since e is an order unit, there exists a $\lambda \in [0, \infty)$ such that $0 \leq |e| \leq \lambda e$, so $e \geq 0$. Also, by definition of an order unit $\|\cdot\|_e$ is well defined, since the given set over which we need to compute the infimum is not empty. For $\lambda \in [0, \infty)$ and $x \in E$ we have that $\lambda e \leq x \leq \lambda e$ if and only if $|x| \leq \lambda e$, so we can rewrite $\|\cdot\|_e$ to

$$\|x\|_e = \inf\{\lambda \in [0, \infty) : \; |x| \leq \lambda e\}.$$

Clearly, we have that $\|x\|_e \geq 0$ for all $x \in E$. If $x = 0$, then $x = 0e$, so $\|x\|_e = 0$. Conversely, if $x \in E$ is such that $\|x\|_e = 0$, then $|x| \leq \frac{1}{n}e$ for all $n \in \mathbb{N}$, so $n|x| \leq e$ for all $n \in \mathbb{N}$ and since E is Archimedean, it follows that $|x| \leq 0$. This implies that $|x| = 0$, so $x = 0$. Let $x, y \in E$ and choose $\lambda, \mu \in [0, \infty)$ such that $|x| \leq \lambda e$ and $|y| \leq \mu e$. We have that $x \leq |x|$ and $y \leq |y|$, so $x + y \leq |x| + |y|$. Analogously, we have that $-x \leq |x|$ and $-y \leq |y|$, so $-(x + y) \leq |x| + |y|$; hence $|x + y| \leq |x| + |y|$. This yields the inequality $|x + y| \leq |x| + |y| \leq (\lambda + \mu)e$, so $\|x + y\|_e \leq \lambda + \mu$. Since λ and μ were arbitrary, we now get $\|x + y\|_e \leq \|x\|_e + \|y\|_e$. Let $\alpha \in \mathbb{R} \setminus \{0\}$ and $0 \neq x \in E$. If $\alpha > 0$, then we have

$$|\alpha x| = (\alpha x) \vee (-\alpha x) = \alpha(x \vee (-x)) = \alpha|x| = |\alpha||x|$$

and if $\alpha < 0$, then

$$|\alpha x| = (\alpha x) \vee (-\alpha x) = (-\alpha(-x)) \vee (-\alpha x) = -\alpha((-x) \vee x) = -\alpha|x| = |\alpha||x|.$$

So, if $\lambda \in [0,\infty)$ is such that $|\alpha x| \le \lambda e$, then $|x| \le |\alpha|^{-1}\lambda e$ and this yields $\|x\|_e \le |\alpha|^{-1}\lambda$, so $|\alpha|\|x\|_e \le \|\alpha x\|_e$ since λ was arbitrary. Finally, if $\lambda \in [0,\infty)$ is such that $|x| \le \lambda e$, then the inequality $|\alpha x| = |\alpha||x| \le |\alpha|\lambda e$ implies that $\|\alpha x\|_e \le |\alpha|\lambda$ and since λ was arbitrary, we obtain $\|\alpha x\|_e \le |\alpha|\|x\|_e$. We conclude that $\|\cdot\|_e$ defines a norm on E. ∎

The norm $\|\cdot\|_e$ is called the **order unit norm** on E and is used in the proof for Kakutani's representation theorem. Before we state this theorem, we need a lemma.

Lemma 3.16 *Let X be a compact Hausdorff space and $\phi : C(X) \to \mathbb{R}$ a non-zero Riesz homomorphism. then there exist uniquely defined $\alpha > 0$ and $x \in X$ such that $\phi(f) = \alpha f(x)$ for all $f \in C(X)$.*

Proof: Since ϕ is positive, the Riesz representation theorem implies that there exists a unique regular Borel measure μ on X such that

$$\phi(f) = \int_X f(x)d\mu(x) \qquad (f \in C(X)).$$

We claim that the support of μ, that is,

$$\text{supp}(\mu) := \{x \in X : \mu(U) > 0 \text{ for all open sets } U \text{ with } x \in U\},$$

is a singleton. Suppose that for every $x \in X$ there is an open set U_x of x such that $\mu(U_x) = 0$. Then $X = \bigcup_{x\in X}$ is an open covering and X being compact, we have finite subcover $X = \bigcup_{k=1}^n U_{x_k}$. Since $\phi \neq 0$, we must have that $\phi(1_X) = \mu(X) > 0$, but now we have

$$0 < \mu(X) = \mu\left(\bigcup_{k=1}^n U_{x_k}\right) \le \sum_{k=1}^n \mu(U_{x_k}) = 0,$$

which is impossible; hence $\text{supp}(\mu) \neq \emptyset$. Now suppose that $x, y \in \text{supp}(\mu)$ with $x \neq y$. Since X is Hausdorff, we may take disjoint open sets U_x of x and V_y of y. By Urysohn's lemma there are $f, g \in C(X)$ with $f(x) = g(y) = 1$ and $f(U_x^c) = g(V_y^c) = 0$. This implies that

$$0 = \phi(f \wedge g) = \min\{\phi(f), \phi(g)\} = \min\left\{\int_X f(x)d\mu(x), \int_X g(x)d\mu(x)\right\} > 0,$$

which is also impossible; hence $\text{supp}(\mu)$ is a singleton. Let $\text{supp}(\mu) = \{x\}$ and suppose U is a Borel set with $x \notin U$. For each $y \in U$ there exist an open set O_y of y such that $\mu(O_y) = 0$ and

$$\left(\bigcup_{y\in U} O_y\right) \cup \overline{U}^c$$

is an open covering of X. Since X is compact, there exists a finite subcover $(\bigcup_{k=1}^n O_{y_k}) \cup \overline{U}^c$ of X. Because $U \cap \overline{U}^c = \emptyset$, it follows that $\bigcup_{k=1}^n O_{y_k}$ covers U. We conclude that

$$\mu(U) \le \mu\left(\bigcup_{k=1}^n O_{y_k}\right) \le \sum_{k=1}^n \mu(O_{y_k}) = 0.$$

Conversely, suppose U is a Borel set such that $x \in U$. Then we have shown that $\mu(U^c) = 0$ and therefore, we must have that

$$\phi(1_X) = \mu(X) = \mu(U) + \mu(U^c) = \mu(U);$$

hence $\mu = \phi(1_X)\delta_x$. We conclude that

$$\phi(f) = \int_X f(y)d\mu(y) = \int_X f(y)d\phi(1_X)\delta_x(y) = \phi(1_X)f(x)$$

for all $f \in C(X)$. This proves the assertion, since both μ and x are unique. ∎

Theorem 3.17 (Kakutani's representation theorem) *Suppose E is an Archimedean Riesz space with order unit $e \geq 0$. Then there exists a compact Hausdorff space X such that E is isomorphic, as a Riesz space, to a uniformly dense subspace of $C(X)$ containing the constant function 1_X.*

Proof: Let Ω be the set of all positive linear maps $\varphi : E \to \mathbb{R}$ such that $\varphi(e) = 1$. It is a straightforward verification to show that Ω is convex. Fix $x \in E$ and let $\lambda \geq 0$ be such that $|x| \leq \lambda e$. Then we have that $|\varphi(x)| \leq \varphi(|x|) \leq \lambda\varphi(e)$, so as λ was arbitrary, we find that $\|\varphi\| \leq 1$ and since $\|e\|_e = 1$, it follows that $\|\varphi\| = 1$ for all $\varphi \in \Omega$. Now if $(\varphi_n)_{n\geq 1}$ is a sequence in Ω such that $\varphi_n \overset{*}{\to} \varphi$ for some $\varphi \in E^*$, then we have that $\varphi(e) = \lim_{n\to\infty} \varphi_n(e) = 1$ and $\varphi(x) = \lim_{n\to\infty} \varphi_n(x) \geq 0$ for all $x \in E^+$; hence Ω is w^*- closed. The Banach-Alaoglu theorem now implies that Ω is w^*- compact. Define $\psi : \mathrm{Sp}\{e\} \to \mathbb{R}$ by $\psi(\alpha e) := \alpha$ and $p : E \to \mathbb{R}$ by $p(x) := \|x^+\|_e$. If $x, y \in E$ with $0 \leq x \leq y$, then for $\lambda \geq 0$ such that $y \leq \lambda e$, we have that $x \leq \lambda e$, so $\|y\|_e \leq \lambda$. Since λ was arbitrary, we find that this implies the inequality $\|y\|_e \leq \|x\|_e$. We have that $(x + y)^+ \leq x^+ + y^+$ for all $x, y \in E$, so

$$p(x + y) = \|(x + y)^+\|_e \leq \|x^+ + y^+\| \leq \|x^+\|_e + \|y^+\|_e = p(x) + p(y)$$

for all $x, y \in E$. If $\alpha \geq 0$, then we have $(\alpha x)^+ = \alpha x^+$, so

$$p(\alpha x) = \|(\alpha x)^+\|_e = \|\alpha x^+\|_e = \alpha\|x^+\|_e = \alpha p(x).$$

Finally, if $\alpha > 0$ we have that

$$\psi(\alpha e) = \alpha = \|\alpha e\|_e = \|(\alpha e)^+\|_e = p(\alpha e)$$

and if $\alpha < 0$, then it follows that

$$\psi(\alpha e) = \alpha \leq 0 = \|(\alpha e)^+\|_e = p(\alpha e).$$

By the Hahn-Banach theorem, there exists a linear extension $\varphi : E \to \mathbb{R}$ of ψ such that $\varphi \leq p$. Now let $x \in E$ with $x \geq 0$. We find that

$$-\varphi(x) = \varphi(-x) \leq p(-x) = \|(-x)^+\|_e = 0,$$

so $\varphi(x) \geq 0$ and we conclude that φ is positive. Since $\varphi(e) = \psi(e) = 1$, it follows that $\varphi \in \Omega$, so Ω is not empty. Since E^* endowed with the w^*- topology is locally convex, it follows from the Krein-Milman theorem that $\mathrm{ext}(\Omega)$ is not empty and $\Omega = \overline{\mathrm{co}}(\mathrm{ext}(\Omega))$. Suppose that $\varphi \in \mathrm{ext}(\Omega)$ and let $x, y \in E$ be such that $x \wedge y = 0$. Define the map $\psi : E^+ \to \mathbb{R}$ by

$$\psi(z) := \sup_{n\geq 1} \varphi(z \wedge nx) = \lim_{n\to\infty} \varphi(z \wedge nx) \leq \varphi(z).$$

If $z_1, z_2 \in E^+$, then $(z_1 + z_2) \wedge nx = z_1 \wedge nx + z_2 \wedge nx$ for all $n \geq 1$ and it follows that ψ is additive. Clearly, we also have that ψ respects scalar multiplication for $\alpha \geq 0$.

Consequently, since $z = z^+ - z^-$ for all $z \in E$, we can extend ψ to a positive map $\psi' : E \to \mathbb{R}$ and find that $\varphi - \psi'$ is positive. Define $\beta := \psi'(e)$. If $\beta = 0$, then $\psi' = 0 = \beta\varphi$ and if $\beta = 1$, then $\varphi - \psi' = 0$ because $(\varphi - \psi')(e) = 0$, so $\psi' = \varphi = \beta\varphi$. If $0 < \beta < 1$, then both $\beta^{-1}\psi'$ and $(1-\beta)^{-1}(\varphi - \psi')$ are elements of Ω and since

$$\varphi = \beta\beta^{-1}\psi' + (1-\beta)(1-\beta)^{-1}(\varphi - \psi),$$

we find that $\psi' = \beta\varphi$, since φ is an extreme point of Ω. These findings yield the identities $\varphi(x) = \psi'(x) = \beta\varphi(x)$ and $\beta\varphi(y) = \psi'(y) = 0$ for all $0 \leq \beta \leq 1$, so $\varphi(x) \wedge \varphi(y) = 0$. For all $x, y \in E$ we have that $(x - x \wedge y) \wedge (y - x \wedge y) = 0$, so $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$ for all $x, y \in E$ and using the identity $x \vee y = x + y - x \wedge y$, we find that

$$\varphi(x \vee y) = \varphi(x) + \varphi(y) - \varphi(x \wedge y) = \varphi(x) + \varphi(y) - \varphi(x) \wedge \varphi(y) = \varphi(x) \vee \varphi(y);$$

hence φ is a Riesz homomorphism. Conversely, suppose that $\varphi : E \to \mathbb{R}$ is a Riesz homomorphism with $\varphi(e) = 1$. Since $\varphi(x) = 0$ implies that $0 = |\varphi(x)| = \varphi(|x|)$ for all $x \in \ker(\varphi)$, it follows for $y \in E$ and $x \in \ker(\varphi)$ with $|y| \leq |x|$ that $|\varphi(y)| \leq \varphi(|x|) = 0$, so $\ker(\varphi)$ is an ideal in E. Let $\varphi_1, \varphi_2 \in \Omega$ and $0 < \beta < 1$ and assume that $\varphi = \beta\varphi_1 + (1-\beta)\varphi_2$. This implies that $\beta\varphi_1 \leq \varphi$ and if $x \in \ker(\varphi)$, then $|\beta\varphi_1(x)| \leq \beta\varphi_1(|x|) \leq \varphi(|x|) = 0$, so $\ker(\varphi) \subset \ker(\varphi_1)$. For $y \in E$ it now follows that $y - \varphi(y)e \in \ker(\varphi_1)$, so $\varphi_1(y) = \varphi(y)$ for all $y \in E$ and we conclude that $\varphi_1 = \varphi_2$; hence φ is an extreme point of Ω. Now if $(\varphi_n)_{n\geq 1}$ is a sequence in $\text{ext}(\Omega)$ such that $\varphi_n \xrightarrow{*} \varphi \in \Omega$, then for all $x, y \in E$, we find that

$$\begin{aligned}\varphi_n(x \vee y) = \varphi_n(x) \vee \varphi_n(y) &= \tfrac{1}{2}(\varphi_n(x) + \varphi_n(y) - |\varphi_n(x) - \varphi_n(y)|) \\ &\to \tfrac{1}{2}(\varphi(x) + \varphi(y) - |\varphi(x) - \varphi(y)|) = \varphi(x) \vee \varphi(y),\end{aligned}$$

so $0 = \varphi_n(x) \vee \varphi_n(y) - \varphi_n(x \vee y) \to \varphi(x \vee y) - \varphi(x) \vee \varphi(y)$; hence $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$ for all $x, y \in E$ and we conclude that φ is a Riesz homomorphism, so $\varphi \in \text{ext}(\Omega)$. This implies that $\text{ext}(\Omega)$ is w^*- compact. Define the map $\Phi : E \to C(\text{ext}(\Omega))$ by $\Phi(x)(\varphi) := \varphi(x)$. By Lemma 3.15, it follows that Φ is well defined and it is straightforward to verify that Φ is a positive linear function. Now suppose that $x \in E$ is such that $\Phi(x) \neq 0$. Then it follows from the Hahn-Banach theorem that

$$0 < \sup_{\varphi\in\text{ext}(\Omega)} |\Phi(x)(\varphi)| \leq \sup_{\varphi\in\Omega} |\Phi(x)(\varphi)| \leq \sup_{\varphi\in E^*, \|\varphi\|=1} |\Phi(x)(\varphi)| = \|x\|_e;$$

hence $x \neq 0$ and we find that Φ is injective. Furthermore, since

$$\Phi(|x|)(\varphi) = \varphi(|x|) = |\varphi(x)| = |\Phi(x)(\varphi)| = |\Phi(x)|(\varphi)$$

for all $\varphi \in \text{ext}(\Omega)$, we find that Φ is a Riesz homomorphism, so $\Phi(E)$ is a Riesz subspace of $C(\text{ext}(\Omega))$. This Riesz subspace $\Phi(E)$ contains the constant functions, because we have that $\Phi(e) = 1_{\text{ext}(\Omega)}$ and if $\varphi_1, \varphi_2 \in \text{ext}(\Omega)$ with $\varphi_1 \neq \varphi_2$, then there must be an element $x \in E$ such that $\varphi_1(x) \neq \varphi_2(x)$, so $\Phi(x)(\varphi_1) \neq \Phi(x)(\varphi_2)$ from which we conclude that $\Phi(E)$ separates the points of $\text{ext}(\Omega)$. Finally, the Stone-Weierstrass theorem now implies that $\Phi(E)$ is uniformly dense in $C(\text{ext}(\Omega))$. ∎

