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An analytical approach to a stochastic process that underlies a class of structured population models

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Contents

1 General Introduction

This master thesis considers a topic that is located at the intersection of probability theory, functional analysis and semigroup theory, and inspired by biology. It presents the construction of a stochastic process that appears in biological population, as a kind of a 'stochastically switched stochastic process', that we will describe below in more detail. The work was motivated by a cell-cycle model, as found in [11] and [4], and its generalizations, as given in [5] and [3]. These articles are about deterministic structured population models, and although they assume implicitly the existence of an underlying stochastic process, it is not clear whether there is one or not. A full description of the conditions for the process is given in Section 3, but here we will give a rough outline.

Suppose one has a population of living organisms (cells, bacteria), capable of growing and dividing, depending on nutrients. One can ask in what amounts new nutrients should be applied to optimize the growth of these organisms.

Every individual of this population is assumed to have a state. For instance, when individual means cell, a possible state could be its size. To make analysis possible, we assume that the individual's state is, at every time $t \in \mathbf{R}_+$, an element of a state space S, which is a separable metric space.

In *A cell cycle model*, the state is assumed to be the cell size, and the following assumptions are made:

(1) The growth of a cell from its birthsize is deterministic.

(2) The time at which a cell either divides or dies is random.

(3) Given that a cell divides, its size is halved and the size of the daughter cell is halve the size of the mother just before birth. In case of death, the individual attains a special dead-state.

In [11], [4], an integral equation is derived for the time evolution of the measure on the state space S, (that is, the 'size space',) say μ_t, such that for any measurable $\Gamma \subset S$, $\mu_t(\Gamma)$ is the expected number of individuals with state (i.e. size) in Γ at time t. Although not explicitly stated, essentially, using [5] and [3], it is a variation of constants formula, of the form

$$\mu_t(\Gamma) = (T(t)\mu_0)(\Gamma) + \int_0^t (T(t-s)F(\mu_s))(\Gamma)ds, \tag{1}$$

where $(T(t))_{t\geq 0}$ is a semigroup in the space of finite Borel measures on S, denoted by $\mathcal{M}(S)$, and $F : \mathcal{M}(S) \to \mathcal{M}(S)$ a map. The semigroup $(T(t))_{t\geq 0}$ and the perturbation F are are based on a description of parts (1), (2) and (3) above.

Two issues arose. On the one hand, equation 1 is a pointwise equation, that is, all measures involved are evaluated in a measurable set Γ. It is customary to consider such equations in an appropriate Banach space of measures. One of the objectives was to understand equation 1 as an equation in a Banach sapce of measures, using the Bochner integral, such that the pointwise version 1 follows. In this we succeeded, see Proposition 5.9, using the Banach space S_{BL} that contains the set $\mathcal{P}(S)$ of all probability measures on S. This Banach space is described in Section 2.1.3.

On the other hand, we were interested in the way equation 1 was derived directly from processes (1), (2) and (3), using the details of the stochastic processes involved. It seemed that this path had not been taken into account in [11], [4], [5] and [3]. After some time we discovered that this is still an open problem. We decided to proceed further, and investigate the situation.

Our idea, in line with [5] and [3], to get an understanding of the dynamics of the population, is to first make a model of the evolution of the law of the stochastic process for an individidual. Then one may try to "sum those" to obtain a suitable population description.

This thesis is only about the first part, that is, to give a description for the law of a stochastic process that is build for the assumptions (1),(2) and (3), above. However, while constructing a model of the evolution of the individual we keep in mind that there is an environment, partially consisting of the population, surrounding the individual.

Since we only consider one individual, we will only look at one of the two daughter cells, in case a cell divides. Or, one may argue that we only follow the fate of the mother before and after division.

Although the assumptions are made in *A cell cycle mode*, our assumptions will be more general. Generalizations are:

- The deterministic growth process is replaced by a stochastic growth process, with càdlàg sample trajectories.
- The deterministic jump in state space, assumption (3), may be random as well, i.e., is a measurable function.

The outline of this thesis is as follows. Section 2 consists of two subjects. First metric spaces and Lipschitz functions are introduced, and how measures can be viewed as functionals. The second subject is about products, finite and infinite, of probability spaces, and how one can define probability measures on those spaces. In Section 3, it is explained how the space of all trajectories is viewed as an infinite product of probability spaces, and how a probability measure P, the law of the underlying stochastic process, is constructed. The two main objectives in the remainder of that section is first to show that a certain subset R of the collection of all trajectories, this subset being the collection of all realistic trajectories, is measurable. Second, that the law P is concentrated on this subset R, that is, we will prove that we have $P(R) = 1$. In Section 4 the model is simplified, using techniques of Bochner integration.

2 Preliminaries

2.1 Metric spaces

In this section we will give a definition and some examples of metric spaces. Metric spaces are important in this thesis, since the state space we will be working at is one. We will also show some ways of how to make new metric space from a given metric space. We will be using this several times further on.

Definition 2.1 *A metric d on a set S is a function $d : S \times S \to \mathbf{R}$ such that for all $x, y, z \in S$*

i) $d(x, y) = 0 \iff x = y$;

ii) $d(x, y) = d(y, x)$;

iii) $d(x, z) \leq d(x, y) + d(y, z)$.

In this case, the pair (S, d) is called a metric space.

The second condition is often called symmetry, and the third is called the triangle inequality.

Lemma 2.2 *Let (S, d) be a metric space. We have $d(x, y) \geq 0$ for all $x, y \in S$.*

Proof: For $x, y \in S$ set $z = x$ in $iii)$. We see that

$$0 = d(x, x) \leq d(x, y) + d(y, x) = 2d(x, y).$$

Hence $d(x, y) \geq 0$ ∎

Examples are $S = \mathbf{R}$ with metric $d(x, y) = |x - y|$, or more general $S = \mathbf{R}^n$ with $d(x, y) = \sqrt{\sum_{k=1}^{n}(x_i - y_i)^2}$. It should be familiar that any normed space X is a metric space with metric $d(x, y) = \|x - y\|$. In the following lemma's we will show how to make new metrics out of existing ones.

Lemma 2.3 *Let (S, d) be a metric space, and let $c \in \mathbf{R}_+ = (0, \infty)$. Then d_c defined by $d_c(x, y) = \min(d(x, y), c)$ is a metric on S.*

Proof: It is obvious that $d_c(x, y) = 0 \iff x = y$ and that $d_c(x, y) = d_c(y, x)$. For the triangle inequality, suppose $d_c(x, z) = c$. It follows that $c \leq d(x, z) \leq d(x, y) + d(y, z)$. So

$$d_c(x, y) + d_c(y, z) = \begin{cases} d(x, y) + d(y, z) & \text{if} \quad d(x, y) \leq c, \quad d(y, z) \leq c, \\ d(x, y) + c & \text{if} \quad d(x, y) \leq c, \quad d(y, z) > c, \\ d(y, z) + c & \text{if} \quad d(x, y) > c, \quad d(y, z) \leq c, \\ 2c & \text{if} \quad d(x, y) > c, \quad d(y, z) > c. \end{cases}$$

In particular, it follows that $d_c(x, y) + d_c(y, z) \geq c$. Hence in this case, the triangle inequality follows. On the other hand, if $d_c(x, z) < c$, then

$$d_c(x, z) = d(x, z) \leq d(x, y) + d(y, z) = d_c(x, y) + d_c(y, z).$$

So also in this case, the triangle inequality follows. Hence d_c is a metric on S. ∎

Definition 2.4 *Given a metric space (S, d) we define the diameter of a subset $A \subset S$ as*

$$\text{diam}(A) = \sup\{d(x, y) : \ x, y \in A\} \in [0, \infty].$$

If (S, d) is a metric space, then one can always define a new metric such that $\text{diam}(S) \le 1$ for this new metric. First we will show how to create a new metric via a function.

Lemma 2.5 *Let (S, d) be a metric space. If $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ satisfies the following three conditions*

i) $\varphi(x) = 0 \iff x = 0$*;*

ii) φ *is non-decreasing;*

iii) φ *is subadditive, that is,* $\varphi(x + y) \le \varphi(x) + \varphi(y)$ *for all* $x, y \in \mathbf{R}_+$*;*

then d_φ *defined by* $d_\varphi(x, y) = \varphi(d(x, y))$ *is a metric on* S*.*

Proof: We have

$$d_\varphi(x, y) = 0 \iff \varphi(d(x, y)) = 0 \iff d(y, x) = 0 \iff x = y.$$

Symmetry is also immediate; $d_\varphi(x, y) = \varphi(d(x, y)) = \varphi(d(y, x)) = d_\varphi(y, x)$. The triangle inequality follows from assumptions *ii)* and *iii)*. Indeed, we have

$$\begin{aligned} d_\varphi(x, z) &= \varphi(d(x, z)) \le \varphi(d(x, y) + d(y, z)) \\ &\le \varphi(d(x, y)) + \varphi(d(y, z)) = d_\varphi(x, y) + d_\varphi(y, z). \quad \blacksquare \end{aligned}$$

The following lemma gives us certain conditions on a function f, so that it is non-decreasing and subadditive.

Lemma 2.6 *If $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ has a continuous derivative, $\varphi(0) = 0$ and φ' is non-increasing, then φ is non-decreasing and subadditive.*

Proof: That φ is non-decreasing follows from the condition that φ' is non-increasing. Suppose, by contradiction, that there are $a, b \in \mathbf{R}_+$ such that $a \ge b$ and $\varphi(b) < \varphi(a)$. Then, by the Mean Value Theorem, there exists $x \in [a, b]$ such that $\varphi'(x) = \frac{\varphi(b)-\varphi(a)}{b-a} < 0$. But since the image of φ is bounded from below, it is obvious that there must exist a $y > x$ such that $\varphi'(y) > \varphi'(x)$, but this cannot happen since φ' is non-increasing. Therefore φ is non-decreasing.
Since we assume that $\varphi(0) = 0$ we have by the Fundamental Theorem of Calculus,

$$\int_0^x \varphi'(t)dt = \varphi(x).$$

Now let $x, y \in \mathbf{R}_+$. Without loss of generality we may assume that $x \geq y$. Since φ' is non-increasing it follows that

$$\begin{aligned}
\varphi(x+y) &= \int_0^{x+y} \varphi'(t)dt = \int_0^x \varphi'(t)dt + \int_x^{x+y} \varphi'(t)dt \\
&= \int_0^x \varphi'(t)dt + \int_0^y \varphi'(u+x)du \\
&\leq \int_0^x \varphi'(t)dt + \int_0^y \varphi'(u)du \\
&= \varphi(x) + \varphi(y).
\end{aligned}$$

Hence φ is subadditive. ∎

It is now clear that if $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ satisfies the conditions in Lemma 2.6, together with the condition $\varphi(x) = 0 \iff x = 0$, then d_φ as in Lemma 2.5 is a metric. It is now possible, given a metric d, to create a new metric d' such that $\mathrm{diam}(S) \leq 1$ for d'.

Corollary 2.7 *If (S, d) is a metric space, then the map $d' : S \times S \to \mathbf{R}$ defined by*

$$d'(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

is a metric on S such that $\mathrm{diam}(S) \leq 1$.

Proof: Note that d' is exactly d_φ where

$$\varphi(x) = \frac{x}{1+x}.$$

It is clear that $\varphi(0) = 0$ and $\varphi'(x) = \frac{1}{(1+x)^2}$ is continuous and non-increasing. It follows from Lemma 2.5 and Lemma 2.6 that d_φ is a metric. It is obvious that $d_\varphi(x,y) < 1$ for all $x, y \in S$, so $\mathrm{diam}(S) \leq 1$. ∎

Although the space with its new metric d_φ could become bounded, as long as φ is strictly increasing and right-continuous at 0, the collection of open sets has not changed.

Lemma 2.8 *$U \subset S$ is open with respect to d if and only if U is open with respect to d_φ, whenever φ is strictly increasing.*

Proof: Let U be open with respect to the metric d. If $x \in U$, then there exists an $r > 0$ such that $\{y :\in S : \; d(x,y) < r\} \subset E$. Set $r_\varphi = \varphi(r)$. By the assumptions on φ we have

$$\begin{aligned}
\{y \in S : \; d_\varphi(x,y) < r_\varphi\} &= \{y \in S : \; \varphi(d(x,y)) < \varphi(r)\} \\
&= \{y \in S : \; d(x,y) < r\} \subset U.
\end{aligned}$$

So we have that U is open with respect to d'.
Since we assumed that φ is strictly increasing, it follows that φ is injective. So one

can consider its inverse φ^{-1} defined on domain $\varphi(\mathbf{R}_+)$. Suppose that $U \subset S$ is open with respect to d_φ. If $x \in U$, then there is an $r \in \varphi(\mathbf{R}_+)$ such that $\{y \in X : d_\varphi(x, y) < r\} \subset U$. Set $r_{\varphi^{-1}} = \varphi^{-1}(r)$. We have

$$\{y \in S : \ d(x, y) < r\} = \{y \in S : \ d_\varphi(x, y)) < \varphi(r_{\varphi^{-1}}) = r\} \subset U. \quad \blacksquare$$

Note that the new metric in Corollary 2.7 does not change the underlying topology, since the φ used in the proof is strictly increasing, so the previous lemma 2.8 holds. For a metric space (S, d) with a finite diameter, such that S consists of more than one point, it is possible to add a point ∞, such that the new space $S' = S \cup \{\infty\}$ is again metric and the new point ∞ has distance $c = \frac{\text{diam}(S)}{2}$ to all other points. The new metric d' will be defined as

$$d'(x, y) = \begin{cases} d(x, y) & \text{if } x, y \in S, \\ c & \text{if } x \in S, y = \infty, \\ c & \text{if } y \in S, x = \infty, \\ 0 & \text{if } x = y = \infty. \end{cases}$$

A metric space (S, d) which consists of only one point can also be extended in the same way, by taking an arbitrary positive number c.
Given a metric space (S, d), possibly with infinite diameter, it is also possible to add a point, say ∞, and extend the metric. This can be done by applying Corollary 2.7, add infinity and construct d' as above. Although after applying Corollary 2.7 the underlying topology does not change, sometimes it is better not to change the metric d in the first place. In that case, one can still add a point infinity and make $S \cup \{\infty\}$ into a metric space. Therefore, fix $c \in S$, and define

$$d'(x, y) = \begin{cases} d(x, y) & \text{if } x, y \in S, \\ d(x, c) + 1 & \text{if } x \in S, y = \infty, \\ d(y, c) + 1 & \text{if } y \in S, x = \infty, \\ 0 & \text{if } x = y = \infty. \end{cases}$$

In the next lemma it is explained that a product of n metric spaces is again a metric space.

Lemma 2.9 *Let $(S_1, d_1), \ldots, (S_n, d_n)$ be n metric spaces. Then $S_1 \times \ldots \times S_n$ is a metric space with metric d defined as*

$$d((x_1, \ldots, x_n), (y_1 \ldots, y_n)) = \max\{d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\}.$$

Proof: Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$ and $z = (z_1, \ldots, z_n)$. It is obvious that $d(x, y) \geq 0$ and that $d(x, y) = 0$ if and only if $x = y$. It is also obvious that $d(x, y) = d(y, x)$. By the triangle inequality at each d_i, it follows that

$$d_i(x_i, z_i) \leq \max\{d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\} + \max\{d_1(y_1, z_1), \ldots, d_n(y_n, z_n)\},$$

for every $i = 1, \ldots, n$. Hence

$$d(x, z) = \max\{d_1(x_1, z_1), \ldots, d_n(x_n, z_n)\} \leq d(x, y) + d(y, z). \quad \blacksquare$$

The metric d defined as in the previous Lemma 2.9 is called the **product metric**. A sequence $\{x_n\}$ in a metric space (S, d) is called **convergent** if there is an element $x \in S$ such that $d(x, x_n) \to 0$ as $n \to \infty$. A sequence $\{x_n\}$ is called a **Cauchy sequence** when $d(x_m, x_n) \to 0$ as $n \to \infty$.
A subset $A \subset S$ is called **dense** when its closure $\overline{A}$ equals S.

Definition 2.10 *A metric space (S, d) is called **complete** when every Cauchy sequence is convergent.*

Definition 2.11 *A metric space (S, d) is called **separable** when there exists a countable dense subset A.*

2.1.1 Lipschitz functions

A Lipschitz function is a special kind of continuous functions, namely one that cannot change too fast. Lipschitz functions will be used when creating a certain metric, described in the next section. Also, the set of all bounded Lipschitz functions turns out to be a vector space. Its dual space plays a central role in section 2.1.3.

Definition 2.12 *Let (S, d) be a metric space. A function $f : S \to \mathbf{R}$ is called a (globally) Lipschitz continuous function if there exists a constant L such that*

$$|f(x) - f(y)| \leq Ld(x, y)$$

for all $x, y \in S$. The smallest L for which the inequality holds will be denoted by $|f|_L$ and is called the Lipschitz constant.

One easily verifies that

$$|f|_L = \sup\left\{\frac{|f(x) - f(y)|}{d(x, y)} : \ x \neq y, \ x, y \in S\right\}.$$

We will now give a few examples of Lipschitz functions, and how to create new Lipschitz functions from existing ones.

Lemma 2.13 *Given an element $x_0 \in S$, the function $f : S \to \mathbf{R}$ defined by $f(x) = d(x, x_0)$ is Lipschitz continuous with $|f|_L \leq 1$.*

Proof: From the triangle-inequality it follows that

$$d(x, x_0) - d(y, x_0) \leq d(x, y),$$
$$d(y, x_0) - d(x, x_0) \leq d(x, y).$$

Given $x, y \in S$, we thus have

$$|f(x) - f(y)| = |d(x, x_0) - d(y, x_0)| \leq d(x, y). \quad \blacksquare$$

Lemma 2.14 *Given a closed subset $C \subset S$, the function $f : S \to \mathbf{R}$ defined by $f(x) = d(x, C) = \inf_{y \in C} d(x, y)$ is Lipschitz continuous with $|f|_L \leq 1$.*

Proof: Suppose $x \neq y$. By definition we have

$$\frac{|f(x) - f(y)|}{d(x,y)} = \frac{|\inf_{z \in C} d(x,z) - \inf_{z \in C} d(y,z)|}{d(x,y)}.$$

Fix $z \in C$. Observe that

$$\begin{aligned}
\inf_{z' \in C} d(x,z') - \inf_{z' \in C} d(y,z') &\leq d(x,z) - inf_{z' \in C} d(y,z') \\
&= \inf_{z' \in C} [d(x,z) - d(y,z')] \\
&\leq \inf_{z' \in C} [d(x,z') + d(z,z') - d(y,z')] \\
&\leq \inf_{z' \in C} [d(x,y) + d(z,z')] \\
&= d(x,y) + \inf_{z' \in C} d(z,z') \\
&= d(x,y)
\end{aligned}$$

So $\inf_{z' \in C} d(x,z') - \inf_{z' \in C} d(y,z') \leq d(x,y)$. It also follows that $\inf_{z' \in C} d(x,z') - \inf_{z' \in C} d(y,z') \geq -d(x,y)$, since this is true if and only if $\inf_{z' \in C} d(y,z') - \inf_{z' \in C} d(x,z') \leq d(x,y)$ which is obviously true. We conclude that

$$\frac{|\inf_{z \in C} d(x,z) - \inf_{z \in C} d(y,z)|}{d(x,y)} \leq 1. \quad \blacksquare$$

Let $\mathrm{Lip}(S)$ be the set of all Lipschitz continuous functions from S to $\mathbf{R}$.

Lemma 2.15 $\mathrm{Lip}(S)$ *is a vector space and* $|\cdot|_L : \mathrm{Lip}(S) \to \mathbf{R}_+$ *is a seminorm.*

Proof: Suppose f is Lipschitz with Lipschitz constant $|f|_L$ and g is Lipschitz with Lipschitz constant $|g|_L$. Then, given $x, y \in S$.

$$\begin{aligned}
|(f+g)(x) - (f+g)(y)| &= |f(x) - f(y) + g(x) - g(y)| \\
&\leq |f(x) - f(y)| + |g(x) - g(y)| \\
&\leq (|f|_L + |g|_L) d(x,y).
\end{aligned}$$

Hence $f + g$ is Lipschitz. Clearly, if f is Lipschitz with constant $|f|_L$, then for every $\lambda \in \mathbf{R}$ we have that λf is again Lipschitz with constant $|\lambda f|_L = |\lambda||f|_L$. $\blacksquare$

We will conclude this section with giving examples of Lipschitz functions as well as showing how to make new Lipschitz functions out of existing ones.

Definition 2.16 *Given two functions* f *and* g*, both from* S *to* $\mathbf{R}$*, we define* $f \vee g$ *and* $f \wedge g : S \to \mathbf{R}$ *as*

$$(f \vee g)(x) = \max(f(x), g(x)),$$

and

$$(f \wedge g)(x) = \min(f(x), g(x)).$$

Lemma 2.17 *For any finite sequence* $f_1, \ldots, f_n$ *of Lipschitz functions, we have that both* $f = f_1 \vee \ldots \vee f_n$ *and* $g = f_1 \wedge \ldots \wedge f_n$ *are Lipschitz functions with Lipschitz constants* $|f|_L \leq \max(|f_1|_L, \ldots |f_n|_L)$ *and* $|g|_L \leq \max(|f_1|_L, \ldots |f_n|_L)$.

