An introduction to and comparison of three notions of dimension in metric spaces

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July 11, 2013

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Introduction

Most commonly used to discriminate between 'fractal sets' (however one wishes to define them[1]), the notions of dimension occurring in this thesis allow for discussion free from application. This is what the majority of this text will focus on. I shall introduce the Hausdorff dimension, the box-counting dimension and the correlation dimension, respectively. Of the first two, we shall see multiple definitions as well as a number of examples. All three notions are related to each other in the sense of inequalities holding under various conditions.

On the Hausdorff dimension - the topic of the first chapter - and the closely related Hausdorff measures, there has been written a great amount. Examples include [2] and [9]. The results presented in Chapter 1 can, indeed, all be found in literature other than this thesis. What appears harder to find, though, are rigorous proofs for said results. I have therefor made it a point of attention to be precise and complete in verifying my claims, complementing proofs found in other literature where necessary.

For the topic of Chapter 2, the box-counting dimension, similar things hold. Plenty of results on this notion are known and collected in a variety of books, e.g. in [2]; precise expositions of the math behind these claims appear scarce. Hence through the second chapter, I maintain the same policy of complementing proofs where needed.

This way of working is supported greatly by Appendix A. As the definition of the box-counting dimension involves calculations with $\liminf$ and $\limsup$, we partly obtain our results by manipulating expressions containing these. Thus, we need some 'elementary facts' about $\liminf$ and $\limsup$ to do our job; the question is: are these facts elementary enough to omit their proofs? I think not, and because proofs for the claims in question seemed nowhere to be found, I provide them in Appendix A.

The third chapter, concludingly, covers the correlation dimension. This notion is fundamentally different from the ones covered in the first two

[1] Though many appear to have some intuitive idea on what a fractal set is, defining it unambiguously turns out an involved task - see [2], pages xx-xxi.

chapters. Originally introduced by Procaccia, Grassberger and Hentschel (in [8]) as a procedure to measure the chaotic behaviour of a dynamical system, a result by Pesin (see [7]) allows us to omit any mention of a dynamical system in its definition. As a result, the correlation dimension demands as input a measure, as opposed to the Hausdorff and the box-counting dimension, which require sets as input.

This way of defining the correlation dimension is not new (see for example [7]). As in the continuation of the chapter I wanted to relate the notion to the Hausdorff dimension, though, I stumbled upon some difficulties that ultimately had me introduce new terminology. These difficulties were the following:

- For x in some metric space, $B(x, r)$ a closed ball of radius r centered at x, and μ a Borel measure, is the function

$$x \mapsto \mu(B(x, r))$$

  Borel measurable?

If the metric space in question is $\mathbb{R}^n$, then the affirmative answer sometimes seems to be taken for granted (see e.g. [10], Proposition 2.1). It has appeared to me, however, not at all trivial to verify this claim. In a general metric space, it is not even necessarily true, and for this reason I introduce the notion of *semi-geodesity.* We will see that in a semi-geodesic metric space, the asserted claim is valid (see Appendix B).

- If $s > 1$, U is a subset of some metric space, μ is a Borel measure on this space and a 'δ-cover of U' is a cover consisting of sets with diameter at most δ, does it automatically hold that

$$\inf \left\{ \sum_i \mu(C_i)^s \colon \{C_i\}_i \text{ is a } \delta\text{-cover of } U \right\}$$

  approaches zero if we let δ do so?

Again, this appears all but trivial to verify, and I coin two new terms to support the investigation of the problem: the *Young property* - as Young, with [10], inspired me to look into the matter - to address those pairs (μ, U) for which the claim holds true, and the *fundamental property* as a property of metric spaces, which in essence is the abstraction of a property of $\mathbb{R}^n$ that, provided the measure in question also behaves 'nicely', is shown to imply the Young property. See Section 3.4 for more details.

Having dealt with these difficulties, the last section of Chapter 3 utilizes the new terminology and the accompanying results to obtain a relation of the desired kind. This relation, thus, is not derived from other literature.

Notation and conventions

In this text we use the following symbols:

- $\mathbb{Z}$, denoting the set of integers.
- $\mathbb{Z}_{>n}$ (for a given $n \in \mathbb{Z}$), denoting the set of integers strictly greater than n. We define $\mathbb{Z}_{\geq n}$, $\mathbb{Z}_{<n}$, and $\mathbb{Z}_{\leq n}$ analogously.
- $n\mathbb{Z}$ (for a given $n \in \mathbb{Z}$), denoting the set $\{nz : z \in \mathbb{Z}\}$.
- $\mathbb{R}$, denoting the set of real numbers.
- $\mathbb{R}_{>r}$ (for a given $r \in \mathbb{R}$), denoting the set of real numbers strictly greater than r. We define $\mathbb{R}_{\geq r}$, $\mathbb{R}_{<r}$, and $\mathbb{R}_{\leq r}$ analogously.
- (r, ∞) (for a given $r \in \mathbb{R}$), as an alternative for $\mathbb{R}_{>r}$. We use the symbols $[r, \infty)$, $(-\infty, r)$ and $(-\infty, r]$ analogously.
- $(r, \infty]$ (for a given $r \in \mathbb{R}$), defined by $[r, \infty) \cup \{\infty\}$. We define $[r, \infty]$, $[-\infty, r)$ and $[-\infty, r]$ analogously.
- $\mathcal{P}(X)$ (for a given set X), denoting the power set of X.

Furthermore, we say that $f : A \to B$ is a *map* whenever $f \subseteq A \times B$ and $(a, b), (a, c) \in f \Rightarrow b = c$. The term *function* is reserved for the special case in which B is a field.

If we say that (X, d) is a *metric space*, we mean that X is any non-empty set and that $d : X \times X \to \mathbb{R}$ is a metric on X.

For a metric space (X, d), $c \in X$ and $\epsilon \geq 0$, the symbol $B(c, \epsilon)$ denotes the *closed* ball centered at c with radius ϵ, i.e.:

$$B(c, \epsilon) = \{x \in X : d(c, x) \leq \epsilon\}.$$

For (X, d) a metric space and $U \subseteq X$, the symbol $\overline{U}$ denotes the closure of U.

If not mentioned otherwise, when we speak of $\mathbb{R}^n$ as a metric space, we assume it to be fitted with the Euclidean metric.

Utilizing infima and suprema, we sometimes prefer to write the condition on the set over which these are taken below the infima and suprema symbols:

$$\inf_{x \in X} f(x) = \inf\{f(x) : x \in X\} \qquad \text{and} \qquad \sup_{x \in X} f(x) = \sup\{f(x) : x \in X\}$$

for any set X and any function f on X.

We adopt the common convention that the infimum and the supremum of the empty set are infinity and zero, respectively:

$$\inf \emptyset = \infty \qquad \text{and} \qquad \sup \emptyset = 0.$$

If we say that (X, Σ) is a *measurable space*, we mean that X is some non-empty set and that Σ is a σ-algebra of subsets of X.

Lastly, whenever we speak of a measure, we assume it to be non-negative.

Chapter 1

Hausdorff dimension

1.1 Introduction

In this chapter we introduce a tool widely used for measuring fractal sets: the *Hausdorff dimension*. An advantage of this notion as compared to, for example, the box-counting dimension, is the fact that it behaves roughly like one would expect such a notion to behave: countable sets have Hausdorff dimension zero, the real line gets assigned the number 1, and in general the Hausdorff dimension is countably stable. In 'practice', though, it often appears difficult to compute it - see Section 1.5 for illustration of this.
We introduce the Hausdorff dimension by first defining the *Hausdorff measures* and deriving some of their useful properties. The definitions in this chapter match those of most authors: Pesin, for example, identically defines shared notions in [7], and Falconer does the same in [2] (specialised for $\mathbb{R}^n$). Although the results are well known, some of them seem to consistently lack a decent proof. That is to say, it appears that whenever these well known facts are mentioned, their proofs are either sketchy or omitted. Most notable among these results is the Hausdorff dimension of the middle-third Cantor set[1] being equal to $\log 2/\log 3$. We give it a rigorous proof in the last section of this chapter.
Lastly, we would like to point out that most authors focus on bounded sets when calculating Hausdorff dimensions. The notion, in fact, certainly allows for unbounded sets, and we illustrate this by showing the Hausdorff dimension of $\mathbb{R}$ to equal 1 - a calculation not as trivial as one might expect.

[1]See Section 1.5 for a definition of the Middle-third Cantor set.

1.2 Hausdorff measures

Definition 1.1. For (X, d) a metric space and $U \subseteq X$, the *diameter* of U is

$$d(U) = \begin{cases} \sup\{d(x,y) : x, y \in U\} & \text{if } U \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

which attains values in $\mathbb{R}$ when U is bounded and equals ∞ otherwise.

Definition 1.2. For (X, d) a metric space, $U \subseteq X$ and $\delta > 0$, a δ-*cover* of U is a countable family $\{C_i\}_i$ of subsets $C_i \subseteq X$ such that $\bigcup_{i=1}^{\infty} C_i \supseteq U$ and $d(C_i) \leq \delta$ for every i.

Remark. It is worth noting that for δ sufficiently small, metric spaces without δ-covers exist. One can, for example, equip $\mathbb{R}$ with the discrete metric d, given by

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases},$$

and pick $\delta = \frac{1}{2}$. Since in this case any $U \subseteq \mathbb{R}$ with $d(U) \leq \delta$ is a singleton, it is impossible to cover $\mathbb{R}$ with only countably many such sets. We therefor suppose from this point on that all metric spaces admit δ-covers for any $\delta > 0$. This is, for instance, the case for separable metric spaces.

Definition 1.3. For (X, d) a metric space, $U \subseteq X$, $\delta > 0$ and $s \geq 0$, we define the quantity $\mathcal{H}^s_\delta(U)$ by

$$\mathcal{H}^s_\delta(U) = \inf \left\{ \sum_{i=1}^{\infty} d(C_i)^s : \{C_i\}_{i=1}^{\infty} \text{ is a } \delta\text{-cover of } U \right\}.$$

Note that for $U \subseteq X$ and $s \geq 0$ fixed, $\mathcal{H}^s_\delta(U)$ is a decreasing function of δ, and hence $\lim_{\delta \to 0} \mathcal{H}^s_\delta(U)$ either exists or diverges to ∞. The following definition is thus justified.

Definition 1.4. For (X, d) a metric space and $s \geq 0$, we define the function $\mathcal{H}^s$ by

$$\begin{aligned} \mathcal{H}^s : \mathcal{P}(X) &\to [0, \infty] \\ U &\mapsto \lim_{\delta \to 0} \mathcal{H}^s_\delta(U). \end{aligned}$$

The following result is essentially Theorem 4 from [9], adjusted slightly to suit our needs.

Theorem 1.5. Let (X, d) be a metric space and let $s \geq 0$. The function $\mathcal{H}^s$ is an outer measure on X. That is, $\mathcal{H}^s : \mathcal{P}(X) \to [0, \infty]$ satisfies the following properties:

a) $\mathcal{H}^s(\emptyset) = 0$,

b) $\mathcal{H}^s(U) \leq \mathcal{H}^s(V)$ whenever $U \subseteq V \subseteq X$,

c) $\mathcal{H}^s(\bigcup_i U_i) \leq \sum_i \mathcal{H}^s(U_i)$ for every countable family $\{U_i\}_i \subseteq \mathcal{P}(X)$.

Proof. Let $\delta > 0$. We shall verify properties (a)-(c) for $\mathcal{H}^s_\delta$ instead of $\mathcal{H}^s$. The desired results then follow from the arbitrariness of δ.

a) The empty family $\emptyset \subset \mathcal{P}(X)$ is a δ-cover of $\emptyset$. Summing over the diameters, risen to the power of s, of the members of this family results in the empty sum, its value being 0.

b) If $U \subseteq V \subseteq X$, any δ-cover of V is a δ-cover of U. It follows directly that $\mathcal{H}^s_\delta(U) \leq \mathcal{H}^s_\delta(V)$.

c) Let $\{U_i\}_i \subseteq \mathcal{P}(X)$ be countable. The claim is trivial when $\sum_i d(U_i)^s = \infty$, so let us assume that $\sum_i d(U_i)^s$ is finite. This means that $d(U_i)^s$ is finite for every $i \in \mathbb{Z}_{>0}$. Hence for these i and some fixed $\epsilon > 0$, we can find a δ-cover $\{C_{ij}\}_j$ of U_i satisfying

$$\sum_j d(C_{ij})^s \leq \mathcal{H}^s_\delta(U_i) + \epsilon \cdot 2^{-i}.$$

Noting that $\bigcup_{i,j} C_{ij}$ is a δ-cover of $\bigcup_i U_i$, we thus have

$$\begin{aligned} \mathcal{H}^s_\delta\left(\bigcup_i U_i\right) &\leq \sum_{i,j} d(C_{ij})^s \\ &= \sum_i \left(\sum_j d(C_{ij})^s\right) \\ &\leq \sum_i \mathcal{H}^s_\delta(U_i) + \epsilon \cdot 2^{-i} \\ &= \left(\sum_i \mathcal{H}^s_\delta(U_i)\right) + \epsilon. \end{aligned}$$

As ϵ may be any positive number, this yields the claim. □

It now follows from basic measure theory that for arbitrary $s \geq 0$ the family of $\mathcal{H}^s$-measurable sets is a σ-algebra Σ_s in (X, d) and that $\mathcal{H}^s$ is a measure on Σ_s.

Definition 1.6. Let (X, d), s and Σ_s be as above. The function

$$\mathcal{H}^s : \Sigma_s \to [0, \infty]$$

is called the *s-dimensional Hausdorff measure* on Σ_s.

It can moreover be shown that every closed set $U \subseteq X$ is $\mathcal{H}^s$-measurable (we refer to Theorem 23 from [9] for a proof[2]). Since the closed sets of (X, d) generate the Borel σ-algebra in (X, d), we have the following result.

Corollary 1.7. Let (X, d), s and Σ_s be as above. The σ-algebra Σ_s includes the Borel σ-algebra of (X, d), and hence every Borel subset of X is $\mathcal{H}^s$-measurable. □

1.3 Basic properties of Hausdorff measures

As the title suggests, in this section we derive some elementary properties of Hausdorff measures. We shall gratefully make us of them when proving similar results for the yet to be defined Hausdorff dimension, in section 1.4.

Theorem 1.8. Let $(X, d_X), (Y, d_Y)$ be metric spaces, let $c, \alpha > 0$ and let $f : X \to Y$ be a map satisfying

$$d_Y(f(x_1), f(x_2)) \leq c \cdot d_X(x_1, x_2)^\alpha$$

for every pair $x_1, x_2 \in X$. Then, for every $s \geq 0$, $U \subseteq X$:

$$\mathcal{H}^{s/\alpha}(f(U)) \leq c^{s/\alpha}\mathcal{H}^s(U).$$

Proof. Let $\delta > 0$ and let $\{C_i\}_i$ be a δ-cover of U. For $\epsilon := c\delta^\alpha$, observe that $\{f(C_i)\}_i$ is an ϵ-cover of $f(U)$, and that

$$\sum_i d_Y(f(C_i))^{s/\alpha} \leq c^{s/\alpha} \sum_i d_X(C_i)^s,$$

giving $\mathcal{H}_\epsilon^{s/\alpha}(f(U)) \leq c^{s/\alpha}\mathcal{H}_\delta^s(U)$. It follows that

$$\begin{aligned} \mathcal{H}^{s/\alpha}(f(U)) &= \lim_{\epsilon \to 0} \mathcal{H}_\epsilon^{s/\alpha}(f(U)) \\ &= \lim_{\delta \to 0} \mathcal{H}_\epsilon^{s/\alpha}(f(U)) \\ &\leq \lim_{\delta \to 0} c^{s/\alpha}\mathcal{H}_\delta^s(U) \\ &= c^{s/\alpha}\mathcal{H}^s(U). \end{aligned}$$

□

Definitions 1.9. Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f : X \to Y$ be a map.

