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Generalising canonical extensions of posets to categories

Master's thesis, 21 December 2015

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Contents

1 Introduction

A poset is a set together with a reflexive, antisymmetric and transitive relation. One can see any poset P as a category $\mathcal{P}$, by setting the objects of $\mathcal{P}$ as the elements of the set P and by defining there to be a unique arrow from one object to another if and only if the first object is smaller than or equal to the second. We will call such a category a poset category, not to be confused with the category of posets. The category of posets is a full subcategory of the category of small categories Cat.

1.1 Overview of the thesis

Chapter 2 through Paragraph 5.1.1 are a review of already existing work.

In this thesis we will see categories as more general versions of posets, and we will investigate for specific completions of posets whether these completions have a generalisation to categories.

In Chapter 2 we will look at the well-known lower set completion and the category of presheaves which will be shown to be the free cocompletion of posets and the free cocompletion of categories, respectively.

Secondly, in Chapter 3 we will explore the ideal completion of posets and the Ind-completion of categories. We will find in Theorem 3.9 that the ideal completion of a poset is the directed join completion of that poset and in Theorem 3.20 we will see that the Ind-completion of a given category is the filtered colimit completion of that category.

Then, in Chapter 4 we will explore the relations between the Dedekind-MacNeille completion and its less well-known categorical counterpart: the reflexive completion.

In chapter 5.1 will try to find a generalisation of the canonical extension to categories. In Paragraph 5.2.1 will generalise the explicit construction of posets found in Paragraph 5.1.2 to a canonical extension of categories. In Paragraph 5.2.3 we will suggest a generalisation to categories of the characterisation of canonical extensions of posets given in Paragraph 5.1.1. These two notions coincide in the case of posets. In Paragraph 5.2.4 we will investigate whether there are conditions under which these notions coincide in the case of categories as well.

Lastly, in Chapter 6 we will investigate which properties of the canonical extension of posets can be generalised to the second definition we give of canonical extensions of posets (defined in Paragraph 5.2.1). First we will see in Paragraph 6.1 that like canonical extensions of posets, canonical extensions of categories do not seem to be functorial. In Paragraph 6.2 we will see that like in the poset case, taking the canonical extension of categories commutes with taking the opposite. Lastly we investigate in Paragraph 6.3 whether taking the canonical extension of categories commutes with taking the product, but we do not find a definite answer.

To summarise, consider the following sketch

Lower set completion	$\rightsquigarrow$	Category of presheaves
Ideal completion and filter completion	$\rightsquigarrow$	Ind-completion and Pro-completion
Dedekind-MacNeille completion	$\rightsquigarrow$	Reflexive completion
Canonical extension	$\rightsquigarrow$	Canonical extension of categories

The above notions heavily depend on each other. Indeed, to define the reflexive completion and to show existence and unicity of the Dedekind-MacNeille completion we will need the category of presheaves and the lower set completion, respectively. To define canonical extension of categories we will need the category of presheaves, the Ind-completion and the Pro-completion and to show existence and unicity of the canonical extension of posets we will need the lower set completion, the ideal completion and the filter completion.

1.2 Acknowledgements

I am very grateful to Benno van den Berg for supervising me. I enjoyed working on the interesting topic that he proposed, a topic which I would not have discovered without him. He guided me by suggesting ideas worth investigating, gave me the freedom to explore them on my own, but also helped me solve specific problems when I asked for help.

Furthermore I would like to thank the reading committee consisting of Benno van den Berg, Robin de Jong and Ronald van Luijk for reading my thesis on such a short notice.

Lastly I would like to thank Niels uit de Bos and Raoul Wols for reading and discussing parts of my thesis with me.

1.3 Convention, notation and terminology

We will assume that the reader is familiar with the notion of limits, colimits, adjoint functors, presheaves, the Yoneda embedding and the Yoneda lemma.

The following are some notations that we will use. We denote the objects of a category $\mathcal{D}$ by $\mathcal{D}_0$ and the arrows by $\mathcal{D}_1$. For two specific objects $d, d' \in \mathcal{D}_0$ we will denote the arrows between them by $\mathcal{D}(d, d')$ and we will denote the representable functor $\mathcal{D}^{\text{op}} \to \text{Sets}$ corresponding to d by $\mathcal{D}(-, d)$. If $\mathcal{C}$ and $\mathcal{D}$ are categories, then we will denote the functors from $\mathcal{C}$ to $\mathcal{D}$ by $[\mathcal{C}, \mathcal{D}]$.

We will ignore size issues.

2 Lower sets and Presheaves

In this chapter we will investigate (free) (co)completions of posets and categories. We will see that for posets the notions of completeness and of cocompleteness coincide. The category of presheaves of a given category is the free cocompletion of that category, see Theorem 2.27. We will see that the lower set completion of a poset has many similarities with the free cocompletion of a category, but they do not coincide: the free cocompletion of the associated category is not in general the same as the category associated to the lower sets.

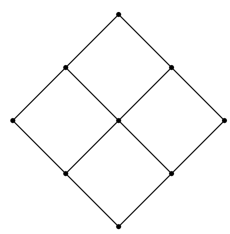
2.1 Lower sets

We will start with the definition of a lower set, also called downset, downward closed set or decreasing set.

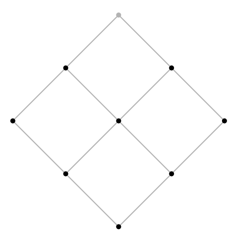
Definition 2.1 (Lower set)**.** Let P be a poset. A subset $I \subseteq P$ is a lower set, if for all $x \in I$ and $y \in P$, $y \leq x$ implies $y \in I$.

Example 2.2. We will represent posets by their Hasse-diagram, which means that we only denote the “generating” relations and that we leave the relations that follow by transitivity and reflexivity implicit. The elements of the poset are represented by points and a line drawn between two points signifies that the lower element is smaller than the higher element.

Consider the following poset



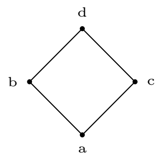
The black dots in the diagram below form a lower set.



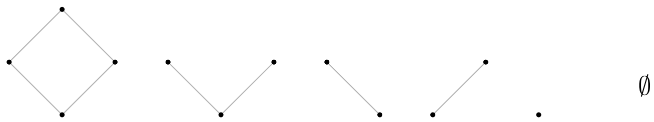
Definition 2.3 (Principal lower set)**.** Let P be a poset. A principal lower set is a lower set with a maximum. Any $p \in P$ is the maximum of a unique principal lower set denoted by $\downarrow p := \{q \in P | q \leq p\}$.

Definition 2.4 (Lower set completion)**.** Let P be a poset. Then the lower set completion $\mathrm{Low}(P)$ is defined as the poset consisting of all lower sets, ordered by inclusion.

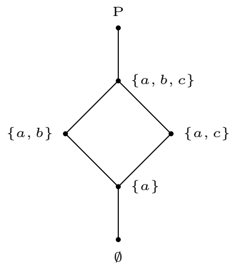
Example 2.5. Consider the following poset P



The lower sets of this poset are



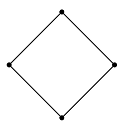
We find that the lower set completion of P is the following poset



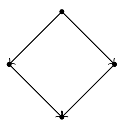
Dually we can define the upper set completion $\mathrm{Up}(P)$ of a poset which consists of all upper sets of P and is ordered by reverse-inclusion.

We can see any lower set as a functor from the opposite of the categorified poset to the category **2**. Here the category **2** is defined as having two objects 0 and 1 and only one non-identity arrow $0 \to 1$. The correspondence works as follows. Let P be a poset and $\mathcal{P}$ its corresponding poset category. To a lower set L of P we associate the functor $F\colon \mathcal{P}^{\mathrm{op}} \to \mathbf{2}$ that maps all objects that come from elements of L to 1 and all other objects to 0. There is only one possibility for the arrows: arrows that come from relations between elements of L are mapped to the identity on 1 and arrows that come from relations between elements outside of L are mapped to 0, while arrows that come from a relation between an element bigger than all elements in L and an element in L are mapped to $0 \to 1$.

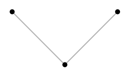
Example 2.6. As an example of the above correspondence, consider the poset



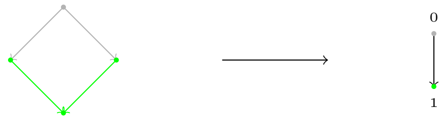
The opposite of the categorified poset can be sketched as follows



The lower set



corresponds to the functor that sends all gray objects to 0 and all green objects to 1 in the following sketch



This comes down to the same as saying that the monomorphisms with as codomain a fixed terminal object (called subterminals) in the category of presheaves $[\mathcal{P}^{\mathrm{op}}, \mathrm{Sets}]$ form exactly $\mathrm{Low}(P)$, as we explain here. The terminal objects in $[\mathcal{P}^{\mathrm{op}}, \mathrm{Sets}]$ are the constant functors mapping to a singleton. A natural transformation from a presheaf to such a terminal presheaf is monic if and only if it is monic on all its components, i.e. if and only if it is injective on all its components. A map to a singleton can only be injective if its domain is the empty set or a singleton. Obviously we can identify $\mathbf{2}$ with the category having as objects the empty set and a singleton and one arrow from the empty set to the singleton.

It follows that there are many more presheaves on the categorification of a poset than lower sets in that poset. Thus, as we remarked in the introduction, the presheaves do not coincide with the lower set completion.

In the same way we can see any representable functor as a principal lower set. Let $\mathcal{P}$ be a poset category. For $p \in \mathcal{P}$ the representable functor $\mathcal{P}(-, p)$ maps $q \leq p$ to $\{*\}$ and $r \not\leq p$ to the emptyset. Hence we can identify $\mathcal{P}(-, p)$ with the principal lower set $\downarrow p$. In the same way we can identify any principal lower set with a representable functor.

As we have seen above we can embed any poset into its lower set completion by the "Yoneda-embedding"

$$
\begin{aligned}
y \colon P &\to \mathrm{Low}(P) \\
p &\mapsto \downarrow p.
\end{aligned}
$$

Dually, we have an order-preserving embedding $P \to \mathrm{Up}(P)$.

Remark 2.7. Any lower set can be written as the union of principal lower sets. To see this, let L be a lower set. Then $L = \cup_{l \in L} \downarrow l$. Since taking unions of lower sets is the same as taking colimits in the poset category corresponding to $\mathrm{Low}(P)$, any lower set can be seen as the colimit of representable functors.

Definition 2.8 (Meet of a subset)**.** Let P be a poset and $A \subseteq P$ a subset. The meet $\wedge A$ of A if it exists, is an element m of P such that m is a lower bound of A (i.e., for all $p \in A$ we have $m \leq p$) and for all lower bounds b of A, we have $m \geq b$.

The meet of a subset can be seen as the infimum of the subset. Dually the join $\vee A$ of a subset A is its supremum. The join and meet of a subset correspond to the categorical product and coproduct of a subset of objects of the poset category, respectively. In a poset category there is at most one arrow between any two elements, so any diagram of arrows commutes. Hence any (co)limit of a diagram of a poset category can be seen as a (co)product.

Definition 2.9 (Complete poset)**.** A complete poset is a poset in which all subsets have a meet.

Dually, we define a cocomplete poset as a poset in which all subsets have a join. However the following proposition says that the notions of a complete and cocomplete poset coincide.

Proposition 2.10. *A poset is complete if and only if it is cocomplete.*

Proof. See [3]. Let P be a poset and $A \subseteq P$ a subset.

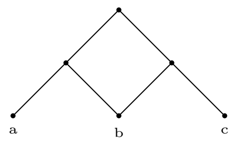
$$\Longrightarrow$$

Suppose that P is complete. We easily see that $\wedge\{p \in P | \forall a \in A : p \geq a\}$ is the join of A. (This simply expressses the fact that $\vee A$ is the smallest upper bound.) Hence $\vee A \in P$.

$$\Longleftarrow$$

Suppose that P is cocomplete. Dually, we see that $\vee\{p \in P | \forall a \in A : p \leq a\}$ is the meet of A. Hence $\wedge A \in P$. ☐

Example 2.11. We will give an example of a poset P that has almost all joins, so it looks cocomplete, but it is not.



The empty join (the smallest element) does not exist, which is why the smallest three elements do not have a meet.

Lemma 2.12. *Let P be a poset and $A \subset \mathrm{Low}(P)$ any subset. Then $\wedge A$ exists and equals $\bigcap A$; and similarly $\vee A$ exists and equals $\bigcup A$.*

Proof. We first prove $\wedge A = \bigcap A$. It is clear that the only difficulty is in proving that $\bigcap A$ is a lower set. Take any $a \in \bigcap A$ and $b \in P$ with $b \leq a$. Since every $T \in A$ is a lower set, $a \in T$ implies $b \in T$. We conclude $b \in T$ for all $T \in A$, hence $b \in \bigcap A$.

For the second part, the difficulty is again only in proving $\bigcup A$ is a lower set, so take any $a \in \bigcup A$ and $b \in P$ with $b \leq a$. Then there is a $T \in A$ with $a \in T$, hence $b \in T$ and therefore $b \in \bigcup A$. ☐

Corollary 2.13. *For any poset P the lower set completion* $\mathrm{Low}(P)$ *is complete.*

Dually, $\mathrm{Up}(P)$ is complete as well.

Definition 2.14 (Extension of a poset)**.** Let P be a poset. An extension of P is an order-embedding $e\colon P \to Q$, i.e. an order-preserving injection.

Definition 2.15 (Completion of a poset)**.** Let P be a poset. A completion of P is an extension $e\colon P \to Q$ such that Q is complete.