Proof: Set $f = f_1 \vee \ldots \vee f_n$, and let $x, y \in S$. Then we have $f(x) = f_i(x), f(y) = f_j(y)$ for some i, j. First suppose that $i = j$. Then we have

$$|f(x) - f(y)| = |f_i(x) - f_i(y)| \leq |f_i|_L d(x, y).$$

Now suppose $i \neq j$. Then we have either $f_j(y) > f_i(x), f_j(y) < f_i(x)$, or $f_j(y) = f_i(x)$. In the first case, we have $|f_i(x) - f_j(y)| \leq |f_j(x) - f_j(y)| \leq |f_j|_L d(x, y)$ while in the second case we have $|f_i(x) - f_j(y) \leq |f_i(x) - f_i(y)| \leq |f_i|_L d(x, y)$. In the third case we already have $|f_i(x) - f_j(y)| = 0$. From this we conclude that

$$|f(x) - f(y)| \leq \max(|f_i|_L, |f_j|_L) d(x, y).$$

When x and y vary over S, i and j will vary in $\{1, 2, \ldots, n\}$. Therefore, for any $x, y \in S$, we have

$$|g(x) - g(y)| \leq \max(|f_i|_L, \ldots, |f_n|_L) d(x, y).$$

A similar argument holds for $g = f_1 \wedge \ldots \wedge f_n$. Indeed, let $x, y \in S$. Then there are i and j such that $g(x) = f_i(x)$ and $g(y) = f_j(y)$. If $i = j$, then $|g(x) - g(y)| = |f_i(x) - f_i(y)| \leq |f_i|_L d(X, y)$. If $i \neq j$, then we have in case $f_i(x) < f_j(y)$ that

$$|f_i(x) - f_j(y)| \leq |f_i(x) - f_i(y)| \leq |f_i|_L d(x, y),$$

and in case $f_i(x) > f_j(y)$,

$$|f_i(x) - f_j(y)| \leq |f_j(x) - f_j(y)| \leq |f_j|_L d(x, y).$$

So for any $x, y \in S$, $|g(x) - g(y)| \leq \max(|f_1|_L, \ldots, |f_n|_L) d(x, y)$ ∎

Lemma 2.18 *If* $\operatorname{diam}(S) < \infty$*, then given an element* $x_0 \in S$*, the function* $f : S \to \mathbf{R}$ *defined by* $f(x) = d(x, x_0)$ *is bounded. Otherwise,* f *will not be bounded.*

Proof: Suppose $\operatorname{diam}(S) < \infty$. Then $d(x, x_0) \leq \operatorname{diam}(S) < \infty$. If $\operatorname{diam}(S)$ is not bounded, then for every M there will be a pair $x, y \in S$ such that $d(x, y) > 2M$. Because of the triangle-inequality $d(x, y) \leq d(x, x_0) + d(y, x_0)$, we conclude that at least one of the terms, $d(x, x_0)$ or $d(y, x_0)$ is greater than M. Hence f is not bounded. ∎

Lemma 2.19 *For every* $c \in \mathbf{R}$ *and* $x_0 \in S$*, the function* $f : S \to \mathbf{R}$ *defined by* $f(x) = c \wedge d(x, x_0)$*, is a Lipschitz function which is also bounded.*

Proof: For any $c \in \mathbf{R}$, the constant function $g : S \to \mathbf{R}, g(x) = c$ for all $x \in S$, is a Lipschitz functions, since $|g(x) - g(y)| = |c - c| = 0$. The functions $h : S \to \mathbf{R}$ defined by $h(x) = d(x, x_0)$ is also a Lipschitz by Lemma 2.13. Hence, by Lemma 2.17, it follows that f is Lipschitz. f is also bounded, because $|f(x)| \leq c$. ∎

Lemma 2.20 *If* $f : \mathbf{R} \to \mathbf{R}$ *and* $g : \mathbf{R} \to \mathbf{R}$ *are both Lipschitz continuous functions, then* $f \circ g$ *is also a Lipschitz continuous function.*

Proof: Just note that

$$|f(g(x)) - f(g(y))| \leq |f|_L |g(x) - g(y)| \leq |f|_L |g|_L |x - y|. \quad ∎$$

2.1.2 Càdlàg functions

Definition 2.21 *Let (S, d) be a metric space. A function $f : E \to S$, where $E \subset \mathbf{R}$, is a càdlàg function if for every $t \in E$ we have*

- *the left limit $f(t-) := \lim_{s \uparrow t} f(s)$ exists;*
- *the right limit $f(t+) := \lim_{s \downarrow t} f(s)$ exists and $f(t+) = f(t)$.*

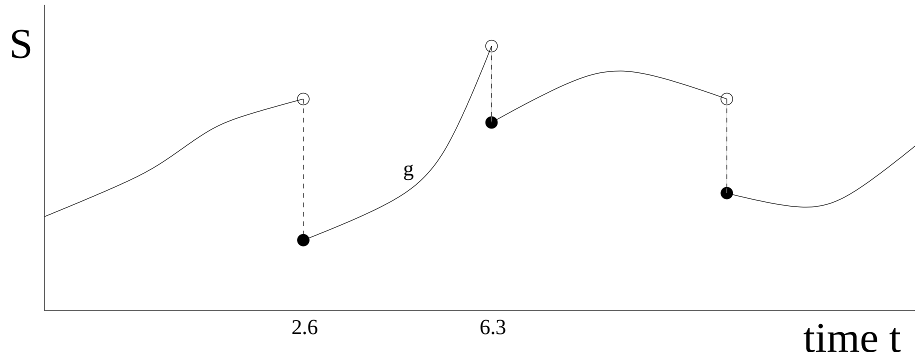

Figure 1: A possible càdlàg-function

So f is right-continuous with left limits. The term càdlàg stands for the French " continue à droite, limite à gauche".

Càdlàg functions are important in this thesis. We denote by $\mathcal{D}_S$ the space of all càdlàg-functions $f : \mathbf{R}_+ \to S$.

We are interested in making $\mathcal{D}_S$ a measurable space. Fortunately, there exists a metric on the space, called the **Skorohod metric**, such that it is separable and complete, as long as S is separable and complete. The construction of this metric is rather complicated, and the proof that the constructed function d is indeed a metric is omitted. For a proof and a more detailed treatment, we refer to [9]
Let d' be a new metric on S defined by $d'(x, y) = d(x, y) \wedge 1$. By Lemma 2.3, it follows that d' is a metric on S. Consider the collection Λ' of strictly increasing surjective functions $f : [0, \infty) \to [0, \infty)$. Note that f is continuous, $f(0) = 0$ and $\lim_{t \to \infty} f(t) = \infty$. Consider the subset $\Lambda \subset \Lambda'$ of Lipschitz continuous functions f such that

$$\gamma(f) = \operatorname*{ess\,sup}_{t \geq 0} |\log f'(t)| = \sup_{s > t \geq 0} \left| \log \frac{f(s) - f(t)}{s - t} \right| < \infty.$$

For $x, y \in \mathcal{D}_S$, $f \in \Lambda$ and $u \in [0, \infty)$, define

$$\varphi(x, y, f, u) = \sup_{t \geq 0} d'(x(t \wedge u), y(f(t) \wedge u)).$$

For $x, y \in \mathcal{D}_S$, define $d_{\mathcal{D}}$ on $\mathcal{D}_S$ as

$$d_{\mathcal{D}}(x, y) = \inf_{f \in \Lambda} \left[\gamma(f) \vee \int_0^\infty e^{-u} d'(x, y, f, u) du \right].$$

Theorem 2.22 *Let (S, d) be a metric space. The space $\mathcal{D}_S$ together with $d_{\mathcal{D}}$ defined above is a metric space, such that the following properties hold.*

i) If S is separable, then $\mathcal{D}_S$ is separable;

ii) If S is complete, then $\mathcal{D}_S$ is complete;

iii) If S is separable, then the Borel-σ-algebra of $\mathcal{D}_S$ equals the smallest σ-algebra such that for every $t \geq 0$, the evaluation function $\mathrm{ev}_t : \mathcal{D}_S \to S$, defined by $\mathrm{ev}_t(f) := f(t)$, is measurable. In other words, the Borel-σ-algebra $\mathcal{B}(\mathcal{D}_S)$ is generated by

$$\{\mathrm{ev}_t^{-1}(E) : \ t \geq 0, \ E \subset S \ \textit{measurable}\}$$

2.1.3 Measures on metric spaces viewed as functionals

Definition 2.23 *Let (S, d) be a metric space. The space of bounded Lipschitz functions, denoted by $BL(S, d)$, is the normed space*

$$BL(S, d) = \{f : S \to \mathbf{R} : \ f \textit{ is Lipschitz continuous and } \|f\|_\infty < \infty\},$$
$$\|f\|_{BL} = \max(\|f\|_\infty, \ |f|_L).$$

Since $BL(S, d)$ is a normed space, its dual $BL(S, d)^*$ is a Banach space. Consider the Dirac measure δ_x defined by

$$\delta_x(A) = \begin{cases} 1 & x \in A; \\ 0 & x \notin A. \end{cases}$$

Then we have, for every measurable function $f : S \to \mathbf{R}$,

$$\int_S f(y) \ d\delta_x(y) = f(x).$$

One can prove this by beginning with the characteristic function $f = \mathbf{1}_A$ on a measurable set A, noting that $\mathbf{1}_A(x) = \delta_x(A)$. Then use linearity of the integral and the Monotone Convergence Theorem.

We can view the measure δ_x as a function from $BL(S)$ to $\mathbf{R}$ by writing

$$\delta_x(f) := \int_S f(y) \ d\delta_x(y) = f(x), \quad f \in BL(S).$$

Lemma 2.24 *For every $x \in S$, we have that $\delta_x \in BL(S)^*$ and $\|\delta_x\|_{BL^*} = 1$.*

Proof: Let $x \in S$, and $\varphi \in BL(S)$. Since $\varphi(x) < \infty$ for all $\varphi \in BL(S)$, it follows that δ_x is bounded, and hence $\delta_x \in BL(S)^*$. Now suppose $\|\varphi\|_{BL} = 1$. In particular, we have that $\|\varphi\|_\infty \leq 1$ and hence

$$\|\delta_x\|_{BL^*} = \sup_{\|\varphi\|_{BL}=1} |\delta_x(\varphi)| = \sup_{\|\varphi\|_{BL}=1} |\varphi(x)| \leq 1.$$

Now take $\varphi \in BL(S)$ defined by $\varphi(x) = 1$ for all x. Then $\|\varphi_{BL}\| = 1$ and

$$|\delta_x(\varphi)| = |\varphi(x)| = 1,$$

so it follows that $\|\delta_x\|_{BL^*} = 1$. ∎

Lemma 2.25 *For every $x, y \in S$, we have*

$$1 \wedge d(x,y) \leq \|\delta_x - \delta_y\|_{BL^*} \leq 2 \wedge d(x,y).$$

Proof: Otherwise trivially, suppose that $x \neq y$. In any case we have

$$\|\delta_x - \delta_y\|_{BL^*} \leq \|\delta_x\|_{BL^*} + \|\delta_y\|_{BL^*} = 2.$$

And, for any φ with norm 1, it automatically follows that $|\varphi|_L \leq 1$, from which it follows that

$$\|\delta_x - \delta_y\|_{BL^*} = \sup_{\|\varphi\|_{BL}=1} |\delta_x(\varphi) - \delta_y(\varphi)| = \sup_{\|\varphi\|_{BL}=1} |\varphi(x) - \varphi(y)| \leq d(x,y).$$

From these two inequalities we have the first inequality

$$\|\delta_x - \delta_y\|_{BL^*} \leq 2 \wedge d(x,y).$$

Now consider the function $\varphi : s \mapsto d(s,y) \wedge 1$ from S to $\mathbf{R}$. By Lemma 2.19, we have that $\varphi \in BL(S)$. Furthermore we claim that $\|\varphi\|_{BL} \leq 1$. To see this, first note that by Lemma 2.17 we have $|\varphi|_L \leq \max(|d(\cdot,y)|_L, |1|_L)$, where $|d(\cdot,y)|_L$ is the Lipschitz constant of the function $s \mapsto d(s,y)$. Since $|1|_L = 0$, it follows from Lemma 2.13 that $|\varphi|_L \leq 1$. It is also clear that $\|\varphi\|_\infty \leq 1$. Hence $\|\varphi\|_{BL} \leq 1$. From this, we get the following result: If $d(x,y) < 1$, then this will give us $|\varphi(x) - \varphi(y)| = |d(x,y) - d(y,y)| = d(x,y)$. Otherwise, if $d(x,y) \geq 1$, then $|\varphi(x) - \varphi(y)| = |1 - d(y,y)| = 1$. From this we conclude the first inequality

$$1 \wedge d(x,y) = |\varphi(x) - \varphi(y)| \leq \sup_{\|\varphi\| \leq 1} |(\delta_x - \delta_y)(\varphi)| = \|\delta_x - \delta_y\|_{BL^*}. \quad \blacksquare$$

Corollary 2.26 *If $diam(S) \leq 1$ then $\|\delta_x - \delta_y\|_{BL^*} = d(x,y)$.*

Now define

$$D := \text{span}_{\mathbf{R}}\{\delta_x : \ x \in S\},$$
$$D^+ := \text{span}_{\mathbf{R}_+}\{\delta_x : \ x \in S\},$$

furthermore define S_{BL} as the closure of D in $BL(S)^*$ and S_{BL}^+ as the closure of D^+.

Proposition 2.27 *If S is a separable metric space, then S_{BL} is separable.*

Proof: Let A be a countable set such that its closure $\overline{A}$ equals S. Consider

$$\Gamma = \text{span}_{\mathbf{Q}}\{\delta_y : \ y \in A\} = \left\{ \sum_{k=1}^{n} \alpha_k \delta_{y_k}, \ n \in \mathbf{N}, \ \alpha_k \in \mathbf{Q}, \ y_k \in A \right\}.$$

We will prove that Γ is countable and $\overline{\Gamma} = S_{BL}$. First we will prove that Γ is countable. Therefore note that

$$\Gamma = \bigcup_{n=1}^{\infty} \left\{ \sum_{k=1}^{n} \alpha_k \delta_{y_k}, \ \alpha_k \in \mathbf{Q}, \ y_k \in A \right\}.$$

We have that $\{\alpha_k \delta_{y_k}, \alpha_k \in \mathbf{Q}, y_k \in A\}$ is countable, since this set equals $\mathbf{Q} \times A$. So it follows that for every $n \in \mathbf{N}$ the set $\{\sum_{k=1}^n \alpha_k \delta_{y_k}, \alpha_k \in \mathbf{Q}, y_k \in A\}$ is countable. Hence also $\cup_{n=1}^\infty \{\sum_{k=1}^n \alpha_k \delta_{y_k}, \alpha_k \in \mathbf{Q}, y_k \in A\}$. So we proved that Γ is countable.

Now we prove $\overline{\Gamma} = S_{BL}$. We obviously have $\Gamma \subset S_{BL}$, and since S_{BL} is closed, we have $\overline{\Gamma} \subset S_{BL}$. So we only need to prove that $S_{BL} \subset \overline{\Gamma}$. Therefore, let $x \in S_{BL}$. Then either $x = \sum_{k=1}^n \alpha_k \delta_{y_k}$, with $n \in \mathbf{N}$, $\alpha_k \in \mathbf{R}$ and $y_k \in S$, or x is a limit point of such elements. First suppose that $x = \sum_{k=1}^n \alpha_k \delta_{y_k}$. Then, for every k there exists a sequence $(\beta_j^{(k)})$ such that $\beta_j^{(k)} \to \alpha_k$, $j \to \infty$, and a sequence $(z_j^{(k)})$ such that $\delta_{z_j^{(k)}} \to \delta_{y_k}$, $j \to \infty$. The latter convergence is in $BL(S)^*$. So for every k we have

$$\begin{aligned}\|\beta_j^{(k)} \delta_{z_j^{(k)}} - \alpha_k \delta_{y_k}\|_{BL^*} &\le \|\beta_j^{(k)} \delta_{z_j^{(k)}} - \alpha_k \delta_{z_j^{(k)}}\| + \|\alpha_k \delta_{z_j^{(k)}} - \alpha_k \delta_{y_k}\| \\ &= \|(\beta_j^{(k)} - \alpha_k) \delta_{z_j^{(k)}}\| + \|\alpha_k (\delta_{z_j^{(k)}} - \delta_{y_k})\| \\ &= |(\beta_j^{(k)} - \alpha_k)| \|\delta_{z_j^{(k)}}\| + |\alpha_k| \|\delta_{z_j^{(k)}} - \delta_{y_k}\|.\end{aligned}$$

The last term converges to 0 as $j \to \infty$, since $\|\delta_{z_j^{(k)}}\|_{BL^*} = 1$ for all j, by Lemma 2.24. It follows that the sequence $(\beta_j^{(k)} \delta_{z_j^{(k)}})$ is a sequence in Γ converging to $\alpha_k \delta_{y_k}$. So we have $x \in \overline{\Gamma}$.
Now suppose $x \in S_{BL}$ where $x = \lim_{n \to \infty} x_m$, with $x_m = \sum_{k=1}^n \alpha_k^{(m)} \delta_{y_k}^{(m)}$. We already proved that $x_m \in \overline{\Gamma}$ for every m, so the sequence (x_m) is actually a sequence in $\overline{\Gamma}$. Since $\overline{\Gamma}$ is closed, it follows that its limit point x is an element of $\overline{\Gamma}$. We conclude that $S_{BL} \subset \overline{\Gamma}$, and hence $\overline{\Gamma} = S_{BL}$. Since Γ is countable, it follows that S_{BL} is separable. ∎

Let $\mathcal{M}_s^+(S)$ the set of all positive Borel measures on S, such that there exists a separable Borel measurable subset $E \subset S$ with $\mu(S \setminus E) = 0$. The following theorem is taken from [13, Theorem 3.9, p11]. The proof can be found there.

Theorem 2.28 $\mathcal{M}_s^+(S) \subset S_{BL}^+$.

Corollary 2.29 *Let (S, d) be a separable metric space. The space $\mathcal{P}(S)$ of all probability measures on S, is a subset of S_{BL}.*

Proof: Since S is separable, we have $\mathcal{M}_s^+(S) = \mathcal{M}^+(S)$. We always have $\mathcal{P}(S) \subset \mathcal{M}^+(S)$, so by Theorem 2.28 it follows that

$$\mathcal{P}(S) \subset \mathcal{M}^+(S) = \mathcal{M}_s^+(S) \subset S_{BL}^+ \subset S_{BL}. \quad ∎$$

It is worth noticing that there exists a metric on $\mathcal{P}(S)$, called the Prokhorov metric [14]. However, we will work with the metric induced by the norm on S_{BL}. These two metric are equivalent, provided that S is separable (see [7, Theorem 11.3.3, p395])
The following theorem gives a relation between the spaces S_{BL}^* and $BL(S)$. The proof can be found in [13, Theorem 3.7, p10].

Theorem 2.30 *S_{BL}^* is isometrically isomorphic to $BL(S)$ under the map $\psi \mapsto T\psi$, where $T\psi(x) = \psi(\delta_x)$.*

2.2 Measures on products of probability spaces

2.2.1 A finite product of probability spaces

In this section we will show that a finite product of measurable spaces is again measurable, and we will state some features. We will also show how to construct a certain kind of probability measure defined on a product of two probability spaces.

Suppose $(X_i, \mathcal{M}_i, \mu_i)$ are measurable spaces for $i = 1, 2, \ldots, n$, and consider $X = \prod_{i=1}^n X_i$. The **product-σ-algebra** is the σ-algebra $\mathcal{M}$ on X generated by

$$\mathcal{E} = \left\{ \prod_{i=1}^{n} E_i : \; E_i \in \mathcal{M}_i \right\}.$$

Lemma 2.31 *The set $\mathcal{E}$ defined above is a semiring.*

Proof: It is obvious that $\emptyset \in \mathcal{E}$. Let $n = 2$. It is clear that for two elements $E = E_1 \times E_2$ and $F = F_1 \times F_2$ we have $E \cap F = E_1 \cap F_1 \times E_2 \cap F_2 \in \mathcal{E}$. Furthermore,

$$E \setminus F = (E_1 \times E_2) \setminus (F_1 \times F_2) = ((E_1 \setminus F_1) \times E_2) \cup ((E_1 \cap F_1) \times (E_2 \setminus F_2)),$$

so the difference is a union of elements in $\mathcal{E}$. Using induction, it should be clear that the equality also holds for larger n. So $\mathcal{E}$ is a semiring. ∎

Lemma 2.32 *Let $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ be two measurable spaces. Suppose that $\mathcal{M}_X$ is generated by a set $\mathcal{E}_X$ and that $\mathcal{M}_Y$ is generated by a set $\mathcal{E}_Y$. If $X \in \mathcal{E}(X)$ and $Y \in \mathcal{E}(Y)$, then the product σ-algebra $\mathcal{M}_{X\times Y}$ of subsets of $X \times Y$ is generated by*

$$\mathcal{E}_{X\times Y} = \{E \times F \mid E \in \mathcal{E}_X, \; F \in \mathcal{E}_Y\}.$$

Proof: The product σ-algebra in $X \times Y$ is by definition

$$\mathcal{M}_{X\times Y} := \mathcal{M}(\{E \times F \mid E \in \mathcal{M}_X, F \in \mathcal{M}_Y\}).$$

So we need to show that $\mathcal{M}(\mathcal{E}_{X\times Y}) = \mathcal{M}_{X\times Y}$. First of all, since $\mathcal{E}_{X\times Y} \subset \mathcal{M}_{X\times Y}$, it follows from Lemma A.3 that $\mathcal{M}(\mathcal{E}_{X\times Y}) \subset \mathcal{M}_{X\times Y}$. So we need to show that

$$\{E \times F | \; E \in \mathcal{M}_X, \; F \in \mathcal{M}_Y\} \subset \mathcal{M}(\mathcal{E}_{X\times Y}).$$

Now note that it suffices to show that

$$\{E \times Y | \; E \in \mathcal{M}_X\} \subset \mathcal{M}(\mathcal{E}_{X\times Y}),$$
$$\{X \times F | \; F \in \mathcal{M}_Y\} \subset \mathcal{M}(\mathcal{E}_{X\times Y}).$$

We will prove the inclusion $\{E \times Y | \; E \in \mathcal{M}_X\} \subset \mathcal{M}(\mathcal{E}_{X\times Y})$, the other inclusion can be shown similarly.

First note that $\{E \times Y | \; E \in \mathcal{E}_X\} \subset \mathcal{E}_{X\times Y}$, since we assumed that $Y \in \mathcal{E}_Y$. So we have

$$\mathcal{M}(\{E \times Y | \; E \in \mathcal{E}_X\}) \subset \mathcal{M}(\mathcal{E}_{X\times Y}).$$

Then note that, by assumption, $\{E \times Y |\ E \in \mathcal{M}_X\} = \mathcal{M}(\mathcal{E}_X) \times \{Y\}$. To prove the inclusion, we will prove the equality

$$\mathcal{M}(\{E \times Y |\ E \in \mathcal{E}_X\}) = \mathcal{M}(\mathcal{E}_X) \times \{Y\}.$$

Set $\mathcal{M}' = \mathcal{M}(\{E \times Y |\ E \in \mathcal{E}_X\})$ and $\Sigma = \{G \subset X |\ G \times Y \in \mathcal{M}'\}$.
We will show that Σ is a σ-algebra. Suppose $E \in \Sigma$. Then E is of the form $E = G \times Y$, and so $E^c = G^c \times Y$. Since $G^c \in \mathcal{M}'$, it follows that $E^c \in \Sigma$. Next, suppose $E_1, E_2, \ldots \in \Sigma$. Then E_i is of the form $E_i = G_i \times Y$ and $\cup_{i=1}^{\infty} G_i = (\cup_{i=1}^{\infty} G_i) \times Y$, so $\cup_{i=1}^{\infty} E_i \in \Sigma$. So indeed, Σ is a σ-algebra. We claim that $\mathcal{M}(\mathcal{E}_X) \subset \Sigma$. Indeed, for $E \in \mathcal{E}_X$, we have that $E \times Y \in \mathcal{M}'$ and hence $E \in \Sigma$. From Lemma A.3, it follows that $\mathcal{M}(\mathcal{E}_X) \subset \Sigma$. We find that

$$\mathcal{M}(\mathcal{E}_X) \times \{Y\} \subset \Sigma \times \{Y\} \subset \mathcal{M}'.$$

So we have proved one inclusion. For the other inclusion, note that $\mathcal{M}(\mathcal{E}_X) \times \{Y\}$ is a σ-algebra. The prove of this statement is the same as how we proved that Σ is a σ-algebra. Furthermore, we have $\{E \times Y |\ E \in \mathcal{E}_X\} \subset \mathcal{M}(\mathcal{E}_X) \times \{Y\}$. So by Lemma A.3, we get the other inclusion

$$\mathcal{M}(\{E \times Y |\ E \in \mathcal{E}_X\} \subset \mathcal{M}(\mathcal{E}_X) \times \{Y\}.$$

We conclude that we indeed have an equality

$$\mathcal{M}(\{E \times Y |\ E \in \mathcal{E}_X\} = \mathcal{M}(\mathcal{E}_X) \times \{Y\}.$$

So, since we have this equality, we have

$$\{E \times Y |\ E \in \mathcal{M}_X\} \subset \mathcal{M}(\mathcal{E}_{X \times Y}).$$

Note that in the proof of the other inclusion, $\{X \times F |\ F \in \mathcal{M}_Y\} \subset \mathcal{M}(\mathcal{E}_{X \times Y})$, we use that $X \in \mathcal{E}_X$. Now these two inclusions give us that $\mathcal{M}_{X \times Y} \subset \mathcal{M}(\mathcal{E}_{X \times Y})$, and we conclude that $\mathcal{M}_{X \times Y} = \mathcal{M}(\mathcal{E}_{X \times Y})$. ∎

Since we have $\mathcal{M}_{X_1 \times X_2 \times X_3} = \mathcal{M}_{(X_1 \times X_2) \times X_3}$, it is now obvious that Lemma 2.32 can be extended to any finite product.