1. The map f is called *Lipschitz* if there is a $c > 0$ such that for every $x_1, x_2 \in X$:
$$d_Y(f(x_1), f(x_2)) \leq c \cdot d_X(x_1, x_2).$$
The number c is called a *Lipschitz constant* of f.

[2] Rogers' result is, in fact, a lot more general than this claim. The proof, consequently, is a bit too comprehensive to include here.

2. The map f is called *bi-Lipschitz* if there are positive real numbers b, c such that for every $x_1, x_2 \in X$:

$$b \cdot d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq c \cdot d_X(x_1, x_2).$$

The numbers b and c are called *lower* and *upper Lipschitz constants* of f, respectively.

Theorem 1.10. Let $(X, d_X,), (Y, d_Y)$ be metric spaces and let $f : X \to Y$ be bi-Lipschitz with lower and upper Lipschitz constants $b, c > 0$, respectively. The map f is injective. If f is surjective, its inverse $f^{-1} : Y \to X$ is bi-Lipschitz with lower and upper Lipschitz constants c^{-1}, b^{-1}, respectively.

Proof. Let $p, q \in X$ be such that $f(p) = f(q)$. Then

$$b \cdot d_X(p, q) \leq d_Y(f(p), f(q)) = 0,$$

yielding $d_X(p, q) = 0$ and thus $p = q$. Now suppose f is surjective. Then for an arbitrary pair $y_1, y_2 \in Y$, there is a unique pair $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. By the Lipschitz condition, we have

$$d_Y(f(x_1), f(x_2)) \leq c \cdot d_X(x_1, x_2),$$

or

$$c^{-1} \cdot d_Y(y_1, y_2) \leq d_X(f^{-1}(y_1), f^{-1}(y_2)).$$

In a completetely analogous way, we find

$$d_X(f^{-1}(y_1), f^{-1}(y_2)) \leq b^{-1} \cdot d_Y(y_1, y_2).$$ □

Corollary 1.11. Let $(X, d_X), (Y, d_Y)$ be metric spaces, let $f : X \to Y$ be a map and let $s \geq 0$. If f is Lipschitz with Lipschitz constant $c > 0$, then

$$\mathcal{H}^s(f(X)) \leq c^s \mathcal{H}^s(X).$$

If f is bi-Lipschitz with lower and upper Lipschitz constants $b, c > 0$ respectively, then

$$b^s \mathcal{H}^s(X) \leq \mathcal{H}^s(f(X)) \leq c^s \mathcal{H}^s(X).$$

Proof. The first result is a special case of Theorem 1.8. To prove the second claim, note that without loss of generality we may assume f to be surjective. Hence by Theorem 1.10 and the first result of this corollary:

$$\mathcal{H}^s(X) = \mathcal{H}^s(f^{-1}(f(X))) \leq b^{-s} \mathcal{H}^s(f(X)),$$

upon which the claim follows. □

1.4 Hausdorff dimension

Having defined the Hausdorff measures, we need one more lemma before we can define the Hausdorff dimension.

Lemma 1.12. Let (X, d) be a metric space and let $U \subseteq X$. The function

$$s \mapsto \mathcal{H}^s(U)$$

is decreasing. There is at most one $s \in \mathbb{R}_{\geq 0}$ for which $\mathcal{H}^s(U)$ is real and non-zero.

Proof. Let $\delta \in (0, 1)$. Then $\mathcal{H}^s_\delta(U)$ is a decreasing function of s, from which the first claim follows by definition of $\mathcal{H}^s$. Furthermore, if $\{C_i\}_i$ is a δ-cover of U and if $t > s \geq 0$, then

$$\sum_i d(C_i)^t = \sum_i d(C_i)^{t-s} d(C_i)^s \leq \delta^{t-s} \sum_i d(C_i)^s,$$

giving $\mathcal{H}^t_\delta(U) \leq \delta^{t-s}\mathcal{H}^s_\delta(U)$. Letting δ tend to zero yields $\mathcal{H}^t(U) = 0$ whenever $\mathcal{H}^s(U)$ is finite, proving the second claim. □

Definition 1.13. Let (X, d) be a metric space and let $U \subseteq X$. The *Hausdorff dimension* of U is

$$\dim_H U = \sup\{s \in \mathbb{R}_{\geq 0} : \mathcal{H}^s(U) = \infty\} = \inf\{s \in \mathbb{R}_{\geq 0} : \mathcal{H}^s(U) = 0\}.$$

Note that if X is a metric space and if $U \subseteq X$ is such that it does not have δ-covers for arbitrary small $\delta > 0$, then $\dim_H U = \infty$ necessarily.
We continue to derive some basic properties of the Hausdorff dimension.

Proposition 1.14. Let (X, d) be a metric space. The Hausdorff dimension satisfies the following properties:

a) The function $\dim_H$ is monotone with respect to inclusion. More precisely, if $U \subseteq V \subseteq X$, then $\dim_H U \leq \dim_H V$.

b) The function $\dim_H$ is countably stable. That is to say, if $\{U_k\}_k \subseteq \mathcal{P}(X)$ is countable, then $\dim_H(\bigcup_k U_k) = \sup\{\dim_H U_k\}_k$.

Proof.

a) This is a direct consequence of Theorem 1.5 (b).

b) By part (a) of this proposition: $\dim_H(\bigcup_i U_i) \geq \dim_H U_k$ for all k, giving $\dim_H(\bigcup_i U_i) \geq \sup\{\dim_H(U_k)\}_k$. Noting that the converse statement is trivial when $\sup\{\dim_H(U_k)\}_k = \infty$, let $s > \sup\{\dim_H(U_k)\}_k \in \mathbb{R}$. Then $\mathcal{H}^s(U_k) = 0$ for all k, yielding $\mathcal{H}^s(\bigcup_k U_k) = 0$ and thus $s \geq \dim_H(\bigcup_k U_k)$. The claim follows by definition of s. □

Theorem 1.15. Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f : X \to Y$ be a map for which there are constants $c, \alpha > 0$ such that $d_Y(f(x_1), f(x_2)) \leq c \cdot d_X(x_1, x_2)^\alpha$ for every pair $x_1, x_2 \in X$. Then for every $U \subseteq X$:

$$\dim_H f(U) \leq \dim_H U/\alpha.$$

Proof. Let $U \subseteq X$ and let $s > \dim_H U$. By Theorem 1.8:

$$\mathcal{H}^{s/\alpha}(f(U)) \leq c^{s/\alpha}\mathcal{H}^s(U) = 0.$$

The desired result follows by definition of s. □

Corollary 1.16. Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f : X \to Y$ be a map. If f is Lipschitz, then $\dim_H f(X) \leq \dim_H X$. If f is bi-Lipschitz, then $\dim_H f(X) = \dim_H X$.

Proof. The first statement is a special case of Theorem 1.15. Alternatively, one can turn to Corollary 1.11, from which both claims of this corollary follow immediately. □

Definition 1.17. Two metric spaces (X, d_X), (Y, d_Y) are called *Lipschitz equivalent* if there exists a surjective bi-Lipschitz function $f : X \to Y$.

Thus, we have seen that Lipschitz equivalent spaces have the same Hausdorff dimension. The Hausdorff dimension thus provides us with information on what spaces are (not) Lipschitz equivalent, namely, spaces with differing Hausdorff dimensions are not Lipschitz equivalent. We shall see that the box-counting dimension will grant us information on this topic in a similar way.
We conclude this section by deriving an alternative definition of the Hausdorff dimension, which will prove itself useful in Chapter 3. It turns out that in the process of defining the Hausdorff dimension, it is of no matter if we restrict ourselves to δ-covers consisting of only balls.

Definition 1.18. Let (X, d) be a metric space, let $U \subseteq X$ and let $\delta > 0$. A *ball δ-cover* of U is a δ-cover $\{C_i\}_i$ of U such that for every $C_i \in \{C_i\}_i$, there are $x_i \in U$ and $\epsilon_i > 0$ such that $C_i = B(x_i, \epsilon_i)$.

Lemma 1.19. Let (X, d) be a metric space and let $U \subseteq X$. For any $s \geq 0$ and $\delta > 0$, let the quantity $\mathcal{B}^s_\delta(U) \in [0, \infty]$ be given by

$$\mathcal{B}^s_\delta(U) = \inf\left\{\sum_i d(C_i)^s : \{C_i\}_i \text{ is a ball } \delta\text{-cover of } U\right\}.$$

The function $s \mapsto \mathcal{B}^s(U) := \lim_{\delta \to 0} \mathcal{B}^s_\delta(U)$ is decreasing. There is at most one $s \in \mathbb{R}$ for which $\mathcal{B}^s(U)$ is real and non-zero. The quantity

$$\sup\{s \in \mathbb{R}_{\geq 0} : \mathcal{B}^s(U) = \infty\} = \inf\{s \in \mathbb{R}_{\geq 0} : \mathcal{B}^s(U) = 0\}$$

is equal to the Hausdorff dimension of U.

Proof. The first two statements can be proven analogously to how Lemma 1.12 was justified. It follows from these that

$$\sup\{s \in \mathbb{R}_{\geq 0} : \mathcal{B}^s(U) = \infty\} = \inf\{s \in \mathbb{R}_{\geq 0} : \mathcal{B}^s(U) = 0\} =: \beta(U).$$

To show that this quantity equals the Hausdorff dimension of U, let $s \geq 0$, $\delta > 0$, and note that since any ball δ-cover of U is a δ-cover of U, it holds that $\mathcal{H}^s_\delta(U) \leq \mathcal{B}^s_\delta(U)$. It follows that $\mathcal{H}^s(U) \leq \mathcal{B}^s(U)$, in turn yielding $\dim_H U \leq \beta(U)$.
Aiming to prove the reverse inequality, let $\{C_i\}_i$ be a δ-cover of U. Without restricting generality, we may for all i assume $C_i \cap U$ to be non-empty. Hence for every i, we can pick a point $c_i \in C_i \cap U$ [3] and consider the set

$$D_i = \{x \in X : d(x, c_i) \leq d(C_i)\}.$$

The family $\{D_i\}_i$ thus introduced is a ball 2δ-cover of U. Moreover, since the diameter of a ball is at most twice its radius, we have

$$\sum_i d(D_i)^s \leq \sum_i (2d(C_i))^s = 2^s \sum_i d(C_i)^s.$$

It follows that $\mathcal{B}^s_{2\delta}(U) \leq 2^s \mathcal{H}^s_\delta(U)$ and, by the arbitrariness of δ, $\mathcal{B}^s(U) \leq 2^s \mathcal{H}^s(U)$. In particular, $\mathcal{B}^s(U)$ is finite whenever $\mathcal{H}^s(U)$ is finite, proving what was to be proven. □

1.5 Three examples

We start this section with a (relatively) simple example. The result is to be compared with Proposition 2.6.

Proposition 1.20. The Hausdorff dimension of the set

$$U := \{0\} \cup \{1/n : n \in \mathbb{Z}_{\geq 1}\}$$

equals 0.

Proof. It suffices to show that for any $s > 0$ it holds that $\mathcal{H}^s(U) = 0$. Hence, it suffices to show that for all $s, \delta > 0$, it holds that $\mathcal{H}^s_\delta(U) = 0$. For this last statement to be true, it is enough if for any s, δ, $\epsilon > 0$, there is a δ-cover $\{C_i\}_i$ of U such that

$$\sum_i d(C_i)^s < \epsilon.$$

[3]Note that we may have to appeal to the axiom of choice here.

Thus, let $s, \delta, \epsilon > 0$ and consider the interval $C_1 := [0, \min\{\delta, (\epsilon/2)^{1/s}\}]$. Clearly, only finitely many points of U, say k, are not contained in C_1. For each of these k points $p_i \in \{p_i : i \in \{2, \dots, k+1\}\}$, let C_i be an interval of diameter $\min\{\delta, (\epsilon/2k)^{1/s}\}$ in which p_i is contained. For the δ-cover $\{C_i\}_{i=1}^{k+1}$ of U, it holds that

$$\sum_{i=1}^{k+1} d(C_i)^s \leq \frac{\epsilon}{2} + \sum_{i=2}^{k+1} d(C_i)^s \leq \frac{\epsilon}{2} + \sum_{i=2}^{k+1} \frac{\epsilon}{2k} = \epsilon,$$

as desired. □

We would like to calculate the Hausdorff dimension of the real line. As we will see, the following lemma helps us achieve this goal.

Lemma 1.21. If $\{C_i\}_i$ is a countable collection of bounded subsets of $\mathbb{R}$ such that $\bigcup_i C_i$ is connected, then

$$d\left(\bigcup_i C_i\right) \leq \sum_i d(C_i).$$

Proof. Let $\{C_i\}_i$ be as in the statement of the lemma. Without losing any generality, we assume all C_i to be non-empty. Out of $\{C_i\}_i$, we craft the family $\{D_i\}_i$ by defining

$$D_i = [\inf C_i, \sup C_i].$$

Clearly, we have $d(D_i) = d(C_i)$ for all i. Moreover, since the D_i are intervals, each $d(D_i)$ equals the Lebesgue measure $\lambda(D_i)$ of D_i. Let us shift focus to the quantity $d\left(\bigcup_i D_i\right)$. We claim that this also equals the Lebesgue measure of $\bigcup_i D_i$. To prove this, it is clearly enough to show that $\bigcup_i D_i$ is an interval, i.e., that $\bigcup_i D_i$ is connected. To this end, assume the opposite. Then there is a triplet of numbers a, x, b such that $a, b \in \bigcup_i D_i$, $x \notin \bigcup_i D_i$ and $a < x < b$. Let i, j be such that $a \in D_i$ and $b \in D_j$. Then certainly $\sup C_i = \sup D_i < x$, since otherwise $x \in D_i$. Likewise we have $\inf C_j > x$. But now it holds for any $c \in C_i \neq \emptyset$, $c' \in C_j \neq \emptyset$ that $c < x < c'$. Since obviously $x \notin \bigcup_i C_i$, it follows that $\bigcup_i C_i$ is disconnected, contradicting our assumption. Thus $d(\bigcup_i D_i) = \lambda(\bigcup_i D_i)$.
We now have

$$\begin{aligned} d\left(\bigcup_i C_i\right) \leq d\left(\bigcup_i D_i\right) = \lambda\left(\bigcup_i D_i\right) &\leq \sum_i \lambda(D_i) \\ &= \sum_i d(D_i) \\ &= \sum_i d(C_i), \end{aligned}$$

where, indeed, we invoke the sub-additive property that the Lebesgue measure is generally known to have. □

Proposition 1.22. The Hausdorff dimension of $\mathbb{R}$ equals 1.