Proposition 2.16. *The "Yoneda-embedding"*

$$y\colon P \hookrightarrow \mathrm{Low}(P)$$
$$p \mapsto \downarrow p$$

is a completion of P.

Proof. The map is obviously injective. Let $p, q \in P$. It is clear that $p \leq q$ if and only if $\downarrow p \subseteq \downarrow q$. Furthermore, we have seen that $\mathrm{Low}(P)$ is complete. $\square$

Theorem 2.17. *Let $F\colon P \to Q$ be an order-preserving map with Q a complete poset. Then there exists a unique join- and order-preserving map U such that the following diagram commutes*

$$\begin{tikzcd} P \arrow[rr, "F"] \arrow[dr, "y"'] & & Q \\ & \mathrm{Low}(P) \arrow[ur, dotted, "U"'] & \end{tikzcd}$$

Proof. Since every element of $\mathrm{Low}(P)$ is the join of principal lower sets, the unique map to make this diagram commute is given by

$$U\colon \mathrm{Low}(P) \to Q$$
$$A = \vee_{a\in A} \downarrow a \mapsto \vee_{a\in A} F(a).$$

Since Q is complete, $\vee_{a\in A} F(a)$ is an element of Q.

The map is order-preserving: from $A \subseteq B$ clearly follows $\vee_{a\in A} F(a) \leq \vee_{b\in B} F(b)$.

To prove the preservation of joins, let $J \subset \mathrm{Low}(P)$ be any subset. Then $\vee J = \bigcup_{j\in J} j = \bigcup_{j\in J} \left(\bigcup_{a\in j} \downarrow a\right) = \vee_{a\in\bigcup J} \downarrow a$ (see Lemma 2.12 for the first equality and Remark 2.7 for the second). This is mapped to $\vee_{a\in\bigcup J} F(a)$ which is indeed equal to $\vee_{j\in J} U(j) = \vee_{j\in J}(\vee_{a\in j} F(a)) = \vee_{a\in\bigcup J} F(a)$. $\square$

As we will see in the following subsection, there exists a similar theorem for categories.

2.2 Some facts about presheaves and co-presheaves

We will refer to elements of $[\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}]$ as presheaves and to elements of $[\mathcal{C}, \mathrm{Sets}]^{\,\mathrm{op}}$ as copresheaves. (It makes sense to look at $[\mathcal{C}, \mathrm{Sets}]^{\,\mathrm{op}}$ instead of at $[\mathcal{C}, \mathrm{Sets}]$ because the co-yoneda embedding $\mathcal{C} \to [\mathcal{C}, \mathrm{Sets}]$ given by $c \mapsto \mathrm{Hom}(c, -)$ reverses arrows.) However we will often consider $[\mathcal{C}, \mathrm{Sets}]$ instead of $[\mathcal{C}, \mathrm{Sets}]^{\,\mathrm{op}}$ because the objects are the same anyway and because it looks more friendly. We will denote the Yoneda embedding by $y\colon \mathcal{C}^{\,\mathrm{op}} \to [\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}]$.

The following is a well-known fact, the proof of which can be found in many references.

Theorem 2.18. *Let $\mathcal{C}$ be a small category. Any presheaf $P \in [\mathcal{C}^{\,op}, \mathrm{Sets}]$ is a colimit of representable functors in $[\mathcal{C}^{\,op}, \mathrm{Sets}]$.*

Proof. See [11] Paragraph I.5. The proof goes as follows. Given a presheaf P we construct some diagram of which the colimit is the same as P. We check that P is the same as this colimit by seeing that P satisfies the universal property of the colimit.

Define the index category J by taking objects

$$J_0 := \{(T,x) | T \in \mathcal{C}_0, x \in P(T)\}$$

and arrows

$$J_1\big((T',x'),(T,x)\big) := \{(f\colon T' \to T) \in \mathcal{C}_1 | P(f)(x) = x'\}$$

i.e. we define J as the category of elements of P. (One can see the category of elements as the comma category $* \downarrow P$, where $*$ denotes the functor from the category with only one object and one arrow to the category Sets that maps its object to a singleton.)

We define the diagram as

$$\begin{aligned} D\colon J &\to [\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}] \\ (T,x) &\mapsto \mathcal{C}(-,T) \\ \big(f\colon (T',x') \to (T,x)\big) &\mapsto \big(y(f)\colon \mathcal{C}(-,T') \to \mathcal{C}(-,T)\big). \end{aligned}$$

A natural transformation $P \to F$ consists of functions $P(T) \to F(T)$ for all $T \in \mathcal{C}$, such that for any arrow $f\colon T' \to T$ in $\mathcal{C}$ the following diagram must commute

$$\begin{tikzcd} P(T) \arrow[r] \arrow[d, "P(f)"'] & F(T) \arrow[d, "F(f)"] \\ P(T') \arrow[r] & F(T') \end{tikzcd}$$

Note that by Yoneda's lemma there is a bijection between $F(T)$ and $[\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}]\big(\mathcal{C}(-,T), F\big)$. Hence the above description of an arrow $P \to F$ corresponds bijectively to arrows

$$\begin{aligned} P(T) &\to [\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}]\big(\mathcal{C}(-,T), F\big) \cong F(T) \\ x &\mapsto (\eta_x \colon \mathcal{C}(-,T) \to F). \end{aligned}$$

for all $T \in \mathcal{C}$ such that for any arrow $f\colon T' \to T$ in $\mathcal{C}$ the following diagram commutes

$$\begin{tikzcd} P(T) \arrow[r] \arrow[d, "P(f)"'] & F(T) \arrow[d, "F(f)"] \\ P(T') \arrow[r] & F(T') \end{tikzcd}$$

Such a family of arrows corresponds bijectively to a family of pairs

$$\big((T,x), \eta_x\big)_{T \in \mathcal{C}, x \in P(T)}$$

such that for any arrow $f\colon T' \to T$ in $\mathcal{C}$ and any $x \in T$ we have $\eta_x \circ y(f) = F(f)(\eta_x) = \eta_{P(f)(x)}$.

Such a family of pairs corresponds bijectively to a family of pairs

$$\big(D(T,x),\eta_x\big)_{T\in\mathcal{C},x\in P(T)}$$

such that for any arrow $f\colon T'\to T$ in $\mathcal{C}$ and any $x\in T$ we have $\eta_x\circ y(f)=\eta_{P(f)(x)}$.

Such a family of pairs corresponds bijectively to a family of arrows

$$\big(D(T,x)\to F\big)_{T\in\mathcal{C},x\in P(T)}$$

such that for every $f\colon T'\to T$ the following diagram commutes

$$\begin{tikzcd}
\mathcal{C}(-,T') \arrow[r, "y(f)"] \arrow[dr, "\eta_{P(f)(x)}"'] & \mathcal{C}(-,T) \arrow[d, "\eta_x"] \\
 & F
\end{tikzcd}$$

Note that we can apply the Yoneda lemma to P and view the arrow $P\to F$ as a family of arrows

$$[\mathcal{C}^{\text{op}},\text{Sets}]\big(\mathcal{C}(-,T),P\big)\to[\mathcal{C}^{\text{op}},\text{Sets}]\big(\mathcal{C}(-,T),F\big)$$
$$(\epsilon_x\colon\mathcal{C}(-,T)\to P)\mapsto(\eta_x\colon\mathcal{C}(-,T)\to F).$$

We use h as name for the map $P\to F$. With this notation the following diagram commutes for all $x\in P(T)$

$$\begin{tikzcd}
\mathcal{C}(-,T) \arrow[r, "\epsilon_x"] \arrow[dr, "\eta_x"'] & P \arrow[d, "h"] \\
 & F
\end{tikzcd}$$

since ϵ_x is the unique natural transformation that maps id_T to $x\in P(T)$ and η_x is the unique natural transformation that maps id_T to $h(x)\in F(T)$.

We find that a family of arrows $\big(D(T,x)\to F\big)_{T\in\mathcal{C},x\in P(T)}$ such that for every $f\colon T'\to T$ the following diagram commutes

$$\begin{tikzcd}
\mathcal{C}(-,T') \arrow[r, "y(f)"] \arrow[dr, "\eta_{P(f)(x)}"'] & \mathcal{C}(-,T) \arrow[d, "\eta_x"] \\
 & F
\end{tikzcd}$$

corresponds bijectively to a family of arrows

$$\big(D(T,x)\to F\big)_{T\in\mathcal{C},x\in P(T)}$$

such that for every $f\colon T'\to T$ the following diagram commutes

$$\begin{tikzcd}
\mathcal{C}(-,T') \arrow[r] \arrow[dr] \arrow[ddr, bend right, "\eta_{P(f)(x)}"'] & \mathcal{C}(-,T) \arrow[d] \arrow[dd, bend left, "\eta_x"] \\
 & P \arrow[d] \\
 & F
\end{tikzcd}$$

Since this holds for any functor F, we can conclude that P has the universal property of the colimit of the diagram D. Hence P is isomorphic to the colimit of the diagram D. $\square$

Corollary 2.19. *Let $\mathcal{C}$ be a small category. Any functor $Q \in [\mathcal{C}, \text{Sets}]$ can be written as a colimit of representable functors in $[\mathcal{C}, \text{Sets}]$.*

Proof. Note that $\mathcal{C} = (\mathcal{C}^{\text{op}})^{\text{op}}$. The above theorem now gives us the statement that we want. $\square$

Corollary 2.20. *Let $\mathcal{C}$ be a small category. Any copresheaf $Q \in [\mathcal{C}, \text{Sets}]^{op}$ can be written as a limit of representable functors in $[\mathcal{C}, \text{Sets}]^{op}$.*

Definition 2.21 (Cocontinuous)**.** We will call a functor $F \colon \mathcal{C} \to \mathcal{D}$ cocontinuous if it preserves colimits: for any diagram $D \colon J \to \mathcal{C}$, if $\operatorname{colim}_{j\in J} D(j)$ exists, then $\operatorname{colim}_{j\in J} F(D(j))$ exists and we have $F(\operatorname{colim}_{j\in J} D(j)) \cong \operatorname{colim}_{j\in J} F(D(j))$.

Note that we do not demand that $\mathcal{C}$ has all colimits.

Corollary 2.22. *Let $\mathcal{C}$ be a small category. Any cocontinuous functor from $[\mathcal{C}^{op}, \text{Sets}]$ to any category $\mathcal{D}$ is uniquely determined by the images of the representable functors in $[\mathcal{C}^{op}, \text{Sets}]$.*

Proof. Any object in $[\mathcal{C}^{\text{op}}, \text{Sets}]$ can be written as a colimit of the objects of $[\mathcal{C}^{\text{op}}, \text{Sets}]$ that are representable functors. Hence a cocontinuous functor is determined by its image of representable functors. $\square$

Analogously any continuous functor from $[\mathcal{C}, \text{Sets}]^{\text{op}}$ to any category $\mathcal{D}$ is uniquely determined by the images of the representable functors.

Let $\mathcal{C}$ be a category. We will see that $[\mathcal{C}^{\text{op}}, \text{Sets}]$ is the free cocompletion of $\mathcal{C}$.

Definition 2.23 (Complete category)**.** A category $\mathcal{C}$ is complete, if it has all small limits.

Dually a category is cocomplete if it has all small colimits.

Note that a category can be complete but not cocomplete and vice versa.

The following is an interesting fact.

Proposition 2.24. *Let $\mathcal{C}$ be a small and complete category. Then $\mathcal{C}$ is a preorder.*

Proof. The following proof is due to Peter Freyd. Let $A, B \in \mathcal{C}_0$. Because $\mathcal{C}$ is complete, the product $\prod_{\mathcal{C}_1} B$ exists in $\mathcal{C}$. Suppose there exist two different arrows $A \to B$. Then there exist $2^{\mathcal{C}_1}$ many arrows $A \to \prod_{\mathcal{C}_1} B$. However $\sharp 2^{\mathcal{C}_1} > \sharp \mathcal{C}_1$, which gives a contradiction. $\square$

Hence if $\mathcal{C}$ is a small category and not a preorder and we embed $\mathcal{C}$ into a complete category $\mathcal{D}$, then $\mathcal{D}$ is not small.

The following is a well-known fact.

Proposition 2.25. *Let $\mathcal{D}$ be a small category. Limits and colimits in $[\mathcal{D}, \text{Sets}]$ are computed pointwise and therefore $[\mathcal{D}, \text{Sets}]$ is complete and cocomplete.*

To illustrate this, consider the example of two representable functors $\mathcal{D}(-,d_1)$ and $\mathcal{D}(-,d_2)$ and the functor that maps an object d to the limit $\mathcal{D}(d,d_1)\times_{\text{Sets}}\mathcal{D}(d,d_2)$. Let $G\colon\mathcal{D}\to\text{Sets}$ be a functor and $G\to\mathcal{D}(-,d_1)$ and $G\to\mathcal{D}(-,d_2)$ natural transformations. Then by the universal property of $\mathcal{D}(d,d_1)\times_{\text{Sets}}\mathcal{D}(d,d_2)$ there exists a unique map η_d that makes the following diagram commute

$$\begin{tikzcd}
 & G(d) \arrow[ddl, bend right] \arrow[d, dotted] \arrow[ddr, bend left] & \\
 & \mathcal{D}(d,d_1)\times_{\text{Sets}}\mathcal{D}(d,d_2) \arrow[dl] \arrow[dr] & \\
\mathcal{D}(d,d_1) & & \mathcal{D}(d,d_2)
\end{tikzcd}$$

These η_d form a natural transformation $G\to\mathcal{D}(-,d_1)\times\mathcal{D}(-,d_2)$.

