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Concurrent exceptional curves on del Pezzo surfaces of degree one

ALGANT Master’s thesis
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9th July, 2014

Università degli studi di Padova        Universiteit Leiden
Acknowledgements

“If your research adviser gives you a problem involving del Pezzo surfaces of degree 2 and 1, it means he really hates you.”

Peter Swinnerton-Dyer.

First and foremost I want to thank my advisor Ronald van Luijk for his patience in explaining everything, his support and encouragement, and for so much of his time. Contrary to what the quote above might suggest, I have really enjoyed working on this thesis with him.

I want to thank the ALGANT program, and in particular the people at the University of Padova and at the Mathematisch Instituut in Leiden for being understanding, helpful and flexible when the situation asked for it. This really made me feel supported during a hard time. A special thanks to Peter Stevenhagen, who thought with me about how to continue my studies in the best way possible.

Last but not least, I want to thank all my ALGANT friends, both from Padova and from the two years I spent in Leiden. The nights spent at the faculty will never be forgotten... che gioia!
Introduction

A del Pezzo surface is a projective, non-singular, geometrically integral surface with ample anticanonical divisor. The degree of a del Pezzo surface is the self-intersection number of the canonical divisor, and this is at most 9. Over an algebraically closed field, del Pezzo surfaces of degree \( d \) are isomorphic to \( \mathbb{P}^2 \) blown up at \( 9 - d \) points in general position for \( d \neq 8 \), and to \( \mathbb{P}^1 \times \mathbb{P}^1 \) or \( \mathbb{P}^2 \) blown up in one point for \( d = 8 \). For degree at least three, del Pezzo surfaces can be embedded as surfaces of degree \( d \) in \( \mathbb{P}^d \). A famous example is given by del Pezzo surfaces of degree three, which are exactly the smooth cubic surfaces in \( \mathbb{P}^3 \). For a del Pezzo surface of degree two, the linear system of the anticanonical divisor gives the surface the structure of a double cover of \( \mathbb{P}^2 \) ramified over a smooth curve of degree four, and for del Pezzo surfaces of degree one, the linear system of the bianticanonical divisor gives the surface the structure of a double cover of a cone \( Q \) in \( \mathbb{P}^3 \), ramified over a smooth curve that is cut out by a cubic surface.

Let \( X \) be a del Pezzo surface of degree \( d \) over an algebraically closed field \( k \), and let \( K_X \) be the canonical divisor on \( X \). An exceptional curve on \( X \) is an irreducible projective curve \( C \subset X \) such that \( C^2 = C \cdot K_X = -1 \). For \( d \geq 3 \), the exceptional curves on \( X \) are exactly the lines on the model of degree \( d \) in \( \mathbb{P}^d \). For \( d = 3 \) this gives a description of the 27 lines on a cubic surface. A lot is known about the exceptional curves on del Pezzo surfaces. For example, we know that there is a one-to-one correspondence between exceptional curves on \( X \) and their classes in \( \text{Pic} X \), and we know what their images under the blow-up in \( \mathbb{P}^2 \) are, see Theorem 2.8. We also know how many exceptional curves there are.

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Now assume \( X \) is of degree one. Let \( \varphi \) be the morphism associated to \( | -2K_X | \). In this thesis we prove the following two theorems.

**Theorem 1.** Let \( P \) be a point on the ramification curve of \( \varphi \). The number of exceptional curves that go through \( P \) is at most ten if \( \text{char } k \neq 2 \), and at most sixteen if \( \text{char } k = 2 \).

**Theorem 2.** Let \( R \) be a point outside the ramification curve of \( \varphi \). The number of exceptional curves that go through \( R \) is at most twelve. If \( \text{char } k = 0 \), it is at most ten.

In [SvL14], various examples of del Pezzo surfaces are given where ten exceptional curves go through one point outside the ramification curve, showing that the upper bound for \( \text{char } k = 0 \) in Theorem 2 is sharp. In Example 4.23 and Example 4.24 we show that the upper bounds given in Theorem 1 are sharp, too.

It is well known that on del Pezzo surfaces of degree three, the maximal number of exceptional curves through one point is three. The fact that three is an upper bound
can be seen by looking at the maximal size of full subgraphs of the graph on the 27 exceptional curves. A geometrical argument can be found for instance in [Rei88], on page 102. A point on a del Pezzo surface of degree three that is contained in three exceptional curves is called an Eckardt point.

On a del Pezzo surface of degree two, the maximal number of exceptional curves through one point is four. As in the case of degree three, this upper bound is given by the graph on the 56 exceptional curves. A geometric argument why four is the upper bound is given in [TVAV09], Lemma 4.1. An example where this upper bound is reached is given in [STVAar], Example 7. A point on a del Pezzo surface of degree two that lies on four exceptional curves is called a generalized Eckardt point.

For del Pezzo surfaces of degree one, the situation is a little different. First of all, for char \( k \neq 2 \), the maximal size of full subgraphs of the graph on the 240 exceptional curves, which we will show is sixteen, is not equal to the maximal number of exceptional curves that can go through one point. Secondly, contrary to del Pezzo surfaces of degree two, where all generalized Eckardt points are outside the ramification curve, in the case of degree one we compute the maximum both for points on the ramification curve, as well as for points outside the ramification curve.

In Section 1, we define del Pezzo surfaces and study their main properties. We look more closely at del Pezzo surfaces of degree one in Subsection 1.1.

In sections 2, 3 and 4 we work over an algebraically closed field.

In Section 2, we study the exceptional curves on del Pezzo surfaces. We look more closely at the exceptional curves on del Pezzo surfaces of degree one in Subsection 2.1 and show that they relate to hyperplanes in \( \mathbb{P}^3 \) that are tritangent to the branch curve of \( \varphi \), and do not contain the vertex of the cone \( Q \). This will later allow us to make the distinction between exceptional curves through one point on the ramification curve of \( \varphi \), and exceptional curves through one point outside the ramification curve of \( \varphi \).

In Section 3, we study the group \( G \) of permutations of the set \( E \) of exceptional classes in \( \text{Pic} \ X \) that preserve the intersection pairing. We prove various results about the action of \( G \) on \( E \), that we will use in the fourth section.

In Section 4, we show that an upper bound for the number of exceptional curves through one point in \( X \) is sixteen. We show moreover that if the elements in a maximal set of exceptional curves that all intersect each other go through one point, then that point lies on the ramification curve of \( \varphi \) if and only if the set contains at least two curves that intersect with multiplicity three.

In Subsection 4.1 we focus on the number of exceptional curves through one point on the ramification curve. For char \( k \neq 2 \), we first show that this is at most twelve. Then we show that ten is a sharp upper bound. To this end, we define the following curves.

Let \( Q_1, \ldots, Q_8 \) be eight points in \( \mathbb{P}^2 \) such that no three of them lie on a line, and no six of them lie on a conic. For \( i \in \{1, 2, 3, 4\} \), let \( L_i \) be the line through
For $i, j \in \{1, \ldots, 8\}$, let $C_{i,j}$ the unique cubic through $Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_8$ that is singular in $Q_j$.

We show that if the elements of a set of twelve exceptional curves go through one point on the ramification curve, we can reduce to a set containing the curves $L_1, L_2, L_3, L_4, C_7, C_8, C_8, C_7, C_6, C_5$. The following proposition is therefore the key to the proof of Theorem 1.

**Proposition 3.** Let $\text{char } k \neq 2$. Assume that the four lines $L_1, L_2, L_3$ and $L_4$ all intersect in one point $P$. Then the three cubics $C_7, C_8, C_5$ do not all go through $P$.

Finally we show that for $\text{char } k = 2$, sixteen is a sharp upper bound.

In Subsection 4.2 we focus on exceptional curves through one point outside the ramification curve. We first show that it is at most twelve, by showing that every set of exceptional curves of size bigger than twelve contains at least two curves intersecting with multiplicity three. To compute a sharp upper bound in the case $\text{char } k = 0$, we define the following.

Let $Q_1, \ldots, Q_8$ be eight points in $\mathbb{P}^2$ such that no three of them lie on a line, and no six of them lie on a conic. Define the following curves.

$L_1$ is the line through $Q_1$ and $Q_2$;
$L_2$ is the line through $Q_3$ and $Q_4$;
$C_1$ is the conic through $Q_1, Q_3, Q_5, Q_6$ and $Q_7$;
$C_2$ is the conic through $Q_1, Q_4, Q_5, Q_6$ and $Q_8$;
$C_3$ is the conic through $Q_2, Q_3, Q_5, Q_7$ and $Q_8$;
$C_4$ is the conic through $Q_2, Q_4, Q_6, Q_7$ and $Q_8$;
$D_1$ is the quartic through all eight points with singular points in $Q_1, Q_7$ and $Q_8$;
$D_2$ is the quartic through all eight points with singular points in $Q_2, Q_5$ and $Q_6$;
$D_3$ is the quartic through all eight points with singular points in $Q_3, Q_6$ and $Q_8$;
$D_4$ is the quartic through all eight points with singular points in $Q_4, Q_5$ and $Q_7$.

As in the case of points on the ramification curve, we show that for a set of eleven or twelve exceptional curves going through one point outside the ramification curve, we can reduce to a set containing these ten curves. From the following proposition we can then deduce Theorem 2.

**Proposition 4.** Assume that $\text{char } k = 0$. Then

$$L_1, L_2, C_1 \ldots C_4, D_1, \ldots, D_4$$

do not all go through one point.
1 Del Pezzo surfaces

In this section we define del Pezzo surfaces and state their main properties. In Subsection 1.1 we will be more specific and focus on del Pezzo surfaces of degree one. We assume that the reader has a basic knowledge of algebraic geometry, and is familiar with concepts as variety, divisor, and Picard group. Most results in this section, as well as more information on del Pezzo surfaces, can be found in [Man74], Chapter IV, and [Kol96], Section III.3.

**Definition 1.1.** Let $k$ be a field, and $X$ a variety over $k$. Then we say that $X$ is nice if it is projective, smooth, and geometrically integral.

**Definition 1.2.** A del Pezzo surface is a nice surface $X$ with ample anticanonical divisor $-K_X$.

Let $X$ be a del Pezzo surface with very ample anticanonical divisor $-K_X$. The linear system $|−K_X|$ determines an embedding $i: X \hookrightarrow \mathbb{P}^n$ for some $n$. If $H$ is a hyperplane in $\mathbb{P}^n$, we have $i^*H \sim -K_X$. Therefore, the degree of $i(X)$ is equal to $(i^*H)^2 = (-K_X)^2 = K_X^2$. This leads to the following definition.

**Definition 1.3.** The degree of a del Pezzo surface $X$ is the self-intersection number $K_X^2$.

**Proposition 1.4.** The degree of a del Pezzo surface $X$ is positive.

*Proof.* Since $-K_X$ is ample, $-nK_X$ is very ample for some $n > 0$, hence determines an embedding of $X$ into some projective space. Then $(-nK_X)^2$ is the degree of the image of $X$ under this embedding, hence $n^2K_X^2 = (-nK_X)^2 > 0$. It follows that $K_X^2 > 0$. \hfill $\square$

**Definition 1.5.** Let $r \leq 8$, and let $P_1, \ldots, P_r$ be points in $\mathbb{P}^2$. Then we say that $P_1, \ldots, P_r$ are in general position if no three of them lie on a line, no six of them lie on a conic, and no eight of them lie on a singular cubic with one of these eight points at the singularity.

**Theorem 1.6.** For $r \leq 8$, let $P_1, \ldots, P_r$ be points in general position in $\mathbb{P}^2$. Let $X$ be the blow-up of $\mathbb{P}^2$ in these points. Then $-K_X$ is ample, and very ample if $r \leq 6$.

*Proof.* See [Man74], Theorem 24.5, and [Dem80], Theorem 1. \hfill $\square$

**Theorem 1.7.** Let $k$ be an algebraically closed field, and let $X$ be a del Pezzo surface over $k$. Then $X$ is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$, in which case $X$ is of degree 8, or to $\mathbb{P}^2$ blown up at $r \leq 8$ points in general position, in which case the degree of $X$ is $9 - r$.

*Proof.* See [Man74], Theorem 24.4, Theorem 26.2, and Remark 26.3. \hfill $\square$

**Remark 1.8.** The previous two theorems give us an explicit description of all del Pezzo surfaces over algebraically closed fields; they are exactly those surfaces that
are isomorphic to the blow-ups of $\mathbb{P}^2$ in $r \leq 8$ points in general position, and the surface $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover, Theorem 1.7 implies that the degree of a del Pezzo surface over an algebraically closed field is at most 9, and a del Pezzo surface of degree 9 is just $\mathbb{P}^2$.

Since the anticanonical divisor of a del Pezzo surface is ample, a del Pezzo surface can be embedded in some projective space by a multiple of its anticanonical divisor $-K$.

To study the various rational maps and morphisms given by multiples of $-K$, we need a couple of classical results.

**Theorem 1.9.** (Nakai-Moishezon criterion). Let $X$ be a nonsingular projective surface over an algebraically closed field. Then a divisor $D$ on $X$ is ample if and only if $D^2 > 0$ and $D \cdot C > 0$ for all irreducible curves $C$ in $X$.

*Proof.* See [Har77], Theorem V.1.10.

**Theorem 1.10.** (Riemann-Roch for surfaces). Let $X$ be a nonsingular projective surface over an algebraically closed field $k$. Then for any divisor $D$ on $X$ we have

$$l(D) - s(D) + l(K - D) = \frac{1}{2}D(D - K) + 1 + p_a,$$

where $l(D)$ is the dimension of the vector space $L(D)$ of rational functions on $X$ with poles at most at $D$, $s(D) = \dim H^1(X, L(D))$, the superabundance of $D$, and $p_a$ is the arithmetic genus of $X$.

*Proof.* See [Har77], Theorem V.1.6.

**Lemma 1.11.** Let $X$ be a del Pezzo surface with canonical divisor $K_X$. Then we have $\dim H^1(X, \mathcal{L}(-mK_X)) = 0$ for all $m \geq 0$.

*Proof.* See [Kol96], Corollary 3.2.5.1.

The following lemma is well known, and can be found for instance in [Kol96], Corollary 3.2.5.2.

**Lemma 1.12.** Let $X$ be a del Pezzo surface of degree $d$ over an algebraically closed field. Then for all positive integers $m$ we have $l(-mK_X) = 1 + \frac{1}{2}m(m + 1)d$.

*Proof.* Let $m > 0$. Since $X$ is geometrically rational, we have $p_a(X) = 0$ (see for example [Har77], Example II.8.20.2). Moreover, by the previous lemma we have $s(-mK_X) = 0$. Since $-K_X$ is ample we have $-K_X \cdot C > 0$ for all irreducible curves $C$ in $X$ by Nakai-Moishezon, so $(m + 1)K_X \cdot C < 0$, hence $l((m + 1)K_X) = 0$. From Riemann-Roch for surfaces it follows that

$$l(-mK_X) = \frac{1}{2}((-mK_X)^2 - mK_X^3) + 1 = \frac{1}{2}(m^2d - md) + 1 = 1 + \frac{1}{2}m(m + 1)d.$$
Remark 1.13. If $X$ is a del Pezzo surface of degree $d \geq 3$, then $-K_X$ is very ample by Theorem 1.6. Therefore, the linear system $|-K_X|$ determines an embedding in $\mathbb{P}^n$, with $n = \frac{1}{2} \cdot 2 \cdot d = d$ by Lemma 1.12, and the image of $X$ under this embedding has degree $(-K_X)^2 = d$. So for $d \geq 3$, a del Pezzo surface of degree $d$ is isomorphic to a surface of degree $d$ in $\mathbb{P}^d$.

Example 1.14. Let $k$ be an algebraically closed field, and $X$ a del Pezzo surface of degree 4 over $k$. Then $X$ is isomorphic to $\mathbb{P}^2$ blown up in 5 points in general position. The anticanonical divisor $-K_X$ is very ample, and by Lemma 1.12 we have $l(-K_X) = 5$, so $-K_X$ determines an embedding $\varphi : X \hookrightarrow \mathbb{P}^4$. The image $\varphi(X)$ has degree 4, and it is the complete intersection of two quadric hypersurfaces in $\mathbb{P}^4$.

To see this, let $\{v, w, x, y, z\}$ be a basis for $\mathcal{L}(-K_X)$. Let $V = \text{Sym}^2(\mathcal{L}(-K_X))$ be the symmetric square of $\mathcal{L}(-K_X)$. Then $V$ has dimension $\binom{5}{2} = 15$, and there is a canonical map $f : V \to \mathcal{L}(-2K_X)$. By Lemma 1.12 we have $l(-2K_X) = 13$, so the dimension of $\ker f$ is at least two, which means that there are two linearly independent quadratic forms vanishing on $\varphi(X)$. This means that $\varphi(X)$ is contained in the intersection of the two quadric hypersurfaces defined by these quadratic forms. Since their intersection has degree 4, which is the degree of $\varphi(X)$, we conclude that $\varphi(X)$ is in fact equal to this intersection.

Let $k$ be an algebraically closed field, and let $X$ be a del Pezzo surface of degree $d$ over $k$. If $X$ is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, then we know from Theorem 1.7 that $X$ is isomorphic to $\mathbb{P}^2$ blown up in 9 points in general position. In this case we know a lot about the Picard group Pic $X$ of $X$. For a divisor $D$ on $X$, we denote its class in Pic $X$ by $[D]$.

Proposition 1.15. Let $Y$ be a smooth surface over an algebraically closed field. Let $\tilde{Y}$ be the blow-up of $Y$ at a point $P$, with corresponding map $\pi : \tilde{Y} \to Y$. Let $E$ be the exceptional curve above $P$. Then $E$ is isomorphic to $\mathbb{P}^1$, and we have $E^2 = -1$. Moreover, we have an isomorphism $\text{Pic} \; \tilde{Y} \oplus \mathbb{Z} \to \text{Pic} \; \tilde{Y}$ sending $(D, n)$ to $\pi^* D + n[E]$. For all $C, D \in \text{Pic} \; Y$ we have $(\pi^* C) \cdot (\pi^* D) = C \cdot D$, and $(\pi^* C) \cdot [E] = 0$. Finally, we have $K_{\tilde{Y}} \sim \pi^* K_Y + E$.

Proof. See [Har77], Propositions V.3.1, V.3.2, and V.3.3.

Proposition 1.16. Let $k$ be an algebraically closed field. For $1 \leq d \leq 8$, let $Y$ be the blow-up of $\mathbb{P}^2$ in $r = 9 - d$ points $P_1, \ldots, P_r$ in general position. Let Pic $Y$ be the Picard group of $X$, then we have $\text{Pic} \; Y \cong \mathbb{Z}^{10-d}$. More specifically, if $E_i$ is the class of the exceptional curve corresponding to $P_i$, and $L$ the class of the pullback of a line $l$ in $\mathbb{P}^2$ not passing through any of the $P_i$, then $\{L, E_1, \ldots, E_r\}$ forms a basis for Pic $Y$.

Proof. This follows from the previous proposition and the fact that Pic $\mathbb{P}^2 = \langle [l] \rangle$.  

Remark 1.17. Keeping the notation of the previous proposition, we have

\[ E_i^2 = -1 \text{ for all } i; \]
\[ E_i \cdot E_j = 0 \text{ for } i \neq j; \]
\[ L^2 = 1; \]
\[ L \cdot E_i = 0 \text{ for all } i. \]

Since the canonical divisor \( K_{\mathbb{P}^2} \) of \( \mathbb{P}^2 \) is linearly equivalent to \(-3l\), we have
\[ [-K_X] = 3L - \sum_{i=1}^m E_i. \] It follows that \([-K_X] \cdot E_i = 1 \) for all \( i \).

1.1 Del Pezzo surfaces of degree one

Let \( X \) be a del Pezzo surface of degree one over an algebraically closed field \( k \) with anticanonical divisor \(-K_X\). In this subsection we define the anticanonical model of \( X \) and see that this describes \( X \) as a smooth sextic surface in the weighted projective space \( \mathbb{P}(2,3,1,1) \). Moreover, we will see that the linear system \([\mathcal{L}(2K_X)]\) realizes \( X \) as a double cover of a quadric cone in \( \mathbb{P}^3 \). The linear system \([\mathcal{L}(K_X)]\) defines a rational map that is not a morphism, but by blowing up \( X \) we can extend this map to an elliptic fibration. The results in this subsection can be found in \([VA]\) and \([CO99]\).

The anticanonical model of \( X \)

Definition 1.18. The anticanonical ring of \( X \) is the graded ring
\[ R(X, -K_X) = \bigoplus_{m \geq 0} \mathcal{L}(-mK_X). \]

Definition 1.19. The anticanonical model of \( X \) is the scheme \( \text{Proj } R(X, -K_X) \).