Now, let $X_1, \ldots, X_n$ be the compact Hausdorff spaces such that $E_i \subset C(X_i)$ for all $1 \leq i \leq n$ according to Theorem 3.17 and consider the map

$$\psi : C(X_1) \otimes \cdots \otimes C(X_n) \to C(X_1 \times \cdots \times X_n)$$

defined by

$$\sum_{k=1}^{m} f_1^{(k)} \otimes \cdots \otimes f_n^{(k)} \mapsto \sum_{k=1}^{m} f_1^{(k)} \cdots f_n^{(k)} \qquad (f_i^{(k)} \in C(X_i),\ 1 \le i \le n)$$

where $f_1^{(k)} \cdots f_n^{(k)}(x_1, \ldots, x_n) := f_1^{(k)}(x_1) \cdots f_n^{(k)}(x_n)$ for all $1 \le k \le m$ and $x_i \in X_i$. This map is well defined, because a finite product of such continuous functions is again continuous with respect to the product topology. Clearly, the map ψ is linear and suppose that

$$\mathbf{x} := \sum_{k=1}^{m} f_1^{(k)} \otimes \cdots \otimes f_n^{(k)} \neq 0.$$

Without loss of generality, we may assume that $(f_n^{(k)})_{k=1}^m$ is linearly independent and that $f_1^{(1)}, \ldots, f_{n-1}^{(1)} \neq 0$, so we may chose $x_i \in X_i$ for $1 \le i \le n-1$ such that $f_i^{(1)}(x_i) \neq 0$, so

$$\sum_{k=1}^{m} (f_1^{(k)}(x_1) \cdots f_{n-1}^{(k)}(x_{n-1})) f_n^{(k)} \neq 0$$

and therefore, there also exists an element $x_n \in X_n$ such that

$$\sum_{k=1}^{m} f_1^{(k)}(x_1) \cdots f_{n-1}^{(k)}(x_{n-1}) f_n^{(k)}(x_n) \neq 0;$$

hence $\psi(\mathbf{x}) \neq 0$ and we conclude that ψ is injective. This implies that $C(X_1) \otimes \cdots \otimes C(X_n)$ can be regarded as a linear subspace of $C(X_1 \times \cdots \times X_n)$ and, by construction, we have the same result for $E_1 \otimes \cdots \otimes E_n$ under ψ. It is a straightforward verification to show that $C(X_1) \otimes \cdots \otimes C(X_n)$ is a subalgebra of $C(X_1 \times \cdots \times X_n)$ and clearly, it contains the constant functions. Also, if $(x_1, \ldots, x_n) \neq (y_1, \ldots, y_n)$, then there must be a $1 \le k \le n$ such that $x_k \neq y_k$, so if we put $f_i := 1_{X_i}$ whenever $i \neq k$ and chose $f_k \in C(X_k)$ such that $f_k(x_k) \neq f_k(y_k)$, which can be done by Urysohn's lemma, it follows that $f_1 \cdots f_n(x_1, \ldots, x_n) \neq f_1 \cdots f_n(y_1, \ldots, y_n)$; hence $C(X_1) \otimes \cdots \otimes C(X_n)$ separates the points of $X_1 \times \cdots \times X_n$ and the Stone-Weierstrass theorem for subalgebras now implies that $C(X_1) \otimes \cdots \otimes C(X_n)$ is uniformly dense in $C(X_1 \times \cdots \times X_n)$. Since E_i is also uniformly dense in $C(X_i)$ for all $1 \le i \le n$, an inductive argument together with the triangle inequality for norms now shows that $E_1 \otimes \cdots \otimes E_n$ must be uniformly dense in $C(X_1 \times \cdots \times X_n)$ too.

The last thing that should be discussed concerning the order unital Archimedean Riesz spaces E_i, before we start focussing on constructing a Riesz space from $E_1 \otimes \cdots \otimes E_n$, is that they are also order dense in $C(X_i)$ for all $1 \le i \le n$. For if $0 < f \in C(X_i)$ and $x, y \in E_i$ are such that $\gamma := \|f - x\| < \frac{1}{4}\|f\|$ and $\|y - 1_{X_i}\| < \frac{1}{2}$, then for

$$z := (x - 2\gamma y)^+ \in E_i$$

it follows that $z > 0$, because otherwise we would have

$$f - 2\gamma = f - x + x - 2\gamma < \tfrac{1}{4}\|f\| + 2\gamma y - 2\gamma < \tfrac{1}{4}\|f\| + \gamma < \tfrac{1}{2}\|f\|,$$

so $f < \frac{1}{2}\|f\| + 2\gamma < \|f\|$ and this is impossible, since X_i is compact. Moreover, for $t \in X_i$ we also find that

$$\begin{aligned} x(t) - 2\gamma y(t) &= x(t) - f(t) - 2\gamma y(t) + f(t) \leq \gamma - 2\gamma y(t) + f(t) \\ &= 2\gamma - 2\gamma y(t) + f(t) - \gamma \\ &< \gamma + f(t) - \gamma \\ &= f(t), \end{aligned}$$

so $0 < z \leq f$; hence E_i is order dense in $C(X_i)$.

In the case of the algebraic tensor product n-linear functions are considered, but since we are dealing with Riesz spaces, these mappings need additional structure.

Definition: Let $E_1, \ldots, E_n$ and F be Archimedean Riesz spaces. An n-linear map

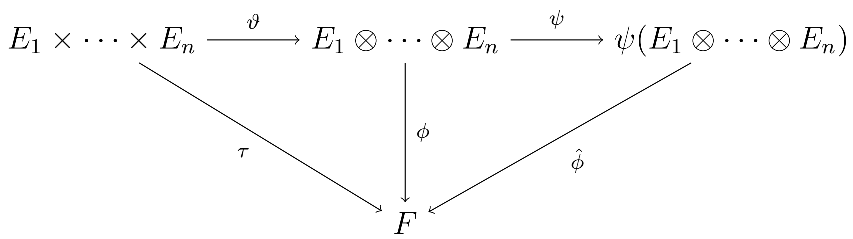
$$\varphi : E_1 \times \cdots \times E_n \to F$$

is said to be a **Riesz n-morphism** if $\varphi(|x_1|, \ldots, |x_n|) = |\varphi(x_1, \ldots, x_n)|$ for all $x_i \in X_i$ with $1 \leq i \leq n$.

Now, let G be the Riesz subspace of $C(X_1 \times \cdots \times X_n)$ which is generated by $E_1 \otimes \cdots \otimes E_n$. For the canonical n-linear map ϑ used for defining the algebraic tensor product, the composition $\psi \circ \vartheta$ satisfies

$$\psi \circ \vartheta(|x_1|, \ldots, |x_n|) = \psi(|x_1| \otimes \cdots \otimes |x_n|) = |\psi(x_1 \otimes \cdots \otimes x_n)| = |\psi \circ \vartheta(x_1, \ldots, x_n)|,$$

so it is a Riesz n-morphism and represents $E_1 \otimes \cdots \otimes E_n$ as a linear subspace of G. For if F is a linear space and $\tau : E_1 \times \cdots \times E_n \to F$ is an n-linear map and $\tilde{\phi} : \psi(E_1 \otimes \cdots \otimes E_n) \to F$ defines a linear map such that $\tau = \tilde{\phi} \circ (\psi \circ \vartheta)$, then, by the universal property of the algebraic tensor product, there is a unique linear map ϕ for which the diagram



commutes and we find that $\tilde{\phi} \circ \psi = \phi$, so $\tilde{\phi}$ uniquely depends on ϕ. Having this, denote by θ the canonical n-linear map for $E_1 \otimes \cdots \otimes E_n$ when viewed as a subspace of G. We will show that this Archimedean Riesz space G satisfies a universal property, but we need a lemma for the proof.

Lemma 3.18 *Let E be an order unital Archimedean Riesz space. Then the non-zero Riesz homomorphisms mapping E into $\mathbb{R}$ separate the points of E.*

Proof: By Theorem 3.17 we can view E as a uniformly dense subspace of $C(X)$ for some compact Hausdorff space X. Let $f_1, f_2 \in E$ be such that $f_1 \neq f_2$. Then there is an element $x \in X$ such that $f_1(x) \neq f_2(x)$ and define the map $\phi_x : C(X) \to \mathbb{R}$ by $\phi_x(g) := g(x)$. Clearly, this map is linear and it also satisfies the property

$$\phi_x(|g|) = |g|(x) = |g(x)| = |\phi_x(g)| \qquad (g \in C(X)),$$

so ϕ_x is a Riesz homomorphism. Since we have $\phi_x(f_1) = f_1(x) \neq f_2(x) = \phi_x(f_2)$, the desired result is obtained. ∎

Theorem 3.19 *Let $E_1, \ldots, E_n$ be order unital Archimedean Riesz spaces and H be an Archimedean Riesz space. Then for every Riesz n-morphism $\varphi : E_1 \times \cdots \times E_n \to H$ there exists a unique Riesz homomorphism $\phi : G \to H$ such that $\phi \circ \theta = \varphi$.*

Proof: By Theorem 3.17, we can view E_i as uniformly dense subspaces of $C(X_i)$ where X_i is a compact Hausdorff space for all $1 \leq i \leq n$. If $\varphi = 0$, then we can take $\phi = 0$ and this gives the desired result. On the other hand, if $\varphi \neq 0$, then we must have that $\varphi(1_{X_1}, \ldots, 1_{X_n}) \neq 0$ and define H_0 to be the ideal of H that is generated by $\varphi(1_{X_1}, \ldots, 1_{X_n})$, that is,

$$H_0 := \{x \in H : \exists \lambda \geq 0 \text{ such that } |x| \leq \lambda \varphi(1_{X_1}, \ldots, 1_{X_n})\}.$$

We find that $\varphi(1_{X_1}, \ldots, 1_{X_n})$ is an order unit in H_0. Let $x_i \in E_i$ and $\lambda_i \geq 0$ be such that $|x_i| \leq \lambda 1_{X_i}$ for $1 \leq i \leq n$. By writing $x_i = x_i^+ - x_i^-$ and using an inductive argument, it follows that we have the inequality

$$|\varphi(x_1, \ldots, x_n)| \leq \varphi(|x_1|, \ldots, |x_n|) \leq \left(\prod_{i=1}^{n} \lambda_i\right) \varphi(1_{X_1}, \ldots, 1_{X_n});$$

hence $\varphi(E_1 \times \cdots \times E_n) \subset H_0$. Let Ω be the set of all non-zero Riesz homomorphisms that map H_0 into $\mathbb{R}$. Then for $\eta \in \Omega$ we have that $\eta \circ \varphi : E_1 \times \cdots \times E_n \to \mathbb{R}$ is a non-zero and n-linear map, because $\eta \circ \varphi(1_{X_1}, \ldots, 1_{X_n}) = 0$ would imply that $\eta = 0$. Furthermore, the identity

$$\eta \circ \varphi(|x_1|, \ldots, |x_n|) = \eta(|\varphi(x_1, \ldots, x_n)|) = |\eta(\varphi(x_1, \ldots, x_n))| = |\eta \circ \varphi(x_1, \ldots, x_n)|$$

yields that $\eta \circ \varphi$ is a Riesz n-morphism. We also find that

$$|\eta \circ \varphi(x_1, \ldots, x_n)| = \eta \circ \varphi(|x_1|, \ldots, |x_n|) \leq \left(\prod_{i=1}^{n} \|x_i\|\right) \eta \circ \varphi(1_{X_1}, \ldots, 1_{X_n});$$

hence $\eta \circ \varphi$ is continuous. Fix $y_i \in E_i$ for $2 \leq i \leq n$ and let $f \in C(X_1)$. Then there is a sequence $(x_k)_{k \geq 1}$ in E_1 such that $\|f - x_k\| \to 0$ and we find that $(\eta \circ \varphi(x_k, y_2, \ldots, y_n))_{k \geq 1}$ is a Cauchy sequence in $\mathbb{R}$, so we can define a map

$$\xi_1 : C(X_1) \times E_2 \times \cdots \times E_n \to \mathbb{R}$$

by $\xi_1(f, x_2, \ldots, x_n) := \lim_{k \to \infty} \eta \circ \varphi(x_k, x_2, \ldots, x_n)$. It is a straightforward verification to show that ξ_1 is n-linear and continuous, since taking the norm of an element is continuous. Furthermore, since the inequality $||f| - |x_k|| \leq |f - x_k|$ implies that $\||f| - |x_k|\| \to 0$, it follows from the identity

$$\begin{aligned} \xi_1(|f|, |x_2|, \ldots, |x_n|) &= \lim_{k \to \infty} \eta \circ \varphi(|x_k|, |x_2|, \ldots, |x_n|) = \lim_{k \to \infty} |\eta \circ \varphi(x_k, x_2, \ldots, x_n)| \\ &= \left|\lim_{k \to \infty} \eta \circ \varphi(x_k, x_2, \ldots, x_n)\right| = |\xi_1(f, x_2, \ldots, x_n)| \end{aligned}$$

that ξ_1 is a Riesz n-morphism, so we find that ξ_1 is the unique extension of $\eta \circ \varphi$ on $C(X_1) \times E_2 \times \cdots \times E_n$ which is a Riesz n-morphism, since this extension must be continuous. Analogously, we can find a unique extension ξ_2 of ξ_1 on $C(X_1) \times C(X_2) \times E_3 \times \cdots \times E_n$ that is a Riesz n-morphism and, by repeating this process, there is a unique extension ξ_n of $\eta \circ \varphi$ on $C(X_1) \times \cdots \times C(X_n)$ which is a Riesz n-morphism. Consider the map $\zeta_1 : C(X) \to \mathbb{R}$

defined by $\zeta_1(f) := \xi_n(f, 1_{X_2}, \ldots, 1_{X_n})$. This map clearly is a Riesz homomorphism, so by Lemma 3.16 there are uniquely defined $\alpha > 0$ and $x_1 \in X$ such that $\zeta_1(f) = \alpha f(x_1)$ for all $f \in C(X_1)$. Note that $\alpha = \xi_n(1_{X_1}, \ldots, 1_{X_n})$. Similarly, for $\zeta_2 : C(X_1) \times C(X_2) \to \mathbb{R}$ with $\zeta_2(f, g) := \xi_n(f, g, 1_{X_3}, \ldots, 1_{X_n})$ we have a unique $x_2 \in X_2$ such that $\zeta_2(1_{X_1}, f) = \alpha f(x_2)$ for all $f \in C(X_2)$ and it follows that

$$\begin{aligned}|\zeta_2(f_1, f_2) - \alpha f_1(x_1) f_2(x_2)| &= |\zeta_2(f_1, f_2) - \zeta_2(f_1, f_2(x_2)1_{X_2})| = |\zeta_2(f_1, f_2 - f_2(x_2)1_{X_2})| \\ &= \zeta_2(|f_1|, |f_2 - f_2(x_2)1_{X_2}|) \leq \|f\| \zeta_2(1_{X_1}, |f_2 - f_2(x_2)1_{X_2}|) \\ &= \alpha \|f\| |f_2 - f_2(x_2)1_{X_2}|(x_2) = 0,\end{aligned}$$

so $\zeta_2(f_1, f_2) = \alpha f_1(x_1) f_2(x_2)$ for all $f_1 \in C(X_1)$ and $f_2 \in C(X_2)$. Suppose that for $2 \leq k < n$ we have unique $x_k \in X_k$ such that $\zeta_k(f_1, \ldots, f_k) = \alpha f_1(x_1) \cdots f_k(x_k)$ and consider ζ_{k+1}. Since the map $f_{k+1} \mapsto \zeta_{k+1}(1_{X_1}, \ldots, 1_{X_k}, f_{k+1})$ is a Riesz homomorphism mapping $C(X_{k+1})$ into $\mathbb{R}$, again Lemma 3.16 lets us pick an $x_{k+1} \in X_{k+1}$ such that $\zeta_{k+1}(1_{X_1}, \ldots, 1_{X_k}, f_{k+1}) = \alpha f_{k+1}(x_{k+1})$ for all $f_{k+1} \in C(X_{k+1})$. For $M := \prod_{i=1}^{k} \|f_i\|$, we now obtain the inequality

$$\begin{aligned}\left|\zeta_{k+1}(f_1, \ldots, f_{k+1}) - \alpha \prod_{i=1}^{k+1} f_i(x_i)\right| &= \left|\zeta_{k+1}(f_1, \ldots, f_{k+1}) - \zeta_{k+1}(f_1, \ldots, f_k, f_{k+1}(x_{k+1})1_{X_{k+1}})\right| \\ &= |\zeta_{k+1}(f_1, \ldots, f_k, f_{k+1} - f_{k+1}(x_{k+1})1_{X_{k+1}})| \\ &\leq M \zeta_{k+1}(1_{X_1}, \ldots, 1_{X_k}, |f_{k+1} - f_{k+1}(x_{k+1})1_{X_{k+1}}|) \\ &= M |f_{k+1} - f_{k+1}(x_{k+1})1_{X_{k+1}}|(x_{k+1}) = 0,\end{aligned}$$

which inductively proves that $\xi_n(f_1, \ldots, f_n) = \zeta_n(f_1, \ldots, f_n) = \alpha f_1(x_1) \cdots f_n(x_n)$ for all $f_1 \in C(X_1), \ldots, f_n \in C(X_n)$. This now allows us to define the set

$$G_0 := \{f \in C(X_1 \times \cdots \times X_n) : \exists!\ z \in H_0,\ \eta(z) = \alpha_\eta f(x_1^\eta, \ldots, x_n^\eta) \text{ for all } \eta \in \Omega\}.$$