Proof. We will give an idea of the proof. Let $D\colon J\to[\mathcal{D},\text{Sets}]$ be a diagram. Note that the category Sets is complete and cocomplete. We claim that the presheaf

$$\begin{aligned} P\colon \mathcal{D} &\to \text{Sets} \\ d &\mapsto \lim_{j\in J} D(j)(d) \end{aligned}$$

is the limit of D. Let $F\colon\mathcal{D}\to\text{Sets}$ together with natural transformations be a cone of D. Then we can define a natural transformation $\eta\colon P\Rightarrow F$ by using the universal property of $\lim_{j\in J}D(j)(d)$ and letting η_d be the morphism induced by the universal property

$$\eta_d\colon \lim_{j\in J} D(j)(d)\to F(d).$$

Note that the naturality of η follows by the uniqueness of the limit. We have now proved that limits in $[\mathcal{D},\text{Sets}]$ are computed pointwise. Analogously, colimits in $[\mathcal{D},\text{Sets}]$ are computed pointwise. Because Sets is complete and cocomplete and because limits and colimits are computed pointwise, $[\mathcal{D},\text{Sets}]$ is complete and cocomplete. $\square$

Proposition 2.26. *Let $\mathcal{D}$ be a category and $D\in\mathcal{D}_0$. Let $\operatorname{colim}_{j\in J}D_j$ be a colimit in $\mathcal{D}$. Then $\mathcal{D}(\operatorname{colim}_{j\in J}D_j,D)=\lim_{j\in J}\mathcal{D}(D_j,D)$ where the limit is taken in* Sets. *Analogously, let $\lim_{i\in I}D_i$ be a limit in $\mathcal{D}$, then $\mathcal{D}(D,\lim_{i\in I}D_i)\cong\lim_{i\in I}\mathcal{D}(D,D_i)$.*

Proof. By the universal property of the limit an arrow $D\to\lim_{i\in I}D_i$ corresponds to a family of arrows $\{\alpha_i\colon D\to D_i\}_{i\in I}$ such that if there exists an arrow $f\colon D_i\to D_j$ then $\alpha_j=f\circ\alpha_i$. This is the same as taking the limit of the $\mathcal{D}(D,D_i)$. Dually, to prove $\mathcal{D}(\operatorname{colim}_{j\in J}D_j,D)=\lim_{j\in J}\mathcal{D}(D_j,D)$, we can use the universal property of the colimit. $\square$

From this it follows immediately that the Yoneda-embedding preserves limits.

Theorem 2.27. *Let $\mathcal{C}$ be a small category. The category of presheaves $[\mathcal{C}^{op},\text{Sets}]$ is the free cocompletion of $\mathcal{C}$ in the sense that*

1. *the Yoneda-embedding* $y\colon \mathcal{C} \to [\mathcal{C}^{op}, \mathrm{Sets}]$ *is an embedding;*

2. $[\mathcal{C}^{op}, \mathrm{Sets}]$ *is cocomplete; and*

3. *it has the following universal property: for any cocomplete category* $\mathcal{D}$ *and any functor* $F\colon \mathcal{C} \to \mathcal{D}$*, there exists a unique (up to isomorphism) cocontinuous functor* U *such that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "y"'] & & \mathcal{D} \\ & {[\mathcal{C}^{op}, \mathrm{Sets}]} \arrow[ur, dotted, "U"'] & \end{tikzcd}$$

Proof. 1. The full- and faithfulness follow because

$$\mathcal{C}(c, c') \cong [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]\big(\mathcal{C}(-, c), \mathcal{C}(-, c')\big)$$

is a particular case of Yoneda's lemma.

2. We already proved this in Proposition 2.25.

3. See [11] paragraph I.5. The idea of the proof is as follows. Since presheaves can be written as a colimit of representable functors, (up to isomorphism) the only functor that can make the diagram commute, is the following one

$$\begin{aligned} U\colon [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}] &\to \mathcal{D} \\ \mathrm{colim}_{j \in J}\, \mathcal{C}(-, c_j) &\mapsto \mathrm{colim}_{j\in J}\, F(c_j). \end{aligned}$$

Since $\mathcal{D}$ is cocomplete, the colimit $\mathrm{colim}_{j\in J}\, F(c_j)$ always exists. Because of the universal property of colimits, this functor is defined on arrows in $[\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]$ as well.

To prove that U is cocontinuous one proves that U is a left adjoint of the following functor

$$\begin{aligned} R\colon \mathcal{D} &\to [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}] \\ d &\mapsto \big(c \mapsto \mathcal{D}(F(c), d)\big). \end{aligned}$$

□

Remark 2.28. Let $\mathcal{C}$ be a small category. By Proposition 2.25, the free cocompletion $[\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]$ is complete and the free completion $[\mathcal{C}, \mathrm{Sets}]^{\mathrm{op}}$ is cocomplete.

Hence by definition of a free cocompletion, there exists a unique cocontinuous functor $[\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}] \to [\mathcal{C}, \mathrm{Sets}]^{\mathrm{op}}$, and dually, there exists a unique continuous functor $[\mathcal{C}, \mathrm{Sets}]^{\mathrm{op}} \to [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]$.

Remark 2.29. Let $\mathcal{C}$ be a small category. Then by Proposition 2.24 and the above $[\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]$ is not small, since it is clearly not a pre-order.

The Yoneda-embedding does not in general preserve colimits. Indeed, there is no reason it should: the universal property of a colimit specifies the arrows going out of it, while the Yoneda-embedding tests an object by the arrows going into it.

Example 2.30. Consider the following category $\mathcal{C}$

$$\begin{tikzcd}
 & d & \\
b \arrow[ur] & & c \arrow[ul] \\
 & a \arrow[ul, bend left] \arrow[ul, bend right] \arrow[ur, bend left] \arrow[ur, bend right] &
\end{tikzcd}$$

with all arrows $a \to d$ coinciding. The coproduct of b and c in this category is d. However $\mathcal{C}(-,b)+\mathcal{C}(-,c) \ncong \mathcal{C}(-,d)$, to see this consider $\mathcal{C}(a,b)+\mathcal{C}(a,c)$, which consists of four elements, while $\mathcal{C}(a,d)$ only consists of one element. Hence we have found an example of a category $\mathcal{C}$ and a (finite) colimit that is not preserved by the Yoneda-embedding.

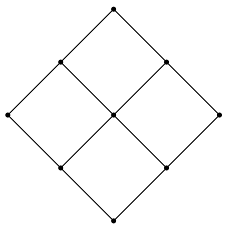
3 Ideal completion and Ind-completion

In this chapter we will explore the ideal completion of posets and the Ind-completion of categories. We will find in Theorem 3.9 that the ideal completion of a poset is the directed join completion of that poset and in Theorem 3.20 we will see that the Ind-completion of a given category is the filtered colimit completion of that category.

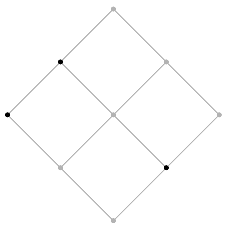
3.1 Ideal completion

Definition 3.1 ((Upward) directed set)**.** Let P be a poset. A subset $D \subseteq P$ is an (upward) directed set, if it is nonempty and for all $x, y \in D$ there exists a $z \in D$ such that $x \leq z$ and $y \leq z$.

Example 3.2. Consider again the following poset



The three black elements form an upward directed set.

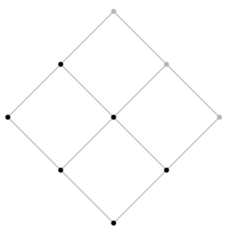


A downward directed set is defined dually: for all $x, y \in D$ there exists a $z \in D$ such that $x \geq z$ and $y \geq z$.

Definition 3.3 (Ideal in a poset)**.** Let P be a poset. A non-empty subset $I \subseteq P$ is an ideal, if it is (upwards) directed and it is a lower set.

The dual notion of an ideal is called a filter.

Example 3.4. Considering again the poset from the previous example, we see that the black elements in the diagram below form an ideal, but not a filter since the set is not an upper set.



Again we have a notion of principality.

Definition 3.5 (Principal ideal)**.** Let P be a poset. A principal ideal is an ideal with a maximal element. Any $p \in P$ is the maximum of a unique principal ideal $\downarrow p := \{q \in P | q \leq p\}$.

Remark that any finite ideal is principal.

Definition 3.6 (Ideal completion)**.** Let P be a poset. The ideal completion $\mathrm{Idl}(P)$ of P is the poset of all ideals of P with inclusion as the ordering.

Dually we have the filter completion which has reverse-inclusion as ordering. Note that the filter completion $\mathrm{Filt}(P)$ of a poset P is equal to the order reverse of the ideal completion $\mathrm{Idl}(P^{\mathrm{op}})$ of the order reverse of P.

We can embed any poset into its ideal completion by the "Yoneda-embedding"

$$\begin{aligned} y \colon P &\to \mathrm{Idl}(P) \\ p &\mapsto \downarrow p. \end{aligned}$$

Definition 3.7 (Directed join)**.** Let P be a poset. A directed join in P is the join of a directed subset $A \subset P$.

Lemma 3.8. *Let P a poset and $A \subset \mathrm{Idl}(P)$ an upward directed subset. Then $\vee A = \bigcup A$.*

Proof. This lemma is similar to Lemma 2.12, but we have the additional condition of directedness to check.

We prove $\vee A = \bigcup A$. It is clear that the only difficulty is in proving that $\bigcap A$ is an ideal. We already know that it is a lower set by Lemma 2.12. Take any $a, a' \in \bigcup A$. There are $I, I' \in A$ with $a \in I$ and $a' \in I'$. Because A is directed, there exists $J \in A$ with $I, I' \subseteq J$ and hence $a, a' \in J$. Because J is directed, there exists $b \in J \subset \bigcup A$ with $a, a' \leq b$. ☐

Theorem 3.9. *Let P and Q be posets, $F \colon P \to Q$ an order-preserving map and assume that Q has all directed joins. Then there exists a unique directed join- and order-preserving map U such that the following diagram commutes*

$$\begin{tikzcd} P \arrow[rr, "F"] \arrow[dr, "y"'] & & Q \\ & \mathrm{Idl}(P) \arrow[ur, dotted, "U"'] & \end{tikzcd}$$

Proof. Since every element of $\mathrm{Idl}(P)$ is the directed join of principal ideals, (indeed, let I be an ideal, then $I = \vee_{i \in I} \downarrow i$) the unique map to make this diagram commute is given by

$$\begin{aligned} U \colon \mathrm{Idl}(P) &\to Q \\ A = \vee_{a \in A} \downarrow a &\mapsto \vee_{a \in A} F(a). \end{aligned}$$

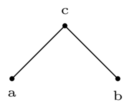
Since Q has all directed joins, $\vee_{a \in A} F(a)$ is an element of Q.

The map is clearly order-preserving.

Suppose we are given a directed subset $A \subset \mathrm{Idl}(P)$. The directed join of these ideals is the union of these ideals, as we saw in Lemma 3.8. The proof then continues as in Theorem 2.17. ☐

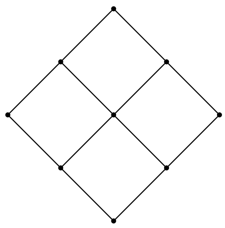
Note however that in the above theorem the map U need not preserve all joins, as the following example shows.

Example 3.10. Consider the following poset P

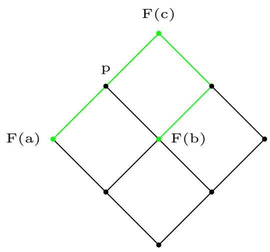


The ideals of this poset are

Note that the following poset Q is complete



We construct a map $F\colon P \to Q$ that sends the relations and elements of P to the green relations and elements as follows



Clearly the join of $F(a)$ and $F(b)$ in Q is p and not $F(c)$. The embedding $y\colon P \to \mathrm{Idl}(P)$ does preserve the join $a \vee b$, since $y(a \vee b) = y(c) = y(a) \vee y(b)$. Hence since F does not preserve the join $a \vee b$, the map U cannot preserve the join $y(c) = y(a) \vee y(b)$. (Indeed, $\downarrow c$ is mapped to $F(c)$ and not to p.)

Lemma 3.11. *Let P be a poset that has all finite joins and let $A \subset \mathrm{Idl}(P)$ be a subset. Then $\wedge A$ exists and equals $\bigcap A$.*

Proof. Since $\mathrm{Idl}(P)$ is ordered by inclusion, it is clear that $\bigcap A$ is the meet, provided that it is an ideal. We will show that this is indeed the case. We already showed that the intersection of lower sets is again a lower set, so we just have to verify that the intersection of directed sets is again a directed set. Let a and b be two elements in the intersection. Because P has finite joins, their join exists in P. Because the ideals of which we took the intersection are lower sets, they must contain the join of a and b. We can conclude that the intersection is a directed set. $\square$

Proposition 3.12. *Let P be a poset that has all finite joins. Then the embedding $y\colon P \to \mathrm{Idl}(P)$ preserves meets.*

Proof. Let m be the meet of the subset $A \subset P$. We need to show

$$\downarrow m = \wedge_{a\in A} \downarrow a.$$

By Lemma 3.11, $\wedge_{a\in A} \downarrow a = \bigcap_{a\in A} \downarrow a$. Since $m \leq a$ for all $a \in A$, $m \in\downarrow a$ for all $a \in A$ and hence $\downarrow m \subset \bigcap_{a\in A} \downarrow a$. Conversely, for any $b \in \bigcap_{a\in A} \downarrow a$, we have $b \leq a$ for all $a \in A$ and therefore $b \leq m$ by definition of meet. □

3.2 Inductive completion

The Ind-completion of a category $\mathcal{C}$ is defined to be the free cocompletion under small filtered colimits (see Theorem 3.20). We construct it in Definition 3.13. It turns out to be a subcategory of the category $[\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]$ of presheaves on $\mathcal{C}$ (see Lemma 3.21) and if $\mathcal{C}$ has all finite colimits, then it is even a reflective subcategory (Proposition 3.32).