Since \(-K_X\) is ample, \( X \) is isomorphic to its anticanonical model. We compute the anticanonical model of \( X \) as follows. By Lemma 1.12, we have \( l(-K_X) = 2 \). Let \( \{z, w\} \) be a basis for \( \mathcal{L}(-K_X) \). By Proposition 2.3 in \([CO99]\), for all \( m \geq 1 \) the elements \( z^m, z^{m-1}w, \ldots, zw^{m-1}, w^m \) are linearly independent in \( \mathcal{L}(-mK_X) \). So \( z^2, zw, w^2 \) are linearly independent elements of \( \mathcal{L}(-2K_X) \). Since \( l(-2K_X) = 4 \), we can choose an element \( x \in \mathcal{L}(-2K_X) \) such that \( \{z^2, zw, w^2, x\} \) forms a basis for \( \mathcal{L}(-2K_X) \). Now \( z^3, z^2w, zw^2, w^3, zx, wx \) are elements of \( \mathcal{L}(-3K_X) \) and linearly independent by the arguments in \([CO99]\), page 1200. Since \( l(-3K_X) = 7 \) we can therefore choose an element \( y \in \mathcal{L}(-3K_X) \) to obtain a basis \( \{z^3, z^2w, zw^2, z^2x, wx, y\} \) of \( \mathcal{L}(-3K_X) \). We have \( l(-4K_X) = 11 \) and \( l(-5K_X) = 16 \), and together with the arguments in \([CO99]\), page 1200 this implies that

\[ \{z^4, z^3w, z^2w^2, zw^3, w^4, x^2, xz^2, xw^2, xzw, yz, yw\} \]

is a basis for \( \mathcal{L}(-4K_X) \), and

\[ \{z^5, z^4w, z^3w^2, z^2w^3, zw^4, w^5, x^2w, x^2z, xz^3, wx^3, xz^2w, xzw^2, xy, yz^2, yw^2, yzw\} \]

is a basis for \( \mathcal{L}(-5K_X) \). Finally, since \( l(-6K_X) = 22 \), the 23 elements

\[ z^6, z^5w, z^4w^2, z^3w^3, z^2w^4, zw^5, w^6, x^3, x^2z^2, x^2w^2, x^2zw, x^4w, xz^3w, \\
xzw^2, xzw^3, xw^4, xy, xzw, y^2, yz^3, yw^2, yzw^2, yw^3 \]

are a basis for \( \mathcal{L}(-6K_X) \).
Let \( k \) be the graded \( k \)-algebra with grading defined by \( \deg z = \deg w = 1 \), \( \deg x = 2 \), and \( \deg y = 3 \). Then by Proposition 2.5 in [CO99] there exists a natural isomorphism between the anticanonical ring of \( X \) and \( k[x, y, z, w]/(h) \). Therefore, \( X \) is isomorphic to the zero locus of \( h \) in the weighted projective space \( \mathbb{P}(2, 3, 1, 1) \).

For the rest of this section we assume that \( \text{char}(k) \neq 2, 3 \), and identify \( X \) with its anticanonical model inside \( \mathbb{P}(2, 3, 1, 1) \).

### The linear system \( | -2K_X | \)

Let \( p : \mathbb{P}(2, 3, 1, 1) \to \mathbb{P}(2, 1, 1) \) be the projection sending a point \((x : y : z : w)\) to \((x : z : w)\). This is a rational map that is well defined on \( X \). The restriction to \( X \) is a morphism of degree 2. Let \( i : \mathbb{P}(2, 1, 1) \to \mathbb{P}^3(a_0, a_1, a_2, a_3) \) be the 2-uple embedding, sending \((x : z : w)\) to \((x : z^2 : zw : w^2)\). Then \( i(\mathbb{P}(2, 1, 1)) \) is a quadric cone \( Q \) given by \( a_2^2 = a_1a_3 \), with vertex \((1 : 0 : 0 : 0)\). The composition \( \varphi = i \circ p : X \to \mathbb{P}^3 \) is the morphism defined by \( -2K_X \). It is a double covering of \( Q \). The preimage of the vertex \((1 : 0 : 0 : 0)\) of \( Q \) under this morphism is the point \((1 : 1 : 0 : 0) = (1 : -1 : 0 : 0) \) in \( X \). We define \( \tilde{X} \) to be the blow-up of \( X \) in this point with associated map \( \pi : \tilde{X} \to X \). Moreover, we define \( \tilde{Q} \) to be the blow-up of \( Q \) in the vertex, with associated map \( \rho : \tilde{Q} \to Q \). Then \( \varphi \) induces a morphism \( \psi : \tilde{X} \to \tilde{Q} \). The morphism \( \psi \) is ramified at the exceptional curve \( E \) in \( \tilde{X} \) above \((1 : 1 : 0 : 0)\), and at those points in \( \mathbb{P}(2, 3, 1, 1) \) where \( y = 0 \), which are the points \((x : y : z : w)\) for which \( x^3 + f(z, w)x + g(z, w) = 0 \). The latter defines a surface in \( \mathbb{P}(2, 3, 1, 1) \), whose image under \( \psi \) defines a cubic surface in \( \mathbb{P}^3 \). The branch curve of \( \varphi \) is therefore the union of the vertex \( V \) of \( Q \) and a curve \( B \) that is contained in the intersection of the cubic surface with \( Q \). Since \( X \) is smooth it follows that \( B \) is too. Moreover, \( B \) is irreducible and reduced, so it is a smooth curve of degree six and genus four, see Proposition 3.1 in [CO99].

### The linear system \( |-K_X| \)

The linear system \( |-K_X| \) defines a rational map \( \mu : X \to \mathbb{P}^1 \), sending \((x : y : z : w)\) to \((z : w)\). This is not defined in the point \((1 : 1 : 0 : 0) \in X \), which is the unique basepoint of \( |-K_X| \). As \( \tilde{X} \) is the blow-up of \( X \) in this point, the rational map \( \mu \) induces a morphism \( \nu : \tilde{X} \to \mathbb{P}^1 \). The fiber under \( \nu \) above a point \((z_0 : w_0) \in \mathbb{P}^1 \) is isomorphic to the set of points \((x : y : z_0 : w_0) \in X \) with \( y^2 = x^3 + xf(z_0, w_0) + g(z_0, w_0) \). This is an elliptic curve for almost all \((z_0, w_0)\), so \( \nu \) is an elliptic fibration.

The morphisms described above are shown in the following commutative diagram.
2 Exceptional curves

Let \( k \) be an algebraically closed field, and let \( X \) be a del Pezzo surface of degree \( d \) over \( k \) that is isomorphic to \( \mathbb{P}^2 \) blown up at \( r = 9 - d \) points \( \{ P_1, \ldots, P_r \} \) in general position. Let \( -K_X \) be the anticanonical divisor of \( X \). Let \( \pi : X \rightarrow \mathbb{P}^2 \) denote the blow-up. For all \( i \), the inverse image \( \pi^{-1}(P_i) \) of \( P_i \) is an exceptional curve on \( X \). From Proposition 1.15 and Remark 1.17, we know that \( \pi^{-1}(P_i) \) is isomorphic to \( \mathbb{P}^1 \), and \( (\pi^{-1}(P_i))^2 = K_X \cdot \pi^{-1}(P_i) = -1 \). As we will see, \( X \) contains more curves with these properties. In this section we define the general notion of an exceptional curve on a surface and describe the exceptional curves on a del Pezzo surface. In Subsection 2.1 we consider exceptional curves on del Pezzo surfaces of degree one, which have a very nice geometrical description. All results in this section can be found in [Man74], unless stated otherwise.

**Definition 2.1.** Let \( Y \) be a nice surface. An exceptional curve on \( Y \) is an irreducible projective curve \( C \subset Y \) such that \( C^2 = C \cdot K_Y = -1 \).

The following proposition is a very classical result.

**Proposition 2.2.** (Adjunction formula). Let \( Y \) be a nice surface over an algebraically closed field with canonical divisor \( K_Y \), and \( C \) an irreducible projective curve on \( Y \). Then

\[
2p_a(C) - 2 = C \cdot (C + K_Y),
\]

where \( p_a(C) \) is the arithmetic genus of \( C \).

**Proof.** See [Har77], Proposition V.1.5. \( \square \)

From the Adjunction formula it follows that for an exceptional curve \( C \) on \( X \) we have \( 2p_a(C) - 2 = -2 \), hence \( p_a(C) = 0 \), so \( C \cong \mathbb{P}^1 \).

If \( X \) has degree \( d \geq 3 \), then \( X \) has very ample anticanonical divisor \( -K_X \), which determines an embedding in \( \mathbb{P}^n \) for some \( n \). The image under this embedding of an exceptional curve \( C \) on \( X \) has degree \( -K_X \cdot C = 1 \), hence it is a line.

On a del Pezzo surface, every irreducible curve with negative self-intersection is in fact an exceptional curve. The following proposition can be found for instance in [Man74], Theorem 24.3.

**Proposition 2.3.** Let \( Y \) be a del Pezzo surface over an algebraically closed field, and \( C \) an irreducible curve on \( Y \) with \( C^2 < 0 \). Then \( C \) is an exceptional curve.

**Proof.** Since \( -K_Y \) is ample and \( C \) is irreducible, we have \( -K_Y \cdot C > 0 \) by Theorem 1.9, so \( K_Y \cdot C < 0 \). Moreover, since \( C \) is irreducible we have \( g_a(C) \geq 0 \). From the adjunction formula it follows that

\[
-2 \leq 2g_a(C) - 2 = C \cdot (C + K_Y) = C^2 + C \cdot K_Y \leq -2,
\]

so equality must hold, hence \( C^2 = K_Y \cdot C = -1 \), so \( C \) is exceptional. \( \square \)
We can now give the following condition for points in \( \mathbb{P}^2 \) to be in general position.

**Proposition 2.4.** Let \( Q_1, \ldots, Q_8 \) be eight points in \( \mathbb{P}^2 \) and let \( \pi : Y \rightarrow \mathbb{P}^2 \) be the blow-up in these points. Then \( Q_1, \ldots, Q_8 \) are in general position if and only if \( Y \) is a del Pezzo surface.

**Proof.** The fact that \( Y \) is a del Pezzo surface if \( Q_1, \ldots, Q_8 \) are in general position is Theorem 1.6. For the converse, assume that three points \( Q_j, Q_k \) and \( Q_l \) are on a line \( M \) in \( \mathbb{P}^2 \). Let \( M' \) be the strict transform of \( M \) on \( Y \) and let \( D_i \) be the exceptional curve above \( Q_i \) for all \( i \). Then we have

\[
\pi^* M = M' + D_j + D_k + D_l,
\]

so

\[
1 = M^2 = (\pi^* M)^2 = M'^2 + 2M' \cdot (D_j + D_k + D_l) + D_j^2 + D_k^2 + D_l^2 = M'^2 + 6 - 3,
\]

hence \( M'^2 = -2 \), which contradicts Proposition 2.3. Analogously, a conic containing six of the \( Q_i \) and a singular cubic through seven of the \( Q_i \) with one of them at the singularity would have a strict transform on \( Y \) with self-intersection \( \leq -2 \), contradicting Proposition 2.3. We conclude that \( Q_1, \ldots, Q_8 \) are in general position.

Exceptional curves can be ‘blown down’, as is described in the well-known theorem by Castelnuovo.

**Theorem 2.5.** (Castelnuovo). If \( C \) is a curve on a nice surface \( Y \) over an algebraically closed field such that \( C^2 = -1 \) and \( C \cong \mathbb{P}^1 \), then there exists a morphism \( f : Y \rightarrow Y_0 \) to a nonsingular projective surface \( Y_0 \), and a point \( P \in Y_0 \), such that \( Y \) is the blow-up of \( Y_0 \) at \( P \), and \( C \) is the exceptional curve above \( P \).

**Proof.** See [Har77], Theorem V.5.7.

After blowing down an exceptional curve on a del Pezzo surface, we obtain again a del Pezzo surface. Proposition 2.6 can be found in [Pie], Lemma 4.20.

**Proposition 2.6.** Let \( Y \) be a del Pezzo surface of degree \( d \leq 8 \) over an algebraically closed field that is the blow-up of \( r = 9 - d \) points in \( \mathbb{P}^2 \), and let \( C \) be an exceptional curve on \( Y \). Let \( f : Y \rightarrow Y_0 \) be a morphism to a nonsingular projective surface \( Y_0 \), such that \( Y \) is the blow-up of \( Y_0 \) in a point \( P \), and such that \( C \) is the exceptional curve above \( P \). Then \( Y_0 \) is a del Pezzo surface of degree \( d + 1 \).

**Proof.** Let \( K_Y, K_{Y_0} \) be the canonical divisors of \( Y, Y_0 \), respectively. By Proposition 1.15 we have \( K_Y \sim f^* K_{Y_0} + C \), so, using Proposition 1.15 we have

\[
K_{Y_0}^2 = (f^* K_{Y_0})^2 = (K_Y - C)^2 = K_Y^2 - 2K_Y \cdot C + C^2 = d + 2 - 1 = d + 1 > 0.
\]

Let \( D \) be an irreducible curve on \( Y_0 \) containing \( P \) with multiplicity \( m \), and let \( D' \) be its strict transform on \( Y \). Then \( D' \) is an irreducible curve on \( Y \), so \(-K_Y \cdot D' > 0\)
by Nakai-Moishezon. Therefore we have, using Proposition 1.15,

\[-K_Y \cdot D = f^*(-K_Y) \cdot f^*D = (-K_Y + C) \cdot f^*D = -K_Y \cdot f^*D - C \cdot f^*D = -K_Y \cdot (D' + mC) = -K_Y \cdot D' + m > 0.\]

From Nakai-Moishezon it follows that \(-K_Y\) is ample, so \(Y_0\) is a del Pezzo surface. Its degree is \(K_{Y_0}^2 = d + 1\).

Let \(C\) be an exceptional curve in \(X\). Then the class of \(C\) in \(\text{Pic} X\) satisfies

\([C]^2 = [C] \cdot [K_X] = -1.\]

We call a class in \(\text{Pic} X\) satisfying these conditions \textit{exceptional}. We will describe the exceptional classes in \(\text{Pic} X\) and show that there is a one-to-one correspondence between exceptional classes in \(\text{Pic} X\) and exceptional curves on \(X\).

As we have seen, \(\text{Pic} X\) has a basis \(\{L, E_1, \ldots, E_r\}\), where \(E_i\) is the class of the exceptional curve above \(P_i\), and \(L\) is the class of the strict transform of a line in \(\mathbb{P}^2\) not going through any of the \(P_i\). If \(D\) is a class in \(\text{Pic} X\) given by \(D = aL - \sum_{i=1}^r b_i E_i\), then \(D\) is an exceptional class if and only if \(D^2 = D \cdot [K_X] = -1\), or, using the results in Remark 1.17

\[a^2 - \sum_{i=1}^r b_i^2 = -1,\]

and

\[3a - \sum_{i=1}^r b_i = 1.\]

Using the fact that \(a\) and all \(b_i\) are integers, we can solve these two equations and find all exceptional classes in \(\text{Pic} X\).

**Proposition 2.7.** The exceptional classes in \(\text{Pic} X\) are the classes of the form \(aL - \sum_{i=1}^r b_i E_i\) where \((a, b_1, \ldots, b_r)\) is given by one of the rows of the following table, where all \(b_i\) can be permuted (we only consider the rows where \(b_i = 0\) for all \(i > r\)).

<table>
<thead>
<tr>
<th>a</th>
<th>b_1</th>
<th>b_2</th>
<th>b_3</th>
<th>b_4</th>
<th>b_5</th>
<th>b_6</th>
<th>b_7</th>
<th>b_8</th>
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<td>0</td>
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<td>2</td>
</tr>
</tbody>
</table>

**Proof.** See [Man74], Proposition 26.1.

**Proposition 2.7** gives us a very explicit description of all exceptional classes in \(\text{Pic} X\). The following theorem relates exceptional classes to exceptional curves on \(X\).
Theorem 2.8.

(i) There is a one-to-one correspondence between the set of exceptional curves on $X$ and the set of exceptional classes in $\text{Pic} X$, given by the map sending an exceptional curve in $X$ to its class in $\text{Pic} X$.

(ii) Let $f : X \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ in the points $P_1, \ldots, P_r$. Then the image $f(C)$ of an exceptional curve $C \subset X$ is one of the following types.

(a) One of the points $P_i$;
(b) a line passing through two of the points $P_i$;
(c) a conic passing through five of the points $P_i$;
(d) a cubic passing through seven of the points $P_i$ such that one of them is a double point;
(e) a quartic passing through eight of the points $P_i$ such that three of them are double points;
(f) a quintic passing through eight of the points $P_i$ such that six of them are double points;
(g) a sextic passing through eight of the points $P_i$ such that seven of them are double points and one is a triple point.

(For $d = 2$, only $(a) - (d)$ hold; for $d = 3, 4$, only $(a) - (c)$ hold; for $d = 5, 6, 7$, only $(a) - (b)$ hold; for $d = 8$, only $(a)$ holds.)

Proof. See [Man74], Theorem 26.2.

Remark 2.9. Theorem 2.8.(ii) gives a geometrical description of the table in Proposition 2.7. An exceptional class of the form $C = aL - \sum_{i=1}^{r} b_i E_i$, with $(a, b_1, \ldots, b_8)$ a solution given by Proposition 2.7, is either one of the $E_i$, or it is the class of the strict transform of a curve in $\mathbb{P}^2$ of degree $a_i$ going through $P_i$ with multiplicity $b_i$ for each $i$. Moreover, Theorem 2.8 tells us that these are in one-to-one correspondence with all exceptional curves on $X$. We can therefore count the exceptional curves on $X$ using the table in Proposition 2.7 and obtain the following table.

\[
\begin{array}{c|cccccccc}
d & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\text{exceptional curves on } X & 240 & 56 & 27 & 16 & 10 & 6 & 3 & 1
\end{array}
\]

2.1 Exceptional curves on del Pezzo surfaces of degree one

Let $X$ be a del Pezzo surface of degree one over an algebraically closed field $k$. Let $E$ be the set of exceptional curves on $X$. We have $|E| = 240$. As in Subsection 1.1, let $\varphi : X \to \mathbb{P}^3(a_0, a_1, a_2, a_3)$ be the morphism corresponding to the linear system $| - 2K_X|$. We have seen that this is a double covering of a quadric cone $Q$ given by $a_2^2 = a_1a_3$ in $\mathbb{P}^3$, that branches over a sextic curve $B$ and an isolated branch point at the vertex of $Q$. In this subsection we show that the exceptional curves on $X$ are related to hyperplane sections of $Q$ that do not pass through the vertex of $Q$, and are tritangent to $B$. We start by studying the elements in $| - K_X|$.

Proposition 2.10 and Proposition 2.12 can both be found in [CO99].
Proposition 2.10. For every element \( D \in |-K_X| \), its image \( \varphi(D) \) is a line in \( Q \) passing through the vertex of \( Q \). Conversely, a line through the vertex of \( Q \) pulls back under \( \varphi \) to an element of \( |-K_X| \).

Proof. As we saw in Subsection 1.1, \( \mathcal{L}(-K_X) \) is generated by two elements \( z \) and \( w \). Let \( D \in |-K_X| \), then \( D \) is of the form \( \alpha z + \beta w = 0 \), with \( \alpha, \beta \in k \). Without loss of generality we can assume that \( \alpha \neq 0 \). Then \( z = -\frac{\beta}{\alpha}w \), and \( \varphi(D) \) is contained in the two hyperplanes \( a_1 = \frac{\beta^2}{\alpha^2}a_3 \) and \( a_1 = -\frac{\beta}{\alpha}a_2 \) in \( \mathbb{P}^3 \), both containing the vertex of \( Q \). Since \( \varphi \) is finite, \( \varphi(D) \) is equal to their intersection.

Conversely, let \( M \) be a line in \( Q \) through the vertex of \( Q \). Then \( M \) is the intersection of two hyperplanes \( \gamma a_1 + \delta a_2 + \varepsilon a_3 = 0 \) and \( \lambda a_1 + \mu a_2 + \nu a_3 = 0 \) in \( \mathbb{P}^3 \). Keeping the notation of Subsection 1.1, we identify \( Q \) with \( \mathbb{P}(2,1,1) \). Under this identification, \( M \) is given by a linear relation in \( z \) and \( w \). Therefore \( M \) projects under the map \( p': \mathbb{P}(2,1,1) \rightarrow \mathbb{P}(1,1) \) to a point in \( \mathbb{P}^1 \). The fiber of \( \nu \) above a point in \( \mathbb{P}^1 \) is an element of \( |-K_X| \), so \( \varphi^*M \) is an element of \( |-K_X| \).

To prove the following proposition, we first need a Lemma.

Lemma 2.11. Let \( Y, Z \) be two normal projective varieties, and \( f: Y \rightarrow Z \) a finite morphism of degree \( d \). Let \( D, D' \) be two divisors on \( Z \). Then \( f^*D \cdot f^*D' = d(D \cdot D') \), and for a divisor \( C \) on \( Y \) we have \( f^*D \cdot C = D \cdot f_\ast C \).

Proof. See [HS00], Theorem A.2.3.2, and [Kol96], Proposition VI.2.11.

Proposition 2.12.

(i) If \( e \) is an exceptional curve on \( X \), then \( \varphi(e) \) is a smooth conic in \( Q \) not containing the vertex of \( Q \). Moreover \( \varphi|_e: e \rightarrow \varphi(e) \) is one-to-one.

(ii) If \( H \) is a hyperplane in \( \mathbb{P}^3 \) not containing the vertex of \( Q \), then \( \varphi^*H \) has an exceptional curve as component if and only if it has at least three (maybe infinitely near) singular points. If this is the case, then \( \varphi^*H = e_1 + e_2 \) with \( e_1, e_2 \) exceptional curves, and \( e_1 \cdot e_2 = 3 \).

Proof.

(i) Let \( H \) be a hyperplane in \( \mathbb{P}^3 \), then we have \( \deg \varphi(e) = H \cdot \varphi(e) \) and \( \varphi^*H \sim -2K_X \). Let \([k(e) : k(\varphi(e))] = n\), then \( \varphi_*(e) = n\varphi(e) \), so by Lemma 2.11 we have

\[
H \cdot n\varphi(e) = H \cdot \varphi_*(e) = \varphi^*H \cdot e = -2K_X \cdot e = 2,
\]
hence \( \deg \varphi(e) = \frac{2}{n} \). Therefore, \( n \) is either 1 or 2. If \( n = 2 \), then \( \deg \varphi(e) = 1 \), so \( \varphi(e) \) is a line \( M \) in \( Q \) and \( \varphi|_e: e \rightarrow M \) is 2:1. Then \( \varphi^*M = e \). But \( \varphi^*M \) is an element in \( |-K_X| \) by Proposition 2.10, which gives a contradiction. Therefore we have \( n = 1 \), so \( \varphi|_e: e \rightarrow \varphi(e) \) is one-to-one and \( \deg \varphi(e) = 2 \). Since \( \varphi(e) \) is irreducible, it is a smooth conic in \( Q \).