Corollary 2.33 *Let $(X_i, \mathcal{M}_i)$, $i = 1, \ldots, n$ be n measurable spaces, such that $\mathcal{M}_i$ is generated by a set $\mathcal{E}_i$. Set $X = \prod_{i=1}^{n} X_i$. If $X_i \in \mathcal{E}_i$ for all i, then the product σ-algebra $\mathcal{M}_X$ in X is generated by*

$$\mathcal{E}_X = \left\{ \prod_{i=1}^{n} E_i |\ E_i \in \mathcal{E}_i \right\}.$$

Given the product-σ-algebra, the **coordinate map** $\pi_i : X \to X_i$ defined by $\pi_i((x_n)) = x_i$, is a measurable function. This is so, since for $E_i \in \mathcal{M}_i$ we have $\pi_i^{-1}(E_i) = X_1 \times X_2 \times \ldots \times X_{i-1} \times E_i \times X_{i+1} \times \ldots \times X_n$, and the right hand side is an element of $\mathcal{E}_X$, hence $\pi_i^{-1}(E_i) \in \mathcal{M}(\mathcal{E}_X)$.

Proposition 2.34 *Let $(X_1, d_1), \dots, (X_n, d_n)$ be metric spaces, and consider (X, d) where $X = \prod_{i=1}^n X_i$, and d the product metric as in Lemma 2.9. If $X_1, \dots, X_n$ are separable, then the Borel-σ-algebra of X equals the product σ-algebra of the Borel-σ-algebras of the X_i, $i = 1, \dots, n$.*

Proof: Let $\mathcal{E}$ be the collection of open sets in X, and $\mathcal{E}_j$ the collection of open sets in X_j. Suppose that C_j is a countable dense subset in X_j. Let $\mathcal{F}_j$ be the collection of all open balls in X_j with center $x_j \in C_j$ and radius $r \in \mathbf{Q}$. Then observe that $\mathcal{F}_j$ is countable. Set

$$\mathcal{F} = \left\{ \prod_{j=1}^n F_j : \; F_j \in \mathcal{F}_j \right\}.$$

We will show that $\mathcal{M}_{X_j}$ is generated by $\mathcal{F}_j$. First note that $\mathcal{F}_j \subset \mathcal{E}_j$. So $\mathcal{M}(\mathcal{F}_j) \subset \mathcal{M}(\mathcal{E}_j)$. Note that every open set in X_j is a countable union of elements in $\mathcal{F}_j$. From this it follows that $\mathcal{E}_j \subset \mathcal{M}(\mathcal{F}_j)$. Since by definition we have that $\mathcal{M}(\mathcal{E}_j)$ is the smallest σ-algebra containing $\mathcal{E}_j$, we have that $\mathcal{M}(\mathcal{E}_j) = \mathcal{M}(\mathcal{F}_j)$. So indeed $\mathcal{M}_{X_j}$ is generated by $\mathcal{F}_j$. It follows from the previous Corollary 2.33 that $\mathcal{M}_X$ is generated by $\mathcal{F}$.

To finish the proof, we will show that $\mathcal{B}(X)$ is also generated by $\mathcal{F}$. This is similar as the first part of the proof, and we will first show that $\mathcal{F} \subset \mathcal{E}$. Suppose $E \subset X$ is of the form $E = \prod_{j=1}^n E_j$ with $E_j \in \mathcal{E}_j$. Let $x = (x_1, \dots, x_n) \in E$. Then, for every $j = 1, \dots, n$, there is a positive number $r_j > 0$ such that $B(r_j, x_j) \subset E_j$. Set $r = \min_{1 \le j \le n} r_j$. It follows that $B(r, x) \subset E$, so we have that $E \in \mathcal{E}$. The inclusion now follows since

$$\mathcal{F} \subset \left\{ \prod_{j=1}^n E_j : \; E_j \in \mathcal{E}_j \right\} \subset \mathcal{E}.$$

So we have $\mathcal{M}(\mathcal{F}) \subset \mathcal{M}(\mathcal{E})$. Let C be the collection of elements in X of the form $x = (x_1, \dots, x_n)$ where $x_j \in C_j$ for all j. Note that C is a countable dense subset of X. It follows that every open set $E \in \mathcal{E}$ is a union of open balls with radius $r \in \mathbf{Q}$ and center $x \in C$. This is again a countable union. Furthermore, note that every such open ball is an element in $\mathcal{F}$. It follows that $\mathcal{E} \subset \mathcal{M}(\mathcal{F})$, and we conclude that $\mathcal{M}(\mathcal{F}) = \mathcal{M}(\mathcal{E})$. Hence

$$\mathcal{M}_X = \mathcal{M}(\mathcal{F}) = \mathcal{M}(\mathcal{E}) = \mathcal{B}(X). \quad \blacksquare$$

For a subset E of a product $X_1 \times X_2$ of measurable spaces, and $x_1 \in X_1$, we define the x**-section** E_{x_1} of E by

$$E_{x_1} := \{x_2 \in X_2 \mid (x_1, x_2) \in E\}.$$

Now we will show that there exists a certain measure μ on a product of two probability spaces, which will be important later. First a lemma. For any two measurable spaces $(X_1, \mathcal{M}_1)$ and $(X_2, \mathcal{M}_2)$ we will denote the product-σ-algebra on $X_1 \times X_2$ by $\mathcal{M}_1 \otimes \mathcal{M}_2$.

Lemma 2.35 *Let $(X_1, \mathcal{M}_1)$ and $(X_2, \mathcal{M}_2)$ be two measurable spaces. Suppose that for every $x \in X_1$ there is a probability measure μ_x on X_2 such that for every $E \in \mathcal{M}_2$ we have that the function $f : X_1 \to [0, \infty]$ given by $f(x) = \mu_x(E)$ is measurable. Then for every $B \in \mathcal{M}_1 \otimes \mathcal{M}_2$ the function $g : X_1 \to [0, \infty]$ given by $g(x) = \mu_x(B_x)$ is measurable.*

Proof: Set

$$\Sigma = \{B \in \mathcal{M}_1 \otimes \mathcal{M}_2 : \ x \mapsto \mu_x(B_x) \text{ is measurable}\}.$$

Then observe that if $B = E_1 \times E_2$ with $E_1 \in \mathcal{M}_1$, $E_2 \in \mathcal{M}_2$, then $B \in \Sigma$, by assumption. We will show that Σ is a σ-algebra.
It is clear that $\emptyset \in \Sigma$, since $\mu_x(\emptyset) = 0$ for all $x \in X$, hence the map $x \mapsto \mu_x(\emptyset)$ is a constant function, and thus measurable. Now suppose $B_n \in \Sigma$, for $n = 1, 2, \ldots$, where the B_n are disjoint. It is obvious that $(\bigcup_{n=1}^{\infty} B_n)_x = \bigcup_{n=1}^{\infty} (B_n)_x$, where the latter is also a disjoint union. It follows that

$$\mu_x\left(\left(\bigcup_{n=1}^{\infty} B_n\right)_x\right) = \mu_x\left(\bigcup_{n=1}^{\infty} (B_n)_x\right) = \sum_{n=1}^{\infty} \mu_x((B_n)_x).$$

Since the map $x \mapsto \sum_{n=1}^{\infty} \mu_x((B_n)_x)$ is measurable, we see that $\bigcup_{n=1}^{\infty} B_n \in \Sigma$.
If $B \in \Sigma$, then $(B_x)^c = (B^c)_x$. Since no confusion can arise, we will write B_x^c for either set. Because $\mu_x(B_x^c) + \mu_x(B_x) = \mu_x(X_2) = 1$ we see that the function $x \mapsto \mu_x(B_x^c) = 1 - \mu_x(B_x)$ is measurable. Note that we really need that $\mu_x(X_2) < \infty$, otherwise we could get problems when $\mu_x(B_x) = \infty$. Anyhow, Σ is a σ-algebra. But since $\mathcal{M}_1 \otimes \mathcal{M}_2$ is by definition the smallest σ-algebra which includes rectangles, we conclude that $\Sigma = \mathcal{M}_1 \otimes \mathcal{M}_2$. This means that the map $x \mapsto \mu_x(B_x)$ is measurable for every $B \in \mathcal{M}_1 \otimes \mathcal{M}_2$. ∎

The following proposition, as mentioned before, proves the existence of a certain measure defined on a product of probability space.

Proposition 2.36 *Let $(X_1, \mathcal{M}_1, P)$ be a probability space and $(X_2, \mathcal{M}_2)$ a measurable space. Suppose that for each $x \in X_1$, μ_x is a probability measure on $(X_2, \mathcal{M}_2)$ and that the function $x \mapsto \mu_x(E)$ is measurable for each $E \in \mathcal{M}_2$. Then for any $B \in \mathcal{M}_1 \otimes \mathcal{M}_2$ the repeated integral*

$$I_B = \int_{X_1} \int_{X_2} \mathbf{1}_B(x_1, x_2) \, d\mu_{x_1}(x_2) \, dP(x_1)$$

is well-defined and there exists a measure μ on $(X_1 \times X_2, \mathcal{M}_1 \otimes \mathcal{M}_2)$ that extends $B \mapsto I_B$.

Proof: First note that if $x_1 \in X_1$ is fixed, then

$$\mathbf{1}_B(x_1, x_2) = \begin{cases} 1 & \text{if } x_2 \in B_{x_1} \\ 0 & \text{if } x_2 \notin B_{x_1} \end{cases}$$

It follows that

$$\int \mathbf{1}_B(x_1, x_2) d\mu_{x_1}(x_2) = \mu_{x_1}(B_{x_1}).$$

By the previous Lemma 2.35, we have that $x \mapsto \mu_{x_1}(B_x)$ is measurable, thus the integral

$$\int \mu_{x_1}(B_x) dP(x)$$

is well-defined. This means that I_B is well-defined. We will show that μ defined as $\mu(B) = I_B$, $B \in \mathcal{M}_1 \otimes \mathcal{M}_2$, is a measure. It is obvious that $\mu(\emptyset) = 0$. Let B_n, $n = 1, 2, \ldots$ be disjoint measurable sets, and set $B = \bigcup_{n=1}^{\infty} B_n$. By the Monotone Convergence Theorem, we have

$$\mu(B) = \sum_{n=1}^{\infty} \mu(B_n).$$

We conclude that μ is a measure. ∎

There is also another way to show the existence of the measure μ using Proposition A.9. Let $\mathcal{E}$ be the collection of all sets of the form $E_1 \times E_2$, where $E_1 \in \mathcal{M}_1$ and $E_2 \in \mathcal{M}_2$. By Lemma 2.31, it follows that $\mathcal{E}$ is a semiring. Define $\alpha : \mathcal{E} \to [0, \infty]$ as

$$\alpha(E) = \iint \mathbf{1}_E(x_1, x_2)\ d\mu_{x_1}(x_2)\ dP(x_1).$$

Suppose $E = \cup_{j=1}^{n} E_j$, with $E_j \in \mathcal{E}$ disjoint. Then we have $\mathbf{1}_E(x_1, x_2) = \sum_{j=1}^{n} \mathbf{1}_{E_j}(x_1, x_2)$, and by linearity of the integral, we get

$$\begin{aligned} \alpha(E) &= \iint \mathbf{1}_E(x_1, x_2)\ d\mu_{x_1}(x_2)\ dP(x_1) \\ &= \sum_{j=1}^{n} \iint \mathbf{1}_{E_j}(x_1, x_2)\ d\mu_{x_1}(x_2)\ dP(x_1) = \sum_{j=1}^{n} \alpha(E_j). \end{aligned}$$

Thus α is additive. By the Monotone Convergence Theorem, α is also countably additive. Now using Proposition A.9, we conclude that there exists a measure μ_1 defined on the σ-algebra generated by $\mathcal{E}$ that extends α on E.
So far we have shown that there exists a measure μ_1 which has the desired form for elements of the form $E_1 \times E_2$. However, it is not clear that it has the same desired form for arbitrary sets $B \in \mathcal{M}_1 \otimes \mathcal{M}_2$., i.e., it is not clear that we have

$$\mu_1(B) = \int_{X_1} \int_{X_2} \mathbf{1}_B(x_1, x_2)\ d\mu_{x_1}(x_2)\ dP(x_1).$$

Fortunately, we can use Lemma A.13, since both measures coincide on the π-system of sets of the form $E_1 \times E_2$, $E_1 \in \mathcal{M}_1$ and $E_2 \in \mathcal{M}_2$, to conclude that $\mu = \mu_1$. So the measure μ_1 constructed using Proposition A.9 is the same measure as constructed in the above proof.

2.2.2 An infinite product of probability spaces

In this section we will show how to construct a certain kind of probability measure defined on an infinite product of probability spaces. For $n = 1, 2, \ldots$, let $(X_n, \mathcal{M}_n, \mu_n)$ be probability spaces. Let X be the Cartesian product $X = \prod_{n=1}^{\infty} X_n$. We will call a subset $\prod_{j=1}^{\infty} A_j \subset X$ a **rectangle** when $A_n = X_n$ for all but finitely many n. Let $\mathcal{D}$ be the set of all rectangles.

Lemma 2.37 *The set of all rectangles $\mathcal{D}$ together with $\emptyset$ is a semiring. The algebra $\mathcal{A}$ generated by $\mathcal{D}$ equals the collection of finite disjoint unions of elements of $\mathcal{D}$.*

Proof: First, consider the collection $\mathcal{E}$ of all rectangles such that $\prod_{j=1}^{\infty} A_j$ such that $A_j = X_j$ for $j = 3, 4, \ldots$. The conditions of a semiring are then fulfilled, as is proved in Lemma 2.31. With induction, one can prove that this holds for arbitrary rectangles. ∎

The ring $\mathcal{R}$ generated by $\mathcal{D}$ is the set of all finite disjoint unions of rectangles.
We will define the **product σ-algebra** on X as the σ-algebra generated by $\mathcal{D}$. This σ-algebra is the same as the σ-algebra generated by

$$\mathcal{E} = \left\{ \prod_{i=1}^{\infty} E_i : \ E_i \in \mathcal{M}_i \right\}.$$

To see this, first note that we have $\mathcal{D} \subset \mathcal{E}$, so $\mathcal{M}(\mathcal{D}) \subset \mathcal{M}(\mathcal{E})$. For the other inclusion, let $E \in \mathcal{E}$. Then E is of the form $E = \prod_{i=1}^{\infty} E_i$. Then $E = \cap_{i=1}^{\infty} F_i$ where $F_i = E_1 \times E_2 \times \ldots \times E_i \times X_{i+1} \times X_{i+2} \times \ldots$. By construction, $F_i \in \mathcal{D}$. So we have $E \in \mathcal{M}(\mathcal{D})$. It follows that $\mathcal{E} \subset \mathcal{M}(\mathcal{D})$, and hence, by Lemma A.3, $\mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{D})$. We conclude that $\mathcal{M}(\mathcal{E}) = \mathcal{M}(\mathcal{D})$. We will write $\mathcal{M}_X$ for the σ-algebra generated by rectangles on X.

Also on an infinite product of probability spaces, the coordinate map $\pi_i : X \to X_i$ is measurable. For $E_i \in \mathcal{M}_i$ we have $\pi_i^{-1}(E_i) = X_1 \times X_2 \times \ldots \times X_{i-1} \times E_i \times X_{i+1} \times \ldots$, and the right hand side is a rectangle, hence $\pi_i^{-1}(E_i) \in \mathcal{M}(\mathcal{E}_X)$.

The main Theorem is now given, to assure us the existence of a measure P defined on an infinite product of probability spaces.

Theorem 2.38 *Let $(X_j, \mathcal{S}_j)$ be measurable spaces for $j = 1, 2, \ldots$, and $P^{(1)}$ be a probability measure on $(X_1, \mathcal{S}_1)$. Suppose that for each $n = 1, 2, \ldots$ we have that $p(x_n, \cdot)$ is a probability measure on $(X_{n+1}, \mathcal{S}_{n+1})$, such that for each $E \subset \mathcal{S}_{n+1}$, the function $x \mapsto p(x, E)$ is measurable. Then the map $P^{(2)}$ defined on $\mathcal{S}_1 \otimes \mathcal{S}_2$ as*

$$P^{(2)}(B) = \iint \mathbf{1}_B(x_1, x_2) p(x_1, dx_2) dP^{(1)}(x_1)$$

is a probability measure. Furthermore, for larger n, the function $P^{(n+1)}$ on $\mathcal{S}_1 \otimes \ldots \otimes \mathcal{S}_{n+1}$ defined recursively as

$$P^{(n+1)}(B) = \iint \mathbf{1}_B(x_1, \ldots, x_n) p(x_n, dx_{n+1}) dP^{(n)}(x_1, \ldots x_n)$$

are probability measures for all n. Also, there exists a unique probability measure P on the product space $X = \prod_{j=1}^{\infty} X_n$ such that for each n and each $B \in \mathcal{S}_1 \otimes \ldots \otimes \mathcal{S}_n$ we have

$$P(B \times X_{n+1} \times X_{n+2} \times \ldots) = P^{(n)}(B).$$

Proof: For every n, we already know that $P^{(n)}$ is a probability measure by Proposition 2.36. We only need to show that P is a well-defined measure. Therefore, consider the algebra $\mathcal{A}$ generated by the collection of rectangles $\mathcal{D}$. By Lemma 2.37, an element $A \in \mathcal{A}$ is a finite disjoint union of rectangles. So $A = \bigcup_{k=1}^{p} B_k$, with B_k a rectangle. So for every k, B_k is a product of sets $B_{kr} \in \mathcal{S}_r$ such that $B_{kr} = X_r$ for all $r \geq n(k)$ with $n(k) < \infty$. Set $m = \max(n(1), \ldots, n(p))$. Then A can be written as

$$A = A^{(m)} \times X_{m+1} \times X_{m+2} \times \ldots,$$

where $A^{(m)} \in \mathcal{S}_1 \otimes \ldots \otimes \mathcal{S}_m$. For this A, define

$$P_0(A) = P^{(m)}(A^{(m)}).$$

This definition is not ambiguous, since we have

$$\begin{aligned} P^{(m+1)}(A^{(m)} \times X_{m+1}) &= \iint \mathbf{1}_{A^{(m)} \times X_{m+1}} p(x_m, dx_{m+1}) dP^{(m)}(x_1, \ldots, x_m) \\ &= \int_{A^{(m)}} p(x_m, X_{m+1}) dP^{(m)}(x_1, \ldots, x_m) \\ &= P^{(m)}(A^{(m)}), \end{aligned}$$

and so, more general, we have

$$\begin{aligned} P^{(m+k)}(A^{(m)} \times X_{n+1} \times \ldots \times X_{m+k}) &= P^{(m+k-1)}(A^{(m)} \times X_{n+1} \times \ldots \times X_{m+k-1}) \\ &= \ldots \\ &= P^{(m)}(A^{(m)}). \end{aligned}$$

It is obvious that $P_0(\emptyset) = 0$. Suppose $(A_j)_{j=1}^{\infty}$ is a sequence of finite disjoint sets, such that $A = \cup_{j=1}^{\infty} A_j \in \mathcal{A}$. Then A is again a union of rectangles, and there is again an $m \in \mathbf{N}$ such that $A = A^{(m)} \times X_{m+1} \times X_{m+2} \times \ldots$, with $A^{(m)} \in \mathcal{S}_1 \otimes \ldots \otimes \mathcal{S}_m$. Also, each A_j has a similar expression, with the same m, say $A_j = A_j^{(m)} \times X_{m+1} \times X_{m+2} \times \ldots$. Since $P^{(m)}$ is a measure, it follows that

$$P_0(A) = P^{(m)}(A^{(m)}) = P^{(m)}\left(\bigcup_{j=1}^{\infty} A_j^{(m)}\right) = \sum_{j=1}^{\infty} P^{(m)}(A_j^{(m)}) = \sum_{j=1}^{\infty} P_0(A_j).$$

So we see that P_0 is a premeasure on $\mathcal{A}$. Now, by Theorem A.16 there is a measure P on the σ-algebra generated by $\mathcal{A}$, such that $P = P_0$ on $\mathcal{A}$. We claim that P is the desired measure. So we need to show that, given $B \in \mathcal{S}_1 \otimes \ldots \otimes \mathcal{S}_n$, we have

$$P(B \times X_{n+1} \times \ldots) = P^{(n)}(B).$$

Fix n, and consider the collection Σ of sets of the form $B \times \prod_{k>n} X_k$, with $B \in \mathcal{S}_1 \otimes \ldots \otimes \mathcal{S}_n$. Since $\mathcal{S}_1 \otimes \ldots \otimes \mathcal{S}_n$ is a σ-algebra, it follows directly that Σ is a σ-algebra. It is a sub-σ-algebra of the σ-algebra on $\prod_{n=1}^{\infty} X_n$, and P_Σ, P restricted to Σ, is still a probability measure. Furthermore, we have $P_\Sigma = P^{(n)}$ on the set of n-dimensional rectangles, which is a π-system, hence by Lemma A.13, $P_\Sigma = P^{(n)}$ on $\mathcal{S}_1 \otimes \ldots \otimes \mathcal{S}_n$. Since we have taken n arbitrarily, we conclude that

$$P(B \times X_{n+1} \times \ldots) = P^{(n)}(B)$$

for $B \in \mathcal{S}_1 \otimes \ldots \otimes \mathcal{S}_n$. If there is another probability measure $\tilde{P}$ with the above property, then $P = \tilde{P}$ by Lemma A.13, since they coincide on the π-system of rectangles. ▮

2.3 A lemma concerning joint measurability

Let $(X, \mathcal{M})$ be a measurable space, (S_1, d_1) and (S_2, d_2) be two metric spaces such that S_1 is separable. For a function $f : X \times S_1 \to S_2$ to be measurable, it is in general not sufficient that $s \mapsto f(x, s)$ for fixed $x \in X$ and $x \mapsto f(x, s)$ for fixed $s \in S_1$ are measurable. A counterexample can e.g. be found in [1, exercise 3.10.49]. The following lemma states that, for $S_1 = I$, where $I \subset \mathbf{R}$ is a left-open interval, it is sufficient to assume that the function $s \mapsto f(x, s)$ is left-continuous, instead of only measurable. The same holds for a right-continuous function when I is a right-open interval.

Lemma 2.39 *Let $(X, \mathcal{M})$ be a measurable space, and let (S, d) be a metric space. Let I be a left-open interval, i.e., $I = (a, b)$, $(a, b]$ or (a, ∞), $b \in \mathbf{R}$. Suppose a function $f : X \times I \to S$ satisfies the following conditions.*

i) For every $t \in I$, the function $x \mapsto f(x, t)$ is measurable;

ii) For every $x \in X$, the function $t \mapsto f(x, t)$ is left-continuous.

Then the function f is measurable in the product σ-algebra of $\mathcal{M} \otimes \mathcal{B}(I)$ in $X \times I$.

Proof: Suppose that $t \mapsto f(x, t)$ is left-continuous for all $x \in X$. For every $n \in \mathbf{N}$, we can make a partition of the interval $(0, 1]$ into 2^n disjoint intervals: $(0, 1] = \bigcup_{k=1}^{2^n} ((k-1)2^{-n}, k2^{-n}]$. Set

$$f_n(x, t) = f(x, m + k2^{-n}), \text{ if } t \in \big(m + k2^{-n}, m + (k+1)2^{-n}\big] \cap I,$$

where $m \in \mathbf{Z}, k = 0, \ldots, 2^n - 1$. We claim that $\lim_{n\to\infty} f_n(x, t) = f(x, t)$ for all (x, t). To see this, fix (x, t) and let $\varepsilon > 0$. Since $t \mapsto f(x, t)$ is left-continuous, there must exist a $\delta > 0$, such that for all $s \in (t - \delta, t]$ we have $d(f(x, t), f(x, s)) < \varepsilon$. Fix N such that $2^{-N} < \delta$. Then for each $n \geq N$, there exist $m_n \in \mathbf{Z}$ and $k_n \in \{0, \ldots, 2^n - 1\}$ such that

$$t - \delta < m_n + k_n 2^{-n} < t \leq m + (k+1)2^{-n}.$$

From this observation, it follows that $d(f(x, t), f_n(x, t)) < \varepsilon$ for all $n \geq N$. Thus indeed $\lim_{n\to\infty} f_n(x, t) = f(x, t)$.
Now we will show that f_n is measurable in the product space $X \times I$ for every n.

