Leiden University

Bachelor project

Theorems on ultrafilters

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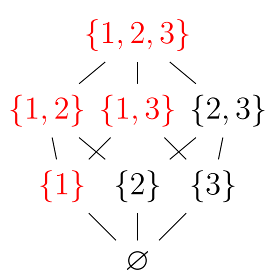

August 25, 2013

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Preface

In this thesis we explore some interesting theorems about ultrafilters. We start by defining filters and ultrafilters on a set X as a collection of subsets of X, satisfying properties in such a way that we can intuitively think of the sets in the filter as being 'large'.

The notion of (ultra)filters and many of the results we will encounter are abstract and set-theoretic (e.g., some proofs use infinite combinatorics). However, filters are known to be applicable in a wide variety of mathematical areas, most notably in topology and model theory. Ideals of Boolean rings are the dual notion of filters on these sets, as an example of an occurence in algebra. Also, in functional analysis they may apply as a method of convergence, similar to the notion of nets (more specifically, for every filter base an associated net can be constructed).

Despite these interesting applications, we will focus more on properties of ultrafilters themselves, but sometimes it will be hinted why it is useful to get a specific result. For example, knowing the number of ultrafilters on a set (see section 2) provides enough information to easily compute the number of topologies on these sets. In a different view, we can define a natural topology on the set of all ultrafilters on a given set X, this is called the Stone-Čech compactification of X. Using ultrafilters on X is not the only method to obtain this topological space. This compactification (denoted by βX) has a very rich mathematical structure, for which we refer the reader to [3] for more details.

In the first section we will give some fundamental definitions and examples, followed by a few basic, yet important corollaries and theorems. In the next part, we study independent sets and derive a method to compute the number of ultrafilters on any given set. The third section is concerned with the Rudin-Keisler ordering on ultrafilters; the main objective here is to prove that the ordering is non-linear. More specifically, we will indicate two non-comparable *uniform* ultrafilters. Finally, we devote the last section to a proof of the existence of a good, countably incomplete ultrafilter on any infinite set. The latter has some useful consequences in model theory, particularly because they make ultrapowers saturated.

Preliminary knowledge required to read this thesis includes naive set theory, e.g., ordinals, cardinals, transfinite induction/recursion and the corresponding notation. We assume the

Axiom of Choice (AC), but will not excessively use cardinal arithmetic. The following is an example of a statement that I did not find necessary to prove, indicating the overall level: "The family $[\kappa]^{<\aleph_0}$ of finite subsets of an infinite cardinal number κ has cardinality κ."[1]

Before we start, I would like to thank my supervisor K.P. Hart for many enlightening conversations about the subject and for his general support throughout this year. While not concretely referred to anywhere in this thesis, the source [1] has proven to be most useful in the last two sections, therefore many credits are directed to this article.

[1]The proof is just $|\kappa^{<\aleph_0}| = \left|\bigcup_{n\in\aleph_0}[\kappa]^n\right| = \sum_{n\in\aleph_0}|[\kappa]^n| = \kappa\cdot\aleph_0 = \kappa$, where $[\kappa]^n$ denotes the set of (finite) subsets of κ which have exactly n elements.

1 Fundamental Matters

Informally, given a set X, we would like to provide a way of deciding whether a subset $Y \subset X$ is 'large'.

Definition 1.1. *Let X be a set. A (proper) filter on X is a nonempty collection $\mathscr{F} \subset \mathcal{P}(X)$ such that*

1. $\varnothing \notin \mathscr{F}$;
2. *If $A, B \in \mathscr{F}$ then $A \cap B \in \mathscr{F}$;*
3. *If $A \in \mathscr{F}$ and $A \subset B \subset X$ then $B \in \mathscr{F}$.*

First let us examine a few elementary consequences of this definition. By an induction argument it easily follows from (2) that a filter is closed under finite intersections. Also, from (3) it follows that X is a member of any filter on X. Note that $\mathcal{P}(X)$ is not a filter on X. However, it satisfies (2) and (3). Therefore it is sometimes called the improper filter on X, explaining the optional word 'proper' in the definition. Conversely, if $\mathscr{F}$ is a collection of subsets of X containing the empty set and satisfying (2) and (3), it follows from (3) that $\mathscr{F} = \mathcal{P}(X)$, i.e., $\mathscr{F}$ is improper. Let us continue by defining the most central objects in this thesis.

Definition 1.2. *Let $\mathscr{F}$ be a proper filter on a set X. Then $\mathscr{F}$ is an ultrafilter on X if there is no proper filter $\mathscr{G}$ on X such that $\mathscr{F} \subsetneq \mathscr{G}$.*

In other words, if we denote by $\Omega(X)$ the set of all filters on X, then $(\Omega(X), \subset)$ is a partially ordered set and ultrafilters are its maximal elements. If X is understood, we will simply speak of '(ultra)filters' instead of '(ultra)filters on X'. Frequently used symbols for ultrafilters are $\mathscr{U}$ and $\mathscr{V}$. There are many equivalent ways to define ultrafilters and throughout the thesis, the following is used frequently.

Proposition 1.3. *Let $\mathscr{F}$ be a proper filter on a set X. Then $\mathscr{F}$ is an ultrafilter if and only if for every $A \subset X$ either $A \in \mathscr{F}$ or $X \backslash A \in \mathscr{F}$.*

Proof. Suppose $\mathscr{F}$ is not an ultrafilter. Let $\mathscr{G}$ be a filter such that $\mathscr{F} \subsetneq \mathscr{G}$ and take $A \in \mathscr{G} \backslash \mathscr{F}$. Then $X \backslash A \notin \mathscr{F}$, for otherwise we would have $\varnothing = A \cap (X \backslash A) \in \mathscr{G}$ by (2).

Conversely, if there is $A \subset X$ such that neither A nor its complement $X \backslash A$ is a member of $\mathscr{F}$, it suffices to point out a proper filter $\mathscr{G}$ which extends $\mathscr{F}$. Take $\mathscr{G} = \{G \subset X : (\exists F \in \mathscr{F})\ F \cap A \subset G\}$. First we need to show that $\mathscr{G}$ is actually a filter:

1. If $\varnothing$ would be a member of $\mathscr{G}$ then there is $F \in \mathscr{F}$ such that $F \cap A = \varnothing$. But then $F \subset X \backslash A$, so $X \backslash A \in \mathscr{F}$ because $\mathscr{F}$ is closed under supersets, contradiction. We conclude $\varnothing \notin \mathscr{G}$.
2. If $G_1, G_2 \in \mathscr{G}$ then there are $F_1, F_2 \in \mathscr{F}$ such that $F_1 \cap A \subset G_1$ and $F_2 \cap A \subset G_2$. Now $F_1 \cap F_2 \in F$ because $\mathscr{F}$ is closed under (finite) intersections and $(F_1 \cap F_2) \cap A \subset G_1 \cap G_2$. Consequently, $G_1 \cap G_2 \in \mathscr{G}$.

3. If $G \in \mathscr{F}$, take $F \in \mathscr{F}$ such that $F \cap A \subset G$. Now $G \subset H \subset X$ implies $F \cap A \subset H$, hence $H \in \mathscr{G}$ for any H containing G.

Because $F \cap A \subset F$ whenever $F \in \mathscr{F}$, we have $\mathscr{F} \subset \mathscr{G}$. Also, $A \in \mathscr{G} \backslash \mathscr{F}$ (take, for example, $F = X$ in the definition of $\mathscr{G}$). Hence, $\mathscr{G}$ is a proper filter on X strictly containing $\mathscr{F}$. $\square$

This proposition shows that ultrafilters actually 'choose' between subsets and their complements on a set. Certain intuitive properties are achieved by this selection, e.g. ultrafilters induce a 2-valued measure on the set.

Generally, filters can be generated by certain collections of subsets. In the previous proof, we implicitly used the notion of a generated filter already. In fact, any collection of subsets enjoying the finite intersection property generates a filter as we will see in a moment.

Definition 1.4. *Let X be a set and $\mathscr{A} \subset \mathcal{P}(X)$ a collection of subsets. Then $\mathscr{A}$ has the finite intersection property (FIP) if any finite intersection of sets in $\mathscr{A}$ is nonempty. In other words, $\mathscr{A}$ has (FIP) if and only if for every natural number $n \in \mathbb{N}$ and any sequence $\{A_k\}_{k \leqslant n} \subset \mathscr{A}$ we have $\bigcap_{k \leqslant n} A_k \neq \varnothing$.*

Clearly, filters have (FIP) because they do not contain the empty set. Also, if $\mathscr{A}$ has (FIP) then it can not contain the empty set itself. Note that $\bigcap \mathscr{A}$ might be empty if $\mathscr{A}$ is infinite. For example, $\mathscr{A} = \{\mathbb{N} \backslash \{n\} : n \in \mathbb{N}\} \subset \mathcal{P}(\mathbb{N})$ has (FIP) but $\bigcap \mathscr{A} = \varnothing$.