(ii) Let \( H \) be a hyperplane in \( \mathbb{P}^3 \) not containing the vertex of \( Q \), so that \( C = H \cap Q \) is a smooth conic section of \( Q \). First assume that \( \varphi^*H \) has
an exceptional curve $e_1$ as component. If $\varphi^*H = me_1$ for some $m \geq 1$, then $2 = \varphi^*H \cdot e_1 = -m$, which is a contradiction. Therefore, $\varphi^*H$ is not irreducible. Since $\deg \varphi = 2$ and $\varphi^*H$ is not in the ramification divisor of $\varphi$, it follows that we have $\varphi^*H = e_1 + e_2$, where $e_2$ is irreducible and distinct form $e_1$. But then we have $e_1 \cdot e_2 = e_1 \cdot \varphi^*H - e_1^2 = e_1 \cdot -2K_X - e_1^2 = 3$. Therefore, $\varphi^*H$ has three (maybe infinitely near) singular points.
Conversely, assume that $\varphi^*H$ has at least three (maybe infinitely near) singular points. We have $(\varphi^*H)^2 = 4$ and $\varphi^*H \cdot K_X = -2K_X^2 = -2$. If $\varphi^*H$ were irreducible, then, by the adjunction formula, we would have

$$2p_a(\varphi^*H) - 2 = \varphi^*H(\varphi^*H + K_X) = 4 - 2 = 2,$$

so $p_a(\varphi^*H) = 2$. Since $\varphi^*H$ has at least three (maybe infinitely near) singularities, this would imply that it has genus at most $g(\varphi^*H) \leq 2 - 3 < 0$, which is impossible. We conclude that $\varphi^*H$ is not irreducible. Therefore, since $\deg \varphi = 2$ and $\varphi^*H$ is not the ramification divisor, we have $\varphi^*H = D_1 + D_2$, where $D_1$ and $D_2$ are irreducible and $D_1$ is distinct from $D_2$. Since $C$ is smooth, the singular points of $\varphi^*H$ are the intersections between $D_1$ and $D_2$, so $D_2 \cdot D_2 \geq 3$. Since $\varphi(D_1) = \varphi(D_2)$, the automorphism of $X$ sending a point $(x : y : z : w)$ to $(x : -y : z : w)$ is an involution that interchanges $D_1$ and $D_2$, so $D_1 \cdot K_X = D_2 \cdot K_X$ and $D_1^2 = D_2^2$. Hence from

$$-2 = \varphi^*H \cdot K_X = D_1 \cdot K_X + D_2 \cdot K_X$$

it follows that $D_1 \cdot K_X = D_2 \cdot K_X = -1$. Finally, we have

$$4 = (-2K_X)^2 = D_1^2 + D_2^2 + 2D_1 \cdot D_2 = 2D_1^2 + 2D_1 \cdot D_2,$$

so $2D_1^2 = 4 - 2D_1 \cdot D_2 \leq -2$. Therefore $D_1^2 < 0$, hence from Proposition 2.3 it follows that $D_1^2 = D_2^2 = -1$ and so $D_1 \cdot D_2 = 3$. We conclude that $D_1$ and $D_2$ are exceptional curves with intersection multiplicity three.

\[ \square \]

**Remark** 2.13. From the previous proposition we can conclude that if $e_1, e_2$ are exceptional curves on $X$ such that $e_1 \cdot e_2 = 3$, the points in the intersection $e_1 \cap e_2$ are exactly the points in the intersection of $e_i$ with the ramification curve of $\varphi$, for $i = 1, 2$. We conclude that there is a bijection between the sets

$$\{ \{e_1, e_2\} \mid e_1, e_2 \in E; e_1 \cdot e_2 = 3 \}$$

and

$$\{ H \mid H \subset \mathbb{P}^3 \text{ hyperplane tritangent to } B; (1 : 0 : 0 : 0) \notin H \}.$$
3 The Weyl group acting on exceptional curves

To count the maximal number of exceptional curves through one point, we will make a lot of use of the group that permutes the exceptional classes in the Picard group while preserving the intersection pairing. In this section we describe this group and study its action on the exceptional classes on a del Pezzo surface of degree one. All results in this section about root systems and the Weyl group can be found in [Man74].

Let $X$ be a del Pezzo surface of degree $d$ over an algebraically closed field $k$, such that $X$ is isomorphic to $\mathbb{P}^2$ blown up in $r = 9 - d$ points $P_1, \ldots, P_r$. Let $E_i \in \text{Pic} X$ be the class of the exceptional curve above $P_i$ for all $i$, and let $L$ be the class of the strict transform of a line not going through any of the $P_i$. Let $K_X$ be the class of the canonical divisor on $X$. As we have seen, $\text{Pic} X$ is a free abelian group of rank $r + 1$. Consider the $R$-vectorspace $R \otimes \mathbb{Z} \text{Pic} X$. Since $\{L, E_1, \ldots, E_r\}$ is a basis for the Picard group, the set $\{1 \otimes L, 1 \otimes E_1, \ldots, 1 \otimes E_r\}$ is a basis for $R \otimes \mathbb{Z} \text{Pic} X$.

Lemma 3.1. For $0 < r \leq 8$, the intersection number $(\cdot, \cdot)$ is negative-definite on the orthogonal complement $K_X^\perp$ of $K_X$ in $R \otimes \mathbb{Z} \text{Pic} X$.

Proof. Let $D = aL - \sum_{i=1}^{r} b_i E_i \in \text{Pic} X$. Then we have

$$K_X \cdot D = \left(-3L + \sum_{i=1}^{r} E_i\right) \cdot \left(aL - \sum_{i=1}^{r} b_i E_i\right) = -3a + \sum_{i=1}^{r} b_i,$$

so $K_X \cdot D = 0$ if and only if $3a = \sum_{i=1}^{r} b_i$. Now assume $D \in K_X^\perp$. Note that $D$ has self-intersection $a^2 - \sum_{i=1}^{r} b_i^2$. By Cauchy-Schwarz we have

$$\sum_{i=1}^{r} b_i^2 = \frac{1}{r} \sum_{i=1}^{r} b_i^2 \sum_{i=1}^{r} 1^2 \geq \frac{1}{r} \left(\sum_{i=1}^{r} b_i\right)^2,$$

so

$$a^2 - \sum_{i=1}^{r} b_i^2 \leq a^2 - \frac{1}{r} \left(\sum_{i=1}^{r} b_i\right)^2 = a^2 - \frac{9}{r} a^2 < 0.$$

We conclude that $D^2 < 0$, so the intersection number is negative definite on $K_X^\perp$.

Definition 3.2. We define $\left(K_X^\perp, \langle \cdot, \cdot \rangle\right)$ to be the vector space in $R \otimes \mathbb{Z} \text{Pic} X$ with inner product $\langle \cdot, \cdot \rangle = -(\cdot, \cdot)$. Note that this inner product is positive-definite by Lemma 3.1.

We now give the definition of a root system.

Definition 3.3. Let $V$ be a finite-dimensional vector space over a field $l \subseteq R$ with a positive-definite inner product $(\cdot, \cdot)$. A root system in $V$ is a finite set $R$ of non-zero vectors, called roots, that satisfy the following conditions:

(i) the roots span $V$;
(ii) for all \( r \in R \), we have \( \lambda r \in R \implies \lambda = \pm 1 \);

(iii) for all \( r, s \in R \), we have \( s - 2r \frac{\langle r, s \rangle}{\langle r, r \rangle} \in R \);

(iv) for all \( r, s \in R \), the number \( 2 \frac{\langle r, s \rangle}{\langle r, r \rangle} \) is an integer.

Define the set

\[ R_r = \{ D \in \text{Pic} \, X \mid D^2 = -2; \, D \cdot K_X = 0 \} \]

**Proposition 3.4.** The set \( R_r \) is a root system of rank \( r \) in \( (K^+_X, \langle \cdot, \cdot \rangle) \).

**Proof.** See [Man74], Proposition 25.2.

From now on we assume that \( X \) is a del Pezzo surface of degree one, so \( r = 8 \).

**Proposition 3.5.** The root system \( R_8 \) is isomorphic to the classical root system \( E_8 \). Moreover, a basis for \( R_8 \) is given by the elements \( r_1, \ldots, r_8 \), given by

\[
E_1 - E_2, \, E_2 - E_3, \ldots, \, E_7 - E_8, \, L - E_1 - E_2 - E_3.
\]

**Proof.** See [Man74], Theorem 25.4 and Proposition 25.5.6.

**Definition 3.6.** The Weyl group \( W(R_8) \) is the group of permutations of the roots of \( R_8 \) generated by the reflections with respect to \( r_1, \ldots, r_8 \) (the reflection with respect to \( r_i \) is given by \( s \mapsto s - 2r_i \frac{\langle s, r_i \rangle}{\langle r_i, r_i \rangle} \) for all \( s \in R_8 \)).

**Theorem 3.7.** The following groups are isomorphic:

(i) the group of automorphisms of \( \text{Pic} \, X \) preserving \( K_X \) and the intersection pairing;

(ii) the group of permutations of the exceptional classes in \( \text{Pic} \, X \) preserving their pairwise intersection multiplicities;

(iii) the Weyl group \( W(R_8) \).

The order of the group \( W(R_8) \) is \( 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7 \).

**Proof.** See [Man74], Theorem 23.9 and 26.6.

Let \( E \) be the set of exceptional classes in \( \text{Pic} \, X \). Recall that \( E \) is in one-to-one correspondence with the set of exceptional curves on \( X \). For the rest of this thesis we refer to \( W(R_8) \), the group of permutations of \( E \) preserving the intersection pairing, as \( G \). Since \( G \) preserves the intersection pairing on \( E \), we can use results about the action of \( G \) on \( E \) when computing the maximal number of exceptional curves that go through one point. The following proposition will be used a lot.

**Proposition 3.8.** Let \( E \) and \( G \) be as above. Then we have:

(i) the group \( G \) acts transitively on \( E \);
(ii) for all $r \leq 8$ such that $r \neq 7$, the group $G$ acts transitively on the set
\[
\{(e_1, \ldots, e_r) \in E^r \mid \forall i \neq j : e_i \cdot e_j = 0\}.
\]

**Proof.** See [Man74], Corollaries 26.7 and 26.8. □

Let $U$ be the set
\[
\{(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8) \in E^8 \mid \forall i \neq j : e_i \cdot e_j = 0\}.
\]

The group $G$ acts transitively on $U$ by Proposition 3.8. We show some results about $U$ that will be useful later.

**Lemma 3.9.** For $u = (e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8) \in U$, there exists a morphism $f : X \to \mathbb{P}^2$, and points $Q_1, \ldots, Q_8 \in \mathbb{P}^2$ that are in general position, such that $X$ is the blow-up of $\mathbb{P}^2$ at $Q_1, \ldots, Q_8$, and for all $i$, the element $e_i$ is the class in $\text{Pic} X$ of the exceptional curve above $Q_i$.

**Proof.** Set $Y_0 = X$. Since all $Q_i$ are disjoint, by Theorem 2.5, for each $i \in \{1, \ldots, 8\}$ there is a nonsingular projective surface $Y_i$, and a morphism $f_i : Y_{i-1} \to Y_i$ that is the blow-up of $Y_i$ in $Q_i$, where $e_i$ is the class in $\text{Pic} X$ of the exceptional curve above $Q_i$. Since $k$ is algebraically closed, by Proposition 2.6, the surface $Y_i$ is a del Pezzo surface of degree $i + 1$ for all $i$. It follows that $Y_8 = \mathbb{P}^2$, and the composition of the $f_i$ is a morphism $f : X \to \mathbb{P}^2$ that is the blow-up in $Q_1, \ldots, Q_8$. Since $X$ is a del Pezzo surface, from Proposition 2.4 it follows that $Q_1, \ldots, Q_8$ are in general position. □

**Corollary 3.10.** For $u = (e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8) \in U$, there is a unique element $l \in \text{Pic} X$ such that $K_X = -3l + \sum_{i=1}^8 e_i$. Moreover, the set $\{l, e_1, \ldots, e_8\}$ forms a basis for $\text{Pic} X$.

**Proof.** Let $u = (e_1, \ldots, e_8) \in U$. By the previous lemma there exists a morphism $f : X \to \mathbb{P}^2$, and points $Q_1, \ldots, Q_8 \in \mathbb{P}^2$ that are in general position, such that $X$ is the blow-up of $\mathbb{P}^2$ at $Q_1, \ldots, Q_8$, and $e_i$ is the class of the exceptional curve above $Q_i$ for all $i$. By Remark 1.17 we have $K_X = -3l + \sum_{i=1}^8 e_i$, where $l$ is the class of the strict transform of a line in $\mathbb{P}^2$ not containing any of the $Q_i$. By Proposition 1.16 we know that $\{l, e_1, \ldots, e_8\}$ forms a basis for $\text{Pic} X$. □

**Remark 3.11.** Let $u = (e_1, \ldots, e_8) \in U$, and let $l$ be the unique element such that $K_X = -3l + \sum_{i=1}^8 e_i$, which exists by Corollary 3.10. Let $g_1, g_2 \in G$ be such that $g_1(u) = g_2(u)$. This implies $g_1(l) = g_2(l)$, since $g_1, g_2$ fix $K_X$. Therefore, $g_1$ and $g_2$ act the same on the basis $\{l, e_1, \ldots, e_8\}$ of $\text{Pic} X$, hence they act the same on $\text{Pic} X$. Since $G$ acts faithfully on $\text{Pic} X$ we conclude that $g_1 = g_2$. Therefore, the action of $G$ on $U$ is free. Since $G$ acts transitively on $U$ and $U$ is not empty, this implies that $|U| = |G|$.

**Remark 3.12.** Let $A$ be the set of 240 vectors $(a, b_1, \ldots, b_8)$ that are in the table in Proposition 2.7 (where the $b_i$ can be permuted). We have a map
\[
f : U \to \text{Hom}_{\text{Set}}(E, A)
\]
as follows. Given $u = (e_1, \ldots, e_8) \in U$, let $l$ be the unique element such that $K_X = -3l + \sum_{i=1}^{8} e_i$, which exists by Corollary 3.10. Then we define $f(u)$ as follows.

$$f(u) : E \rightarrow A, \ e \mapsto (e \cdot l, e \cdot e_1, \ldots, e \cdot e_8).$$

For $\alpha = (a, b_1, \ldots, b_8) \in A$ and $e = al - \sum_{i=1}^{8} b_i e_i \in E$ we have $f(u)(e) = \alpha$, hence $f(u)$ is surjective. Since $E$ and $A$ have the same cardinality, it follows that $f(u)$ is a bijection.

To study $G$ and its action on $E$ further, we first look at the intersection multiplicities on $E$. These results will be useful later.

**Lemma 3.13.** For every $e \in E$ there is a unique $c \in E$ such that $e \cdot c = 3$.

*Proof.* Since $G$ acts transitively on $E$, it is enough to check this for $e = E_1$. Let $c = aL - \sum_{i=1}^{8} b_i E_i \in E$. Then $c$ intersects $e$ with multiplicity three if and only if $b_1 = 3$. By looking at the table in Proposition 2.7, we find that there is one solution for $c$ with $b_1 = 3$, given by $c = 6L - 3E_1 - \sum_{i=2}^{8} 2E_i$. \hfill \Box

**Remark 3.14.** Since for every element $e \in E$ there is a unique element intersecting $e$ with multiplicity three, and $G$ acts transitively on $E$, the group $G$ acts transitively on the set

$$\{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = 3\}.

**Lemma 3.15.** There is a bijection between the sets

$$Z = \{(e_0, e_1) \in E^2 \mid e_0 \cdot e_1 = 0\} \quad \text{and} \quad Z' = \{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = 2\},$$

given by

$$f : Z \rightarrow Z', \ (e_0, e_1) \mapsto (e_1, e_2), \text{ where } e_2 \text{ is such that } e_0 \cdot e_2 = 3.$$

*Proof.* Let $e_0 = E_1$ and $e_1 = E_2$. Then $(e_0, e_1)$ is an element in $Z$. By looking at the table in Proposition 2.7 we see that there is exactly one exceptional class intersecting $e_0$ with multiplicity three, which is $e_2 = 6L - 3E_1 - \sum_{i=2}^{8} 2E_i$. We have $e_1 \cdot e_2 = 2$. Since $G$ acts transitively on $Z$ by Proposition 3.8 it follows that for all elements $(c_0, c_1) \in Z$, the unique element $c_2$ such that $c_0 \cdot c_2 = 3$ has the property that $c_1 \cdot c_2 = 2$. Therefore, $f$ is well defined. Let $(c_1, c_2) \in Z'$. Then $f^{-1}((c_1, c_2))$ consists of all elements $(c, c_1) \in Z$ such that $c \cdot c_2 = 3$. From Lemma 3.13 we know that there is one such $c$, say $c_0$. We conclude that $f^{-1}((c_1, c_2)) = \{(c_0, c_1)\}$, so all fibers of $f$ have cardinality one. Therefore, $f$ is a bijection. \hfill \Box

**Corollary 3.16.** $G$ acts transitively on the set

$$Z = \{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = 2\}.

*Proof.* This follows from Lemma 3.15 and Proposition 3.8 \hfill \Box

**Proposition 3.17.** Any two elements in $E$ intersect each other with multiplicity at most three. Let $e \in E$ be an exceptional class in Pic $X$. Then there are exactly 56 exceptional classes disjoint from $e$, there are 126 exceptional classes intersecting
$e$ with multiplicity one, and there are 56 exceptional classes intersecting $e$ with multiplicity two.

**Proof.** Since $G$ acts transitively on $E$, it is enough to check this for $e = E_1$. Let $c = aL - \sum_{i=1}^8 b_iE_i \in E$. Then $e \cdot c = b_1$. We can now easily compute the results case by case.

We have $e \cdot c = 0$ if and only if $b_1 = 0$. Looking at the table in Proposition 2.7, we find the following possibilities.

<table>
<thead>
<tr>
<th>$a$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of possibilities for $c$</td>
<td>7</td>
<td>21</td>
<td>21</td>
<td>7</td>
</tr>
</tbody>
</table>

This gives a total of 56 exceptional classes that are disjoint from $e$. By the bijection in Lemma 3.15, this gives also 56 exceptional classes intersecting $e$ with multiplicity two.

Similarly, $c$ intersects $e$ with multiplicity one if and only if $b_1 = 1$. This gives the following possibilities.

<table>
<thead>
<tr>
<th>$a$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of possibilities for $c$</td>
<td>7</td>
<td>35</td>
<td>42</td>
<td>35</td>
<td>7</td>
</tr>
</tbody>
</table>

Therefore we find a total of 126 exceptional classes intersecting $e$ with multiplicity one.

From Lemma 3.13 we know that there is one $c$ such that $e \cdot c = 3$. Since we have a total of 240 exceptional classes, we conclude that these are all the possibilities.

**Lemma 3.18.** Let $e_1, e_2 \in E$ such that $e_1 \cdot e_2 = 0$. Then there are exactly 72 elements of $E$ intersecting both $e_1$ and $e_2$ with multiplicity one.

**Proof.** We know that $G$ acts transitively on the set $\{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = 0\}$ from Proposition 3.8, so it is enough to check this for $e_1 = E_1$ and $e_2 = E_2$. Let $e = aL - \sum_{i=1}^8 b_iE_i \in E$. Then $E_1 \cdot e = b_1$ and $E_2 \cdot e = b_2$, so $E_1 \cdot e = E_2 \cdot e = 1$ if and only if $b_1 = b_2 = 1$. Looking at the table in Proposition 2.7, we find the following possibilities.

<table>
<thead>
<tr>
<th>$a$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of possibilities for $e$</td>
<td>1</td>
<td>20</td>
<td>30</td>
<td>20</td>
<td>1</td>
</tr>
</tbody>
</table>

This gives a total of 72 exceptional classes that intersect both $E_1$ and $E_2$ with multiplicity one.

We are now able to prove a couple of propositions about the action of $G$ on various subsets of $E$. These propositions are very useful for our purpose in the next section. First, we need some lemmas.

**Lemma 3.19.** Let $V$ be the set

$$V = \{(e_0, e_1, e_2) \in E^3 \mid e_0 \cdot e_1 = e_0 \cdot e_2 = 1; e_1 \cdot e_2 = 0\}.$$

Then $|V| = 967680$, and $G$ acts transitively on $V$. 

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Proof. Since $G$ preserves intersection multiplicities, it acts on $V$. Fix $e_1 \in E$. By Proposition 3.17 there are exactly 56 exceptional classes disjoint from $e$, and by Lemma 3.18 for each $e_2$ of those 56 there are exactly 72 exceptional classes intersecting both $e_1$ and $e_2$ with multiplicity one. Therefore we have

$$|V| = 240 \cdot 56 \cdot 72 = 967680.$$ 

Let $e_0 = L - E_1 - E_2$, $e_1 = E_1$, and $e_2 = E_2$. Then $v = (e_0, e_1, e_2)$ is an element in $V$. Let $G_v$ be the stabilizer of $v$ in $G$ and $Gv$ the orbit of $v$ in $V$, then we have $|G : G_v| = |Gv| \leq |V|$. We want to show that the latter is an equality. Let $W_v$ be the set

$$W_v = \{e \in E | e \cdot e_0 = e \cdot e_1 = e \cdot e_2 = 0\}.$$ 

For $e = aL - \sum_{i=1}^{r} b_iE_i \in W_v$, the condition $e \cdot e_0 = e \cdot e_1 = e \cdot e_2 = 0$ is equivalent to $a = b_1 = b_2 = 0$. Looking at the table in Proposition 2.7 we see that there are only 6 possibilities for $e$, which are $E_3, \ldots, E_8$. So we have $W_v = \{E_3, \ldots, E_8\}$. Since $G$ preserves intersection multiplicities, $G_v$ acts on $W_v$. Let $g \in G_v$. If $gE_i = E_i$ for $3 \leq i \leq 8$, then $g$ fixes $E_1, \ldots, E_8$. But then $g$ fixes every element in $E$ and since $G$ acts faithfully on $E$ this implies that $g$ is the identity. Therefore, $G_v$ acts faithfully on $W_v$, hence $G_v \subseteq S_6$, so $|G_v| \leq 720$. We now have

$$967680 = \frac{2^{14} \cdot 3^5 \cdot 5^2 \cdot 7}{720} \leq \frac{|G|}{|G_v|} = |Gv| \leq |V| = 967680,$$

so we have equality everywhere and so $Gv = V$. We conclude that $G$ acts transitively on $V$. 

\[\Box\]

**Lemma 3.20.** Let $H$ be a group, let $A, B$ be $H$-sets, and $f : A \to B$ a morphism of $H$-sets. Then the following hold.