The following definition is taken from [8].

Definition 3.13 (Ind-completion of a category)**.** Let $\mathcal{C}$ be a small category. The objects of the Ind-completion $\mathrm{Ind}(\mathcal{C})$ of $\mathcal{C}$ are defined as small filtered diagrams $(A\colon I \to \mathcal{C})$. To define the morphisms between two Ind-objects $(A\colon I \to \mathcal{C})$ and $(B\colon J \to \mathcal{C})$ we first introduce an equivalence relation on morphisms $(f\colon A(i) \to B(j)) \in \mathcal{C}_1$. We say that $f\colon A(i) \to B(j)$ is equivalent to $f'\colon A(i) \to B(j')$ if and only if there exist morphisms $j \to j'', j' \to j'' \in J_1$ such that the following diagram commutes

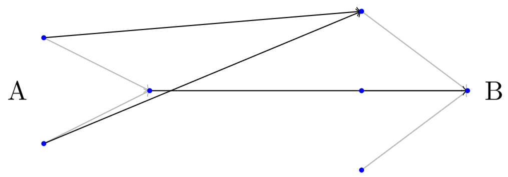
$$\begin{tikzcd} A(i) \arrow[r, "f"] \arrow[d, "f'"'] & B(j) \arrow[d] \\ B(j') \arrow[r] & B(j'') \end{tikzcd}$$

The morphisms $(A\colon I \to \mathcal{C}) \to (B\colon J \to \mathcal{C})$ are now defined as families $(\phi_i)_{i\in I}$, where each ϕ_i is an equivalence class of morphisms $f\colon A(i) \to B(j)$, with the compatibility condition that for any arrow $g\colon i \to i'$ the arrow $\phi_{i'} \circ A(g)\colon A(i) \to A(i')$ is equivalent to $\phi_i\colon A(i) \to B(j)$. Composition of arrows in $\mathrm{Ind}(\mathcal{C})$ is determined by composition of arrows in $\mathcal{C}$.

We will call objects in the Ind-completion Ind-objects or Ind-systems.

Example 3.14. To illustrate what arrows in $\mathrm{Ind}(\mathcal{C})$ look like, we consider the following two Ind-objects

and an example of an arrow between them



Note that for every object a in the Ind-system A the family of arrows $A \to B$ has an arrow in $\mathcal{C}$ with a as domain.

From the above description, it follows that

$$\mathrm{Ind}(\mathcal{C})\big((A\colon I \to \mathcal{C}), (B\colon J \to \mathcal{C})\big) \cong \lim_{i\in I} \mathrm{colim}_{j\in J}\, \mathcal{C}(A(i), B(j)).$$

Note that two Ind-objects $(A\colon I \to \mathcal{C})$ and $(A'\colon I' \to \mathcal{C})$ that happen to have equal colimits, are isomorphic, since for any Ind-object $(B\colon J \to \mathcal{C})$ we have

$$\begin{aligned}
\mathrm{Ind}(\mathcal{C})\big((A\colon I \to \mathcal{C}), (B\colon J \to \mathcal{C})\big) &\cong \lim_{i\in I} \mathrm{colim}_{j\in J}\, \mathcal{C}(A(i), B(j)) \\
&\cong \mathrm{colim}_{j\in J}\, \mathcal{C}(\mathrm{colim}_{i\in I}\, A(i), B(j)) \\
&= \mathrm{colim}_{j\in J}\, \mathcal{C}(\mathrm{colim}_{i\in I'}\, A'(i), B(j)) \\
&\cong \lim_{i\in I'} \mathrm{colim}_{j\in J}\, \mathcal{C}(A'(i), B(j)) \\
&\cong \mathrm{Ind}(\mathcal{C})\big((A'\colon I' \to \mathcal{C}), (B\colon J \to \mathcal{C})\big).
\end{aligned}$$

The construction dual to the Ind-completion is called the projective completion; it will turn out to be the free completion under small cofiltered limits. We define it as the category with as objects the small cofiltered diagrams in $\mathcal{C}$ and as hom-sets

$$\mathrm{Pro}(\mathcal{C})\big((A\colon I \to \mathcal{C}), (B\colon J \to \mathcal{C})\big) \cong \lim_{j\in J} \mathrm{colim}_{i\in I}\, \mathcal{C}(A(i), B(j)).$$

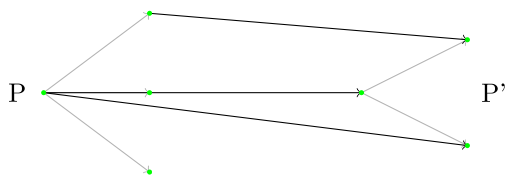
Proposition 3.15. *Let $\mathcal{C}$ be a small category. Then* $\mathrm{Pro}(\mathcal{C}) \cong \mathrm{Ind}(\mathcal{C}^{\,\mathrm{op}})^{\,op}$.

Proof. The objects of $\mathrm{Ind}(\mathcal{C}^{\,\mathrm{op}})^{\,\mathrm{op}}$ are small filtered diagrams in $\mathcal{C}^{\,\mathrm{op}}$, which is the same as small cofiltered diagrams in $\mathcal{C}$. It remains to check that the hom-sets are what they should be. Let $A\colon I \to \mathcal{C}^{\,\mathrm{op}}$ and $B\colon J \to \mathcal{C}^{\,\mathrm{op}}$ be small filtered diagrams. Then

$$\begin{aligned}
\mathrm{Ind}(\mathcal{C}^{\,\mathrm{op}})^{\,\mathrm{op}}\big((A\colon I \to \mathcal{C}^{\,\mathrm{op}}), (B\colon J \to \mathcal{C}^{\,\mathrm{op}})\big) &= \mathrm{Ind}(\mathcal{C}^{\,\mathrm{op}})\big((B\colon J \to \mathcal{C}^{\,\mathrm{op}}), (A\colon I \to \mathcal{C}^{\,\mathrm{op}})\big) \\
&\cong \lim_{j\in J} \mathrm{colim}_{i\in I}\, \mathcal{C}^{\,\mathrm{op}}(B(j), A(i)) \\
&\cong \lim_{j\in J} \mathrm{colim}_{i\in I}\, \mathcal{C}(A(i), B(j)).
\end{aligned}$$

$\square$

Example 3.16. The following is an example of an arrow $P \to P'$ in $\mathrm{Pro}(\mathcal{C})$



Note that for every object p' in the Pro-system P' the family of arrows $P \to P'$ has an arrow in $\mathcal{C}$ with p' as codomain.

Proposition 3.17. *The construction of the Ind-completion* $\mathrm{Ind}(\mathcal{C})$ *out of a small category* $\mathcal{C}$ *is functorial.*

Proof. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is mapped to the functor

$$\begin{aligned} \mathrm{Ind}(F)\colon\ \mathrm{Ind}(\mathcal{C}) &\to \mathrm{Ind}(\mathcal{D}) \\ (A\colon I \to \mathcal{C}) &\mapsto F \circ A. \end{aligned}$$

Clearly $\mathrm{Ind}(\mathrm{Id}_{\mathcal{C}}) = \mathrm{Id}_{\mathcal{C}}$ and $\mathrm{Ind}(G \circ F) = \mathrm{Ind}(G) \circ \mathrm{Ind}(F)$ for all functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$. □

Definition 3.18 (Embedding)**.** We will call a functor an embedding if it is full and faithful.

It is not hard to see that the embedding

$$\begin{aligned} \mathcal{C} &\hookrightarrow \mathrm{Ind}(\mathcal{C}) \\ c &\mapsto (\bar{c}\colon 1 \to \mathcal{C}) \end{aligned}$$

is full and faithful.

Proposition 3.19. *The embedding* $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ *has a left adjoint if and only if* $\mathcal{C}$ *has all filtered colimits.*

Proof. See [8] Lemma C.4.2.5. The proof goes as follows.

$$\Longrightarrow$$

Let $(C_j | j \in J)$ be a filtered diagram, so an Ind-object, and let $D \in \mathcal{C}_0$. Then because the inclusion has a (colimit preserving) left-adjoint L we have

$$\begin{aligned} \mathcal{C}(L(C_j | j \in J), D) &\cong \mathrm{Ind}(\mathcal{C})((C_j | j \in J), i(D)) \\ &= \lim_{j \in J} \mathcal{C}(C_j, D). \end{aligned}$$

Hence $L(C_j | j \in J) \in \mathcal{C}_0$ is a colimit of the filtered diagram $(C_j | j \in J)$.

$$\Longleftarrow$$

Define

$$L\colon\ \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$$

to be the functor that maps a filtered diagram to its colimit. Then

$$\begin{aligned} \mathcal{C}(L(C_j | j \in J), D) &= \mathcal{C}(\mathrm{colim}_{j \in J}(C_j | j \in J), D) \\ &\cong \mathrm{Ind}(\mathcal{C})((C_j | j \in J), i(D)). \end{aligned}$$

□

Recall the definition of a free completion and that the category of presheaves is the free cocompletion, see Theorem 2.27. The Ind-completion is in the following similar way a filtered cocompletion.

Theorem 3.20. *Let $\mathcal{C}$ be a small category. The Ind-completion* $\mathrm{Ind}(\mathcal{C})$ *is the free filtered cocompletion of $\mathcal{C}$ in the sense that*

1. $y\colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$*, defined by mapping $A \in \mathcal{C}_0$ to a diagram over the terminal category 1, is an embedding;*
2. $\mathrm{Ind}(\mathcal{C})$ *has all filtered colimits; and*
3. *it has the following universal property: if $\mathcal{D}$ is a category that has all filtered colimits and if $F\colon \mathcal{C} \to \mathcal{D}$ is a functor there exists a unique (up to isomorphism) filtered colimit preserving functor U such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "y"'] & & \mathcal{D} \\ & \mathrm{Ind}(\mathcal{C}) \arrow[ur, dotted, "U"'] & \end{tikzcd}$$

Proof. 1. We have $\mathrm{Ind}(\mathcal{C})(y(A), y(B)) \cong \lim \operatorname{colim} \mathcal{C}(A, B)$.

2. A filtered colimit of Ind-objects is a filtered colimit of filtered colimits, so it is an Ind-object.

3. Because $\mathcal{D}$ has all filtered colimits we can define the functor $\mathrm{Ind}(\mathcal{D}) \to \mathcal{D}$ that maps a filtered diagram to its colimit. If we precompose this functor with the filtered colimits preserving functor $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$, we get a filtered colimits preserving functor $\mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$.

   Since the objects of $\mathrm{Ind}(\mathcal{C})$ are filtered colimits of images of y, there can only be one filtered colimits preserving functor to make the diagram commute.

□

Lemma 3.21. *The functor*

$$\begin{aligned} \mathrm{Ind}(\mathcal{C}) &\hookrightarrow [\mathcal{C}^{\,op}, \mathrm{Sets}] \\ (A\colon I \to \mathcal{C}) &\mapsto \operatorname{colim}_I y(A(i)) \end{aligned}$$

is an embedding.

Proof. See [7] Chapter VI page 226. Recall that arrows between Ind-objects can be characterised as follows

$$\mathrm{Ind}(\mathcal{C})\big((A\colon I \to \mathcal{C}), (B\colon J \to \mathcal{C})\big) \cong \lim_{i \in I} \operatorname{colim}_{j \in J} [\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}]\big(y(A(i)), y(B(j))\big)$$

We check that the functor is full and faithful

$$\begin{aligned} \mathrm{Ind}(\mathcal{C})\big((A\colon I \to \mathcal{C}), (B\colon J \to \mathcal{C})\big) &\cong \lim_{i \in I} \operatorname{colim}_{j \in J} [\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}]\big(y(A(i)), y(B(j))\big) \\ &\cong \lim_{i \in I} [\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}]\big(y(A(i)), \operatorname{colim}_{i \in I} y(A(i))\big) \\ &\cong [\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}]\big(\operatorname{colim}_{i \in I} y(A(i)), \operatorname{colim}_{i \in I} y(A(i))\big). \end{aligned}$$

□

Proposition 3.22. *Let $\mathcal{C}$ be a small category that has finite colimits. Then a presheaf $P\colon \mathcal{C}^{\,op} \to \mathrm{Sets}$ is a filtered colimit of representables if and only if it preserves all finite limits which exist in $\mathcal{C}^{\,op}$.*

Proof. See [7] Chapter VI proposition 1.3. □

We can conclude that in the case that $\mathcal{C}$ has finite colimits, the Ind-completion of $\mathcal{C}$ is equivalent to the category of finite limit preserving functors $\mathcal{C}^{\,\mathrm{op}} \to \mathrm{Sets}$.

Notation 3.23. We denote the category of finite limit preserving functors $\mathcal{C}^{\,\mathrm{op}} \to \mathrm{Sets}$ as $\mathrm{Cart}[\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}]$.

3.2.1 Reflective subcategories

Definition 3.24 (Reflective subcategory)**.** A full subcategory $\mathcal{C}$ of $\mathcal{D}$ is reflective if the inclusion $\mathcal{C} \hookrightarrow \mathcal{D}$ has a left adjoint.

Example 3.25. Let $\mathcal{C}$ be a small category. The category of sheaves on $\mathcal{C}$ is a reflective subcategory of the category of presheaves on $\mathcal{C}$. The left adjoint is given by sheafification.