Therefore, let E be open in S. By proposition A.11, it suffices to show that $f_n^{-1}(E)$ is measurable in $X\times I$. For $m \in \mathbf{Z}$ and $k \in \{0,\dots,2^n-1\}$ we have by the measurability of $x \mapsto f(x,t)$ that

$$E_{mk} = \{x \in X : f(x, m+k2^{-n}) \in E\}$$

is measurable. it follows that

$$f_n^{-1}(E) = \bigcup_{m\in\mathbf{Z}} \bigcup_{k=0}^{2^n-1} \left(E_{mk} \times [m+k2^{-n}, m+(k+1)2^{-n})\right),$$

hence $f_n^{-1}(E)$ is measurable in $X \times I$. So f_n is measurable, and since the limit of a sequence of measurable functions with values in a metric space is again measurable (see [2, Cor. 6.2.6, p11]), we conclude that f is measurable. ∎

As mentioned above, a similar result holds for a right-continuous function $x \mapsto f(x,t)$, whenever I is right-open. The proof is similar.
When one replaces the left-open (or right-open) interval I by a general separable metric space (S, d_1), the statement remains valid once the right-continuity or left-continuity is replaced by continuity. The proof goes analogously, only one has to replace the intervals $(m+k2^{-n}, m+(k+1)2^{-n}]$ by a countable number of disjoint sets that cover S. It is possible to find such a cover, since S is assumed to be separable.

3 The stochastic process of an individual's fate

A brief outline of what is to be discussed in this section is now given. In section 3.1, we will give the data for the process and give conditions on them. The data are in the form of one stochastic process and two probability distributions. The two distributions have hidden stochastic processes behind them. Using the data, we expect the sample trajectories of the total process to be càdlàg trajectories. The stochastic process that describes the fate of an individual is a map

$$X : \mathbf{R}_+ \times \mathcal{D}_S \to S,$$

such that for every $t \in \mathbf{R}_+$, the function

$$X(t, \bullet) : \mathcal{D}_S \to S,$$

is measurable, together with a probability measure $\tilde{P}$ on $(\mathcal{D}_S, \mathcal{B}(\mathcal{D}_S))$. The measurability of the map $X(t, \bullet)(f) := \mathrm{ev}_t(f)$ follows from Theorem 2.22. The process satisfies the conditions of the data by construction of $\tilde{P}$. This is to be seen as follows. The process X induces a function $\Phi_X : \mathcal{D}_S \to S^{\mathbf{R}_+}$, where $S^{\mathbf{R}_+}$ is the set of all functions from $\mathbf{R}_+$ to S, such that

$$(\Phi_X(f))(t) := X_t(f).$$

The law $\mathcal{L}_X$ of the process, which is in fact the probability distribution of the sample trajectories, is by definition the measure defined on $S^{\mathbf{R}_+}$ (which is a measurable space), such that for $E \subset S^{\mathbf{R}_+}$ measurable,

$$\mathcal{L}_X(E) := \tilde{P}(\Phi_X^{-1}(E)).$$

In our case, since $X_t(f) = f(t)$, it follows that Φ_X is the identity map, hence $\mathcal{L}(E) = \tilde{P}(E)$.

This means that the existence of the stochastic process X which satisfies the conditions in Section 3.1, relies on the existence of the probability measure $\tilde{P}$ which satisfies the same conditions. The existence of this measure will be proved by constructing it explicitly. In fact, the existence of another measure P will be proved using Theorem 2.38, and the existence of $\tilde{P}$ will follow from the existence of P.

In Section 3.2 we will code the càdlàg trajectories between the jumps as elements $(f, \tau, t) \in X = \mathcal{D}_S \times (0, \infty] \times \mathbf{R}_+$, where

- f is the sample trajectory between jumps, as element in $\mathcal{D}_S$;
- τ the duration of the trajectory, that is, the time difference between the two jumps;
- t the absolute starting time of the trajecory.

In this way, we get a discrete time Markov process of sample trajectories, since the n-th trajectory only depends on the $(n-1)$-th trajectory.
The total trajectory will be viewed as an element in the measurable space $X_\partial^{\mathbf{N}}$, defined in Section 3.2, and it will be discussed that this trajectory actually belongs

to a subset $R \subset X_\partial^{\mathbf{N}}$, R standing for the set of realistic trajectories, and it will be proved in Section 3.2.3 that this subset is measurable. Then in Section 3.2.4 the concatenation map will be defined and it will be proved that this map is measurable. In Section 3.3 the discrete time Markov process of sample trajectories will be made rigorous by constructing its law P, as mentioned before, using Theorem 2.38. It will be proved that P is concentrated on the set R of realistic trajectories, by which we mean that $P(R) = 1$.
In Section 3.4 we will give a summary of the two processes thus far, and we will give a description for the distribution μ_t. In words, given an initial distribution μ_0 on S and a measurable $E \subset S$, $\mu_t(E)$ is the probability that a trajectory will go through E at time t.

3.1 Fundamental data for the process and its technical conditions

As explained in the introduction, the individual is subject to three stochastic processes:

(1) Process 1 is called the partial trajectory. For any initial distribution μ_0 of the state, (i.e. $\mu_0 \in \mathcal{P}(S)$), there is a stochastic process in $\mathbf{R}_+$ on S, with càdlàg sample trajectories. The law of this process is defined on the càdlàg functions $\mathcal{D}_S$ and will be denoted as P_{μ_0}.

(2) Process 2 is called the jump time. Given a sample trajectory $f \in \mathcal{D}_S$, there is a distribution $p_f^\varepsilon \in \mathcal{P}((0,\infty])$ of the jump time. This is actually a random variable on $(0,\infty]$, but can be considered as the stopping time of some hidden stochastic process. For this reason we stick to call it a stochastic process, although this may be somewhat misleading.
Furthermore, p_f^ε does not only depend on f, but also on its 'environment' ε. This environment itself is another state space, but we will not use this space in any way, since we only consider the evolution of the individual. The environment will be important when 'summing' the models of the individuals in an appropriate way.

(3) Process 3 is called the jump in state space. Given a jump-time $t \in \mathbf{R}$ (note that we take $t < \infty$), and the state $s \in S$ right before the jump, there is a distribution $\nu = \nu(t,s)$ on the state space S in such a way that $\nu(E)$, $E \subset S$ measurable, is the probability that the new sample trajectory starts in E.
Since this new initial distribution depends on the jump time and on the state before the jump, there exists a function $\Psi : \mathbf{R}_+ \times S \to \mathcal{P}(S)$ such that $\Psi(t,s) = \nu$ as above.
This random variable may be viewed as a stochastic process, with càdlàg sample trajectories, which is stopped almost immediately after it started . The random variable is obtained as the duration of the process tends to 0.

Note that the distribution of the jump time $p_f^\varepsilon \in \mathcal{P}((0,\infty])$, is dependent of the sample trajectory $f \in \mathcal{D}_S$, while the future of the evolution of the cell should not be of any influence regarding the cell division. Therefore, we could introduce the

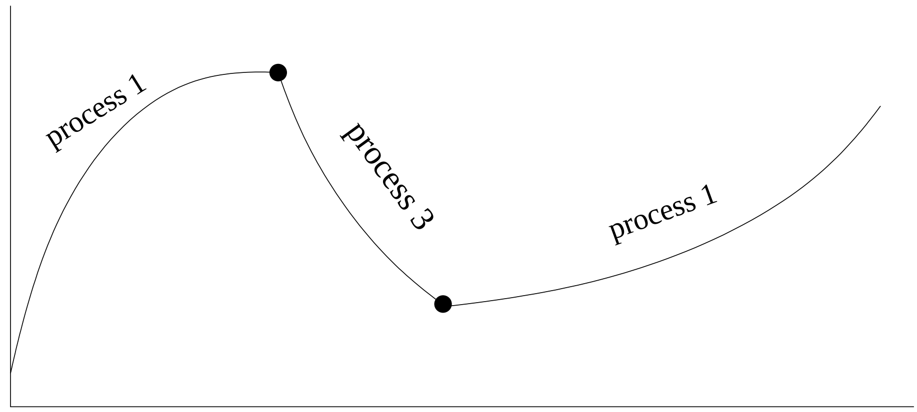

Figure 2: Process 3 as a 'slow' process

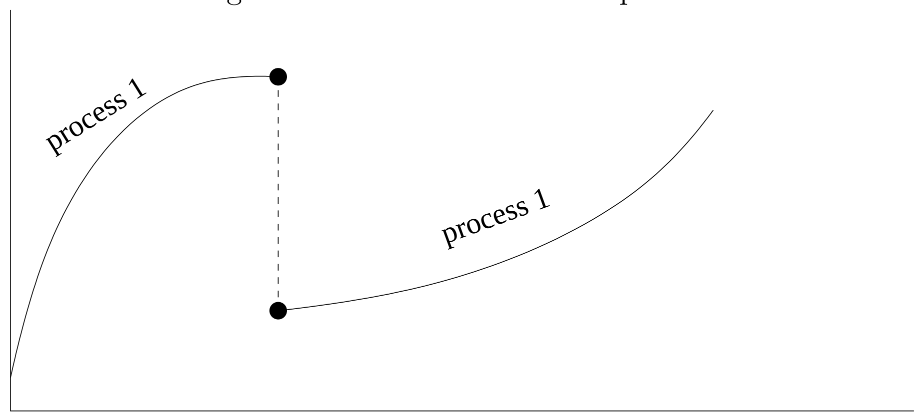

Figure 3: Process 3 as a 'fast' process

'no anticipation of the future' rule. That is, if $f(t) = g(t)$ for all $t \in (0, T)$, then for all measurable $E \subset (0, T)$ we have $p_f^{\varepsilon}(E) = p_g(E)$. However, in the sequel, this extra assumption does not simplify any of our computations. We did feel obliged to mention it, since this may be useful in any further research.

So, summarizing, there is a random sample trajectory, a random jump time, and a random state at which the individual starts over, after jumping. The second process stops the first process and initiates the third process. The third process restarts process one. We refer to a sample trajectory of process one, stopped by process two, as a **partial trajectory**.

Next, we will give some technical conditions on the given processes.

- We assume that for every measurable $E \subset \mathcal{D}_S$, the function $\nu \mapsto P_\nu(E)$ from $\mathcal{P}(S) \to \mathbf{R}_+$ is measurable. This assumption can be achieved by, for instance, assuming that the function $\nu \mapsto P_\nu$ from $\mathcal{P}(S)$ to $(\mathcal{D}_S)_{BL}$ is strongly measurable. (See Section 5.1)

- The first condition on p_f^{ε} is the following. For any environment ε and any function $f \in \mathcal{D}_S$, the distribution p_f^{ε} will be such that almost surely finitely many jumps will be made within a finite amount of time. To realize this, we assume that there exist numbers $T > 0$ and $0 \leq \delta < 1$ such that for all environments ε and all $f \in \mathcal{D}_S$ we have

$$p_f^\varepsilon((0,T]) \leq \delta. \tag{2}$$

- The second condition on p_f^ε is that for every measurable $E \subset (0, \infty]$, the function $f \mapsto p_f^\varepsilon(E)$ from $\mathcal{D}_S$ to $[0,1]$ is measurable. This assumption can achieved by, for instance, assuming that the map $f \mapsto p_f^\varepsilon$, from $\mathcal{D}_S$ to $\mathcal{P}((0,\infty])$ is strongly measurable. Here, the space $\mathcal{P}((0,\infty])$ is a measurable space when equipped with the Borel-σ-algebra of the Banach space $((0,\infty])^+_{BL}$. See Section 2.1.3 and in particular Corollary 2.29 for details.
- On $\Psi : \mathbf{R}_+ \times S \to \mathcal{P}(S) \subset S_{BL}$ we assume that the function is jointly measurable. A sufficient condition is that

  i) $\Psi(t, \cdot) : S \to \mathcal{P}(S)$ is measurable,
  ii) $\Psi(\cdot, x) : \mathbf{R}_+ \to \mathcal{P}(S)$ is right-continuous.

  One can then apply Lemma 2.39 to show that Ψ is measurable. Although the proof of this Lemma is for left-open intervals, with left-continuous functions, the result holds also for right-open intervals and right-continuous functions.

3.2 Coding of sample trajectories

In the sequel, when we mention 'jump', we will mean a jump caused by process 2 and 1, and not the discontinuities in the partial càdlàg sample trajectories of process 1. When several jumps occur, the orbit, as a function $f : \mathbf{R}_+ \to S$ is an element of $\mathcal{D}_S$. The part between two jumps will play an important role. We could assume that the part between two jumps is continuous, but since the metric on $\mathcal{D}_S$ is complicated, we do not want to make things 'worse' by considering the subset $\mathcal{C}_S$ of continuous functions instead of the full space $\mathcal{D}_S$. Furthermore, it is more natural to allow these discontinuities at a smaller scale.

3.2.1 The spaces X_∂ and $X_\partial^{\mathbf{N}}$

We define the space $X := \mathcal{D}_S \times (0,\infty] \times \mathbf{R}_+$ of partial trajectories, so that $x = (f, \tau, t)$ codes for an orbit f with duration τ and absolute starting time t. A full trajectory is then considered as a sequence of elements in X. So, for instance, the second partial trajectory in Figure 3 will be coded as $x_2 = (g, 3.7, 2.6)$. Some conditions need to be met. Because we do not want to exclude a priori that there may be a partial trajectory with $\tau = \infty$, while keeping the convention of coding a full trajectory by a sequence, we add a point ∂ to X which codes for a terminal state after a trajectory of infinite length. The space $X \cup \{\partial\}$ will be denoted by X_∂.

So every trajectory, that is, every càdlàg function, can be viewed as a sequence in X_∂; every trajectory is an element of $X_\partial^{\mathbf{N}}$.

We will first show that X_∂ is a measurable space, in fact a metric space, and that

$X_\partial^{\mathbf{N}}$ is a measurable space. Then we will give a useful lemma which can be used to prove measurability of certain subsets of $X_\partial^{\mathbf{N}}$.
In Section 2.1.2 it is described that $\mathcal{D}_S$ is a metric space, with metric d_1. The space $(0,\infty]$ is also a metric space with metric d_2, where d_2 is defined as

$$d_2(x,y) = \begin{cases} \frac{|x-y|}{1+|x-y|} & \text{if } x,y < \infty \\ 1 & \text{if either } y = \infty \text{ or } x = \infty \\ 0 & \text{if } x,y = \infty \end{cases}$$

Let d_3 be the standard metric on $\mathbf{R}_+$. Then consider the product metric d, as in Lemma 2.9. In section 2.1 we gave two ways to extend the metric d to the space X_∂. One of them uses Corollary 2.7, and note that this involves changing the metric d on X. But changing d on X means changing the metric on $\mathcal{D}_S$. Since we prefer not changing this metric, we use the other method. Therefore, let $c \in X$ be arbitrary, and define

$$d'(x,y) = \begin{cases} d(x,y) & \text{if } x,y \in X, \\ d(x,c)+1 & \text{if } x \in X, y = \partial, \\ d(y,c)+1 & \text{if } y \in X, x = \partial, \\ 0 & \text{if } x = y = \partial. \end{cases}$$

So X_∂ is a metric space, and hence a measurable space when equipped with the Borel-σ-algebra. In Section 2.2 it is described that $X_\partial^{\mathbf{N}}$ is again a measurable space. The σ-algebra on $X_\partial^{\mathbf{N}}$ is the σ-algebra generated by

$$\mathcal{E} = \left\{ \prod_{i=1}^{\infty} E_i : \ E_i \in \mathcal{B}(X_\partial) \right\}.$$

Lemma 3.1 *Let $E \subset X_\partial^m$ and $F \subset X_\partial^{\mathbf{N}}$ be measurable. Then $E \times F \subset X_\partial^{\mathbf{N}}$ is measurable.*

To prove this proposition, we will use Lemma 2.32.

Proof: Consider the map

$$\begin{aligned} \varphi_m : X_\partial^{\mathbf{N}} &\to X_\partial^m \times X_\partial^{\mathbf{N}}, \\ (x_n) &\mapsto ((x_1, x_2 \ldots, x_m), (x_{m+1}, x_{m+2}, \ldots)). \end{aligned}$$

We will prove that φ_m is a measurable function. The σ-algebra on $X_\partial^m \times X_\partial^{\mathbf{N}}$ is the product-σ-algebra generated by sets of the form $E \times F$, where $E \subset X_\partial^m$ is measurable, and $F \subset X_\partial^{\mathbf{N}}$ is measurable. The result follows if we can prove that this σ-algebra is generated by

$$\mathcal{E} = \left\{ \left(\prod_{n=1}^{m} E_n \right) \times \left(\prod_{n=1}^{\infty} F_n \right) \middle| \ E_i, F_i \subset X_\partial \text{ measurable} \right\},$$

since then by Proposition A.11 we have that φ_m is measurable if $\varphi_m^{-1}(E)$ is measurable in $X_\partial^{\mathbf{N}}$ for every $E \in \mathcal{E}$. And the latter is indeed the case.

By definition, the σ-algebra in X_∂^m is generated by sets of the form $\prod_{n=1}^m E_n$, which is a π-system containing X_∂^m. Also, the σ-algebra in $X_\partial^{\mathbf{N}}$ is generated by sets of the form $\prod_{n=1}^\infty E_n$, which is also a π-system containing $X_\partial^{\mathbf{N}}$. Now we can use Lemma 2.32 to conclude that the σ-algebra in $X_\partial^m \times X_\partial^{\mathbf{N}}$ is generated by $\mathcal{E}$. So we find that φ_m is measurable, and the result follows. ∎

Define the set $\Delta(\Gamma, s)$ for fixed $\Gamma \subset S$ and $s \in \mathbf{R}_+$ by

$$\Delta(\Gamma, s) := \{(f, \tau, t) \in X : \ t \le s < t + \tau, \ f(s-t) \in \Gamma\}.$$

In words, $\Delta(\Gamma, s)$ is the set of all partial trajectories which start at time $t \le s$ and run through Γ at time s. the next lemma shows that $\Delta(\Gamma, s)$ is measurable whenever $\Gamma \subset S$ is measurable.

Lemma 3.2 *Let $s \in \mathbf{R}_+$. If $\Gamma \subset S$ is measurable, then $\Delta(\Gamma, s)$ is measurable in X.*

Proof: Fix $s \in \mathbf{R}_+$ and let $\Gamma \subset S$ be measurable. Consider the function

$$\begin{aligned} \varphi_s : \mathcal{D}_S \times (0,\infty] \times (0,\infty] &\to S \times (0,\infty] \times \mathbf{R}_+ \\ (f, \tau, t) &\mapsto \begin{cases} (f(s-t), \tau, t) & \text{if } s - t \ge 0, \\ (x_0, \tau, t) & \text{if } s - t < 0. \end{cases} \end{aligned}$$

where x_0 is some fixed element in S. We claim that φ_s is a measurable function. To prove this, we use Lemma 2.39. Therefore we will prove the following claims:

i) For fixed $t \in (0,\infty]$, the function $(f, \tau) \mapsto \varphi_s(f, \tau, t)$ is measurable,

ii) For fixed $(f, \tau) \in \mathcal{D}_S \times (0,\infty]$, the function $t \mapsto \varphi_s(f, \tau, t)$ is left-continuous.

To prove the second claim, note that, for fixed $f \in \mathcal{D}_S$, the function

$$t \mapsto \begin{cases} f(s-t) & \text{if } s - t \ge 0, \\ x_0 & \text{if } s - t < 0, \end{cases}$$

is left-continuous. The same holds for the mapping into the second and the third coordinate, these are in fact continuous. By the definition of the product metric, it follows that the function $t \mapsto \varphi_s(f, \tau, t)$, for fixed (f, τ), is left-continuous.
Consider next the first claim. Fix $t \in (0,\infty]$. To prove measurability of $\phi_t : (f, \tau) \mapsto \varphi_s(f, \tau, t)$, it is sufficient to prove that $\phi_t^{-1}(E_1 \times E_2 \times E_3)$ is measurable in $\mathcal{D}_S \times (0,\infty] \times (0,\infty]$ for any rectangle $E_1 \times E_2 \times E_3$. First assume that $s - t \ge 0$. Let $E_1 \times E_2 \times E_3$ be any rectangle. Observe that $\phi_t^{-1}(E_1 \times E_2 \times E_3) = \mathrm{ev}_{s-t}^{-1}(E_1) \times E_2$ whenever $t \in E_3$, and $\phi_t^{-1}(E_1 \times E_2 \times E_3) = \emptyset$ otherwise. In either case, the set belongs to the σ-algebra for $\mathcal{D}_S \times (0,\infty] \times (0,\infty]$, because by Theorem 2.22, $\mathrm{ev}_{s-t}^{-1}(E_1)$ is measurable in $\mathcal{D}_S$. On the other hand, assume that $s - t < 0$. Then $\phi_t^{-1}(E_1 \times E_2 \times E_3) = \mathcal{D}_S \times E_2$ when $x_0 \in E_1$, and $\phi_t^{-1}(E_1 \times E_2 \times E_3) = \emptyset$ when $x_0 \notin E_1$. Again, $\phi_t^{-1}(E_1 \times E_2 \times E_3)$ is measurable. So it follows that ϕ_t is measurable. Now we can apply Lemma 2.39 in order to obtain that φ_s is measurable.
Consider the set

$$A_s = \{(\tau, t) \in (0,\infty] \times (0,\infty] : \ t \le s < \tau + t\}.$$

It is not hard to see that A_s is measurable, for instance, by considering that

$$A_s^c = (0, \infty] \times (s, \infty) \cup \{(\tau, t) : \; t + \tau \leq s\}.$$

Consider $\tilde{\varphi}_s : \mathcal{D}_S \times (0, \infty] \times \mathbf{R}_+ \to S \times (0, \infty] \times \mathbf{R}_+$ such that $\tilde{\varphi}_s = \varphi_s$ on $\mathcal{D}_S \times (0, \infty] \times (0, \infty]$, and $\tilde{\varphi}_s = (x_0, 0, 0)$, for some $x_0 \in \mathcal{D}_S$, on $\mathcal{D}_S \times (0, \infty] \times \{0\}$. Since $\mathcal{D}_S \times (0, \infty] \times \{0\}$ is of measure zero in the product measure, it follows that $\tilde{\varphi}_s$ is measurable.
We can now conclude that $\Delta(\Gamma, s)$ is measurable, since we proved that $\tilde{\varphi}_s$ is measurable, and $\Delta(\Gamma, s) = \tilde{\varphi}_s^{-1}(\Gamma \times A_s)$. ∎

We will conclude this section with showing that a certain function, which will be important when constructing the probability measure P, is measurable. For a measure $\nu \in \mathcal{P}(\mathcal{D}_S)$, define the distribution $Q_\nu \in \mathcal{P}(\mathcal{D}_S \times (0, \infty])$ on rectangles by

$$Q_\nu(E \times F) = \int_E p_f^\varepsilon(F) dP_\nu(f).$$

This definition extends to a measure, since we assumed in Section 3.1 that for every measurable $E \subset (0, \infty]$, the map $f \mapsto p_f^\varepsilon(E)$ is measurable from $\mathcal{D}_S$ to $[0, 1]$, so one can apply Proposition 2.36.
Here, we will write $\nu = \nu(f, \tau, t) = \Psi(\tau + t, \lim_{\sigma \uparrow (\tau + t)} f(\sigma))$.