We claim that Ω is not empty. Clearly, for all $\varphi(f_1, \ldots, f_n) \in \varphi(E_1 \times \ldots \times E_n)$ we have the identity $\eta \circ \varphi(f_1, \ldots, f_n) = \alpha_\eta f_1(x_1^\eta) \cdots f_n(x_n^\eta)$ for all $\eta \in \Omega$ as was shown above, and since Ω separates the points of H_0 by Lemma 3.18, it follows that $\varphi(f_1, \ldots, f_n)$ is the unique element of H_0 with this property. Let $f_1, f_2 \in G_0$ and $\eta \in \Omega$. For the unique elements $z_1, z_2 \in H_0$ with $\eta(z_1) = \alpha_\eta f_1(x_1^\eta, \ldots, x_n^\eta)$ and $\eta(z_2) = \alpha_\eta f_2(x_1^\eta, \ldots, x_n^\eta)$ we now have that

$$\eta(z_1 + z_2) = \alpha_\eta (f_1 + f_2)(x_1^\eta, \ldots, x_n^\eta)$$

and since η was arbitrary, it follows that $z_1 + z_2 \in H_0$ is the unique element with this property by Lemma 3.18, so $f_1 + f_2 \in G_0$. Similarly, one shows that $\lambda f_1 \in G_0$ for all $\lambda \in \mathbb{R}$, so G_0 is a linear subspace of $C(X_1 \times \cdots \times X_n)$. Also, because η is a Riesz homomorphism, we find that

$$\eta(|z_1|) = |\eta(z_1)| = \alpha_\eta |f_1(x_1^\eta, \ldots, x_n^\eta)| = \alpha_\eta |f_1|(x_1^\eta, \ldots, x_n^\eta)$$

and analogously, we conclude that $|f_1| \in G_0$; hence G_0 is a Riesz subspace. Moreover, the identity

$$\eta \circ \varphi(f_1, \ldots, f_n) = \alpha_\eta f_1(x_1^\eta) \cdots f_n(x_n^\eta) = \alpha_\eta \theta(f_1, \ldots, f_n)(x_1^\eta, \ldots, x_n^\eta)$$

for all $\eta \in \Omega$ and $f_1 \in E_1, \dots, f_n \in E_n$ implies that we have the inclusion $G \subset G_0$ and we can define the map $\phi : G \to H_0$ by letting $\phi(f)$ be the unique element such that

$$\eta(\phi(f)) = \alpha_\eta f(x_1^\eta, \dots, x_n^\eta) \qquad (\eta \in \Omega).$$

Again, by using Lemma 3.18, we can show that ϕ is a Riesz homomorphism in a similar way and also, since

$$\eta(\phi(\theta(f_1, \dots, f_n))) = \alpha_\eta \theta(f_1, \dots, f_n)(x_1^\eta, \dots, x_n^\eta) = \alpha_\eta f_1(x_1^\eta) \cdots f_n(x_n^\eta) = \eta(\varphi(f_1, \dots, f_n))$$

for all $\eta \in \Omega$, the identity $\phi \circ \theta = \varphi$ is satisfied. Finally, if $\phi' : G \to H$ is a Riesz homomorphism such that $\phi' \circ \theta = \varphi$, then ϕ' and ϕ coincide on $E_1 \otimes \cdots \otimes E_n$ and if $f \in G$, then we can write $f = \bigvee_{i=1}^k f_i - \bigvee_{j=1}^m g_j$ for some $f_i, g_j \in E_1 \otimes \cdots \otimes E_n$ for all $1 \le i \le k$ and $1 \le j \le m$. So, since ϕ' and ϕ are Riesz homomorphisms, it follows that

$$\phi'(f) = \bigvee_{i=1}^k \phi'(f_i) - \bigvee_{j=1}^m \phi'(g_j) = \bigvee_{i=1}^k \phi(f_i) - \bigvee_{j=1}^m \phi(g_j) = \phi(f);$$

hence $\phi' = \phi$ and we conclude that ϕ is the unique Riesz homomorphism that satisfies the identity $\phi \circ \theta = \varphi$. ∎

Now that we have established this elegant universal property for G, the next thing we wish to investigate, in order to acquire a more graphic understanding of the situation, is what order structure related properties the embedding of $E_1 \otimes \cdots \otimes E_n$ in G has. The following lemma states which prove to be useful:

Lemma 3.20 *If $f \in G$, then there exist $x_i \in E_i$ for all $1 \le i \le n$ such that for every $\delta > 0$ there is a $g \in E_1 \otimes \cdots \otimes E_n$ such that $|f - g| \le \delta\theta(x_1, \dots, x_n)$. Moreover, if $f > 0$, then there are $x_i \in E_i^+$ for all $1 \le i \le n$ such that the inequalities $0 < \theta(x_1, \dots, x_n) \le f$ are satisfied.*

Proof: Let $f \in G$ and suppose that there is a $\delta > 0$ such that

$$|f - g| \not\le \delta\theta(1_{X_1}, \dots, 1_{X_n}) = \delta 1_{X_1 \times \cdots \times X_n}$$

for all $g \in E_1 \otimes \cdots \otimes E_n$. Then there are $x_i \in X_i$ for all $1 \le i \le n$ for which we have that $|f - g|(x_1, \dots, x_n) > \delta$ for all $g \in E_1 \otimes \cdots \otimes E_n$, so $\|f - g\| > \delta$ for all $g \in E_1 \otimes \cdots \otimes E_n$ and this is impossible, since we have shown that $E_1 \otimes \cdots \otimes E_n$ is uniformly dense in $C(X_1 \times \cdots \times X_n)$; hence this proves the first statement. As for the case where $f > 0$, we can pick $x_i \in X_i$ for all $1 \le i \le n$ such that $f(x_1, \dots, x_n) > 0$ and chose $\sigma > 0$ such that $\sigma f(x_1, \dots, x_n) > 1$. By the continuity of f, there exists an open neighborhood U of $(x_1, \dots, x_n)$ in $X_1 \times \cdots \times X_n$ such that $\sigma f(U) > 1$. For the projections $\pi_{X_i}(U) \subset X_i$ with $x_i \in \pi_{X_i}(U)$ where $1 \le i \le n$ there exist functions $f_i \in C(X_i)$ with $0 \le f_i \le 1$ such that

$$f_i(\xi) := \begin{cases} 1 & \text{if } \xi = x_i \\ 0 & \text{if } \xi \notin \pi_{X_i}(U) \end{cases}$$

for all $1 \le i \le n$ by Urysohn's lemma. It follows that

$$0 \le f_1(y_1) \cdots f_n(y_n) \le 1 < \sigma f(y_1, \dots, y_n)$$

for all $(y_1, \dots, y_n) \in U$ and $0 = f_1(y_1) \cdots f_n(y_n) \le \sigma f(y_1, \dots, y_n)$ for all $(y_1, \dots, y_n) \notin U$; hence $\sigma^{-1}\theta(f_1, \dots, f_n) \le f$. We have shown above that E_i is order dense in $C(X_i)$ for all $1 \le i \le n$, so there are $0 < x_i \le f_i$ in E_i for all $1 \le i \le n$ and this implies that $0 < \sigma^{-1}\theta(x_1, \dots, x_n) \le \sigma^{-1}\theta(f_1, \dots, f_n) \le f$, which proves the second assertion. ∎

When $E_1, \dots, E_n$ are arbitrary Archimedean Riesz spaces, the situation is more elaborate. It is still possible however, to represent them as a quotient of a Riesz space constructed by specific types of functions. For a set X, let $\mathbb{R}^X$ denote the set consisting of functions $f : X \to \mathbb{R}$. Under pointwise addition, scalar multiplication and ordering, we find that this set $\mathbb{R}^X$ is a Riesz space. Moreover, if $f, g \in \mathbb{R}^X$ are such that $nf \leq g$ for all $n \geq 1$, then $nf(x) \leq g(x)$ for all $n \geq 1$ and as $\mathbb{R}$ is Archimedean, it follows that $f(x) \leq 0$ for all $x \in X$; hence $f \leq 0$, so $\mathbb{R}^X$ is Archimedean. A non-empty subset I of the powerset $\mathcal{P}(X)$ of X is said to be an **ideal** of X if for $A, B \in I$ we have that $A \cup B \in I$ and if $C \subset A$, then $C \in I$. Now, for an ideal I of X let N_I be the subset of functions $f \in \mathbb{R}^X$ with the property that

$$\{x \in X : f(x) \neq 0\} \in I.$$

Clearly, this subset N_I is not empty, since for $A \in I$ we can define the function $f_A : X \to \mathbb{R}$ by $f_A(x) = 1$ for all $x \in A$ and $f_A(x) = 0$ for all $x \in X \setminus A$; hence $f_A \in N_I$. Let $f, g \in N_I$. Then the inclusion

$$\{x \in X : f(x) = 0\} \cap \{x \in X : g(x) = 0\} \subset \{x \in X : f(x) + g(x) = 0\}$$

implies that $f + g \in N_I$ and since $\emptyset \in I$, the zero function is a member of N_I, and for $\alpha \in \mathbb{R} \setminus \{0\}$ we have that

$$\{x \in X : \alpha f(x) \neq 0\} = \{x \in X : f(x) \neq 0\},$$

so $\alpha f \in N_I$ for all $\alpha \in \mathbb{R}$. Furthermore, if $f \in N_I$ and $g \in \mathbb{R}^X$ is such that $|g| \leq |f|$, then

$$\{x \in X : g(x) \neq 0\} \subset \{x \in X : f(x) \neq 0\};$$

hence $g \in N_I$, and we conclude that N_I is an ideal in $\mathbb{R}^X$. This allows us to consider the quotient Riesz space $\mathbb{R}^X/N_I$ and if I is a σ-ideal of X, that is, it is closed under countable unions, then we can derive a useful property:

Lemma 3.21 *If I is a σ-ideal of X, then the quotient Riesz space $\mathbb{R}^X/N_I$ is Archimedean.*

Proof: Let $\overline{f}, \overline{g} \in \mathbb{R}^X/N_I$ be such that $n\overline{f} \leq \overline{g}$ for all $n \geq 1$. Without loss of generality, we may assume that $g \geq 0$. Then for every $n \geq 1$ there is an element $f_n \in \overline{f}$ such that $f_n \leq \frac{1}{n}g$. Since we have that $f_1 \wedge f_n \in \overline{f}$ for all $n \geq 1$, we may reconfigure f_n to be such that $f_n \leq f_1$ for all $n \geq 1$ and $f_1 \geq 0$ and still have that $f_n \leq \frac{1}{n}g$ for all $n \geq 1$. This implies that $0 \leq f_1 - \frac{1}{n}g \leq f_1 - f_n \in N_I$, so $f_1 - \frac{1}{n}g \in N_I$ for all $n \geq 1$. Now if $x \in X$ is such that $f_1(x) > 0$, then there exists an $m \geq 1$ such that $f_1(x) > \frac{1}{m}g(x) \geq 0$, so $f_1(x) - \frac{1}{m}g(x) > 0$ and if $x \in X$ is such that $f_1(x) - \frac{1}{m}g(x) > 0$, then we definitely have that $f_1(x) > 0$, so as I is a σ-ideal in X, we find that

$$\{x \in X : f_1(x) > 0\} = \bigcup_{n=1}^{\infty} \{x \in X : f_1(x) - \tfrac{1}{n}g(x) > 0\} \in I;$$

hence $f_1 \in N_I$ and we conclude that $g \in N_I$. But this implies that $\overline{f} \leq 0$, so $\mathbb{R}^X/N_I$ is Archimedean. ∎

The representation of an Archimedean Riesz space is stated in the following theorem and for the proof we refer to [3, Thm. 3].

Theorem 3.22 *Let E be an Archimedean Riesz space. Then there exists a set X and a σ-ideal I in X such that E is Riesz isomorphic to a Riesz subspace of $\mathbb{R}^X/N_I$.* ∎

A key tool in constructing a proof for the general case depends on Theorem 3.22 which, just as in the order unital case, allows us to view the algebraic tensor product $E_1 \otimes \cdots \otimes E_n$ as a subspace of an Archimedean Riesz space.