Lemma 3.26. *Let $\mathcal{C}$ be a reflective subcategory of $\mathcal{D}$ with inclusion i, left adjoint L, unit η and counit ϵ. Let $D \in \mathcal{D}$. Then the following two conditions are equivalent*

1. *there exists a $C \in \mathcal{C}$ such that $i(C) \cong D$*
2. *for all $X \in \mathcal{D}$ precomposition with the unit*
$$- \circ \eta_X \colon \mathrm{Hom}(iL(X), D) \to \mathrm{Hom}(X, D)$$
is a bijection.

Proof.

$$\Longleftarrow$$

Suppose for all $X \in \mathcal{D}$ precomposition with the unit

$$- \circ \eta_X \colon \mathrm{Hom}(iL(X), D) \to \mathrm{Hom}(X, D)$$

is a bijection. In particular

$$- \circ \eta_D \colon \mathrm{Hom}(iL(D), D) \to \mathrm{Hom}(D, D)$$

is a bijection, so there exists a unique $h\colon iL(D) \to D$ such that $h \circ \eta_D = \mathrm{id}_D$. From this we can obtain the equality $\eta_D \circ h \circ \eta_D = \eta_D = \mathrm{id}_{iL(D)} \circ \eta_D$.

The universal property of units says that for any arrow $f\colon Y \to X$ in $\mathcal{D}$ there exists a unique $g\colon L(Y) \to X$ such that the following diagram commutes

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\eta_Y"'] & i(X) \\ iL(Y) \arrow[ur, "i(g)"'] & \end{tikzcd}$$

so since i is faithful, in particular there exists a unique arrow such that the following diagram commutes

$$\begin{tikzcd} D \arrow[r, "\eta_D"] \arrow[d, "\eta_D"'] & i(L(D)) \\ iL(D) \arrow[ur, dotted] & \end{tikzcd}$$

We can conclude that $\eta_D \circ h = \mathrm{id}_{iL(D)}$, so η_D is an isomorphism and we are done.

$$\Longrightarrow$$

Let $C \in \mathcal{C}$ be such that $i(C) \cong D$. Let $X \in \mathcal{D}$.

Because $D \cong i(C)$ there is a bijection

$$\mathrm{Hom}(iL(X), D) \to \mathrm{Hom}(iL(X), i(C)).$$

Because i is a full embedding

$$\mathrm{Hom}(iL(X), i(C)) \to \mathrm{Hom}(L(X), C)$$
$$f \mapsto i^{-1}(f)$$

is a bijection. Because i and L are adjoint the following is a bijection

$$\mathrm{Hom}(L(X), C) \to \mathrm{Hom}(X, i(C))$$
$$f \mapsto i(f) \circ \eta_X$$

Again, because $D \cong i(C)$ there is a bijection

$$\mathrm{Hom}(X, i(C)) \to \mathrm{Hom}(X, D).$$

We obtain a bijection

$$\mathrm{Hom}(iL(X), D) \to \mathrm{Hom}(X, D)$$

given by precomposition with the unit. □

Proposition 3.27 (If $\mathcal{C}$ is a reflective subcategory of $\mathcal{D}$, then $\mathcal{C}$ has all limits that $\mathcal{D}$ has.)**.** *Let $\mathcal{C}$ be a reflective subcategory of $\mathcal{D}$ with inclusion i, left adjoint L, unit η and counit ϵ. Let $F \colon J \to \mathcal{C}$ be a diagram. If the limit $\lim_{j \in J} i(F_j)$ exists in $\mathcal{D}$ then the diagram F has a limit in $\mathcal{C}$ as well.*

This proposition is exercise 7 of Chapter IV paragraph 3 Reflective subcategories of [10].

Proof. Let I be an index-category. Let $C_i \in \mathcal{C}$ for all $i \in I$ and let $\lim_{i \in I} Y_i \in \mathcal{D}$. Then for all $D \in \mathcal{D}$ we have

$$\mathrm{Hom}(X, \lim_{i \in I} C_i) = \lim_{i \in I} \mathrm{Hom}(X, C_i).$$

Because the C_i are in $\mathcal{C}$ and because of the lemma there is bijection

$$\mathrm{Hom}(X, C_i) \to \mathrm{Hom}(iL(X), X_i)$$

for all $i \in I$, so there is a bijection

$$\lim_{i \in I} \operatorname{Hom}(X, C_i) \to \lim_{i \in I} \operatorname{Hom}(iL(X), X_i).$$

Furthermore

$$\lim_{i \in I} \operatorname{Hom}(iL(X), X_i) = \operatorname{Hom}(iL(X), \lim_{i \in I} X_i).$$

$\square$

Lemma 3.28. *Let $\mathcal{C}$ be a reflective subcategory of $\mathcal{D}$ with inclusion i, left adjoint L, unit η and counit ϵ. Then the counit $\epsilon\colon Li \to 1_{\mathcal{D}}$ is a natural isomorphism.*

Proof. See paragraph IV.3 of [10]. $\square$

Proposition 3.29 (If $\mathcal{C}$ is a reflective subcategory of $\mathcal{D}$, then $\mathcal{C}$ has all colimits that $\mathcal{D}$ has.)**.** *Let $\mathcal{C}$ be a reflective subcategory of $\mathcal{D}$ with inclusion i, left adjoint L, unit η and counit ϵ. Let $F\colon J \to \mathcal{C}$ be a diagram. If the colimit $\operatorname{colim}_{j \in J} i(F_j)$ exists in $\mathcal{D}$, then the diagram F has a colimit in $\mathcal{C}$ as well.*

Proof. Consider a diagram $F\colon J \to \mathcal{C}$. Suppose the colimit $\operatorname{colim}_{j \in J} i(F_j)$ exists in $\mathcal{D}$. Since L is a left adjoint, it preserves colimits. Hence

$$L(\operatorname{colim}_{j \in J} i(F_j)) = \operatorname{colim}_{j \in J} Li(F_j)$$

is a colimit in $\mathcal{C}$. Since $\epsilon\colon Li \to 1_{\mathcal{D}}$ is a natural isomorphism, we know that $\operatorname{colim}_{j \in J} Li(F_j)$ is naturally isomorphic to $\operatorname{colim}_{j \in J} F_j$ and hence this colimit exists in $\mathcal{C}$. $\square$

However colimits need to pass through the reflective functor L. This is the same phenomenon as in algebraic geometry where limits of sheaves are formed as in the category of presheaves, but colimits in the category of presheaves first need to be sheafified for them to be colimits in the category of sheaves. In other words, the inclusion preserves limits and limits of sheaves can be calculated in the category of presheaves, but a colimit in the category of sheaves cannot be computed by taking the colimit in the category of presheaves.

3.2.2 The embedding $\mathcal{C} \hookrightarrow \operatorname{Ind}(\mathcal{C})$

Proposition 3.30. *Let $\mathcal{C}$ be a small category that has finite colimits. Then the embedding $\mathcal{C} \hookrightarrow \operatorname{Ind}(\mathcal{C})$ preserves limits.*

Proof. See [7] Chapter VI proposition 1.7.ii. Since $\mathcal{C}$ has finite colimits, $\operatorname{Ind}(\mathcal{C})$ is equivalent to the subcategory $\operatorname{Cart}[\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]$ of $[\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]$, see Notation 3.23. In [9] it is proved that $\operatorname{Cart}[\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]$ is a reflective subcategory of $[\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]$. Since right adjoints preserve limits the embedding $\operatorname{Cart}[\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}] \to [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]$ preserves limits, so the embedding $\operatorname{Ind}(\mathcal{C}) \to [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]$ preserves limits. Hence, $\mathcal{C} \hookrightarrow \operatorname{Ind}(\mathcal{C})$ preserves limits because the Yoneda-embedding $\mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]$ preserves limits. $\square$

Proposition 3.31. *Let $\mathcal{C}$ be a small category with finite colimits. The embedding $\mathcal{C} \hookrightarrow \operatorname{Ind}(\mathcal{C})$ preserves finite colimits.*

Proof. See [7] Chapter VI proposition 1.6. $\square$

We can summarise that if $\mathcal{C}$ has finite limits and finite colimits, then we are in the following situation.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "\text{Preserves colimits and finite limits}"] \arrow[d, hook, "\text{Preserves limits and finite colimits}"'] & \mathrm{Pro}(\mathcal{C}) \\
\mathrm{Ind}(\mathcal{C}) &
\end{tikzcd}$$

3.2.3 The embedding $\mathrm{Ind}(\mathcal{C}) \hookrightarrow [\mathcal{C}^{\,\mathbf{op}}, \mathrm{Sets}]$

Proposition 3.32. *Let $\mathcal{C}$ be a small category with finite colimits. The category $\mathrm{Ind}(\mathcal{C})$ is a reflective subcategory of $[\mathcal{C}^{\,op}, \mathrm{Sets}]$.*

Proof. Since $\mathcal{C}$ has finite colimits $\mathrm{Ind}(\mathcal{C})$ is equivalent to $\mathrm{Cart}[\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}]$. In [9] it is proved that $\mathrm{Cart}[\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}]$ is a reflective subcategory of $[\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}]$. □

Corollary 3.33. *Let $\mathcal{C}$ be a small category with finite colimits. The embedding $\mathrm{Ind}(\mathcal{C}) \hookrightarrow [\mathcal{C}^{\,op}, \mathrm{Sets}]$ preserves limits.*

Proof. All right adjoints preserve limits. □

Remark 3.34. A big difference between the way $\mathrm{Ind}(\mathcal{C})$ lies reflectively in $[\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}]$ and the way the category of sheaves lies reflectively in $[\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}]$ is that the left adjoint of the latter embedding preserves finite limits and the former does not (in general).

Proposition 3.35. *Let $\mathcal{C}$ be a small category that has finite colimits. The embedding $\mathrm{Ind}(\mathcal{C}) \hookrightarrow [\mathcal{C}^{\,op}, \mathrm{Sets}]$ preserves filtered colimits.*

Proof. See [7] Chapter VI proposition 1.4. Since $\mathcal{C}$ has finite colimits $\mathrm{Ind}(\mathcal{C})$ is equivalent to $\mathrm{Cart}[\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}]$. The embedding $\mathrm{Cart}[\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}] \to [\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}]$ preserves colimits. Indeed a filtered colimit of finite limit preserving functors with codomain Sets is again a finite limit preserving functor since (filtered) (co)limits of functors are calculated pointwise, so the image of a finite limit under a filtered colimit of finite limit preserving functors is a filtered colimit of finite limits and this is a finite limit of filtered colimits since filtered colimits commute with finite limits in Sets. □

Remark 3.36. Suppose that $\mathcal{C}$ has all finite colimits. Then any diagram $J \to \mathcal{C}$ can be made into a filtered diagram without changing the colimit. However the embedding $\mathrm{Ind}(\mathcal{C}) \hookrightarrow [\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}]$ does not preserve finite colimits. Which is why not every colimit of representable functors can be made into a filtered colimit.

We can summarise that if $\mathcal{C}$ has finite limits and finite colimits, the following

diagram applies.

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "\text{Preserves colimits and finite limits}"] \arrow[d, hook, "\text{Preserves limits and finite colimits}"'] & \mathrm{Pro}(\mathcal{C}) \arrow[dr, hook, "\text{Preserves colimits and cofiltered limits}"] & \\
\mathrm{Ind}(\mathcal{C}) \arrow[dr, hook, "\text{Preserves limits and filtered colimits}"] & & {[\mathcal{C}, \mathrm{Sets}]}^{\mathrm{op}} \\
& {[\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]} &
\end{tikzcd}
$$

4 Dedekind-MacNeille completion and a categorical generalisation of it

In this chapter we will investigate the Dedekind-MacNeille completion and its less well-known categorical counterpart: the reflexive completion. Do not confuse reflective subcategories with the reflexive completion.

4.1 Dedekind-MacNeille completion of a poset

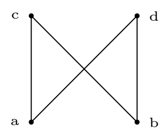
Definition 4.1 (Join-dense)**.** Let Q be an extension of a poset P. Then P is join-dense in Q, if all elements of Q are the join of a subset of P.

Dually, we define a poset P to be meet-dense in Q if all elements of Q are the meet of a subset of P.

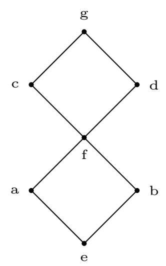
Definition 4.2 (Dedekind-MacNeille completion)**.** Let P be a poset. The Dedekind-MacNeille completion $\mathrm{DM}(P)$ of P is a completion $e\colon P \to \mathrm{DM}(P)$ of P, such that P is both join-dense and meet-dense in $\mathrm{DM}(P)$.

We will prove that every poset has a unique Dedekind-MacNeille completion, but first we will give an example.

Example 4.3. Consider the following finite poset P



This poset is not complete. The following poset Q is complete.



The poset P is also join- and meet-dense in Q, as we will now show. It is already clear that a, b, c and d are both joins and meets of subsets of P. The element g is the join of P, the element f is a join of $\{a, b\}$ and e is the join of the empty set. The element g is the meet of the empty set, the element f is a meet of $\{c, d\}$ and e is the meet of P.

4.1.1 Existence and unicity of Dedekind-MacNeille completions

Notation 4.4 (A^u and A^l)**.** Let P be a poset and A be a subset of P. Then we will denote the set of upper bounds of A by A^u and the set of lower bounds of A by A^l.

Proposition 4.5. *Let P be a poset. Then P has a unique Dedekind-MacNeille completion, given by the set $\{A \subseteq P | (A^u)^l = A\}$ ordered by inclusion.*

Proof. We claim that the map

$$\begin{aligned} P &\hookrightarrow \{A \subseteq P | (A^u)^l = A\} \\ p &\mapsto \downarrow p \end{aligned}$$

is an extension.