(i) if $H$ acts transitively on $A$, then $H$ acts transitively on $f(A)$;

(ii) if $H$ acts transitively on $A$ and $A$ is finite, then all non-empty fibers of $f$ have the same cardinality, say $n$, and $|f(A)| = \frac{|A|}{n}$;

(iii) if $H$ acts transitively on $B$, then all fibers of $f$ have the same cardinality.

Proof.

(i) Let $f(a), f(a') \in f(A)$ with $a, a' \in A$. Assume that $H$ acts transitively on $A$, then there is an $h \in H$ such that $ha = a'$. Since $f$ is a morphism of $H$-sets, we have $hf(a) = f(ha) = f(a')$, so $H$ acts transitively on $f(A)$.

(ii) Let $b, b' \in B$ such that $f^{-1}(b)$ and $f^{-1}(b')$ are non-empty. Then we have $b, b' \in f(A)$, so there is an element $h \in H$ such that $hb = b'$ by (i). But then $|f^{-1}(b')| = |f^{-1}(hb)| = |hf^{-1}(b)| = |f^{-1}(b)|$. We conclude that all non-empty fibers have the same cardinality, say $n$. It is now immediate that $|A| = |f^{-1}(B)| = \sum_{b \in f(A)} n = n|f(A)|$, so $|f(A)| = \frac{|A|}{n}$.

(iii) Let $b, b' \in B$. Since $H$ acts transitively on $B$, there is an $h \in H$ such that $hb = b'$, so $|f^{-1}(b')| = |f^{-1}(hb)| = |hf^{-1}(b)| = |f^{-1}(b)|$.

\[\Box\]
Lemma 3.21. Let \( e_1 = E_1 \) and \( e_2 = L - E_1 - E_2 \). Then there are 32 elements \( e \) in \( E \) such that \( e_1 \cdot e = 1 \) and \( e_2 \cdot e = 0 \).

Proof. Let \( e = aL - \sum_{i=1}^{8} b_i E_i \in E \), then \( e_1 \cdot e = 1 \) and \( e_2 \cdot e = 0 \) if and only if \( b_1 = 1 \) and \( a - b_1 - b_2 = 0 \). Looking at the table in Proposition 2.7, we find the following possibilities.

<table>
<thead>
<tr>
<th>( a )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of possibilities for ( e )</td>
<td>6</td>
<td>20</td>
<td>6</td>
</tr>
</tbody>
</table>

This gives a total of 32 possibilities for \( e \).

Proposition 3.22. \( G \) acts transitively on the set

\[
W = \{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = 1\}.
\]

Proof. Consider the set \( V = \{(e_1, e_2, e_3) \in E^3 \mid e_1 \cdot e_2 = e_1 \cdot e_3 = 1; \ e_2 \cdot e_3 = 0\} \). We have a projection \( f : V \longrightarrow W \) on the first two coordinates. Consider the elements \( e_1 = E_1 \) and \( e_2 = L - E_1 - E_2 \). Then \( w = (e_1, e_2) \) is an element of \( W \). Let \( e \in E \), then \( (e_1, e_2, e) \in V \) if and only if \( e_1 \cdot e = 1 \) and \( e_2 \cdot e = 0 \). By Lemma 3.21 this gives 32 possibilities for \( e \), so \( |f^{-1}((e_1, e_2))| = 32 \). Since \( G \) acts transitively on \( V \) by Lemma 3.19 it follows from Lemma 3.20 that all non-empty fibers of \( f \) have cardinality 32, and \( |f(V)| = \frac{|V|}{32} = 30240 \). By Proposition 3.17 we have \( |W| = 240 \cdot 126 = 30240 \). We conclude that \( f(V) = W \), hence \( f \) is surjective. Therefore, \( G \) acts transitively on \( W \) by Lemma 3.20.

Now that we know that \( G \) acts transitively on all pairs in \( E \) that intersect with multiplicity one, we can easily get more results on the intersection multiplicities in \( E \).

Lemma 3.23. For each pair \( (e_1, e_2) \in E^2 \) such that \( e_1 \cdot e_2 = 1 \) there are exactly 60 exceptional classes \( e \in E \) such that \( e_1 \cdot e = e_2 \cdot e = 1 \).

Proof. By Proposition 3.22 it is enough to check this for one pair. Let \( e_1 = E_1 \) and \( e_2 = L - E_1 - E_2 \). Then \( e_1 \cdot e_2 = 1 \). Now let \( e = aL - \sum_{i=1}^{r} b_i E_i \in E \), then \( e_1 \cdot e = 1 \) if and only if \( b_1 = 1 \), and \( e_2 \cdot e = 1 \) if and only if \( a - b_1 - b_2 = 1 \). Combining this we have \( e_1 \cdot e = e_2 \cdot e = 1 \) if and only if \( b_1 = 1 \) and \( a - b_2 = 2 \). Looking at the table in Proposition 2.7 we find all following possibilities.

<table>
<thead>
<tr>
<th>( a )</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of possibilities for ( e )</td>
<td>15</td>
<td>30</td>
<td>15</td>
</tr>
</tbody>
</table>

This gives a total of 60 exceptional classes intersecting both \( e_1 \) and \( e_2 \) with multiplicity one.

The following graph shows some of the information we obtained so far about the intersection multiplicities in \( E \). The vertexes are subsets of \( E \) and the number in a vertex is the cardinality of the subset. The numbers on the edges between the vertexes are the intersection multiplicities between the elements in the two subsets. By Lemma 3.19 replacing the elements \( L - E_1 - E_2, E_1, E_2 \) by any other triple \( e_0, e_1, e_2 \) with \( e_0 \cdot e_2 = e_0 \cdot e_2 = 1 \) and \( e_1 \cdot e_2 = 0 \) will give the same graph.
We conclude this section by looking at the action of $G$ on the sets

$$V = \{(e_1, e_2, e_3) \in E^3 \mid \forall i \neq j : e_i \cdot e_j = 1\}$$

and

$$W = \{(e_1, e_2, e_3, e_4) \in E^4 \mid \forall i \neq j : e_i \cdot e_j = 1\}.$$  

$G$ acts on both $V$ and $W$, since $G$ preserves the intersection pairing on $E$.

We start by introducing some notation. Let

$$Z = \{\{(e_1, e_2), (e_3, e_4), (e_5, e_6), (e_7, e_8)\} \mid \forall i : e_i \in E; \forall i \neq j : e_i \cdot e_j = 0\}.$$  

Recall the set $U$ defined by

$$U = \{(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8) \in E^8 \mid \forall i \neq j : e_i \cdot e_j = 0\}.$$  

We have $|Z| = \frac{|U|}{24}$, so from Remark 3.11 it follows that $|Z| = \frac{|G|}{24} = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7$.

Moreover, from Proposition 3.8, it follows that $G$ acts transitively on $Z$.

Let $f : W \to V$ be the projection on the first three coordinates. We define a map $g : Z \to W$ as follows. For $z = \{(e_1, e_2), (e_3, e_4), (e_5, e_6), (e_7, e_8)\} \in Z$, let $l$ be the unique element in $E$ such that $K_X = -3l + \sum_{i=1}^8 e_i$. Then we set

$$g(z) = (l - e_1 - e_2, l - e_3 - e_4, l - e_5 - e_6, l - e_7 - e_8).$$

Let $h \in G$, then

$$K_X = hK_X = -3hl + \sum_{i=1}^8 he_i,$$

so

$$g(hz) = (hl - he_1 - he_2, hl - he_3 - he_4, hl - he_5 - he_6, hl - he_7 - he_8) = hg(z).$$
Therefore, the map $g$ is a morphism of $G$-sets.

Let $Y$ be the image of $g$. The following commutative diagram shows the maps and sets that are defined.

![Diagram](image)

**Lemma 3.24.** The map $g$ is injective.

**Proof.** Consider the elements $e_1 = L - E_1 - E_2$, $e_2 = L - E_3 - E_4$, $e_3 = L - E_5 - E_6$ and $e_4 = L - E_7 - E_8$. Then $w = (e_1, e_2, e_3, e_4)$ is an element in $W$. The fiber of $g$ above $w$ consists of the elements $(\{c_1, c_2\}, \{c_3, c_4\}, \{c_5, c_6\}, \{c_7, c_8\}) \in Z$ such that

\[
\begin{align*}
  c_1 \cdot e_1 &= c_2 \cdot e_1 = 1 \text{ and } c_1 \cdot e_i = c_2 \cdot e_i = 0 \text{ for all } i \neq 1; \\
  c_3 \cdot e_2 &= c_4 \cdot e_2 = 1 \text{ and } c_3 \cdot e_i = c_4 \cdot e_i = 0 \text{ for all } i \neq 2; \\
  c_5 \cdot e_3 &= c_6 \cdot e_3 = 1 \text{ and } c_5 \cdot e_i = c_6 \cdot e_i = 0 \text{ for all } i \neq 3; \\
  c_7 \cdot e_4 &= c_8 \cdot e_4 = 1 \text{ and } c_7 \cdot e_i = c_8 \cdot e_i = 0 \text{ for all } i \neq 4.
\end{align*}
\]

Clearly, $c_i$ and $c_{i+1}$ are interchangeable for $i \in \{1, 3, 5, 7\}$. Let $c_1 = aL - \sum_{i=1}^{8} b_i E_i$. Then (1) implies $a - b_1 - b_2 = 1$ and $a = b_3 + b_4 = b_5 + b_6 = b_7 + b_8$. Looking at the table in Proposition 2.7, the only possibilities for $c_1$ and $c_2$ are $E_1$ and $E_2$. Analogously we find that the only possibilities for $c_3$ and $c_4$ are $E_3$ and $E_4$, the only possibilities for $c_5$ and $c_6$ are $E_5$ and $E_6$, and the only possibilities for $c_7$ and $c_8$ are $E_7$ and $E_8$. Therefore we have $g^{-1}(w) = \{(E_1, E_2), (E_3, E_4), (E_5, E_6), (E_7, E_8)\}$, hence the fiber above $w$ has cardinality one. Since $G$ acts transitively on $Z$, we conclude from Lemma 3.20 that all non-empty fibers of $g$ have cardinality one, so $g$ is injective.

**Remark 3.25.** By the previous proposition, the map $g : Z \to Y$ is a bijection. Since $g$ is a $G$-map, it follows that $Y$ is a $G$-set, and that $G$ acts transitively on $Y$. 

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Lemma 3.26. Consider the elements of $E$ given by

$$e_1 = L - E_1 - E_2; \quad c_1 = 3L - \sum_{i=1}^{6} E_i - 2E_7$$

$$e_2 = L - E_3 - E_4; \quad c_2 = 3L - \sum_{i=1}^{6} E_i - 2E_8.$$  

$$e_3 = L - E_5 - E_6;$$

Then $w_1 = (e_1, e_2, e_3, c_1)$ and $w_2 = (e_1, e_2, e_3, c_2)$ are elements in $W$ that are not in $Y$.

Proof. It is easy to check that $w_1$ and $w_2$ are in $W$. We want to show that the fibers of $g$ above $w_1$ and $w_2$ are empty. Let $z = ([d_1, d_2], [d_3, d_4], [d_5, d_6], [d_7, d_8]) \in Z$, and write $d_1 = rL - \sum_{i=1}^{6} s_i E_i$. Then $z \in g^{-1}(w_1)$ implies that $d_1 \cdot e_1 = 1$ and $d_1 \cdot e_2 = d_1 \cdot e_3 = d_1 \cdot c_1 = 0$, which is equivalent to $r - s_1 - s_2 = 1$, $r = s_3 + s_4 = s_5 + s_6$, and $3r - \sum_{i=1}^{6} s_i - 2s_7 = 0$. But this implies

$$0 = 3r - \sum_{i=1}^{6} s_i - 2s_7 = 3r - (s_1 + s_2) - 2r - 2s_7 = r - (s_1 + s_2) - 2s_7 = 1 - 2s_7,$$

and since $s_7$ is an integer this has no solutions. We conclude that the fiber of $g$ above $w_1$ is empty and analogously the fiber of $g$ above $w_2$ is empty. This proves that $w_1$ and $w_2$ are not in $Y$. \qed

Proposition 3.27. Let $v = (e_1, e_2, e_3)$ be an element of $V$. The following hold.

(i) The group $G$ acts transitively on $V$.

(ii) We have $|f^{-1}(v)| = 26$, and $|f^{-1}(v) \cap Y| = 24$.

(iii) For $e \in f^{-1}(v) \cap Y$ and $\{c_1, c_2\} = f^{-1}(v) \setminus Y$, we have $e \cdot c_1 = e \cdot c_2 = 1$, and $c_1 \cdot c_2 = 3$.

Proof.

(i) Consider the map $\lambda = f \circ g : Z \to V$. Note that $\lambda$ is a $G$-map, since both $f$ and $g$ are. We want to show that $\lambda$ is surjective. Let $e_1 = L - E_1 - E_2$, $e_2 = L - E_3 - E_4$, $e_3 = L - E_5 - E_6$.

Then $v = (e_1, e_2, e_3) \in V$. Note that

$$\lambda((\{E_1, E_2\}, \{E_3, E_4\}, \{E_5, E_6\}, \{E_7, E_8\})) = v,$$

so the fiber of $\lambda$ above $v$ is not empty. To compute this fiber, we first compute the fiber of $f$ above $v$. Let $e = aL - \sum_{i=1}^{6} E_i \in E$. The conditions $e \cdot e_1 = 1$, $e \cdot e_2 = 1$ and $e \cdot e_3 = 1$ are equivalent to $a - b_1 - b_2 = a - b_3 - b_4 = a - b_5 - b_6 = 1$. By looking at the table in Proposition 2.7, we find all possibilities.

<table>
<thead>
<tr>
<th>$a$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of possibilities for $e$</td>
<td>1</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>1</td>
</tr>
</tbody>
</table>
We find a total of 26 possibilities for e, so \( |f^{-1}(v)| = 26 \). Since \( \lambda^{-1}(v) \) is not empty, by Lemma 3.20, we have \( |\lambda(Z)| = \frac{|Z|}{|\lambda^{-1}(v)|} \). Since we have
\[
\lambda(Z) \leq |V| = 240 \cdot 126 \cdot 60 = 1814400
\]
this implies
\[
\lambda^{-1}(v) \geq \frac{|Z|}{1814400} = 24. \tag{2}
\]
Since \( g : Z \to Y \) is a bijection we have \( \lambda^{-1}(v) \leq 26 \). Consider the elements \( c_1 = 3L - \sum_{i=1}^{6} E_i - 2E_7 \) and \( c_2 = 3L - \sum_{i=1}^{6} E_i - 2E_8 \), then \( w_1 = (e_1, e_2, e_3, c_1) \) and \( w_2 = (e_1, e_2, e_3, c_2) \) are both elements in \( f^{-1}(v) \). By Lemma 3.26 we know that the fibers of \( g \) above \( w_1 \) and \( w_2 \) are empty. It follows that \( \lambda^{-1}(v) \leq 24 \), which together with (2) implies \( \lambda^{-1}(v) = 24 \). This means that
\[
|\lambda(Z)| = \frac{|Z|}{24} = |V|
\]
so \( \lambda \) is surjective. Since \( G \) acts transitively on \( Z \), we conclude that \( G \) acts transitively on \( V \), too.

(ii) In part (i) we showed that \( |f^{-1}(v)| = 26 \) and \( |\lambda^{-1}(v)| = 24 \). Since \( g \) is a bijection, we have \( |f^{-1}(v) \cap Y| = |\lambda^{-1}(v)| = 24 \). Since \( f \) is a \( G \)-map, and \( G \) acts transitively on \( V \), the result holds for all elements in \( V \).

(iii) This is an easy check, after writing down the 26 elements we found in part (i).

\[\square\]

**Proposition 3.28.** The set \( W \) has two orbits under the action of \( G \).

**Proof.** From Remark 3.25 it follows that \( Y \) is an orbit under the action of \( G \) on \( W \). Therefore \( O = W - Y \) is also a \( G \)-set. Consider the restriction of \( f \) to \( O \),
\[
f|_O : O \to V.
\]
Let \( e_1, e_2, e_3, c_1, c_2 \) be as in Lemma 3.26, and let \( v = (e_1, e_2, e_3) \), \( w_1 = (e_1, e_2, e_3, c_1) \), and \( w_2 = (e_1, e_2, e_3, c_2) \). Then we have \( v \in V \), and \( w_1, w_2 \in f|_O^{-1}(v) \) by Lemma 3.26. From Proposition 3.27 we know that \( |f^{-1}(v) \cap Y| = 24 \), so \( |f|_O^{-1}(v) \cap Y| = 2 \). This implies \( f|_O^{-1}(v) = \{w_1, w_2\} \). For \( r_7 = E_7 - E_8 \in R_8 \), the reflection with respect to \( r_7 \) is an element in \( G \), say \( h \). Since \( e_1 \cdot r_7 = e_1 \cdot r_7 = e_3 \cdot r_7 = 0 \), for \( i \in \{1, 2, 3\} \) we have
\[
h e_i = e_i - 2r_7 e_i \cdot r_7 r_7 = e_i,
\]
so \( h \) is contained in the stabilizer \( G_v \) of \( v \) in \( G \). We have
\[
h c_1 = c_1 - 2r_7 c_1 \cdot r_7 = c_1 + 2r_7 \sum_{i=1}^{6} E_i - 2E_8 = c_2,
\]
so \( h \) interchanges \( w_1 \) and \( w_2 \), hence \( G_v \) acts transitively on \( f|_O^{-1}(v) \). Since \( G \) acts transitively on \( V \), this holds for all elements in \( V \). Now let \( o, o' \in O \). Let \( a = f(o) \)

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and $b = f(o')$. Since $G$ acts transitively on $V$ there is an element $h_1 \in G$ such that $b = h_1a$. Then $f(o') = h_1f(o) = f(h_1o)$, so $o'$ and $h_1o$ are in the same fiber of $f$. Since $G_b$ acts transitively on this fiber, there is an element $h_2 \in G_b$ such that $o' = h_2h_1o$. We conclude that $G$ acts transitively on $O$. It follows that $W$ consists of the orbits $O$ and $Y$. \hfill \Box
In this section we prove Theorem 1 and Theorem 2. Recall that we defined $X$ to be a del Pezzo surface of degree one over an algebraically closed field $k$, and $E$ the set of exceptional classes in Pic $X$. We want to compute the maximal number of exceptional curves on $X$ that go through one point $P$. Let $Q$ be the quadratic cone such that $X$ is a double cover of $Q$ under the map $\varphi$, branched over a smooth curve $B$ of degree 6. We distinguish two cases. In Subsection 4.1 we will consider the case when $\varphi(P)$ lies on $B$, and prove Theorem 1. In Subsection 4.2 we consider the case when $\varphi(P)$ does not lie on $B$, and prove Theorem 2.

Since there is a one-to-one correspondence between exceptional curves on $X$ and exceptional classes in Pic $X$, a set of exceptional curves can go through one point only if the corresponding set of exceptional classes pairwise intersect with multiplicity greater than zero. Therefore, we start by looking at sets of exceptional curves that pairwise intersect positively, and we do this by studying the graph on $E$.

**Definition 4.1.** By $G$ we denote the weighted graph whose vertices are the elements of $E$, and where two vertices are connected by an edge of weight $n$ if and only if the corresponding elements of $E$ intersect with multiplicity $n > 0$.

**Remark 4.2.** Since $G$ is the group of automorphisms of $E$ that preserve intersection multiplicities, it follows that $G$ is the automorphism group of $G$.

**Definition 4.3.** A clique in $G$ is a weighted subgraph of $G$ in which every two vertices are connected by an edge. The size of a clique is the number of vertices contained in it.

As we mentioned before, a number of exceptional curves go through one point only if the corresponding classes form a clique in $G$. Of course, the converse is not always true. However, the maximal size of the cliques in $G$ does give us a first upper bound for the maximal number of exceptional curves that go through one point. To compute this, we first need a couple of lemmas.

**Definition 4.4.** A maximal clique in $G$ is a clique that is maximal with respect to inclusion.

**Lemma 4.5.** The size of a clique in $G$ that contains no edges of weight one is at most three.

**Proof.** Let $K$ be a maximal clique in $G$ without edges of weight one. We distinguish two cases.

First assume that $K$ contains an edge of weight two. Then by Proposition 3.16, we can without loss of generality assume that $K$ contains the vertices corresponding to $e_1 = E_1$ and $e_2 = 3L - 2E_1 - \sum_{i=2}^7 E_i$. Let $e = aL - \sum_{i=1}^8 b_i E_i \in E$, then $e \cdot e_1 > 1$ and $e \cdot e_2 > 1$ if and only if $b_1 > 1$ and $3a - 2b_1 - \sum_{i=2}^7 b_i > 1$. By looking at the table in Proposition 2.7 we find only one possibility for $e$, which is $e = 6L - 2 \sum_{i=1}^7 E_i - 3E_8$. We conclude that $K$ consists of the edges corresponding...
to $e_1$, $e_2$ and $e$ and thus has size three.

Now assume that $K$ contains an edge of weight three. Then by Remark 3.14, we can without loss of generality assume that $K$ contains the vertices corresponding to $c_1 = E_1$ and $c_2 = 6L - 3E_1 - 2 \sum_{i=2}^{8} E_i$. Let $c = aL - \sum_{i=1}^{8} b_i E_i \in E$, then $c \cdot c_1 > 1$ and $c \cdot c_2 > 1$ if and only if $b_1 > 1$ and $6a - 3b_1 - \sum_{i=2}^{8} b_i > 1$. This has no solutions in the table in Proposition 2.7, so $K$ consists of only the vertices corresponding to $c_1$ and $c_2$. \hfill \Box

**Lemma 4.6.** Let $e_1$, $e_2$ be elements in $E$ that are not disjoint, and let $c_1, c_2$ be such that $e_1 \cdot c_1 = e_2 \cdot c_2 = 3$. Then the following are equivalent.