Definition 1.5. *Let X be a set and $\mathscr{A} \subset \mathcal{P}(X)$ a collection of subsets. The (im)proper filter generated by $\mathscr{A}$ is the set*

$$\langle \mathscr{A} \rangle = \bigcap \big(\{\mathscr{F} \subset \mathcal{P}(X) : \mathscr{F} \supset \mathscr{A} \text{ and } \mathscr{F} \text{ is a(n) (im)proper filter on } X\} \big).$$

So $\langle \mathscr{A} \rangle$ is the intersection of all (im)proper filters on X that contain $\mathscr{A}$. Here follows an equivalent characterization that is often easier to work with.

Proposition 1.6. *Let X be a set and $\mathscr{A} \subset \mathcal{P}(X)$ a collection of subsets. Then*

$$\langle \mathscr{A} \rangle = \{F \subset X : (\exists n \in \mathbb{N})(\exists A_1, \dots, A_n \in \mathscr{A})\, A_1 \cap \dots \cap A_n \subset F\}.$$

Proof. Let us denote the set on the right hand side in the proposition by $\mathscr{G}$. To prove $\mathscr{G} \subset \langle \mathscr{A} \rangle$, take $G \in \mathscr{G}$ and let $\mathscr{F}$ be an (im)proper filter on X containing $\mathscr{A}$. Because $G \in \mathscr{G}$ there are $A_1, \dots, A_n \in \mathscr{A}$ such that $A_1 \cap \dots \cap A_n \subset G$. Since $\mathscr{A} \subset \mathscr{F}$ we have $A_1 \cap \dots \cap A_n \in \mathscr{F}$. Hence $G \in F$ because $\mathscr{F}$ is closed under supersets. Since $\mathscr{F}$ was arbitrary, we have $G \in \langle \mathscr{A} \rangle$ and therefore $\mathscr{G} \subset \langle \mathscr{A} \rangle$.

Conversely, by definition of $\langle \mathscr{A} \rangle$ it suffices to show that $\mathscr{G}$ is an (im)proper filter on X containing $\mathscr{A}$. Let $G_1, \dots, G_n \in \mathscr{G}$. Then there exist, for every $1 \leqslant i \leqslant n$, an $n_i \in \mathbb{N}$ and a sequence $\{A^i_j\}_{1 \leqslant j \leqslant n_i} \subset \mathscr{A}$ such that $\bigcap_{1 \leqslant j \leqslant n_i} A^i_j \subset G_i$. We have

$$\bigcap_{1 \leqslant i \leqslant n} \bigcap_{1 \leqslant j \leqslant n_i} A^i_j \quad \subset \quad \bigcap_{1 \leqslant i \leqslant n} G_i,$$

and because the intersection on the left is finite, this yields $G_1 \cap \dots \cap G_n \in \mathscr{F}$. Closure of $\mathscr{F}$ under supersets is trivial as in proposition 1.3. $\square$

Clearly, if $\mathscr{A}$ has (FIP) in the context above then $\varnothing \notin \langle \mathscr{A} \rangle$. Thus, $\langle \mathscr{A} \rangle$ is a proper filter on X if and only if $\mathscr{A}$ has (FIP).

Let us look at some natural examples of (ultra)filters and definine free, principal and non-principal filters.

Let X be a set.

- $\{X\}$ is called the trivial filter on X. For nonempty X this is a proper filter, but it is an ultrafilter (if and) only if X is a singleton set.
- For any subset $\varnothing \neq A \subset X$, the set $\{Y \subset X : A \subset Y\}$ is the filter generated by A; it equals $\langle \{A\} \rangle$. If A is a singleton subset, i.e., $A = \{a\}$ for some $a \in X$, then the *principal filter* generated by A consists of the subsets containing a. In this case, $\langle \{A\} \rangle$ is easily verified to be an ultrafilter by making use of proposition 1.6.
- The Fréchet filter on X, denoted by FR_X, is the collection of all subsets with finite complement, i.e. cofinite subsets. If X is finite then $\mathrm{FR}_X = \mathcal{P}(X)$ is not a proper filter. For this reason it is only defined for infinite X in most texts. When X is infinite, clearly $\varnothing \notin \mathrm{FR}_X$ and if A, B are finitely complemented in X, then $X \backslash (A \cap B) = X \backslash A \cup X \backslash B$ is finite as well. Supersets of elements in FR_X have an even smaller complement, so FR_X is indeed a filter on infinite X. From now on, we will always assume X to be infinite if we work with the Fréchet filter on X.
- A filter $\mathscr{F}$ on X is free if $\bigcap \mathscr{F} = \varnothing$. Since filters are closed under finite intersections, no filter on a finite set is free. Indeed, any filter on a finite set is generated by the intersection of its members.

The Fréchet filter on X is free. Otherwise, take $a \in \bigcap \mathrm{FR}_X$ and take any $Y \in \mathrm{FR}_X$. Then $a \in Y$, but $a \notin Y \backslash \{a\} \in \mathrm{FR}_X$, contradiction. Hence, FR_X is also non-principal. Another interesting property is that every free filter contains the Fréchet filter: let $\mathscr{F}$ be a free filter on X and let $Y \in \mathrm{FR}_X$. For $x \in X \backslash Y$, take $F_x \in \mathscr{F}$ such that $x \notin F_x$. Since $X \backslash Y$ is finite, the intersection $F := \bigcap_{x \in X \backslash Y} F_x$ is a member of $\mathscr{F}$ and $F \subset Y$ by construction. Thus $Y \in \mathscr{F}$. If we assume (AC), however, the Fréchet filter is never an ultrafilter on X, because in that case we can write $X = X_1 \sqcup X_2$ where $|X_1| = |X_2| = |X|$. As a consequence, neither X_1 nor its complement, X_2, is a member of FR_X.

Note that there are non-principal filters that are not free either. The filter $\mathscr{F} = \langle \{\mathbb{N} \backslash \{2k\} : k \in N\} \rangle$ provides an example: we have $\bigcap \mathscr{F} = \mathbb{N} \backslash E$ where E denotes the set of even natural numbers. Any $F \in \mathscr{F}$ has a finite complement, so $\mathbb{N} \backslash E \notin \mathscr{F}$. Thus, $\mathscr{F}$ is neither free nor principal.

An interesting question is whether there exist non-principal ultrafilters on an infinite set. The (positive) answer to this requires the Axiom of Choice in the form of Zorn's lemma. Under this assumption, we will prove that any collection with (FIP) can be extended to an ultrafilter. On the other hand, this so-called ultrafilter lemma does not imply (AC). Indeed, it is equivalent to the Boolean prime ideal theorem (BPIT), a well-known intermediate point between the axioms of Zermelo-Fraenkel set theory (ZF) and the theory that augments (ZF) with the Axiom of Choice, (ZFC).

Theorem 1.8. (Ultrafilter lemma) *Let X be a set and suppose $\mathscr{A} \subset \mathcal{P}(X)$ has (FIP). Then there is an ultrafilter $\mathscr{U}$ on X which contains all of $\mathscr{A}$.*

Proof. We will apply Zorn's lemma to the set $\mathscr{B}$ consisting of all proper filters on X containing $\mathscr{A}$, partially ordered by set inclusion. Note that $\mathscr{B}$ is nonempty because $\langle \mathscr{A} \rangle \in \mathscr{B}$. Let $\mathscr{C}$ be a chain in $\mathscr{B}$. We will prove $\bigcup \mathscr{C} \in \mathscr{B}$. First of all, $\varnothing \notin \bigcup \mathscr{C}$ since it is not included in any element of $\mathscr{C}$. If $A, B \in \bigcup \mathscr{C}$, then there are $C_1, C_2 \in \mathscr{C}$ such that $A \in C_1$ and $B \in C_2$. Since $\mathscr{C}$ is a chain, we have $C_1 \subset C_2$ without loss of generality. Consequently, A and B are elements of C_2 and since C_2 is a filter, we have $A \cap B \in C_2 \subset \bigcup \mathscr{C}$. It is a trivial matter to verify that $\bigcup \mathscr{C}$ is closed under supersets, so we have $\bigcup \mathscr{C} \in \mathscr{B}$ indeed. This union is obviously an upper bound of $\mathscr{C}$ in $\mathscr{B}$. According to Zorn's lemma, $\mathscr{B}$ has maximal elements. Let $\mathscr{U}$ be a maximal element of $\mathscr{B}$. If $\mathscr{F} \supset \mathscr{U}$ is a filter, then $\mathscr{A} \subset \mathscr{F}$. Hence $\mathscr{F} \subset \mathscr{U}$ by maximality of $\mathscr{U}$ and we have $\mathscr{U} = \mathscr{F}$. So $\mathscr{U}$ is an ultrafilter and it contains all of $\mathscr{A}$. $\square$

This theorem directly implies the existence of free ultrafilters on infinite X by extending $\mathscr{A} = \{X \backslash \{x\} : x \in X\}$ to an ultrafilter, for example.