Lemma 3.23 *Let $E_1, \ldots, E_n$ be Archimedean Riesz spaces. Then there is an Archimedean Riesz space H and a Riesz n-morphism $\varphi : E_1 \times \cdots \times E_n \to H$ such that the induced linear map $\hat{\varphi} : E_1 \otimes \cdots \otimes E_n \to H$ is injective.*

Proof: By Theorem 3.22 we have that E_i is Riesz isomorphic to a Riesz subspace of $\mathbb{R}^{X_i}/N_{I_i}$ where X_i is a set and I_i is a σ-ideal in X_i for all $1 \leq i \leq n$. Now define

$$J := \left\{ A : \exists A_i \in I_i \text{ such that } A \subset \bigcup_{i=1}^{n} X_1 \times \cdots \times X_{i-1} \times A_i \times X_{i+1} \times \cdots \times X_n \right\}.$$

It is a straightforward verification to show that J is a σ-ideal in $X_1 \times \cdots \times X_n$, so it follows from Lemma 3.21 that

$$H := \mathbb{R}^{X_1 \times \cdots \times X_n}/N_J$$

is an Archimedean Riesz space. Analogous to the order unital case, the multiplication map $\psi : \mathbb{R}^{X_1} \times \cdots \times \mathbb{R}^{X_n} \to \mathbb{R}^{X_1 \times \cdots \times X_n}$ defined by $\psi(f_1, \ldots, f_n) := f_1 \cdots f_n$ with

$$\psi(f_1, \ldots, f_n)(x_1, \ldots, x_n) := f_1(x_1) \cdots f_n(x_n) \qquad (x_i \in X_i,\ 1 \leq i \leq n)$$

is a Riesz n-morphism. By definition of J it follows that we have a well defined n-linear map

$$\varphi : \mathbb{R}^{X_1}/N_{I_1} \times \cdots \times \mathbb{R}^{X_n}/N_{I_n} \to H.$$

Furthermore, by definition of de lattice order on the quotients $\mathbb{R}^{X_i}$ for all $1 \leq i \leq n$ and on H, we have that

$$\begin{aligned}\varphi(|\overline{f_1}|, \ldots, |\overline{f_n}|) &= \varphi(\overline{|f_1|}, \ldots, \overline{|f_n|}) = \overline{|f_1| \cdots |f_n|} = \overline{|f_1 \cdots f_n|} = |\overline{f_1 \cdots f_n}| \\ &= |\varphi(\overline{f_1}, \ldots, \overline{f_n})|,\end{aligned}$$

so φ is a Riesz n-morphism. Now let $\hat{\varphi} : \mathbb{R}^{X_1}/N_{I_1} \otimes \cdots \otimes \mathbb{R}^{X_n}/N_{I_n} \to H$ be the induced linear map and suppose that $\mathbf{f} \in \mathbb{R}^{X_1}/N_{I_1} \otimes \cdots \otimes \mathbb{R}^{X_n}/N_{I_n}$ is such that $\mathbf{f} \neq 0$. We may write

$$\mathbf{f} = \sum_{k=1}^{m} \overline{f_1}^{(k)} \otimes \cdots \otimes \overline{f_n}^{(k)}$$

where the set $(\overline{f_n}^{(k)})_{k=1}^m$ is linearly independent and $f_i^{(1)} \notin N_{I_i}$ for $1 \leq i \leq n-1$. Let $A \in J$. Then there are $A_i \in I_i$ for all $1 \leq i \leq n$ such that

$$A \subset \bigcup_{i=1}^{n} X_1 \times \cdots \times X_{i-1} \times A_i \times X_{i+1} \times \cdots \times X_n.$$

There are elements $x_i \in X_i$ such that $f_i^{(1)}(x_i) \neq 0$ with $\{x_i\} \notin I_i$ for all $1 \leq i \leq n-1$ and it follows that $x_i \notin A_i$ for all $1 \leq i \leq n-1$. Now we must have that

$$\sum_{k=1}^{m} f_1^{(k)}(x_1) \cdots f_{n-1}^{(k)}(x_{n-1}) \overline{f_n}^{(k)} \neq 0,$$

so there is an element $x_n \notin A_n$ such that such that

$$\left(\sum_{k=1}^{m} f_1^{(k)} \cdots f_n^{(k)}\right)(x_1, \ldots, x_n) = \sum_{k=1}^{m} f_1^{(k)}(x_1) \cdots f_n^{(k)}(x_n) \neq 0.$$

As $(x_1, \ldots, x_n) \notin A$ and $A \in J$ was arbitrary, it follows that

$$\left\{(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n : \left(\sum_{k=1}^{m} f_1^{(k)} \cdots f_n^{(k)}\right)(x_1, \ldots, x_n) \neq 0\right\} \notin J;$$

hence

$$\sum_{k=1}^{m} f_1^{(k)} \cdots f_n^{(k)} \notin N_J$$

and we conclude that $\hat{\varphi}(\mathbf{f}) \neq 0$, so $\hat{\varphi}$ is injective and the restriction of $\hat{\varphi}$ to $E_1 \otimes \cdots \otimes E_n$ together with H now satisfy the desired properties. ∎

With these results, we are now ready to attack the general case.

Theorem 3.24 *Let $E_1, \ldots, E_n$ be Archimedean Riesz spaces. Then, up to a Riesz isomorphism, there exists a unique Archimedean Riesz space G and a Riesz n-morphism $\phi : E_1 \times \cdots \times E_n \to G$ such that*

- *i) whenever H is an Archimedean Riesz space and $\psi : E_1 \times \cdots \times E_n \to H$ is a Riesz n-morphism, there is a unique Riesz homomorphism $\xi : G \to H$ such that $\xi \circ \phi = \psi$;*
- *ii) ϕ induces an embedding of $E_1 \otimes \cdots \otimes E_n$ in G;*
- *iii) $E_1 \otimes \cdots \otimes E_n$ is dense in G in the sense that for every $w \in G$ there exist $x_i \in E_i$ for all $1 \leq i \leq n$ such that for every $\delta > 0$ we have a $v \in E_1 \otimes \cdots \otimes E_n$ that satisfies the inequality $|w - v| \leq \delta\phi(x_1, \ldots, x_n)$;*
- *iv) if $w > 0$ in G, then there exist $x_i \in E_i^+$ for all $1 \leq i \leq n$ such that we have the inequality $0 < \phi(x_1, \ldots, x_n) \leq w$.*

Proof *i)***:** Let Λ be the set of all Archimedean Riesz spaces that have as underlying point set a subset of the Cartesian product $E_1 \times \cdots \times E_n$. Clearly, this set is not empty, since it contains $\{0\} \times \cdots \times \{0\}$ and $E_1 \times \cdots \times E_n$ itself. Let F be an Archimedean Riesz space with $\operatorname{card}(F) \leq \operatorname{card}(E_1 \times \cdots \times E_n)$. Then there is an injection $J : F \hookrightarrow E_1 \times \cdots \times E_n$. Since F has a basis $(e_\alpha)_\alpha$, the injection J induces a linear map by defining

$$J\left(\sum_{k=1}^{m} \lambda_k e_k\right) := \sum_{k=1}^{m} \lambda_k J(e_k).$$

We can also construct a vector space order on $J(F)$ by putting

$$J(x) \leq J(y) :\Longleftrightarrow x \leq y.$$

Now for $x, y \in F$ we have that $J(x) \leq J(x \vee y)$ and $J(y) \leq J(x \vee y)$, so if $J(x) \leq J(x)$ and $J(y) \leq J(z)$, then $x \vee y \leq z$ and therefore we must have $J(x \vee y) \leq J(z)$; hence $J(x \vee y)$ is the supremum of $J(x)$ and $J(y)$ and we conclude J is a Riesz homomorphism.

It follows that F is Riesz isomorphic to some member of Λ. Let I be set of all pairs (F,φ) where $F \in \Lambda$ and $\varphi : E_1 \times \cdots \times E_n \to F$ is a Riesz n-morphism. Again, this set is nt empty, because $(\{0\} \times \cdots \times \{0\}, \mathbf{0}), (E_1 \times \cdots \times E_n, \mathrm{id}) \in I$. Now, for each $i \in I$ define F_i and φ_i by setting $i := (F_i, \varphi_i)$. Let G' be the product space

$$\prod_{i \in I} F_i$$

and consider the map $\phi : E_1 \times \cdots \times E_n \to G'$ defined by

$$\phi(x_1, \ldots, x_n) := (\varphi_i(x_1, \ldots, x_n))_{i \in I},$$

which is clearly a Riesz n-morphism. Let G be the Riesz subspace of G' that is generated by $\phi(E_1 \times \cdots \times E_n)$. Suppose that H is an Archimedean Riesz space and $\psi : E_1 \times \cdots \times E_n \to H$ is a Riesz n-morphism. If we define H_0 to be the Riesz subspace of H that is generated by $\psi(E_1 \times \cdots \times E_n)$, then we claim that $\mathrm{card}(H_0) \leq \mathrm{card}(E_1 \times \cdots \times E_n)$. For if $E_i = \{0\}$ for all $1 \leq i \leq n$, then $\psi(E_1 \times \cdots \times E_n) = \{0\}$ so $H_0 = \{0\}$ and the desired inequality holds. On the other hand, since

$$\psi : E_1 \times \cdots \times E_n \to \psi(E_1 \times \cdots \times E_n)$$

is a surjection, we must have that $\mathrm{card}(\psi(E_1 \times \cdots \times E_n)) \leq \mathrm{card}(E_1 \times \cdots \times E_n)$ and $\mathrm{card}(E_1 \times \cdots \times E_n) \geq 2^{\aleph_0}$. By adding the linear structure, scalar multiplication and lattice operations to $\psi(E_1 \times \cdots \times E_n)$ in order to obtain H_0, its cardinality does not exceed

$$\begin{aligned}
\mathrm{card}(H_0) &\leq \aleph_0 \cdot \aleph_0 \cdot 2^{\aleph_0} \cdot 2^{\aleph_0} \cdot 2^{\aleph_0} \cdot \mathrm{card}(\psi(E_1 \times \cdots \times E_n)) \\
&\leq \aleph_0 \cdot \aleph_0 \cdot 2^{\aleph_0} \cdot 2^{\aleph_0} \cdot 2^{\aleph_0} \cdot \mathrm{card}(E_1 \times \cdots \times E_n) \\
&= 2^{\aleph_0} \cdot \mathrm{card}(E_1 \times \cdots \times E_n) \\
&= \mathrm{card}(E_1 \times \cdots \times E_n).
\end{aligned}$$

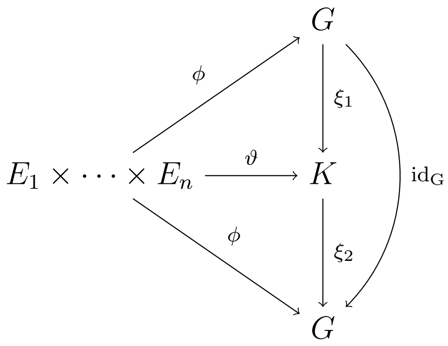
So, there exists an $F \in \Lambda$ which is isomorphic to H_0. Let $\zeta : H_0 \to F$ be a Riesz isomorphism and consider the map $\zeta \circ \psi : E_1 \times \cdots \times E_n \to F$. This composition defines a Riesz n-morphism and therefore, we have $\kappa = (F, \zeta \circ \psi) \in I$. We can now define the Riesz homomorphism $\xi : G \to H$ by $\xi((f_i)_{i \in I}) := \zeta^{-1}(f_\kappa)$. Moreover, it follows that

$$\begin{aligned}
\xi \circ \phi(x_1, \ldots, x_n) &= \xi((\phi_i(x_1, \ldots, x_n))) = \zeta^{-1}(\phi_\kappa(x_1, \ldots, x_n)) = \zeta^{-1}(\zeta(\psi(x_1, \ldots, x_n))) \\
&= \psi(x_1, \ldots, x_n);
\end{aligned}$$

hence $\xi \circ \phi = \psi$. As for the uniqueness of ξ, if $\xi' : G \to H$ is a Riesz homomorphism that satisfies the identity $\xi' \circ \phi = \psi$, then the set

$$G_0 := \{w \in G : \xi(w) = \xi'(w)\} \subset G$$

clearly is a Riesz subspace of G that contains $\phi(E_1 \times \cdots \times E_n)$, so $G \subset G_0$ which implies that $G_0 = G$. Analogous to showing that the algebraic tensor product is unique up to a linear isomorphism, we find that an Archimedean Riesz space K together with a Riesz n-morphism $\vartheta : E_1 \times \cdots \times E_n \to K$ that satisfies property $i)$ yields the commutative diagram



from which it follows that K must be Riesz isomorphic to G. ∎

Proof *ii)***:** By Lemma 3.23 there exists an Archimedean Riesz space F together with a Riesz n-morphism $\varphi : E_1 \times \cdots \times E_n \to F$ such that the induced linear map

$$\hat{\varphi} : E_1 \otimes \cdots \otimes E_n \to F$$

is injective. It follows from part $i)$ that there also is a Riesz homomorphism $\xi : G \to F$ such that $\xi \circ \phi = \varphi$. So for the canonical n-linear map ϑ defining the algebraic tensor product and the corresponding linear map $\tilde{\varphi} : E_1 \otimes \cdots \otimes E_n \to G$ such that $\tilde{\varphi} \circ \vartheta = \phi$ we now find that

$$\xi \circ \tilde{\varphi}(\mathbf{x}) = \xi \circ \tilde{\varphi}\left(\sum_{k=1}^{m} \vartheta(x_1^{(k)}, \ldots, x_n^{(k)})\right) = \xi\left(\sum_{k=1}^{m} \phi(x_1^{(k)}, \ldots, x_n^{(k)})\right) = \sum_{k=1}^{m} \varphi(x_1^{(k)}, \ldots, x_n^{(k)})$$
$$= \hat{\varphi}\left(\sum_{k=1}^{m} \vartheta(x_1^{(k)}, \ldots, x_n^{(k)})\right) = \hat{\varphi}(\mathbf{x});$$

Which implies that $\xi \circ \tilde{\varphi}$ is injective; hence $\tilde{\varphi}$ must be injective too and we conclude that $\tilde{\varphi}$ embeds $E_1 \otimes \cdots \otimes E_n$ in G. ∎

Proof *iii)***:** Suppose that $w \in G$. Since $w = \bigvee_{j=1}^{k} a_j - \bigvee_{j=1}^{m} b_j$ where

$$a_j, b_j \in \mathrm{Sp}\{\phi(E_1 \times \cdots \times E_n)\},$$

there must be finite sets $F_i \subset E_i$ for $1 \leq i \leq n$ such that w is an element of the Riesz subspace of G which is generated by $\phi(F_1 \times \cdots \times F_n)$. Let E_i' be the Riesz subspace of E_i that is generated by F_i for all $1 \leq i \leq n$. Define

$$e_i := \sum_{x \in F_i} |x| \qquad (1 \leq i \leq n)$$

and pick $y \in \mathrm{Sp}\{F_i\}$. Then there clearly exists a number $\lambda_y \geq 0$ such that $|y| \leq \lambda_y e_i$, so if we now chose

$$y = \bigvee_{j=1}^{k} a_j - \bigvee_{j=1}^{m} b_j \in E_i',$$

then the inequalities

$$|y| \leq \left|\bigvee_{j=1}^{k} a_j\right| + \left|\bigvee_{j=1}^{m} b_j\right| \leq \bigvee_{j=1}^{k} \lambda_{a_j} e_i + \bigvee_{j=1}^{m} \lambda_{b_j} e_i \leq \left(\max_{1 \leq j \leq k} \lambda_{a_j} + \max_{1 \leq j \leq m} \lambda_{b_j}\right) e_i$$

imply that E_i' have order units for all $1 \leq i \leq n$. Let G_0 together with the Riesz n-morphism $\theta : E_1' \times \cdots \times E_n' \to G_0$ now correspond to what was mentioned above. Since $\phi : E_1' \times \cdots \times E_n' \to G$ also is a Riesz n-morphism, it follows from Theorem 3.19 that there exists a Riesz homomorphism $\xi' : G_0 \to G$ such that $\xi' \circ \theta = \phi$. As $\xi'(G_0)$ is a Riesz subspace of G that contains $\phi(E_1' \times \cdots \times E_n')$ and therefore, also includes $\phi(F_1 \times \cdots \times F_n)$, we must have that $w \in \xi'(G_0)$, so let $v \in G_0$ be such that $w = \xi'(v)$. We know from Lemma 3.20 that for every $\delta > 0$ there is a v_0 in the linear span of $\theta(E_1' \times \cdots \times E_n')$ such that $|w - v_0| \leq \delta\theta(e_1, \ldots, e_n)$, so $\xi'(v_0)$ lies in the linear span of $\phi(E_1 \times \cdots \times E_n)$ and we obtain the inequality

$$\xi'(|v - v_0|) = |\xi'(v) - \xi'(v_0)| = |w - \xi'(v_0)| \leq \delta\phi(e_1, \ldots, e_n).$$

Since δ was arbitrarily chosen, this proves the assertion. ∎

Proof *iv)***:** Suppose that $0 < w \in G$. Then, by part *iii)*, it follows that $\xi'(v) = w \neq 0$, so $v \neq 0$; hence $|v| > 0$ and we have the identity

$$\xi'(|v|) = |\xi'(v)| = |w| = w.$$

By Lemma 3.20 there exist $0 < x_i \in E_i'$ for all $1 \leq i \leq n$ such that $0 < \theta(x_1, \ldots, x_n) \leq |v|$ and we therefore have the inequality $\phi(x_1, \ldots, x_n) \leq w$. Part *ii)* finally implies that $\phi(x_1, \ldots, x_n) \neq 0$ and we conclude that $0 < \phi(x_1, \ldots, x_n) \leq w$. ∎

The Archimedean Riesz space G constructed in Theorem 3.24 corresponding to the Archimedean Riesz spaces $E_1, \ldots, E_n$ shall be denoted by $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ and is referred to as the **Fremlin tensor product** of $E_1, \ldots, E_n$. Via its canonical Riesz n-morphism ϕ we can view $E_1 \otimes \cdots \otimes E_n$ as a linear subspace of $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ and will consider the elementary tensors $x_1 \otimes \cdots \otimes x_n$ as if they are of the form $\phi(x_1, \ldots, x_n)$.