(i) $e_1 \cdot e_2 = 1$;

(ii) $e_1 \cdot c_2 > 0$ and $e_2 \cdot c_1 > 0$;

(iii) $e_1 \cdot c_2 = e_2 \cdot c_1 = 1$.

*Proof.* First assume that $e_1 \cdot e_2 = 1$. Since $G$ acts transitively on the set of pairs of exceptional classes intersecting with multiplicity one, we can assume without loss of generality that $e_1 = E_1$ and $e_2 = L - E_1 - E_2$. Then $c_1 = 6L - 3E_1 - 2 \sum_{i=2}^{8} E_i$ and $c_2 = 5L - E_1 - E_2 - 2 \sum_{i=3}^{8} E_i$. It is straightforward to check that $e_1 \cdot c_2 = e_2 \cdot c_1 = 1$, so (i) implies (iii).

Since (iii) obviously implies (ii), it is now enough to prove that (ii) implies (i). To this end, assume that $e_1 \cdot c_2 > 0$ and $e_2 \cdot c_1 > 0$. If $e_1 \cdot e_2 = 2$ then from the bijection in Lemma 3.15 we have $e_1 \cdot e_2 = 0$, which is a contradiction. Therefore we have $e_1 \cdot c_2 = 1$. Without loss of generality, we take $e_1 = E_1$ and $c_2 = L - E_1 - E_2$. Then $e_2 = 5L - E_1 - E_2 - 2 \sum_{i=3}^{8} E_i$, so $e_1 \cdot e_2 = 1$. \hfill \Box

The previous lemma states that if we have two pairs $(e_1, c_1)$ and $(e_2, c_2)$ of exceptional classes intersecting with multiplicity three, the four classes together form a clique in $G$ if and only if $e_1 \cdot e_2 = e_1 \cdot c_2 = e_2 \cdot c_1 = c_1 \cdot c_2 = 1$. We call the pair $\{(e_1, c_1), (e_2, c_2)\}$ an intersecting pair.

**Corollary 4.7.** $G$ acts transitively on the set

$$S = \{(e_1, e_2, c_1, c_2) \in E^4 \mid (\{e_1, c_1\}, \{e_2, c_2\}) \text{ is an intersecting pair}\}.$$

*Proof.* Consider the set $T = \{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = 1\}$ and the map

$$\lambda : T \rightarrow S, \ (e_1, e_2) \mapsto (e_1, e_2, c_1, c_2),$$

where $c_1 \cdot e_1 = c_2 \cdot e_2 = 3$. Note that $\lambda$ is well defined by Lemma 4.6. Let $(e_1, e_2, c_1, c_2)$ be an element in $S$. Then $\{(e_1, c_1), (e_2, c_2)\}$ is an intersecting pair, so $e_1 \cdot e_2 = 1$, and $\lambda((e_1, e_2)) = (e_1, e_2, c_1, c_2)$. We conclude that $\lambda$ is surjective. The statement now follows from the fact that $G$ acts transitively on $T$. \hfill \Box

**Corollary 4.8.** Every maximal clique $K$ in $G$ of size bigger than two that does not contain any edge of weight two is of the form

$$K = \{e_1, \ldots, e_n, c_1, \ldots, c_n \mid \forall i \neq j : (\{e_i, c_i\}, \{e_j, c_j\}) \text{ is an intersecting pair}\}.$$
Proof. Let $K$ be a maximal clique not containing any edge of weight two. If $K$ would not contain any edge of weight one, it would consist of two vertexes connected by an edge of weight three, hence it would have size two. Therefore we can assume that $K$ contains at least one edge of weight one. Let $e_1, \ldots, e_n$ be a subclique of maximal size in $K$ that only contains edges of weight one. By Lemma 4.6, for all $i \neq j$, the unique elements $c_i, c_j \in E$ such that $e_i \cdot c_i = e_j \cdot c_j = 3$ satisfy $e_i \cdot c_j = e_j \cdot c_i = c_i \cdot c_j = 1$. Since $K$ is maximal, it follows that $K$ also contains the $n$ elements $c_1, \ldots, c_n$. If there would be another element $d \in K$, then, since there is only one element intersecting $d$ with multiplicity three, either $d \cdot e_i = 1$ for all $i$, or $d \cdot c_i = 1$ for all $i$. But this contradicts the fact that the set $\{e_1, \ldots, e_n\}$ is maximal. We conclude that $K = \{e_1, \ldots, e_n, c_1, \ldots, c_n\}$.

Lemma 4.9. The maximal size of the cliques in $\mathcal{G}$ that contain an edge of weight two is thirteen.

Proof. Let $K$ be a clique of maximal size in $\mathcal{G}$ that contains an edge of weight two. By Proposition 3.16 we can without loss of generality assume that $K$ contains $e_1 = E_1$ and $e_2 = 3L - 2E_1 - \sum_{i=2}^{7} E_i$. Let $e = aL - \sum_{i=1}^{8} E_i \in E$. The conditions $e \cdot e_1 \geq 1$ and $e \cdot e_2 \geq 1$ are equivalent to $b_1 \geq 1$ and $3a - 2b_1 - \sum_{i=2}^{7} b_i \geq 1$. By looking at the table in Proposition 2.7 we find all possibilities.

<table>
<thead>
<tr>
<th>a</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of possibilities for e</td>
<td>1</td>
<td>20</td>
<td>36</td>
<td>41</td>
<td>22</td>
<td>7</td>
</tr>
</tbody>
</table>

We find a total of 127 possibilities. As it is too tedious to compute all pairwise intersection multiplicities by hand, we do this with MAGMA and find that the maximal size of a clique in the graph on these 127 exceptional curves is eleven. This gives a total of 13 elements in $K$. We conclude that the maximal size of a clique that contains an edge of weight two is thirteen.

Let $V, W, Z, f, g$ and $Y$ be as in the diagram on page 25. Let $\mathcal{S}$ be the set of all cliques of size sixteen. The following proposition gives us a first upper bound for the maximal number of exceptional curves going through one point.

Proposition 4.10. The following hold.

(i) The maximal size of a clique in $\mathcal{G}$ is sixteen.

(ii) Every clique of size sixteen is of the form

$$\{e_1, \ldots, e_8, c_1, \ldots, c_8 \mid \forall i \neq j : (\{e_i, c_i\}, \{e_j, c_j\}) \text{ is an intersecting pair}\},$$

and every clique that has no edges of weight two is contained in a clique of size sixteen.

(iii) For $y = (e_1, \ldots, e_4) \in Y$, there is a unique element in $\mathcal{S}$ containing $e_1, \ldots, e_4$. This gives rise to a map $s : Y \rightarrow \mathcal{S}$, which is surjective.

Proof.

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(i) Let $K$ be a maximal clique in $\mathcal{G}$. We consider two cases. First assume that $K$ contains an edge of weight two. Then $K$ has size at most thirteen by Lemma 4.9.

Now assume that $K$ contains no edges of weight two. From Corollary 4.8 it follows that $K$ is of the form

$$K = \{e_1, \ldots, e_n, c_1, \ldots, c_n \mid \forall i \neq j : (\{e_i, c_i\}, \{e_j, c_j\}) \text{ is an intersecting pair}\}.$$  

By Corollary 4.7 we can without loss of generality assume that $K$ contains the four exceptional classes

$$e_1 = L - E_1 - E_2; \quad e_2 = L - E_3 - E_4; \quad c_1 = 5L - E_1 - E_2 - 2 \sum_{i=3}^{8} E_i; \quad c_2 = 5L - 2E_1 - 2E_2 - E_3 - E_4 - 2 \sum_{i=5}^{8} E_i.$$ 

Let $e_3$ be a different element in $K$. Then $e_1 \cdot e_3 = e_2 \cdot e_3 = 1$, so by Proposition 3.27 we can take without loss of generality $e_3 = L - E_5 - E_6$. Then $K$ also contains the unique exceptional curve intersecting $e_3$ with multiplicity three, which is $e_3 = 5L - 2 \sum_{i=1}^{4} E_i - E_5 - E_6 - 2E_7 - 2E_8$.

Let $e_4$ be a different element in $K$. Then $e_1 \cdot e_4 = e_2 \cdot e_4 = e_4 \cdot e_3 = 1$, so by Proposition 3.27 there are 26 possibilities for $e_4$. They are

$$L - E_7 - E_8; \quad 2L - E_i - E_j - E_k - E_7 - E_8 \text{ for } i \in \{1, 2\}, j \in \{3, 4\}, k \in \{5, 6\}; \quad 3L - 2E_i - \sum_{j \in \{1, \ldots, i-1, i+2, \ldots, 8\}} E_j \text{ for } i \in \{1, 3, 5, 7\}; \quad 3L - 2E_i - \sum_{j \in \{1, \ldots, i-2, i+1, \ldots, 8\}} E_j \text{ for } i \in \{2, 4, 6, 8\}; \quad 4L - 2E_i - 2E_j - 2E_k - \sum_{i \notin \{i, j, k\}} E_i \text{ for } i \in \{1, 2\}, j \in \{3, 4\}, k \in \{5, 6\}; \quad 5L - 2 \sum_{i=1}^{6} E_i - E_7 - E_8.$$  

The elements $e_4 = 3L - \sum_{i=1}^{6} E_i - 2E_7$ and $c_4 = 3L - \sum_{i=1}^{6} E_i - 2E_8$ intersect all other 24 elements with multiplicity one and satisfy $e_4 \cdot c_4 = 3$, so they are both in $K$. As we have seen in Proposition 3.27 all other 24 elements $e$ have the property that $(e_1, e_2, e_3, e)$ is in $Y$, so by Proposition 3.28 without loss of generality we can assume that $e_5 = L - E_7 - E_8$ is in $K$. Then $K$ also contains $e_5 = 5L - 2 \sum_{i=1}^{6} E_i - E_7 - E_8$, since $e_5 \cdot c_5 = 3$. Of the remaining 22 elements, the only elements intersecting $e_5$ with multiplicity one are

$$3L - 2E_1 - \sum_{i=3}^{8} E_i; \quad 3L - E_1 - E_2 - 2E_3 - \sum_{i=5}^{8} E_i; \quad 3L - 2E_2 - \sum_{i=3}^{8} E_i; \quad 3L - \sum_{i=1}^{4} E_i - 2E_5 - E_7 - E_8; \quad 3L - E_1 - E_2 - 2E_3 - \sum_{i=5}^{8} E_i; \quad 3L - \sum_{i=1}^{4} E_i - 2E_6 - E_7 - E_8.$$ 

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These intersect pairwise with multiplicity three or one, so they are all contained in $K$. This gives sixteen elements in $K$.

(ii)-(iii) These points follow from the construction of the clique in part (i).

**Corollary 4.11.** The number of exceptional curves on $X$ that go through one point is at most sixteen.

*Proof.* This follows directly from Proposition 4.10.

**Corollary 4.12.** $G$ acts transitively on $S$, and $|S| = 2025$.

*Proof.* By Proposition 4.10, there is a surjective map $s : Y \to G$. Therefore, $G$ acts transitively on $S$ by Lemma 3.20. Let $K$ be a clique of size sixteen. From Proposition 4.10 (ii), it follows that there are $16 \cdot 14 \cdot 12$ tuples $(c_1, c_2, c_3) \in K^3$ with $c_1 \cdot c_2 = c_1 \cdot c_3 = c_2 \cdot c_3 = 1$. Moreover, for a fixed tuple $(e_1, e_2, e_3) \in K^3$ with $e_1 \cdot e_2 = e_1 \cdot e_3 = e_2 \cdot e_3 = 1$ there are ten exceptional classes in $K$ intersecting $e_1$, $e_2$ and $e_3$ with multiplicity one. From Proposition 3.27 (iii) and the fact that all cliques of size sixteen are maximal, it follows that two of those ten, say $d_1$ and $d_2$, are such that $(e_1, e_2, e_3, d_i) \notin Y$. The other eight $e$ are such that $(e_1, e_2, e_3, e)$ is an element of $Y$. We conclude that $|s^{-1}(K)| = 16 \cdot 14 \cdot 12 \cdot 8 = 21504$. Since $s$ is surjective, we have $|S| = |s(Y)| = \frac{|Y|}{16 \cdot 14 \cdot 12 \cdot 8} = 2025$.

### 4.1 Points on the ramification curve

By Proposition 2.12, a hyperplane section $H$ of $Q$ that intersects $B$ with multiplicity two in three (possibly infinitely near) points and does not contain the vertex of $Q$ pulls back under $\varphi$ to the sum of two exceptional curves that intersect in three (possibly infinitely near) points. From now on, we call two exceptional curves or exceptional classes intersecting with multiplicity three a *pair*.

Proposition 4.14 will give an upper bound for the maximal number of exceptional curves going through one point on the ramification curve of $\varphi$. The proof uses the following well-known proposition.

**Proposition 4.13.** (Hurwitz). Let $f : C \to D$ be a finite separable morphism of complete, nonsingular curves over an algebraically closed field $k$. Let $n = \deg f$. Then

$$2g(C) - 2 = n \cdot (2g(D) - 2) + \deg R,$$

where $R$ is the ramification divisor of $f$. We have $\deg R \geq \sum_{P \in C} (e_P - 1)$, with equality if $f$ only has tame ramification.

*Proof.* See [Har77], Proposition IV.2.2 and Corollary IV.2.4.

I got the idea for Proposition 4.14 from Niels Lubbes, who was so kind to share this with us. Later I found that part of it was also done in [TVAV09].
Proposition 4.14. Assume that \( \text{char } k \neq 2 \). Then the number of exceptional curves that go through one point on the ramification curve of \( \varphi \) is at most twelve.

Proof. Fix a point \( p \in B \), and let \( M \) be the tangent line to \( B \) at \( p \). The set of planes through \( M \) is a pencil \( P \) of planes in \( \mathbb{P}^3 \), hence can be parametrized by \( \mathbb{P}^1 \). Let \( \lambda : B \dashrightarrow \mathbb{P}^1 \) be the rational map sending every point \( x \notin M \) to the unique plane through \( M \) containing \( x \). Then since \( B \) is smooth, this extends to a morphism \( \lambda : B \rightarrow \mathbb{P}^1 \). As is shown in Lemma 4.5 (1) in [TVAV09], the morphism \( \lambda \) is separable, and \( \deg \lambda = 4 \). Therefore, by Proposition 4.13 we have

\[
2g(B) - 2 = (\deg \lambda)(2g(\mathbb{P}^1) - 2) + \deg R,
\]

where \( R \) is the ramification divisor of \( \lambda \). We have \( g(B) = 4 \), so this gives

\[
\sum_{x \in B} (e_x - 1) \leq \deg R = 6 - (4 \cdot -2) = 14.
\]

Let \( H \) be a plane through \( p \) that is tritangent to \( B \). Then \( H \) contains two points where \( \lambda \) ramifies with ramification degree 2, or one point where \( \lambda \) ramifies with ramification degree four, hence \( H \) contributes 2 or 3 to the degree of \( R \). Therefore, there are at most 7 tritangent planes going through \( p \), which is Lemma 4.5 (1) in [TVAV09]. Note that \( P \) contains exactly one plane \( H' \) containing the vertex of \( Q \). The intersection of \( H' \) with \( Q \) is a double line, each component intersecting \( B \) with multiplicity three. Therefore, the morphism \( \lambda \) branches over \( H' \), hence we counted \( H' \) as one of the 7 tritangent planes through \( p \). We conclude that there are at most 6 planes tritangent to \( B \) and not going through the vertex of \( Q \). By the bijection in Remark 2.13 this gives an upper bound of twelve exceptional curves going through \( \varphi^{-1}(p) \).

We will later give a sharper upper bound for the number of exceptional curves through one point on \( B \).

Remark 4.15. Let \( K \) be a set of exceptional curves all going through a point \( P \) on the ramification curve of \( \varphi \). Let \( e \) be an exceptional curve in \( K \). There is a unique exceptional curve \( c \) intersecting \( e \) with multiplicity three, and by Remark 2.13 their intersection \( c \cap e \) consists exactly of those points in \( e \) that are on the ramification curve of \( \varphi \). We conclude that \( P \) is also contained in \( c \).

From the previous remark we conclude that a maximal clique that corresponds to a set of exceptional curves all going through one point \( P \) on the ramification curve of \( \varphi \) is a union of pairs. Therefore, by Proposition 4.10 and Lemma 4.6 such a maximal clique is contained in a clique of size sixteen.

Let \( S \) be a clique of size sixteen in \( G \), and \( G_S \) the stabilizer of \( S \) in \( G \). Consider the sets

\[
I = \{(e_1, e_2, e_3) \in S^3 \mid e_1 \cdot e_2 = e_1 \cdot e_3 = e_2 \cdot e_3 = 1\},
\]

and

\[
J = \{(e_1, e_2) \in S^2 \mid e_1 \cdot e_2 = 1\}.
\]

Proposition 4.16. The group \( G_S \) acts transitively on \( I \).
Proof. Since \( S \) consists of eight pairs, we have \( |I| = 16 \cdot 14 \cdot 12 = 2688 \). Fix an element \( \iota = (e_1, e_2, e_3) \) in \( I \). We want to show that the orbit \( G_{S\iota} \) has size 2688. Let \( G_{S\iota} \) be the stabilizer of \( \iota \) in \( G_S \). We have \( G_{S\iota} = [G_S : G_{S\iota}] \), and

\[
[G : G_{S\iota}] = [G : G_S][G_S : G_{S\iota}].
\]

By Corollary 4.12 we have \( [G : G_S] = GS = 2025 \). Moreover, we have

\[
[G : G_{S\iota}] = [G : G_S][G_S : G_{S\iota}].
\]

By Proposition 3.27 we have \( [G : G_S] = G_\iota = 240 \cdot 126 \cdot 60 = 1814400 \). We now compute \( [G_\iota : G_{S\iota}] = G_\iota S \). Since \( G_{\iota} \) acts transitively on the 24 exceptional curves \( e \) such that \( (e_1, e_2, e_3, e) \) is an element in \( Y \), and every element in \( y \) is contained in a unique clique of size sixteen, the orbit \( G_\iota S \) contains all different cliques that are the images under \( s \) of these 24 elements in \( Y \), where \( s \) is the map in Proposition 4.10. Now fix \( e \) such that \( y = (e_1, e_2, e_3, e) \) is in \( Y \), and let \( K = s(y) \). By Proposition 3.27 the clique \( K \) contains the two exceptional classes \( d_1, d_2 \) such that \( (e_1, e_2, e_3, d_1) \) and \( (e_1, e_2, e_3, d_2) \) are in \( W \setminus Y \), and we have \( d_1 \cdot d_2 = 3 \). By Proposition 4.10 (ii), we know that \( K \) also contains the unique \( c_1, c_2, c_3 \in E \) such that for \( i \in \{1, 2, 3\} \) we have \( c_i \cdot c_i = 3 \). We conclude that the seven other elements in \( K \) are among the 24 exceptional classes \( f \) such that \( (e_1, e_2, e_3, f) \) is an element in \( Y \). Therefore, they determine the same unique clique of size sixteen as \( e \). We conclude that there are \( 24 \cdot 3 = 72 \) different cliques containing \( \iota \). So we have \( [G, S] \geq 3 \), and we conclude that

\[
[G : G_{S\iota}] \geq 240 \cdot 126 \cdot 60 \cdot 3 = 5443200
\]

It follows that \( [G_S : G_{S\iota}] \geq \frac{5443200}{2025} = 2688 \). Since \( [G_S : G_{S\iota}] = |G_{S\iota}| \leq 2688 \), this finishes the proof. \( \square \)

**Corollary 4.17.** The group \( G_S \) acts transitively on \( J \).

**Proof.** We have a projection map \( \lambda : I \rightarrow J \) on the first two coordinates. Since \( S \) consists of eight pairs, if we fix two elements \( e_1, e_2 \) such that \( (e_1, e_2) \in J \), there are \( 16 - 4 = 12 \) elements \( e \in S \) such that \( (e_1, e_2, e) \in I \). Therefore, \( \lambda \) is surjective. From Proposition 4.11 it follows that \( G_S \) acts transitively on \( J \). \( \square \)

**Corollary 4.18.** The group \( G_S \) acts transitively on \( S \).

**Proof.** We have a projection map \( \lambda : J \rightarrow S \) on the first coordinate. For every element \( e \) in \( S \) there are 14 elements \( c \) such that \( (e, c) \in J \), so \( \lambda \) is surjective. From Corollary 4.17 it follows that \( G_S \) acts transitively on \( S \). \( \square \)

The following proposition allows us to say something about all cliques of a certain type by considering only one of them, which is very useful.

**Proposition 4.19.** For \( n \in \{2, 3, 5, 6, 7, 8\} \), \( G \) acts transitively on the set

\[
D_n = \left\{ \{e_1, \ldots, e_n, c_1, \ldots, c_n\} \mid \forall i \neq j : (\{e_i, c_i\}, \{e_j, c_j\}) \text{ is an intersecting pair} \right\}.
\]
Proof. The case \( n = 2 \) follows from Corollary 4.7 and \( n = 3 \) follows from Proposition 3.27(i) and Lemma 4.6. By Proposition 4.16 and Lemma 4.6, the stabilizer \( G_S \) of \( S \) in \( G \) acts transitively on the set

\[
\{(e_1, e_2, e_3, c_1, c_2, c_3) \in S^6 \mid \forall i \neq j : \{\{e_i, c_i\}, \{e_j, c_j\}\} \text{ is an intersecting pair.}\}
\]

Since \( S \) consists of eight pairs, the cliques of five pairs in \( S \) are the complements of the cliques of three pairs in \( S \), so this implies that \( G_S \) acts transitively on the set of cliques of five pairs in \( S \). By Corollary 4.12 this implies the statement for \( n = 5 \). The cases \( n = 6 \) and \( n = 7 \) are proved analogously since we showed that \( G_S \) acts transitively on \( J \) and on \( S \). Finally, \( n = 8 \) is Corollary 4.12.

\[
\text{prop 4.20. There are two orbits under the action of } G \text{ on the set}
\]

\[
\{(e_1, \ldots, e_4, c_1, \ldots, c_4) \in E^8 \mid \forall i \neq j : \{\{e_i, c_i\}, \{e_j, c_j\}\} \text{ is an intersecting pair.}\}
\]

Proof. This follows from Proposition 3.28 and Lemma 4.6.