Proof. Let us begin to prove that $\mathcal{H}^s(\mathbb{R}) = \infty$ for any $s \in [0,1]$. To this end, let $\{C_i\}_i$ be a cover of $\mathbb{R}$ such that $d(C_i) \leq 1$ for all $C_i \in \{C_i\}_i$. Then, by Lemma 1.21, $\sum_i d(C_i) = \infty$. But for any $s \in [0,1]$:

$$\sum_i d(C_i)^s \geq \sum_i d(C_i) = \infty,$$

yielding $\mathcal{H}^s_1(\mathbb{R}) = \infty$. Since $\delta \mapsto \mathcal{H}^s_\delta(\mathbb{R})$ is decreasing, it follows that $\mathcal{H}^s(\mathbb{R}) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(\mathbb{R}) \geq \mathcal{H}^s_1(\mathbb{R}) = \infty$. Hence $\dim_H \mathbb{R} \geq 1$.
In order to prove the reverse inequality, we define for every pair $m, i \in \mathbb{Z}_{>0}$ the interval

$$C_i^m = \left[\frac{1}{m}\sum_{k=1}^{i-1}\frac{1}{k}, \quad \frac{1}{m}\sum_{k=1}^{i}\frac{1}{k}\right].$$

Observe that for any $m \in \mathbb{Z}_{>0}$, $\{C_i^m\}_i$ is a cover of $\mathbb{R}_{\geq 0}$. Moreover, for all C_i^m it holds that $d(C_i^m) = \frac{1}{mi}$. For arbitrary $s \in (1,\infty)$ and $\delta > 0$, there is hence an $m \in \mathbb{Z}_{>0}$ such that $\{C_i^m\}_i$ is a δ-cover of $\mathbb{R}_{\geq 0}$. For such an m, we have

$$\sum_i d(C_i^m)^s = \sum_i \left(\frac{1}{mi}\right)^s = \left(\frac{1}{m}\right)^s \zeta(s),$$

where ζ denotes the Riemann zeta function. It follows that $\mathcal{H}^s_\delta(\mathbb{R}_{\geq 0}) \leq \left(\frac{1}{m}\right)^s \zeta(s)$ for all $m \geq 1/\delta$, implying $\mathcal{H}^s_\delta(\mathbb{R}_{\geq 0}) = 0$. By the arbitrariness of δ, we thus have $\mathcal{H}^s(\mathbb{R}_{\geq 0}) = 0$. Now because the map $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\leq 0}$ given by $f(s) = -s$ is a bi-Lipschitz map with lower and upper Lipschitz constants both equal to 1, appealing to Corollary 1.11 yields $\mathcal{H}^s(\mathbb{R}_{\leq 0}) = \mathcal{H}^s(\mathbb{R}_{\geq 0}) = 0$. Finally, invoking the σ-additivity of $\mathcal{H}^s$ (granted by Corollary 1.7) gives us

$$\mathcal{H}^s(\mathbb{R}) = \mathcal{H}^s(\mathbb{R}_{\geq 0}) + \mathcal{H}^s(\mathbb{R}_{<0}) \leq \mathcal{H}^s(\mathbb{R}_{\geq 0}) + \mathcal{H}^s(\mathbb{R}_{\leq 0}) = 0.$$

We conclude $\dim_H \mathbb{R} = 1$. □

Thus, we see once again a set with differing Hausdorff and box-counting dimensions: see Proposition 2.7 for the fact that the box-counting dimension of $\mathbb{R}$ equals ∞.
Next, we consider a set more exotic.

Definition 1.23. Let the set $E_1 \subset [0,1]$ be acquired from $E_0 := [0,1]$ by removing the middle third of E_0, so that $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. For $k = 2, 3, 4, \ldots$, recursively define E_{k+1} by removing the middle thirds of the intervals of E_k, so that E_k consists of 2^k intervals, each of diameter 3^{-k}. We call these intervals *basic intervals*. The *middle-third Cantor set* F is defined as

$$F = \bigcap_{k \geq 0} E_k.$$

In general, a set constructed analogously by acquiring E_{k+1} from E_k by removing the middle-α part (for some $\alpha \in [0,1]$) of the intervals of E_k is called a *middle-α Cantor set*.

In the following proposition we collect some well-known properties of the middle-third Cantor set (for a proof, see for example [3], Proposition 1.22, page 38).

Proposition 1.24. Let F denote the middle-third Cantor set.

1. F does not contain any intervals.
2. The cardinality of F equals that of $\mathbb{R}$. In particular, F is uncountable.
3. F is a Lebesgue-null set. □

We would like to calculate the Hausdorff dimension of the middle-third Cantor set. As this is quite an involved task, we first introduce a lemma to better organize our exposition.

Lemma 1.25. Let F be the middle-third Cantor set and let $s = \log 2/\log 3$. For any finite cover $\{C_i\}_{i=1}^n$ of F:

$$\sum_{i=1}^{n} d(C_i)^s \geq 1. \tag{1.1}$$

Proof. Let $\{C_i\}_{i=1}^n$ be a finite cover of F. The desired result is obvious if $d(C_i) \geq 1$ for some C_i, so let us assume that $d(C_i) < 1$ for all C_i. Furthermore, as in the proof of Lemma 1.21, we may assume every C_i to be a closed interval, say $[a_i, b_i]$.
A less trivial assumption is for the endpoints of every C_i to be in the complement of F. We justify this by supposing the opposite: let the endpoints of some $C_i = [a_i, b_i]$ be in F. We pick an arbitrary $\epsilon > 0$, and note that since F does not contain any intervals, there is a pair $\epsilon_i, \epsilon_i' \in [0, \frac{1}{2}(\epsilon/n)^{\frac{1}{s}}]$ such that the endpoints of $D_i := [a_i - \epsilon_i, b_i + \epsilon_i']$ are not in F. We thus acquire a cover $\{D_i\}_{i=1}^n$ of F for which it holds that

$$\begin{aligned}
\sum_{i=1}^{n} d(D_i)^s &\leq \sum_{i=1}^{n} \left(d(C_i) + (\epsilon/n)^{\frac{1}{s}} \right)^s \\
&\leq \sum_{i=1}^{n} d(C_i)^s + \epsilon/n \qquad (1.2) \\
&= \left(\sum_{i=1}^{n} d(C_i)^s \right) + \epsilon,
\end{aligned}$$

where (1.2) is justified by Lemma A.1. By the arbitrariness of ϵ, to prove (1.1) it suffices to show that $\sum_{i=1}^{n} d(D_i) \geq 1$. This justifies our assumption. Now for every $k \in \mathbb{Z}_{\geq 0}$, let E_k be as defined in Definition 1.23 and let $C_i \in \{C_i\}_i$. Because the endpoints of C_i are not in F, there exists a number $m(i) \in \mathbb{Z}_{\geq 0}$ such that for every basic interval B of $E_{m(i)}$ either $B \subseteq C_i$ or $B \cap C_i = \emptyset$, so that $C_i \cap E_{m(i)} = \bigcup_j B_j$ for some collection of basic intervals B_j of $E_{m(i)}$. We claim that

$$d(C_i)^s \geq \sum_j d(B_j)^s. \tag{1.3}$$

To see this, let $k(i) \in \mathbb{Z}_{\geq 0}$ be such that $3^{-k(i)-1} \leq d(C_i) < 3^{-k(i)}$ (such a $k(i)$ exists, since $d(C_i) < 1$). Note that C_i intersects at most one of the basic intervals of $E_{k(i)}$, since the seperation of these intervals is at least $3^{-k(i)}$. It follows that C_i intersects either one or two basic intervals of $E_{k(i)+1}$. If C_i intersects only one basic interval B of $E_{k(i)+1}$, it is clear that $d(C_i)^s \geq d(C_i \cap B)^s$. If C_i intersects two basic intervals B and B' of $E_{k(i)+1}$, then, with $B'' := C_i \backslash (B \cup B')$:

$$\begin{aligned}
d(C_i)^s &= \big(d(C_i \cap B) + d(C_i \cap B') + d(B'')\big)^s & (1.4)\\
&\geq \left(\frac{3}{2}(d(B \cap C_i) + d(B' \cap C_i))\right)^s & (1.5)\\
&= 2\left(\frac{1}{2}(d(B \cap C_i) + d(B' \cap C_i))\right)^s & \\
&\geq 2\left(\frac{1}{2}d(B \cap C_i)^s + \frac{1}{2}d(B' \cap C_i)^s\right) & (1.6)\\
&= d(B \cap C_i)^s + d(B' \cap C_i)^s, &
\end{aligned}$$

where (1.5) follows from the fact that $d(B'') = d(B') = d(B)$, and hence

$$d(B'') = \frac{1}{2}(d(B) + d(B')) \geq \frac{1}{2}(d(B \cap C_i) + d(B' \cap C_i)),$$

and (1.6) is justified by the fact that $x \mapsto x^s$ is a concave function. Since $B \cap C_i$ and $B' \cap C_i$ are again closed intervals, the inequality can be applied over and over, until one obtains a finite collection $\{B_j\}_j$ of basic intervals of $E_{m(i)}$, maintaining inequality (1.3).

Having found for every $C_i \in \{C_i\}_i$ a number $m(i) \in \mathbb{Z}_{\geq 0}$ such that $C_i \cap E_{m(i)}$ is a union of basic intervals of $E_{m(i)}$, let $m = \max\{m(i) : 1 \leq i \leq n\}$ and define the cover $\{D_j\}_j$ of F to be the collection of all basic intervals of E_m. Observe that for every C_i: $C_i \cap E_m = \bigcup_{j \in J(i)} D_j$ for some $J(i) \subseteq \{1, ..., 2^m\}$. Moreover, since $\{C_i\}_i$ is a cover of F, for every $D_j \in \{D_j\}_j$ there is at least one $C_i \in \{C_i\}_i$ such that $D_j \subseteq C_i$. By (1.3), we conclude

$$\sum_{i=1}^{n} d(C_i)^s \geq \sum_{i=1}^{n} \left(\sum_{j \in J(i)} d(D_j)^s \right) \geq \sum_{j=1}^{2^m} d(D_j)^s = 2^m \cdot 3^{-ms} = 1.$$

□

Note that in the proof of Lemma 1.25 we exploited the particular structure of $\mathbb{R}$ at least twice: by assuming the C_i to be closed intervals, and in justifying (1.4).

Proposition 1.26. The Hausdorff dimension of the middle-third Cantor set equals $\log 2/\log 3$.

Proof. We start off with some notation: let F denote the middle-third Cantor set, let $s := \log 2/\log 3$, and for all $k \in \mathbb{Z}_{\geq 0}$, let E_k be as defined in Definition 1.23. Now let us first prove s to be an upper bound for $\dim_H F$. To this end, note that for any $k \in \mathbb{Z}_{\geq 0}$, we can cover F by E_k, that is, the family $\{C_i\}_i$ consisting of the 2^k basic intervals of E_k is a 3^{-k}-cover of F. This yields

$$\mathcal{H}^s_{3^{-k}}(F) \leq \sum_i d(C_i)^s = 2^k 3^{-ks} = 1.$$

Because $\mathcal{H}^s_\delta(F)$ is a decreasing function of δ, letting k tend to infinity hence yields $\mathcal{H}^s(F) \leq 1$, and so $\dim_H F \leq s = \log 2/\log 3$.
In order to prove the reverse inequality, let $\{C_i\}_{i\in I}$ be a δ-cover of F, for some $\delta > 0$. As shown in the proof of Lemma 1.21, we may replace each C_i by a closed interval $[a_i, b_i]$ of the same diameter, and such that the collection of intervals $[a_i, b_i]$ covers F. Out of $\{[a_i, b_i]\}_{i\in I}$, we craft a new cover $\{D_i\}_{i\in I}$ of F by letting $\epsilon > 0$ and defining $D_i = (a_i(1+\epsilon)^{1/s}, b_i(1+\epsilon)^{1/s})$ for every $i \in I$. Because F is compact and $\{D_i\}_{i\in I}$ is an open cover of F, there is a finite subset J of I such that $\{D_j\}_{j\in J}$ is a cover of F. But by Lemma 1.25, it holds that

$$\sum_{j\in J} d(D_j)^s \geq 1,$$

from which it follows that

$$\sum_{i\in I} d(C_i)^s = \frac{1}{1+\epsilon}\sum_{i\in I} d(D_i)^s \geq \frac{1}{1+\epsilon}\sum_{j\in J} d(D_j)^s \geq \frac{1}{1+\epsilon}.$$

Since ϵ can be arbitrarily close to zero, this implies $\sum_{i\in I} d(C_i)^s \geq 1$. For any $\delta > 0$, we thus have $\mathcal{H}^s_\delta(F) \geq 1$, in turn yielding $\mathcal{H}^s(F) \geq 1$, and hence $\dim_H F \geq s = \log 2/\log 3$. □

The preceding examples illustrate that computation of the Hausdorff dimension can be quite involved. In applications, therefore, other notions of dimension are often utilized, offering more computational convenience. We shall continue discussing one such notion: the box-counting dimension.

Chapter 2

Box-counting dimension

2.1 Introduction

Besides the Hausdorff dimension, another popular notion of dimension is the *box-counting dimension.* It has the advantage of being easy 'in practice', i.e., computation of the box-counting dimension of a specific set often appears easier than computing its Hausdorff dimension; its behaviour, regrettably, is less desirable: while one would expect any countable set to have dimension 0 for any notion of dimension, the box-counting dimension of the set $\{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ is not (see Proposition 2.6). Moreover, the box-counting dimension is not countably stable.
The definitions and results in the very next section generalize those in [2], where the box-counting dimension is defined on $\mathbb{R}^n$. In Section 2.4, we show that under 'mild' conditions, our definition equals that in [7], opening the way for a derivation of a relation between the box-counting and the Hausdorff dimension. Although this relation is commonly known, detailed proofs appear to be scarce, as we haven't succeeded in finding one.

2.2 Definition and basic properties

Definition 2.1. Let (X, d) be a metric space, let $\delta > 0$ and let $U \subseteq X$. The quantity $N_\delta(U)$ is given by

$$N_\delta(U) = \min \{\#\{C_i\}_i : \{C_i\}_i \text{ is a } \delta\text{-cover of } U\}.$$

We shall write $\#\{C_i\}_i = \infty$ if $\{C_i\}_i$ is infinite.

Definitions 2.2. Let (X, d) be a metric space and let $U \subseteq X$.

1. Provided $N_\delta(U) \neq \infty$ for all $\delta > 0$, the *lower box-counting dimension*

of U and the *upper box-counting dimension* of U are

$$\begin{aligned}\underline{\dim}_B U &= \liminf_{\delta\to 0} \frac{\log N_\delta(U)}{\log \delta^{-1}} \\ \overline{\dim}_B U &= \limsup_{\delta\to 0} \frac{\log N_\delta(U)}{\log \delta^{-1}}\end{aligned}$$

respectively. If $N_\delta(U) = 0$ for some $\delta > 0$, we say that $\underline{\dim}_B U = \overline{\dim}_B U = 0$. If $N_\delta(U) = \infty$ for some $\delta > 0$, we say that $\underline{\dim}_B U = \overline{\dim}_B U = \infty$.

2. If $\underline{\dim}_B U$ and $\overline{\dim}_B U$ are equal, the *box-counting dimension* of U is

$$\dim_B U = \underline{\dim}_B U = \overline{\dim}_B U.$$

In many cases, it is much more convenient to work with sequential limits instead of continuous ones. Indeed, the following lemma will prove itself useful (see for example Proposition 2.8).

Lemma 2.3. Let (X, d) be a metric space, let $U \subseteq X$ be such that $\underline{\dim}_B U \neq \infty \neq \overline{\dim}_B(U)$ and let $(\delta_k)_k$ be a decreasing sequence in $(0, \infty)$ satisfying $\delta_{k+1} \geq c \cdot \delta_k$ for all $k \in \mathbb{Z}_{>0}$ and some fixed $c \in (0, 1)$. If $\lim_{k\to\infty} \delta_k = 0$, then

$$\underline{\dim}_B U = \liminf_{k\to\infty} \frac{\log N_{\delta_k}(U)}{\log \delta_k^{-1}} \qquad \text{and} \qquad \overline{\dim}_B U = \limsup_{k\to\infty} \frac{\log N_{\delta_k}(U)}{\log \delta_k^{-1}}.$$

Proof. Without loss of generality, we assume $(\delta_k)_k$ to be a sequence in $(0, 1)$. Noting moreover that for any $\delta \in (0, 1)$, it holds that

$$\frac{\log N_\delta(U)}{\log \delta^{-1}} = \frac{\log(N_\delta(U)^{-1})}{\log \delta}$$

and that $N_\delta(U)^{-1}$ is an increasing function of δ attaining values in $(0, 1]$, we can apply Lemma A.6 to directly obtain what is desired. □

The box-counting dimension turns out to have some properties in common with the Hausdorff dimension.

Proposition 2.4. $\underline{\dim}_B$ and $\overline{\dim}_B$ are monotone with respect to inclusion.