2 Independent sets and the number of ultrafilters

This section is devoted to introduce the concept of independent sets and families of large oscillation. After defining them and proving their existence, we will use them to compute the number of ultrafilters on a given set. From now on, we will look at ultrafilters on cardinal numbers. Under the assumption of the Axiom of Choice, every set X can be identified with a cardinal number κ by means of a bijective map. Thus, the ultrafilters on X and κ are essentially the same in this situation.

Definition 2.1. *Let α, β and κ be cardinals and let $\mathscr{S} \subset \beta^\alpha$. The family $\mathscr{S}$ is a family of κ-large oscillation if the following condition is satisfied: if $\lambda < \kappa$, $\{f_\zeta : \zeta < \lambda\}$ are distinct elements of $\mathscr{S}$, and $\{\eta_\zeta : \zeta < \lambda\}$ are elements of β, then there is $\xi \in \alpha$ such that $f_\zeta(\xi) = \eta_\zeta$ for all $\zeta < \lambda$.*

In this thesis we only need families of $\aleph_0$-large oscillation.

Theorem 2.2. *Let κ be an infinite cardinal. Then there is a family $\mathscr{S} \subset \kappa^\kappa$ of $\aleph_0$-large oscillation such that $|\mathscr{S}| = 2^\kappa$.*

Proof. We define

$$\mathscr{F} := \{(F, G, s) : F \in [\kappa]^{<\aleph_0}, G \in \mathcal{P}(\mathcal{P}(F)) \text{ and } s \in \kappa^G\}.$$

If $|F| < \aleph_0$ then $|\mathcal{P}(\mathcal{P}(F))| = 2^{(2^{|F|})} < \aleph_0 \leqslant \kappa$. Also, since every $G \in \mathcal{P}(\mathcal{P}(F))$ is finite, we have $|\kappa^G| = \kappa$ for G as such. Thus $|\mathscr{F}| = \kappa$. Therefore, it is sufficient (by identifying $\mathscr{F}$ with κ) to find $\mathscr{S} \subset \kappa^{\mathscr{F}}$ such that $|\mathscr{S}| = 2^\kappa$ and $\mathscr{S}$ is of $\aleph_0$-large oscillation. We define $f : \mathcal{P}(\kappa) \to \kappa^{\mathscr{F}}$ by $A \mapsto f_A$ where $f_A : \mathscr{F} \to \kappa$ is given by

$$f_A(F, G, s) = \begin{cases} s(A \cap F) & \text{if } A \cap F \in G \\ 0 & \text{if } A \cap F \notin G. \end{cases}$$

To see that f is injective, let $A \neq B \in \mathcal{P}(\kappa)$. Then there is $x \in A \triangle B$, say $x \in A \backslash B$. Take $F = \{x\}$, $G = \{F\}$ and $s(F) = 1$. Then

$$f_A(F, G, s) = s(A \cap F) = s(F) = 1 \neq 0 = f_B(F, G, s),$$

hence $f_A \neq f_B$.

Therefore, the set $\mathscr{S} = \{f_A : A \in \mathcal{P}(\kappa)\}$ has cardinality 2^κ. It remains to prove that $\mathscr{S}$ is a family of $\aleph_0$-large oscillation. Let $n < \aleph_0$, $\{A_m : m < n\}$ be a family of distinct subsets of κ and $\{\eta_m : m < n\}$ a subset of κ. For $\ell < m < n$ we choose $a_{\ell,m} \in A_\ell \triangle A_m$ and we set $F = \{a_{\ell,m} : \ell < m < n\}$. We define $G = \{A_m \cap F : m < n\}$, and a function $s : G \to \kappa$ by $s(A_m \cap F) = \eta_m$. This function is well-defined, for if $\ell \neq m$, then $a_{\ell,m}$ is a member of only one of $A_\ell \cap F$ and $A_m \cap F$, which shows that $A_\ell \cap F \neq A_m \cap F$. Clearly, $(F, G, s) \in \mathscr{F}$ and for $m < n$ we have

$$f_{A_m}(F, G, s) = s(A_m \cap F) = \eta_m. \qquad \square$$

More generally, for cardinals λ and κ, we can show the existence of a 2^κ-power subset of κ^κ of λ-large oscillation if a certain condition is satisfied. This condition is that κ can be written as the sum of all κ^ξ, where $\xi < \lambda$. This assumption is vital for $\mathscr{F}$ to be of cardinality κ in the proof above (see [3, p. 76] for details). Note that the condition is automatically satisfied when $\lambda = \aleph_0$. Another, closely related concept will be needed in the next section.

Definition 2.3. *Let $\mathscr{S} \subset \mathcal{P}(X)$. Then $\mathscr{S}$ is said to be independent (over X) if for any $n, m \geq 0$ and any finite sequence $A_1, \ldots, A_n, B_1, \ldots, B_m \in \mathscr{S}$ of distinct sets,*

$$A_1 \cap \cdots \cap A_n \cap (X \backslash B_1) \cap \cdots \cap (X \backslash B_m) \neq \varnothing.$$

The existence of independent sets is a fairly straight consequence of theorem 2.2;

Corollary 2.4. *Let κ be an infinite cardinal. Then there is an independent $\mathscr{S} \subset \mathcal{P}(\kappa)$ of cardinality 2^κ.*

Proof. Let $\mathscr{T}$ be a family of $\aleph_0$-large oscillation of functions $\kappa \to \{0,1\}$ such that $|\mathscr{T}| = 2^\kappa$. Put $\mathscr{S} = \{f^{-1}(\{0\}) : f \in \mathscr{S}\}$. Let $A_1, \ldots, A_n, B_1, \ldots, B_m \in \mathscr{S}$ be distinct. Then there are distinct $f_1, \ldots, f_n$ and $g_1, \ldots, g_m$ in $\mathscr{T}$ such that $A_i = f_i^{-1}(\{0\})$ and $B_j = g_j^{-1}(\{0\})$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Since $\mathscr{T}$ is of $\aleph_0$-large oscillation, there is $\eta \in \kappa$ such that $f_i(\eta) = 0$ and $g_i(\eta) = 1$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Using that $\kappa \backslash B_j = g_j^{-1}(\{1\})$ for all $1 \leqslant j \leqslant m$, we see

$$A_1 \cap \cdots \cap A_n \cap (\kappa \backslash B_1) \cap \cdots \cap (\kappa \backslash B_m) \neq \varnothing,$$

because η is an element of the intersection. Hence, $\mathscr{S}$ is independent over κ. Finally, assume $f \neq g \in \mathscr{S}$. Then there is $\xi \in \kappa$ such that $f(\xi) \neq g(\xi)$. Since $\xi \in f^{-1}(\{0\}) \triangle g^{-1}(\{0\})$, these preimages are distinct. Thus, $|\mathscr{S}| = 2^\kappa$. $\square$

We denote by $U(X)$ the set of all ultrafilters on X.

Theorem 2.5. *Let κ be a cardinal. Then $|U(\kappa)| = \kappa$ if $\kappa < \aleph_0$, and $|U(\kappa)| = 2^{(2^\kappa)}$ if $\kappa \geq \aleph_0$.*

Proof. If κ is a finite cardinal then every ultrafilter $\mathscr{U}$ on κ is generated by a singleton subset $\{a\}$ ($a \in \kappa$). To see this, assume the contrary. Then for all $a \in \kappa$, $\kappa \backslash \{a\} \in \mathscr{U}$. Since κ is finite, the empty intersection of all $\kappa \backslash \{a\}$ is an element of $\mathscr{U}$, which is impossible. So $|U(\kappa)| \leqslant \kappa$. If $a \neq b$ then $\langle \{a\} \rangle \neq \langle \{b\} \rangle$, so we have $|U(\kappa)| \geq \kappa$ as well.

For infinite κ, $|U(\kappa)| \leqslant 2^{(2^\kappa)}$ is clear from $U(\kappa) \subset \mathcal{P}(\mathcal{P}(\kappa))$. To prove the other inequality, we will exhibit a family of $2^{(2^\kappa)}$ distinct ultrafilters on κ.