From Proposition 4.22 we will deduce a sharp upper bound for the number of exceptional curves going through one point on the ramification curve of \( \varphi \) if \( \text{char } k \neq 2 \). We need a lemma first. This lemma will also be used in the next subsection.

Let \( \mathbb{P}^2 \) be the projective plane over \( k \) with coordinates \( x, y, z \). Let \( R_1, \ldots, R_9 \) be nine points in \( \mathbb{P}^2 \), with \( R_i = (x_i : y_i : z_i) \) for \( i \in \{1, \ldots, 9\} \). We define the following lists of polynomials in \( x, y, z \).

\[
\begin{align*}
\text{Mon}_1 &= [x, y, z]; \\
\text{Mon}_2 &= [x^2, xy, xz, y^2, yz, z^2]; \\
\text{Mon}_3 &= [x^3, x^2y, x^2z, xy^2, xzy, zx^2, y^3, y^2z, yz^2, z^3]; \\
\text{Mon}_4 &= [x^4, x^3y, x^3z, x^2y^2, x^2yz, x^2z^2, xy^3, xy^2z, xyz^2, xz^3, y^4, y^3z, y^2z^2, yz^3, z^4].
\end{align*}
\]

For \( i = 3, 4 \), let \( \text{Mon}_i^1 \) be the list of derivatives of \( \text{Mon}_i \) with respect to \( x \), let \( \text{Mon}_i^2 \) be the list of derivatives of \( \text{Mon}_i \) with respect to \( y \), and let \( \text{Mon}_i^3 \) be the list of derivatives of \( \text{Mon}_i \) with respect to \( z \). Define the following matrices, where each row is defined up to scaling.

\[
\begin{align*}
M &= (a_{i,j})_{i,j \in \{1,2,3\}} \quad \text{with } a_{i,j} = \text{Mon}_i[j](R_i); \\
N &= (b_{i,j})_{i,j \in \{1,\ldots,6\}} \quad \text{with } b_{i,j} = \text{Mon}_2[j](R_i); \\
L &= (c_{i,j})_{i,j \in \{1,\ldots,10\}} \quad \text{with } c_{i,j} = \begin{cases} 
\text{Mon}_3[j](R_i) & \text{for } i \leq 8 \\
\text{Mon}_3[j](R_8) & \text{for } i = 9 \\
\text{Mon}_3[j](R_8) & \text{for } i = 10
\end{cases} \\
H &= (d_{i,j})_{i,j \in \{1,\ldots,15\}} \quad \text{with } d_{i,j} = \begin{cases} 
\text{Mon}_4[j](R_i) & \text{for } i \leq 6 \\
\text{Mon}_4^{i-6}[j](R_7) & \text{for } i \in \{7,8,9\} \\
\text{Mon}_4^{i-9}[j](R_8) & \text{for } i \in \{10,11,12\} \\
\text{Mon}_4^{i-12}[j](R_9) & \text{for } i \in \{13,14,15\}
\end{cases}.
\end{align*}
\]

Lemma 4.21. The following hold.
Proposition 4.22

(i) The points $R_1, R_2,$ and $R_3$ are collinear if and only if $\det(M) = 0.$

(ii) The points $R_1, \ldots, R_6$ are on a conic if and only if $\det(N) = 0.$

(iii) If $y_8 \neq 0,$ then the points $R_1, \ldots, R_8$ are on a cubic with a singular point at $R_8$ if and only if $\det(L) = 0.$

(iv) If char $k = 0,$ then the points $R_1, \ldots, R_9$ are on a quartic that is singular at $R_7,$ $R_8$ and $R_9$ if and only if $\det(H) = 0.$

Proof.

(i) The determinant of $M$ is zero if and only if there is a non-zero element in the nullspace of $M,$ that is, there is a non-zero vector $(m_1, m_2, m_3)$ such that for all $i \in \{1, 2, 3\},$ we have $m_1 a_{i,1} + m_2 a_{i,2} + m_3 a_{i,3} = 0.$ But this vector exists if and only if the line defined by $m_1 x + m_2 y + m_3 z$ contains all three points.

(ii) This proof goes analogously to the proof of (i).

(iii) The determinant of $L$ is zero if and only if there is a non-zero vector $(l_1, \ldots, l_{10})$ such that for all $i \in \{1, \ldots, 10\},$ we have $l_1 c_{i,1} + \cdots + l_{10} c_{i,10} = 0.$ This is the case if and only if the cubic $C$ defined by $\lambda : \sum_{i=1}^{10} l_i Mon_3[i]$ contains all eight points, and moreover, the derivatives $\lambda_x, \lambda_z$ of $\lambda$ with respect to $x$ and $z$ vanish in $R_8.$ Since we have $x \lambda_x + y \lambda_y + z \lambda_z = 3\lambda$ and $y_8 \neq 0,$ this implies that also the derivative $\lambda_y$ of $\lambda$ with respect to $y$ vanishes in $R_8,$ hence $C$ is singular in $R_8.$

(iv) The determinant of $H$ is zero if and only if there is a non-zero vector $(h_1, \ldots, h_{15})$ such that for all $i \in \{1, \ldots, 15\},$ we have $h_1 d_{i,1} + \cdots + h_{15} d_{i,15} = 0.$ This is the case if and only if the quartic $K$ defined by $\lambda : \sum_{i=1}^{15} h_i Mon_4[i]$ contains $R_1, \ldots, R_6,$ and moreover, the derivatives $\lambda_x, \lambda_y, \lambda_z$ of $\lambda$ with respect to $x, y,$ and $z$ vanish in $R_7, R_8,$ and $R_9.$ Since we have $x \lambda_x + y \lambda_y + z \lambda_z = 4\lambda$ and char $k = 0,$ this implies that also $R_7, R_8,$ and $R_9$ are in contained in $\lambda.$

\[
\square
\]

**Proposition 4.22.** Assume that char $k \neq 2.$ Let $Q_1, \ldots, Q_8$ be eight points in $\mathbb{P}^2$ in general position. Let $L_i$ be the line through $Q_{2i}$ and $Q_{2i-1}$ for $i \in \{1, 2, 3, 4\},$ and $C_{i,j}$ the unique cubic through $Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_8$ that is singular in $Q_j.$ Assume that the four lines $L_1,$ $L_2,$ $L_3$ and $L_4$ all intersect in one point $P.$ Then the three cubics $C_{7,8}, C_{8,7},$ and $C_{6,5}$ do not all go through $P.$

**Proof.** Assume that $C_{7,8}, C_{8,7},$ and $C_{6,5}$ go through $P.$ First note that if $P$ were equal to one of the $Q_i,$ then three of the eight $Q_i$ would be on a line, which would contradict the fact that $Q_1, \ldots, Q_8$ are in general position. We conclude that $P$ is not equal to one of the $Q_i.$

Let $(x : y : z)$ be the coordinates in $\mathbb{P}^2.$ Without loss of generality we can choose four points in general position in $\mathbb{P}^2,$ and we set

\[
\begin{align*}
Q_1 &= (0 : 1 : 1); & Q_3 &= (1 : 0 : 1) \\
Q_5 &= (1 : 1 : 1); & P &= (0 : 0 : 1).
\end{align*}
\]

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Then we have the following.

\[ \begin{align*}
L_1 & \text{ is the line given by } x = 0; \\
L_2 & \text{ is the line given by } y = 0; \\
L_3 & \text{ is the line given by } x = y. \\
\end{align*} \]

Since \( L_4 \) contains \( P \), and is unequal to \( L_1 \) and \( L_2 \), there is an \( m \in k^* \) such that \( L_4 \) is the line \( my = x \). Since \( Q_2, Q_7 \) and \( Q_8 \) are not in \( L_2 \), and \( Q_4 \) is not in \( L_1 \), there are \( a, b, c, u, v \in k \) such that

\[ \begin{align*}
Q_2 &= (0 : 1 : a); \\
Q_4 &= (1 : 0 : b); \\
Q_6 &= (1 : 1 : u); \\
Q_7 &= (m : 1 : v); \\
Q_8 &= (m : 1 : c). \\
\end{align*} \]

We define \( \mathbb{A}^6 \) to be the affine space with coordinate ring \( T_6 = k[a, b, c, m, u, v] \). Points in \( \mathbb{A}^6 \) correspond to configurations of the points \( Q_1, \ldots, Q_8 \). The fact that \( C_{6,5}, C_{7,8} \) and \( C_{8,7} \) go through \( P \) gives polynomial equations in these six variables, hence defines an algebraic set \( A_0 \) in \( \mathbb{A}^6 \). We define \( S_0 \) to be the algebraic set of all points in \( \mathbb{A}^6 \) that correspond to the configurations where three of the points \( Q_1, \ldots, Q_8 \) lie on a line, or six of the points lie on a conic. We want to show that \( A_0 \) is contained in \( S_0 \).

**Claim 4.22.1**: There is a unique cubic through \( Q_1, Q_2, Q_3, Q_4, Q_5, Q_8 \) and \( P \) that is singular in \( Q_5 \).

**Proof**: The vector space spanned by all monomials in \( x, y, z \) of degree three has dimension ten, so cubics in \( \mathbb{P}^2 \) correspond to points in \( \mathbb{P}^9 \). Requiring that a point lies on a cubic defines a hyperplane in \( \mathbb{P}^9 \). Requiring that a cubic contains and is singular in a point gives three linear conditions. We conclude that all cubics through these seven points with a singularity at one of them are in the intersection of 9 hyperplanes of \( \mathbb{P}^9 \), which gives at least one point, so at least one cubic. Assume that this cubic is not unique. Then there are two linearly independent cubics \( D_1 \) and \( D_2 \) that go through these seven points with a singularity at \( Q_5 \). Let \( l_i \) be the tangent line to \( D_i \) at \( P \) for \( i = 1, 2 \).

If the equations defining \( l_1 \) and \( l_2 \) are not linearly independent, then there is a linear combination \( F \) of \( D_1 \) and \( D_2 \) that is singular in \( P \). But then the line \( L_3 \) through \( P, Q_5 \) and \( Q_6 \) intersects \( F \) in at least four points counted with multiplicity, which implies that \( F \) has \( L_3 \) as a component, hence is reducible. If \( F \) splits in three lines, then, since the five points \( Q_1, Q_2, Q_3, Q_4 \) and \( Q_8 \) are not on \( L_3 \), they are contained in the other two lines. But then there would be at least three points on a line, contradicting the fact that the points are in general position. On the other hand, if \( F \) splits in a line and a smooth conic \( C \), then, since \( Q_5 \) is a singular point of \( F \), it is in the intersection of \( L_3 \) and \( C \). The five points \( Q_1, Q_2, Q_3, Q_4 \) and \( Q_8 \) are not contained in \( L_3 \), hence they are in \( C \), too. But then there are six points in general position on the smooth conic \( C \), which gives a contradiction. We conclude that \( l_1 \) and \( l_2 \) must be linearly independent.

Since the equations defining \( l_1 \) and \( l_2 \) are linearly independent, the two lines span the whole plane, so there is a linear combination \( G \) of \( D_1 \) and \( D_2 \) such that \( L_1 \) is the tangent line to \( G \) at \( P \). But then \( L_1 \) intersects \( G \) in four points counted
Let $f$ with multiplicity, so it is contained in $G$. Therefore $G$ is reducible. If $G$ splits in three lines, then, since the points $Q_3, Q_4, Q_5, Q_8$ are not contained in $L_1$, each of the other two lines contains two of these five points. But since $Q_5$ is a singular point of $G$, it is contained in the intersection of two lines, so there is a line that contains three of the $Q_i$. On the other hand, if $G$ splits in $L_1$ and a smooth conic, then, since $Q_5$ is a singular point of $G$, it lies in the intersection of the conic and $L_1$, hence $L_1$ contains $Q_1, Q_2$ and $Q_5$. In both cases, the points $Q_1, \ldots, Q_8$ are not in general position, leading to a contradiction. We conclude that there is a unique cubic through $Q_1, Q_2, Q_3, Q_4, Q_5, Q_8$ and $P$ that is singular in $Q_5$. ($\square$)

Let $D$ be the unique cubic through $Q_1, Q_2, Q_3, Q_4, Q_5, Q_8$ and $P$ that is singular in $Q_5$. By uniqueness, it must be equal to $C_{6,5}$. Note that $D$ intersects $L_4$ in $Q_8$ and in $P$. If $L_4$ were contained in $D$, then by the same reasoning as used in Claim 4.21 there would be either three of the $Q_i$ on $L_4$, or six of the $Q_i$ on a smooth conic, which is not possible. Therefore $L_4$ is not contained in $D$, so $Q_7$ is the third point of intersection of $L_4$ with $D$. By Lemma 4.21 the equation expressing that $Q_7$ is in $D$ is given by $\det(L) = 0$, where $L$ is the matrix associated to $(R_1, \ldots, R_8) = (Q_1, Q_2, Q_3, Q_4, Q_7, Q_8, P, Q_5)$. We have

$$\det(L) = m(m-1)(c-v)(b-1)(a-1)f,$$

where

$$f = (a - ac - bc + bm)v + b(a-1)m^2 + b(c - 2a)m + a(b + c - 1).$$

The first five factors of $\det(L)$ define subsets of $S_0$, hence do not correspond to configurations where $Q_1, \ldots, Q_8$ are in general position. Therefore, $C_{6,5}$ goes through $P$ if and only if $f = 0$. Define $U = Z(a - ac - bc + bm)$.

**Claim 4.22.2:** $U \cap A_0$ is contained in $S_0$.

**Proof:** Let $(a_0, b_0, c_0, m_0, u_0, v_0) \in A_0$ be such that $a_0 - a_0c_0 - b_0c_0 + b_0m_0 = 0$. Then, since $f(a_0, b_0, c_0, m_0, u_0, v_0) = 0$, we have also

$$b_0(a_0 - 1)m_0^2 + b_0(c_0 - 2a_0)m_0 + a_0(b_0 + c_0 - 1) = 0.$$

But then $f(a, b, c, m, u, v) = 0$ for every $v$, so the whole line $L_4$ is contained in $D$. As we have seen before, this implies that the points $Q_1, \ldots, Q_8$ are not in general position. ($\square$)

Analogously, the fact that $C_{7,8}$ goes through $P$ is expressed by $\det(L')$, where $L'$ is the matrix $L$ associated to $(R_1, \ldots, R_8) = (Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, P, Q_8)$ in Lemma 4.21. We have

$$\det(L') = m(u-1)(m-1)(b-1)(a-1)g,$$

where $g = \beta u + \gamma$ with

$$\beta = bm^3 + (1 - bc - c)m^2 + (c^2 - 2c + 1)m + a(1 - c) + c^2 - c,$$

and

$$\gamma = -abm^3 + (abc + ab + ac - a + b - 2bc)m^2 + (ab - 2abc + a + 2bc^2 - b - ac^2 + 2c^2 - 2e)m$$

$$+ a(bc - b + 2c^2 - 2c) - bc^2 + bc - 2c^3 + 2c^2.$$
The first five factors of det($L'$) correspond to configurations where the eight points are not in general position, so $C_{7,8}$ contains $P$ if and only if $g = 0$. Define $V = Z(\beta)$.

By the same reasoning as in Claim 4.22.2 we have $V \cap A_0 \subseteq S_0$. Set

$$v' = \frac{-b(a - 1)m^2 + b(c - 2a)m + a(b + c - 1)}{a - ac - bc + bm} \quad \text{and} \quad u' = \frac{-\gamma}{\beta}. $$

Define $A^4$ to be the affine space with coordinate ring $T_4 = k[m, a, b, c]$, and let $K_4 = \text{Frac}(T_4)$ be the field of rational fractions of elements in $T_4$. Consider the ring homomorphism $T_5 \rightarrow K_4$ defined by

$$(m, a, b, c, u, v) \mapsto (m, a, b, c, u', v').$$

This defines an injective rational map $i : A^4 \rightarrow A^6$, which is a section of the projection $A^6 \rightarrow A^4$ on the first four coordinates. Let $A'_0 = A_0 \setminus ((A_0 \cap U) \cup (A_0 \cap V))$.

Showing that $A_0 \subseteq S_0$ is equivalent to showing that $A'_0 \subseteq S_0$. Note that, since $i$ is defined outside the subvarieties of $A^4$ defined by $a - ac - bc + bm$ and $\beta$, we have $i^{-1}(A'_0) \cong A'_0$. Let $A_1 = i^{-1}(A'_0)$ and $S_1 = i^{-1}(S_0)$, then $A'_0 \subseteq S_0$ is equivalent to $A_1 \subseteq S_1$.

Let $L''$ be the matrix $L$ associated to $(R_1, \ldots, R_8) = (Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, P, Q_7)$ in Lemma 4.21. Similarly to $C_{7,8}$, the fact that $C_{8,7}$ contains $P$ is expressed by $\det(L'')$. We have

$$\det(L'') = -2abm(m - 1)^2(b - 1)(a - 1)(a + b - 1)f_1 \cdot f_2 \cdot f_3,$$

with

$$f_1 = ac - a + bcm - bm^2 - c^2 + cm + c - m,$$

$$f_2 = abm^2 - 2abm + ab - ac^2 + 2ac - a - bc^2 + 2bcm - bm^2,$$

and

$$f_3 = abcm^2 - 2abcm + abc - abm^3 + abm^2 + abm - ab - ac^2m + 2ac^2$$

$$+ acm^2 - 3ac - am^2 + am + a + 2bc^2m - bc^2 - 3bcm^2 + bc + bm^3$$

$$+ bm^2 - bm - 2c^3 + 3c^2m + 3c^2 - cm^2 - 4cm - c + m^2 + m.$$ 

Since char $k \neq 2$, the determinant of $L''$ equals zero if and only if at least one of the non-constant factors equals zero. We can show that all non-constant factors of $\det(L'')$ define subvarieties of $S_1$. If $a = 0$, then $Q_2, Q_3$ and $Q_5$ are contained in the line $x = z$. If $b = 0$, then $Q_3, Q_4$ and $Q_5$ are on the line $y - z = 0$, and $a + b - 1 = 0$ implies that $Q_1, Q_4$ and $Q_5$ are on the line $x(a - 1) - ay + z = 0$. If $m = 0$ then $L_4 = L_2$, and $m = 1$ implies $L_4 = L_3$, so in both cases there are four points on a line. If $a = 1$ or $b = 1$, then two points would be the same. Let $(R_1, \ldots, R_6) = (Q_3, \ldots, Q_8)$, and let $N$ be the corresponding matrix from Lemma 4.21. We compute the determinant of $N$ and find that $f_1f_2f_3$ divides $\det(N)$. This means that $f_1, f_2$, as well as $f_3$ define subsets of $S_1$. We conclude that all irreducible components of $A_1$ are contained in $S_1$, which finishes the proof. □
We can now prove Theorem 1.

**Proof of Theorem 1.** First note that by Corollary 4.11, the number of exceptional curves through any point in $X$ is at most sixteen in all characteristics.

Now assume $\text{char } k \neq 2$. Consider the clique $K = \{e_1, \ldots, e_6, c_1, \ldots, c_6\}$, where

\[
\begin{align*}
e_1 &= L - E_1 - E_2; \\
e_2 &= L - E_3 - E_4; \\
e_3 &= L - E_5 - E_6; \\
e_4 &= L - E_7 - E_8; \\
e_5 &= 3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - 2E_8; \\
e_6 &= 3L - E_1 - E_2 - E_3 - E_4 - 2E_5 - E_7 - E_8,
\end{align*}
\]

and $c_i$ is the unique class in $E$ such that $e_1 \cdot c_i = 3$, for all $i \in \{1, \ldots, 6\}$. By Remark 2.9, the classes $e_1, \ldots, e_6, c_5$ are the strict transforms of the four lines through $P_i$ and $P_{i+1}$ for $i \in \{1, 3, 5, 7\}$, and the unique cubics through $P_1, \ldots, P_6$ and $P_8$ respectively $P_1, \ldots, P_5, P_7$ and $P_8$ respectively $P_1, \ldots, P_6$ and $P_7$, that are singular in $P_8$ respectively $P_3$ respectively $P_7$.

Now let $K'$ be a clique in $G$ consisting of at least six pairs, and let

\[
\{(f_1, d_1), \ldots, (f_6, d_6)\}
\]

be a set of six pairs in $K'$. Since $G$ acts transitively on the set of cliques of six pairs in $E$ by Proposition 4.19, after changing the indexes and interchanging $f_i$'s and $d_i$'s if necessary, there is an element $g \in G$ such that $f_i = g(e_i)$ and $d_i = g(c_i)$ for $i \in \{1, \ldots, 6\}$. Let $E'_i = g(E_i)$. Then, since the $E'_i$ are pairwise disjoint, by Lemma 3.9 we can blow down $E'_1, \ldots, E'_8$ to points $Q_1, \ldots, Q_8 \in \mathbb{P}^2$ that are in general position, such that $X$ is the blow-up of $\mathbb{P}^2$ at $Q_1, \ldots, Q_8$, and $E'_i$ is the class in Pic $X$ of the exceptional curve above $Q_i$ for all $i$. By the bijection in Remark 3.12, the element $f_i$ is the class of the strict transform of the line through $Q_{2i-1}$ and $Q_i$ for $i \in \{1, \ldots, 4\}$, the elements $f_5$ and $f_6$ are the classes of the strict transforms of the unique cubics through $Q_1, \ldots, Q_6$ and $Q_8$ respectively $Q_1, \ldots, Q_5, Q_7$ and $Q_8$, that are singular in $Q_8$ respectively $Q_3$, and $d_i$ is the unique class in $E$ intersecting $f_i$ with multiplicity three for all $i$. From Proposition 4.22 it follows that the curves corresponding to $f_1, \ldots, f_6, d_5$ and $d_6$ can not all go through one point.