Proof. This follows from the fact that for a metric space (X, d), subsets $U \subseteq V$ of X and any $\delta > 0$, it holds that $N_\delta(U) \leq N_\delta(V)$. □

Theorem 2.5. Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f : X \to Y$ be a map. If f is Lipschitz, then

$$\underline{\dim}_B f(X) \leq \underline{\dim}_B X \qquad \text{and} \qquad \overline{\dim}_B f(X) \leq \overline{\dim}_B X.$$

If f is bi-Lipschitz, then

$$\underline{\dim}_B f(X) = \underline{\dim}_B X \qquad \text{and} \qquad \overline{\dim}_B f(X) = \overline{\dim}_B X.$$

Proof. The claims are immediate if $\underline{\dim}_B X = \infty$, so let us assume $\underline{\dim}_B X$ to be real. Suppose furthermore that f is Lipschitz with Lipschitz constant $c > 0$. Let $\delta > 0$, and observe that if $\{C_i\}_i$ is a δ-cover of X, then $\{f(C_i)\}_i$ is a $c\delta$-cover of $f(X)$. This yields $N_{c\delta}(f(X)) \leq N_\delta(X)$. It follows that

$$\begin{aligned}
\underline{\dim}_B f(X) &= \liminf_{\delta\to 0} \frac{\log N_\delta(f(X))}{\log(\delta^{-1})} \\
&= \liminf_{\delta\to 0} \frac{\log N_{c\delta}(f(X))}{\log((c\delta)^{-1})} \\
&\leq \liminf_{\delta\to 0} \frac{\log N_\delta(X)}{\log((c\delta)^{-1})} \\
&= \liminf_{\delta\to 0} \frac{\log N_\delta(X)}{\log(\delta^{-1})} \cdot \frac{1}{1 + \frac{\log c}{\log \delta}} \\
&= \liminf_{\delta\to 0} \frac{\log N_\delta(X)}{\log(\delta^{-1})} \qquad (2.1) \\
&= \underline{\dim}_B X,
\end{aligned}$$

where we refer to Lemma A.5 for a justification of (2.1). Proving that $\overline{\dim}_B f(X) \leq \overline{\dim}_B X$ can be done completely analogously.
Now suppose f is bi-Lipschitz. Without restricting generality, we assume f to be surjective, upon which Theorem 1.10 guarantees $f^{-1} : f(X) \to X$ to be bi-Lipschitz. In particular, f^{-1} is Lipschitz, yielding

$$\underline{\dim}_B X = \underline{\dim}_B f^{-1}(f(X)) \leq \underline{\dim}_B f(X)$$

by the very first result of this theorem. One can show that $\overline{\dim}_B f(X) = \overline{\dim}_B X$ by the same method. □

2.3 Three examples

We shall calculate the box-counting dimension of the same trio of sets for which we calculated the Hausdorff dimension. The very first example immediately shows how the two notions of dimension may differ.

Proposition 2.6. The box-counting dimension of the set

$$U := \{0\} \cup \{1/n : n \in \mathbb{Z}_{\geq 1}\}$$

equals 1/2.

Proof. Consider the sequence $(\delta_k)_k := (2^{-k})_k$ and the function $\psi : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$, defined to satisfy

$$\psi(k)\,(\psi(k) - 1) \leq 2^k < \psi(k)\,(\psi(k) + 1)$$

for all $k \in \mathbb{Z}_{\geq 0}$. Since for any $k \in \mathbb{Z}_{\geq 1}$ the points of $\{1, \frac{1}{2} \ldots, \frac{1}{\psi(k)}\}$ are seperated at least $((\psi(k)(\psi(k)-1))^{-1}$, any δ_k-cover of U contains at least $\psi(k)$ elements. We thus have

$$\frac{\log N_{\delta_k}(U)}{\log \delta_k^{-1}} \geq \frac{\log \psi(k)}{\log(\psi(k)(\psi(k)+1))}.$$

Taking lim inf on both sides and invoking Lemma 2.3 thus yields $\underline{\dim}_B U \geq 1/2$.

Conversely, for $k \in \mathbb{Z}_{\geq 1}$ arbitrary, the interval $[0, \psi(k)]$ can be covered by $\psi(k)+1$ intervals of diameter δ_k. The remaining $\psi(k)-1$ points of U can be covered by $\psi(k)-1$ intervals. It follows that $N_{\delta_k}(U) \leq 2\psi(k)$, and so

$$\frac{\log N_{\delta_k}(U)}{\log \delta_k^{-1}} \leq \frac{\log(2\psi(k))}{\log(\psi(k)(\psi(k)-1))}.$$

Taking lim sup on both sides and invoking Lemma 2.3 yields $\overline{\dim}_B(U) \leq 1/2$. Combining this with our preceding result completes the proof. □

Note that it follows from this that the box-counting dimension is not countably stable: it is easy to see that $\dim_B\{x\} = 0$ for any singleton $\{x\}$ in any metric space.

Proposition 2.7. The box-counting dimension of $\mathbb{R}$ equals ∞.

Proof. For any $\delta > 0$, it holds that $N_\delta(\mathbb{R}) = \infty$. The claim follows by definition of the box-counting dimension. □

Proposition 2.8. The box-counting dimension of the middle-third Cantor set equals $\log 2/\log 3$.

Proof. Let F denote the middle-third Cantor set. For any $k \in \mathbb{Z}_{>0}$, note that an interval of length 3^{-k-1} intersects at most one basic interval of length 3^{-k}. Since there are 2^k such basic intervals, we need at least 2^k intervals of length 3^{-k-1} to cover F. Defining the sequence $(\delta_k)_k$ by $\delta_k = 3^{-k-1}$ for all $k \in \mathbb{Z}_{>0}$, this translates to $N_{\delta_k}(F) \geq 2^k$. Invoking Lemma 2.3 thus yields

$$\begin{aligned} \liminf_{\delta \to 0} \frac{\log N_\delta(F)}{\log \delta^{-1}} &= \liminf_{k \to \infty} \frac{\log N_{\delta_k}(F)}{\log \delta_k^{-1}} \\ &\geq \liminf_{k \to \infty} \frac{\log 2^k}{\log 3^{k+1}} \\ &= \frac{\log 2}{\log 3}. \end{aligned}$$

Conversely, define the sequence $(\epsilon_k)_k$ by $\epsilon_k = 3^{-k}$ for all $k \in \mathbb{Z}_{>0}$ and observe that for these k, F can be covered by 2^k intervals of length 3^{-k}. We see thus that $N_{\epsilon_k}(F) \leq 2^k$, and again appealing to Lemma 2.3 yields

$$\begin{aligned}
\limsup_{\delta \to 0} \frac{\log N_\delta(F)}{\log \delta^{-1}} &= \limsup_{k \to \infty} \frac{\log N_{\epsilon_k}(F)}{\log \epsilon_k^{-1}} \\
&\leq \limsup_{k \to \infty} \frac{\log 2^k}{\log 3^k} \\
&= \frac{\log 2}{\log 3}.
\end{aligned}$$

For the middle-third Cantor set F, we conclude $\dim_B F = \log 2 / \log 3 = \dim_H F$. □

2.4 Relating the box-counting dimension to the Hausdorff dimension

It turns out that the box-counting dimension can be defined in a way very similar to how we introduced the Hausdorff dimension. This *capacity dimension*, as we shall call it, opens the way for an easy derivation of a relation between the Hausdorff and the box-counting dimension. Some authors, like Pesin ([7]), use this as the canonical definition of the box-counting dimension.

As with the Hausdorff dimension, we need some pillars for our notion to rest on.

Definition 2.9. Let (X, d) be a metric space, let $U \subseteq X$ and let $\delta > 0$. A *strict δ-cover* of U is a countable family $\{C_i\}_i \subseteq \mathcal{P}(X)$ such that $\bigcup_i C_i \supseteq U$ and $d(C_i) = \delta$ for all i.

Definition 2.10. Let (X, d) be a metric space, let $\delta > 0$ and let $U \subseteq X$. The quantity $N'_\delta(U)$ is given by

$$N'_\delta(U) = \min \left\{ \#\{C_i\}_i : \{C_i\}_i \text{ is a strict } \delta\text{-cover of } U \right\},$$

with the conventions that $\#\{C_i\}_i = \infty$ whenever $\{C_i\}_i$ is infinite, and $\min\{\infty\} = \min \emptyset = \infty$.

Definition 2.11. Let (X, d) be a metric space, let $U \subseteq X$ and let $s > 0$. The *s-dimensional lower capacity measure* and the *s-dimensional upper capacity measure* of U are

$$\begin{aligned}
\underline{\mathcal{D}}^s(U) &= \liminf_{\delta \to 0} \inf \left\{ \sum_i d(C_i)^s : \{C_i\}_i \text{ is a strict } \delta\text{-cover of } U \right\}, \\
\overline{\mathcal{D}}^s(U) &= \limsup_{\delta \to 0} \inf \left\{ \sum_i d(C_i)^s : \{C_i\}_i \text{ is a strict } \delta\text{-cover of } U \right\}
\end{aligned}$$

respectively.

Lemma 2.12. Let (X, d) be a metric space and let $U \subseteq X$. The functions

$$s \mapsto \underline{\mathcal{D}}^s(U) \qquad \text{and} \qquad s \mapsto \overline{\mathcal{D}}^s(U)$$

are decreasing. There is at most one $s \in \mathbb{R}_{\geq 0}$ for which $\underline{\mathcal{D}}^s(U)$ is real and non-zero. The same holds for $\overline{\mathcal{D}}^s(U)$.

Proof. This can be shown almost completely analogously to the way we proved Lemma 1.12. $\square$

Definition 2.13. Let (X, d) be a metric space and let $U \subseteq X$. The *lower and upper capacity dimensions* of U are

$$\begin{aligned} \underline{\dim}_D(U) &= \sup\{s \in \mathbb{R}_{\geq 0} : \underline{\mathcal{D}}^s(U) = \infty\} = \inf\{s \in \mathbb{R}_{\geq 0} : \underline{\mathcal{D}}^s(U) = 0\}, \\ \overline{\dim}_D(U) &= \sup\{s \in \mathbb{R}_{\geq 0} : \overline{\mathcal{D}}^s(U) = \infty\} = \inf\{s \in \mathbb{R}_{\geq 0} : \overline{\mathcal{D}}^s(U) = 0\} \end{aligned}$$

respectively.

Proposition 2.14. Let (X, d) be a metric space. For any $U \subseteq X$:

$$\dim_H U \leq \underline{\dim}_D U \leq \overline{\dim}_D U.$$

Proof. Let $U \subseteq X$. Since any strict δ-cover of U is a δ-cover of U, it holds for all $s \geq 0$ that $\mathcal{H}^s(U) \leq \underline{\mathcal{D}}^s(U)$, from which it follows that $\dim_H U \leq \underline{\dim}_D U$. The other inequality is immediate from the definitions of lim inf and lim sup. $\square$

Having related the capacity to the Hausdorff dimension, we shall continue to show that under mild conditions it equals the box-counting dimension. The following two lemmas hence comprise the essence of this section.

Lemma 2.15. Let (X, d) be a metric space, let $U \subseteq X$ be nonempty and for $d := \overline{\dim}_D(U)$, assume that $\overline{\mathcal{D}}^d(U) \neq \infty$. The following equations hold:

$$\underline{\dim}_D(U) = \liminf_{\delta \to 0} \frac{\log N'_\delta(U)}{\log \delta^{-1}}, \qquad \overline{\dim}_D(U) = \limsup_{\delta \to 0} \frac{\log N'_\delta(U)}{\log \delta^{-1}}.$$

Proof. Let us first note that for any $\delta > 0, s \geq 0$ and $U \subseteq X$, it holds that

$$\inf\left\{\sum_i d(C_i)^s : \{C_i\}_i \text{ is a strict } \delta\text{-cover of } U\right\} = N'_\delta(U)\delta^s,$$

so that

$$\overline{\mathcal{D}}^s(U) = \limsup_{\delta \to 0} N'_\delta(U)\delta^s. \tag{2.2}$$

Furthermore, observe that our assumption on U yields that $N'_\delta(U) > 0$ for any $\delta > 0$. For if $N'_\delta(U) = 0$ for some $\delta > 0$, then U would admit the empty cover and hence U itself would be empty. Combining this with (2.2), it follows that there is a $\delta' > 0$ such that for all $\delta \in (0, \delta')$:

$$N'_\delta(U)\delta^d =: c(\delta) \in (0, \infty).$$

Thus for those δ:

$$d = \frac{\log N'_\delta(U) - \log c(\delta)}{\log(\delta^{-1})}.$$

Now since $\liminf_{\delta\to 0} c(\delta) = \underline{\mathcal{D}}^d(U) \in (0, \infty)$, this yields

$$\underline{\dim}_D(U) = d = \liminf_{\delta\to 0} \frac{\log N'_\delta(U) - \log c(\delta)}{\log(\delta^{-1})} = \liminf_{\delta\to 0} \frac{\log N'_\delta(U)}{\log(\delta^{-1})}.$$

The analogue equation for $\overline{\dim}_D(U)$ can be proven in very much the same way. □

Lemma 2.16. Let (X, d) be a metric space and let $U \subseteq X$ be such that $N_\delta(U) \neq \infty \neq N'_\delta(U)$ for all δ in some interval $(0, \delta')$. The following equations hold:

$$\liminf_{\delta\to 0} \frac{\log N'_\delta(U)}{\log \delta^{-1}} = \underline{\dim}_B U, \qquad \limsup_{\delta\to 0} \frac{\log N'_\delta(U)}{\log \delta^{-1}} = \overline{\dim}_B U.$$

Proof. Observe that for $\delta > 0$, it holds that $N'_\delta(U) \geq N_\delta(U)$. Thus

$$\liminf_{\delta\to 0} \frac{\log N'_\delta(U)}{\log \delta^{-1}} \geq \underline{\dim}_B U.$$

Conversely, let $\delta \in (0, 1)$ and let $\{C_i\}_i$ be a $\frac{1}{2}\delta$-cover of U. Replacing every C_i by a closed ball of radius $\frac{1}{2}\delta$ centerered at any point of C_i, we obtain a strict δ-cover $\{D_i\}_i$ of U with $\#\{D_i\}_i = \#\{C_i\}_i$. It follows that $N'_\delta(U) \leq N_{\frac{\delta}{2}}(U)$, and hence

$$\frac{N'_\delta(U)}{\log \delta^{-1}} \leq \frac{N_{\frac{\delta}{2}}(U)}{\log \delta^{-1}}.$$

Te arbitrariness of δ then yields

$$\begin{aligned} \inf_{\epsilon\leq\delta} \frac{N'_\epsilon(U)}{\log \epsilon^{-1}} &\leq \inf_{\epsilon\leq\delta} \frac{N_{\frac{\epsilon}{2}}(U)}{\log \epsilon^{-1}} \\ &= \inf_{\epsilon\leq\frac{\delta}{2}} \frac{N_\epsilon(U)}{\log(2\epsilon^{-1})} \\ &= \inf_{\epsilon\leq\frac{\delta}{2}} \left(\frac{N_\epsilon(U)}{\log(\epsilon^{-1})} \cdot \frac{1}{1 + \frac{\log 2}{\log \epsilon^{-1}}} \right), \end{aligned}$$

whereupon Lemma A.4 shows

$$\begin{aligned}
\liminf_{\delta\to 0} \frac{N'_\delta(U)}{\log(\delta^{-1})} &= \lim_{\delta\to 0} \inf_{\epsilon\le\delta} \frac{N'_\epsilon(U)}{\log \epsilon^{-1}} \\
&\le \lim_{\delta\to 0} \inf_{\epsilon\le\frac{\delta}{2}} \left(\frac{N_\epsilon(U)}{\log(\epsilon^{-1})} \cdot \frac{1}{1+\frac{\log 2}{\log \epsilon^{-1}}} \right) \\
&= \lim_{\delta\to 0} \inf_{\epsilon\le\frac{\delta}{2}} \frac{N_\epsilon(U)}{\log(\epsilon^{-1})} \\
&= \liminf_{\delta\to 0} \frac{N_\delta(U)}{\log(\delta^{-1})} \\
&= \underline{\dim}_B U,
\end{aligned}$$

as desired. The second claim can be proven in very much the same way. □

Theorem 2.17. Let (X, d) be a metric space and suppose $U \subseteq X$ is such that for $d := \underline{\dim}_D(U)$ and $d' := \overline{\dim}_D(U)$:

$$0 < \underline{\mathcal{D}}^d(U) \le \overline{\mathcal{D}}^{d'}(U) < \infty.$$

Then the lower and upper capacity dimensions of U equal the lower and upper box-counting dimensions of U, respectively.