Let $\mathscr{S}$ be a family of $\aleph_0$-large oscillation of functions $\kappa \to \{0,1\}$ such that $|\mathscr{S}| = 2^\kappa$. For $\mathscr{C} \in \mathcal{P}(\mathscr{S})$, we set

$$\mathscr{B}(\mathscr{C}) = \{f^{-1}(\{0\}) : f \in \mathscr{C}\} \cup \{f^{-1}(\{1\}) : f \in \mathscr{S} \backslash \mathscr{C}\}.$$

If $B_1, \ldots, B_n \in \mathscr{B}(\mathscr{C})$ then there are $f_1, \ldots, f_n \in \mathscr{S}$ and $\delta_1, \ldots, \delta_n \in \{0,1\}$ such that $B_i = f_i^{-1}(\{\delta_i\})$ for all $1 \leqslant i \leqslant n$. Because $\mathscr{S}$ is a family of ω-large oscillation, there is $\eta \in \kappa$ such that $f_i(\eta) = \delta_i$ for all $1 \leqslant i \leqslant n$. This shows $\bigcap_{1 \leqslant i \leqslant n} B_i \neq \varnothing$, so $\mathscr{B}(\mathscr{C})$ has the finite intersection property. Thus, for all $\mathscr{C} \in \mathcal{P}(\mathscr{S})$ we can extend $\mathscr{B}(\mathscr{C})$ to an ultrafilter $\mathscr{U}(\mathscr{C})$ on κ. If $\mathscr{C}_1 \neq \mathscr{C}_2 \in \mathcal{P}(\mathscr{S})$ then there is $f \in \mathscr{C}_1 \backslash \mathscr{C}_2$, without loss of generality. For such f we have $f^{-1}(\{0\}) \in \mathscr{B}(\mathscr{C}_1) \subset \mathscr{U}(\mathscr{C}_1)$ and $f^{-1}(\{1\}) \in \mathscr{B}(\mathscr{C}_2) \subset \mathscr{U}(\mathscr{C}_2)$, hence $\mathscr{U}(\mathscr{C}_1) \neq \mathscr{U}(\mathscr{C}_2)$ because $f^{-1}(\{0\}) = \kappa \backslash f^{-1}(\{0\})$.

Since $|\mathcal{P}(\mathscr{S})| = 2^{(2^\kappa)}$, the subfamily

$$\{\mathscr{U}(\mathscr{C}) : \mathscr{C} \in \mathcal{P}(\mathscr{S})\}$$

of $U(\kappa)$ has cardinality $2^{(2^\kappa)}$ and this shows that $|U(\kappa)| \geq 2^{(2^\kappa)}$, i.e., $|U(\kappa)| = 2^{(2^\kappa)}$. □

Corollary 2.6. If $|X| = \kappa \geq \aleph_0$, then the number of topologies on X is also $2^{(2^\kappa)}$. This cardinal being an upper bound follows in the same way as for ultrafilters. Since $\mathscr{U} \cup \{\varnothing\}$ is a topology on X whenever $\mathscr{U}$ is an ultrafilter on X, $2^{(2^\kappa)}$ is a lower bound as well. For finite X, however, computing the number of possible topologies on X is more involved (it is equal to the number of pre-orders[2] on the set, see [6] for a relatively new article that proves this claim and uses graph theory in these computations).

[2] A preorder on X is a relation $\leqslant \subset X \times X$ that is reflexive and transitive.

3 Nonlinearity of the Rudin-Keisler ordering

In this section we will study an ordering on ultrafilters, defined independently by M.E. Rudin and H.J. Keisler around 1970. The fundament of this definition is a rather categorical concept between collections of ultrafilters.

Definition 3.1. *Let X and Y be infinite sets and let $f\colon X \to Y$ be a function. Then we define $f_*\colon U(X) \to U(Y)$ by $\mathscr{U} \mapsto \{A \subset Y : f^{-1}(A) \in \mathscr{U}\}$.*

Thus, if we identify $\mathscr{U}$ with a 2-valued measure on X, $f_*(\mathscr{U})$ is the induced measure on Y in the usual measure-theoretic sense. However, we will not examine these measure properties in this thesis.

Proposition 3.2. *Let X, Y, Z be infinite sets and let $f\colon X \to Y$ and $g : Y \to Z$ be functions. Then $(g \circ f)_* = g_* \circ f_*$.*

Proof. Let $\mathscr{U} \in U(X)$. Then we have

$$\begin{aligned}(g \circ f)_*(\mathscr{U}) &= \{A \subset Z : (g \circ f)^{-1}(A) \in \mathscr{U}\}\\ &= \{A \subset Z : f^{-1}(g^{-1}(A)) \in \mathscr{U}\}\\ &= \{A \subset Z : g^{-1}(A) \in f_*(\mathscr{U})\}\\ &= (g_* \circ f_*)(\mathscr{U}).\end{aligned}$$ □

Definition 3.3. **(Rudin-Keisler order)** *Let X and Y be infinite sets and let $\mathscr{U} \in U(X), \mathscr{V} \in U(Y)$. Then $\mathscr{U} \leqslant_{\mathrm{RK}} \mathscr{V}$ if there exists a function $f : X \to Y$ such that $\mathscr{U} = f_*(\mathscr{V})$.*

More often than not, the subscript 'RK' is omitted. This relation is transitive by proposition 3.2, which implies that the relation $\approx$ defined by

$$\mathscr{U} \approx \mathscr{V} \quad \Leftrightarrow \quad \mathscr{U} \leqslant \mathscr{V} \text{ and } \mathscr{V} \leqslant \mathscr{U}$$

is an equivalence relation. The following theorem (see the appendix) shows that $\approx$ is a reasonable notion of equivalence:

Theorem 3.4. *Let X and Y be infinite sets and let $\mathscr{U} \in U(X)$ and $\mathscr{V} \in U(Y)$. Then*

1. *$\mathscr{U} \approx \mathscr{V}$ if and only if there are $A \in \mathscr{U}$, $B \in \mathscr{V}$ and $f : X \to Y$ such that $\mathscr{V} = f_*(\mathscr{U})$ and $f \restriction A$ is injective and onto B;*
2. *If $\mathscr{U} \approx \mathscr{V}$ and $|X| = |Y|$ then there is a bijection $f : X \to Y$ such that $\mathscr{V} = f_*(\mathscr{U})$ and $\mathscr{U} = (f^{-1})_*(\mathscr{V})$.*

The generalized Fréchet filter on an infinite set X is $\mathrm{FR}_X = \{A \subset X : |X \backslash A| < |X|\}$. Note that we use the same notation as in section 1. An ultrafilter $\mathscr{U}$ on a set X is said to be *uniform* if $|A| = |X|$ for all $A \in \mathscr{U}$.

Proposition 3.5. *Let X be an infinite set. Then an ultrafilter $\mathscr{U}$ on X is uniform if and only if it contains FR_X.*

Proof. Suppose $\mathscr{U}$ does not contain FR_X. Then there is $A \in \mathrm{FR}_X \backslash \mathscr{U}$, so $X\backslash A \in \mathscr{U}$ but $|X\backslash A| < |X|$. Therefore, $\mathscr{U}$ is not uniform. Conversely, if $\mathscr{U}$ is not uniform, take $A \in \mathscr{U}$ such that $|A| < |X|$. Then $|X\backslash(X\backslash A)| = |A| < |X|$, so $X\backslash A \in \mathrm{FR}_X\backslash\mathscr{U}$. Hence, $\mathscr{U}$ does not contain FR_X. □

The following proposition justifies the choice of specifying our investigation in these uniform ultrafilters.

Proposition 3.6. *Let X be an infinite set. If $\mathscr{U} \in U(X)$ is not uniform and $A \in \mathscr{U}$ is of least cardinality, then $\mathscr{U} \approx \mathscr{U} \cap \mathcal{P}(A) \in U(A)$.*

Proof. It is a trivial matter to verify that $\mathscr{U} \cap \mathcal{P}(A)$ is an ultrafilter on A. Next, we prove $\mathscr{U} \leqslant \mathscr{U} \cap \mathcal{P}(A)$. Let $f\colon A \to X$ be the inclusion of A into X. Then

$$f_*(\mathscr{U} \cap \mathcal{P}(A)) = \{Y \subset X : f^{-1}(Y) \in U \cap \mathcal{P}(A)\} = \{Y \subset X : Y \cap A \in \mathscr{U} \cap \mathcal{P}(A)\} = \mathscr{U}.$$

To show the other inequality, we take an arbitrary $a \in A$ and define $g\colon X \to A$ by $g(x) = x$ if $x \in A$ and $g(x) = a$ if $x \notin A$. We need to prove the equality $\{Y \subset A : g^{-1}(Y) \in \mathscr{U}\} = \mathscr{U} \cap \mathcal{P}(A)$. Suppose there is $Y \subset A$ such that $g^{-1}(Y) \in \mathscr{U}$ but $Y \notin \mathscr{U}$. Then $X\backslash Y \in \mathscr{U}$, and so is $g^{-1}(Y) \cap A \cap (X\backslash Y) = \{x \in A\backslash Y : g(x) \in Y\} = \varnothing$, contradiction. For the other inclusion, let $Y \in \mathscr{U} \cap \mathcal{P}(A)$ and suppose $g^{-1}(Y) \notin \mathscr{U}$. Then its complement is a member of $\mathscr{U}$ and so is $Y \cap (X\backslash g^{-1}(Y)) = \{x \in Y : \neg((\exists y \in Y) g(y) = x)\} = \varnothing$, contradiction. □

We denote by $U_{\mathrm{u}}(X)$ the set of uniform ultrafilters over X. The main proof in this section reveals that $\leqslant_{\mathrm{RK}}$ restricted to $U_{\mathrm{u}}(X)$ is not a linear relation, i.e.