Lemma 3.3 *Under the conditions given in Section 3.1, we have that for every measurable* $E \subset \mathcal{D}_S \times (0, \infty]$*, the function* $f_E : X^{n-1} \to [0, 1]$ *defined by*

$$f_E(x_1, x_2 \ldots, x_{n-1}) = Q_{\nu(f_{n-1}, \tau_{n-1}, \tau_1 + \ldots + \tau_{n-2})}(E),$$

where $x_i = (f_i, \tau_1, t_i)$*, is measurable.*

Proof: We will proceed in steps.

step 1: The map $(x_1, \ldots, x_{n-1}) \mapsto (f_{n-1}, \tau_{n-1}, \tau_1 + \ldots + \tau_{n-2})$ from X^{n-1} to X is measurable,

step 2: the map $(f, \tau, t) \mapsto \Psi\left(t + \tau, \lim_{\sigma \uparrow (t + \tau)} f(\sigma)\right) = \nu(f, \tau, t)$ from X to $\mathcal{P}(\mathcal{S}) \subset S_{BL}$ is measurable,

step 3: for every measurable $E \subset \mathcal{D}_S \times (0, \infty]$ the map $\mu \mapsto Q_\mu(E)$ from $\mathcal{P}(S) \to [0, 1]$ is measurable.

Once we proved this, the function f_E is indeed measurable, since then it can be written as a composition of three measurable functions.
Step one is rather easy. The function $(x_1, \ldots, x_{n-1}) \mapsto (f_{n-1}, \tau_{n-1}, \tau_1 + \ldots + \tau_{n-2})$ is a composition of projections and additions, which are all measurable (in fact continuous).
We proceed with step two. Consider the function $(f, \tau, t) \mapsto (f, \tau + t)$ from X to $\mathcal{D}_S \times (0, \infty]$. This function is measurable. Then fix τ and consider the function $f \mapsto \lim_{\sigma \uparrow \tau} f(\sigma)$ as a function from $\mathcal{D}_S$ to S. Let (σ_n) be a sequence in $(0, \tau)$ such that $\sigma_n \uparrow \tau$. Then for every n, the function $f \mapsto f(\sigma_n)$ is measurable, by Theorem

2.22. Hence the function $f \mapsto \lim_{\sigma \uparrow \tau} f(\sigma)$ is measurable, since it can be written as a limit of a sequence of measurable functions. Note that in Section 3.1 we have assumed that Ψ is measurable. Then $(f, \tau, t) \mapsto \Psi(\tau + t, \lim_{\sigma \uparrow (\tau+t)} f(\sigma))$ is measurable, since this is a composition of the measurable functions just considered.
Finally, we will show that the map $\mu \mapsto Q_\mu(E)$ is measurable for every measurable $E \subset \mathcal{D}_S \times (0, \infty]$. Since $Q_\mu(E) = \int_{\mathcal{D}_S} p_f^\varepsilon(E_f) dP_\mu(f)$, where E_f is the f-section of E, it is equivalent to show that

$$\mu \mapsto \int_{\mathcal{D}_S} p_f^\varepsilon(E_f) P_\mu(df)$$

is measurable. Since we have assumed that $f \mapsto p_f^\varepsilon(F)$ is measurable when $F \subset (0, \infty]$ is measurable, it follows that $f \mapsto p_f^\varepsilon(E_f)$ is measurable, where E_f is as above. Now consider

$$\mathcal{H} = \left\{ \psi : \mathcal{D}_S \to \mathbf{R} | \ \mu \mapsto \int_{\mathcal{D}_S} \psi(f) \ dP_\mu(f) \text{ is measurable} \right\}.$$

It is not hard to see that $\mathcal{H}$ satisfies the three conditions in Theorem A.14. It also contains every indicator function $\psi = \mathbf{1}_A$, where $A \in \mathcal{B}(\mathcal{D}_S)$, and $\mathcal{B}(\mathcal{D}_S)$ is certainly a π-system. So $\mathcal{H}$ contains every measurable function, in particular the map $f \mapsto p_f^\varepsilon(E_f)$. Hence $\mu \mapsto Q_\mu(E)$ is measurable. ∎

3.2.2 Realistic trajectories

Since the space $X_\partial^{\mathbf{N}}$ is larger than the space $\mathcal{D}_S$ we need to make some conditions on the sequence space, so that the trajectory does not become unrealistic. For instance, when the n-th partial trajectory has absolute starting time t_0 and duration $\tau_0 < \infty$, then we want the $(n+1)$-th partial trajectory to have absolute starting time $t_0 + \tau_0$. This particular condition is described in condition 3 below. Four conditions on $(x_n) \in X_\partial^{\mathbf{N}}$ are:

1. If $x_{n,2} = \infty$, then $x_{n+1} = \partial$;
2. if $x_n = \partial$ then $x_{n+1} = \partial$;
3. if $x_{n+1} \neq \partial$ then $x_{n+1,3} = x_{n,2} + x_{n,3}$;
4. $\sum_{x_n \neq \partial} x_{n,2} = \infty$.

The fourth condition is to exclude the possibility of getting infinitely many jumps within a finite amount of time. In that case we would get a function $f : [0, T] \to S$ instead of $f : \mathbf{R}_+ \to S$.
Since we will always want these four conditions to hold, we define the set

$$R = \{(x_n) \in X_\partial^{\mathbf{N}} \mid \text{every } x_n \text{ satisfies the above four conditions}\}.$$

The set R will be called the set of realistic trajectories.

3.2.3 Measurability of the set of realistic trajectories

In this section we will, as the title suggests, prove that the set R of realistic trajectories is measurable. Let R_m, $m \in \mathbf{N}$, be the collection of realistic trajectories consisting of exactly m partial trajectories, without the terminal state ∂. So the first, second and fourth condition given in Section 3.2.2 are not considered; R_m is the set of trajectories satisfying condition 3.

Proposition 3.4 *For each m, the set R_m is measurable.*

Proof: Consider the map $\psi_m : X^m \to [0, \infty]$ defined by

$$\psi_m((x_n)) = \sum_{j=1}^{m-1} | \, x_{m,2} + x_{m,3} - x_{m+1,3}|, \quad (x_n) \in X^m.$$

So ψ_m measures the total time between the end time of one trajectory and the starting time of its successor. As explained in section 2.2.1, we have that the coordinate map is measurable. So the map $\pi_2 : X \to (0, \infty]$ defined by $\pi_2(x) = x_2$ is measurable. Similar to Lemma B.1 and Lemma B.2, we have that $f : X \to (0, \infty]$ is measurable if and only if $f^{-1}((a, \infty]) \in \mathcal{M}$ for all $0 < a < \infty$. From this, it follows that if $f : X \to (0, \infty]$ is measurable, then also the same function considered as $f : X \to [-\infty, \infty]$. The other inclusion is also true: if $f : X \to [-\infty, \infty]$ is measurable with range in $(0, \infty]$, then f is measurable as function from X to $(0, \infty]$.
Now, using Proposition B.3 several times, we see that ψ_m is measurable. Furthermore, since $\{0\}$ is closed in $[0, \infty]$, it is measurable, and hence $\psi_m^{-1}(\{0\})$ is measurable. The result now follows, since $R_m = \psi_m^{-1}(\{0\})$. ∎

Let R_∞ be the set of realistic trajectories consisting of infinitely many trajectories, without considering the terminal state ∂ and without considering the fourth condition. So if $(x_n) \in R_\infty$, then $x_{n,2} < \infty$ for all n.

Corollary 3.5 *The set R_∞ is measurable.*

Proof: Observe that

$$R_\infty = \bigcap_{m=1}^{\infty} (R_m \times X^{\mathbf{N}}).$$

Also note that X is open in X_∂, so X is measurable. So we have that $X^{\mathbf{N}}$ is measurable in $X_\partial^{\mathbf{N}}$, by definition of the σ-algebra in $X_\partial^{\mathbf{N}}$. Then by Lemma 3.1, it follows that $R_m \times X^{\mathbf{N}}$ is measurable for every m, and hence R_∞ is measurable. ∎

Next, define

$$F_m := [\mathcal{D}_S \times (0, \infty) \times \mathbf{R}_+]^{m-1} \times [\mathcal{D}_S \times \{\infty\} \times \mathbf{R}_+] \times \{\partial\}^{\mathbf{N}}.$$

In words, F_m is the set of trajectories where the first $m - 1$ are partial trajectories with finite length and the m-th an infinite one. Note that we do not exclude the possibility of two or more trajectories not connecting in the sense of condition 3. What we need, and what turns out to be the case, is that F_m is measurable for every m.

Lemma 3.6 *For each m, the set F_m is measurable.*

Proof: The set $\mathcal{D}_S\times(0,\infty)\times\mathbf{R}_+$ is open in X with the product metric d. This follows from the definition of the product metric. In fact, when $E_1\subset\mathcal{D}_S$, $E_2\subset(0,\infty]$, and $E_3\subset\mathbf{R}_+$ are all open, then its product $E_1\times E_2\times E_3$ is open in X. It is easily seen that $\mathcal{D}_S\times(0,\infty)\times\mathbf{R}_+$ is open in X_∂ with respect to the metric d' given above. Also, we have that $\{\partial\}$ is closed in X_∂. It follows that both $(\mathcal{D}_S\times(0,\infty)\times\mathbf{R}_+)$ and $\{\partial\}$ are measurable, and hence also

$$\mathcal{D}_S\times\{\infty\}\times\mathbf{R}_+=X_\partial\setminus([\mathcal{D}_S\times(0,\infty)\times\mathbf{R}_+]\cup\{\partial\}).$$

Applying Lemma 3.1 several times, it follows that F_m is measurable. ∎

Now consider the set R' of trajectories satisfying condition 1, 2 and 3. This set is measurable

Corollary 3.7 *The set R' is measurable*

Proof: Just note that

$$R'=R_\infty\cup\bigcup_{m=1}^{\infty}((R_m\times X_\partial^{\mathbf{N}})\cap F_m). \quad ∎$$

We are now ready to state the fundamental result of this section.

Theorem 3.8 *The set R of realistic trajectories is measurable.*

Proof: Consider the function

$$\begin{aligned}\psi: X_\partial &\to [-\infty,\infty]\\ x&\mapsto\begin{cases}x_2 & \text{if } x\neq\partial,\\ 0 & \text{if } x=\partial.\end{cases}\end{aligned}$$

We will first show that this function is measurable. In section 2.2 it is explained that the coordinate map $\pi_2: X\to[-\infty,\infty]$ given by $\pi_2(x)=x_2$ is measurable. By Lemma B.2, we have that $\pi_2^{-1}((a,\infty])$ is measurable in X for all $0\leq a<\infty$. It is also open in X, since $\pi_2^{-1}((a,\infty])=\mathcal{D}_S\times(a,\infty]\times\mathbf{R}_+$. It follows that $\pi_2^{-1}((a,\infty])$ is also open in X_∂, and hence measurable in X_∂. It follows that $\psi^{-1}((a,\infty])$ for $0\leq a<\infty$. But for $a<0$ we have that $\psi^{-1}((a,\infty])=X_\partial$. Again by Lemma B.2, it follows that ψ is measurable. Also, by similar reasoning, we see that the function $\pi_m: X_\partial^{\mathbf{N}}\to X_\partial$ given by $\pi_m((x_n))=x_m$ is measurable for every m.
Consider the function $\varphi: X_\partial^{\mathbf{N}}\to[-\infty,\infty]$ given by

$$\varphi((x_n))=\sum_{n:\ x_n\neq\partial}x_{n,2}.$$

Observe that

$$\varphi=\sum_{n=1}^{\infty}\psi\circ\pi_n,$$

so φ is a limit of measurable functions, and hence measurable. This gives us that $\varphi^{-1}(\{\infty\})\subset X_\partial^{\mathbf{N}}$ is measurable, and the result follows since

$$R=\varphi^{-1}(\{\infty\})\cap R'. \quad ∎$$

3.2.4 The concatenation map

The total trajectory can now be viewed as the image of a concatenation map defined as

$$\begin{aligned} \gamma : R &\to \mathcal{D}_S, \\ (f_n, \tau_n, t_n)_{n=1}^{\infty} &\mapsto f_n(t), \text{ if } t_n \leq t < \tau_n + t_n. \end{aligned}$$

that pieces together the subsequent partial trajectories. For the process $\tilde{X}$ and in particular the distribution $\tilde{P}$ to make sense, we need that $\gamma : R \to \mathcal{D}_S$ is measurable.

For a given subset Γ of S, we will write $E(\Gamma, t)$ for $\mathrm{ev}_t^{-1}(\Gamma)$.

Proposition 3.9 *The concatenation map $\gamma : R \to \mathcal{D}_S$ is measurable.*

Proof: Fix $s \in \mathbf{R}_+$ and define

$$X_s^{(1)} := \{(f, \tau, 0) \in X : \tau < s\},$$

the set of first partial trajectories which end before time s. And, inductively,

$$X_s^{(n+1)} := \{(f_i, \tau_i, t_i)_{i=1}^{n+1} \in X^{n+1} : \ (f_i, \tau_i, t_i)_{i=1}^{n} \in X_s^{(n)}, \ t_{n+1} = t_n + \tau_n, \ t_{n+1} + \tau_n \leq s\}.$$

We have

$$\gamma^{-1}(E(\Gamma, s)) = \left(\Delta(\Gamma, s) \times X_\partial^{\mathbf{N}} \cup \bigcup_{m=2}^{\infty} X_s^{(m-1)} \times \Delta(\Gamma, s) \times X_\partial^{\mathbf{N}} \right) \cap R. \tag{3}$$

This equation should be read as follows. Suppose $(x_n) \in \gamma^{-1}(E(\Gamma, s))$, then there is precisely one m such that $x_m \in \Delta(\Gamma, s)$. If $m = 1$, then $(x_n) \in \Delta(\Gamma, s) \times X_\partial^{\mathbf{N}}$. If $m > 1$, then $(x_n) \in X_s^{(m-1)} \times \Delta(\Gamma, s) \times X_\partial^{\mathbf{N}}$. It follows that the union in (3) is pairwise disjoint.
We can rewrite equation (3) to

$$\begin{aligned} \gamma^{-1}(E(\Gamma, s)) &= \left(\left(\Delta(\Gamma, s) \times X_\partial^{\mathbf{N}} \right) \cap R \right) \cup \bigcup_{m=2}^{\infty} \left(\left(X_s^{(m-1)} \times \Delta(\Gamma, s) \times X_\partial^{\mathbf{N}} \right) \cap R \right) \\ &= \left(\left(\Delta(\Gamma, s) \times X_\partial^{\mathbf{N}} \right) \cap R \right) \cup \bigcup_{m=2}^{\infty} \left(\left(X^{m-1} \times \Delta(\Gamma, s) \times X_\partial^{\mathbf{N}} \right) \cap R \right) \end{aligned}$$

Note that in the last equation we replaced $X_s^{(m-1)}$ by X^{m-1}. By lemma 3.2, it follows that for measurable $\Gamma \subset S$, we have that $\gamma^{-1}(E(\Gamma, s))$ is measurable in $X_\partial^{\mathbf{N}}$. Also, by Proposition A.11, γ is measurable if $\gamma^{-1}(A)$ is measurable for every A in a set that generates the σ-algebra on $\mathcal{D}_S$. By Theorem 2.22, such a set is

$$\{E(\Gamma, t) \mid t \in \mathbf{R}_+, \Gamma \subset S \text{ measurable}\}.$$

We conclude that γ is indeed measurable. ∎

3.3 Construction and properties of the law

We want to make a probability measure P on the product space $X_\partial^{\mathbf{N}}$ which, given a measurable $E \subset X_\partial^{\mathbf{N}}$, calculates the probability that a random outcome is an element of E. As we have seen in Section 3.2, a partial trajectory is coded as an element of X_∂, and it follows that the process on $X_\partial^{\mathbf{N}}$ is actually a discrete time Markov process. Given a fixed $x \in X_\partial$, there is a transition function $p_\varepsilon(x, \bullet) \in \mathcal{P}(X_\partial)$. The ε-subscript is to note that this transition function is dependent on the environment ε. More on the environment can be found at Section 3.1. Conditions one to three at Section 3.2.2 have some influence on this transition function, namely

1. $p_\varepsilon(\partial, E) = \delta_\partial(E)$,
2. $p_\varepsilon(\partial, \{\partial\}) = 1$,
3. $p_\varepsilon((f, \infty, t), E) = \delta_\partial(E)$,
4. $p_\varepsilon((f, \tau, t), \cdot)$, (given that $\tau < \infty$) is concentrated on trajectories with starting time $t + \tau$.

To be able to construct the law, we need to make an assumption concerning p_ε. For any measurable set $E \subset X_\partial$, we assume that $x \mapsto p_\varepsilon(x, E)$ is measurable. It then satisfies the condition stated in Theorem 2.38. One way to achieve this assumption, is by assuming that the map $x \mapsto p_\varepsilon(x, \bullet)$, from $X_\partial \to \mathcal{P}(X_\partial)$, is strongly measurable, see Definition 5.4. This claim is proved in Proposition 5.8.

3.3.1 Construction of P

Suppose $x = (f, \tau, t) \in X$ with $\tau < \infty$, and suppose $E \times F \times G \subset X$ is a rectangle. Let ν be the distribution such that

$$\nu = \nu(f, \tau, t) = \Psi(t + \tau, \lim_{s \uparrow (t+\tau)} f(s)).$$

Then we have an expression for p_ε, namely

$$p_\varepsilon((f, \tau, t), E \times F \times G) = \delta_{t+\tau}(G) \int_E p_g^\varepsilon(F)\ dP_{\nu(f,\tau,t)}(g) \tag{4}$$

We will show that p_ε satisfies the given condition of Theorem 2.38, that is, we will show that for a given measurable $A \subset X_\partial$, $x \mapsto p_\varepsilon(x, A)$ is measurable. First notice that X_∂ is generated by rectangles $E \times F \times G$ where $E \in \mathcal{B}(\mathcal{D}_S)$, $F \in \mathcal{B}((0, \infty])$ and $G \in \mathcal{B}(\mathbf{R}_+)$, together with the terminal state ∂. First suppose that A is a rectangle, $A = E \times F \times G$. Then the function $x \mapsto p_\varepsilon(x, A)$ can be written as

$$x \mapsto \mathbf{1}_X(x)\mathbf{1}_G(x_2 + x_3) f_{F \times G}(x),$$

where f is as in Lemma 3.3. By the same lemma, we see that $x \mapsto p_\varepsilon(x, A)$ is measurable. Next suppose that $A = \{\partial\}$. Then the function $x \mapsto p_e(x, A)$ can be written as

$$x \mapsto \mathbf{1}_{\{\partial\}}(x) + \mathbf{1}_X(x)\mathbf{1}_{\{\infty\}}(x_2).$$

This function is again measurable. Since $p_\varepsilon(x,\cdot)$ is a measure, it follows that $x \mapsto p_\varepsilon(x,A)$ is measurable for any measurable $A \subset X_\partial$.
Let μ_0 and ε be given. Let $P^{(1)} = P^\varepsilon_{\mu_0} \in \mathcal{P}(X)$ be the distribution of the first trajectory, defined, in a similar way as p_ε, by

$$P^{(1)}(E \times F \times G) = \delta_0(G) \int_E p^\varepsilon_g(F)\; dP_{\mu_0}(g).$$

Next, we will define the probability measure $P^{(2)}$ as the joint distribution of the first two trajectories. So, given $B \in \mathcal{B}(X^2_\partial)$, we have

$$P^{(2)}(B) = \iint \mathbf{1}_B p_\varepsilon(x_1, dx_2) dP^{(1)}(x_1)$$

For higher n we define the joint distribution recursively as a probability measure $P^{(n+1)}$, so that given $B \in \mathcal{B}(X^n_\partial)$ we have

$$P^{(n+1)}(B) = \iint \mathbf{1}_B p_\varepsilon(x_n, dx_{n+1}) dP^{(n)}(x_1, x_2, \ldots, x_n)$$

All these probability measures are well-defined by Theorem 2.38. Note that $P^{(n)}$ is defined in such a way that for a rectangle $E_1 \times \ldots \times E_{n+1}$, we have

$$P^{(n+1)}(E_1 \times \ldots \times E_{n+1}) = \int_{E_1\times\ldots\times E_n} p_\varepsilon(x_n, E_{n+1}) dP^{(n)}(x_1, \ldots, x_n).$$

Also, by Theorem 2.38, there is a probability measure P defined on the product space $X^{\mathbf{N}}_\partial$, such that for $B \subset X^n_\partial$ measurable, P satisfies

$$P(B \times X^{\mathbf{N}}_\partial) = P^{(n)}(B).$$

3.3.2 P is concentrated on R

Our goal in the rest of this section is to prove the following theorem.

Theorem 3.10 *P is concentrated on the set R of realistic trajectories; $P(R) = 1$.*

We proceed in steps. First, define $F_\infty = (\mathcal{D}_S \times (0,\infty] \times \mathbf{R}_+)^{\mathbf{N}}$ and set

$$F := F_\infty \cup \bigcup_{n=1}^{\infty} F_n.$$

F is measurable since F_∞ is measurable, and by Lemma 3.6, F_m is measurable for every m. Step one is to prove that P is concentrated on F, and for this we will use a small lemma.

Lemma 3.11 *Let X be a set, and $A \subset X$. For every $n \in \mathbf{N}$ we have that $(A^n)^c$ equals the union of all rectangles of the form $\prod_{i=1}^n Z_i$, where $Z_i \in \{A, A^c\}$ except the rectangle A^n.*

Proof: Suppose $(x_n) \in X^n$. Then either $x_i \in A$ or $x_i \in A^c$. It follows that $x \in \prod_{i=1}^n Z_i$ where $Z_i \in \{A, A^c\}$. The result now follows. ∎

Proposition 3.12 *P is concentrated on F, that is, $P(F) = 1$.*

Proof: We will prove that $P(F^c) = 0$. Note that $F^c = F_\infty^c \cap \bigcap_{m=1}^\infty F_m^c$. Set

$$E_1 = \mathcal{D}_S \times (0, \infty) \times \mathbf{R}_+$$
$$E_2 = \mathcal{D}_S \times \{\infty\} \times \mathbf{R}_+$$
$$E_3 = \{\partial\}$$

We now have $F_m = E_1^{m-1} \times E_2 \times E_3^{\mathbf{N}}$. Observe that $E_1 \cup E_2 \cup E_3 = X_\partial$, and that this is a disjoint union. We claim that

$$\left(\bigcup_{j=1}^\infty F_j\right)^c = \bigcup_{j=1}^\infty \left(E_1^{j-1} \times E_3 \times X_\partial^{\mathbf{N}}\right) \cup \bigcup_{j=1}^\infty \left(E_1^{j-1} \times E_2 \times (E_3^{\mathbf{N}})^c\right) \cup F_\infty. \qquad (5)$$

First we will prove the easy part, namely the inclusion $\supset$.
Suppose $x \in \bigcup_{j=1}^\infty \left(E_1^{j-1} \times E_3 \times X_\partial^{\mathbf{N}}\right)$. By definition of F_j it should be clear that $x \notin F_j$ for every j. The same for $x \in \bigcup_{j=1}^\infty \left(E_1^{j-1} \times E_2 \times (E_3^{\mathbf{N}})^c\right)$. Also, if $x \in F_\infty$, then we immediately have $x \notin F_j$ for all j. So it should be clear that the inclusion $\supset$ holds.
Now suppose $x \in \left(\bigcup_{j=1}^\infty F_j\right)^c = \bigcap_{j=1}^\infty F_j^c$. Consider F_m for a fixed m. Observe that

$$F_m^c = ((E_1^{m-1})^c \times X_\partial \times X_\partial^{\mathbf{N}}) \cup (E_1^{m-1} \times E_2^c \times X_\partial^{\mathbf{N}}) \cup (E_1^{m-1} \times E_2 \times (E_3^{\mathbf{N}})^c).$$

Suppose that $x \in ((E_1^{m-1})^c \times X_\partial \times X_\partial^{\mathbf{N}})$. By Lemma 3.11 we have that $(E_1^{m-1})^c$ equals the union of all rectangles $\prod_{i=1}^{m-1} Z_i$ where $Z_i \in \{E_1, E_2, E_3\}$, with the exception of the rectangle E_1^{m-1}. So x is an element of one of these rectangles $\prod_{i=1}^{m-1} Z_i$. This means that there is at least one i such that $Z_i \neq E_1$. Consider $j = \min\{i : Z_i \neq E_i\}$. Then either $Z_j = E_2$ or $Z_j = E_3$.
First suppose that $Z_j = E_2$. Then we have either $x \in E_1^{j-1} \times E_2 \times E_3^{\mathbf{N}}$ or $x \in E_1^{j-1} \times E_2 \times (E_3^{\mathbf{N}})^c$. Obviously, in the first case we have $x \in F_j$, which gives a contradiction. In the second case we have in particular that

$$x \in \bigcup_{j=1}^\infty (E_1^{j-1} \times E_2 \times (E_3^{\mathbf{N}})^c),$$

so then we have that x is an element of the right-hand side of (5).
Next, suppose that $Z_j = E_3$. Then we have $x \in E_1^{j-1} \times E_3 \times X_\partial^{\mathbf{N}}$, so then x is again an element of the right-hand side of (5).