Proof. As shown in the proof of Lemma 2.15, our hypothesis on U yields a $\delta' > 0$ such that for all $\delta \in (0, \delta')$:

$$N_\delta(U) \le N'_\delta(U) < \infty.$$

We can hence combine Lemmas 2.15 and 2.16 to directly obtain the desired result. □

Corollary 2.18. Let (X, d) be a metric space and suppose $U \subseteq X$ is as in Theorem 2.17. Then

$$\dim_H U \le \underline{\dim}_B U \le \overline{\dim}_B U.$$

Proof. This is immediate from the conjunction of Proposition 2.14 and Theorem 2.17. □

Chapter 3

Correlation dimension

3.1 Introduction

Another popular notion of dimension is the *correlation dimension.* Originally introduced by Procaccia, Grassberger and Hentschel (in [8]) as a numerical procedure to measure the chaotic behaviour of a dynamical system, one considers for a metric space (X, d), 'small' $r \in \mathbb{R}_{>0}$, 'large' $n \in \mathbb{Z}_{>0}$ and some sequence $(x_i)_i$ in X the quantity

$$C(n, r) := \frac{1}{n^2} \# \{(i, j) \in [0, n) \times [0, n) : d(x_i, x_j) < r\} .$$

It is then assumed that $C(r) := \lim_{n\to\infty} C(n, r)$ exists, and that this function is asymptotically proportional to r^α for 'small' r and herewith determined $\alpha \in \mathbb{R}$. The number α is then called the *correlation dimension*, thus defined by

$$\alpha = \lim_{r\to 0} \lim_{n\to\infty} \frac{\log C(n, r)}{\log r}. \tag{3.1}$$

Seeking to justify this definition, we will need to investigate whether the limits in (3.1) exist. This, however, requires a change in perspective. While the approach by Procaccia, Grassberger and Hentschel ([8]) could be considered a statistical approach to investigate 'correlation' in experimental time series data, which is of course always a finite collection, one needs to make assumptions on the 'origin' of the (infinite) time series to be able to prove results. We will make the assumption, like Pesin ([7]), that the sequence $(x_i)_i$ is generated by a Borel measurable map $f : X \to X$, i.e., $x_{i+1} := f(x_i)$.

3.2 Preliminaries on ergodic measures

Elementary as the following notions are, this section mainly serves to avoid any ambiguity.

Definition 3.1. Let X be a nonempty set and let $f : X \to X$ be a map. A subset $U \subseteq X$ is called *f-invariant* if $f(U) \subseteq U$.

Definition 3.2. Let (X, Σ) be a measurable space and let $f : X \to X$ be a (Σ, Σ)-measurable map. A measure μ on Σ is called *f-invariant* if $\mu(f^{-1}(U)) = \mu(U)$ for all $U \in \Sigma$.

Definition 3.3. Let (X, Σ) be a measurable space and let $f : X \to X$ be a (Σ, Σ)-measurable map. An f-invariant measure μ on Σ is called *ergodic with respect to f* if for all f-invariant sets $U \in \Sigma$ either $\mu(U) = 0$ or $\mu(X \backslash U) = 0$.

Definition 3.4. Let (X, Σ) be a measurable space and let μ be a measure on Σ. The *support* of μ is

$$\operatorname{supp}(\mu) = \{x \in X : \mu((B(x, \epsilon)) > 0 \text{ for all } \epsilon > 0\}$$

Definition 3.5. For a metric space (X, d) and a given $r \in \mathbb{R}_{>0}$, we denote by S_r the r-neighbourhood of the diagonal in $X \times X$, i.e.:

$$S_r = \{(x, y) \in X \times X : d(x, y) \leq r\}.$$

Definition 3.6. For a metric space (X, d), a map $f : X \to X$ and given $x \in X$, $n \in \mathbb{Z}_{>0}$ and $r \in \mathbb{R}_{>0}$, the quantity $C(x, n, r)$ is given by

$$C(x, n, r) = \frac{1}{n^2} \# \left\{(i, j) \in [0, n) \times [0, n) : (f^i(x), f^j(x)) \in S_r\right\}.$$

3.3 Defining the correlation dimension

Lemma 3.7. Let (X, d) be a separable metric space, let Σ denote the Borel σ-algebra in (X, d) and let μ be a finite measure on Σ. For every pair $\delta, \epsilon > 0$, there is a finite partition $\mathcal{A} = \{A_n : 1 \leq n \leq N\}$ of X such that $\mu(A_N) \leq \delta$ and $d(A_n) \leq \epsilon$ for all $n \in \{1, ..., N - 1\}$.

Proof. Let $\delta, \epsilon > 0$ and let $\{q_i\}_i$ be a denumerable, dense subset of X. The family $\{C_i\}_i := \{B(q_i, \epsilon/2)\}_i$ obviously covers X, and moreover $d(C_i) \leq \epsilon$ for all i. Out of $\{C_i\}_i$, we craft a partition $\{A_i\}_i$ of X by defining recursively:

$$A_i = C_i \backslash \left(\bigcup_{j=1}^{i-1} C_j \right).$$

Observe that $\sum_{i=1}^{\infty} \mu(A_i) = \mu(X) \in \mathbb{R}$, and that there is hence an $N \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=N}^{\infty} \mu(A_i) \leq \delta$. It follows that the family

$$\mathcal{A} := \{A_i : 1 \leq i < N\} \cup \left(\bigcup_{i=N}^{\infty} A_i \right)$$

satisfies the properties indicated in the lemma. □

The following theorem was originally obtained by Pesin ([7]). The proof utilized here is from [6].

Theorem 3.8. Let (X, d) be a separable metric space and let Σ be the Borel σ-algebra in (X, d). Let $f : X \to X$ be a measurable map and let μ be an f-invariant, ergodic (with respect to f) probability measure on Σ. Let ν denote the product measure $\mu \times \mu$ and let $\phi : [0, \infty] \to [0, \infty]$ be given by $\phi(r) = \nu(S_r)$. There is a $Y \in \Sigma$ with $\mu(Y) = 1$ such that for every $x \in Y$ and all $r > 0$ in which ϕ is continuous:

$$\lim_{n\to\infty} C(x, n, r) = \phi(r).$$

Proof. For every $m \in \mathbb{Z}_{>0}$, let $\mathcal{A}^m = \{A^m_j : 1 \leq j \leq M(m)\}$ be a finite partition of X such that $\mu(A^m_1) \leq 2^{-m}$ and $d(A^m_j) \leq 2^{-m}$ for all $j \in \{2, ..., M(m)\}$ (these partitions exist by Lemma 3.7). Next, fix an $m \in \mathbb{Z}_{>0}$, let $r > 0$ be such that ϕ is continuous in r, and define the families $\mathcal{C} = \{C \in \mathcal{A}^m \times \mathcal{A}^m : C \subseteq S_r\}$ and $\mathcal{C}' = \{C \in \mathcal{A}^m \times \mathcal{A}^m : C \cap S_r \neq \emptyset\}$. Observe that $\bigcup_{C\in\mathcal{C}} C \subseteq S_r \subseteq \bigcup_{C\in\mathcal{C}'} C$, and hence for all $x \in X$, $n \in \mathbb{Z}_{>0}$:

$$\begin{aligned}
& \sum_{C\in\mathcal{C}} \frac{1}{n^2} \# \left\{(i, j) \in [0, n) \times [0, n) : (f^i(x), f^j(x)) \in C\right\} \\
= \; & \frac{1}{n^2} \# \left\{(i, j) \in [0, n) \times [0, n) : (f^i(x), f^j(x)) \in \bigcup_{C\in\mathcal{C}} C\right\} \\
\leq \; & C(x, n, r) \\
\leq \; & \frac{1}{n^2} \# \left\{(i, j) \in [0, n) \times [0, n) : (f^i(x), f^j(x)) \in \bigcup_{C\in\mathcal{C}'} C\right\} \\
= \; & \sum_{C\in\mathcal{C}'} \frac{1}{n^2} \# \left\{(i, j) \in [0, n) \times [0, n) : (f^i(x), f^j(x)) \in C\right\}. \qquad (3.2)
\end{aligned}$$

Moreover, one can check that

$$\begin{aligned}
& S_{r-2^{-m+1}} \setminus ((A^m_1 \times X) \cup (X \times A^m_1)) \subseteq \bigcup_{C\in\mathcal{C}} C \\
\subseteq & \bigcup_{C\in\mathcal{C}'} C \subseteq S_{r+2^{-m+1}} \cup ((A^m_1 \times X) \cup (X \times A^m_1)). \qquad (3.3)
\end{aligned}$$

As guaranteed by the Birkhoff ergodic theorem, there is a $Y \subseteq X$ of full measure such that for all $x \in Y$, $A \in \mathcal{A}^m$:

$$\lim_{n\to\infty} \frac{1}{n} \# \left\{i \in [0, n) : f^i(x) \in A\right\} = \mu(A).$$

Fixing an $x \in Y$, we can hence choose for every $m \in \mathbb{Z}_{>0}$, $A \in \mathcal{A}^m$ an $N(m, A) \in \mathbb{Z}_{>0}$ such that for all $n > N(m, A)$:

$$\left| \frac{1}{n} \# \left\{ i \in [0, n) : f^i(x) \in A \right\} - \mu(A) \right| < \frac{2^{-m-1}}{M(m)^2}. \tag{3.4}$$

Recalling that all $\mathcal{A}^m$ are finite, for every $m \in \mathbb{Z}_{>0}$ let $N(m) = \max\{N(m, A) : A \in \mathcal{A}^m\}$. Now observe that by (3.4), for any $m \in \mathbb{Z}_{>0}$, every $C = A \times A' \in \mathcal{A}^m \times \mathcal{A}^m$ and all $n > N(m)$ it holds that

$$\begin{aligned} & \left| \frac{1}{n^2} \# \left\{ (i, j) \in [0, n) \times [0, n) : (f^i(x), f^j(x)) \in C \right\} - \nu(C) \right| \\ = & \left| \frac{1}{n} \# \left\{ i \in [0, n) : f^i(x) \in A \right\} \cdot \frac{1}{n} \# \left\{ i \in [0, n) : f^i(x) \in A' \right\} - \mu(A)\mu(A') \right| \\ < & \frac{2^{-m}}{(M(m))^2}. \end{aligned}$$

In combination with (3.2), this yields for all $m \in \mathbb{Z}_{>0}$, $n > N(m)$:

$$\sum_{C \in \mathcal{C}} \left(\nu(C) - \frac{2^{-m}}{(M(m))^2} \right) < C(x, n, r) < \sum_{C \in \mathcal{C}'} \left(\nu(C) + \frac{2^{-m}}{(M(m))^2} \right).$$

Appealing to (3.3) and invoking the σ-additivity of ν, we thus obtain for all $m \in \mathbb{Z}_{>0}$ and $n > N(m)$:

$$\nu(S_{r-2^{-m+1}}) - 2^{-m} - 2 \cdot 2^{-m} < C(x, n, r) < \nu(S_{r+2^{-m+1}}) + 2^{-m} + 2 \cdot 2^{-m}.$$

We acquire the desired result by letting m tend to infinity and recalling that $\nu(S_r) = \phi(r)$ is continuous at r. □

Theorem 3.8, along with (3.1), motivates the following definition. It is due to Ruelle (unpublished; treated rigorously in [7]).

Definition 3.9. Let (X, d) be a metric space, let μ be a finite Borel measure on X and let ν denote the product measure $\mu \times \mu$ on $X \times X$. We call the quantities

$$\underline{\dim}_C(\mu) = \liminf_{r \to 0} \frac{\log \nu(S_r)}{\log r} \qquad \text{and} \qquad \overline{\dim}_C(\mu) = \limsup_{r \to 0} \frac{\log \nu(S_r)}{\log r}$$

the *lower* and *upper correlation dimension* of μ, respectively. If they are equal, we call

$$\dim_C(\mu) = \underline{\dim}_C(\mu) = \overline{\dim}_C(\mu)$$

the *correlation dimension* of μ.

3.4 Relating the correlation dimension to the Hausdorff dimension

3.4.1 Introduction

In an attempt to find a relation between the Hausdorff and Correlation dimension in the vein of Corollary 2.18, a result from [10] - namely, Proposition 2.1 - turned out useful. We treat it (generalized from $\mathbb{R}^n$ in [10] to metric spaces here) in the very next subsection.
A certain claim made in the proof, however, appeared difficult to verify. Introducing the *Young Property* to adress those measures and sets for which the claim is satisfied, the third subsection is devoted to finding sets and measures that have this property.
The fourth subsection, concludingly, utilizes the result from the second subsection to obtain a relation of the desired kind.

3.4.2 Local measure dimension

Definitions 3.10. Let (X, d) be a metric space, let μ be a Borel probability measure on X and let $x \in X$ be in the support of μ. The *lower and upper local measure dimensions of μ at x* are

$$\underline{d}_\mu(x) = \liminf_{\epsilon \to 0} \frac{\log \mu(B(x,\epsilon))}{\log \epsilon} \qquad \text{and} \qquad \overline{d}_\mu(x) = \limsup_{\epsilon \to 0} \frac{\log \mu(B(x,\epsilon))}{\log \epsilon}$$

respectively. If they are equal, we call $d_\mu(x) := \overline{d}_\mu(x) = \underline{d}_\mu(x)$ the *local measure dimension of μ in x*.

Names for the above definitions vary slightly. In [7], they are called the *(lower and upper) pointwise measure dimensions at x* and the *(lower and upper) local dimensions at x*. Note that the condition $x \in \text{supp}(\mu)$ assures $\underline{d}_\mu(x)$ and $\overline{d}_\mu(x)$ to be well defined: $\mu(B(x,\epsilon)) > 0$ for all $\epsilon > 0$.

The following two theorems generalize Proposition 2.1 in [10], which is set in $\mathbb{R}^n$. Ideally, one would like to prove the analogue result for metric spaces. As it turns out, though, this seems difficult at the very least: certain claims made in Youngs proof fail to hold in general metric spaces. For this reason, the following theorems involve a particular property of metric spaces, which we decided to call *semi-geodesity*. We refer to Appendix B for the definition and a number of results regarding this notion. Most notably, in a semi-gedesic metric space (X, d) it holds that for any $r > 0$ the map

$$\begin{array}{rcl} X & \to & \mathcal{P}(X) \\ x & \mapsto & B(x,r) \end{array}$$

is Lipschitsch continuous with respect to the Hausdorff pseudometric (Lemma B.12). In particular, it is continuous, from which the measurability of the function

$$\begin{array}{rcl} X & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & \mu(B(x,r)) \end{array}$$

follows.
Aside from the fact that a 'full' generalization seems impossible, the proof of Young's proposition also includes a claim not easily verified even in $\mathbb{R}^n$. As satisfaction of the claim is dependent on both the measure and the metric space one deals with, we use the *Young property* to address those measures and sets for which the claim in question is correct. The *fundamental property* is a property of metric spaces, and implies the Young property if the local measure dimension behaves 'nicely'. We refer to the next subsection for the exact definitions and details surrounding these notions (both coined by the author). As a short summary: in a metric space having the fundamental property, there is a positive integer N such that for arbitrarily small δ, any bounded set U has a δ-cover for which all N-fold intersections are empty. If a measure μ has the Young Property on a set U, then for all $s > 1$:

$$\liminf_{\delta \to 0} \left\{ \sum_i \mu(C_i)^s : \{C_i\}_i \text{ is a } \delta\text{-cover of } U \right\} = 0.$$

Lemma 3.11. Let (X, d) be a semi-geodesic metric space with $\#X \geq 2$ having the fundamental property, let μ be a Borel probability measure on (X, d) and let $U \subseteq \operatorname{supp}(X)$ be bounded and have positive μ-measure. Suppose there are $a, b \in [0, \infty)$ and $\delta \in (0, 1)$ such that for every ball δ-cover $\{B(x_i, r_i)\}_i$ of U, it holds for all i that

$$a \leq \frac{\log \mu(B(x_i, r_i))}{\log r_i} \leq b. \tag{3.5}$$

Then

$$a \leq \dim_H U \leq b.$$

Proof. Let $\alpha > 0$ be as in the statement of Lemma B.14, let $\{B(x_i, r_i)\}_i$ be a ball $\min\{\alpha, \delta\}$-cover of U, and note that by the first inequality of (3.5), it holds that

$$\sum_i r_i^a \geq \sum_i \mu(B(x_i, r_i)) \geq \mu(U).$$

Turning to Lemma B.14, it follows that

$$\sum_i d(B(x_i, r_i))^a \geq \mu(U).$$

By Lemma 1.19 and the fact that $\mu(U) > 0$, we conclude $\dim_H U \geq a$.