Theorem 3.7. *If X is infinite, there are $\mathscr{U}, \mathscr{V} \in U_{\mathrm{u}}(X)$ such that $\mathscr{U} \nleqslant \mathscr{V}$ and $\mathscr{V} \nleqslant \mathscr{U}$.*

Proof. We must construct $\mathscr{U}, \mathscr{V} \in U_{\mathrm{u}}(X)$ such that $\mathscr{V} \neq f_*(\mathscr{U})$ and $\mathscr{U} \neq f_*(\mathscr{V})$ for every function $f : X \to X$. So we must have, for every such f,

$$(\exists A \in \mathscr{U})(X\backslash f^{-1}(A) \in \mathscr{V}) \quad \text{and} \quad (\exists B \in \mathscr{V})(X\backslash f^{-1}(B) \in \mathscr{U}). \tag{$*$}$$

Say $|X| = \kappa$. Fix an enumeration $\{f_\eta : \eta < 2^\kappa\}$ of all functions $X \to X$. Using transfinite recursion, the construction will be carried out over the ordinals $\eta < 2^\kappa$. We shall construct two increasing sequences of filters $\{\mathscr{F}_\eta\}_{\eta<2^\kappa}$, $\{\mathscr{G}_\eta\}_{\eta<2^\kappa}$ and take $\mathscr{U}, \mathscr{V}$ to be ultrafilters extending their respective unions. To be precise, we will insure that the following five properties hold for all $\eta < 2^\kappa$:

1. $\mathscr{F}_\eta$ and $\mathscr{G}_\eta$ are filters on X;
2. If $\xi < \eta$, then $\mathscr{F}_\xi \subset \mathscr{F}_\eta$ and $\mathscr{G}_\xi \subset \mathscr{G}_\eta$;
3. $\mathscr{F}_0 = \mathscr{G}_0 = \mathrm{FR}_X$;
4. If η is a limit ordinal, we set $\mathscr{F}_\eta = \bigcup_{\xi<\eta} \mathscr{F}_\xi$ and $\mathscr{G}_\eta = \bigcup_{\xi<\eta} \mathscr{G}_\xi$;
5. $(\exists A \in \mathscr{F}_{\eta+1})(X\backslash f_\eta^{-1}(A) \in \mathscr{G}_{\eta+1})$ and $(\exists B \in \mathscr{G}_{\eta+1})(X\backslash f_\eta^{-1}(B) \in \mathscr{F}_{\eta+1})$.

Conditions (1)–(4) would not necessarily present a problem, but (5) may become impossible at some stage η. For example, if η is a limit ordinal, the construction before stage η determines what $\mathscr{F}_\eta$ and $\mathscr{G}_\eta$ must be, and it might happen that they are already ultrafilters (which cannot be extended anymore) and that $\mathscr{F}_\eta = (f_\eta)_*(\mathscr{G}_\eta)$. To prevent such failure, we enlist the aid of a generalized concept of independent sets from section 2.

Definition 3.8. *If $\mathscr{S} \subset \mathcal{P}(X)$ and $\mathscr{F}$ is a filter on X, then $\mathscr{S}$ is independent modulo $\mathscr{F}$ if and only if whenever $n, m \geq 0$ and $A_1, \dots, A_n, B_1, \dots B_m$ are distinct elements of $\mathscr{S}$, we have $(X\backslash A_1) \cup \dots \cup (X\backslash A_n) \cup B_1 \dots \cup B_m \notin \mathscr{F}$.*

Note that $\mathscr{S}$ is independent over X (see section 2) if and only if $\mathscr{S}$ is independent modulo $\{X\}$. If $\mathscr{S} \neq \varnothing$ is independent modulo $\mathscr{F}$, then $\mathscr{F}$ is not an ultrafilter because for any $S \in \mathscr{S}$, we would have $S \notin \mathscr{F}$ and $X\backslash S \notin \mathscr{F}$ (see proposition 1.3). We formulate the next property in a lemma.

Lemma 3.9. *If $\mathscr{S}$ is independent modulo $\mathscr{F}$ and $\mathscr{A} \subset \mathscr{S}$, then $\mathscr{S}\backslash\mathscr{A}$ is independent modulo $\langle \mathscr{F} \cup \mathscr{A} \rangle$.*

Proof. We first need to show that $\mathscr{F} \cup \mathscr{A}$ enjoys (FIP), so that $\mathscr{G} := \langle \mathscr{F} \cup \mathscr{A} \rangle$ is a filter. Let $A_1, \dots, A_n \in \mathscr{A}$ and $F \in \mathscr{F}$. If $A_1 \cap \dots \cap A_n \cap F = \varnothing$, then $F \subset (X\backslash A_1) \cup \dots \cup (X\backslash A_n)$, so $(X\backslash A_1) \cup \dots \cup (X\backslash A_n) \in \mathscr{F}$ as a superset of $F \in \mathscr{F}$. But this is impossible since $\mathscr{S}$ is independent modulo $\mathscr{F}$.

To continue, we will show that $\mathscr{S}\backslash\mathscr{A}$ is independent modulo $\mathscr{G}$. If we assume the contrary, then there are $B_1, \dots, B_\ell, C_1, \dots, C_m \in \mathscr{S}\backslash\mathscr{A}$ (distinct), $A_1, \dots A_n \in \mathscr{A}$ and $F \in \mathscr{F}$ such that

$$(X\backslash B_1) \cup \dots \cup (X\backslash B_\ell) \cup C_1 \cup \dots \cup C_m \supset F \cap A_1 \cap \dots \cap A_n.$$

By first taking complements and then intersecting with $A_1 \cap \dots \cap A_n$ on both sides, we get

$$B_1 \cap \dots \cap B_\ell \cap (X\backslash C_1) \cap \dots \cap (X\backslash C_m) \cap A_1 \cap \dots \cap A_n \subset X\backslash F.$$

Taking complements again and using that $\mathscr{F}$ is closed under supersets shows that the complement of the left-hand side is a member of $\mathscr{F}$, which is a contradiction since all A_i, B_j and C_k are distinct and $\mathscr{S}$ is independent modulo $\mathscr{F}$. □

Furthermore, an independent set $\mathscr{S} \subset \mathcal{P}(X)$ of cardinality 2^κ exists by corollary 2.4 and it can be taken to be independent modulo FR_X. To see this, we take $g\colon X \to X$ such that $|g^{-1}(\{x\})| = \kappa$ for all $x \in X$. The set $\mathscr{S}' = \{g^{-1}(S) : S \in \mathscr{S}\}$ is easily seen to be of cardinality 2^κ. Take $S_1, \dots, S_n, T_1, \dots, T_m \in \mathscr{S}$ to be distinct. Then

$$X\backslash\Big[(X\backslash g^{-1}(S_1)) \cup \dots \cup (X\backslash g^{-1}(S_n)) \cup g^{-1}(T_1) \cup \dots \cup g^{-1}(T_m)\Big]$$

is just

$$g^{-1}(S_1 \cap \dots \cap S_n \cap (X\backslash T_1) \cap \dots \cap (X\backslash T_m)),$$

where the intersection is nonempty, since $\mathscr{S}$ is independent. Therefore, the cardinality of the inverse image is κ and

$$(X\backslash g^{-1}(S_1)) \cup \dots \cup (X\backslash g^{-1}(S_n)) \cup g^{-1}(T_1) \cup \dots \cup g^{-1}(T_m) \notin \mathrm{FR}_X.$$

So $\mathscr{S}'$ is independent modulo FR_X indeed.

To continue the proof, we keep a large family of sets $\mathscr{S}_\eta$, independent modulo $\mathscr{F}_\eta$ and $\mathscr{G}_\eta$. This prevents the filters on every level of the construction from being ultrafilters. Thus, in addition to (1)–(5), we arrange for the following to hold for all $\eta < 2^\kappa$:

6. $\mathscr{S}_\eta$ is independent modulo $\mathscr{F}_\eta$ and $\mathscr{G}_\eta$;
7. If $\xi < \eta < 2^\kappa$, then $\mathscr{S}_\xi \supset \mathscr{S}_\eta$;
8. $|\mathscr{S}_\eta| = 2^\kappa$;
9. If η is a limit ordinal, we set $\mathscr{S}_\eta = \bigcap_{\xi<\eta} \mathscr{S}_\xi$;
10. $\mathscr{S}_\eta \backslash \mathscr{S}_{\eta+1}$ is finite.

Note that (8) is assured by (9) and (10), because we have $|\mathscr{S}_0| = 2^\kappa$ by the previous argument. For limit ordinals $\eta < 2^\kappa$, S_η will be independent modulo $\mathscr{F}_\eta$ and $\mathscr{G}_\eta$: assuming the contrary, we find, without loss of generality, $\zeta < \eta$ and distinct sets $A_1, \dots, A_n, B_1, \dots, B_m \subset X$ such that for all $\xi < \eta$ we have $A_1, \dots, A_n, B_1, \dots, B_m \in \mathscr{S}_\xi$ and

$$(X\backslash A_1) \cup \cdots \cup (X\backslash A_n) \cup B_1 \cup \cdots \cup B_m \in \mathscr{F}_\zeta.$$

This is not possible because $\mathscr{S}_\zeta$ is independent modulo $\mathscr{F}_\zeta$.