Suppose that $x \in (E_1^{m-1} \times E_2^c \times X_\partial^{\mathbf{N}})$. Then we have either $x \in E_1^{m-1} \times E_1 \times X_\partial^{\mathbf{N}}$ or $x \in E_1^{m-1} \times E_3 \times X_\partial^{\mathbf{N}}$. For the first case, observe that

$$E_1^{m-1} \times E_1 \times X_\partial^{\mathbf{N}} = E_1^{\mathbf{N}} \cup (E_1^m \times (E_1^{\mathbf{N}})^c) = F_\infty \cup (E_1^m \times (E_1^{\mathbf{N}})^c).$$

This means that either $x \in F_\infty$ or $x \in (E_1^m \times (E_1^{\mathbf{N}})^c)$. For the last case, note that

$$E_1^m \times (E_1^{\mathbf{N}})^c \subset \bigcup_{j=m}^{\infty} (E_1^j \times E_3 \times X_\partial^{\mathbf{N}}) \cup \bigcup_{j=m}^{\infty} (E_1^j \times E_2 \times X_\partial^{\mathbf{N}})$$

$$= \bigcup_{j=m}^{\infty} (E_1^j \times E_3 \times X_\partial^{\mathbf{N}}) \cup \bigcup_{j=m}^{\infty} (E_1^j \times E_2 \times (E_3^{\mathbf{N}})^c) \cup \bigcup_{j=m}^{\infty} F_j.$$

But since we assumed that $x \notin F_j$ for all j, it follows that if $x \in E_1^m \times (E_1^{\mathbf{N}})^c$, then we have again that x is an element of the right-hand side of (5).

Suppose that $x \in (E_1^{m-1} \times E_2 \times (E_3^{\mathbf{N}})^c)$, we have immediately that x is an element of the right-hand side of (5).

Since we now have considered all possible cases, we can conclude that if $x \in (\bigcup_{j=1}^\infty F_j)^c$ then x is in the right-hand side of (5), hence we have the inclusion $\subset$. Since $F^c = F_\infty^c \cap \bigcap_{m=1}^\infty F_m^c$ we now have

$$F^c = \bigcup_{j=1}^{\infty} (E_1^{j-1} \times E_3 \times X_\partial^{\mathbf{N}}) \cup \bigcup_{j=1}^{\infty} (E_1^{j-1} \times E_2 \times (E_3^{\mathbf{N}})^c). \tag{6}$$

Also, the union in the right-hand side of (6) is a disjoint union, so we particularly have

$$P(F^c) = \sum_{j=1}^{\infty} P(E_1^{j-1} \times E_3 \times X_\partial^{\mathbf{N}}) + \sum_{j=1}^{\infty} P(E_1^{j-1} \times E_2 \times (X_3^{\mathbf{N}})^c).$$

We want to prove that $P(F^c) = 0$, so we will prove that the right-hand side of the above equation equals 0. By definition of P we have $P(E_1^{j-1} \times E_3 \times X_\partial^{\mathbf{N}}) = P^{(j)}(E_1^{j-1} \times E_3)$. The set $E_1^{j-1} \times E_3$ is a rectangle, so

$$P^{(j)}(E_1^{j-1} \times E_3) = \int_{E_1^{j-1}} p_\varepsilon(x_{j-1}, E_3) dP^{(j-1)}(x_1, \ldots, x_{j-1}).$$

But by condition 4 in the list of conditions on p_ε, we see that $p_\varepsilon(x_{j-1}, \{\partial\}) = 0$ for all $x_{j-1} \in E_1 = (\mathcal{D}_S \times (0, \infty) \times \mathbf{R}_+)$. So it follows that $P^{(j)}(E_1^{j-1} \times E_3) = 0$. Thus $P(E_1^{j-1} \times E_3 \times X_\partial^{\mathbf{N}}) = 0$.
On the other hand, observe that

$$(E_1^{j-1} \times E_2 \times (E_3^{\mathbf{N}})^c) = \bigcap_{n=1}^{\infty} (E_1^{j-1} \times E_2 \times (X_\partial^n \setminus E_3^n) \times X_\partial^{\mathbf{N}}).$$

Note that we have $(E_1^{j-1} \times E_2 \times (X_\partial^n \setminus E_3^n) \times X_\partial^{\mathbf{N}}) \supset (E_1^{j-1} \times E_2 \times (X_\partial^m \setminus E_3^m) \times X_\partial^{\mathbf{N}})$ if $m > n$. For every n we have

$$\begin{aligned} P(E_1^{j-1} \times E_2 \times (X_\partial^n \setminus E_3^n) \times X_\partial^{\mathbf{N}}) &\le P(E_1^{j-1} \times E_2 \times E_3^c \times X_\partial^{n-1} \times X_\partial^{\mathbf{N}}) \\ &= P^{(j+2)}(E_1^{j-1} \times E_2 \times E_3^c) \\ &= \int_{E_1^{j-1} \times E_2} p_\varepsilon(x_j, E_3^c) dP^{(j+1)}(x_1, \ldots, x_j). \end{aligned}$$

But by condition 3 on p_ε it follows that $p_\varepsilon(x_j, E_3^c)$ for all $x_j \in E_2$. So, for every n we have $P(E_1^{j-1} \times E_2 \times (X_\partial^n \setminus E_3^n)) = 0$. By Theorem A.5, we now get

$$\begin{aligned} P(E_1^{j-1} \times E_2 \times (E_3^{\mathbf{N}})^c) &= P\left(\bigcap_{n=1}^{\infty}(E_1^{j-1} \times E_2 \times (X_\partial^n \setminus E_3^n) \times X_\partial^{\mathbf{N}})\right) \\ &= \lim_{n\to\infty} P\left(E_1^{j-1} \times E_2 \times (X_\partial^n \setminus E_3^n) \times X_\partial^{\mathbf{N}}\right) \\ &= 0. \end{aligned}$$

We conclude that

$$P(F^c) = \sum_{j=1}^{\infty} P(E_1^{j-1} \times E_3 \times X_\partial^{\mathbf{N}}) + \sum_{j=1}^{\infty} P(E_1^{j-1} \times E_2 \times (X_3^{\mathbf{N}})^c) = 0. \quad \blacksquare$$

Next, consider the collection G of all sequences in F which do not connect well, that is,

$$G = \{x_n \in F : \text{ there is an } n \in \mathbf{N} \text{ such that } x_{n+1} \neq \partial,\ x_{n+1,3} \neq x_{n,2} + x_{n,3}\}$$

Step two is to prove that P is concentrated on G^c.

Proposition 3.13 *P is concentrated G^c, that is, $P(G) = 0$.*

Proof: For every $x \in G$, there is a smallest n for which the first connection fails. Let G_n be the subset of G consisting of all sequences for which n is that smallest integer. It is obvious that $G = \cup_{m=1}^{\infty} G_m$. Then observe that $G_m = (R_m \times X) \setminus R_{m+1}$. We have

$$P^{(m+1)}((R_m \times X) \setminus R_{m+1}) = P^{(m+1)}(R_m \times X) - P^{(m+1)}(R_{m+1})$$

Also, by condition 4 on P_ε, we get

$$\begin{aligned} P^{(m+1)}(R_m \times X) &= \int_{R_m} p_\varepsilon(x_m, X) dP^{(m)}(x_1, \ldots, x_m) \\ &= \int_{R_m} p_\varepsilon(x_m, (\mathcal{D}_S \times (0, \infty] \times \{x_{m,2} + x_{m,3}\})) dP^{(m)}(x_1, \ldots, x_m) \\ &= P^{(m+1)}(R_{m+1}). \end{aligned}$$

This gives us $P^{(m+1)}((R_m \times X) \setminus R_{m+1}) = 0$, so $P(G_m) = 0$ for every m. We conclude

$$P(G) = \sum_{m=1}^{\infty} P(G_m) = 0. \quad \blacksquare$$

Proposition 3.14 *P is concentrated on R', that is, $P(R') = 1$.*

Proof: Recall that

$$R' = R_\infty \cup \bigcup_{m=1}^{\infty} ((R_m \times X_\partial^{\mathbf{N}}) \cap F_m).$$

We set

$$\begin{aligned}E^{(0)} &= \mathcal{D}_S \times (0,\infty) \times \{0\};\\ E_\infty &= \mathcal{D}_S \times \{\infty\} \times \mathbf{R}_+;\\ E_\infty^{(0)} &= \mathcal{D}_S \times \{\infty\} \times \{0\}.\end{aligned}$$

First we will prove that

$$\begin{aligned}&P((R_1 \times X_\partial^{\mathbf{N}}) \cap F_1) = P^{(1)}(E_\infty^{(0)}),\\ &P((R_n \times X_\partial^{\mathbf{N}}) \cap F_n) = P^{(n)}(E^{(0)} \times (X \setminus E_\infty)^{n-2} \times E_\infty).\end{aligned}$$

Since $(R_1 \times X_\partial^{\mathbf{N}}) \cap F_1 = E_\infty^{(0)} \times \{\partial\}^{\mathbf{N}}$ it makes sense to define $A_1 = E_\infty^{(0)} \times X_\partial^{\mathbf{N}}$ and $A_n = E_\infty^{(0)} \times \{\partial\}^{n-1} \times X_\partial^{\mathbf{N}}$ for $n = 2, 3, \ldots$. Namely, we then have $A_1 \supset A_2 \supset \ldots$, and $(R_1 \times X_\partial^{\mathbf{N}}) \cap F_1 = \cap_{j=1}^\infty A_j$. Observe that $P(A_1) = P^{(1)}(E_\infty^{(0)})$. Next,

$$\begin{aligned}P(A_2) = P^{(2)}(E_\infty^{(0)} \times \{\partial\}) &= \int_{E_\infty^{(0)}} p_\varepsilon(x_1, \{\partial\}) dP^{(1)}(x_1)\\ &= \int_{E_\infty^{(0)}} \delta_\partial(\{\partial\}) dP^{(1)}(x_1)\\ &= P^{(1)}(E_\infty^{(0)}).\end{aligned}$$

Now suppose that $P(A_n) = P^{(1)}(E_\infty^{(0)})$. It then follows that

$$\begin{aligned}P(A_{n+1}) &= P^{(n+1)}(E_\infty^{(0)} \times \{\partial\}^n \times \{\partial\})\\ &= \int_{E_\infty^{(0)} \times \{\partial\}^n} p_\varepsilon(x_n, \{\partial\}) dP^{(n)}(x_1, \ldots, x_n)\\ &= \int_{E_\infty^{(0)} \times \{\partial\}^n} \delta_\partial(\{\partial\}) dP^{(n)}(x_1, \ldots, x_n)\\ &= P^{(n)}(E_\infty^{(0)} \times \{\partial\}^n) = P(A_n) = P^{(1)}(E_\infty^{(0)}).\end{aligned}$$

So we have $P(A_n) = P^{(1)}(E_\infty^{(0)})$ for all n. Due to Theorem A.5 we have

$$P((R_1 \times X_\partial^{\mathbf{N}}) \cap F_1) = \lim_{n\to\infty} P(A_n) = P^{(1)}(E_\infty^{(0)}).$$

For $n \geq 2$, we have

$$(R_n \times X_\partial^{\mathbf{N}}) \cap F_n = E^{(0)} \times (X \setminus E_\infty)^{n-2} \times E_\infty \times \{\partial\}^{\mathbf{N}}.$$

Similar to the case $n = 1$, we now define $B_1 = E^{(0)} \times (X \setminus E_\infty)^{n-2} \times E_\infty \times X_\partial^{\mathbf{N}}$ and $B_j = E^{(0)} \times (X \setminus E_\infty)^{n-2} \times E_\infty \times \{\partial\}^j \times X_\partial^{\mathbf{N}}$. Using the same arguments as above, we can conclude that

$$P((R_n \times X_\partial^{\mathbf{N}}) \cap F_n) = P^{(n)}(E^{(0)} \times (X \setminus E_\infty)^{n-2} \times E_\infty).$$

Note that we have $E_\infty^c = X \setminus E_\infty \cup \{\partial\}$. For $n \in \mathbf{N}$, we write

$$(E_\infty^c)^n = ((X \setminus E_\infty) \cup \{\partial\})^n = (X \setminus E_\infty)^n \cup A^{(n)},$$

where $A^{(n)}$ is the collection of all sets of the form $A^{(n)} = A_1 \times \ldots \times A_n$ where $A_i = \{\partial\}$ for at least one i. By the previous Proposition 3.12 we have

$$\begin{aligned} P^{(n)}(E_1 \times (E_\infty^c)^{n-2} \times E_\infty) &= P(E_1 \times (E_\infty^c)^{n-2} \times E_\infty \times \{\partial\}^{\mathbf{N}}) \\ &= P(E_1 \times (X \setminus E_\infty)^{n-2} \times E_\infty \times \{\partial\}^{\mathbf{N}}) \\ &\quad + P(E_1 \times A^{(n-2)} \times E_\infty \times \{\partial\}^{\mathbf{N}}) \\ &= P(E_1 \times (X \setminus E_\infty)^{n-2} \times E_\infty \times \{\partial\}^{\mathbf{N}}) + 0 \\ &= P^{(n)}(E_1 \times (X \setminus E_\infty)^{n-2} \times E_\infty). \end{aligned}$$

Now we will prove the equality

$$\sum_{j=1}^{n} P((R_j \times X_\partial^{\mathbf{N}}) \cap F_j) = 1 - P^{(n)}(E^{(0)} \times (E_\infty^c)^{n-1}).$$

We will prove this with induction, so first let $n = 1$. We have

$$P((R_1 \times X_\partial^{\mathbf{N}}) \cap F_1) = P^{(1)}(E_\infty^{(0)}) = P^{(1)}(\mathcal{D}_S \times (0, \infty] \times \{0\}) - P^{(1)}(E^{(0)}).$$

Furthermore, the first partial trajectory always starts at time $t = 0$, so $P^{(1)}(\mathcal{D}_S \times (0, \infty] \times \{0\}) = 1$. So the equality is justified; we indeed have

$$P((R_1 \times X_\partial^{\mathbf{N}}) \cap F_1) = 1 - P^{(1)}(E^{(0)}).$$

For $n = 2$, observe that

$$\begin{aligned} P((R_2 \times X_\partial^{\mathbf{N}}) \cap F_2) &= P^{(2)}(E^{(0)} \times E_\infty) \\ &= P^{(2)}(E^{(0)} \times X_\partial) - P^{(2)}(E^{(0)} \times E_\infty^c) \\ &= P^{(1)}(E^{(0)}) - P^{(2)}(E^{(0)} \times E_\infty^c). \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{i=1}^{2} P((R_i \times X_\partial^{\mathbf{N}}) \cap F_i) &= 1 - P^{(1)}(E^{(0)}) + P^{(1)}(E^{(0)}) - P^{(2)}(E^{(0)} \times E_\infty^c) \\ &= 1 - P^{(2)}(E^{(0)} \times E_\infty^c). \end{aligned}$$

Now suppose that the equality holds for n, and consider the case $n + 1$. We have already seen that

$$\begin{aligned} P((R_{n+1} \times X_\partial^{\mathbf{N}}) \cap F_{n+1}) &= P^{(n+1)}(E^{(0)} \times (X \setminus E_\infty)^{n-1} \times E_\infty) \\ &= P^{(n+1)}(E^{(0)} \times (E_\infty^c)^{n-1} \times E_\infty). \end{aligned}$$

And again we have

$$\begin{aligned} P^{(n+1)}(E^{(0)} \times (E_\infty^c)^{n-1} \times E_\infty) &= P^{(n+1)}(E^{(0)} \times (E_\infty^c)^{n-1} \times X_\partial) \\ &\quad - P^{(n+1)}(E^{(0)} \times (E_\infty^c)^{n-1} \times E_\infty^c), \end{aligned}$$

so we conclude that

$$\sum_{i=1}^{n+1} P((R_i \times X_\partial^{\mathbf{N}}) \cap F_i) = 1 - P^{(n+1)}(E^{(0)} \times (E_\infty^c)^{n-1} \times X_\partial)$$
$$+ P^{(n+1)}(E^{(0)} \times (E_\infty^c)^{n-1} \times X_\partial) - P^{(n+1)}(E^{(0)} \times (E_\infty^c)^n)$$
$$= 1 - P^{(n+1)}(E^{(0)} \times (E_\infty^c)^n).$$

Now, by Theorem 3.12 again, we have for every n that

$$P^{(n+1)}(E^{(0)} \times (E_\infty^c)^n) = P^{(n+1)}(E^{(0)} \times (X \setminus E_\infty)^n).$$

Set $C_1 = E^{(0)} \times (X \setminus E_\infty) \times X_\partial^{\mathbf{N}}$, for higher n set $C_n = E^{(0)} \times (X \setminus E_\infty)^n \times X_\partial^{\mathbf{N}}$, and set $C = E^{(0)} \times (X \setminus E_\infty)^{\mathbf{N}}$. Then $C_1 \supset C_2 \supset \ldots$, and $C = \cap_{j=1}^\infty C_j$. From Theorem A.5 we have

$$P(C) = \lim_{n\to\infty} P(C_j) = \lim_{n\to\infty} P^{(n)}(E^{(0)} \times (X \setminus E_\infty)^n)$$
$$= \lim_{n\to\infty} P^{(n)}(E^{(0)} \times (E_\infty^c)^n).$$

So we get

$$\sum_{i=1}^{\infty} P((R_i \times X_\partial^{\mathbf{N}}) \cap F_i) = 1 - P(C).$$

By the previous Proposition 3.13 we have that

$$P(C) = P((E^{(0)} \times (X \setminus E_\infty)^{\mathbf{N}}) \cap (F \setminus G)) = P(R_\infty).$$

So we can conclude that

$$P(R') = P\left(R_\infty \cup \bigcup_{i=1}^{\infty}((R_i \times X_\partial^{\mathbf{N}}) \cap F_i)\right)$$
$$= P(R_\infty) + \sum_{i=1}^{\infty} P((R_i \times X_\partial^{\mathbf{N}}) \cap F_i)$$
$$= P(C) + 1 - P(C)$$
$$= 1 \quad \blacksquare$$

Now consider the collection

$$R'_\infty = \left\{(x_n) \in R_\infty : \sum_{n=1}^{\infty} x_{n,2} < \infty\right\},$$

that is, the collection of trajectories that make infinitely many jumps within a finite amount of time. Step three is to prove that P is concentrated on $(R'_\infty)^c$.

Proposition 3.15 *The set R'_∞ is measurable and $P(R'_\infty) = 0$.*

Proof: To prove that R'_∞ is measurable, consider the function $\varphi : X_\partial^{\mathbf{N}} \to [-\infty, \infty]$ given by

$$\varphi((x_n)) = \sum_{n:\ x_n \neq \partial} x_{n,2}.$$

That φ is a measurable function is explained in the proof of Theorem 3.8. Then $\varphi^{-1}((0,\infty))$ is a measurable set, and note that

$$R'_\infty = \varphi^{-1}((0,\infty)) \cap R_\infty.$$

Remember that there exist a $T > 0$ and a $0 \leq \delta < 1$ such that for all environments ε and $f \in \mathcal{D}_S$ we have $p_f^\varepsilon((0,T]) \leq \delta$. Define the set

$$G_T := (\mathcal{D}_S \times (0,T] \times \mathbf{R}_+) .$$

Then $G_T^{\mathbf{N}}$ is measurable in $X_\partial^{\mathbf{N}}$ since $G_T^{\mathbf{N}} = \bigcap_{m=1}^\infty G_T^m \times X_\partial^{\mathbf{N}}$. Next, define

$$G'_T := \bigcup_{m=1}^\infty X_\partial^m \times G_T^{\mathbf{N}}.$$

By similar reasoning it should be clear that G'_T is also measurable. We claim that $P(G'_T) = 0$. We will prove this by showing that $P(X_\partial^m \times G_T^{\mathbf{N}}) = 0$ for every m. By Theorem A.5, part ii), it follows that

$$P(X_\partial^m \times G_T^{\mathbf{N}}) = \lim_{n\to\infty} P(X_\partial^m \times G_T^n \times X_\partial^{\mathbf{N}}) = \lim_{n\to\infty} P^{(m+n)}(X_\partial^m \times G_T^n).$$

Observe that if $x = (f, \tau, t) \in X$ with $\tau < \infty$, then, using equation (4),

$$p_\varepsilon(x, G_T) = \delta_{\tau+t}(\mathbf{R}_+) \int_{\mathcal{D}_S} p_g((0,T]) dP_\nu(g),$$

where ν is the distribution right after the jump. Since, by assumption (2), we have $p_g((0,T]) \leq \delta$, we have that $p_\varepsilon(x, G_T) \leq \delta$. By condition 2. and 3. on p_ε we see that $p_\varepsilon(x, G_T) \leq \delta$ for all $x \in X_\partial$.
Let $n = 1$. We then have

$$\begin{aligned} P^{(m+1)}(X_\partial^m \times G_T) &= \iint \mathbf{1}_{X_\partial^m \times G_T} p_\varepsilon(x_m, dx_{m+1}) dP(x_1, \ldots, x_m) \\ &= \int_{X_\partial^m} p_\varepsilon(x_m, G_T) dP(x_1, \ldots, x_m) \\ &\leq \int_{X_\partial^m} \delta \ dP(x_1, \ldots, x_m) \\ &= \delta. \end{aligned}$$

Using induction, suppose that $P^{(m+n)}(X_\partial^m \times G_T^n) \leq \delta^n$. It follows that

$$\begin{aligned} P^{(m+n+1)}(X_\partial^m \times G_T^{n+1}) &= \iint \mathbf{1}_{(X_\partial^m \times G_T^{n+1})} p_\varepsilon(x_{m+n}, dx_{m+n+1}) dP(x_1, \ldots, x_{m+n}) \\ &= \int_{X_\partial^m \times G_T^n} p_\varepsilon(x_{m+n}, G_T) dP(x_1, \ldots, x_{m+n}) \\ &\leq \int_{X_\partial^m \times G_T^n} \delta \; dP(x_1, \ldots, x_{m+n}) \\ &= \delta P^{(m+n)}(X_\partial^m \times G_T^n) \\ &\leq \delta^{n+1}. \end{aligned}$$

So given m we have for every $n \in \mathbf{N}$ that $P^{(m+n)}(X_\partial^m \times G_T^n) \leq \delta^n$. Hence, given m, we have

$$\begin{aligned} P(X_\partial^m \times G_T^{\mathbf{N}}) &= \lim_{n\to\infty} P^{(m+n)}(X_\partial^m \times G_T^n) \\ &\leq \lim_{n\to\infty} \delta^n = 0. \end{aligned}$$

It follows that $P(G_T') = 0$.
Now for a sequence $(x_n) \in R_\infty'$, it is clear that there must exist an m such that $(x_n) \in X_\partial^m \times G_T^{\mathbf{N}}$. So $R_\infty' \subset G_T'$ and we conclude that $P(R_\infty') = 0$. ∎

We are now ready to prove the theorem.