If we now choose $E_1, \ldots, E_n$ to be Banach lattices, then in particular, they are Archimedean Riesz spaces, so we can use the previous results in order to construct a Banach lattice from the algebraic tensor product $E_1 \otimes \cdots \otimes E_n$. First, we consider the map $\varrho : E_1 \bar{\otimes} \cdots \bar{\otimes} E_n \to \mathbb{R}$ defined by

$$\varrho(\mathbf{x}) := \inf \left\{ \sum_{k=1}^{m} \|x_1^{(k)}\| \cdots \|x_n^{(k)}\| : x_i^{(k)} \in E_i^+,\ 1 \leq k \leq m,\ |\mathbf{x}| \leq \sum_{k=1}^{m} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)} \right\}.$$

Let $\mathbf{x} \in E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$. By Theorem 3.24 there exist elements $x_i^{(0)} \in E_i^+$ for all $1 \leq i \leq n$ such that

$$\left| \mathbf{x} - \sum_{k=1}^{m} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)} \right| \leq x_1^{(0)} \otimes \cdots \otimes x_n^{(0)}$$

for some $x_i^{(k)} \in E_i$ for all $1 \leq k \leq n$ and this implies that we have the inequality

$$|\mathbf{x}| \leq \sum_{k=0}^{m} |x_1^{(k)}| \otimes \cdots \otimes |x_n^{(k)}|,$$

so ϱ is well defined. Clearly, this map is a Riesz seminorm and in fact, even more is true:

Lemma 3.25 *Let $E_1, \ldots, E_n$ be Banach lattices. Then ϱ defines a norm on $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ such that*

$$\varrho(x_1 \otimes \cdots \otimes x_n) = \|x_1\| \cdots \|x_n\| \qquad (x_i \in E_i,\ 1 \leq i \leq n).$$

Proof: It follows immediately that we have the inequality

$$\varrho(x_1 \otimes \cdots \otimes x_n) \leq |||x_1||| \cdots |||x_n||| = \|x_1\| \cdots \|x_n\| \qquad (x_i \in E_i,\ 1 \leq i \leq n).$$

Conversely, suppose $y_i^{(k)} \in E_i^+$ for all $1 \leq i \leq n$ and $1 \leq k \leq m$ satisfy the inequality

$$|x_1| \otimes \cdots \otimes |x_n| \leq \sum_{k=1}^{m} y_1^{(k)} \otimes \cdots \otimes y_n^{(k)}.$$

Then, by the Hahn-Banach theorem, there are functionals $f_i \in E_i^*$ such that $\|f_i\| = 1$ and $f_i(|x_i|) = \|x_i\|$ for all $1 \leq i \leq n$. Since E_i^* is a Banach lattice for all $1 \leq i \leq n$, the inequalities

$$\|x_i\| = |f_i(|x_i|)| \leq |f_i|(|x_i|) \leq |||f_i|||\,|||x_i||| = \|f_i\|\|x_i\| = \|x_i\|$$

imply that $|f_i|(|x_i|) = \|x_i\|$ for all $1 \leq i \leq n$. We now find that

$$\begin{aligned}\|x_1\| \cdots \|x_n\| = |f_1| \otimes \cdots \otimes |f_n|(|x_1| \otimes \cdots \otimes |x_n|) &\leq \sum_{k=1}^{m} |f_1|(y_1^{(k)}) \cdots |f_n|(y_n^{(k)}) \\ &\leq \|f_1\| \cdots \|f_n\| \sum_{k=1}^{m} \|y_1^{(k)}\| \cdots \|y_n^{(k)}\| \\ &= \sum_{k=1}^{m} \|y_1^{(k)}\| \cdots \|y_n^{(k)}\|;\end{aligned}$$

hence $\|x_1\| \cdots \|x_n\| \leq \varrho(x_1 \otimes \cdots \otimes x_n)$. Now, for $\mathbf{x} \neq 0$ in $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ there must exist $0 < x_i \in E_i$ for all $1 \leq i \leq n$ such that $0 < x_1 \otimes \cdots \otimes x_n \leq |\mathbf{x}|$ by Theorem 3.24, so

$$\varrho(\mathbf{x}) \geq \varrho(x_1 \otimes \cdots \otimes x_n) = \|x_1\| \cdots \|x_n\| > 0$$

and we conclude that ϱ is a norm on $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$. ∎

Let G be the closure of $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ under ϱ and define the relation

$$\mathbf{x} \geq 0 :\Longleftrightarrow \mathbf{x} \in \overline{C}_F \qquad (\mathbf{x} \in G)$$

where $\overline{C}_F$ is the closure under ϱ of the positive cone C_F in $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$. Clearly, we have that $\overline{C}_F + \overline{C}_F \subset \overline{C}_F$ and $\alpha \overline{C}_F \subset \overline{C}_F$ for all $\alpha \geq 0$. If $\mathbf{x} \geq 0$ and $-\mathbf{x} \geq 0$, then there are sequences $(\mathbf{x}_k)_{k \geq 1}$ and $(\mathbf{y}_k)_{k \geq 1}$ in C_F such that $\varrho(\mathbf{x} - \mathbf{x}_k) \to 0$ and $\varrho(\mathbf{x} + \mathbf{y}_k) \to 0$. But since $0 \leq \mathbf{x}_k \leq \mathbf{x}_k + \mathbf{y}_k$ and $\varrho(\mathbf{x}_k + \mathbf{y}_k) \leq \varrho(\mathbf{x} - \mathbf{x}_k) + \varrho(\mathbf{x} + \mathbf{y}_k) \to 0$, it follows that

$$\varrho(\mathbf{x}) \leq \varrho(\mathbf{x} - \mathbf{x}_k) + \varrho(\mathbf{x}_k) \to 0,$$

so $\mathbf{x} = 0$; hence $\overline{C}_F$ is a cone in G. Furthermore, for $\mathbf{x}, \mathbf{y} \in G$ let $(\mathbf{x}_k)_{k \geq 1}$ and $(\mathbf{y}_k)_{k \geq 1}$ be sequences in $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ such that $\varrho(\mathbf{x} - \mathbf{x}_k) \to 0$ and $\varrho(\mathbf{y} - \mathbf{y}_k) \to 0$. Since we have the inequality $|\mathbf{x}_k \vee \mathbf{y}_k - \mathbf{x}_m \vee \mathbf{y}_m| \leq |\mathbf{x}_k - \mathbf{x}_m| + |\mathbf{y}_k - \mathbf{y}_m|$, it follows that $(\mathbf{x}_k \vee \mathbf{y}_k)_{k \geq 1}$ is a Cauchy sequence and define $\mathbf{z} := \lim_{k \to \infty} \mathbf{x}_k \vee \mathbf{y}_k$. Moreover, this inequality also implies that $\mathbf{z}$ does not depend on the choice of the sequences that converge to $\mathbf{x}$ and $\mathbf{y}$. The inequality $\mathbf{x}_k \vee \mathbf{y}_k - \mathbf{x}_k \geq 0$ for all $k \geq 1$ together with

$$\varrho(\mathbf{z} - \mathbf{x} - (\mathbf{x}_k \vee \mathbf{y}_k - \mathbf{x}_k)) \leq \varrho(\mathbf{z} - \mathbf{x}_k \vee \mathbf{y}_k) + \varrho(\mathbf{x} - \mathbf{x}_k) \to 0$$

imply that $\mathbf{z}-\mathbf{x}\geq 0$, so $\mathbf{x}\leq\mathbf{z}$. Analogously, we find that $\mathbf{y}\leq\mathbf{z}$. Suppose that $\mathbf{w}\in G$ is such that $\mathbf{x}\leq\mathbf{w}$ and $\mathbf{y}\leq\mathbf{w}$. Let $(\mathbf{u}_k)_{k\geq 1}$ and $(\mathbf{v}_k)_{k\geq 1}$ be sequences in C_F such that $\varrho(\mathbf{w}-\mathbf{x}-\mathbf{u}_k)\to 0$ and $\varrho(\mathbf{w}-\mathbf{y}-\mathbf{v}_k)\to 0$ and $(\mathbf{w}_k)_{k\geq 1}$ be a sequence in $E_1\bar{\otimes}\cdots\bar{\otimes}E_n$ such that $\varrho(\mathbf{w}-\mathbf{w}_k)\to 0$. Then $\varrho(\mathbf{x}-(\mathbf{w}_k-\mathbf{u}_k))\to 0$ and $\varrho(\mathbf{y}-(\mathbf{w}_k-\mathbf{v}_k))\to 0$, so we have the convergence $\varrho(\mathbf{z}-((\mathbf{w}_k-\mathbf{u}_k)\vee(\mathbf{w}_k-\mathbf{v}_k)))\to 0$. If we define

$$\mathbf{q}_k := \mathbf{w}_k-(\mathbf{w}_k-\mathbf{u}_k)\vee(\mathbf{w}_k-\mathbf{v}_k) \qquad (k\geq 1),$$

then $\mathbf{q}_k\geq 0$ for all $k\geq 1$ and

$$\varrho(\mathbf{w}-\mathbf{z}-\mathbf{q}_k)\leq\varrho(\mathbf{w}-\mathbf{w}_k)+\varrho(\mathbf{z}-(\mathbf{w}_k-\mathbf{u}_k)\vee(\mathbf{w}_k-\mathbf{v}_k))\to 0,$$

so $\mathbf{z}\leq\mathbf{w}$ and we conclude that $\mathbf{z}=\mathbf{x}\vee\mathbf{y}$ in G; hence G is a Riesz space, because for the infimum we have the identity $\mathbf{x}\wedge\mathbf{y}=-((-\mathbf{x})\vee(-\mathbf{y}))\in G$. We claim that ϱ also defines a lattice norm on G. To this end, let $\mathbf{x},\mathbf{y}\in G$ be such that $|\mathbf{x}|\leq|\mathbf{y}|$. For the sequences $(\mathbf{x}_k)_{k\geq 1}$ and $(\mathbf{y}_k)_{k\geq 1}$ in $E_1\bar{\otimes}\cdots\bar{\otimes}E_n$ converging to $\mathbf{x}$ and $\mathbf{y}$ respectively, we have shown that $\varrho(|\mathbf{x}|-|\mathbf{x}_k|)\to 0$ and $\varrho(|\mathbf{y}|-|\mathbf{y}_k|)\to 0$ and

$$\varrho(|\mathbf{x}|\wedge|\mathbf{y}|-|\mathbf{x}_k|\wedge|\mathbf{y}_k|)=\varrho(|\mathbf{x}|-|\mathbf{x}_k|\wedge|\mathbf{y}_k|)\to 0.$$

Furthermore, since $|\mathbf{x}_k|\wedge|\mathbf{y}_k|\leq|\mathbf{y}_k|$, it follows that $\varrho(|\mathbf{x}_k|\wedge|\mathbf{y}_k|)\leq\varrho(|\mathbf{y}_k|)$ for all $k\geq 1$ and we find that

$$\begin{aligned}\varrho(|\mathbf{x}|)&\leq\varrho(|\mathbf{x}|-|\mathbf{x}_k|\wedge|\mathbf{y}_k|)+\varrho(|\mathbf{x}_k|\wedge|\mathbf{y}_k|)\leq\varrho(|\mathbf{x}|-|\mathbf{x}_k|\wedge|\mathbf{y}_k|)+\varrho(|\mathbf{y}_k|)\\&\leq\varrho(|\mathbf{x}|-|\mathbf{x}_k|\wedge|\mathbf{y}_k|)+\varrho(|\mathbf{y}|-|\mathbf{y}_k|)+\varrho(|\mathbf{y}|)\to\varrho(|\mathbf{y}|),\end{aligned}$$

so $\varrho(|\mathbf{x}|)\leq\varrho(|\mathbf{y}|)$. In a similar way, we can obtain the equality $\varrho(\mathbf{x})=\varrho(|\mathbf{x}|)$; hence G is a Banach lattice.

As for the representation of ϱ on G, for $\mathbf{x}\in G$ define the number $\alpha\geq 0$ by

$$\alpha:=\inf\left\{\sum_{k=1}^{\infty}\|x_1^{(k)}\|\cdots\|x_n^{(k)}\| : x_i^{(k)}\in E_i^+,\ 1\leq i\leq n,\ |\mathbf{x}|\leq\sum_{k=1}^{\infty}x_1^{(k)}\otimes\cdots\otimes x_n^{(k)}\right\}.$$

Now if $(x_i^{(k)})_{k\geq 1}$ are sequences in E_i^+ for all $1\leq i\leq n$ such that

$$\sum_{k=1}^{\infty}\|x_1^{(k)}\|\cdots\|x_n^{(k)}\|<\infty,$$

then we must have that

$$\sum_{k=1}^{\infty}x_1^{(k)}\otimes\cdots\otimes x_n^{(k)}\in G$$

and if

$$|\mathbf{x}|\leq\sum_{k=1}^{\infty}x_1^{(k)}\otimes\cdots\otimes x_n^{(k)},$$

then it follows that we have the inequality

$$\varrho(\mathbf{x})\leq\varrho\left(\sum_{k=1}^{\infty}x_1^{(k)}\otimes\cdots\otimes x_n^{(k)}\right)\leq\sum_{k=1}^{\infty}\varrho(x_1^{(k)}\otimes\cdots\otimes x_n^{(k)})=\sum_{k=1}^{\infty}\|x_1^{(k)}\|\cdots\|x_n^{(k)}\|.$$

Taking the infimum over such series yields $\varrho(\mathbf{x}) \leq \alpha$. On the other hand, for an arbitrary $\varepsilon > 0$, we can choose elements $\mathbf{x}_i \in E_1\bar{\otimes}\cdots\bar{\otimes}E_n$ such that $\varrho(\mathbf{x}-\mathbf{x}_i) < 2^{-i}\varepsilon$ for all $i \geq 1$. By definition of ϱ, we can also construct sequences $(x_j^{(m)})_{m\geq 1}$ in E_j^+ for all $1 \leq j \leq n$ such that

$$|\mathbf{x}_1| \leq \sum_{k=1}^{m_1} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)}, \ \sum_{k=1}^{m_1} \|x_1^{(k)}\| \cdots \|x_n^{(k)}\| \leq \varrho(\mathbf{x}_1) + \varepsilon$$

and

$$|\mathbf{x}_{i+1} - \mathbf{x}_i| \leq \sum_{k=m_i+1}^{m_{i+1}} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)}, \quad \sum_{k=m_i+1}^{m_{i+1}} \|x_1^{(k)}\| \cdots \|x_n^{(k)}\| \leq \varrho(\mathbf{x}_{i+1} - \mathbf{x}_i) + 2^{-i}\varepsilon$$

for all $i > 1$. Since $\varrho(|\mathbf{x}-\mathbf{x}_i|) \to 0$, we now find that

$$|\mathbf{x}| \leq \lim_{i\to\infty} |\mathbf{x}-\mathbf{x}_i| + |\mathbf{x}_1| + \sum_{i=1}^{\infty} |\mathbf{x}_{i+1} - \mathbf{x}_i| \leq \sum_{k=1}^{\infty} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)}$$

and by using the triangle inequality twice, we obtain

$$\begin{aligned}\sum_{k=1}^{\infty} \|x_1^{(k)}\| \cdots \|x_n^{(k)}\| &\leq \varrho(\mathbf{x}_1) + \sum_{k=1}^{\infty} \varrho(\mathbf{x}_{k+1} - \mathbf{x}_k) + 2\varepsilon \leq \varrho(\mathbf{x}) + 2 \cdot \sum_{k=1}^{\infty} \varrho(\mathbf{x}-\mathbf{x}_k) + 2\varepsilon \\ &\leq \varrho(\mathbf{x}) + 4\varepsilon,\end{aligned}$$

so $\alpha \leq \varrho(\mathbf{x}) + 4\varepsilon$ and we conclude that $\alpha \leq \varrho(\mathbf{x})$.