We continue to prove the reverse inequality. To this end, note that by Theorem 3.19 the measure μ has the Young Property on U. It follows from this that for $s > 1$, $\delta_1 \in (0, \delta)$ and $\delta_2 > 0$, there is a ball δ_1-cover $\{B(x_i, r_i)\}_i$ of U satisfying

$$\sum_i \mu(B(x_i, r_i))^s \leq \delta_2.$$

Now by (3.5), it holds for all i that $r_i^b \leq \mu(B(x_i, r_i))$. Noting moreover that in any metric space the diameter of a ball is at most twice its radius, the foregoing yields

$$\sum_i d(B(x_i, r_i))^{sb} \leq \sum_i (2r_i)^{sb} \leq 2^{sb} \sum_i \mu(B(x_i, r_i))^s \leq 2^{sb}\delta_2,$$

and hence $\dim_H U \leq sb$. By definition of s, it follows that $\dim_H U \leq b$. □

Theorem 3.12. Let (X, d) be a semi-geodesic metric space with $\#X \geq 2$ having the fundamental property, let μ be a Borel probability measure on (X, d) and let $V \subseteq \operatorname{supp}(X)$ be bounded and have positive μ-measure. Suppose there are $a, b \in (0, \infty)$ such that for every $x \in V$:

$$a \leq \underline{d}_\mu(x) \leq \overline{d}_\mu(x) \leq b.$$

Then

$$a \leq \dim_H V \leq b.$$

Proof. Let $(r_k)_k$ be a sequence in $(0, \infty)$ such that $\lim_{k\to\infty} r_k = 0$ and $r_{k+1} \geq c \cdot r_k$ for all k and some fixed $c \in (0, 1)$. By Lemma A.6:

$$\begin{aligned} a \leq \underline{d}_\mu(v) \quad &= \liminf_{k\to\infty} \tfrac{\log \mu(B(v,r_k))}{\log(r_k)} \\ &\leq \limsup_{k\to\infty} \tfrac{\log \mu(B(v,r_k))}{\log(r_k)} \quad = \overline{d}_\mu(v) \leq b \end{aligned} \tag{3.6}$$

for all $v \in V$. Now defining for a fixed $\epsilon > 0$ and all $k \in \mathbb{Z}_{>0}$ the set

$$V_k = \left\{ v \in V : a - \epsilon \leq \frac{\log \mu(B(v, \delta_i))}{\log \delta_i} \leq b + \epsilon \text{ for all } i \geq k \right\},$$

observe that by Theorems B.7 and B.13 the function $x \mapsto \mu(B(x, \delta))$ is Borel-measurable and hence every V_k is Borel-measurable. Cearly $V_{k+1} \supseteq V_k$. Noting moreover that by (3.6) we have $\bigcup_k V_k = V$, it follows that $\lim_{k\to\infty} \mu(V_k) = \mu(V)$. Hence for a certain $n \in \mathbb{Z}_{>0}$, it holds for all $k \geq n$ that $\mu(V_k) > 0$. Thus, Lemma 3.11 can be applied to every such V_k, yielding for all $k \geq n$:

$$a - \epsilon \leq \dim_H V_k \leq b + \epsilon,$$

in turn giving

$$a - \epsilon \leq \sup\{\dim_H V_k\}_{k=n}^\infty \leq b + \epsilon.$$

But by Proposition 1.14 (b) and the fact that $V_k \subseteq V_{k+1}$ for all k, we have

$$\sup\{\dim_H V_k\}_{k=n}^{\infty} = \dim_H \left(\bigcup_{k=n}^{\infty} V_k \right) = \dim_H V.$$

The desired result thus follows by the arbitrariness of ϵ. □

3.4.3 The Young Property

We shall in this subsection introduce the Young Property, a term we came up with ourselves. Up to a certain degree, we shall sort out what measures and sets do or do not possess it. To elaborate a bit on the motivation behind this scheme, we recall that Young made a claim in her proof of Theorem 2.1 (from [10]) whose analogue for metric spaces appeared hard to verify. For $U \subseteq \mathbb{R}^n$ and μ a Borel measure, it reads as follows:

> 'If U is measurable and has positive μ-measure, then it is easy to verify that $\alpha(U, \mu) = 1$.'

Using the terminology introduced in the next few pages, the statement '$\alpha(U, \mu) = 1$' translates to 'μ has the Young Property on U'. Hence our goal is to answer the following questions:

- What are the precise conditions for μ to have the Young Property on U if $U \subseteq \mathbb{R}^n$?
- What are the precise conditions for μ to have the Young Property on U if U is a subset of a metric space?

Both of these are partially answered by Theorem 3.19. In particular, the correctness of Youngs claim follows for all bounded $U \subset \mathbb{R}^n$.
As a side remark, we note that the theorem in which Young made her claim was 'essentially borrowed' from [1], in which $U = [0, 1]$. In this last case, the claim *is* indeed not hard to verify.

The following is fundamental to the statement of the property.

Definitions 3.13. Let (X, d) be a metric space, let μ be a Borel measure on (X, d), let $U \subseteq X$ and let $s \geq 0$.

1. For $\delta > 0$, the quantity $\mu_\delta^s(U)$ is given by

$$\mu_\delta^s(U) = \inf \left\{ \sum_i \mu(C_i)^s : \{C_i\}_i \text{ is a ball } \delta\text{-cover of } U \right\}.$$

2. The quantity $\mu^s(U)$ is given by

$$\mu^s(U) = \lim_{\delta \to 0} \mu_\delta^s(U).$$

Similar to the Hausdorff measure, the quantity $\mu^s(U)$ is well defined by the fact that $\mu^s_\delta(U)$ is a decreasing function of δ.

Proposition 3.14. For a metric space (X, d), a Borel measure μ on (X, d) and $s \geq 0$, the function μ^s is an outer measure on X. That is to say, $\mu^s : \mathcal{P}(X) \to [0, \infty]$ satisfies the following properties:

a) $\mu^s(\emptyset) = 0$,

b) $\mu^s(U) \leq \mu^s(V)$ whenever $U \subseteq V \subseteq X$,

c) $\mu^s(\bigcup_i U_i) \leq \sum_i \mu^s(U_i)$ for every countable family $\{U_i\}_i \subseteq \mathcal{P}(X)$.

Proof. This can be verified analogously to how Theorem 1.5 (and hence Theorem 4 from [9]) was proven. □

Definition 3.15 (Young Property)**.** Let (X, d) be a metric space and let $U \subseteq X$. A Borel measure μ on (X, d) has the *Young Property on* U if

$$\mu^s(U) = 0 \quad \text{for all} \quad s > 1.$$

If μ has the Young Property on all $U \subseteq X$, we simply say that μ has the *Young Property.*

We continue to investigate what measures do or do not have the Young Property.

Theorem 3.16. An atomic measure does not have the Young Property.

Proof. Let (X, d) be a metric space and suppose μ is an atomic Borel measure on (X, d), i.e., $\mu(\{x\}) > 0$ for some $x \in X$. Then for any $\delta > 0$, $s \geq 0$ and any ball δ-cover $\{C_i\}_i$ of $\{x\}$:

$$\sum_i \mu(C_i)^s \geq \mu(\{x\})^s > 0.$$

Hence $\mu^s_\delta(\{x\}) \geq \mu(\{x\})^s$ and so $\mu^s(\{x\}) \geq \mu(\{x\})^s > 0$. The result follows by Proposition 3.14 (b). □

Seeking measures that *do* have the Young property, we contrived the following condition. Again, this is the author's terminology.

Definition 3.17. We say that a metric space (X, d) has the *fundamental property* if there is an $N \in \mathbb{Z}_{\geq 0}$ such that for all bounded $U \subseteq X$ and $\delta > 0$, there is a finite ball δ-cover $\{C_i\}_{i=1}^n$ of U satisfying

$$(J \subseteq \{1, \ldots, n\}, \quad \#J \geq N) \quad \Rightarrow \quad \bigcap_{j \in J} C_j = \emptyset. \tag{3.7}$$

Examples of metric spaces with this property are $\mathbb{R}^n$, either equipped with the Euclidean metric, the metric inherited from the p-norm or that from the maximum norm. Infinite dimensional vector spaces often lack it.
We shall continue to show how this condition may lead to satisfaction of the Young property. The proof invokes the generally known *inclusion-exclusion principle*, of which a proof is included in Appendix C.

Lemma 3.18. Let (X,d) be a metric space having the fundamental property and let μ be a finite Borel measure on (X,d). For any bounded $U \subseteq X$:

$$\mu^s(U) < \infty \quad \text{for all} \quad s \geq 1.$$

Proof. Let $U \subseteq X$ be bounded, let $\delta > 0$ and let N be as in the statement of the fundamental property. Then, let $\{C_i\}_{i=1}^n$ be a finite ball δ-cover satisfying (3.7). By the inclusion-exclusion principle:

$$\mu\left(\bigcup_{i=1}^n C_i\right) = \sum_{k=1}^n (-1)^{k-1} \left(\sum_{\substack{I \subseteq \{1,\dots,n\} \\ \#I=k}} \mu(C_I) \right),$$

where we define $C_I = \bigcap_{i \in I} C_i$ for notational reasons. Rewriting this equation and estimating from above yields:

$$\begin{aligned}
\sum_{i=1}^n \mu(C_i) &= \mu\left(\bigcup_{i=1}^n C_i\right) - \sum_{k=2}^n (-1)^{k-1} \left(\sum_{\substack{I \subseteq \{1,\dots,n\} \\ \#I=k}} \mu\left(C_I\right) \right) \\
&\leq \mu\left(\bigcup_{i=1}^n C_i\right) + \sum_{\substack{k=2 \\ k \text{ even}}}^n \left(\sum_{\substack{I \subseteq \{1,\dots,n\} \\ \#I=k}} \mu(C_I) \right).
\end{aligned}$$

For $s \geq 1$, it follows that

$$\begin{aligned}
\sum_{i=1}^n \mu(C_i)^s &\leq \left(\sum_{i=1}^n \mu(C_i) \right)^s \\
&\leq \left(\mu\left(\bigcup_{i=1}^n C_i\right) + \sum_{\substack{k=2 \\ k \text{ even}}}^n \left(\sum_{\substack{I \subseteq \{1,\dots,n\} \\ \#I=k}} \mu(C_I) \right) \right)^s.
\end{aligned}$$

Hence, by the fact that μ (X,d) has the fundamental property:

$$\begin{aligned}\sum_{i=1}^{n}\mu(C_i)^s &\leq \left(\mu\left(\bigcup_{i=1}^{n}C_i\right)+\sum_{\substack{k=2\\k\text{ even}}}^{N}\left(\sum_{\substack{I\subseteq\{1,\ldots,n\}\\\#I=k}}\mu(C_I)\right)\right)^s\\&\leq \left(\mu\left(\bigcup_{i=1}^{n}C_i\right)+\frac{N}{2}m_N\cdot\mu\left(\bigcup_{i=1}^{n}C_i\right)\right)^s,\end{aligned}$$

with $m_N := \max\left\{\binom{N}{k} : k\in\{1,2,\ldots,N\}\right\}$. Now defining c_N by

$$c_N = \left(1+\frac{N}{2}m_N\cdot\mu(X)\right)^s,$$

recalling that μ is finite and observing that $\mu\left(\bigcup_{i=1}^{n}C_i\right)\leq\mu(X)$, we thus have

$$\sum_{i=1}^{n}\mu(C_i)^s\leq c_N^s\mu(X)^s.$$

By definition of μ_δ^s, it follows that $\mu_\delta^s(U)\leq c_N^s\mu(X)^s$. The arbitrariness of δ yields the final result. □

Theorem 3.19. Let (X,d) be a metric space having the fundamental property, let μ be a Borel probability measure on (X,d) and let $U\subseteq\operatorname{supp}(X)$ be bounded. Suppose there are $\delta\in(0,1)$ and $a\in(0,\infty)$ such that for any ball δ-cover $\{B(x_i,r_i)\}_i$ of U, it holds for all i that

$$a\leq\frac{\log\mu(B(x_i,r_i))}{\log r_i}. \qquad (3.8)$$

Then μ has the Young Property on U.

Proof. It suffices to show that

1) if $t>s\geq 0$ and $\mu^s(U)<\infty$, then $\mu^t(U)=0$,

2) $\mu^1(U)<\infty$.

To this end, let δ and a be as in the statement of this theorem, let $\delta_1\in(0,\delta]$ and let $\{B(x_i,r_i)\}_i$ be a ball δ_1-cover of U. Observe that since $r_i\leq\delta_1\leq\delta<1$ for all i, it follows from (3.8) that

$$\mu(B(x_i,r_i))\leq r_i^a\leq\delta_1^a.$$

So if $t > s \geq 0$, then

$$\begin{aligned}\sum_i \mu(B(x_i, r_i))^t &= \sum_i \mu(B(x_i, r_i))^{t-s} \mu(B(x_i, r_i))^s \\ &\leq \delta_1^{a(t-s)} \sum_i \mu((B(x_i, r_i))^s.\end{aligned}$$

By Lemma A.4, it follows that $\mu^t_{\delta_1}(U) \leq \delta_1^{a(t-s)} \mu^s_{\delta_1}(U)$, and letting δ_1 tend to zero shows that $\mu^t(U) = 0$ whenever $\mu^s(U)$ is finite. This proves the first statement. The second statement is a special case of Lemma 3.18. □

3.4.4 Relating under the Young Property

In the next theorem, we shall make grateful use of Jensen's inequality. For the sake of clarity and completeness, we will state here the exact variant we wish to apply (to be found in [5], along with a proof).