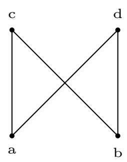
First, we prove that $\downarrow p \in \{A \subseteq P | (A^u)^l = A\}$ for every $p \in P$. In general we have for any subset $A \subseteq P$ that $A \subset (A^u)^l$. Since we clearly have $p \in (\downarrow p)^u$, any $x \in ((\downarrow p)^u)^l$ satisfies in particular $x \leq p$, so we get $(\downarrow p^u)^l \subseteq \downarrow p$.

The set $\{A \subseteq P | (A^u)^l = A\}$ is complete, see Paragraph 7.38 in [1].

Now we prove the poset $\{A \subseteq P | (A^u)^l = A\}$ is join- and meet-dense. Take any such $A \subseteq P$ with $(A^u)^l = A$. Remark that for a family $\{A_i\}_{i \in I}$ of subsets of P we have $(\cup_{i \in I} A_i)^u = \cap_{i \in I} A_i^u$ and $(\cup_{i \in I} A_i)^l = \cap_{i \in I} A_i^l$ Since $A = (A^u)^l$ is a lower set, we know that $A = \bigcup_{a \in A} \downarrow a$. Similarly, A^u is an upper set, so we get $A^u = \bigcup_{a \in A^u} \uparrow a$ and therefore $A = (A^u)^l = (\bigcup_{a \in A^u} \uparrow a)^l = \bigcap_{a \in A^u} \downarrow a$.

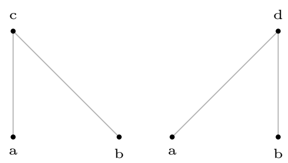
Any completion of P that is join- and meet-dense, is isomorphic to $\{A \subseteq P | (A^u)^l = A\}$ via an isomorphism that fixes P. See Theorem 7.41.ii in [1]. $\square$

Example 4.6. Consider again the following poset P



The upper set of P is empty and the lower set of the empty set is P, so P is in $\mathrm{DM}(P)$.

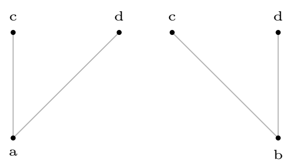
The upper sets of the following subsets



are $\{c\}$ and $\{d\}$, respectively. The lower sets of $\{c\}$ and $\{d\}$ are $\{a, b, c\}$ and $\{a, b, d\}$, respectively. Hence $\{a, b, c\}$ and $\{a, b, d\}$ are in $\mathrm{DM}(P)$.

The upper set of $\{a, b\}$ is $\{c, d\}$ and the lower set of $\{c, d\}$ is $\{a, b\}$, so $\{a, b\}$ is in $\mathrm{DM}(P)$.

The upper set of the singletons $\{a\}$ and $\{b\}$ are the following subsets

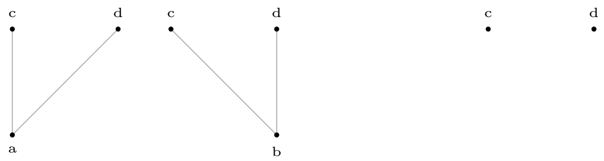


The lower sets of these subsets are $\{a\}$ and $\{b\}$, respectively, so the singletons $\{a\}$ and $\{b\}$ are in $\mathrm{DM}(P)$.

The upper set of the empty set is P and the lower set of P is empty, so the empty set is in $\mathrm{DM}(P)$.

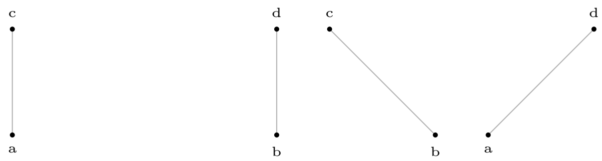
We have now found all elements of $\mathrm{DM}(P)$, which we will check by seeing that all other subsets are not in $\mathrm{DM}(P)$.

The upper sets of the following three subsets



are empty, so their lower sets are P again. Hence these subsets are not in $\mathrm{DM}(P)$.

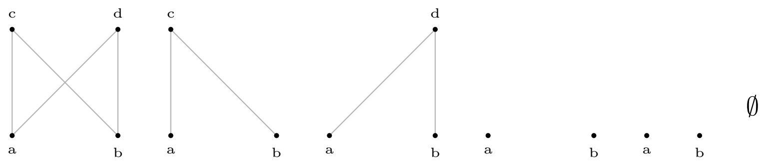
The upper sets of the following four two-element subsets



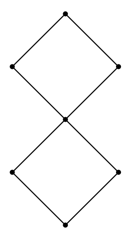
are $\{c\}$, $\{d\}$, $\{c\}$ and $\{d\}$, respectively. The lower sets of $\{c\}$ and $\{d\}$ both contain, as we saw above, three elements. Hence these subsets are not in $\mathrm{DM}(P)$.

The upper sets of the singletons $\{c\}$ and $\{d\}$ are $\{c\}$ and $\{d\}$, respectively. The lower sets of $\{c\}$ and $\{d\}$ are both $\{a, b\}$. Hence the singletons are not in $\mathrm{DM}(P)$.

We can conclude that $\mathrm{DM}(P)$ consists of the following seven elements



If we see these subsets as elements of a poset and order the elements by inclusion, then we indeed get the following poset



4.1.2 Some remarks on Dedekind-MacNeille completions of posets

We can characterise the Dedekind-MacNeille completion of a poset in a way different but equivalent to the definition we gave above. We will introduce this

new characterisation [1] because it can be extended to a Dedekind-MacNeille completion of categories.

Because the poset of upper sets $\mathrm{Up}(P)$ of P is complete and has an order-preserving embedding $P \to \mathrm{Up}(P)$ and because of the universal property of lower sets, see Theorem 2.17, there exists a join-preserving map $-^u\colon \mathrm{Low}(P) \to \mathrm{Up}(P)$. Dually there exists a meet-preserving map $-^l\colon \mathrm{Up}(P) \to \mathrm{Low}(P)$. We will show that these maps are in fact adjoint.

An upper subset B of P can be seen as a functor $B\colon P \to \mathbf{2}$ sending an object $x \in P_0$ to 1 if $x \in B$ and to 0 if $x \notin B$. Dually a lower subset A of P can be seen as a functor $A\colon P^{\,\mathrm{op}} \to \mathbf{2}$. (See Section 2.1.)

For a lower set A and an upper set B we have $A \subseteq B^l$ if and only if $A \times B \subseteq (\leq)$ (where $(\leq) \subset P \times P$ is the set defining the relation) if and only if $A^u \supseteq B$. Hence there exists an arrow from A to B^l in the category that corresponds to $\mathrm{Low}(P)$ if and only if there exists an arrow from B to A^u in the category that corresponds to the poset of upper subsets ordered by inclusion. This gives us an adjunction

$$[P^{\,\mathrm{op}}, \mathbf{2}](A, B^l) \cong [P, \mathbf{2}]^{\,\mathrm{op}}(A^u, B).$$

This restricts to an equivalence of categories (as we explain in more detail after Definition 4.9): the poset of lower sets A for which $(A^u)^l = A$, ordered by inclusion, is isomorphic to the poset of upper sets B for which $(B^l)^u = B$, reverse ordered by inclusion. Note that the poset of lower sets A for which $(A^u)^l = A$ ordered by inclusion is exactly the Dedekind-MacNeille completion of P.

4.2 Reflexive completion

We will now use the above description to generalise the Dedekind-MacNeille completion of posets to categories.

Theorem 4.7. *Let $\mathcal{C}$ be a small category. The functor*

$$\begin{aligned} L\colon [\mathcal{C}^{\,op}, \mathrm{Sets}] &\to [\mathcal{C}, \mathrm{Sets}]^{\,op} \\ P &\mapsto := \big(c \mapsto [\mathcal{C}^{\,op}, \mathrm{Sets}](P, \mathcal{C}(-, c))\big) \end{aligned}$$

and functor

$$\begin{aligned} R\colon [\mathcal{C}, \mathrm{Sets}]^{\,op} &\to [\mathcal{C}^{\,op}, \mathrm{Sets}] \\ Q &\mapsto \hat{Q} := \big(c \mapsto [\mathcal{C}, \mathrm{Sets}](Q, \mathcal{C}(c, -))\big) \end{aligned}$$

form an adjoint pair.

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d, hook] & {[\mathcal{C}, \mathrm{Sets}]^{\,\mathrm{op}}} \arrow[dl, bend right, "R"'] \\ {[\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}]} \arrow[ur, bend right, "L"'] & \end{tikzcd}$$

To be precise, the adjunction is between $[\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}]$ and $[\mathcal{C}, \mathrm{Sets}]^{\,\mathrm{op}}$ i.e. for any $P \in [\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}]$ and any $Q \in [\mathcal{C}, \mathrm{Sets}]^{\,\mathrm{op}}$ the following holds $[\mathcal{C}, \mathrm{Sets}]^{\,\mathrm{op}}(L(P), Q) = [\mathcal{C}, \mathrm{Sets}](Q, L(P)) \cong [\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}](P, R(Q))$.

[1] This characterisation is discussed in http://mathoverflow.net/questions/59291/completion-of-a-category.

Proof. Let $P \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]$ and $Q \in [\mathcal{C}, \mathrm{Sets}]$. Then we can write them as colimits $P = \mathrm{colim}_{j\in J}\, \mathcal{C}(-, p_j)$ and $Q = \mathrm{colim}_{i\in I}\, \mathcal{C}(q_i, -)$. Consider the following two sequences of equalities (written next to each other because the steps are analogous).

$$\begin{array}{ll}
[\mathcal{C}, \mathrm{Sets}](Q, L(P)) & [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}](P, R(Q)) \\
= [\mathcal{C}, \mathrm{Sets}]\big(\mathrm{colim}_{i\in I}\, \mathcal{C}(q_i, -), L(P)\big) & = [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]\big(\mathrm{colim}_{j\in J}\, \mathcal{C}(-, p_j), R(Q)\big) \\
= \lim\limits_{i\in I} [\mathcal{C}, \mathrm{Sets}]\big(\mathcal{C}(q_i, -), L(P)\big) & = \lim\limits_{j\in J} [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]\big(\mathcal{C}(-, p_j), R(Q)\big) \\
= \lim\limits_{i\in I} L(P)(q_i) & = \lim\limits_{j\in J} R(Q)(p_j) \\
= \lim\limits_{i\in I} [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]\big(P, \mathcal{C}(-, q_i)\big) & = \lim\limits_{j\in J} [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]\big(Q, \mathcal{C}(p_j, -)\big) \\
= \lim\limits_{i\in I} [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]\big(\mathrm{colim}_{j\in J}\, \mathcal{C}(-, p_j), \mathcal{C}(-, q_i)\big) & = \lim\limits_{j\in J} [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]\big(\mathrm{colim}_{i\in I}\, \mathcal{C}(q_i, -), \mathcal{C}(p_j, -)\big) \\
= \lim\limits_{i\in I} \lim\limits_{j\in J} [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]\big(\mathcal{C}(-, p_j), \mathcal{C}(-, q_i)\big) & = \lim\limits_{j\in J} \lim\limits_{i\in I} [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]\big(\mathcal{C}(q_i, -), \mathcal{C}(p_j, -)\big)
\end{array}$$

We can conclude that

$$\begin{aligned}
[\mathcal{C}, \mathrm{Sets}](Q, L(P)) &= \lim_{i\in I} \lim_{j\in J} [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]\big(\mathcal{C}(-, p_j), \mathcal{C}(-, q_i)\big) \\
&= \lim_{j\in J} \lim_{i\in I} [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]\big(\mathcal{C}(q_i), \mathcal{C}(p_j, -)\big) \\
&= [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}](P, R(Q)).
\end{aligned}$$

□

This pair is called the Isbell conjugation.

Remark 4.8. Let $\mathcal{C}$ be a small category. Then the functors L and R defined above are cocontinuous and continuous, respectively. In general, for an adjoint pair $F \dashv G$ the functor F is cocontinuous and the functor G is continuous.

Definition 4.9 (Adjoint equivalence)**.** An adjoint pair is called an adjoint equivalence, if the unit and counit are natural isomorphisms.

Remark 4.10. Remark that an adjoint equivalence is an equivalence of categories.

Remark 4.11. We can restrict an arbitrary adjoint pair $(L\colon \mathcal{C} \to \mathcal{D}, R\colon \mathcal{D} \to \mathcal{C})$ to an adjoint equivalence $(L'\colon \mathcal{A} \to \mathcal{B}, R'\colon \mathcal{B} \to \mathcal{A})$ as follows. We let $\mathcal{A}$ be the full subcategory of $\mathcal{C}$ consisting of objects $A \in \mathcal{C}$ for which the unit $\eta_A\colon A \to RL(A)$ is an isomorphism. Similarly, we let $\mathcal{B}$ be the full subcategory of $\mathcal{D}$ consisting of objects $B \in \mathcal{D}$ for which the counit $\epsilon_B\colon LR(B) \to B$ is an isomorphism.

This applies in particular to to the adjoint pair

$$(L\colon [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}] \to [\mathcal{C}, \mathrm{Sets}]^{\mathrm{op}}, R\colon [\mathcal{C}, \mathrm{Sets}]^{\mathrm{op}} \to [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]).$$

In this case, the full subcategory $\mathcal{A} \subset [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]$ is called the reflexive completion and denoted by $\mathrm{Rex}(\mathcal{C})$. The elements of this category are called reflexive presheaves. This is the categorical equivalent of the Dedekind-MacNeille completion.

Similarly to the Dedekind-MacNeille completion being join- and meet-dense, every reflexive presheaf is a colimit and a limit of representable functors.