Just as in the case of the Archimedean Riesz spaces, it possible to establish a universal property for this Banach lattice G. First, we need a lemma for its proof however.

Lemma 3.26 *Let $X_1, \ldots, X_n$ be compact Hausdorff spaces. If $f_i^{(k)} \in C(X_i)$ for $1 \leq i \leq n$ are such that*

$$\sum_{k=1}^{m} f_1^{(k)}(x_1) \cdots f_n^{(k)}(x_n) \geq 0$$

for all $x_i \in X_i$, then we also have that

$$\sum_{k=1}^{m} \psi(f_1^{(k)}, \ldots, f_n^{(k)}) \geq 0$$

for all n-positive functionals $\psi : C(X_1) \times \cdots \times C(X_n) \to \mathbb{R}$.

Proof: Let Ω be the set of n-positive functionals $\varphi : C(X_1)\times\cdots\times C(X_n) \to \mathbb{R}$ such that $\varphi(1_{X_1}, \ldots, 1_{X_n}) = 1$ and Θ be the set of n-linear functionals $\xi : C(X_1)\times\cdots\times C(X_n) \to \mathbb{R}$ with $\xi(1_{X_1}, \ldots, 1_{X_n}) = 1$. Now if we consider $C(X_1) \otimes \cdots \otimes C(X_n)$ endowed with the projective norm $\|\cdot\|_\pi$, we see that Θ is isometrically embedded in $C(X_1)\otimes\cdots\otimes C(X_n)$ by Theorem 3.5. Let $(\theta_k)_{k\geq 1}$ be a sequence in Ω and suppose that $(\xi_k)_{k\geq 1}$ is the corresponding sequence. If $\xi \xrightarrow{*} \xi$ for some bounded $\xi : C(X_1) \otimes \cdots \otimes C(X_n) \to \mathbb{R}$, then it follows that

$$\xi(1_{X_1} \otimes \cdots \otimes 1_{X_n}) = \lim_{k\to\infty} \xi_k(1_{X_1} \otimes \cdots \otimes 1_{X_n}) = \lim_{k\to\infty} \theta_k(1_{X_1}, \ldots, 1_{X_n}) = 1$$

and

$$\xi(f_1 \otimes \cdots \otimes f_n) = \lim_{k\to\infty} \xi_k(f_1 \otimes \cdots \otimes f_n) = \lim_{k\to\infty} \theta_k(f_1, \ldots, f_n) \geq 0 \qquad (f_i \in C(X_i)^+),$$

so Ω is w^*- closed in Θ. Since $\|\xi\| = 1$ for all $\xi \in \Theta$, it follows from the Banach-Alaoglu theorem that Ω is w^*- compact and clearly, we also have that Ω is convex. Now for $\psi : C(X_1)\times\cdots\times C(X_n) \to \mathbb{R}$ a non-zero n-positive functional, the Krein-Milman theorem implies that $\psi' := \|\psi\|^{-1}\psi \in \overline{\text{co}}(\text{ext}(\Theta))$. Suppose that ψ_0 is an extreme point of the set Θ. Fix $1 \le i \le n$ and choose $\varphi_i, \eta_i \in C(X_i)$ such that $\varphi_i \wedge \eta_i = 0$. Define the map $\phi_i' : C(X_1)^+ \times \cdots \times C(X_n)^+ \to \mathbb{R}$ by

$$\begin{aligned}\phi_i'(f_1, \ldots, f_n) &:= \sup_{k\ge1} \psi_0(f_1, \ldots, f_i \wedge k\varphi_i, \ldots, f_n) = \lim_{k\to\infty} \psi_0(f_1, \ldots, f_i \wedge k\varphi_i, \ldots, f_n)\\ &\le \psi_0(f_1, \ldots, f_n).\end{aligned}$$

Clearly, we have that ϕ_i' is additive in the j-th coordinate where $j \neq i$. Moreover, if $g_i \ge 0$, since $\varphi_i, f_i \ge 0$, we have that $(f_i + g_i) \wedge k\varphi_i = f_i \wedge k\varphi_i + g_i \wedge k\varphi_i$ for all $k \ge 1$, so ϕ_i' is also additive in the i-th coordinate and evidently respects scalar multiplication for $\alpha \ge 0$ in all coordinates. Consequently, as is claimed in [9, Prop. 3.5], we find that ϕ_i' has an n-positive extension ϕ_i to all of $C(X_1) \times \cdots \times C(X_n)$ and we find that $\psi_0 - \phi_i$ is also positive. Define $\beta := \phi_i(1_{X_1}, \ldots, 1_{X_n})$. If $\beta = 0$, then $0 = \phi_i = \beta\psi_0$ and if $\beta = 1$, then $(\psi_0 - \phi_i)(1_{X_1}, \ldots, 1_{X_n}) = 0$, so $\phi_i = \psi_0 = \beta\psi_0$ and if $1 < \beta < 1$, then $\beta^{-1}\phi_i$ and $(1-\beta)^{-1}(\psi_0 - \phi_i)$ both belong to Θ and we have that

$$\psi_0 = \beta\beta^{-1}\phi_i + (1-\beta)(1-\beta)^{-1}(\psi_0 - \phi_i).$$

Since ψ_0 is an extreme point, in this case, we also find that $\phi_i = \beta\psi_0$. Now, for any $f_j \in C(X_j)^+$ with $j \neq i$, we have that

$$\psi_0(f_1, \ldots, \varphi_i, \ldots, f_n) = \phi_i(f_1, \ldots, \varphi_i, \ldots, f_n) = \beta\psi_0(f_1, \ldots, \varphi_i, \ldots, f_n)$$

and

$$\beta\psi_0(f_1, \ldots, \eta_i, \ldots, f_n) = \phi_i(f_1, \ldots, \eta_i, \ldots, f_n) = 0$$

for all $0 \le \beta \le 1$, so this yields $\psi_0(f_1, \ldots, \varphi_i, \ldots, f_n) \wedge \psi_0(f_1, \ldots, \eta_i, \ldots, f_n) = 0$; hence the maps $f_i \mapsto \psi_0(g_1, \ldots, f_i, \ldots, g_n)$ are Riesz homomorphisms for all $1 \le i \le n$ for every $g_j \in C(X_j)^+$ where $1 \le j \le n$. Using the inequality $||f| - |g|| \le |f - g|$ inductively, we find that

$$\psi_0(|f_1|, \ldots, |f_n|) \le |\psi_0(f_1, \ldots, f_n)|,$$

so $\psi_0(|f_1|, \ldots, |f_n|) = |\psi_0(f_1, \ldots, f_n)|$ and we conclude that ψ_0 is a Riesz n-morphism. As we have shown in the proof of Theorem 3.19, there are unique $x_i \in X_i$ with $(1 \le i \le n)$ such that

$$\sum_{k=1}^m \psi_0(f_1^{(k)}, \ldots, f_n^{(k)}) = \psi_0(1_{X_1}, \ldots, 1_{X_n}) \sum_{k=1}^m f_1^{(k)}(x_1)\cdots f_n^{(k)}(x_n) \qquad (f_i^{(k)} \in C(X_i)).$$

So, if $f_i^{(k)} \in C(X_i)$ satisfy the hypothesis and $(\zeta_t)_{t\ge1}$ is a sequence in $\text{co}(\text{ext}(\Theta))$ such that $\zeta_t \to \psi'$, it follows that

$$\sum_{k=1}^m \psi'(f_1^{(k)}, \ldots, f_n^{(k)}) = \lim_{t\to\infty} \sum_{k=1}^m \zeta_t(f_1^{(k)}, \ldots, f_n^{(k)}) \ge 0;$$

hence ψ also satisfies this property. ∎

Theorem 3.27 *If H is any Banach lattice and $\psi : E_1 \times \cdots \times E_n \to H$ is a bounded n-positive map, then there is a unique positive linear map $\xi : G \to H$ with $\|\xi\| = \|\psi\|$ such that $\xi \circ \phi = \psi$. Moreover, the map ψ is a Riesz n-morphism if and only if ξ is a Riesz homomorphism.*

Proof: Let H be a Banach lattice and $\psi : E_1 \times \cdots \times E_n \to H$ be an n-positive and bounded map, and $\hat{\psi} : E_1 \otimes \cdots \otimes E_n \to H$ be the corresponding linear map. Then for $\mathbf{x} \in E_1 \otimes \cdots \otimes E_n$ with $\mathbf{x} \geq 0$ in $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$, we consider the finitely generated Archimedean Riesz spaces $E_1', \ldots, E_n'$ such that $\mathbf{x} \in E_1' \otimes \cdots \otimes E_n'$ and we claim that $\mathbf{x} \geq 0$ in $E_1' \bar{\otimes} \cdots \bar{\otimes} E_n'$. For if $\theta' : E_1' \times \cdots \times E_n' \to E_1' \bar{\otimes} \cdots \bar{\otimes} E_n'$ is the canonical Riesz n-morphism, then there is a unique Riesz homomorphism $\xi : E_1' \bar{\otimes} \cdots \bar{\otimes} E_n' \to E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ such that $\xi \circ \theta' = \phi$ by Theorem 3.24 and if $0 \neq \mathbf{y} \in E_1' \bar{\otimes} \cdots \bar{\otimes} E_n'$, it follows, again by Theorem 3.24, that there are $0 < x_i \in E_i'$ for all $1 \leq i \leq n$ such that $\theta'(x_1, \ldots, x_n) \leq |\mathbf{y}|$. This yields

$$0 < \phi(x_1, \ldots, x_n) \leq \xi(|\mathbf{y}|) = |\xi(\mathbf{y})|,$$

so $\xi(\mathbf{y}) \neq 0$; hence ξ induces an embedding and we find that $E_1' \bar{\otimes} \cdots \bar{\otimes} E_n'$ is Riesz isomorphic to $\xi(E_1' \bar{\otimes} \cdots \bar{\otimes} E_n')$, which must be the Riesz subspace of $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ that is generated by $\phi(E_1' \times \cdots \times E_n')$. We conclude that $\mathbf{x} \geq 0$ in $E_1' \bar{\otimes} \cdots \bar{\otimes} E_n'$ whit respect to the induced order of $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$.

By Theorem 3.17 there exist compact Hausdorff spaces $X_1, \ldots, X_n$ such that E_i' is uniformly dense in $C(X_i)$. Let H_0 be the ideal in H which is generated by $\psi(1_{X_1}, \ldots, 1_{X_n})$. Analogous to what was shown above, we find that $\psi(E_1' \times \cdots \times E_n') \subset H_0$, so $\hat{\psi}(\mathbf{x}) \in H_0$ and if $\zeta : H_0 \to \mathbb{R}$ is any non-zero Riesz homomorphism, then $\zeta \circ \psi : E_1' \times \cdots \times E_n' \to \mathbb{R}$ is n-positive which we can extend to an n-positive functional $\zeta' : C(X_1) \times \cdots \times C(X_n) \to \mathbb{R}$. Now, if we write

$$\mathbf{x} = \sum_{k=1}^{m} f_1^{(k)} \otimes \cdots \otimes f_n^{(k)}$$

with $f_i^{(k)} \in E_i'$ for all $1 \leq k \leq m$ and $1 \leq i \leq n$ and regard $E_1' \bar{\otimes} \cdots \bar{\otimes} E_n'$ as a Riesz subspace of $C(X_1 \times \cdots \times X_n)$, then, since $\mathbf{x} \geq 0$ in $E_1' \bar{\otimes} \cdots \bar{\otimes} E_n'$, it follows that

$$\sum_{k=1}^{m} f_1^{(k)}(x_1) \cdots f_n^{(k)}(x_n) = \mathbf{x}(x_1, \ldots, x_n) \geq 0 \qquad (x_i \in X_i,\ 1 \leq i \leq n).$$

By Lemma 3.26 we now find that

$$\zeta(\hat{\psi}(\mathbf{x})) = \sum_{k=1}^{m} \zeta \circ \psi(f_1^{(k)}, \ldots, f_n^{(k)}) = \sum_{k=1}^{m} \zeta'(f_1^{(k)}, \ldots, f_n^{(k)}) \geq 0$$

and since the non-zero Riesz homomorphisms mapping H_0 into $\mathbb{R}$ separate the points of H_0, by Lemma 3.18, we conclude that $\hat{\psi}(\mathbf{x}) \geq 0$. Now, for $\mathbf{x} \in E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$, define the sets

$$A_{\mathbf{x}} := \{\hat{\psi}(\mathbf{y}) : \mathbf{y} \in E_1 \otimes \cdots \otimes E_n,\ \mathbf{y} \leq \mathbf{x}\} \quad \text{and} \quad B_{\mathbf{x}} := \{\hat{\psi}(\mathbf{y}) : \mathbf{y} \in E_1 \otimes \cdots \otimes E_n,\ \mathbf{x} \leq \mathbf{y}\}.$$