Jensen's inequality. Let (X, Σ) be a measurable space and let μ be a probability measure on Σ. If $g : \Sigma \to \mathbb{R}$ is in $L^1(\mu)$ and if $f : \mathbb{R} \to \mathbb{R}$ is convex, then

$$f\left(\int_X g \, d\mu\right) \leq \left(\int_X f \circ g \, d\mu\right).$$

Theorem 3.20. Let (X, d) be a semi-geodesic metric space with $\#X \geq 2$, let μ be a Borel probability measure on X and let $Y \subseteq X$ be the support of μ. If $\underline{d}_\mu(y) = \overline{d}_\mu(y) = d_\mu(y)$ for all $y \in Y$, then

$$\overline{\dim}_C(\mu) \leq \int_Y d_\mu(y) d\mu(y).$$

Proof. From the fact that $r \mapsto \log r$ is a concave function, it follows that $r \mapsto -\log r$ is convex. Noting moreover that for any $r > 0$, the function $X \to \mathbb{R} : x \mapsto \mu(B(x, r))$ is measurable (Lemmas B.7 and B.13) and so, by the finiteness of μ, integrable, we apply Jensen's inequality to obtain for any $r \in (0, 1)$:

$$\begin{aligned}-\log(\mu \times \mu(S_r)) &= -\log\left(\int_Y \mu(B(y, r)) \; d\mu(y)\right) \\ &\leq \int_Y -\log(\mu(B(y, r))) \; d\mu(y),\end{aligned}$$

or

$$\frac{\log((\mu \times \mu)(S_r))}{\log r} \leq \int_Y \frac{\log(\mu(B(y, r)))}{\log r} \; d\mu(y). \tag{3.9}$$

From this last equality, it follows that we may assume the function

$$\frac{\log\left(\mu\left(B(y,r)\right)\right)}{\log r}$$

to be integrable, since our desired result becomes trivially true if we assume the opposite. Now let $(\delta_k)_k$ be a sequence in $(0,1)$ satisfying the conditions stated in Lemma A.6, say $\delta_k := 2^{-k}$ for all $k \in \mathbb{Z}_{\geq 1}$. By this lemma and our assumption on $d_\mu(y)$ (for $y \in Y$), we have

$$\begin{aligned}
\int_Y d_\mu(y)\ d\mu(y) &= \int_Y \lim_{k\to\infty} \frac{\log \mu(B(y,\delta_k))}{\log \delta_k}\ d\mu(y) \\
&= \lim_{k\to\infty} \int_Y \frac{\log \mu(B(y,\delta_k))}{\log \delta_k}\ d\mu(y) \\
&= \limsup_{k\to\infty} \int_Y \frac{\log \mu(B(y,\delta_k))}{\log \delta_k}\ d\mu(y) \\
&\geq \limsup_{k\to\infty} \frac{\log\left((\mu\times\mu)(S_{\delta_k})\right)}{\log \delta_k} \\
&= \overline{\dim}_C(\mu),
\end{aligned}$$

where the inequality is justified by (3.9). □

Corollary 3.21. Let (X,d) be a bounded, semi-geodesic metric space with $\#X \geq 2$ having the fundamental property, let μ be a Borel probability measure on X and let $Y \subseteq X$ denote the support of μ. If there is a $c \in \mathbb{R}$ such that

$$\underline{d}_\mu(y) = \overline{d}_\mu(y) = c$$

for all $y \in Y$, then

$$\underline{\dim}_C(\mu) \leq \overline{\dim}_C(\mu) \leq c = \dim_H X.$$

Proof. This is immediate from Theorems 3.12 and 3.20. □

Appendix A

Useful estimates

Lemma A.1. For any finite set of numbers $\{r_i\}_{i=1}^n \subset \mathbb{R}_{\geq 0}$ and any $s \in [0,1]$:

$$\left(\sum_{i=1}^n r_i\right)^s \leq \sum_{i=1}^n r_i^s.$$

Proof. Let $\{r_i\}_{i=1}^n$ be as in the statement of this lemma, and note that by induction it suffices to verify the inequality for the case $n = 2$. To this end, let $a, b \in \mathbb{R}_{\geq 0}$ and assume $a \leq b$. Observe that, for $s \in [0,1]$:

$$\left(1 + \frac{a}{b}\right)^s \leq 1 + \frac{a}{b} \leq 1 + \left(\frac{a}{b}\right)^s.$$

It follows that

$$(a+b)^s = b^s \left(1 + \frac{a}{b}\right)^s \leq b^s \left(1 + \left(\frac{a}{b}\right)^s\right) = a^s + b^s,$$

as desired. □

Lemma A.2. Let $f : (0,\infty) \to \mathbb{R}$, and let $(\delta_k)_k \subset (0,\infty)$ be such that $\delta_k \to 0$. Then

$$\liminf_{x\to 0} f(x) \leq \liminf_{k\to\infty} f(\delta_k) \qquad \text{and} \qquad \limsup_{x\to 0} f(x) \geq \limsup_{k\to\infty} f(\delta_k).$$

Proof. We start out with the first inequality. We need to prove that

$$\lim_{x\to 0} \inf\{f(y) \colon y \in (0,x)\} \leq \lim_{k\to\infty} \inf\{f(\delta_i) \colon i \geq k\}. \tag{A.1}$$

To this end, we shall show that for every $x > 0$ there is a $k \in \mathbb{Z}_{>0}$ such that

$$\inf\{f(y) \colon y \in (0,x)\} \leq \inf\{f(\delta_i) \colon i \geq k\},$$

since this clearly implies (A.1). Hence let $x > 0$. Since $\delta_i \to 0$ as $i \to \infty$, there is a $k \in \mathbb{Z}_{>0}$ such that $\{\delta_i \colon i \geq k\} \subset (0,x)$. It follows that $\{f(\delta_i) \colon i \geq k\} \subseteq \{f(y) \colon y \in (0,x)\}$, and this yields the result required. The other inequality can be proven analogously. □

Lemma A.3. Let $f : (0, \infty) \to \mathbb{R}$, and let $(\delta_k)_k$ be a real, positive and decreasing sequence such that $\delta_k \to 0$. Suppose that for every $k \in \mathbb{Z}_{>0}$, the numbers $m_k,\ M_k \in \mathbb{R}$ satisfy

$$m_k \leq \inf\{f(x) : \delta_{k+1} \leq x < \delta_k\} \leq \sup\{f(x) : \delta_{k+1} \leq x < \delta_k\} \leq M_k.$$

Then $\liminf_{k\to\infty} m_k \leq \liminf_{x\to 0} f(x)$, and $\limsup_{k\to\infty} M_k \geq \limsup_{x\to 0} f(x)$.

Proof. Observe that

$$\begin{aligned} \liminf_{x\to 0} f(x) &= \lim_{x\to 0} \inf\{f(y) : y \in (0, x)\} \\ &= \lim_{k\to\infty} \inf\{f(y) : y \in (0, \delta_k)\} \\ &\geq \lim_{k\to\infty} \inf\{m_n : n \geq k\} \\ &= \liminf_{k\to\infty} m_k. \end{aligned}$$

The other inequality can be proven analogously. □

Lemma A.4. For any non-empty $A, B \subseteq \mathbb{R}_{\geq 0}$:

a) $\inf A \cdot \inf B = \inf\{ab : a \in A,\ b \in B\}$,

b) $\sup A \cdot \sup B = \sup\{ab : a \in A,\ b \in B\}$.

Proof.

a) Obviously $\inf A \cdot \inf B \leq ab$ for any pair $a \in A,\ b \in B$, giving

$$\inf A \cdot \inf B \leq \inf\{ab : a \in A,\ b \in B\} =: x.$$

To prove the reverse inequality, let us first get rid of the trivial cases. If either $0 \in A$ or $0 \in B$ (or both), then the asserted equality follows immediately. We can therefor assume $A, B \subseteq \mathbb{R}_{>0}$. Moreover, it is easy to see that $\inf B = 0$ implies $x = 0$. Hence we assume $\inf B \neq 0$.

Now let $a \in A$ be arbitrary, and observe that for any $\epsilon > 0$ there is a $b \in B$ such that

$$\frac{b}{\inf B} \leq 1 + \frac{\epsilon}{a},$$

or

$$\frac{ab}{\inf B} \leq a + \epsilon.$$

But by definition of x, it holds for the same a and b that

$$\frac{x}{\inf B} \leq \frac{ab}{\inf B}.$$

Thus by the arbitrariness of ϵ, it follows that

$$\frac{x}{\inf B} \leq a,$$

upon which the arbitrariness of a yields

$$\frac{x}{\inf B} \leq \inf A,$$

which is equivalent to the desired result.

b) This can be shown analogously to how part (a) was proven. □

Lemma A.5. Let $f, g : (0, \infty) \to (0, \infty)$ and let $(\delta_k)_k$ be some real, positive sequence. If either

1) $\liminf_{k\to\infty} f(\delta_k) \neq \infty \neq \liminf_{k\to\infty} g(\delta_k)$, or

2) $\liminf_{k\to\infty} f(\delta_k) = \infty$ and $\liminf_{k\to\infty} g(\delta_k) \neq 0$,

then

$$\liminf_{k\to\infty} f(\delta_k) \cdot \liminf_{k\to\infty} g(\delta_k) \leq \liminf_{k\to\infty} f(\delta_k)g(\delta_k). \tag{A.2}$$

If in addition to (1) the sequence $(f(\delta_k))_k$ converges, then (A.2) holds with equality. Similarly, if either

3) $\limsup_{k\to\infty} f(\delta_k) \neq \infty \neq \limsup_{k\to\infty} g(\delta_k)$, or

4) $\limsup_{k\to\infty} f(\delta_k) = \infty$ and $\limsup_{k\to\infty} g(\delta_k) \neq 0$,

then

$$\limsup_{k\to\infty} f(\delta_k) \cdot \limsup_{k\to\infty} g(\delta_k) \leq \limsup_{k\to\infty} f(\delta_k)g(\delta_k). \tag{A.3}$$

And if, in addition to (3), the sequence $(f(\delta_k))_k$ converges, then (A.3) holds with equality.

Proof. Let us assume case (1). By the definitions of the notions involved, we have:

$$\begin{aligned}
& \liminf_{k\to\infty} f(\delta_k) \cdot \liminf_{k\to\infty} g(\delta_k) \\
= \; & \lim_{k\to\infty} \inf\{f(\delta_i)\colon i \geq k\} \cdot \lim_{k\to\infty} \inf\{g(\delta_i)\colon i \geq k\} \\
= \; & \lim_{k\to\infty} (\inf\{f(\delta_i)\colon i \geq k\} \cdot \inf\{g(\delta_i)\colon i \geq k\}) \\
= \; & \lim_{k\to\infty} \inf\{f(\delta_i)g(\delta_j)\colon i, j \geq k\} \\
\leq \; & \lim_{k\to\infty} \inf\{f(\delta_i)g(\delta_i)\colon i \geq k\} \\
= \; & \liminf_{k\to\infty} f(\delta_k)g(\delta_k),
\end{aligned} \tag{A.4}$$

where (A.4) is a direct consequence of Lemma A.4.
Assume case (2). By the fact that $\liminf_{k\to\infty} g(\delta_k) > 0$, there are $K \in \mathbb{Z}_{>0}$ and $m \in (0, \infty)$ with

$$\inf\{g(\delta_i) \colon i \geq k\} \geq m$$

for all $k \geq K$. In particular, we have $g(\delta_k) \geq m$ for all $k \geq K$. Hence for these k:

$$\inf\{f(\delta_i)g(\delta_i) \colon i \geq k\} \geq \inf\{f(\delta_i)m \colon i \geq k\} = m \cdot \inf\{f(\delta_i) \colon i \geq k\},$$

utilizing Lemma A.4 for the equality. It follows that

$$\begin{aligned}
\liminf_{k\to\infty} f(\delta_k)g(\delta_k) &= \lim_{k\to\infty} \inf\{f(\delta_i)g(\delta_i) \colon i \geq k\} \\
&\geq \lim_{k\to\infty} m \cdot \inf\{f(\delta_i) \colon i \geq k\} \\
&= m \cdot \liminf_{k\to\infty} f(\delta_k) \\
&= \infty,
\end{aligned}$$

which is clearly sufficient.
Let us assume case (1) in conjunction with the sequence $(f(\delta_k))_k$ being convergent. As we have already shown that (1) alone implies

$$\liminf_{k\to\infty} f(\delta_k)g(\delta_k) \geq \liminf_{k\to\infty} f(\delta_k) \cdot \liminf_{k\to\infty} g(\delta_k),$$

it suffices to prove the converse inequality. Let $L = \lim_{k\to\infty} f(\delta_k)$, let $\epsilon > 0$ and let $N \in \mathbb{Z}_{>0}$ be such that for all $k \geq N$ it holds that $0 \leq f(\delta_k) \leq L + \epsilon$. By Lemma A.4, it follows that for these k:

$$\inf_{n\geq k} (f(\delta_n)g(\delta_n)) \leq \inf_{n\geq k} ((L+\epsilon)g(\delta_n)) = (L+\epsilon) \cdot \inf_{n\geq k} g(\delta_n).$$

We observe from this that

$$\liminf_{k\to\infty} f(\delta_k)g(\delta_k) \leq (L+\epsilon) \liminf_{k\to\infty} g(\delta_k),$$

so that the arbitrariness of ϵ yields

$$\liminf_{k\to\infty} f(\delta_k)g(\delta_k) \leq L \cdot \liminf_{k\to\infty} g(\delta_k) = \liminf_{k\to\infty} f(\delta_k) \cdot \liminf_{k\to\infty} g(\delta_k).$$

Lastly, we note that the claims involving lim sup can, as usual, be proven in a very similar way. □

Lemma A.6. Let $f : (0,1) \to (0,1]$ be an increasing function and let $(\delta_k)_k$ be a decreasing sequence in $(0,1)$ satisfying $\delta_k \to 0$ and $\delta_{k+1} \geq c \cdot \delta_k$ for all $k \in \mathbb{Z}_{>0}$ and some fixed $c \in (0,1)$. Then

$$\begin{aligned}
\liminf_{\delta\to 0} \frac{\log f(\delta)}{\log \delta} &= \liminf_{k\to\infty} \frac{\log f(\delta_k)}{\log \delta_k}, \\
\limsup_{\delta\to 0} \frac{\log f(\delta)}{\log \delta} &= \limsup_{k\to\infty} \frac{\log f(\delta_k)}{\log \delta_k}.
\end{aligned}$$

Proof. By Lemma A.2:

$$\liminf_{\delta\to 0} \frac{\log f(\delta)}{\log \delta} \leq \liminf_{k\to\infty} \frac{\log f(\delta_k)}{\log \delta_k}. \tag{A.5}$$

Aiming to prove the reverse inequality, let $k \in \mathbb{Z}_{>0}$ and observe that for any $\delta \in [\delta_{k+1}, \delta_k)$:

$$\begin{aligned}
\frac{\log f(\delta)}{\log \delta} &\geq \frac{\log f(\delta_k)}{\log \delta_{k+1}} \\
&= \frac{\log f(\delta_k)}{\log \delta_k + \log(\delta_{k+1}/\delta_k)} \\
&\geq \frac{\log f(\delta_k)}{\log \delta_k + \log c} \\
&= \frac{\log f(\delta_k)}{\log \delta_k} \cdot \frac{1}{1 + \frac{\log c}{\log \delta_k}}.
\end{aligned}$$

Thus, we can consecutively invoke Lemmas A.3 and A.5 to obtain

$$\begin{aligned}
\liminf_{\delta\to 0} \frac{\log f(\delta)}{\log \delta} &\geq \liminf_{k\to\infty} \frac{\log f(\delta_k)}{\log \delta_k} \cdot \frac{1}{1 + \frac{\log c}{\log \delta_k}} \\
&\geq \liminf_{k\to\infty} \frac{\log f(\delta_k)}{\log \delta_k},
\end{aligned}$$

together with (A.5) proving the first claim of the lemma. The second statement can be verified in much the same way. □

Appendix B

Hausdorff pseudometric and (semi-)geodesicness

In this appendix, we introduce a common notion known as the *Hausdorff pseudometric* as well as some useful results in which it plays a central role. Up until Definition B.6, the definitions and results are commonly known and to be found, for example, in [3]. From Theorem B.7 on, we utilize these results to walk a path not found in other literature.

Definitions B.1. Let (X, d) be a metric space. With the symbol $B(X)$, we denote the family of all non-empty, bounded $U \subseteq X$. With the symbol $BC(X)$, we denote the family of all closed $U \in B(X)$.

Definitions B.2. Let (X, d) be a metric space, let $x \in X$ and let $U, V \in B(X)$.

1. The *distance between x and V* is given by

$$d(x, V) = \inf\{d(x, v) : v \in V\}.$$

2. The *Hausdorff semidistance between U and V* is given by

$$\delta(U, V) = \sup\{d(u, V) : u \in U\}.$$

Lemma B.3. Let (X, d) be a metric space. For any triplet $U, V, W \in B(X)$:

a) $\delta(U, V) = 0 \Leftrightarrow U \subseteq \overline{V}$,

b) $\delta(U, W) \leq \delta(U, V) + \delta(V, W)$,

c) $|\delta(U, V) - \delta(U, W)| \leq \max\{\delta(V, W), \delta(W, V)\}$,

d) $U \subseteq V \Rightarrow \delta(W, U) \geq \delta(W, V)$.