The following lemma glues everything together; by applying it twice (once to both filters) in every stage, we can actually carry out the construction.

Lemma 3.10. *Let $\mathscr{H}$ and $\mathscr{K}$ be filters over X. Let $\mathscr{T} \subset \mathcal{P}(X)$ be infinite and independent modulo $\mathscr{H}$ and $\mathscr{K}$. Let $f : X \to X$. Then there are filters $\mathscr{H}' \supset \mathscr{H}$, $\mathscr{K}' \supset \mathscr{K}$, and a family $\mathscr{T}' \subset \mathscr{T}$ such that $\mathscr{T}'$ is independent modulo $\mathscr{H}'$ and $\mathscr{K}'$, $\mathscr{T}\backslash\mathscr{T}'$ is finite, and, for some $B \in \mathscr{H}'$, $X\backslash f^{-1}(B) \in \mathscr{K}'$.*

Proof. Fix $A \in \mathscr{T}$. We distinguish two cases:

1. $\mathscr{T}\backslash\{A\}$ is independent modulo $\langle \mathscr{K} \cup \{X\backslash f^{-1}(A)\}\rangle$. We can take $\mathscr{T}' = \mathscr{T}\backslash\{A\}$, $\mathscr{K}' = \langle \mathscr{K} \cup \{X\backslash f^{-1}(A)\}\rangle$, $\mathscr{H}' = \langle \mathscr{H} \cup \{A\}\rangle$, and $B = A$. Note that $\mathscr{T}'$ is independent modulo $\mathscr{H}'$ by a straightforward application of lemma 3.9.
2. $\mathscr{T}\backslash\{A\}$ is not independent modulo $\langle \mathscr{K} \cup \{X\backslash f^{-1}(A)\}\rangle$. Then then are distinct A_1, $\dots, A_n, B_1, \dots, B_m \in \mathscr{S}\backslash\{A\}$ and $K \in \mathscr{K}$ such that

   $$(X\backslash A_1) \cup \cdots \cup (X\backslash A_n) \cup B_1 \cup \cdots \cup B_m \supset K \cap (X\backslash f^{-1}(A)).$$

   Hence,

   $$A_1 \cap \cdots \cap A_n \cap (X\backslash B_1) \cap (X\backslash B_m) \cap K \subset f^{-1}(A).$$

   Take $\mathscr{K}' = \langle \mathscr{K} \cup \{A_1, \dots, A_n, X\backslash B_1, \dots, X\backslash B_m\}\rangle$, $\mathscr{H}' = \langle \mathscr{H} \cup \{X\backslash A\}\rangle$, $\mathscr{T}' = \mathscr{T}\backslash\{A, A_1, \dots, A_n, X\backslash B_1, \dots, X\backslash B_m\}$ and $B = X\backslash A$. Then $X\backslash f^{-1}(X\backslash A) = f^{-1}(A) \in \mathscr{K}'$. By using an argument similar to the proof of lemma 3.9, we can see that $\mathscr{T}'$ is independent modulo $\mathscr{K}'$ and $\mathscr{H}'$. □

What we have now are sequences $\{\mathscr{F}_\eta : \eta < 2^\kappa\}$ and $\{\mathscr{G}_\eta : \eta < 2^\kappa\}$ satisfying all conditions previously stated. Extend $\mathscr{F} = \bigcup_{\eta<2^\kappa} \mathscr{F}_\eta$ and $\mathscr{G} = \bigcup_{\eta<2^\kappa} \mathscr{G}_\eta$ to ultrafilters $\mathscr{U}$ and $\mathscr{V}$, respectively. Let $f : X \to X$. Then f appears in the enumeration of all functions $X \to X$ we started out with, so there is $\eta < 2^\kappa$ such that $f = f_\eta$. If we take A and B as in condition (5), they obviously suffice for $(*)$ as well. Finally, proposition 3.5 implies that $\mathscr{U}, \mathscr{V} \in U_{\mathrm{u}}(X)$ because they both contain FR_X. This concludes the proof of theorem 3.7. □

4 The existence of good ultrafilters

The interest of good ultrafilters in model theory is that they make ultrapowers saturated. If A and B are two infinite structures of cardinality less than or equal to κ and $\mathscr{U}$ is a good, countably incomplete ultrafilter on κ, then, as Keisler showed in [4], the ultrapowers $A^\kappa/\mathscr{U}$ and $B^\kappa/\mathscr{U}$ have power 2^κ and they are κ^+-saturated. Thus, if $2^\kappa = \kappa^+$ (this is (GCH) at level κ), and A and B are elementarily equivalent, then $A^\kappa/\mathscr{U}$ and $B^\kappa/\mathscr{U}$ are isomorphic. Here we will define and prove the existence of a good, countably incomplete ultrafilter. For details about ultrapowers and saturation in model theory, the reader is referred to [2] and [5].

To understand the main proof in this section, we need to be familiar with the concept of cofinality.

Definition 4.1. *Let α be a limit ordinal. The cofinality of α, written as* $\mathrm{cf}\,(\alpha)$*, is the least ordinal β such that there is an increasing β-sequence $\{\alpha_\xi\}_{\xi<\beta}$ with $\lim_{\xi\to\beta}\alpha_\xi = \alpha$.*

We list a few easy-to-prove consequences and examples here;

- $\mathrm{cf}\,(\omega+\omega) = \mathrm{cf}\,(\aleph_\omega) = \omega$;
- $\mathrm{cf}\,(\mathrm{cf}\,(\alpha)) = \mathrm{cf}\,(\alpha)$ for any limit ordinal α;
- $\mathrm{cf}\,(2^\kappa) > \kappa$ for any infinite cardinal κ.

Proving the last is a little more involved, see corollary 5.12 in [5, p. 54]. We need the following to define whether an ultrafilter is good or not.

Definition 4.2. *Let X be a set and let $\mathscr{A}, \mathscr{B} \subset \mathcal{P}(X)$. A function $p\colon [X]^{<\aleph_0} \to \mathscr{A}$ is*

- *multiplicative, if $p(s\cup t) = p(s)\cap p(t)$ for all $s,t\in[X]^{<\aleph_0}$ and*
- *monotone, if $p(s)\supset p(t)$ whenever $s\subset t\in[X]^{<\aleph_0}$.*

If $p\colon [X]^{<\aleph_0}\to\mathscr{A}$ and $q\colon [X]^{<\aleph_0}\to\mathscr{B}$, we write $p\leqslant q$ iff for all $s\in[X]^{<\aleph_0}$ we have $p(s)\subset q(s)$.

Definition 4.3. *Let X be a set. An ultrafilter $\mathscr{U}$ on X is called good iff whenever $p\colon [X]^{<\aleph_0}\to\mathscr{U}$ is monotone, there is a multiplicative $q\colon [X]^{<\aleph_0}\to\mathscr{U}$ such that $q\leqslant p$.*

To explain this, let us prove that every ultrafilter on a set X of size $\aleph_0$ is good, as an example. It suffices to consider $X=\aleph_0$. Let $\mathscr{U}$ be an ultrafilter on $\aleph_0$ and let $p\colon [\aleph_0]^{<\aleph_0}\to\mathscr{U}$ be monotone. For a finite subset $F\subset\aleph_0$ we set $n(F) = \min\{n\in\aleph_0 : F\subset n\}\in[\aleph_0]^{<\aleph_0}$, and we define $q\colon [\aleph_0]^{<\aleph_0}\to\mathscr{U}$ by $q(F) = p(n(F))$. Since $F\subset n(F)$ and p is monotone we have $q(F) = p(n(F))\subset p(F)$ for $F\in[\aleph_0]^{<\aleph_0}$, i.e., $q\leqslant p$. To prove that q is multiplicative, let $F,G\in[\aleph_0]^{<\aleph_0}$. Then $n(F\cup G) = n(F)\cup n(G) = \max\{n(F),n(G)\}$, and hence

$$\begin{aligned} q(F\cup G) &= p(n(F\cup G)) = p(\max\{n(F),n(G)\}) \\ &= p(n(F))\cap p(n(G)) = q(F)\cap q(G), \end{aligned}$$

since p is monotone. Another property we will need is completeness.

Definition 4.4. *Let α and κ be cardinals. A filter $\mathscr{F}$ on α is κ-complete if $\bigcap \mathscr{A} \in \mathscr{F}$ whenever $\mathscr{A} \subset \mathscr{F}$ and $|\mathscr{A}| < \kappa$. A filter is countably incomplete if it is not $\aleph_0^+$-complete.*

Note that every filter is $\aleph_0$-complete. Now we can state the main theorem of this section:

Theorem 4.5. *On every infinite X there is a good, countably incomplete ultrafilter.*

In this section, we use a notion of independence slightly different from the independent sets in section 3. The following defines independent sets of functions, with respect to a filter.