Proof of Theorem 3.10: Observe that

$$R = R' \setminus R_\infty'.$$

By Propositions 3.14 and 3.15 it follows that

$$\begin{aligned} P(R) &= P(R' \cap (R_\infty')^c) = P(((R')^c \cup R_\infty')^c) = 1 - P\left((R')^c \cup R_\infty'\right) \\ &= 1 - P((R')^c) - P(R_\infty') = 1 - 0 - 0 = 1. \quad ∎ \end{aligned}$$

3.4 The process on S and its one-dimensional distribution

Here we will give a short summary of the process we have constructed. We have a probability space $(X_\partial^{\mathbf{N}}, \Sigma, P)$, where Σ is the product-σ-algebra for $X_\partial^{\mathbf{N}}$. The stochastic process is a family of X_∂-valued random variables $(X_n)_{n=1}^\infty$, where

$$X_n : X_\partial^{\mathbf{N}} \to X_\partial$$

is defined as

$$X_n((x_i)) = \mathrm{ev}_n((x_i)) = x_n.$$

Note that each X_n is indeed a random variable, since the projection onto the n-th coordinate is a measurable function.
We constructed X_∂ in such a way that this process is actually a Markov process,

with transition function p_ε. For every $n \in \mathbf{N}$, the law $\mathcal{L}(X_n)$ is the measure on the Borel-σ-algebra of X_∂ given by

$$\mathcal{L}(X_n) = P \circ X_n^{-1}.$$

By construction, $P(X_n^{-1}(E))$, where $E \subset X_\partial$ is measurable, is the probability that the n-th partial trajectory belongs to the measurable set E.

Using this discrete-time process, we will construct a continuous-time stochastic process on S which satisfies the criteria given at the beginning of Section 3. This process is a collection of random variables $\tilde{X}_t$, $t \in \mathbf{R}_+$, given by

$$\begin{aligned} \tilde{X}_t : \mathcal{D}_S &\to S, \\ f &\mapsto \mathrm{ev}_t(f) = f(t). \end{aligned}$$

In fact, Theorem 2.22 yields that $\tilde{X}_t$ is measurable for every $t \in \mathbf{R}_+$. For the process to satisfy the given criteria, we need a specific measure $\tilde{P}$ on $(\mathcal{D}_S, \mathcal{B}(\mathcal{D}_S))$. This measure $\tilde{P}$ will be such that for every $t \in \mathbf{R}_+$, the law $\mu_t = \tilde{P} \circ \tilde{X}_t^{-1}$ describes the actual flow, by this, we mean that given a measurable $\Gamma \subset S$ the value $\mu_t(\Gamma)$ is exactly the probability that the individual is in state Γ at time t.
Since P is concentrated on R, a trajectory (x_n) is such that concatenation is possible. Consider the concatenation map

$$\begin{aligned} \gamma : R &\to \mathcal{D}_S, \\ (f_n, \tau_n, t_n)_{n=1}^{\infty} &\mapsto f_n(t), \text{ if } t_n \leq t < \tau_n + t_n. \end{aligned}$$

Intuitively, we should have $\tilde{P} = P \circ \gamma^{-1}$. $\tilde{P}$ is indeed a measure, since for a measurable $E \in S$ we have, by Proposition 3.9 that $\gamma^{-1}(E)$ is measurable in $X_\partial^{\mathbf{N}}$. We have that for every $t \in \mathbf{R}_+$ and measurable $\Gamma \subset S$, the law μ_t satisfies

$$\mu_t(\Gamma) = P(\gamma^{-1}(E(\Gamma, t))).$$

Using equation (3), we get

$$\mu_s(\Gamma) = P\left(\left[\Delta(\Gamma, s) \times X_\partial^{\mathbf{N}} \cup \bigcup_{m=2}^{\infty} X_s^{(m-1)} \times \Delta(\Gamma, s) \times X_\partial^{\mathbf{N}}\right] \cap R\right).$$

Since the union in (3) is pairwise disjoint, we get

$$\mu_s(\Gamma) = P\left(\Delta(\Gamma, s) \times X_\partial^{\mathbf{N}}\right) + \sum_{m=2}^{\infty} P\left(X^{m-1} \times \Delta(\Gamma, s) \times X_\partial^{\mathbf{N}}\right). \tag{7}$$

The next objective is to find simplifying, convenient expressions for 7.

4 Expressions for the one-dimensional distribution

Let an initial distribution $\mu_0 \in \mathcal{P}(S)$ be given. We will rewrite the equation (7), by rewriting every m-term in the summation.

Proposition 4.1 *Let $\mu_0 \in \mathcal{P}(S)$ be given. If $\phi_n : X^n \to \mathbf{R}$ is a bounded measurable function, then the integral*

$$\int_{X^n} \phi_n(x_1, \ldots, x_n) dP^{(n)}(x_1, \ldots, x_n)$$

equals the repeated integral

$$\int_{(\mathcal{D}_S \times (0,\infty])^n} \phi_n\left((f_1, \tau_1, 0), (f_2, \tau_2, \tau_1), \ldots, (f_n, \tau_n, \tau_1 + \ldots + \tau_{n-1})\right)$$
$$dQ_{\nu(f_{n-1}, \tau_{n-1}, \tau_1 + \ldots + \tau_{n-2})}(f_n, \tau_n) \ldots dQ_{\nu(f_1, \tau_1, 0)}(f_2, \tau_2) dQ_{\mu_0}(f_1, \tau_1)$$

Proof: We proceed by induction. Take $n = 1$. Suppose $\phi_1(x_1) = \mathbf{1}_{E_1}(x_1)$ with $E_1 = E_{1,1} \times E_{1,2} \times E_{1,3}$. Then

$$\begin{aligned}
\int_X \phi_1(x_1) dP^{(1)}(x_1) = P^{(1)}(E_1) &= \mathbf{1}_{E_{1,3}}(0) \int_{E_{1,1}} p_f^{\varepsilon}(E_{1,2}) dP_{\mu_0}(f) \\
&= \mathbf{1}_{E_{1,3}}(0) Q_{\mu_0}(E_{1,1} \times E_{1,2}) \\
&= \int_{\mathcal{D}_S \times (0,\infty]} \mathbf{1}_{E_{1,3}}(0) \mathbf{1}_{E_{1,1} \times E_{1,2}}(f_1, \tau_1) dQ_{\mu_0}(f_1, \tau_1) \\
&= \int_{\mathcal{D}_S \times (0,\infty]} \phi_1(f_1, \tau_1, 0) dQ_{\mu_0}(f_1, \tau_1).
\end{aligned}$$

Using linearity and the Dominated Convergence Theorem, it follows that the same result holds for arbitrary bounded measurable functions ϕ_1. So for $n = 1$, the proposition holds.
Assume the equality holds for $n-1$. Suppose $\phi_n(x_1, \ldots, x_n) = \mathbf{1}_{E_1 \times \ldots \times E_n}(x_1, \ldots, x_n)$, and $E_n = E_{n,1} \times E_{n,2} \times E_{n,3}$. We get

$$\begin{aligned}
&\int_{X^n} \phi_n(x_1, \ldots, x_n) dP^{(n)}(x_1, \ldots, x_n) = dP^{(n)}(E_1 \times \ldots \times E_n) \\
&= \int_{E_1 \times \ldots \times E_{n-1}} p_\varepsilon(x_{n-1}, E_n) dP^{(n-1)}(x_1, \ldots, x_{n-1}).
\end{aligned}$$

Since P is concentrated on R, we will write $x_{n-1} = (f_{n-1}, \tau_{n-1}, t_{n-1})$, where $t_{n-1} = \tau_1 + \ldots + \tau_{n-2}$. Since equations are going to be large, we will write $\nu_{n-1} = \nu(f_{n-1}, \tau_{n-1}, t_{n-1})$.

Using (4) we get

$$\int_{X^n} \phi_n(x_1,\ldots,x_n)dP^{(n)}(x_1,\ldots,x_n)$$
$$= \int_{E_1\times\ldots\times E_{n-1}} \delta_{t_{n-1}}(E_{n,3}) \left[\int_{E_{n,1}} p_g(E_{n,2})dP_{\nu_{n-1}}(g)\right] dP^{(n-1)}(x_1,\ldots,x_{n-2},(f_{n-1},\tau_{n-1},t_{n-1}))$$
$$= \int_{E_1\times\ldots\times E_{n-1}} \mathbf{1}_{E_{n,3}}(t_{n-1}) \left[Q_{\nu_{n-1}}(E_{n,1}\times E_{n,2})\right] dP^{(n-1)}(x_1,\ldots,x_{n-2},(f_{n-1},\tau_{n-1},t_{n-1}))$$
$$= \int_{X^n} \mathbf{1}_{E_1\times\ldots\times E_{n-1}}(x_1,\ldots,x_{n-1})\mathbf{1}_{E_{n,3}}(t_{n-1}) \left[\int_{\mathcal{D}_S\times(0,\infty]} \mathbf{1}_{E_{n,1}\times E_{n,2}}(f_n,\tau_n)dQ_{\nu_{n-1}}\right] dP^{(n-1)}$$
$$= \int_{X^n}\int_{\mathcal{D}_S\times(0,\infty]} \phi_n(x_1,\ldots,x_{n-1},(f_n,\tau_n,\tau_1+\ldots+\tau_{n-1}))dQ_{\nu_{n-1}}(f_n,\tau_n)dP^{(n-1)}.$$

Again, by linearity and the Dominated Convergence Theorem, the above equalities hold for arbitrary bounded measurable functions ϕ_n.

Also, the function $\psi : X^{n-1} \to [0,1]$ given by

$$\psi(x_1,\ldots,x_{n-1}) = \int_{\mathcal{D}_S\times(0,\infty]} \phi_n(x_1,\ldots,x_{n-1},(f_n,\tau_n,\tau_1+\ldots+\tau_{n-1}))dQ_{\nu_{n-1}}(f_n,\tau_n)$$

is measurable. This can be seen as follows. In the previous Lemma 3.3, we have seen that for every E in the σ-algebra of $\mathcal{D}_S\times(0,\infty]$ the function $f_E : X^{n-1} \to [0,\infty]$ defined by

$$f_E(x_1,\ldots,x_{n-1}) = Q_{\nu(f_{n-1},\tau_{n-1},\tau_1+\ldots+\tau_{n-2})}(E)$$

is measurable. Then one can use Lemma 2.35 to prove that ψ is measurable whenever $\phi(x_1,\ldots,x_n) = \mathbf{1}_{E_1\times\ldots\times E_n}(x_1,\ldots,x_n)$. It follows that ψ is measurable when ϕ is a simple function, because ψ is then a linear combination of measurable functions. Also, by a limiting argument, ψ is measurable for any measurable function ϕ. Since the inner integral is measurable, using the induction hypotheses we conclude the proof. ∎

Now we will use this result to rewrite equation (7). We will call $P(\Delta(\Gamma,s)\times X_\partial^{\mathbf{N}})$ the first term and $P(X^{m-1}\times\Delta(\Gamma,s)\times X_\partial^{\mathbf{N}})$ the m-th term, for $m=2,3,\ldots$. Consider the first term. For the first partial trajectory x_1 we have $x_1\in\Delta(\Gamma,s)$ if and only if $x_1 \in E(\Gamma,s)\times(s,\infty]\times\{0\}$, where $E(\Gamma,s) = \mathrm{ev}_t^{-1}(\Gamma)$, as usual. The first term can be rewritten as

$$\begin{aligned} P(\Delta(\Gamma,s)\times X_\partial^{\mathbf{N}}) &= P^{(1)}\left(\Delta(\Gamma,s)\right) = P^{(1)}\left(E(\Gamma,s)\times(s,\infty]\times\{0\}\right) \\ &= \int_{E(\Gamma,s)} p_f^{\varepsilon}((s,\infty])dP_{\mu_0}(f) = Q_{\mu_0}(E(\Gamma,s)\times(s,\infty]) \\ &= \int_{\mathcal{D}_S\times(0,\infty]} \mathbf{1}_{E(\Gamma,s)\times(s,\infty]}(f_1,\tau_1)dQ_{\mu_0}(f_1,\tau_1). \end{aligned}$$

Next, consider the $(m+1)$-th term. We have

$$\begin{aligned}
&P(X^m \times \Delta(\Gamma, s) \times X_\partial^{\mathbf{N}}) = P^{(m+1)}(X^m \times \Delta(\Gamma, s)) \\
&= \int_{X^m} p_\varepsilon(x_m, \Delta(\Gamma, s)) dP^{(m)}(x_1, \ldots, x_m) \\
&= \int_{(\mathcal{D}_S \times (0,\infty])^m} p_\varepsilon((f_m, \tau_m, \tau_1 + \ldots + \tau_{m-1}), \Delta(\Gamma, s)) dQ_{\nu_{m-1}}(f_m, \tau_m) \ldots dQ_{\mu_0}(f_1, \tau_1).
\end{aligned}$$

Now note that we can always add an inner integral $\int_{\mathcal{D}_S \times (0,\infty]} dQ_{\nu_m}(f_{m+1}, \tau_{m+1})$. The result is then, that we integrate the $(m+1)$-th term over the same area as the $(m+2)$-nd term.
Or, more general, the sum of the first m terms can be written as an integral over $(\mathcal{D}_S \times (0, \infty])^m$. Before we make this statement clear, we will write

$$\phi_n(x_1, \ldots, x_n, s)(\Gamma) = \mathbf{1}_{E(\Gamma,s)\times(s,\infty]}(f_1, \tau_1) + \sum_{m=2}^{n} p_\varepsilon((f_m, \tau_m, \tau_1 + \ldots + \tau_{m-1}), \Delta(\Gamma, s)),$$

to avoid enormous equations. The last statement means that, for $n = 2, 3, \ldots$,

$$\begin{aligned}
&P(\Delta(\Gamma, s) \times X_\partial^{\mathbf{N}}) + \sum_{m=2}^{n} P(X^{m-1} \times \Delta(\Gamma, s) \times X_\partial^{\mathbf{N}}) \\
&= \int_{\mathcal{D}_S \times (0,\infty])^n} \phi_n(x_1, \ldots, x_n)(\Gamma) dQ_{\nu_{n-1}}(f_n, \tau_n) \ldots dQ_{\nu_1}(f_2, \tau_2) dQ_{\mu_0}(f_1, \tau_1).
\end{aligned}$$

It follows that

$$\mu_s(\Gamma) = \lim_{n\to\infty} \int_{\mathcal{D}_S \times (0,\infty])^n} \phi_n(x_1, \ldots, x_n)(\Gamma) dQ_{\nu_{n-1}}(f_n, \tau_n) \ldots dQ_{\nu_1}(f_2, \tau_2) dQ_{\mu_0}(f_1, \tau_1), \tag{8}$$

where

$$\phi_n(x_1, \ldots, x_n)(\Gamma) = \mathbf{1}_{E(\Gamma,s)\times(s,\infty]}(f_1, \tau_1) + \sum_{m=2}^{n} p_\varepsilon\left((f_m, \tau_m, \tau_1 + \ldots + \tau_{m-1}), \Delta(\Gamma, s)\right).$$

5 Conclusion and further progression of the model

As we have seen in the previous section, the existence of a stochastic process which satisfies the given conditions is guaranteed since we have constructed the measure $\tilde{P}$. The model we created is more general than the one given in *A cell cycle model* [11], [4]. We concluded the previous section with an integral representation 8 of the distribution μ_t. This representation can possibly be viewed as a converging series in a Banach space, as will be explained in the next section. We will conclude that section with a sufficient condition, so that the representation can indeed be viewed as a converging series.

5.1 Bochner integration

This section covers only basic theory of Bochner integration. A more detailed cover can for instance be found at [12], [6]. First we will define the notion of a vector-valued simple function, then we will define what we mean by a Bochner integrable function.

Definition 5.1 *Let (Ω, Σ, μ) be a measure space and X be a Banach space. A function $f : \Omega \to X$ is called a* ***vector-valued simple function*** *if f is of the form*

$$f = \sum_{k=1}^{n} a_k \mathbf{1}_{A_k},$$

where $a_k \in X$ and A_k measurable sets in $\mathbf{R}$. In this case, the Bochner Integral is defined by

$$\int_\Omega f \; d\mu = \sum_{k=1}^{n} a_k \mu(A_k) \in X.$$

Definition 5.2 *Let (Ω, Σ, μ) be a measure space and X a Banach space. A function $f : \Omega \to X$ is called* ***Bochner integrable*** *if there exists a sequence of simple functions f_n such that*

$$\lim_{n\to\infty} \int_\Omega \|f(x) - f_n(x)\| d\mu(x) = 0.$$

In this case, the integral is defined as

$$\int_\Omega f(x) d\mu(x) = \lim_{n\to\infty} \int_\Omega f_n(x) d\mu(x).$$

The following lemma states that a bounded linear operator and the Bochner integral can be interchanged. The proof is straightforward, and will be omitted.

Lemma 5.3 *Let (Ω, Σ, μ) be a measure space and X, Y be Banach spaces. Let $T : X \to Y$ be a bounded operator. If $f : \Omega \to X$ is Bochner integrable, then $Tf : \Omega \to Y$ is Bochner integrable and*

$$\int_\Omega (Tf)(x) d\mu(x) = T\left(\int_\Omega f(x) d\mu(x)\right).$$

Definition 5.4 *A function $f : \Omega \to X$ is called* ***strongly measurable*** *if there exists a sequence of simple functions (φ_n) such that $\varphi_n(\omega) \to f(\omega)$ for almost all $\omega \in \Omega$.*

Definition 5.5 *A function $f : \Omega \to X$ is called* ***weakly measurable*** *if for every $x^* \in X^*$ the function $\langle f(\omega), x^* \rangle : \Omega \to \mathbf{R}$ is measurable.*

Definition 5.6 *A function $f : \Omega \to X$ is called* ***almost separable valued*** *if there exists a measurable set A such that $f(A)$ is a separable set in X and $\mu(A^c) = 0$.*

Strongly measurability is 'stronger' than weakly measurability, as stated in the next theorem. A proof can be found at [12].

Theorem 5.7 (Pettis Theorem) *A function $f : \Omega \to X$ is strongly measurable if and only if it is weakly measurable and almost separable valued.*

The following proposition relates strongly measurable functions and functions that satisfy the hypotheses in Lemma 2.35, Proposition 2.36 and Theorem 2.38.

Proposition 5.8 *Let $(X, d_1), (Y, d_2)$ be two metric spaces, where Y is separable, and $f : X \to \mathcal{P}(Y)$. Then the following are equivalent.*

i) When viewing f as a function $f : X \to Y_{BL}$, then f is strongly measurable, in the Bochner sense;

ii) For all $A \in \Sigma_Y$, the σ-algebra generated by Y, the map $\varphi_C : X \to \mathbf{R}$, given by $x \mapsto f(x)(A)$ is measurable.

Proof: First, we prove $i) \Rightarrow ii)$. Suppose f is strongly measurable, and let $C \subset Y$ be closed. We will first prove that φ_C is measurable when C is closed. Consider the sequence of functions $q_n(x) = (1 - nd(x, C))$. By the previous Lemmas 2.19, 2.14 and 2.15 we see that the function q_n is Lipschitz continuous. By Lemma 2.17 the function $p_n = \max(q_n, 0)$ is Lipschitz continuous, and is also bounded. Now note that $p_n \downarrow \mathbf{1}_C$ pointwise. Since f is strongly measurable, it is also weakly measurable, by Pettis Theorem 5.7. This means that for every $x^* \in Y_{BL}^*$ the map $x \mapsto \langle x^*, f(x) \rangle$ is measurable. Now, since Y_{BL}^* is isometrically isomorphic to $BL(Y)$, by Theorem 2.30, for every $x^* \in Y_{BL}^*$ there is exactly one $\varphi \in BL(Y)$ such that

$$\langle x^*, f(x) \rangle = \int_Y \varphi \; d(f(x))(y).$$

So we now know that for every $\varphi \in BL(Y)$, the map

$$x \mapsto \int_Y \varphi \; d(f(x))(y)$$

is measurable.
Since $p_n \in BL(Y)$ we see that in particular the map $\Phi_n : X \to \mathbf{R}$ given by $x \mapsto$

$\int_Y p_n \, d(f(x))(y)$ is measurable. By the Lebesgue Dominated Convergence Theorem, we have for every x that

$$\int_Y p_n \; d(f(x))(y) \to \int_Y \mathbf{1}_C \; d(f(x))(y)$$

as $n \to \infty$. Since a limit of measurable functions is measurable, we conclude that the map

$$x \mapsto \int_Y \mathbf{1}_C \; d(f(x))(y) = f(x)(C)$$

is measurable. So we have the implication $i) \Rightarrow ii)$ for closed sets C. Now consider the class of functions

$$\mathcal{H} = \{\varphi : X \to \mathbf{R} \mid \varphi \text{ is measurable and } x \mapsto \textstyle\int_Y \varphi \; d(f(x))(y) \text{ is measurable.}\}$$

Then $\mathcal{H}$ is obviously a vector space over $\mathbf{R}$. Also, the constant function 1 is an element of $\mathcal{H}$ since 1 is measurable, and $1 = \mathbf{1}_Y$. Note that Y is closed. We already proved that $x \mapsto \int_Y \mathbf{1}_C \; d(f(x))(y)$ is measurable for a closed set C. And, if (φ_n) is a sequence such that $\varphi_n \uparrow \varphi$, then obviously φ is measurable and the map $x \mapsto \int_Y \varphi \; d(f(x))(y)$ is measurable. Hence $\varphi \in \mathcal{H}$. So $\mathcal{H}$ satisfies the conditions stated in the Monotone Class Theorem A.14. Now let

$$\mathcal{I} = \{C \subset Y \mid C \text{ closed}\}.$$

Then $\mathcal{I}$ is a π-system, and $\mathcal{H}$ contains the indicator function of every set in $\mathcal{I}$. From the Monotone Class Theorem A.14 we can conclude that $\mathcal{H}$ contains every measurable $\sigma(\mathcal{I})$ function on X. All open sets are elements of $\sigma(\mathcal{I})$, so we see that $\sigma(\mathcal{I}) = \mathcal{B}$. We conclude that $\mathcal{H}$ contains every indicator function $\mathbf{1}_B$ where $B \in \mathcal{B}$, and hence the map $x \mapsto f(x)(B)$ is measurable.