We conclude that the curves corresponding to the classes in a clique containing at least six pairs can not go through one point. Since any maximal number of exceptional curves going through one point on the ramification curve forms a clique consisting of only pairs, hence of even size, we conclude that this number is at most ten.

The following examples show that the upper bounds in Theorem 1 can be reached.

**Example 4.23.** Define the following eight points in $\mathbb{P}^2_3$. 

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They are in general position, as can be checked by asserting that the determinants of the appropriate matrices in Lemma 4.21 are all nonzero. Therefore, the blow-up of \( \mathbb{P}^2 \) in \((Q_1, \ldots, Q_8)\) is a del Pezzo surface \( S \). We have the following four lines in \( \mathbb{P}^2 \).

- The line \( L_1 \) through \( Q_1 \) and \( Q_2 \), which is given by \( x = 0 \);
- the line \( L_2 \) through \( Q_3 \) and \( Q_4 \), which is given by \( y = 0 \);
- the line \( L_3 \) through \( Q_5 \) and \( Q_6 \), which is given by \( x = y \);
- the line \( L_4 \) through \( Q_7 \) and \( Q_8 \), which is given by \( x = -y \).

On \( S \), we define the four exceptional curves \( e_1, \ldots, e_4 \) to be the strict transforms of \( L_1, \ldots, L_4 \). Let \( C_{7,8} \) be the unique cubic through \( Q_1, \ldots, Q_8 \) that is singular in \( Q_8 \). Let \(( R_1, \ldots, R_8 ) = ( Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8 )\), and let \( L \) be the corresponding matrix from Lemma 4.21. Then the equation defining \( C_{7,8} \) is the determinant of \( L' \), where \( L' \) is equal to \( L \) after replacing the first row by \( \text{Mon}_3 \). We compute this determinant and find

\[
C_{7,8} : x^3 - \frac{3}{4}x^2y - \frac{31}{12}xy^2 + \frac{10}{3}xyz + \frac{8}{3}y^2z - \frac{5}{3}yz^2 = 0.
\]

We define the singular cubic \( C_{8,7} \) analogously and compute its defining equation, which is

\[
C_{8,7} : x^3 + \frac{13}{4}x^2y + \frac{43}{4}xy^2 - 14xyz - xz^2 + 15y^3 - 40y^2z + 25yz^2 = 0.
\]

Let the exceptional curves \( e_5, c_5 \) on \( S \) be the strict transforms of \( C_{7,8} \) and \( C_{8,7} \), respectively. Since \( L_1, \ldots, L_4, L_{7,8}, L_{8,7} \) all go through the point \((0 : 0 : 1)\), the six exceptional curves \( e_1, \ldots, e_5, c_5 \) go through one point \( P \) in \( S \). Moreover, since \( e_5 \cdot c_5 = 3 \), this point \( P \) lies on the ramification curve of \( \varphi \). Therefore, by Remark 4.15 the unique exceptional curves \( e_1, \ldots, e_4 \) such that \( e_i \cdot c_i = 3 \) for \( i \in \{1, \ldots, 4\} \) go through \( P \), too. We conclude that the ten exceptional curves \( e_1, \ldots, e_5, c_1, \ldots, c_5 \) all go through \( P \).

**Example 4.24.** Let \( f = x^5 + x^2 + 1 \in \mathbb{F}_2[x] \), and let \( F \cong \mathbb{F}_2[x]/(f) \) be the finite field of 32 elements defined by adjoining a root \( \alpha \) of \( f \) to \( \mathbb{F}_2 \). Define the following eight points in \( \mathbb{P}_F^2 \).
Let $C$ We compute the defining equations of $C$

Let the exceptional curves all go through the point $C$

4.25

Lemma In this subsection we prove Theorem 2.

4.2 Points outside the ramification curve

In this subsection we prove Theorem 2

Lemma 4.25. For $e_1, e_2 \in E$ with $e_1 \cdot e_2 = 1$, there are 138 elements $e$ in $E$ such that $e \cdot e_1 \geq 1$ and $e \cdot e_2 \geq 1$. 

\begin{align*}
Q_1 &= (0 : 1 : 1); & Q_5 &= (1 : 1 : 1); \\
Q_2 &= (0 : 1 : \alpha^{19}); & Q_6 &= (\alpha^{20} : \alpha^{20} : \alpha^{16}); \\
Q_3 &= (1 : 0 : 1); & Q_7 &= (\alpha^{24} : \alpha^{25} : 1); \\
Q_4 &= (1 : 0 : \alpha^{5}); & Q_8 &= (\alpha^{30} : 1 : \alpha^{5}).
\end{align*}
Proof. From Proposition 3.22 it follows that it is enough to show this for $e_1 = E_1$ and $e_2 = L - E_1 - E_2$. Let $e = aL - \sum_{i=1}^{8} E_i b_i \in E$. Then the conditions $e \cdot e_1 \geq 1$, $e \cdot e_2 \geq 1$ are equivalent to $b_1 \geq 1$ and $a - b_1 - b_2 \geq 1$. By looking at the table in Proposition 2.4 we find all possibilities.

\[
\begin{array}{l|cccccc}
  \text{number of possibilities for } e & 2 & 3 & 4 & 5 & 6 \\
  \hline
  a & 15 & 37 & 50 & 28 & 8
\end{array}
\]

We find a total of 138 possibilities for $e$ such that $e$ intersects both $e_1$ and $e_2$. \qed

**Lemma 4.26.** The maximal size of a clique in $G$ without any pairs is twelve. Moreover, for $e_1, e_2 \in E$ with $e_1 \cdot e_2 = 1$, there are 640 cliques of size twelve without pairs that contain $e_1$ and $e_2$.

Proof. As

\[
\{ L - E_1 - E_2, L - E_3 - E_4, L - E_5 - E_6, L - E_7 - E_8 \}
\]

is a clique of size four without pairs, the maximal size of a clique in $G$ without pairs is bigger than three. So by Proposition 4.5 such a clique contains two exceptional classes intersecting with multiplicity one. Let $K$ be a maximal clique without pairs. By Proposition 3.22 we can assume that $K$ contains $e_1 = E_1$ and $e_2 = L - E_1 - E_2$. By Lemma 4.25 there are 138 exceptional classes that intersect both $e_1$ and $e_2$ positively. But since $K$ contains no pairs, the two exceptional classes that intersect $e_1$ or $e_2$ with multiplicity three are not in $K$. This leaves us with 136 possibilities for elements in $K$. Since it is too tedious to compute all pairwise intersection multiplicities by hand, we compute with MAGMA the maximal size of a clique in the graph on these 136 exceptional curves that does not contain any edges of weight three. This maximum is ten, hence $K$ has size twelve, and there are 640 such cliques. \qed

Lemma 4.26 gives an upper bound for the number of exceptional curves going through one point outside the ramification curve of $\varphi$, which is twelve. We will compute a sharp upper bound.

Let $T$ be the clique consisting of the following twelve elements.

\[
\begin{align*}
  t_1 & = L - E_1 - E_2; \\
  t_2 & = L - E_3 - E_4; \\
  t_3 & = L - E_5 - E_6; \\
  t_4 & = L - E_7 - E_8; \\
  t_5 & = 4L - 2 \sum_{i \in \{1,5\}} E_i - \sum_{j \in \{2,4,6,7,8\}} E_j; \\
  t_6 & = 4L - 2 \sum_{i \in \{1,7\}} E_i - \sum_{j \in \{2,3,5,6,8\}} E_j; \\
  t_7 & = 4L - 2 \sum_{i \in \{1,6,8\}} E_i - \sum_{j \in \{2,3,4,5,7\}} E_j; \\
  t_8 & = 4L - 2 \sum_{i \in \{2,3,8\}} E_i - \sum_{j \in \{1,4,5,6,7\}} E_j; \\
  t_9 & = 4L - 2 \sum_{i \in \{3,6,7\}} E_i - \sum_{j \in \{1,2,4,5,8\}} E_j; \\
  t_{10} & = 4L - 2 \sum_{i \in \{4,5,8\}} E_i - \sum_{j \in \{1,2,3,5,7\}} E_j; \\
  t_{11} & = 4L - 2 \sum_{i \in \{2,4,6\}} E_i - \sum_{j \in \{1,3,5,7,8\}} E_j; \\
  t_{12} & = 4L - 2 \sum_{i \in \{2,5,7\}} E_i - \sum_{j \in \{1,3,4,6,8\}} E_j.
\end{align*}
\]

Let $G_T$ be the stabilizer of $T$ in $G$. 44
Proposition 4.27. Let $\mathcal{T}$ be the set of all cliques in $\mathcal{G}$ of size twelve that do not contain any pair. The following hold.

(i) The group $G$ acts transitively on $\mathcal{T}$;

(ii) we have $T^4 \cap W = T^4 \cap Y$, and $G_T$ acts transitively on $T^4 \cap Y$;

(iii) we have $|\mathcal{T}| = 179200$.

Proof.

(i) Consider the two sets

$$A = \{\{e_1, e_2\} \mid e_1, e_2 \in E; \ e_1 \cdot e_2 = 1\},$$

and

$$C = \{(a, K) \in A \times \mathcal{T} \mid a \subset K\}.$$  

We have $|A| = \frac{240 \cdot 126}{2} = 15120$. To compute the cardinality of $C$, note that by Lemma 4.26, for every element $a \in A$ there are 640 elements $K \in \mathcal{T}$ such that $(a, K) \in C$. We conclude that $|C| = |A| \cdot 640 = 9676800$. We will show that $G$ acts transitively on $C$. Define the set

$$D = \{(e_1, e_2, e_3, e_4, K) \in Y \times \mathcal{T} \mid e_1, \ldots, e_4 \in K\}.$$  

It is an easy check that the clique $T$ is an element in $\mathcal{T}$. Let $F = G(y, T)$ be the orbit of $(y, T)$ under the action of $G$ on $D$. We have a map

$$\lambda : F \longrightarrow C, \ (e_1, e_2, e_3, e_4, K) \longmapsto (\{e_1, e_2\}, K).$$

The group $G$ acts transitively on $F$, and we will show that $\lambda$ is surjective, which implies that $G$ acts transitively on $C$.

First we compute the cardinality of $F$. We have a projection $\gamma : F \longrightarrow Y$ on the first coordinate. Let $y = (t_1, t_2, t_3, t_4) \in Y$. Since $G$ acts transitively on $F$, the stabilizer $G_y$ of $y$ in $G$ acts transitively on the fiber $\gamma^{-1}(y)$. Therefore, we have $|\gamma^{-1}(y)| = |G_y(y, T)| = \frac{|G_y|}{|C_{y,K}|}$. The stabilizer $G_y$ of $y$ in $G$ has cardinality $|G_y| = \frac{|G|}{|G_{y}|} = \frac{|G|}{Y} = 16$, since $G$ acts transitively on $Y$. Note that the stabilizer of $y$ in $G$ contains all permutations of $(E_1, \ldots, E_8)$ that are generated by the permutations of $E_i$ and $E_{i+1}$ for $i \in \{1, 3, 5, 7\}$. There are 16 of these, so that means that $G_y$ consists exactly of these permutations. For every permutation $g$ in $G_y$, we have $gT \neq T$, except for the identity and the permutation permuting all $E_i, E_{i+1}$ for $i \in \{1, 3, 5, 7\}$. Therefore we have $\frac{|G_\gamma|}{|C_{y,K}|} = \frac{16}{2} = 8$, so $\gamma^{-1}(y)$ has size eight. Since $G$ acts transitively on $Y$, all fibers of $\gamma$ have size eight, and $|F| = 8 \cdot |Y| = 348364800$.

Now consider the element $\epsilon = \{(L - E_1 - E_2, L - E_3 - E_4), T\}$ in $C$. We compute the cardinality of $\lambda^{-1}(\epsilon)$. By looking at $T$ we see that for the elements $e_1 = L - E_1 - E_2$ and $e_2 = L - E_3 - E_4$ there are six elements $e_3$ in $T$ such that $e_1 \cdot e_3 = e_2 \cdot e_3 = 1$, and for each of those six $e_3$ there are three elements $e_4$ in $T$ such that $(e_1, \ldots, e_4) \in W$. Since we can interchange $e_1$ and $e_2$, it follows that the fiber above $\epsilon$ has cardinality at most $2 \cdot 6 \cdot 3 = 36$. Since $G$ acts
transitively on $F$, it follows that all nonempty fibers of $\lambda$ have cardinality at most 36. But then we have $|\lambda(F)| \geq \frac{|F|}{36} = 9676800 = |C|$, so we conclude that there is equality everywhere, and $\lambda$ is surjective.

Finally, we have a projection $\delta : C \to T$ on the second coordinate, which is surjective since every element in $T$ contains an element in $A$ by Lemma 4.5. Therefore, $G$ acts transitively on $T$.

(ii) From the fact that the fibers of $\lambda$ have cardinality 36 it follows that all elements in $T^4 \cap W$ are elements in $Y$, so $T^4 \cap W = T^4 \cap Y$. Let $\kappa : F \to T$ be the composition $\lambda \circ \delta$, then $\kappa$ is surjective since both $\lambda$ and $\delta$ are. Since $G$ acts transitively on $F$, the stabilizer $G_T$ of $T$ acts transitively on the fiber $\kappa^{-1}(T)$ above $T$. We have a projection $\kappa^{-1}(T) \to T^4 \cap Y$ on the first coordinate, which is injective. By surjectivity of $\kappa$, we have

$$|\kappa^{-1}(T)| = \frac{|F|}{|T|} = \frac{348364800}{179200} = 1944.$$  

By looking at the intersection multiplicities in $T$ we have

$$|T^4 \cap Y| = |T^4 \cap W| = 12 \cdot 9 \cdot 6 \cdot 3 = 1944.$$  

We conclude that the projection is a bijection, hence $G_T$ acts transitively on $T^4 \cap Y$.

(iii) There are 640 elements in $T$ containing $e_1$ and $e_2$ by Lemma 4.26. By looking at the elements in $T$ we see that for a fixed element $c_1 \in T$ there are 9 elements $c_2$ in $T$ such that $c_1 \cdot c_2 = 1$. Since $G$ acts transitively on $T$ this holds for all elements in $T$, so we have $|T| = \frac{640 \cdot 240 \cdot 126}{12 \cdot 9 \cdot 6} = 179200$.

\[ \square \]

**Corollary 4.28.** $G_T$ acts transitively on $T$.

**Proof.** We have a surjective map $Y^4 \cap Y \to T$ projecting on the first coordinate, so this follows from the previous proposition. \[ \square \]

From the following result, which is again purely geometrical, we will deduce Theorem 2.

Let $Q_1, \ldots, Q_8$ be eight points in general position in $\mathbb{P}^2$. Define the following curves:

- $L_1$ is the line through $Q_1$ and $Q_2$;
- $L_2$ is the line through $Q_3$ and $Q_4$;
- $C_1$ is the conic through $Q_1$, $Q_3$, $Q_5$, $Q_6$ and $Q_7$;
- $C_2$ is the conic through $Q_1$, $Q_4$, $Q_5$, $Q_6$ and $Q_8$;
- $C_3$ is the conic through $Q_2$, $Q_3$, $Q_5$, $Q_7$ and $Q_8$;
- $C_4$ is the conic through $Q_2$, $Q_4$, $Q_6$, $Q_7$ and $Q_8$;
- $D_1$ is the quartic through all eight points with singular points in $Q_1$, $Q_7$ and $Q_8$;
- $D_2$ is the quartic through all eight points with singular points in $Q_2$, $Q_5$ and $Q_6$;
- $D_3$ is the quartic through all eight points with singular points in $Q_3$, $Q_5$ and $Q_8$;
- $D_4$ is the quartic through all eight points with singular points in $Q_4$, $Q_5$ and $Q_7$.  

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**Proposition 4.29.** Assume that char $k = 0$. Then $L_1$, $L_2$, $C_1, \ldots C_4$, $D_1, \ldots, D_4$ do not all go through one point.

**Proof.** We assume that these ten curves go through a common point $P$. First note that if $P$ were equal to one of the $Q_i$, then one of the conics would contain six of the eight $Q_i$, which would contradict the fact that $Q_1, \ldots, Q_8$ are in general position. We conclude that $P$ is not equal to one of the $Q_i$.

Let $(x : y : z)$ be the coordinates in $\mathbb{P}^2$. Without loss of generality we can choose four points in general position in $\mathbb{P}^2$, and we set

\[
Q_1 = (1 : 0 : 1); \quad Q_6 = (0 : -1 : 1);
\]

\[
Q_5 = (0 : 1 : 1); \quad P = (-1 : 0 : 1).
\]

Since $L_1$ contains $Q_1$ and $P$, it is given by $y = 0$, so we can set $Q_2 = (a_1 : 0 : a_3)$ for some $a_1$, $a_3 \in k$. Let $b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3, e_1, e_2, e_3 \in k$ be such that

\[
Q_3 = (b_1 : b_2 : b_3); \quad Q_7 = (d_1 : d_2 : d_3);
\]

\[
Q_4 = (c_1 : c_2 : c_3); \quad Q_8 = (e_1 : e_2 : e_3).
\]

Since the four points $Q_1$, $Q_5$, $Q_6$ and $P$ are in general position, the linear system of quadrics through $Q_1$, $Q_5$, $Q_6$ and $P$ is two-dimensional. Therefore it is generated by two linearly independent quadrics, and we take these to be $x^2 + y^2 - z^2$ and $xy$.

Let $l, m \in k$ be such that

\[
C_1 \text{ is given by } x^2 + y^2 - z^2 = 2lxy;
\]

\[
C_2 \text{ is given by } x^2 + y^2 - z^2 = 2mxy.
\]

Since $Q_3, Q_4, Q_7$, and $Q_8$ are not contained in $L_1$, there are $s, t, u \in k$ such that

\[
L_2 \text{ is given by } sy = x + z;
\]

The line $L_3$ through $P$ and $Q_7$ is given by $ty = x + z$;

The line $L_4$ through $P$ and $Q_8$ is given by $uy = x + z$.

We define $\mathbb{A}^{19}$ to be the affine space with coordinate ring

\[T_{19} = k[l, m, s, t, u, a_1, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3, e_1, e_2, e_3].\]

Points in $\mathbb{A}^{19}$ correspond to configurations of the points $Q_1, \ldots, Q_8$. The fact that all ten curves go through $P$ gives polynomial equations in these 19 variables, hence defines an algebraic set $A_0$ in $\mathbb{A}^{19}$. We define $S_0$ to be the algebraic set of all points in $\mathbb{A}^{19}$ that correspond to the configurations where three of the points $Q_1, \ldots, Q_8$ lie on a line, or six of the points lie on a conic. We want to show that $A_0$ is contained in $S_0$, which would mean that all possibilities for the ten curves to go through $P$ imply that $Q_1, \ldots, Q_8$ are not in general position, giving a contradiction and thus proving our statement. Since the equations defining $A_0$ are very big, we do a couple of reduction steps to obtain something that we can actually compute.

**Step 1**
Let $b'_1 = sb_2 - b_3$, $c'_1 = sc_2 - c_3$, $d'_1 = td_2 - d_3$ and $e'_1 = ue_2 - e_3$. The fact that the
Claim 4.29.1: Respectively.

Let $V$ and consider the ring homomorphism $T_{19} \to T_{15}$ defined by

$$(l, m, s, t, u, a_1, a_3, b_1, b_3, c_1, c_2, c_3, d_1, d_2, d_3, e_1, e_2, e_3)$$

$$\mapsto (l, m, s, t, u, a_1, a_3, b_1', b_3, c_1', c_2, c_3, d_1', d_2, d_3, e_1', e_2, e_3).$$

This corresponds to an embedding $i_1 : \mathbb{A}^{15} \to \mathbb{A}^{19}$, and $A_0$ lies in $i_1(\mathbb{A}^{15})$. Let $A_1$ be $i_1^{-1}(A_0)$ and $S_1 = i_1^{-1}(S_0)$, then $A_0 \subseteq S_0$ is equivalent to $A_1 \subseteq S_1$.

Step 2

Since $Q_3$ and $Q_4$ are in the intersection of $L_2$ with $C_1$ and $C_2$, respectively, we have

$$\begin{align*}
(sb_2 - b_3)^2 + b_2^2 - b_3^2 - 2l(sb_2 - b_3)b_2 &= 0; \\
(sc_2 - c_3)^2 + c_2^2 - c_3^2 - 2m(sc_2 - c_3)c_2 &= 0.
\end{align*}$$

(1) and (2)

Since $P$ is also in the intersection of $L_2$ with $C_1$, we can divide (1) by $b_2$ and obtain the equation

$$(s^2 - 2ls + 1)b_2 + 2(l - s)b_3 = 0. \quad (3)$$

Similarly, we can divide (2) by $c_2$ and obtain

$$(s^2 - 2ms + 1)c_2 + 2(l - s)c_3 = 0. \quad (4)$$

Let $V_1$, $V_2$ be the subvarieties of $\mathbb{A}^{15}$ defined by $s^2 - 2ls + 1 = 0$ and $s^2 - 2ms + 1 = 0$, respectively.

Claim 4.29.1: $V_1 \cap A_1$ and $V_2 \cap A_1$ lie in $S_1$.