Proposition 4.12. *Let $\mathcal{C}$ be small a category and let $F \in \mathrm{Rex}(\mathcal{C})$ be a reflexive presheaf. Then F is a colimit in $\mathrm{Rex}(\mathcal{C})$ of representable functors and a limit in $\mathrm{Rex}(\mathcal{C})$ of representable functors.*

Proof. Indeed, F is a colimit of representable functors, simply because it is a presheaf. On the other hand is $RL(F)$ a limit of representable functors, since $L(F)$ is a limit of corepresentable functors (again simply because $L(F)$ is a copresheaf) and since R preserves limits and maps corepresentable functors to the corresponding representable functors. We can conclude that F is both a colimit and a limit of representable functors, since F is isomorphic to $RL(F)$. $\square$

4.2.1 Universal properties of the reflexive completion

For completeness sake we will include the following results that Tom Avery and Tom Leinster announced in their 31 March 2015 talk at the British Mathematical Colloquium, University of Cambridge, see [13].

Definition 4.13 (Dense)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Then F is called dense if the following functor is full and faithful

$$\begin{aligned} \mathcal{D} &\to [\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}] \\ d &\mapsto \mathcal{D}(F(-), d). \end{aligned}$$

Definition 4.14 (Codense)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Then F is called codense if the following functor is full and faithful

$$\begin{aligned} \mathcal{D} &\to [\mathcal{C}, \mathrm{Sets}]^{\,\mathrm{op}} \\ d &\mapsto \mathcal{D}(d, F(-)). \end{aligned}$$

Definition 4.15 (Snug embedding)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be an embedding. Then F is called a snug embedding if it is dense and codense.

Theorem 4.16. *Let $\mathcal{C}$ be a category. The reflexive completion of $\mathcal{C}$ has the following universal property: if $F\colon \mathcal{C} \to \mathcal{D}$ is a snug embedding there exists a unique (up to isomorphism) snug embedding U such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "y"'] & & \mathcal{D} \\ & \mathrm{Rex}(\mathcal{C}) \arrow[ur, dotted, "U"'] & \end{tikzcd}$$

Definition 4.17 (Reflexively complete)**.** Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ is reflexively complete if every reflexive presheaf in $\mathrm{Rex}(\mathcal{C})$ is a representable functor on $\mathcal{C}$.

Theorem 4.18. *Let $\mathcal{C}$ be a category. Then $\mathcal{C} \to \mathrm{Rex}(\mathcal{C})$ is the unique (up to equivalence) snug embedding of $\mathcal{C}$ into a reflexively complete category.*

5 Canonical extensions of posets and categories

We will first give a characterisation of the canonical extension and in Section 5.1.2 we will find an explicit construction for it. In Section 5.2.1 we will generalise the explicit construction to categories and in Paragraph 5.2.3 we will generalise the characterisation given below to categories. In Paragraph 5.2.4 we will investigate to what extent these two notions of canonical extension of a category coincide.

5.1 Canonical extensions of posets

5.1.1 Characterisation of canonical extension of posets

Definition 5.1 (Closed element with respect to a subset)**.** Let Q be an extension of a poset P. Then an element of Q is closed, if it is the meet in Q of some filter of P.

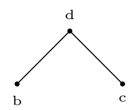
Dually, an open element of an extension Q is the join in Q of some ideal of P.

The following definition is from [4].

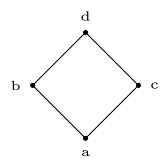
Definition 5.2 (Canonical extension of a poset)**.** Let P be a poset. A canonical extension of P, is a completion $P \hookrightarrow P^\delta$ such that it is

1. dense: every element of P^δ is both the join of all closed elements below it and the meet of all open elements above it; and
2. compact: given a non-empty down-directed subset F of P and a non-empty upward-directed subset I of P such that $\wedge F \leq \vee I$ in P^δ, then there exist $x \in F$ and $y \in I$ with $x \leq y$.

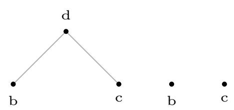
Example 5.3. Consider the following poset P



We want to show that the following completion P^δ of P is dense and compact.



The ideals of P are



The joins of these ideals in P^δ are d, b and c, respectively.

The filters of P are $\{b, d\}$, $\{c, d\}$ and $\{d\}$. The meets of these filters in P^δ are b, c and d, respectively.

Indeed a, b, c and d are all the meets of the open elements above them. It is also clear that b, c and d are the joins of the closed elements below them. The element a is the join of the empty set, so it is the join of all closed elements below itself as well. We can conclude that P^δ is dense.

In this case the non-empty down-directed sets of P are $\{b\}$, $\{c\}$, $\{b, d\}$ and $\{c, d\}$ and the non-empty upward-directed sets of P are $\{b\}$, $\{c\}$, $\{d\}$, $\{b, c, d\}$. The non-empty down-directed sets have meets b, c, b and c in P^δ, respectively. The non-empty upward-directed sets have joins b, c, d and d in P^δ, respectively. We see that in this case for every down-directed set F and upward-directed set I if $\wedge F \leq \vee I$ in P^δ, then there exist $x \in F$ and $y \in I$ with $x \leq y$.

5.1.2 Existence and unicity of the canonical extension, a two-step process

Definition 5.4 (Filter and ideal elements)**.** Let P be a poset. Let P^δ be a canonical extension of P. Define $F(P^\delta) := \{x \in P^\delta | x \text{ is a closed element }\}$ and $I(P^\delta) := \{x \in P^\delta | x \text{ is an open element}\}$. We call the elements of $F(P^\delta)$ filter elements and the elements of $I(P^\delta)$ ideal elements.

The following proposition gives an explanation for the names "filter elements" and "ideal elements".

Proposition 5.5. *Let P be a poset. Let P^δ be a canonical extension of P. The following is an order preserving isomorphism between $F(P^\delta)$ and the filter completion of P*

$$\begin{aligned} F(P^\delta) &\to \mathrm{Filt}(P) \\ x &\mapsto \uparrow x \cap P =: F_x \\ \wedge F &\leftarrow\!\shortmid F \end{aligned}$$

and the following is an order preserving isomorphism between $I(P^\delta)$ and the ideal completion of P

$$\begin{aligned} I(P^\delta) &\to \mathrm{Idl}(P) \\ y &\mapsto \downarrow y \cap P =: I_y \\ \vee I &\leftarrow\!\shortmid I. \end{aligned}$$

Proof. See [4] Theorem 2.5. $\square$

Proposition 5.6. *Let P be a poset. Let P^δ be a canonical extension of P. The order on the subposet $F(P^\delta) \cup I(P^\delta) \subseteq P^\delta$ is as follows. For elements $x, x' \in F(P^\delta)$ and $y, y' \in I(P^\delta)$*

1. *$x \leq x'$ iff $F_{x'} \subseteq F_x$;*
2. *$x \leq y$ iff $F_x \cap I_y \neq \emptyset$;*
3. *$y \leq x$ iff $a \in I_y$, $b \in F_x$ implies $a \leq b$;*
4. *$y \leq y'$ iff $I_y \subseteq I_{y'}$.*

Proof. See [4] Theorem 2.5. Note that 1 and 4 follow from Proposition 5.5.

We will now prove 2. We have $x \leq y$ if and only if $\wedge F_x \leq \vee I_y$ which is by compactness only if $F_x \cap I_y \neq \emptyset$.

For the other direction, suppose $z \in F_x \cap I_y$. Then $z \in I_y$, so $z \leq y$ and $z \in F_x$, so $z \geq x$. Hence $x \leq z \leq y$.

We will now prove 3. Furthermore we have $y \leq x$ if and only if $\vee I_y \leq \wedge F_x$ which is true if and only if $a \in I_y$, $b \in F_x$ implies $a \leq b$. (If $a \in I_y$ then $a \leq \vee I_y$ and if $b \in F_x$ then $b \geq \wedge F_x$.) □

We call $F(P^\delta) \cup I(P^\delta)$ the intermediate structure of the canonical extension P^δ.

Proposition 5.7. *Let P be a poset. Let P^δ be a canonical extension of P. Then P^δ is isomorphic to* $\mathrm{DM}(F(P^\delta) \cup I(P^\delta))$.

Proof. The Dedekind-MacNeille completion of $F(P^\delta) \cup I(P^\delta)$ is join-dense and meet-dense, which means that every element of $\mathrm{DM}(F(P^\delta) \cup I(P^\delta))$ is a join of a subset of $F(P^\delta) \cup I(P^\delta)$ and a meet of a subset of $F(P^\delta) \cup I(P^\delta)$. We want to prove that $F(P^\delta) \cup I(P^\delta)$ is join-dense and meet-dense in the canonical extension. Denseness of the canonical extension says that every element of the canonical extension is both the join of all closed elements below it and the meet of all open elements above it, so in particular every element is the join of a subset of $F(P^\delta) \cup I(P^\delta)$ and the meet of a subset of $F(P^\delta) \cup I(P^\delta)$. Since the Dedekind-MacNeille completion is the unique completion with this property we see that the canonical extension of P must be equal to the Dedekind-MacNeille completion. □

Corollary 5.8. *Let P be a poset. If P has a canonical extension, then it is unique up to an isomorphism that fixes P.*

Proof. See [4] Theorem 2.5. Given two canonical extensions Q and P^δ. Then $F(Q) \cup I(Q)$ and $F(P^\delta) \cup I(P^\delta)$ have the same order, determined by the order on P and on $\mathrm{Idl}(P)$ and on $\mathrm{Filt}(P)$.

The Dedekind-MacNeille completion of a poset is unique. □

We now also have an explicit construction of the canonical extension of a poset.

Corollary 5.9 (Canonical extension as a two-step process for posets)**.** *Let P be a poset. Then P has a canonical extension.*

Proof. See Theorem 2.6 in [4]. Consider the union of the ideals of P and the filters of P.

Define the following relations on $\mathrm{Filt}(P) \cup \mathrm{Idl}(P)$

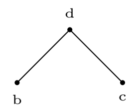
1. $F \leq F'$ iff $F' \subseteq F$;
2. $F \leq I$ iff $F \cap I \neq \emptyset$;
3. $I \leq F$ iff $a \in I$, $b \in F$ implies $a \leq b$;
4. $I \leq I'$ iff $I \subseteq I'$

for filters $F, F' \in \mathrm{Filt}(P)$ and ideals $I, I' \in \mathrm{Idl}(P)$. This relation is reflexive since $F \subseteq F$ and $I \subseteq I$ for all filters $F \in \mathrm{Filt}(P)$ and ideals $I \in \mathrm{Idl}(P)$. To prove that the relation is transitive, first note that inclusion is transitive. If $I \leq F$ and $F \leq I'$ for a filter $F \in \mathrm{Filt}(P)$ and ideals $I, I' \in \mathrm{Idl}(P)$, then $a \in I$, $b \in F$ implies $a \leq b$ and $F \cap I' \neq \emptyset$, so if $z \in F \cap I'$ then $a \leq z$ for all $a \in I$. Hence since I' is a lower set $I \subseteq I'$, so $I \leq I'$. If $F \leq I$ and $I \leq F'$ for filters $F, F' \in \mathrm{Filt}(P)$ and an ideal $I \in \mathrm{Idl}(P)$, then $F \cap I \neq \emptyset$ and $a \in I$, $b \in F'$ implies $a \leq b$, so if $z \in F \cap I$ then $z \leq b$ for all $b \in F'$. Hence since F is an upper set $F' \subseteq F'$, so $F \leq F'$.

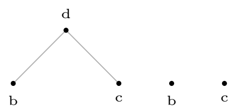
Now we divide this set with reflexive and transitive relation out by an equivalence relation to make it into a poset. The equivalence relation $\sim$ is as follows: for two elements $C, C' \in \mathrm{Filt}(P) \cup \mathrm{Idl}(P)$ we set $C \sim C'$ if and only if $C \leq C'$ and $C' \leq C$. Note that the only case in which $C \neq C'$ but $C \sim C'$ for elements $C, C' \in \mathrm{Filt}(P) \cup \mathrm{Idl}(P)$, is the case in which $F \leq I$ and $I \leq F$ for some filter F and ideal I. In this case there exists an $a \in F \cap I$ since $F \leq I$ and $i \leq a$ for all $i \in I$ because $I \leq F$. Because I is a lower set, $a \in I$ implies $\downarrow a \subset I$. Because $i \leq a$ for all $i \in I$, we know that $I \subset \downarrow a$. Hence $I = \downarrow a$. Analogously $F = \uparrow a$.

The canonical extension of P can be obtained by taking the Dedekind-MacNeille completion of the obtained poset. $\square$

Example 5.10. Consider again the poset P

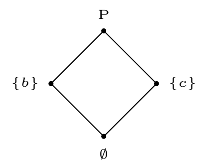


The ideals of P are



The filters of P are $\{b\}$ and $\{c\}$. We find that $\mathrm{Idl}(P) \cup \mathrm{Filt}(P)$ is again P.

Now we want to determine the Dedekind-MacNeille completion of P. The only elements of the Dedekind-MacNeille completion are P, $\{b\}$, $\{c\}$ and $\emptyset$. Hence the Dedekind-MacNeille completion of P, and thus in this case the Dedekind-MacNeille completion of $\mathrm{Idl}(P) \cup \mathrm{Filt}(P)$ is the following poset



5.2 Canonical extension of categories

5.2.1 Explicit construction of a canonical extension of categories $\mathrm{Can}(\mathcal{C})$

To generalise the notion of canonical extensions of posets to the area of category theory, we should translate the idea of the "intermediate object" to category theory.

We can define the canonical extension of a category as the reflexive category of the intermediate object into which the Ind-completion and the Pro-completion embed.