We have already shown, when defining ϱ, that both these sets are not empty. Using Theorem 3.24, we have $x_i \in E_i^+$ for $1 \leq i \leq n$ such that we can define a sequence $(\mathbf{y}_k)_{k \geq 1}$ in $E_1 \otimes \cdots \otimes E_n$ that satisfies $|\mathbf{x} - \mathbf{y}_k| \leq 2^{-k} x_1 \otimes \cdots \otimes x_n$ for all $k \geq 1$. It follows that for $k \geq m$ we have the inequalities

$$-2^{-m+1} x_1 \otimes \cdots \otimes x_n \leq \mathbf{y}_m - \mathbf{y}_k \leq 2^{-m+1} x_1 \otimes \cdots \otimes x_n$$

and since we have shown that $\hat{\psi}$ is positive on $E_1 \otimes \cdots \otimes E_n$, this yields

$$|\hat{\psi}(\mathbf{y}_m) - \hat{\psi}(\mathbf{y}_k)| \leq 2^{-m+1}\psi(x_1, \ldots, x_n)$$

for all $k \geq m$; hence $(\hat{\psi}(\mathbf{y}_k))_{k\geq 1}$ is a Cauchy sequence in H, so there exists an element $\mathbf{z} \in H$ such that $\hat{\psi}(\mathbf{y}_k) \to \mathbf{z}$. Let $\mathbf{y} \in E_1 \otimes \cdots \otimes E_n$ be such that $\mathbf{y} \leq \mathbf{x}$. Then we must have that $\mathbf{y} \leq \mathbf{y}_k + 2^{-k}x_1 \otimes \cdots \otimes x_n$ for all $k \geq 1$, so $\hat{\psi}(\mathbf{y}) \leq \hat{\psi}(\mathbf{y}_k) + 2^{-k}\psi(x_1, \ldots, x_n) \to \mathbf{z}$ and we find that $\mathbf{z}$ is an upper bound for $A_{\mathbf{x}}$. Conversely, suppose that $\mathbf{w} \in H$ is such that $\hat{\psi}(\mathbf{y}) \leq \mathbf{w}$ for all $\mathbf{y} \leq \mathbf{x}$. Then, as $\mathbf{y}_k - 2^{-k}x_1 \otimes \cdots \otimes x_n \leq \mathbf{x}$ for all $k \geq 1$, we must have that

$$\mathbf{w} \geq \hat{\psi}(\mathbf{y}_k) - 2^{-k}\psi(x_1, \ldots, x_n) \to \mathbf{z},$$

so $\mathbf{z} = \sup A_{\mathbf{x}}$. In a similar way, one shows that $\mathbf{z} = \inf B_{\mathbf{x}}$. Since $\mathbf{z}$ is unique, we can define the map $\xi' : E_1 \bar{\otimes} \cdots \bar{\otimes} E_n \to H$ by $\xi'(\mathbf{x}) := \sup A_{\mathbf{x}}$. It is a straightforward verification to show that we have the inequalities

$$\sup A_{\mathbf{x}} + \sup A_{\mathbf{y}} \leq \sup A_{\mathbf{x}+\mathbf{y}} = \inf B_{\mathbf{x}+\mathbf{y}} \leq \inf B_{\mathbf{x}} + \inf B_{\mathbf{y}} = \sup A_{\mathbf{x}} + \sup A_{\mathbf{y}},$$

so ξ' is additive. Via analogous reasoning, we find that $\sup A_{\alpha\mathbf{x}} = \alpha \sup A_{\mathbf{x}}$ for all $\alpha \in \mathbb{R}$ and we conclude that ξ' is linear. It is also easy to see that ξ' extends $\hat{\psi}$ and is positive. Now if $\tilde{\xi} : E_1 \bar{\otimes} \cdots \bar{\otimes} E_n \to H$ is a positive linear extension of $\hat{\psi}$, fix $\mathbf{x} \in \tilde{\xi} : E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ and let $\mathbf{v}, \mathbf{w} \in E_1 \otimes \cdots \otimes E_n$ be such that $\mathbf{v} \leq \mathbf{x}$ and $\mathbf{x} \leq \mathbf{w}$. It follows that

$$\hat{\psi}(\mathbf{v}) = \tilde{\xi}(\mathbf{v}) \leq \tilde{\xi}(\mathbf{x}) \leq \tilde{\xi}(\mathbf{w}) = \hat{\psi}(\mathbf{w}),$$

so $\sup A_{\mathbf{x}} \leq \tilde{\xi}(\mathbf{x}) \leq \inf B_{\mathbf{x}} = \sup A_{\mathbf{x}}$ and we see that ξ' is the unique positive linear extension of $\hat{\psi}$. Since $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ is ϱ-dense in G and $\overline{C}_F$ is the cone in G, it follows that ξ' can uniquely be extended to a positive linear map $\xi : G \to H$. As for the norm of ξ, let $\mathbf{x} \in E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ and $\varepsilon > 0$. By definition of ϱ there are $x_i^{(k)} \in E_i^+$ for $1 \leq i \leq n$ such that

$$|\mathbf{x}| \leq \sum_{k=1}^{m} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)} \quad \text{and} \quad \sum_{k=1}^{m} \|x_1^{(k)}\| \cdots \|x_n^{(k)}\| \leq \varrho(\mathbf{x}) + \varepsilon.$$

This implies that

$$|\xi(\mathbf{x})| \leq \xi(|\mathbf{x}|) \leq \sum_{k=1}^{m} \psi(x_1^{(k)}, \ldots, x_n^{(k)}),$$

so we obtain the inequality

$$\|\xi(\mathbf{x})\| \leq \sum_{k=1}^{m} \|\psi(x_1^{(k)}, \ldots, x_n^{(k)})\| \leq \|\psi\| \sum_{k=1}^{m} \|x_1^{(k)}\| \cdots \|x_n^{(k)}\| \leq \|\psi\|(\varrho(\mathbf{x}) + \varepsilon);$$

hence $\|\xi(\mathbf{x})\| \leq \|\psi\|\varrho(\mathbf{x})$ since $\varepsilon > 0$ was arbitrary and we conclude that $\|\xi\| \leq \|\psi\|$ on $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$. But for $\mathbf{x} \in G$ we have a sequence $(\mathbf{x}_k)_{k\geq 1}$ in $E_1 \bar{\otimes} \cdots \otimes E_n$ such that $\varrho(\mathbf{x}_k) \to \varrho(\mathbf{x})$, so

$$\|\xi(\mathbf{x})\| = \lim_{k\to\infty} \|\xi(\mathbf{x}_k)\| \leq \lim_{k\to\infty} \|\psi\|\varrho(\mathbf{x}_k) = \|\psi\|\varrho(\mathbf{x})$$

and we also have $\|\xi\| \leq \|\psi\|$ on G. On the other hand, the inequality

$$\|\psi(x_1, \ldots, x_n)\| = \|\xi(x_1 \otimes \cdots \otimes x_n)\| \leq \|\xi\|\varrho(x_1 \otimes \cdots \otimes x_n) = \|\xi\|\|x_1\| \cdots \|x_n\|$$

implies that $\|\psi\| \leq \|\xi\|$.

Finally, if ψ is a Riesz n-morphism, then by Theorem 3.24 we have that ξ is a Riesz homomorphism on $E_1\bar{\otimes}\cdots\bar{\otimes}E_n$, so if $\mathbf{x} \in G$ and $(\mathbf{x}_k)_{k\geq 1}$ is a sequence in $E_1\bar{\otimes}\cdots\bar{\otimes}E_n$ such that $\mathbf{x}_k \to \mathbf{x}$, then $|\mathbf{x}_k| \to |\mathbf{x}|$ and the continuity of ξ now yields

$$\xi(|\mathbf{x}|) = \lim_{k\to\infty} \xi(|\mathbf{x}_k|) = \lim_{k\to\infty} |\xi(\mathbf{x}_k)| = |\xi(\mathbf{x})|,$$

so ξ is a Riesz homomorphism on G. Conversely, if ξ is a Riesz homomorphism, then it follows at once that

$$\begin{aligned}|\psi(x_1,\ldots,x_n)| &= |\xi(x_1\otimes\cdots\otimes x_n)| = \xi(|x_1\otimes\cdots\otimes x_n|) = \xi(|x_1|\otimes\cdots\otimes|x_n|)\\ &= \psi(|x_1|,\ldots,|x_n|)\end{aligned}$$

for all $x_i \in E_i$ with $1 \leq i \leq n$; hence ψ is a Riesz n-morphism. ∎

The Banach lattice G is called the **projective Fremlin tensor product** of the Banach lattices $E_1,\ldots,E_n$ and will be denoted by $E_1\breve{\otimes}\cdots\breve{\otimes}E_n$. Theorem 3.27 implies that $E_1\breve{\otimes}\cdots\breve{\otimes}E_n$ is unique up to a Banach lattice isomorphism. Furthermore, we can now give a proof for Theorem 3.14:

Proof: It follows from Theorem 3.27 that we have a bipositive linear bijection between $\mathcal{L}_r(E_1\breve{\otimes}\cdots\breve{\otimes}E_n;F)$, which is a Banach lattice, and $\mathcal{L}_r^n(E_1,\ldots,E_n;F)$. This implies that we can induce a vector space order on $\mathcal{L}_r^n(E_1,\ldots,E_n;F)$ via this bijection, so that it becomes a Riesz space. The same can be done with the norm on $\mathcal{L}_r(E_1\breve{\otimes}\cdots\breve{\otimes}E_n;F)$, which makes it a Banach lattice. Now, if we have $\varphi,\psi \in \mathcal{L}_r^n(E_1,\ldots,E_n;F)$ such that $|\varphi| \leq |\psi|$, then $\pm\varphi \leq |\psi|$, so $\|\varphi\|_r \leq \||\psi|\|$. Suppose that $\xi \in \mathcal{L}_r^n(E_1,\ldots,E_n;F)$ is such that $\pm\psi \leq \xi$. Then $|\psi| \leq \xi$ and for $x_i \in E_i$ with $\|x_i\| \leq 1$ for all $1 \leq i \leq n$, the inequality

$$\pm|\psi|(x_1,\ldots,x_n) \leq |\psi|(|x_1|,\ldots,|x_n|) \leq \xi(|x_1|,\ldots,|x_n|)$$

implies that $||\psi|(x_1,\ldots,x_n)| \leq \xi(|x_1|,\ldots,|x_n|)$, so $\||\psi|(x_1,\ldots,x_n)\| \leq \|\xi\|$ and we find that $\||\psi|\| \leq \|\xi\|$. Taking the infimum now yields $\||\psi|\| \leq \|\psi\|_r$. Conversely, since we also have $\pm\psi \leq |\psi|$, it follows that $\|\psi\|_r \leq \||\psi|\|$, so $\|\varphi\|_r \leq \|\psi\|_r$ and we conclude that $\|\cdot\|_r$ is a Riesz norm. Suppose now that $(\varphi_k)_{k\geq 1}$ ia na increasing sequence of n-positive operators in $\mathcal{L}_r^n(E_1,\ldots,E_n)$ that is Cauchy with respect to $\|\cdot\|_r$. For $k \geq m$ we have that $\|\varphi_k-\varphi_m\| = \||\varphi_k-\varphi_m|\| = \|\varphi_k-\varphi_m\|_r$, so $(\varphi_k)_{k\geq 1}$ is a Cauchy sequence in $\mathcal{L}^n(E_1,\ldots,E_n;F)$ and therefore, there is a $\varphi \in \mathcal{L}^n(E_1,\ldots,E_n;F)$ with $\|\varphi_k-\varphi\| \to 0$. This implies that if $x_i \in E_i^+$ for $1 \leq i \leq n$, we have

$$\sup_{k\geq 1} \varphi_k(x_1,\ldots,x_n) = \lim_{k\to\infty} \varphi_k(x_1,\ldots,x_n) = \varphi(x_1,\ldots,x_n),$$

so φ is n-positive and $\varphi_k \leq \varphi$ for all $k \geq 1$. This yields the convergence

$$\|\varphi-\varphi_k\|_r = \||\varphi-\varphi_k|\| = \|\varphi-\varphi_k\| \to 0$$

and we conclude from Lemma 3.12 that $\mathcal{L}_r^n(E_1,\ldots,E_n;F)$ is a Banach lattice with respect to the norm $\|\cdot\|_r$. Since this implies that the norm on $\mathcal{L}_r(E_1\breve{\otimes}\cdots\breve{\otimes}E_n;F)$ must be equivalent to $\|\cdot\|_r$, we also have that $\mathcal{L}_r(E_1\breve{\otimes}\cdots\breve{\otimes}E_n;F) \cong \mathcal{L}_r^n(E_1,\ldots,E_n;F)$ as Banach lattices. ∎

Let P be the cone in $E_1 \otimes \cdots \otimes E_n$ generated by

$$\{x_1 \otimes \cdots \otimes x_n : x_i \in E_i^+,\ 1 \leq i \leq n\}$$

and consider the intersection $(E_1 \otimes \cdots \otimes E_n)^+ := E_1 \otimes \cdots \otimes E_n \cap C_F$. Then $(E_1 \otimes \cdots \otimes E_n)^+$ is also a cone in $E_1 \otimes \cdots \otimes E_n$ and it is closed with respect to the induced topology induced by ϱ, since for a sequence $(\mathbf{x}_k)_{k\geq 1}$ in $(E_1 \otimes \cdots \otimes E_n)^+$ such that $\varrho(\mathbf{x}_k - \mathbf{x}) \to 0$ for some $\mathbf{x} \in E_1 \otimes \cdots \otimes E_n$ we have the inequality $|\mathbf{x}_k - \mathbf{x}^+| \leq |\mathbf{x}_k - \mathbf{x}|$ for all $k \geq 1$ by Birkhoff's identity, so $\varrho(\mathbf{x}_k - \mathbf{x}^+) \to 0$ and as these limits are unique, we must have that $\mathbf{x} = \mathbf{x}^+ \in C_F$; hence $\mathbf{x} \in (E_1 \otimes \cdots \otimes E_n)^+$. This implies that $\overline{P} \subset (E_1 \otimes \cdots \otimes E_n)^+$. Now suppose that we have an element $\mathbf{x} \in E_1 \otimes \cdots \otimes E_n \setminus \overline{P}$. Then by the Hahn-Banach theorem there exists a functional $\varphi \in (E_1 \otimes \cdots \otimes E_n)^*$ such that $\varphi(\mathbf{x}) < \inf\{\varphi(\mathbf{y}) : \mathbf{y} \in P\}$. Suppose that there is an element $\mathbf{z} \in P$ with $\varphi(\mathbf{z}) < 0$. Then the number α such that $\alpha\varphi(\mathbf{z}) = \varphi(\mathbf{x})$ is positive, so $\alpha\mathbf{z} \in P$, but this implies that $\varphi(\mathbf{x}) \geq \inf\{\varphi(\mathbf{y}) : \mathbf{y} \in P\}$, which is impossible; hence $\varphi(\mathbf{y}) \geq 0$ for all $\mathbf{y} \in P$. Define the map $\psi : E_1 \times \cdots \times E_n \to \mathbb{R}$ by

$$\psi(x_1, \ldots, x_n) := \varphi(x_1 \otimes \cdots \otimes x_n).$$

By Theorem 3.27 there is a positive map $\xi : E_1 \breve{\otimes} \cdots \breve{\otimes} E_n \to \mathbb{R}$ such that

$$\varphi(x_1 \otimes \cdots \otimes x_n) = \xi(x_1 \otimes \cdots \otimes x_n) \qquad (x_i \in E_i,\ 1 \leq i \leq n),$$

so ξ and φ coincide on $E_1 \otimes \cdots \otimes E_n$ and as $\varphi(\mathbf{x}) < 0$, it follows that $\xi(\mathbf{x}) < 0$, so we must have $\mathbf{x} \notin (E_1 \otimes \cdots \otimes E_n)^+$; hence $(E_1 \otimes \cdots \otimes E_n)^+ \subset \overline{P}$. Now pick an element $\mathbf{x} \in \overline{C}_F$ and let $\varepsilon > 0$. Then there exists an element $\mathbf{y} \in C_F$ such that $\varrho(\mathbf{x} - \mathbf{y}) < \varepsilon$ and for this $\mathbf{y}$, by Theorem 3.24, we have a $\mathbf{z} \in E_1 \otimes \cdots \otimes E_n$ and $x_i \in E_i^+$ for $1 \leq i \leq n$, independent of $\mathbf{z}$, such that $|\mathbf{y} - \mathbf{z}| \leq \varepsilon x_1 \otimes \cdots \otimes x_n$. Birkhoff's identity implies that $|\mathbf{y} - \mathbf{z}^+| \leq |\mathbf{y} - \mathbf{z}|$ and so there is an element $\mathbf{w} \in P$ such that $\varrho(\mathbf{z}^+ - \mathbf{w}) < \varepsilon$. Putting this together yields

$$\varrho(\mathbf{x} - \mathbf{w}) \leq \varrho(\mathbf{x} - \mathbf{y}) + \varrho(\mathbf{y} - \mathbf{z}^+) + \varrho(\mathbf{z}^+ - \mathbf{w}) < 2\varepsilon + \varepsilon\|x_1\| \cdots \|x_n\|,$$

so we conclude that P is dense in $\overline{C}_F$.

Now that we have this beautiful theory about the tensor product of Banach lattices, we are ready to return to the Hochschild cohomology groups.