Now we prove the other implication $ii) \Rightarrow i)$. We will use Pettis Theorem 5.7 again. We assumed that the map $x \mapsto f(x)(A)$ is measurable for every measurable A, so it follows that

$$x \mapsto \int_Y \mathbf{1}_A \; d(f(x))(y)$$

is measurable for every A. It then follows that

$$x \mapsto \int_Y \varphi \; d(f(x))(y)$$

is measurable for every measurable function φ. In particular for $\varphi \in BL(Y) = Y_{BL}^*$. It follows that the function f is weakly measurable. By Proposition 2.27, we have that Y_{BL} is separable. Now we can use Pettis Theorem 5.7 to conclude that f is strongly measurable. This concludes the proof. ∎

Proposition 5.9 *Let (X, d_1) and (Y, d_2) be two metric spaces. If $f : X \to \mathcal{P}(Y)$, viewed as a function $f : X \to Y_{\mathrm{BL}}$ is strongly measurable and Bochner integrable, then for every measurable set $E \in \mathcal{B}(Y)$ we have the equality*

$$\left(\int_X f(x) d\mu(x)\right)(E) = \int_X f(x)(E) d\mu(x).$$

Proof: First note that the right-hand side of the above equation is well-defined, by the previous Proposition 5.8; the function $x \mapsto f(x)(E)$ is measurable for every $E \in \mathcal{B}(Y)$.
Let C be closed in Y, and consider the sequence $\phi_n = \max(1 - nd_2(x, C), 0)$. As described in the proof of Proposition 5.8, it follows that ϕ_n is Lipschitz continuous. Remember that $\mathrm{BL}(Y)$ is isometrically isomorphic to Y^*_{BL}. Set $\nu = \int_X f(x)d\mu(x)$ and observe that

$$\left\langle \int_X f(x)d\mu(x), \phi_n \right\rangle = \langle \nu, \phi_n \rangle = \int_Y \phi_n(y)d\nu(y) \to \int_Y \mathbf{1}_C d\nu(y) = \nu(C),$$

as $n \to \infty$. The convergence follows from the Dominated Convergence Theorem. Also, by Lemma 5.3, we see that

$$\left\langle \int_X f(x)d\mu(x), \phi_n \right\rangle = \int_X \langle f(x), \phi_n \rangle d\mu(x).$$

Also by the Dominated Convergence Theorem, we have

$$\int_X \langle f(x), \phi_n \rangle d\mu(x) = \int_X \left[\int_Y \phi_n(y)d(f(x))(y) \right] d\mu(x) \to \int_X f(x)(C)d\mu(x).$$

So for closed $C \subset Y$ we have the desired equality

$$\left(\int_X f(x)d\mu(x) \right)(C) = \int_X f(x)(C)d\mu(x).$$

Consider the collection $\mathcal{H}$ of measurable functions $\varphi : Y \to \mathbf{R}$ such that the equality

$$\int_Y \varphi(y)d\nu(y) = \int_X \left[\int_Y \varphi(y)d(f(x))(y) \right] d\mu(x)$$

holds. We will show that $\mathcal{H}$ satisfies the conditions stated in Theorem A.14. By linearity of the integral and the fact that the set of measurable functions is a vector space, it follows that $\mathcal{H}$ is a vector space. Also the constant function 1 is an element of $\mathcal{H}$, since this is the function $\mathbf{1}_Y$, and Y is closed. Now let φ_n a sequence of non-negative functions in $\mathcal{H}$ such that $\varphi_n \uparrow \varphi$, with φ bounded. Then φ is measurable, and applying the Monotone Convergence Theorem several times we get

$$\begin{aligned}
\int_Y \varphi(y)d\nu(y) &= \int_Y \lim_{n\to\infty} \varphi_n(y)d\nu(y) \\
&= \lim_{n\to\infty} \int_Y \varphi_n(y)d\nu(y) \\
&= \lim_{n\to\infty} \int_X \left[\int_Y \varphi_n(y)d(f(x))(y) \right] d\mu(x) \\
&= \int_X \left[\int_Y \varphi(y)d(f(x))(y) \right] d\mu(x).
\end{aligned}$$

It follows that $\varphi \in \mathcal{H}$. So $\mathcal{H}$ satisfies the three conditions in Theorem A.14. Also, $\mathcal{H}$ contains the indicator function of every set in some π-system $\mathcal{I}$, namely the π-system $\mathcal{I}$ of closed sets in Y. By the Monotone Class Theorem A.14, $\mathcal{H}$ contains every bounded $\mathcal{M}(\mathcal{I})$-measurable function from Y into $\mathbf{R}$, so in particular indicator function $\mathbf{1}_E$ with $E \in \mathcal{B}(Y)$, since $\mathcal{M}(\mathcal{I}) = \mathcal{B}(Y)$. We conclude that for every $E \in \mathcal{B}(Y)$ we have the desired equality

$$\left(\int_X f(x)d\mu(x)\right)(E) = \int_X f(x)(E)d\mu(x). \quad \blacksquare$$

Due to the previous two propositions, the integral representation (8) can be rewritten once we are able to show that $\phi_n : X_n \to S_{BL}$ defined as

$$\phi_n(x_1, \ldots, x_n) = \mathbf{1}_{E(\bullet,s)\times(s,\infty]}(f_1, \tau_1) + \sum_{m=2}^{n} p_\varepsilon\left((f_m, \tau_m, \tau_1 + \ldots + \tau_{m-1}), \Delta(\bullet, s)\right),$$

is strongly measurable and Bochner integrable. This assumption can be reduced by the assumption that the following functions are strongly measurable and Bochner integrable:

$$\begin{aligned} x_1 = (f_1, \tau_1, 0) &\mapsto \mathbf{1}_{E(\bullet,s)\times(s,\infty]}(f_1, \tau_1), \\ (x_1, \ldots, x_m) &\mapsto p_\varepsilon((f_m, \tau_m, \tau_1 + \ldots + \tau_{m-1}), \Delta(\bullet, s)). \end{aligned}$$

5.2 Concluding remarks

Although we did not fully reach the ultimate goal of establishing the variation of constants formula 1 in a Banach space generated by measures from the consideration of the underlying stochastic process, we strongly believe that μ_s given by equation 7, under suitable conditions, can be approximated well by the solution of such a variation of constants formula. We express the hope that this line of research will be followed through and will establish this assertion.

A Fundamental Measure Theory

Definition A.1 *Let X be any set. A **ring** is a subset $\mathcal{A}$ of the power set $\mathcal{P}(X)$, such that $\mathcal{A}$ satisfies the following conditions*

1. $\emptyset \in \mathcal{A}$;

2. *for all $A, B \in \mathcal{A}$ we have $A \cup B \in \mathcal{A}$ and $B \setminus A \in \mathcal{A}$.*

An **algebra** is a ring $\mathcal{A}$ of the power set $\mathcal{P}(X)$ such that $X \in \mathcal{A}$.
There is also another definition of an **algebra**: a subset $\mathcal{A}$ of the power set $\mathcal{P}(X)$ is called an **algebra** if $\emptyset, X \in \mathcal{A}$, and if

1. if $A_j \in \mathcal{A}$, $j = 1, \dots, n$, then $\bigcup_{j=1}^n A_j \in \mathcal{A}$;

2. if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$.

These two definitions are equivalent. This is easy to see. Suppose the first definition holds. Then $\emptyset \in \mathcal{A}$ since $X \setminus X = \emptyset$. The condition $A_j \in \mathcal{A}, j = 1, \dots, n \Rightarrow \cup_{j=1}^n A_j \in \mathcal{A}$ is just applying the condition of a ring simultaneously. The other condition $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$ follows since $A^c = X \setminus A$.
Otherwise, if the second definition holds, then in particular the statement $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$. Furthermore, note that $B \setminus A = B \cap A^c = (B^c \cup A)^c$.

Definition A.2 *Let X be any set. An algebra $\mathcal{A}$ is called a **σ-algebra** if whenever $A_j \in \mathcal{A}$, $j \in \mathbf{N}$, then $\cup_{j=1}^\infty A_j \in \mathcal{A}$.*

A pair $(X, \mathcal{M})$, where X is a set and $\mathcal{M}$ a σ-algebra, is called a **measurable space**. For any subset $\mathcal{E}$ of the power set $\mathcal{P}(X)$, there exists a unique smallest σ-algebra $\mathcal{M}(\mathcal{E})$ such that $\mathcal{E} \subset \mathcal{M}(\mathcal{E})$. This σ-algebra $\mathcal{M}(\mathcal{E})$ is called the σ-algebra **generated by $\mathcal{E}$**.
If X is a topological space, then we denote by $\mathcal{B}(X)$ the σ-algebra generated by open sets.

Lemma A.3 *If $\mathcal{E}$ and $\mathcal{F}$ are subsets of the power set $\mathcal{P}(X)$ with $\mathcal{E} \subset \mathcal{M}(\mathcal{F})$, then we have $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$.*

Proof: By definition, $\mathcal{M}(\mathcal{E})$ is the smallest σ-algebra containing $\mathcal{E}$. Since $\mathcal{M}(\mathcal{F})$ is one of such σ-algebra's, we have either $\mathcal{M}(\mathcal{E}) = \mathcal{M}(\mathcal{F})$ or $\mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{F})$. ∎

Definition A.4 *Let $\mathcal{M}$ be a σ-algebra. A **measure** is a function $\mu : \mathcal{M} \to [0, \infty]$ that satisfies the following conditions*

1. $\mu(\emptyset) = 0$;

2. *if $A_j \in \mathcal{M}$, $j \in \mathbf{N}$, and A_j are pairwise disjoint, then*

$$\mu(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu(A_j).$$

A triple $(X, \mathcal{M}, \mu)$, where X is a set, $\mathcal{M}$ a σ-algebra and μ a measure, is called a **measure space**. A measure is called **finite** when $\mu(X) < \infty$, and is called a **probability measure** when $\mu(X) = 1$. If there exists a countable cover $X = \cup_{j=1}^{\infty} A_j$, such that $\mu(A_j) < \infty$ for all j, then μ is called **σ-finite**. A measure space $(X, \mathcal{M}, \mu)$ with μ a probability measure is called a **probability space**.

One of the probability measures we will be using is the **Dirac measure** δ_x at a point $x \in X$. This measure is defined as

$$\delta_x(A) = \begin{cases} 1 & x \in A; \\ 0 & x \notin A. \end{cases}$$

It is easy to verify that this is indeed a measure.

Theorem A.5 *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $A_j \in \mathcal{M}$, $j = 1, 2, \ldots$. The following two conditions hold*

i) if $A_1 \subset A_2 \subset \ldots$, then $\mu(\cup_{j=1}^{\infty} A_j) = \lim_{j\to\infty} \mu(A_j)$;

ii) if $A_1 \supset A_2 \supset \ldots$ and $\mu(A_1) < \infty$, then $\mu(\cap_{j=1}^{\infty} A_j) = \lim_{j\to\infty} \mu(A_j)$.

Proof: First we prove i). Let $A_0 = \emptyset$ and notice that $B_j := A_j \setminus A_{j-1}$, $j = 1, 2, \ldots$, is a sequence of disjoint sets. Since $A_j \subset A_{j+1}$, we get $A_k = \cup_{j=1}^{k} B_j$. Also, $A_k = \cup_{j=1}^{k} A_k$. It follows that $\cup_{j=1}^{k} A_j = \cup_{j=1}^{k} B_j$, and so

$$\cup_{j=1}^{\infty} A_j = \cup_{j=1}^{\infty} B_j.$$

Because the B_j's form a sequence of disjoint sets, we get

$$\begin{aligned} \lim_{k\to\infty} \mu(A_k) &= \lim_{k\to\infty} \mu(\cup_{j=1}^{k} B_j) = \lim_{k\to\infty} \sum_{j=1}^{k} \mu(B_j) = \sum_{j=1}^{\infty} \mu(B_j) \\ &= \mu(\cup_{j=1}^{\infty} B_j) = \mu(\cup_{j=1}^{\infty} A_j). \end{aligned}$$

This proves i). To prove ii), Set $B_j = A_1 \setminus A_j$. Then we have $B_1 \subset B_2 \subset \ldots$. Also, since B_j and A_j are disjoint sets, and since $A_1 \supset A_2 \supset \ldots$, we get

$$\mu(B_j) + \mu(A_j) = \mu(B_j \cup A_j) = \mu((A_1 \cap A_j^c) \cup A_j) = \mu(A_1).$$

By elementary set theory, we get $\cup_{j=1}^{\infty} B_j = A_1 \setminus (\cap_{j=1}^{\infty} A_j)$. Since B_j and A_j are disjoint, also B_j and $\cap_{j=1}^{\infty} A_j$ are disjoint, and so we have that $\cup_{j=1}^{\infty} B_j$ and $\cap_{j=1}^{\infty} A_j$ are disjoint. Using this and the fact that $\cap_{j=1}^{\infty} A_j \subset A_1$, we get

$$\begin{aligned} \mu\left(\bigcup_{j=1}^{\infty} B_j\right) + \mu\left(\bigcap_{j=1}^{\infty} A_j\right) &= \mu\left(\bigcup_{j=1}^{\infty} B_j \cup \bigcap_{j=1}^{\infty} A_j\right) \\ &= \mu\left(\left(A_1 \cap \left(\bigcup_{j=1}^{\infty} A_j\right)^c\right) \cup \bigcap_{j=1}^{\infty} A_j\right) \\ &= \mu(A_1). \end{aligned}$$

By part i) we see that $\mu(\cup_{j=1}^{\infty} B_j = \lim_{j\to\infty} \mu(B_j)$, and hence

$$\mu(A_1) = \lim_{j\to\infty} \mu(B_j) + \mu\left(\bigcap_{j=1}^{\infty} A_j\right) = \lim_{j\to\infty} (\mu(A_1) - \mu(A_j) + \mu\left(\bigcap_{j=1}^{\infty} A_j\right).$$

Since we assumed that $\mu(A_1) < \infty$, we get

$$\lim_{j\to\infty} \mu\left(\bigcap_{j=1}^{\infty} A_j\right) - \mu(A_j) = 0,$$

and the result follows. ∎

Definition A.6 *Let X be any set. A **semiring** is a subset $\mathcal{D} \subset \mathcal{P}(X)$ that satisfies the following conditions*

1. *$\emptyset \in \mathcal{D}$;*
2. *if $A, B \in \mathcal{D}$ then $A \cap B \in \mathcal{D}$;*
3. *if $A, B \in \mathcal{D}$, then we can write $A \setminus B = \bigcup_{a\leq j\leq n} C_j$, where $C_j \in \mathcal{D}$ are disjoint.*

Note the following. If we have a semiring $\mathcal{D}$, and we add the condition $A, B \in \mathcal{D} \Rightarrow A \cup B \in \mathcal{D}$, then our semiring becomes a ring.

Definition A.7 *For any subset $\mathcal{A}$ of the power set $\mathcal{P}(X)$, the **ring generated by** $\mathcal{A}$ is the intersection of all rings including $\mathcal{A}$.*

Given a semiring $\mathcal{A}$, let $\mathcal{R}$ be the set of all finite disjoint unions of $\mathcal{A}$. Then $\mathcal{R}$ is a ring. For a proof, see [8, §3.2, p.96]. This is obviously the ring generated by $\mathcal{A}$, since there is no smaller ring including $\mathcal{A}$.

Definition A.8 *Let $\mathcal{D}$ be a semiring. A function $\alpha : \mathcal{D} \to [0, \infty)$ is called **additive** when for $D \in \mathcal{D}$ with $D = \cup_{j=1}^{n} D_j$, where the D_j are pairwise disjoint, we have that $\alpha(D) = \sum_{j=1}^{n} \alpha(D_j)$. It is called **countably additive** when for $D \in \mathcal{D}$ with $D = \cup_{j=1}^{\infty}$, where the D_j are pairwise disjoint, we have that $\alpha(D) = \sum_{j=1}^{\infty} \alpha(D_j)$.*

Proposition A.9 *Let $\mathcal{D}$ be a semiring and $\alpha : \mathcal{D} \to [0, \infty]$ an additive function. For disjoint $D_j \in \mathcal{D}$, let*

$$\mu\left(\bigcup_{j=1}^{m} D_j\right) := \sum_{j=1}^{m} \alpha(D_j).$$

Then μ is well-defined and additive on the ring $\mathcal{R}$ generated by $\mathcal{D}$. If α is countably additive on $\mathcal{D}$, so is μ on $\mathcal{R}$, and then μ extends to a measure on the σ-algebra generated by $\mathcal{D}$ of $\mathcal{R}$.

For a proof, see [8, §3.2, Prop. 3.2.4].

Definition A.10 *Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be two measurable spaces. A function $f : X \to Y$ is called **measurable** if $f^{-1}(E) \in \mathcal{M}$ when $E \in \mathcal{N}$.*

Proposition A.11 *Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be two measurable spaces and $f : X \to Y$. If $\mathcal{N}$ is generated by a set $\mathcal{E}$ then f is measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$*

Proof: If f is measurable, then $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$ so in particular for $E \in \mathcal{E}$. On the other hand, suppose that $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$. Consider the set $\Sigma = \{E \subset Y | \ f^{-1}(E) \in \mathcal{M}\}$. We will prove that Σ is a σ-algebra. Let $E \in \Sigma$. Then $f^{-1}(E) \in \mathcal{M}$, and $f^{-1}(E^c) = (f^{-1}(E))^c \in \Sigma$. Let $E_1, E_2 \ldots \in \Sigma$, then $f^{-1}(\cup_{i=1}^{\infty} E_i) = \cup_{i=1}^{\infty} f^{-1}(E_i)$, so it follows that $\cup_{i=1}^{\infty} E_i \in \Sigma$. So Σ is indeed a σ-algebra, and by assumption we have $\mathcal{E} \subset \Sigma$. By Lemma A.3 we have that $\mathcal{N} \subset \Sigma$. The result now follows. ∎

Definition A.12 *Let X be any set. A π**-system** is a subset $\mathcal{I}$ of $\mathcal{P}(X)$ such that whenever $I_1, I_2 \in \mathcal{I}$ then $I_1 \cap I_2 \in \mathcal{I}$.*

The following lemma states that if two probability measures agree on a π-system, then they agree on the σ-algebra generated by that π-system. A proof can be found in [16, A1.4, p.194]

Lemma A.13 *Let $\mathcal{I}$ a π-system, and $\mathcal{M} = \mathcal{M}(\mathcal{I})$. If μ_1, μ_2 are two probability measures such that $\mu_1(B) = \mu_2(B)$ for all $B \in \mathcal{I}$, then $\mu_1(B) = \mu_2(B)$ for all $B \in \mathcal{M}$.*

The following theorem, called the Monotone Class Theorem, which can be used when creating certain measures on product spaces. The proof can be found at [16, A3.1, p205].

Theorem A.14 (Monotone Class Theorem) *Let $\mathcal{H}$ be a class of bounded functions from a set S into $\mathbf{R}$ satisfying the following conditions:*

i) *$\mathcal{H}$ is a vector space over $\mathbf{R}$;*

ii) *the constant function 1 is an element of $\mathcal{H}$;*

iii) *if (f_n) is sequence of non-negative functions in $\mathcal{H}$ such that $f_n \uparrow f$ pointwise, where f is a bounded function on S, then $f \in \mathcal{H}$.*

Then if $\mathcal{H}$ contains the indicator function of every set in some π-system $\mathcal{I}$, then $\mathcal{H}$ contains every bounded $\sigma(\mathcal{I})$-measurable function on S.

We will conclude this section with a technique how to extend a measure from an algebra to a σ-algebra.

Definition A.15 *Let $\mathcal{A}$ be an algebra. A function $\mu_0 : \mathcal{A} \to [0, \infty]$ is called a **premeasure** if*

i) *$\mu_0(\emptyset) = 0$;*

ii) *if $A_j \in \mathcal{A}$ for $j = 1, 2, \ldots$ are pairwise disjoint such that $\cup_{j=1}^{\infty} A_j \in \mathcal{A}$, then*

$$\mu_0\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu_0(A_j).$$

The following theorem states that it is possible to make an extension. The proof can be found in [10, §1.4, p.31]

Theorem A.16 *Let μ_0 be a premeasure on an algebra $\mathcal{A}$, and let $\mathcal{M}$ be the σ-algebra generated by $\mathcal{A}$. Then there exists a measure μ on $\mathcal{M}$ such that the restriction of μ to $\mathcal{A}$ is precisely μ_0.*

B Measurable functions with range in $[-\infty, \infty]$

Let d be a metric on $[-\infty, \infty]$ defined as follows.

$$d(x,y) = \begin{cases} \frac{|x-y|}{1+|x-y|} & \text{if } -\infty < x, y < \infty, \\ \\ 1 & \text{if either } y = \pm\infty \text{ and } x \neq \pm\infty, \\ & x = \pm\infty \text{ and } y \neq \pm\infty, \\ & \text{or } y = -x = \pm\infty \\ \\ 0 & \text{if } y = x = \pm\infty. \end{cases}$$

Lemma B.1 *The σ-algebra $\mathcal{B}([-\infty, \infty])$ is generated by $\mathcal{E} = \{(a, b] : -\infty \leq a < b \leq \infty\}$*

Proof: First note that if $E \subset [-\infty, \infty]$ with $\pm\infty \notin E$ then E is open if and only if E is open in $\mathbf{R}$ with its standard metric. Also, every open set in $\mathbf{R}$ can be written as a countable union of disjoint open intervals (for a proof of this fact, see [15, §1.1, p.6].
Suppose $a < b < \infty$ and consider $(a, b]$. Note that $(a, b] = \cap_{n=1}^{\infty}(a, b + \frac{1}{n})$. If $b = \infty$, then $(a, b]$ is open in $[-\infty, \infty]$. So in either case, $(a, b] \in \mathcal{B}([-\infty, \infty])$. It follows that $\mathcal{E} \subset \mathcal{B}([-\infty, \infty])$ and by Lemma A.3 we have $\mathcal{M}(\mathcal{E}) \subset \mathcal{B}([-\infty, \infty])$. On the other hand, suppose $E \subset [-\infty, \infty]$ is open with $\pm\infty \notin E$. Then E can be written as a countable union of disjoint open intervals. Every such open interval is an element in $\mathcal{M}(\mathcal{E})$. Note that $\{\infty\} = \cap_{n=1}^{\infty}(n, \infty] \in \mathcal{M}(\mathcal{E})$. A similar reasoning gives us $\{-\infty\} \in \mathcal{M}(\mathcal{E})$. So, for an arbitrary open E we have that

$$E = E \setminus (\{-\infty\} \cup \{\infty\}) \cup \{-\infty\} \cup \{\infty\} \in \mathcal{M}(\mathcal{E}).$$

Again by Lemma A.3 it follows that $\mathcal{B}([-\infty, \infty]) \subset \mathcal{M}(\mathcal{E})$. We conclude that $\mathcal{B}([-\infty, \infty]) = \mathcal{M}(\mathcal{E})$. ∎

Lemma B.2 *$\mathcal{B}([-\infty, \infty])$ is generated by $\mathcal{E}_2 = \{(a, \infty] : -\infty \leq a < \infty\}$*

Proof: We will show that $\mathcal{M}(\mathcal{E}) = \mathcal{M}(\mathcal{E}_2)$, where $\mathcal{E} = \{(a, b] : -\infty \leq a < b \leq \infty\}$. It is immediately clear that $\mathcal{M}(\mathcal{E}_2) \subset \mathcal{M}(\mathcal{E})$. To prove the other inclusion, note that if $b < \infty$, then $(a, b] = (a, \infty] \cap ((b, \infty])^c$. It follows that $\mathcal{E} \subset \mathcal{M}(\mathcal{E}_2)$, hence by Lemma A.3, $\mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{E}_2)$. The result now follows from the previous Lemma B.1. ∎

The following proposition shows how to make new measurable functions from existing ones.

Proposition B.3 *Let* $(X, \mathcal{M})$ *be a measurable space. If* $f, g : X \to [-\infty, \infty]$ *measurable and* $\lambda \in \mathbf{R}$*, then*

i) λf *is measurable;*

ii) $f + g$ *is measurable, with* $(f + g)(x) = 0$ *when* $f(x) = -g(x) = \pm\infty$ *;*

iii) $|f|$ *is measurable.*

Proof: By Lemma B.2 and Proposition A.11 it follows that f is measurable if and only if $f^{-1}((a, \infty]) \in \mathcal{M}$ for all $-\infty \leq a < \infty$. For *i*), note that if $\lambda \neq 0$ then $f^{-1}((\lambda^{-1}a, \infty]) = (\lambda f)^{-1}((a, \infty]) \in \mathcal{M}$ for all a. So λf is measurable, when $\lambda \neq 0$. But this is also the case if $\lambda = 0$, since the constant function 0 is measurable. This proves *i*)
For *ii*), note that

$$\begin{aligned}(f + g)^{-1}((a, \infty]) &= \{x \in X : \ (f + g)(x) > a\} \\ &= \cup_{r \in \mathbf{Q}}(\{x \in X : \ f(x) > f\} \cap \{x \in X : \ g(x) > a - r\}) \in \mathcal{M}.\end{aligned}$$

This proves *ii*). For *iii*), note that

$$\{x \in X : \ |f(x)| > a\} = \{x \in X : \ f(x) > a\} \cup \{x \in X : \ -f(x) > a\}.$$

This proves *iii*). ∎

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