Definition 4.6. *Let X be a set, $\mathscr{F}$ a filter over X and $\mathscr{S} \subset X^X = \{f : X \to X\}$. Then $\mathscr{S}$ is independent from $\mathscr{F}$ iff, whenever $f_1, \ldots, f_n$ are distinct members of $\mathscr{S}$ and $x_1, \ldots, x_n \in X$,*

$$X \backslash \{x \in X : f_i(x) = x_i \text{ for all } 1 \leqslant i \leqslant n\} \notin \mathscr{F}.$$

We say $\mathscr{S}$ is independent iff $\mathscr{S}$ is independent from $\{X\}$.[3]

If $\mathscr{S}$ is independent and $\varnothing \subset Y \subset X$, then $\{f^{-1}(Y) : f \in \mathscr{S}\}$ is easily seen to be an independent family of sets. Also, if $\mathscr{S}$ is independent and infinite, S is independent from FR_X. The existence of an independent set $\mathscr{S} \subset X^X$ of power 2^κ follows directly from theorem 2.2 again, which will prove to be important at the base of the recursion we will carry out.

Proof of 4.5. Let $\{A_\eta\}_{\eta<2^\kappa}$ enumerate $\mathcal{P}(X)$ and let $\{p_\eta\}_{\eta<2^\kappa}$ enumerate all monotone functions $[X]^{<\aleph_0} \to \mathcal{P}(X)$ in such a way that each monotone $p\colon [X]^{<\aleph_0} \to \mathcal{P}(X)$ is listed 2^κ times. To prove the theorem, we construct sequences $\{\mathscr{F}_\eta\}_{\eta<2^\kappa}$ and $\{\mathscr{S}_\eta\}_{\eta<2^\kappa}$ which satisfy the following for all $\eta < 2^\kappa$:

1. $\mathscr{F}_\eta$ is a filter over X, $\mathscr{S}_\eta \subset X^X$ and $\mathscr{S}_\eta$ is independent from $\mathscr{F}_\eta$;
2. For $\xi < \eta$, $\mathscr{F}_\xi \subset \mathscr{F}_\eta$ and $\mathscr{S}_\xi \supset \mathscr{S}_\eta$;
3. $|\mathscr{S}_\eta| = 2^\kappa$;
4. If η is a limit ordinal, $\mathscr{F}_\eta = \bigcup_{\xi<\eta} \mathscr{F}_\xi$ and $\mathscr{S}_\eta = \bigcap_{\xi<\eta} \mathscr{S}_\xi$;
5. $\mathscr{S}_\eta \backslash \mathscr{S}_{\eta+1}$ is finite;
6. $\mathscr{F}_0$ is generated by sets B_n $(n < \aleph_0)$ such that $\bigcap_{n<\aleph_0} B_n = \varnothing$;
7. Either $A_\eta \in \mathscr{F}_{\eta+1}$ or $X \backslash A_\eta \in \mathscr{F}_{\eta+1}$;
8. If $p_\eta\colon [X]^{<\aleph_0} \to \mathscr{F}_\eta$, then there is a multiplicative $q\colon [X]^{<\aleph_0} \to \mathscr{F}_{\eta+1}$ such that $q \leqslant p_\eta$.

By (7), $\mathscr{U} := \bigcup_{\eta<2^\kappa} \mathscr{F}_\eta$ will be an ultrafilter. If $p\colon [X]^{<\aleph_0} \to \mathscr{U}$ is monotone, then, since $\mathrm{cf}\, 2^\kappa > \kappa$, $\mathrm{ran}(p) \subset \mathscr{F}_\xi$ for some $\xi < 2^\kappa$. Applying (8) to some $\eta > \xi$ such that $p_\eta = p$ shows that there is a multiplicative $q\colon [X]^{\aleph_0} \to \mathscr{F}_{\eta+1} \subset \mathscr{U}$ such that $q \leqslant p$. Thus, $\mathscr{U}$ will be good. Condition (6) insures that $\mathscr{U}$ will be countably incomplete. To make (6) hold, take $\mathscr{S}_0 \cup \{f\}$ to be independent and of power 2^κ. Take $B_n = \{x \in X : n < f(x) < \aleph_0\}$ and $\mathscr{F}_0 = \langle\{B_n : n < \aleph_0\}\rangle$.

[3]The reason I used the words 'independent *from*', different from the source, is to distinguish between the independent sets from the previous section and independent families of functions w.r.t. a filter.

Conditions (1)–(4) will take care of themselves. To get (5), (7) and (8), we apply 4.6 and 4.7 at each stage $\eta < 2^\kappa$;

Lemma 4.7. *If $\mathscr{S}$ is independent from $\mathscr{F}$ and $A \subset X$, there are $\mathscr{S}' \subset \mathscr{S}$ and $\mathscr{F}' \supset \mathscr{F}$ such that $\mathscr{S}'$ is independent from $\mathscr{F}'$, $\mathscr{S}\backslash\mathscr{S}'$ is finite and either A or $X\backslash A$ is a member of $\mathscr{F}'$.*

Proof. We distinguish two cases.

1. $\mathscr{S}$ is independent from $\langle \mathscr{F} \cup \{A\}\rangle$. Take $\mathscr{S}' = \mathscr{S}$ and $\mathscr{F}' = \langle \mathscr{F} \cup \{A\}\rangle$.
2. $\mathscr{S}$ is not independent from $\langle \mathscr{F} \cup \{A\}\rangle$. Let $f_1, \ldots, f_n$ be distinct members of $\mathscr{S}$ and let $x_1, \ldots, x_n \in X$ such that

$$X\backslash\{x \in X : f_i(x) = x_i \text{ for all } 1 \leqslant i \leqslant n\} \in \langle \mathscr{F} \cup \{A\}\rangle.$$

   Let $\mathscr{S}' = \mathscr{S}\backslash\{f_1, \ldots, f_n\}$ and

$$\mathscr{F}' = \langle \mathscr{F} \cup \{x \in X : f_i(x) = x_i \text{ for all } 1 \leqslant i \leqslant n\}\rangle.$$

   Note that $X\backslash A \in \mathscr{F}'$. □

Lemma 4.8. *If $\mathscr{S}$ is independent from $\mathscr{F}$ and $p\colon [X]^{<\aleph_0} \to \mathscr{F}$ is monotone, then there are $\mathscr{S}' \subset \mathscr{S}$, $\mathscr{F}' \supset \mathscr{F}$, and a multiplicative $q\colon [X]^{<\aleph_0} \to \mathscr{F}'$ such that $\mathscr{S}'$ is independent from $\mathscr{F}'$, $\mathscr{S}\backslash\mathscr{S}'$ is finite, and $q \leqslant p$.*

Proof. Fix $g \in \mathscr{S}$. Let $\mathscr{S}' = \mathscr{S}\backslash\{g\}$. For each $t \in [X]^{<\aleph_0}$ we let

$$q_t(s) = \begin{cases} 0, & \text{if } s \not\subset t \\ p(t), & \text{if } s \subset t. \end{cases}$$

Let t_x $(x \in X)$ enumerate $[X]^{<\aleph_0}$. Let

$$q(s) = \bigcup_{x \in X} \{q_{t_x}(s) \cap g^{-1}(\{x\})\} \quad \text{and} \quad \mathscr{F}' = \langle \mathscr{F} \cup \operatorname{ran}(q)\rangle.$$

It is easily verified that $q \leqslant p$. To show that q is multiplicative, take $r, s \in [X]^{\aleph_0}$. We have

$$\begin{aligned} q(r \cup s) &= \bigcup_{x \in X} \{q_{t_x}(s \cup r) \cap g^{-1}(\{x\})\} \\ &= \bigcup_{x \in X : r,s \subset t_x} \{p(t_x) \cap g^{-1}(\{x\})\} \\ &= \left[\bigcup_{x \in X : r \subset t_x} \{p(t_x) \cap g^{-1}(\{x\})\}\right] \cap \left[\bigcup_{x \in X : s \subset t_x} \{p(t_x) \cap g^{-1}(\{x\})\}\right] \\ &= \left[\bigcup_{x \in X} \{q_{t_x}(r) \cap g^{-1}(\{x\})\}\right] \cap \left[\bigcup_{x \in X} \{q_{t_x}(s) \cap g^{-1}(\{x\})\}\right] \\ &= q(r) \cap q(s). \end{aligned}$$

This concludes the proofs of 4.7 and 4.5. □

5 Appendix (3-set Lemma)

In order to prove Theorem 3.4, we need a combinatorial lemma concerning a specific partitioning of sets. In fact, this lemma is by far more interesting than the proof of the theorem itself, in my opinion. We will apply the Compactness Theorem from Model Theory, which states that a theory has a model if every finite set of axioms in the theory has a model. The latter is often easier to prove by an induction argument, as will be demonstrated below. More generalized versions of the lemma are provable, for which we refer to [7].