3.4 The triviality of $\mathcal{H}_r^n(\mathfrak{A}, E^*)$ for Banach lattice algebras and regular Banach lattice bimodules

It would be a desirable result to have an analogue of Lemma 3.7 for the projective Fremlin tensor product. Along these lines, we need to make the following identification first:

Lemma 3.28 *Let $E_1, \ldots, E_n, F$ be Banach lattices. Then $\mathcal{L}_r^{n+1}(E_1, \ldots, E_n, F; \mathbb{R})$ and $\mathcal{L}_r^n(E_1, \ldots, E_n, F^*)$ are isometrically isomorphic as Banach lattices.*

Proof: It follows from Theorem 3.14 that both these spaces are a Banach lattices. Let $\varphi \in \mathcal{L}_r^{n+1}(E_1, \ldots, E_n, F; \mathbb{R})$. Then we have that $\varphi = \varphi_1 - \varphi_2$ where both φ_1 and φ_2 are $(n+1)$-positive. Now define $\psi : E_1 \times \cdots \times E_n \to F^*$ through

$$\psi(x_1, \ldots, x_n)(y) := \varphi(x_1, \ldots, x_n, y).$$

Clearly, we have that ψ is regular and $\psi(x_1, \ldots, x_n)$ is linear in F for all $x_i \in E_i$ with $1 \leq i \leq n$. Now if $\varphi, \xi \in \mathcal{L}_r^{n+1}(E_1, \ldots, E_n, F; \mathbb{R})$ with η_1, η_2 corresponding to φ_1, φ_2 and

σ_1, σ_2 corresponding to ξ_1, ξ_2 respectively, then for $\Phi(\varphi + \xi) = \psi_1 - \psi_2$ it is analogously checked that $\psi_1 = \eta_1 + \sigma_1$ and $\psi_2 = \eta_2 + \sigma_2$, so $\Phi(\varphi + \xi) = \Phi(\varphi) + \Phi(\xi)$. The fact that $\Phi(\alpha\varphi) = \alpha\Phi(\varphi)$ for all $\alpha \in \mathbb{R}$ should be clear, which implies that Φ is linear. Now for $x_i \in E_i^+$ with $1 \leq i \leq n$ and $y \in F^+$ we find, by using Theorem 3.14, that

$$\begin{aligned}\Phi(\varphi^+)(x_1, \ldots, x_n)(y) &= \varphi^+(x_1, \ldots, x_n, y)\\ &= \sup\{\varphi(y_1, \ldots, y_n, z) : 0 \leq y_i \leq x_i,\ 0 \leq z \leq y,\ 1 \leq i \leq n\}\\ &= \sup\{\Phi(\varphi)(y_1, \ldots, y_n)(z) : 0 \leq y_i \leq x_i,\ 0 \leq z \leq y,\ 1 \leq i \leq n\}\\ &= \sup\left\{\Phi(\varphi)^+(x_1, \ldots, x_n)(z) : 0 \leq z \leq y\right\}\\ &= \Phi(\varphi)^+(x_1, \ldots, x_n)(y),\end{aligned}$$

so $\Phi(\varphi^+) = \Phi(\varphi)^+$ and it follows from

$$\begin{aligned}\Phi(\varphi \vee \psi) &= \Phi((\psi - \varphi)^+ + \varphi) = \Phi(\psi - \varphi)^+ + \Phi(\varphi) = (\Phi(\psi) - \Phi(\varphi))^+ + \Phi(\varphi)\\ &= \Phi(\varphi) \vee \Phi(\psi)\end{aligned}$$

that Φ is a Riesz homomorphism. For all $1 \leq i \leq n$, let $x_i \in E_i$ and $y \in F$ be such that $\|x_i\| \leq 1$ and $\|y\| \leq 1$. Then

$$|\varphi(x_1, \ldots, x_n, y)| = |\Phi(\varphi)(x_1, \ldots, x_n)(y)| \leq \|\Phi(\varphi)(x_1, \ldots, x_n)\| \leq \|\Phi(\varphi)\|,$$

so $\|\varphi\| \leq \|\Phi(\varphi)\|$ and conversely, we now also have $|\Phi(\varphi)(x_1, \ldots, x_n)(y)| \leq \|\varphi\|$; hence $\|\Phi(\varphi)\| \leq \|\varphi\|$. The fact that Φ is a Riesz homomorphism yields

$$\|\varphi\|_r = \||\varphi|\| = \|\Phi(|\varphi|)\| = \||\Phi(\varphi)|\| = \|\Phi(\varphi)\|_r$$

and we conclude that Φ is an isometry. Finally, let $\psi \in \mathcal{L}_r^n(E_1, \ldots, E_n; F^*)$ with regular decomposition $\psi = \psi_1 - \psi_2$ and define the map $\varphi_1 : E_1 \times \cdots \times E_n \times F \to \mathbb{R}$ through

$$\varphi_1(x_1, \ldots, x_n, y) := \psi_1(x_1, \ldots, x_n)(y).$$

Analogously, we find that φ_1 is n-positive with $\|\varphi_1\| \leq \|\psi_1\|$. In a similar way, we have an n-positive map φ_2 corresponding to ψ_2. It follows that $\varphi_1 - \varphi_2 \in \mathcal{L}_r^{n+1}(E_1, \ldots, E_n, F; \mathbb{R})$ and it is a straightforward verification to show that $\Phi(\varphi_1 - \varphi_2) = \psi$; hence Φ is an isometric Riesz isomorphism. ∎

The analogue of Lemma 3.7 is now a consequence of the following statement:

Theorem 3.29 *Let $E_1, \ldots, E_n, F$ be Banach lattices. Then $\mathcal{L}_r^{n+1}(E_1, \ldots, E_n; F^*)$ and the dual space $(E_1 \breve{\otimes} \cdots \breve{\otimes} E_n \breve{\otimes} F)^*$ are isometrically isomorphic as Banach lattices.*

Proof: We clearly have $(E_1 \breve{\otimes} \cdots \breve{\otimes} E_n \breve{\otimes} F)^* = \mathcal{L}_r(E_1 \breve{\otimes} \cdots \breve{\otimes} E_n \breve{\otimes} F; \mathbb{R})$ and we have shown that $\mathcal{L}_r(E_1 \breve{\otimes} \cdots \breve{\otimes} E_n \breve{\otimes} F; \mathbb{R}) \cong \mathcal{L}_r^{n+1}(E_1, \ldots, E_n, F; \mathbb{R})$ as Banach lattices in the proof of Theorem 3.14. Finally, we have that $\mathcal{L}_r^{n+1}(E_1, \ldots, E_n, F; \mathbb{R}) \cong \mathcal{L}_r^{n+1}(E_1, \ldots, E_n; F^*)$ as Banach lattices by Lemma 3.28. ∎

Now let $\mathfrak{A}$ be a Banach lattice algebra and E be a regular Banach lattice $\mathfrak{A}$-bimodule. For $n \in \mathbb{N}_+$ with $n > 1$, define the Banach lattice F by

$$F := \overbrace{\mathfrak{A} \breve{\otimes} \cdots \breve{\otimes} \mathfrak{A}}^{n-1} \breve{\otimes} E$$

and consider the bimodule actions of $\mathfrak{A}$ on F as in the proof of Theorem 3.8 on the elementary tensors. Similarly, we can show that these actions turn F into a regular Banach $\mathfrak{A}$-bimodule. Furthermore, by Lemma 3.28 and Theorem 3.29 the Banach lattice isomorphism

$$\Phi : \mathcal{L}_r^{n-1}(\mathfrak{A}, E^*) \to F^*$$

is given by $\Phi(\varphi)(a_1 \otimes \cdots \otimes a_{n-1} \otimes x) := \varphi(a_1, \ldots, a_{n-1})(x)$ and the bimodule actions of $\mathfrak{A}$ on F^* are defined by

$$(a \cdot \mathbf{f})(\mathbf{x}) := \mathbf{f}(\mathbf{x} \cdot a) \quad \text{and} \quad (\mathbf{f} \cdot a)(\mathbf{x}) := \mathbf{f}(a \cdot \mathbf{x}) \qquad (\mathbf{f} \in F^*,\ \mathbf{x} \in F).$$

Now fix $a \in \mathfrak{A}$. If $a_i \in \mathfrak{A}^+$ for $1 \leq i \leq n-1$ and $x \in E^+$, then we find that

$$\begin{aligned}\Phi(a \cdot \varphi)(a_1 \otimes \cdots \otimes a_{n-1} \otimes x) &= (a \cdot \varphi)(a_1, \ldots, a_{n-1})(x) = \varphi(a_1, \ldots, a_{n-1})(x \cdot a) \\ &= \Phi(\varphi)(a_1 \otimes \cdots \otimes a_{n-1} \otimes x \cdot a) = \Phi(\varphi)((a_1 \otimes \cdots \otimes a_{n-1} \otimes x) \cdot a) \\ &= (a \cdot \Phi(\varphi))(a_1 \otimes \cdots \otimes a_{n-1} \otimes x),\end{aligned}$$

so $\Phi(a \cdot \varphi)$ and $a \cdot \Phi(\varphi)$ coincide on the cone P in $\mathfrak{A} \otimes \cdots \otimes \mathfrak{A} \otimes E$ that is generated by

$$\left\{a_1 \otimes \cdots \otimes a_{n-1} \otimes x : a_i \in \mathfrak{A}^+, x \in E^+, 1 \leq i \leq n-1\right\}.$$

Taking limits now yields that they coincide on $\overline{C}_F$ and since $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-$ for all $\mathbf{x} \in F$, we conclude that $\Phi(a \cdot \varphi) = a \cdot \Phi(\varphi)$. On the other hand, we have

$$\begin{aligned}(\Phi(\varphi) \cdot a)&(a_1 \otimes \cdots \otimes a_{n-1} \otimes x) = \Phi(\varphi)(a \cdot (a_1 \otimes \cdots \otimes a_{n-1} \otimes x)) \\ &= \Phi(\varphi)(aa_1 \otimes \cdots \otimes a_{n-1} \otimes x) + \sum_{k=1}^{n-2}(-1)^k \Phi(\varphi)(a \otimes a_1 \otimes \cdots \otimes a_k a_{k+1} \otimes a_{n-1} \otimes x) \\ &\quad + (-1)^{n-1}\Phi(\varphi)(a \otimes a_1 \otimes \cdots \otimes a_{n-2} \otimes a_{n-1} \cdot x) \\ &= \varphi(aa_1, \ldots, a_{n-1})(x) + \sum_{k=1}^{n-2}(-1)^k \varphi(a, a_1, \ldots, a_k a_{k+1}, a_{n-1})(x) \\ &\quad + (-1)^{n-1}(\varphi(a, a_1, \ldots, a_{n-2}) \cdot a_{n-1})(x) = (\varphi \cdot a)(a_1, \ldots, a_{n_1})(x) \\ &= \Phi(\varphi \cdot a)(a_1 \otimes \cdots a_{n-1} \otimes x),\end{aligned}$$

so analogously, we find that $\Phi(\varphi \cdot a) = \Phi(\varphi) \cdot a$; hence $\mathcal{L}_r^{n-1}(\mathfrak{A}, E^*) \cong F^*$ as regular Banach lattice $\mathfrak{A}$-bimodules. We now conclude from (17) that

$$\mathcal{H}_r^n(\mathfrak{A}, E^*) \cong \mathcal{H}_r^1(\mathfrak{A}, \mathcal{L}_r^{n-1}(\mathfrak{A}, E^*)) \cong \mathcal{H}_r^1(\mathfrak{A}, F^*)$$

for all $n > 1$ and this proves the main result of this section:

Theorem 3.30 *Let $\mathfrak{A}$ be a Banach lattice algebra. Then $\mathfrak{A}$ is regularly amenable if and only if $\mathcal{H}_r^n(\mathfrak{A}, E^*) = \{0\}$ for all regular Banach lattice $\mathfrak{A}$-bimodules E.*

4 Concluding remarks

An important fact we haven't discussed yet is that not every projective tensor product of Banach lattices is a Banach lattice. If this were the case, then for all Banach lattices $E_1, \dots, E_n$ and $\mathbf{x} \in E_1 \hat{\otimes} \cdots \hat{\otimes} E_n$ an arbitrary representation

$$\mathbf{x} = \sum_{k=1}^{\infty} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)}$$

satisfies the inequality

$$\pm\mathbf{x} \leq \sum_{k=1}^{\infty} |x_1^{(k)}| \otimes \cdots \otimes |x_n^{(k)}|,$$

so

$$\varrho(\mathbf{x}) \leq \sum_{k=1}^{\infty} \|x_1^{(k)}\| \cdots \|x_n^{(k)}\|$$

which implies that $\varrho(\mathbf{x}) \leq \|\mathbf{x}\|_{\tilde{\pi}}$. On the other hand, if we have

$$|\mathbf{x}| \leq \sum_{k=1}^{\infty} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)}$$

for $x_i^{(k)} \in E_i^+$ with $1 \leq i \leq n$, then by our assumption, it follows that

$$\|\mathbf{x}\|_{\tilde{\pi}} = \||\mathbf{x}|\|_{\tilde{\pi}} \leq \sum_{k=1}^{\infty} \|x_1^{(k)}\| \cdots \|x_n^{(k)}\|,$$

so $\|\mathbf{x}\|_{\tilde{\pi}} \leq \varrho(\mathbf{x})$. Since the algebraic tensor product $E_1 \otimes \cdots \otimes E_n$ is dense in $E_1 \hat{\otimes} \cdots \hat{\otimes} E_n$ for $\|\cdot\|_{\tilde{\pi}}$ and dense in $E_1 \breve{\otimes} \cdots \breve{\otimes} E_n$ for ϱ, we find that $E_1 \hat{\otimes} \cdots \hat{\otimes} E_n = E_1 \breve{\otimes} \cdots \breve{\otimes} E_n$. As we have shown, this also implies that we have the identification

$$\mathcal{L}_r^{n-1}(E_1, \dots, E_{n-1}; E_n^*) = \mathcal{L}^{n-1}(E_1, \dots, E_{n-1}; E_n^*); \tag{18}$$

hence $\mathcal{H}_r^n(\mathfrak{A}, E^*) = \mathcal{H}^n(\mathfrak{A}, E^*)$ whenever $n \geq 1$ for all Banach lattice algebras and all Banach lattice $\mathfrak{A}$-bimodules E, which makes section 3.3 and section 3.4 a bit superfluous. To this end, we will show that exists a bounded operator mapping from a Banach lattice algebra into the dual of a Banach lattice that is not regular. If we consider the Banach lattices $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$ where $\mathbb{T}$ is the unit circle in $\mathbb{C}$, every $f \in L^2(\mathbb{T})$ can be viewed as a function $\tilde{f}$ with period 2π on $\mathbb{R}$ such that $\tilde{f}(t) := f(e^{it})$ and we have a sequence $(\hat{f}_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, the Fourier coefficients of f, that are defined by

$$\hat{f}_n := \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} \tilde{f}(t)\cos(nt)dt & \text{if } n \geq 0 \\[2ex] \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} \tilde{f}(t)\sin(nt)dt & \text{if } n < 0 \end{cases}$$

which satisfy the property

$$\sum_{n\in\mathbb{Z}} |x_n|^2 = \|f\|.$$

For details, see [14, Ch. 4]. This allows us to define the linear isometry

$$\Phi : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$$

with $\Phi(f) := (\hat{f}_n)_{n\in\mathbb{Z}}$. Now by our assumption, since $\ell^2(\mathbb{Z})^* = \ell^2(\mathbb{Z})$, we have that

$$\mathcal{L}_r(L^2(\mathbb{T}), \ell^2(\mathbb{Z})) = \mathcal{L}(L^2(\mathbb{T}), \ell^2(\mathbb{Z})),$$

so Φ is a regular operator, but $\ell^2(\mathbb{Z})$ is Dedekind complete, so in that case $|\Phi|$ would exist and in particular

$$(|\Phi|(1_{\mathbb{T}}))_n = \sup\left\{\left(\Phi(\hat{f})\right)_n : -1 \leq \tilde{f} \leq 1\right\} \geq \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \cos^2(nt)dt = \frac{1}{2} & \text{if } n \geq 0, \\ \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \sin^2(nt)dt = \frac{1}{2} & \text{if } n < 0; \end{cases}$$

hence we can never have that $|\Phi|(1_{\mathbb{T}}) \in \ell^2(\mathbb{Z})$. Since this contradicts (18), the projective tensor product $L^2(\mathbb{T})\hat{\otimes}\ell^2(\mathbb{Z})$ is not a Banach lattice and this explains why we considered studying the Fremlin tensor product in section 3.3.1, in order to characterize regularly amenable Banach lattice algebras.

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