Proof: Let $(l, m, s, t, u, a_1, a_3, b_1, b_3, c_1, c_2, c_3, d_2, d_3, e_2, e_3)$ be a point in $A_1$ and assume that $s^2 - 2ls + 1 = 0$. Then, by (3), we have $2(l - s)b_3 = 0$, which implies that $C_1$ contains the whole line $L_2$. But then $Q_4 \in C_1$, which means that the six points $Q_1, Q_3, Q_4, Q_5, Q_6$ and $Q_7$ lie on a conic. Since $S_1$ consists of all points in $\mathbb{A}^{15}$ that correspond to configurations of the points $Q_1, \ldots, Q_8$ where three of the points lie on a line or six of the points lie on a conic, we conclude that $V_1 \cap A_1$ lies in $S_1$. The proof for $V_2 \cap A_1$ goes analogously. (□)

Let

$$b_1' = \frac{-2(l - s)b_3}{s^2 - 2ls + 1} \quad \text{and} \quad c_1' = \frac{-2(l - s)c_3}{s^2 - 2ms + 1}.$$ 

Define $\mathbb{A}^{13}$ to be the affine space with coordinate ring

$$T_{13} = k[l, m, s, t, u, a_1, a_3, b_3, c_3, d_2, d_3, e_2, e_3],$$

and let $K_{13} = \text{Frac}(T_{13})$ be the field of rational fractions of elements in $T_{13}$. Consider the ring homomorphism $T_{15} \to K_{13}$ defined by

$$\begin{align*}
(l, m, s, t, u, a_1, a_3, b_1, b_3, c_1, c_2, c_3, d_2, d_3, e_2, e_3)
\mapsto (l, m, s, t, u, a_1, a_3, b_1', b_3, c_1', c_3, d_2, d_3, e_2, e_3).
\end{align*}$$

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This defines an injective rational map $i_2 : \mathbb{A}^{13} \to \mathbb{A}^{15}$. Let

$$A'_1 = A_1 \setminus ((A_1 \cap V_1) \cup (A_1 \cap V_2)).$$

By Claim 4.29.1 showing that $A_1 \subseteq S_1$ is equivalent to showing that $A'_1 \subseteq S_1$.

Note that, since $i_2$ is defined outside the subvarieties of $\mathbb{A}^{13}$ defined by $s^2 - 2ls + 1 = 0$ and $s^2 - 2ms + 1 = 0$, we have $i_2^{-1}(A'_1) \cong A'_1$. Let $A_2 = i_2^{-1}(A'_1)$ and $S_2 = i_2^{-1}(S_1)$, then $A'_1 \subseteq S_1$ is equivalent to $A_2 \subseteq S_2$.

**Step 3**

Since $Q_7$ and $Q_8$ are in the intersection of $L_3$ with $C_1$ and in the intersection of $L_4$ with $C_2$, respectively, we have

\[
(td_2 - d_3)^2 + d_2^2 - d_2^2 - 2ld_2(td_2 - d_3) = 0; \tag{5}
\]

\[
(ue_2 - e_3)^2 + e_2^2 - e_3^2 - 2me_2(ue_2 - e_3) = 0. \tag{6}
\]

Since $P$ is also in the intersection of $L_3$ with $C_1$ and in the intersection of $L_4$ with $C_2$, we can divide (5) by $d_2$ and obtain the equation

\[
(t^2 - 2lt + 1)d_2 + 2ld_3 - 2td_3 = 0. \tag{7}
\]

Similarly, we can divide (6) by $e_2$ and obtain

\[
(u^2 - 2mu + 1)e_2 + 2me_3 - 2ue_3 = 0. \tag{8}
\]

Let $U_1$, $U_2$ be the subvarieties of $\mathbb{A}^{13}$ defined by $t^2 - 2lt + 1 = 0$ and $u^2 - 2mu + 1 = 0$, respectively.

**Claim 4.29.2:** $U_1 \cap A_2$ and $U_2 \cap A_2$ lie in $S_2$.  
**Proof:** Analogously to the proof of Claim 4.29.1 $U_1 \cap A_2$ and $U_2 \cap A_2$ consist of points in $\mathbb{A}^{13}$ corresponding to configurations where $L_3$ is contained in $C_1$, and $L_4$ is contained in $C_2$, respectively. If $L_3$ is contained in $C_1$ then $C_1$ is reducible, so three of the points $Q_1, Q_3, Q_5, Q_6$ and $Q_7$ are on a line. Equivalently, $L_4 \subseteq C_2$ implies that three of the $Q_i$ are on a line. Since $S_2$ contains all points in $\mathbb{A}^{13}$ corresponding to configurations of the $Q_i$ where three of them lie on a line, we conclude that $U_1 \cap A_2$ and $U_2 \cap A_2$ are both in $S_2$. \(\square\)

Let

$$d'_2 = \frac{2td_3 - 2td_3}{t^2 - 2lt + 1} \quad \text{and} \quad e'_2 = \frac{2ue_3 - 2me_3}{u^2 - 2mu + 1}.$$

Define $A^{11}$ to be the affine space with coordinate ring

$$T_{11} = k[l, m, s, t, u, a_1, a_3, b_3, c_3, d_3, e_3],$$

and let $K_{11}$ be the function field of $A_{11}$, consisting of rational fractions of elements in $T_{11}$. Consider the ring homomorphism $T_{13} \to K_{11}$ defined by

$$(l, m, s, t, u, a_1, a_3, b_3, c_3, d_2, d_3, e_2, e_3) \mapsto (l, m, s, t, u, a_1, a_3, b_3, c_3, d'_2, d_3, e'_2, e_3).$$

This defines an injective rational map $i_3 : \mathbb{A}^{11} \to \mathbb{A}^{13}$. Let

$$A'_2 = A_2 \setminus ((A_2 \cap U_1) \cup (A_2 \cap U_2)).$$

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By Claim 4.29.2, showing that $A_2 \subseteq S_2$ is equivalent to showing that $A_2' \subseteq S_2$.

Since $i_3$ is defined outside the subvarieties of $\mathbb{A}^{11}$ defined by $t^2 - 2lt + 1 = 0$ and $u^2 - 2mu + 1 = 0$, we have $i_3^{-1}(A_2') \cong A_2'$. Let $A_3 = i_3^{-1}(A_2')$ and $S_3 = i_3^{-1}(S_2)$, then $A_2' \subseteq S_2$ is equivalent to $A_3 \subseteq S_3$.

Step 4

Since no four of the points $Q_3, Q_5, Q_7, Q_8$ and $P$ are on a line, there is a unique conic $C$ through these five points. Note that $C$ intersects $L_1$ in $P$, and $L_1$ is not contained in $C$ since then $C$ would contain six of the $Q_i$. Since $C_3$ contains all of the five points, we conclude that $C = C_3$, and the second intersection point of $C$ and $L_1$ is $Q_2$. Let $(R_1, \ldots, R_6) = (Q_2, Q_3, Q_5, Q_7, Q_8, P)$, and let $N$ be the corresponding matrix from Lemma 4.21. We have

$$\det(N) = \frac{1}{2} e^2 d^2 b^2 (a_1 + a_3)(u - 1)(t - 1)(s - 1)(s - t)(m - u)(l - t)(l - s)(\alpha a_1 + \beta),$$

with

$$\alpha = l^2 s u + l^2 s - lmsu - lms - \frac{1}{2} lstu - \frac{1}{2} lst + \frac{1}{2} lsu^2 + lsu + \frac{1}{2} ls - \frac{1}{2} tu - \frac{1}{2} l$$

$$+ \frac{1}{2} mst - msu - \frac{1}{2} ms + \frac{1}{2} su^2 + \frac{1}{2} l t + \frac{1}{2} mt + \frac{1}{2} m - \frac{1}{2} u - \frac{1}{2} stu,$$

and

$$\beta = \frac{1}{2} a_3 \left( lstu + lst - lsu^2 + ls + lu + l - mst - ms - mt - m + stu - su^2 - t + u \right).$$

All factors of $\det(N)$ except for $\alpha a_1 + \beta$ define a subset of $S_3$, hence correspond to configurations where the points are not in general position. Therefore, $C_3$ contains $P$ if and only $\alpha a_1 + \beta = 0$. By the same reasoning as before, we have $Z(\alpha) \cap A_3 \subseteq S_3$.

Define $A^{10}$ to be the affine space with coordinate ring

$$T_{10} = k[l, m, s, t, u, a_3, b_3, c_3, d_3, e_3],$$

and let $K_{10} = \text{Frac}(T_{10})$. Consider the ring homomorphism $T_{11} \longrightarrow K_{10}$ defined by

$$(l, m, s, t, u, a_3, b_3, c_3, d_3, e_3) \mapsto \left( l, m, s, t, u, \frac{-\beta}{\alpha}, a_3, b_3, c_3, d_3, e_3 \right).$$

This defines an injective rational map $i_4 : A^{10} \longrightarrow A^{11}$. Let $A'_3 = A_3 \setminus (Z(\alpha) \cap A_3)$. Showing that $A_3 \subseteq S_3$ is equivalent to showing that $A'_3 \subseteq S_3$.

Since $i_4$ is defined everywhere outside the subvariety of $A^{10}$ defined by $\alpha = 0$, we have $i_4^{-1}(A'_3) \cong A'_3$. Let $A_4 = i_4^{-1}(A'_3)$ and $S_4 = i_4^{-1}(S_3)$, then $A'_3 \subseteq S_3$ is equivalent to $A_4 \subseteq S_4$.

Step 5

The equation expressing that $P$ is contained in $C_4$, is the determinant of the matrix $N$ in Lemma 4.21 corresponding to $(R_1, \ldots, R_6) = (Q_2, Q_4, Q_6, Q_7, Q_8, P)$. This determinant is given by

$$\frac{1}{2} e_3^2 d_3^2 b_3^2 a_3^2 (u + 1)(t + 1)(s + 1)(s - u)(m - u)(m - s)(l - t)(l - m)f,$$
This defines an injective rational map \( A \)

Let

We have

Showing that

Since

and consider the ring homomorphism \( P \) to \( C_1 \) and \( C_2 \), respectively. Note that \( \frac{d(x^2)}{d(y^2)} l_1 = l \) and \( \frac{d(x^2)}{d(y^2)} L_3 = t \), and both lines contain \( P \). Therefore \( t = l \) implies \( l_1 = L_3 \), and this means that \( P \) is equal to \( Q_7 \). Similarly, \( u = m \) implies \( P = Q_8 \). Therefore the point \( (t, u) = (l, m) \) corresponds to a subvariety of \( S_4 \), so we can assume \( t - l \neq 0 \) and \( u - m \neq 0 \). So we can parametrize \( D \) by intersecting \( D \) with a line \( M \) through \( (l, m) \), where \( M \) is given by \( v(t - l) = (u - m) \) for a parameter \( v \). Intersecting \( D \) with \( M \) gives

\[
(av^2 + cv + b)t^2 + (2mv - 2v^2l - cvl + m - cv + c)t \\
+ am^2 - 2mvl + v^2l^2 - cm + cvl - (a + b + c) = 0.
\]

Since \( t = l \) is a solution, we can divide by \( t - l \) and obtain

\[
(\frac{a}{v^2} + \frac{c}{v} - b)t - l(\frac{a}{v^2} + \frac{c}{v} - b) + 2((l^2 - 1)(ms - 1)v + m^2 - 1) = 0. \tag{9}
\]

Let \( \gamma = \frac{a}{v^2} + \frac{c}{v} - b \). As before, \( \gamma = 0 \) implies that all \( t \) satisfy \([9]\), which means that \( M \) is contained in \( D \), so \( D \) is reducible. But then, by the same reasoning as before, the points \( Q_1, \ldots, Q_8 \) would not be in general position, giving a contradiction. We conclude that \( \gamma \neq 0 \).

Let \( \mathbb{A}^9 \) be the affine space with coordinate ring \( T_9 = k[l, m, s, a_3, b_3, c_3, d_3, e_3, v] \), and let \( K_9 = \text{Frac}(T_9) \). Let

\[
t' = -2 \left( \frac{(l^2 - 1)(ms - 1)v + m^2 - 1}{av^2 + cv - b} \right) + l,
\]

and consider the ring homomorphism \( T_{10} \to K_9 \) defined by

\[
(l, m, s, t, u, a_3, b_3, c_3, d_3, e_3) \mapsto (l, m, s, t', t'v + m - vl, a_3, b_3, c_3, d_3, e_3).
\]

This defines an injective rational map \( i_5 : \mathbb{A}^9 \to \mathbb{A}^{10} \). Let

\[
A_4' = A_4 \setminus \{A_4 \cap Z(\gamma)\}.
\]

Showing that \( A_4 \subseteq S_4 \) is equivalent to showing that \( A_4' \subseteq S_4 \).

We have \( i_5^{-1}(A_4') \cong A_4' \). Let \( A_5 = i_5^{-1}(A_4') \) and \( S_5 = i_5^{-1}(S_4) \), then \( A_4' \subseteq S_4 \) is equivalent to \( A_5 \subseteq S_5 \).

**Finishing the proof**

For \( i \in \{1, 2, 3, 4\} \), the expression stating that \( P \) is contained in \( D_i \) is given by \( \det(H_i) = 0 \), where \( H_i \) is the matrix \( H \) Lemma [1.21] with

\[
\begin{align*}
H_1 & \text{ is the matrix associated to } (R_1, \ldots, R_9) = (Q_2, Q_3, Q_4, Q_5, Q_6, Q_1, Q_7, Q_8); \\
H_2 & \text{ is the matrix associated to } (R_1, \ldots, R_9) = (Q_1, Q_3, Q_4, Q_7, Q_8, Q_2, Q_5, Q_6); \\
H_3 & \text{ is the matrix associated to } (R_1, \ldots, R_9) = (Q_1, Q_2, Q_4, Q_5, Q_7, Q_3, Q_6, Q_8); \\
H_4 & \text{ is the matrix associated to } (R_1, \ldots, R_9) = (Q_1, Q_2, Q_3, Q_6, Q_8, Q_4, Q_5, Q_7).
\end{align*}
\]
With MAGMA, we compute these determinants. For \( i \in \{1, 2, 3, 4 \} \), we have \( \det(H_i) = \lambda_i g_i \), where \( \lambda_i \) defines a subset of \( S_5 \), and \( g_i \) is irreducible. The algebraic set \( A_5 \) is the zero set of the radical of the ideal \( (g_1, \ldots, g_4) \subset T_9 \). Let \( \delta = v^2(ls - l - ms - m + 2s)(l - m)(l + 1)(m - 1)(l - 1)(m + 1) \in T_9 \). With MAGMA we can compute the Gröbner basis of \( I \), and then it is a relatively easy check that \( \delta \) is contained in \( I \). Therefore, \( A_5 \) is contained in the union of the varieties defined by the factors of \( \delta \). As these are all subsets of \( S_5 \), we conclude that \( A_5 \) is contained in \( S_5 \). This finishes the proof.

**Lemma 4.30.** All maximal cliques of size eleven contain a pair.

**Proof.** By Lemma 4.25 it is sufficient to prove this statement for all cliques of size eleven containing the two elements \( e_1 = L - E_1 - E_2 \) and \( e_2 = L - E_3 - E_4 \). From Lemma 4.25 we know that there are 138 exceptional curves intersecting both \( e_1 \) and \( e_2 \) positively. Since it is too tedious to compute all maximal cliques of size nine in the graph on these 138 exceptional curves by hand, we use MAGMA to compute them, and check that they all contain at least one pair.

**Lemma 4.31.** Let \( K \) be a clique of size eleven without pairs. Then \( K \) is contained in a clique of size twelve without pairs.

**Proof.** The clique \( K \) is not maximal by the previous lemma, so it is contained in a clique of size twelve. Let \( e_1 = L - E_1 - E_2 \) and \( e_2 = L - E_3 - E_4 \). With MAGMA, we compute all (not necessarily maximal) cliques of size twelve in the graph on these 138 curves. There are no cliques of size twelve with only one pair. Therefore, if \( K \) were contained in a clique of size twelve that has a pair, \( K \) would contain a pair too, which is a contradiction.

**Proposition 4.32.** Assume that \( \text{char } k = 0 \). Then the number of exceptional curves that go through one point outside the ramification curve of \( \varphi \) is at most ten.

**Proof.** By Lemma 4.26 we know that it is at most twelve. Consider the twelve classes

\[
\begin{align*}
e_1 &= L - E_1 - E_2; \\
e_2 &= L - E_3 - E_4; \\
e_3 &= 2L - E_1 - E_3 - E_5 - E_6 - E_7; \\
e_4 &= 2L - E_1 - E_4 - E_5 - E_6 - E_8; \\
e_5 &= 2L - E_2 - E_3 - E_5 - E_7 - E_8; \\
e_6 &= 2L - E_2 - E_4 - E_6 - E_7 - E_8; \\
e_7 &= 4L - 2E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - 2E_7 - 2E_8; \\
e_8 &= 4L - E_1 - 2E_2 - E_3 - E_4 - 2E_5 - 2E_6 - E_7 - E_8; \\
e_9 &= 4L - E_1 - E_2 - 2E_3 - E_4 - E_5 - 2E_6 - E_7 - 2E_8; \\
e_{10} &= 4L - E_1 - E_2 - E_3 - 2E_4 - 2E_5 - E_6 - E_7 - E_8; \\
e_{11} &= 5L - 2E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - E_6 - E_7 - 2E_8; \\
e_{12} &= 5L - 2E_1 - 2E_2 - 2E_3 - 2E_4 - E_5 - 2E_6 - 2E_7 - E_8.
\end{align*}
\]

It is straightforward to check that they form a clique without pairs. By Remark 2.9
we know that $e_1, \ldots, e_{10}$ are the classes in Pic $X$ of the strict transforms of the curves $L_1, L_2, C_1, \ldots, C_4, D_1, \ldots, D_4$, defined with respect to $Q_i = P_i$ for $i \in \{1, \ldots, 8\}$. Let $T = \{c_1, \ldots, c_{12}\}$ be a clique of size twelve that does not contain any pairs. By Proposition 4.27, after changing the indexes if necessary, there is an element $g \in G$ such that $c_i = g(e_i)$ for $i \in \{1, \ldots, 12\}$. Let $E'_i = g(E_i)$. Then, since the $E'_i$ are pairwise disjoint, by Lemma 3.9 we can blow down $E'_1, \ldots, E'_8$ to points $R_1, \ldots, R_8 \in \mathbb{P}^2$ that are in general position, such that $X$ is the blow-up of $\mathbb{P}^2$ at $R_1, \ldots, R_8$, and $E'_i$ is the class in Pic $X$ of the exceptional curve above $R_i$ for all $i$. By the bijection in Remark 3.12, the elements $c_1, \ldots, c_{10}$ are the classes of the strict transforms of $L_1, L_2, C_1, \ldots, C_4, D_1, \ldots, D_4$ defined with respect to $Q_i = R_i$ for $i \in \{1, \ldots, 8\}$. From Proposition 4.29 it follows that the curves corresponding to $c_1, \ldots, c_{10}$ can not all go through one point.

Since every set of twelve exceptional curves without pairs corresponds to a clique of size twelve without pairs, we conclude that the number of exceptional curves going through one point outside the ramification curve of $\varphi$ is less than twelve.

Let $K$ be a clique of size eleven without pairs. By Lemma 4.31, $K$ is contained in a clique of size twelve without pairs, say $H$. As we just showed, the clique $H$ contains a set of ten classes $\{d_1, \ldots, d_{10}\}$ such that the corresponding curves do not all go through one point. By Corollary 4.28 there is an element $g \in G_H$ such that $K$ contains $g(d_1), \ldots, g(d_{10})$. But then, as we did above, we can blow down $g(E_1), \ldots, g(E_8)$, such that $g(d_1), \ldots, g(d_{10})$ are the classes of the strict transforms of $L_1, L_2, C_1, \ldots, C_4, D_1, \ldots, D_4$ defined with respect to $Q_i = g(E_i)$ for $i \in \{1, \ldots, 8\}$, which can not all go through one point. We conclude that the elements in $K$ can not all go through one point, so the maximum is less than eleven.

**Proof of Theorem 2.**

By Lemma 4.26, the number of exceptional curves that go through one point outside the branch curve of $X$ is at most twelve. If char $k = 0$, it is at most ten by Proposition 4.32.

The following example is in [SvL14].

**Example 4.33.** Assume that the characteristic of $k$ is unequal to 2,3 and 5. Let $\beta, \delta \in k^*$, and let $S$ be the surface in $\mathbb{P}(2, 3, 1, 1)$ given by

$$y^2 = x^3 + f(z, w)x + g(z, w),$$

where

$$f = -27(\beta^4 + 12\beta^3 + 14\beta^2 - 12\beta + 1)w^4,$$

and

$$g = \delta z^5w + 54(\beta^2 + 1)(\beta^4 + 18\beta^3 + 74\beta^2 - 18\beta + 1)w^6.$$

Assume that $S$ is smooth, so it is a del Pezzo surface of degree one. Consider the point $Q = (x_0 : y_0 : 0 : 1) \in S$ with $x_0 = 3(\beta^2 + 6\beta + 1)$, and $y_0 = 108\beta$. Note that $Q$ is outside that ramification curve of $\varphi$, since $y_0 \neq 0$.

Let $\alpha, \varepsilon$ be in a field extension of $k$ such that $\alpha^2 = \alpha + 1$ and $\delta = -6(\beta + \alpha^5)\varepsilon^5$.

Since char $k \neq 2, 3, 5$, there are ten such pairs $(\alpha, \varepsilon)$. Now consider the curve $C_{\alpha, \varepsilon}$ in $\mathbb{P}(2, 3, 1, 1)$ defined by

$$x = \varepsilon^2 z^2 + 6\alpha\varepsilon zw + x_0 w^2,$$

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\[ y = -\varepsilon^3 z^3 + 3(\beta + 2\alpha + 3)\varepsilon^2 z^2 w + 18\alpha(\beta + 1)\varepsilon zw^2 + y_0 w^3. \]

Let \( \mu \) be the restriction to \( U = \mathbb{P}(2, 3, 1, 1) - \{z = w = 0\} \) of the projection \( \mathbb{P}(2, 3, 1, 1) \rightarrow \mathbb{P}^1_{k(\alpha, \varepsilon)} \) on the last two coordinates. By Lemma 2.1 in [SvL14], the curve \( C_{\alpha, \varepsilon} \) is a section of \( \mu \). Moreover, it is a quick check that \( C_{\alpha, \varepsilon} \) is contained in \( S \), so from Lemma 2.2 in [SvL14] it follows that \( C_{\alpha, \varepsilon} \) is an exceptional curve in \( S \) over \( k(\alpha, \varepsilon) \). It is easy to see that \( Q \) is contained in \( C_{\alpha, \varepsilon} \). Since there are ten pairs \( (\alpha, \varepsilon) \), we conclude there are ten exceptional curves through \( Q \) over a field extension of \( k \).
References