Proof.

a) We break the equivalence up into a series of more trivial ones:

$$\begin{aligned} U \subseteq \overline{V} &\Leftrightarrow u \in \overline{V} \text{ for all } u \in U \\ &\Leftrightarrow d(u, V) = 0 \text{ for all } u \in U \\ &\Leftrightarrow \delta(U, V) = 0. \end{aligned}$$

b) It follows directly from the definitions involved that for all $u \in U, v \in V$ and $w \in W$:

$$\begin{aligned} & d(u, w) \leq d(u, v) + d(v, w) \\ \Rightarrow\ & d(u, w) \leq d(u, V) + d(v, w) \\ \Rightarrow\ & d(u, W) \leq d(u, V) + d(v, W) \\ \Rightarrow\ & \delta(U, W) \leq \delta(U, V) + d(v, W) \\ \Rightarrow\ & \delta(U, W) \leq \delta(U, V) + \delta(V, W). \end{aligned}$$

Since the first of these statements is just the triangle inequality for d, truthness of the last line is guaranteed.

c) The statements

$$\begin{aligned} \delta(U, V) - \delta(U, W) &\leq \delta(W, V) \\ \delta(U, W) - \delta(U, V) &\leq \delta(V, W) \end{aligned}$$

both are equivalent to the triangle inequality for δ, proven in part (b) of this lemma. The desired result is immediate from these.

d) Observe that for all $u \in U$, there is a $v \in V$ such that for all $w \in W$:

$$d(w, u) \geq d(v, w),$$

for given a $u \in U$, we can just pick $v = u$. It follows that for all $w \in W$:

$$d(w, U) \geq d(w, V).$$

Taking suprema on W then yields the claim. □

Definition B.4. Let (X, d) be a metric space. The function

$$\begin{aligned} d_H : B(X) \times B(X) &\rightarrow \mathbb{R} \\ (U, V) &\mapsto \max\{\delta(U, V), \delta(V, U)\} \end{aligned}$$

is called the *Hausdorff pseudometric on X*.

Lemma B.5. Let (X, d) be a metric space. The Hausdorff pseudometric on X is a metric on $BC(X)$.

Proof. The function d_H is symmetric by definition. It follows from this and Lemma B.3 (a) that for $U, V \in B(X)$:

$$d_H(U, V) = 0 \Leftrightarrow U \subseteq \overline{V} \subseteq \overline{U},$$

of which the right side reads $U = V$ whenever U and V are closed. The triangle inequality follows directly from Lemma B.3 (b). $\square$

Definition B.6. Let (X, d) be a metric space. The metric space $(BC(X), d_H)$ is called the *hyperspace* of (X, d).

Theorem B.7. Let (X, d) be a metric space and let $F : X \to BC(X)$ be continuous relative to the metrics d and d_H, respectively. For each finite, positive Borel measure μ on X, the function

$$\begin{array}{ccc} X & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & \mu(F(x)) \end{array}$$

is Borel-measurable.

Proof. For all $n \in \mathbb{Z}_{>0}$ and $x \in X$, let the function $f_{n,x} : X \to \mathbb{R}_{\geq 0}$ be given by

$$f_{n,x}(y) = \max\{0, 1 - n \cdot d(y, F(x))\}.$$

Now let $n \in \mathbb{Z}_{>0}$, let $x, y \in X$ and let $(x_k)_k$ be a sequence in X converging to x. By Lemma B.3 (c) and the continuity of F, we have

$$\lim_{k\to\infty} |d(y, F(x_k)) - d(y, F(x))| \leq \lim_{k\to\infty} d_H(F(x_k), F(x)) = 0.$$

Since $f_{n,x}$ is a continuous function of d, this yields $\lim_{k\to\infty} |f_{n,x_k}(y) - f_{n,x}(y)| = 0$. Thus by Lebesgue's dominated convergence theorem, it follows that the function $\psi_n : X \to \mathbb{R}_{\geq 0}$, given by

$$\psi_n(x) := \int_X f_{n,x}(y) d\mu(y)$$

is continuous in x, and hence Borel-measurable. But since $F(x) \in BC(X)$ is closed, it holds that

$$\lim_{n\to\infty} f_{n,x}(y) = 1_{F(x)}(y),$$

yielding, again appealing to Lebesgue's dominated convergence theorem:

$$\lim_{n\to\infty} \psi_n(x) = \mu(F(x)).$$

We conclude that $x \mapsto \mu(F(x))$ is the pointwise limit of a sequence of Borel-measurable functions, and is hence Borel-measurable. $\square$

In the second half of this appendix, we would like to prove the function

$$\begin{array}{rcl} X & \to & BC(X) \\ x & \mapsto & B(x,r) \end{array}$$

for a metric space (X, d) and any $r > 0$ to be continuous relative to the metrics d and d_H, respectively. This claim, however, appears to be false for certain examples of (X, d). Demanding the metric space to be of the following kind turns out to be sufficient.

Definitions B.8. A metric space is *(uniquely) geodesic* if for each pair $x, y \in X$ there is a (unique) $\gamma : [0, 1] \to X$ such that for all $s, t \in [0, 1]$:

$$d(\gamma(s), \gamma(t)) = |t - s| \cdot d(x, y),$$
$$\gamma(0) = x, \quad \gamma(1) = y.$$

The map γ is called a *geodesic from x to y.*

To illustrate the above condition still admits a reasonable range of metric spaces, we note that any Banach space $(X, \|\cdot\|)$ with a metric d associated to its norm $\|\cdot\|$ is geodesic: given $x, y \in X$, the map $\gamma : t \mapsto (1 - t)x + ty$ is a geodesic from x to y.

Still, the requirement of geodesicness turns out too strong for our needs. Aiming to prove our claim for a broader range of spaces, including certain countable ones, we introduce the following notion[1].

Definition B.9. We call a metric space (X, d) *semi-geodesic* if for each pair of distinct points $x, y \in X$, any $r \in [0, d(x, y)]$ and all $\epsilon > 0$ there is a $z_\epsilon \in X$ such that

$$r - \epsilon < d(x, z_\epsilon) \leq r,$$
$$d(x, z_\epsilon) + d(z_\epsilon, y) \leq d(x, y) + \epsilon.$$

Remark. A related property is *metric convexness* of a metric space. As defined in [4], a metric space (X, d) is metrically convex if for each pair $x, y \in X$ and any $r \in (0, d(x, y))$ there is a $z_r \in X$ such that both $d(x, z_r) = r$ and $d(x, y) = d(x, z_r) + d(z_r, y)$. Clearly, a metrically convex space is semi-geodesic.

Before we proceed to prove the demand of semi-geodesicness sufficient, we show what one might expect considering the choice of names in the above definitions.

Proposition B.10. Every geodesic metric space is semi-geodesic.

[1] I stress that this is my own terminology (to the best of my knowledge): I have not encountered it in other literature.

Proof. Let (X, d) be a geodesic metric space, let $x, y \in X$ and let $r \in [0, d(x, y)]$. For γ a geodesic from x to y and $t = \frac{r}{d(x,y)}$, we have

$$d(x, \gamma(t)) = d(\gamma(0), \gamma(t)) = t \cdot d(x, y) = r,$$
$$d(\gamma(t), y) = d(\gamma(t), \gamma(1)) = (1 - t) \cdot d(x, y),$$

so that

$$d(x, \gamma(t)) + d(\gamma(t), y) = d(x, y).$$

It follows directly that for any $\epsilon > 0$, the point $\gamma(t)$ satisfies the conditions on z_ϵ in Definition B.9. □

We continue to prove the requirement of semi-geodesicness to be strong enough.

Lemma B.11. Let (X, d) be a semi-geodesic metric space and let $x, y \in X$. For any $r \geq 0$:

$$d(y, B(x, r)) = (d(x, y) - r)^+.$$

Proof. Let $r \geq 0$. If $y \in B(x, r)$, then $d(x, y) - r \leq 0$ and hence $(d(x, y) - r)^+ = 0 = d(y, B(x, r))$. Assume thus that $d(x, y) > r$. For $\epsilon > 0$ arbitrary, our assumption on (X, d) yields a $z_\epsilon \in B(x, r) \backslash B(x, r - \epsilon)$ such that

$$d(x, z_\epsilon) + d(z_\epsilon, y) \leq d(x, y) + \epsilon,$$

or

$$r - \epsilon + d(z_\epsilon, y) \leq d(x, y) + \epsilon.$$

It follows that

$$d(y, B(x, r)) \leq d(z_\epsilon, y) \leq d(x, y) - r + 2\epsilon,$$

which, by the arbitrariness of ϵ, yields

$$d(y, B(x, r)) \leq d(x, y) - r.$$

To prove the converse inequality, we note that for any $z \in B(X, r)$:

$$d(y, z) \geq d(x, y) - d(z, x)$$

by the triangle inequality for d. This yields

$$d(y, z) \geq d(x, y) - r,$$

from which the desired result is immediate. □

Lemma B.12. Let (X, d) be a semi-geodesic metric space and let $x, y \in X$. For any $r \geq 0$:

$$d_H(B(x, r), B(y, r)) \leq d(x, y).$$

Proof. Let $r \geq 0$. Observe that

$$\begin{aligned}
\delta(B(x,r), B(y,r)) &= \sup\{d(z, B(y,r)) : z \in B(x,r)\} \\
&= \sup\{(d(z,y) - r)^+ : z \in B(x,r)\} \qquad \text{(B.1)} \\
&= \sup\{0\} \cup \{d(z,y) - r : z \in B(x,r)\} \\
&\leq \sup\{0\} \cup \{d(z,x) + d(x,y) - r : z \in B(x,r)\} \\
&\leq d(x,y),
\end{aligned}$$

where (B.1) is a consequence of Lemma B.11. Interchanging the roles of x and y yields

$$\delta(B(y,r), B(x,r)) \leq d(y,x) = d(x,y).$$

The claim follows by definition of d_H. □

Theorem B.13. Let (X, d) be a semi-geodesic metric space and let $r > 0$. The function

$$\begin{aligned}
X &\to BC(X) \\
x &\mapsto B(x,r)
\end{aligned}$$

is continuous with respect to the metrics d and d_H, respectively.

Proof. This is a direct consequence of Lemma B.12. □

We need one more result to support the main content of this thesis.

Lemma B.14. Let (X, d) be a semi-geodesic metric space containing at least two points. For all $x \in X$, define the number r_x by

$$r_x = \sup_{y \in X} d(x,y) \in (0, \infty].$$

For any $x \in X$ and $r \in (0, r_x)$, it holds that

$$r \leq d(B(x,r)) \leq 2r. \qquad \text{(B.2)}$$

There is an $\alpha > 0$ such that $r_x \geq \alpha$ for all $x \in X$.

Proof. Let $x \in X$. Starting off with the second inequality of (B.2), observe that for any $r > 0$ and each pair $y, z \in B(x,r)$:

$$d(y,z) \leq d(y,x) + d(x,z) \leq 2r,$$

from which the claim follows immediately.
Continuing to prove the first inequality, note that since X contains at least two points, we have $r_x > 0$. Now let $r \in (0, r_x)$ and $\epsilon > 0$ be arbitrary. By definition of r_x, there is a $y_r \in X$ with $d(x, y_r) > r$. The semi-geodesicness of (X, d) then grants us a $z_\epsilon \in X$ with $r - \epsilon \leq d(x, z_\epsilon) \leq r$. Since it

follows from this inequality that $z_\epsilon \in B(x,r)$ and since ϵ is arbitrary, we have $\sup_{z\in B(x,r)} (d(x,z)) \geq r$, and so $d(B(x,r)) \geq r$.
Closing off with a verification of the last statement, note that if (X,d) is unbounded, then $r_x = \infty$ for all $x \in X$ and so the claim is trivial. Assuming therefor that (X,d) is bounded, let

$$\beta = \sup_{x,y\in X} d(x,y) \in (0,\infty).$$

We claim that $\alpha := \beta/2$ satisfies the desired condition. To prove this, let $\epsilon > 0$, let $x, y \in X$ be such that $d(x,y) \geq \beta - 2\epsilon$ and let $z \in X$ be arbitrary. If $d(x,z) \geq \beta/2 - \epsilon$, then by the arbitrariness of ϵ we have $d(x,z) \geq \beta/2 = \alpha$ and hence $r_z \geq \alpha$. If $d(x,z) \leq \beta/2 - \epsilon$, then

$$d(z,y) \geq d(x,y) - d(x,z) \geq (\beta - 2\epsilon) - (\beta/2 - \epsilon) = \beta/2 - \epsilon,$$

hence $r_z \geq \beta/2 - \epsilon$, and so $r_z \geq \beta/2 = \alpha$. □

Observe that the second inequality of (B.2) holds in any metric space: we have not used semi-geodesicness in its verification.

Appendix C

Inclusion-exclusion principle

The following principle supports Lemma 3.18. It is the measure variant of the inclusion-exclusion principle, and thus the most general one.

Theorem C.1 (inclusion-exclusion principle)**.** Let (X, Σ) be a measurable space, let $\mu : \Sigma \to \mathbb{R}_{\geq 0}$ be a measure and let $C \in \Sigma$. Suppose $C = \bigcup_{i=1}^{n} C_i$ for some finite collection $\{C_i\}_{i=1}^{n} \subseteq \Sigma$. Then

$$\mu(C) = \sum_{k=1}^{n} (-1)^{k-1} \left(\sum_{\substack{I \subseteq \{1,\ldots,n\} \\ \#I = k}} \mu\left(\bigcap_{i \in I} C_i \right) \right).$$

Proof. For the sake of simplicity of the actual proof, we start by introducing some terminology. For a set $A \subseteq X$, we define the indicator function $\mathbb{I}_A$ in the well known way

$$\mathbb{I}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise,} \end{cases}$$

and the three binary operators $+$, $-$ and $\cdot$ on indicator functions $\mathbb{I}_A, \mathbb{I}_B$ by

$$(\mathbb{I}_A \circ \mathbb{I}_B)(x) = \mathbb{I}_A(x) \circ \mathbb{I}_B(x),$$

where $\circ$ may represent any of the symbols $+$, $-$ and $\cdot$. By these definitions, it holds that $\mathbb{I}_A \cdot \mathbb{I}_B = \mathbb{I}_{A \cap B}$.
Now consider the identity

$$\prod_{i=1}^{n} (\mathbb{I}_C - \mathbb{I}_{C_i}) \equiv 0.$$

By the observation above, the left-hand side expands to

$$\mathbb{I}_C + \sum_{k=1}^{n} (-1)^k \left(\sum_{\substack{I \subseteq \{1,\ldots,n\} \\ \#I = k}} \mathbb{I}_{\bigcap_{i \in I} C_i} \right),$$

from which it follows that

$$\mathbb{I}_C \equiv \sum_{k=1}^{n} (-1)^{k-1} \left(\sum_{\substack{I \subseteq \{1,\ldots,n\} \\ \#I = k}} \mathbb{I}_{\bigcap_{i \in I} C_i} \right).$$

Integrating both sides with respect to μ yields the desired result. $\square$

Bibliography

[1] Billingsley, P., 1965. *Ergodic Theory and Information.* John Wiley & Sons, Inc., New York.

[2] Falconer, K., 1990. *Fractal Geometry: Mathematical Foundations and Applications.* John Wiley & Sons Ltd., Chichester.

[3] Folland, G.B., 1999. *Real Analysis: Modern Techniques and their Applications (2nd edition).* John Wiley & Sons, New York.

[4] Gacki, H., 2007. *Applications of the Kantorovich-Rubinstein maximum principle in the theory of Markov semigroups.* Institute of Mathematics, Polish Academy of Sciences, Warsaw.

[5] Lieb, E.H. and Loss, M., 2001. *Analysis (2nd ed.): Graduate Studies in Mathematics vol. 14.* American Mathematical Society, Providence.

[6] Manning, A. and Simon, K., 1998. A Short Existence Proof for Correlation Dimension. *Journal of Statistical Physics*, Vol. 90, Nos. 3/4, pages 1047-1049.

[7] Pesin, Ya.B., 1993. On Rigorous Mathematical Definitions of Correlation Dimension and Generalized Spectrum for Dimensions. *Journal of Statistical Physics*, Vol. 71, Nos. 3/4, pages 529-547.

[8] Procaccia, I., Grassberger, P. and Hentschel, V.G.E., 1983. On the characterization of chaotic motions. *Lecture Notes in Physics* No. 179, pages 212-221.

[9] Rogers, C.A., 1970. *Hausdorff Measures.* Cambridge University Press, Cambridge.

[10] Young, L., 1982. Dimension, entropy and Lyapunov exponents. *Ergodic Theory and Dynamical Systems*, No. 2, pages 109-124.