Theorem (3-set lemma). *Let X be an infinite set and let $h\colon X \to X$ be a function. Then there are pairwise disjoint $X_0, X_1, X_2, X_3 \subset X$ with $X = \bigcup_{i \leqslant 3} X_i$ such that $h(x) = x$ if $x \in X_0$ and $h(X_i) \cap X_i = \varnothing$ if $i = 1, 2, 3$.*

Proof. Let $X_0 = \{x \in X : h(x) = x\}$ and $X' := X \backslash X_0$. We consider the language $\mathscr{L}$ with h as the only function letter, for every $x \in X'$ an individual constant symbol c_x (where c_x and c_y are distinct iff x and y are distinct) and three unary predicate letters A_1, A_2 and A_3. We define a theory K of $\mathscr{L}$ by adding, for every $x \in X'$, the following axioms:

- $A_1(c_x) \vee A_2(c_x) \vee A_3(c_x)$;
- $\neg\Big[(A_1(c_x) \wedge A_2(c_x)) \vee (A_1(c_x) \wedge A_3(c_x)) \vee (A_2(c_x) \wedge A_3(c_x)\Big]$;
- $A_i(c_x) \Rightarrow A_{i+1 \bmod 3}(h(c_x)) \vee A_{i+2 \bmod 3}(f(c_x)) \quad (i = 1, 2, 3)$.

Now if K has a model χ, then we can identify χ with X' and interpret A_1, A_2 and A_3 as the subsets X_1, X_2 and X_3 we were looking for. To prove the existence of a model, we show that every finite subset of axioms has a model, which proves the claim because of the compactness theorem [8].

For a finite subset S of axioms, let $F^S \subset X'$ be the set consisting of all $x \in X'$ such that there is an axiom in S which demands a property of c_x. Obviously, F^S is finite and it does not contain fixed points of f. We will use induction on the cardinality of F^S to show that there are $F_1^S, F_2^S, F_3^S \subset F^S$, pairwise disjoint, such that $F^S = F_1^S \cup F_2^S \cup F_3^S$, and $F_i^S \cap f(F_i^S) = \varnothing$ $(i = 1, 2, 3)$. Note that we implicitly use induction on the size of S.

The case $|F^S| = 0$ is trivial. Let $m \geq 1$ and assume that for every subset S such that F^S has cardinality $m-1$, F^S can be partitioned as desired. Let F^T be of cardinality m. Construct a graph Γ whose vertices are the elements of F^T. The vertices g_1 and g_2 are *connected* in Γ if and only if $g_1 = h(g_2)$ or $g_2 = h(g_1)$. Since h is a function, m is an upper bound for the amount of edges in F^T. So there exists a vertex $g \in F^T$ which is connected with at most 2 other vertices. By the induction hypothesis, $F^T \backslash \{g\}$ is the union of three sets G_1^T, G_2^T and G_3^T such that G_i^T is totally disconnected $(i = 1, 2, 3)$. It follows that there is $1 \leqslant j \leqslant 3$ such that g is not connected with any vertex in G_j^T, so g can be added to G_j^T without making new connections. So we take $F_j^T = G_j^T \cup \{g\}$ and $F_i^T = G_i^T$ if $i \neq j$. This completes the induction step. □

Now we can prove the following theorem, which we recall from section 3:

Theorem 3.4. *Let X and Y be infinite sets and let $\mathscr{U} \in U(X)$ and $\mathscr{V} \in U(Y)$. Then*

1. *$\mathscr{U} \approx \mathscr{V}$ if and only if there are $A \in \mathscr{U}$, $B \in \mathscr{V}$ and $f : X \to Y$ such that $\mathscr{V} = f_*(\mathscr{U})$ and $f \restriction A$ is injective and onto B;*
2. *If $\mathscr{U} \approx \mathscr{V}$ and $|X| = |Y|$ then there is a bijection $f : X \to Y$ such that $\mathscr{V} = f_*(\mathscr{U})$ and $\mathscr{U} = (f^{-1})_*(\mathscr{V})$.*

Proof.

1. Suppose $\mathscr{U} \approx \mathscr{V}$. Take $f\colon X \to Y$ and $g\colon Y \to X$ such that $f_*(\mathscr{U}) = \mathscr{V}$ and $g_*(\mathscr{V}) = \mathscr{U}$. Let $h = g \circ f$ and $k = f \circ g$. Let $X_0, X_1, X_2, X_3 \subset X$ and $Y_0, Y_1, Y_2, Y_3 \subset Y$ be pairwise disjoint such that $X = \bigcup_{i \leqslant 3} X_i$, $Y = \bigcup_{i \leqslant 3} Y_i$, $h \restriction X_0 = \mathrm{id}_{X_0}$, $k \restriction Y_0 = \mathrm{id}_{Y_0}$ and $X_i \cap h(X_i) = Y_i \cap k(Y_i) = \varnothing$ for all $1 \leqslant i \leqslant 3$.

   Next we will show $X_0 \in \mathscr{U}$. If $X_i \in \mathscr{U}$ for $i \in \{1, 2, 3\}$ then $h(X_i) \in h_*(\mathscr{U})$. But $h(X_i) \cap X_i = \varnothing$, so $\mathscr{U} \neq h_*(\mathscr{U})$. We conclude $X_i \notin \mathscr{U}$. So $X_1 \cup X_2 \cup X_3 \notin \mathscr{U}$ and $X_0 \in \mathscr{U}$. In a similar fashion, we infer $Y_0 \in \mathscr{V}$.

   For $x \in X_0$ we have $(f \circ g)(f(x)) = f((g \circ f)(x)) = f(k(x)) = f(x)$, so $f(x) \in Y_0$. The same calculation shows that $f(y) \in X_0$ whenever $y \in Y_0$. So $f \restriction X_0$ and $g \restriction Y_0$ are each others inverse. Now $A = X_0$ and $B = Y_0$ satisfy the properties in the theorem.

   For the other implication, it suffices to show the existence of a function $g\colon Y \to X$ with $g_*(\mathscr{V}) = \mathscr{U}$. If A and B are finite then $\mathscr{U}$ and $\mathscr{V}$ are principal ultrafilters, so they are generated by singleton subsets. More specifically, if there is $x \in X$ such that $\mathscr{U}$ consists of all subsets that contain x, then $\mathscr{V}$ is just $\{C \in Y : f(x) \in C\}$, so clearly $\mathscr{U} \approx \mathscr{V}$.

   If A and B are infinite, let $x \in X$ and define $g\colon Y \to X$ by $g(y) = f^{-1}(y)$ if $y \in B$ and $g(y) = x$ if $y \notin B$.

2. Take A, B and f as in part 1. If A and B are finite, the claim is obvious. If they are infinite, then there are subsets $A_0, A_1 \subset A$ such that $A = A_0 \cup A_1$, $A_0 \cap A_1 = \varnothing$, $|A| = |A_0| = |A_1|$ and $A_0 \in \mathscr{U}$. Now $|X \backslash A_0| = |Y \backslash f(A_0)|$. Take a function $\pi\colon X \to Y$ such that $\pi(X \backslash A_0) = Y \backslash f(A_0)$ and $\pi \restriction A_0 = f \restriction A_0$. Then $\pi_*(\mathscr{U}) = f_*(\mathscr{U}) = \mathscr{V}$ and $\pi_*^{-1}(\mathscr{V}) = \mathscr{U}$. $\square$

References

[1] K. Kunen, *Ultrafilters and Independent Sets*, Transactions of the American Mathematical Society, Volume 172, October 1972.
http://www.dm.unipi.it/ dinasso/ULTRABIBLIO/Kunen%20-%20Ultrafilters%20and%20independent%20sets.pdf.

[2] H. Keisler, *The Ultraproduct Construction*,
http://www.logic.amu.edu.pl/images/1/14/Keislerultraproduct.pdf.

[3] W.W. Comfort & S. Negrepontis, *The Theory of Ultrafilters*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen (Band 11), 1974.

[4] C. Chang & H. Keisler, *Model Theory*, Elsevier Science Publishing Company, 1973.

[5] T. Jech, *Set Theory*, Springer Monographs in Mathematics, 1978.

[6] D. Kim, Y. Kwon, J. Lee, *Enumerations of Finite Topologies associated with a finite Graph*, arXiv: 1206.0550v1 [math.co], 2012
http://arxiv.org/pdf/1206.0550.pdf.

[7] N.G. de Bruijn & P. Erdös, *A colour problem for infinite graphs and a problem in the theory of relations*, Koninklijke Nederlandse Akademie van Wetenschappen, Amsterdam, 1951
http://www.renyi.hu/ p_erdos/1951-01.pdf

[8] E. Mendelson, *Introduction to Mathematical Logic*, Discrete Mathematics and its applications, CRC Press, 2009.