Schematically we can represent this as follows

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook] \arrow[d, hook] & \mathrm{Pro}(\mathcal{C}) \arrow[d, hook] & & \\
\mathrm{Ind}(\mathcal{C}) \arrow[r, hook] & \mathrm{Am}(\mathcal{C}) \arrow[rr, hook] \arrow[dr] \arrow[dd, hook] & & {[\mathrm{Am}(\mathcal{C}), \mathrm{Sets}]}^{\mathrm{op}} \arrow[ddll, bend right] \\
& & \mathrm{R}(\mathrm{Am}(\mathcal{C})) \arrow[ur, hook] \arrow[dl, hook] & \\
& {[\mathrm{Am}(\mathcal{C})^{\mathrm{op}}, \mathrm{Sets}]} \arrow[uurr, bend right] & &
\end{tikzcd}$$

5.2.2 Intermediate object of categories

Let $\mathcal{C}$ be a category. We have full embeddings of $\mathcal{C}$ into $\mathrm{Ind}(\mathcal{C})$ and of $\mathcal{C}$ into $\mathrm{Pro}(\mathcal{C})$. We would like to take something like a pushout of these two embeddings, but in such a way that the Ind-objects are embedded into "the pushout" in a manner that to some extent preserves the properties of the Ind-object i.e. the functor from $\mathrm{Ind}(\mathcal{C})$ to "the pushout" should preserve filtered colimits. Of course we also want that the functor from $\mathrm{Pro}(\mathcal{C})$ to "the pushout" preserves cofiltered limits.

We construct such a "pushout" by first taking the pushout in the category of small categories and then adding some arrows.

Proposition 5.11. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{B}'$ be categories and $I_1 \colon \mathcal{A} \to \mathcal{B}$ and $I_2 \colon \mathcal{A} \to \mathcal{B}'$ be embeddings. Then the pushout of I_1 and I_2 is the category P with as objects*

$$P_0 := (\mathcal{B}_0 \setminus I_1(\mathcal{A}_0)) \sqcup ({\mathcal{B}'}_0 \setminus I_2(\mathcal{A}_0)) \sqcup \mathcal{A}_0$$

and as morphisms strings

$$P_1 := \{[\alpha_n, \dots, \alpha_2, \alpha_1] : \alpha_i \in \mathcal{B}_1 \backslash I_1(\mathcal{A}_1), {\mathcal{B}'}_1 \backslash I_2(\mathcal{A}_1), \mathcal{A}_1,\ \mathit{source}(\alpha_{i+1}) = \mathit{target}(\alpha_i)\} / \sim$$

with composition concatenation of strings and the relation $\sim$ being the relation generated by pairs $([\alpha_2, \alpha_1], [\alpha_2 \circ \alpha_1])$ with both α_1 and α_2 in $\mathcal{B}$, or both in $\mathcal{B}'$ or both in $\mathcal{A}$.

Proof. See [12]. □

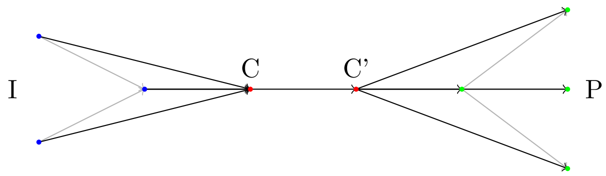
Proposition 5.12. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{B}'$ be categories and $I_1 \colon \mathcal{A} \to \mathcal{B}$ and $I_2 \colon \mathcal{A} \to \mathcal{B}'$ be embeddings and P, J_1, J_2 the pushout of I_1 and I_2. Then the functors $J_1 \colon \mathcal{B} \to P$ and $J_2 \colon \mathcal{B}' \to P$ are faithful.*

Proof. See [12]. □

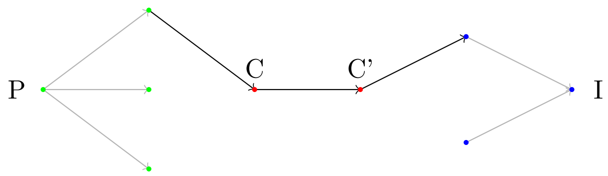
One can think of the pushout of $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$ over $\mathcal{C}$ as the category P with as objects $P_0 := \mathrm{Ind}(\mathcal{C})_0 \sqcup \mathrm{Pro}(\mathcal{C})_0 / \sim$ where an Ind-object and Pro-object are identified if they represent the same object in $\mathcal{C}$, and as morphisms,

morphisms in $\mathrm{Ind}(\mathcal{C})$, $\mathrm{Pro}(\mathcal{C})$ or $\mathcal{C}$ or morphisms between Ind-objects and objects of $\mathcal{C}$ induced by the embedding $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ or morphisms between Pro-objects and objects of $\mathcal{C}$ induced by the embedding $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ or a composition of such arrows.

Example 5.13. We give an example of an arrow from an Ind-object I to a Pro-object P. The objects in Ind-systems are colored blue, the objects in Pro-systems green, the arrows in Ind-systems and in Pro-systems grey and objects of $\mathcal{C}$ red.



Here $C, C' \in \mathcal{C}_0$ and $(C \to C') \in \mathcal{C}_1$. The following is an example of an arrow from P to I



We would like to embed $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$ into a category in such a way that the Ind-objects and Pro-objects still "behave as Ind- and Pro-objects". Therefore we define the intermediate object as follows.

Definition 5.14 (Intermediate object of a category)**.** Let $\mathcal{C}$ be a small category. We define the intermediate object $\mathrm{Am}(\mathcal{C})$ of $\mathcal{C}$ as a (necessarily unique) category (if it exists) together with a filtered colimit preserving functor $\mathrm{Ind}(\mathcal{C}) \to A(\mathcal{C})$ and a cofiltered limit preserving functor $\mathrm{Pro}(\mathcal{C}) \to A(\mathcal{C})$, such that for any category $\mathcal{D}$ with a filtered colimit preserving functor $\mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$ and a cofiltered limit preserving functor $\mathrm{Pro}(\mathcal{C}) \to \mathcal{D}$, there exists a unique functor $\mathrm{Am}(\mathcal{C}) \to \mathcal{D}$ that makes the following diagram commute

$$
\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathrm{Pro}(\mathcal{C}) \arrow[d] \arrow[ddr, bend left] & \\
\mathrm{Ind}(\mathcal{C}) \arrow[r] \arrow[drr, bend right] & \mathrm{Am}(\mathcal{C}) \arrow[dr, dotted] & \\
& & \mathcal{D}
\end{tikzcd}
$$

Definition 5.15 ($\mathrm{A}(\mathcal{C})$)**.** Let $\mathcal{C}$ be a small category. Define a set of arrows E consisting of arrows from Ind-objects to Pro-objects. For an Ind-object $(C\colon I \to \mathcal{C})$ and Pro-object $(L\colon A \to \mathcal{C})$ an arrow $(C\colon I \to \mathcal{C}) \to (L\colon A \to \mathcal{C})$ in E, if it exists, is given by a family of arrows $\{\alpha_{i,a}\colon C(i) \to L(a)\}_{i\in I, a\in A}$ with $\alpha_{i,a} \in \mathcal{C}_1$ for all $i \in I$ and for all $a \in A$, such that for all $i \in I$ and for all $a \in A$ the

following diagram commutes

$$\begin{tikzcd} C(i) \arrow[r] \arrow[d] & L(a) \arrow[d] \\ C(j) \arrow[r] & L(b) \end{tikzcd}$$

Let $\mathrm{Ind}(\mathcal{C}) \amalg_{\mathcal{C}} \mathrm{Pro}(\mathcal{C})$ be the the pushout of the embeddings $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ and $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$. Define $\mathrm{A}(\mathcal{C})$ as the category with as objects

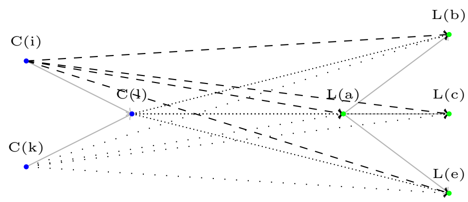
$$\mathrm{A}(\mathcal{C})_0 := \big(\mathrm{Ind}(\mathcal{C}) \amalg_{\mathcal{C}} \mathrm{Pro}(\mathcal{C})\big)_0$$

and for objects $C \in \mathrm{Ind}(\mathcal{C})$ and $L \in \mathrm{Pro}(\mathcal{C})$ we define

$$\mathrm{A}(\mathcal{C})(C, L) := \big(\mathrm{Ind}(\mathcal{C}) \amalg_{\mathcal{C}} \mathrm{Pro}(\mathcal{C})\big)(C, L) \cup E(C, L).$$

Morphisms between Ind-objects, morphisms between Pro-objects and morphisms from a Pro-object to an Ind-object are defined as in $\mathrm{Ind}(\mathcal{C}) \amalg_{\mathcal{C}} \mathrm{Pro}(\mathcal{C})$. The composition is induced by the composition in $\mathcal{C}$.

Example 5.16. We will give an example of an Ind-object $C \colon I \to \mathcal{C}$ and a Pro-object $L \colon A \to \mathcal{C}$ and a family of arrows $C(i) \to L(a)$, such that there exists an extra arrow $C \to L$. If all arrows in the following diagram commute, then the family of arrows $\{C(i) \to L(a)\}_{i \in I, a \in A}$ defines an arrow $C \to L$ in $\mathrm{A}(\mathcal{C})$.



Proposition 5.17. *The embedding of* $\mathrm{Ind}(\mathcal{C})$ *into* $\mathrm{A}(\mathcal{C})$ *preserves filtered colimits and the embedding of* $\mathrm{Pro}(\mathcal{C})$ *into* $\mathrm{A}(\mathcal{C})$ *preserves cofiltered limits.*

Proof. First we prove that $\mathrm{Ind}(\mathcal{C})$ into $\mathrm{A}(\mathcal{C})$ preserves filtered colimits. Let $C \in \mathrm{Ind}(\mathcal{C})$ be the filtered colimit of the diagram $D \colon I \to \mathrm{Ind}(\mathcal{C})$. We need to show that for any object B in $\mathrm{A}(\mathcal{C})$ if there exist arrows $D(i) \to B$ in $\mathrm{A}(\mathcal{C})$ for all $i \in I$ such that for all arrows $i \to i'$ in I the following diagram commutes

$$\begin{tikzcd} D(i) \arrow[r] \arrow[dr] & D(i') \arrow[d] \\ & B \end{tikzcd}$$

then there exists a unique arrow $C \to B$ such that for all $i \in I$ the following diagrams commute

$$\begin{tikzcd} D(i) \arrow[r] \arrow[dr] & C \arrow[d] \\ & B \end{tikzcd}$$

Obviously, since the functor $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Am}(\mathcal{C})$ is an embedding, if the object B is an Ind-object this property is satisfied since C is a filtered colimit in $\mathrm{Ind}(\mathcal{C})$.

Let $(P\colon K \to \mathcal{C}) \in \mathrm{Pro}(\mathcal{C})_0$. Suppose there exist arrows $D(i) \to P$ in $\mathrm{A}(\mathcal{C})$ for all $i \in I$ such that for all arrows $i \to i'$ in I the following diagram commutes

$$\begin{tikzcd} D(i) \arrow[r] \arrow[dr] & D(i') \arrow[d] \\ & P \end{tikzcd}$$

Because of how morphisms from Ind-objects to Pro-objects are defined, there exists an arrow $D(i)(a) \to P(k)$ for all $i \in I$ and every object $D(i)(a)$ and all $k \in K$. We can view $P(k) \in \mathcal{C}$ as an object in $\mathrm{Ind}(\mathcal{C})$ for any $k \in K$, since we can embed $\mathcal{C}$ into $\mathrm{Ind}(\mathcal{C})$, so for all $i \in I$ there exists an arrow of the Ind-object $D(i)$ to an Ind-object $P(k)$ such that for all arrows $i \to i'$ in I the following diagram commutes

$$\begin{tikzcd} D(i) \arrow[r] \arrow[dr] & D(i') \arrow[d] \\ & P(k) \end{tikzcd}$$

By the universal property of the filtered colimit, there must exist arrows $C \to P(k)$ for all $k \in K$. This defines a unique arrow $C \to P$ such that for all $i \in I$ the following diagrams commute

$$\begin{tikzcd} D(i) \arrow[r] \arrow[dr] & C \arrow[d] \\ & P \end{tikzcd}$$

Now we prove that the embedding of $\mathrm{Pro}(\mathcal{C})$ into $\mathrm{A}(\mathcal{C})$ preserves cofiltered limits. Let $L \in \mathrm{Pro}(\mathcal{C})$ be the cofiltered limit of the diagram $D\colon I \to \mathrm{Pro}(\mathcal{C})$.

Obviously, if the object B is a Pro-object this property is satisfied since L is a cofiltered limit in $\mathrm{Pro}(\mathcal{C})$.

Let $(J\colon K \to \mathcal{C}) \in \mathrm{Ind}(\mathcal{C})_0$. Suppose there exist arrows $J \to D(i)$ in $\mathrm{A}(\mathcal{C})$ for all $i \in I$ such that for all arrows $i \to i'$ in I the following diagram commutes

$$\begin{tikzcd} L \arrow[d] \arrow[dr] & \\ D(i) \arrow[r] & D(i') \end{tikzcd}$$

Because of how morphisms from Pro-objects to Ind-objects are defined, for all $i \in I$ there exists a unique arrow $J(k) \to D(i)(a)$ for some object $D(i)(a)$ and some object $k \in K$. We can view $J(k) \in \mathcal{C}$ as an object in $\mathrm{Pro}(\mathcal{C})$. For all $i \in I$ the arrow $J(k) \to D(i)(a)$ in $\mathcal{C}$ defines an arrow between Pro-objects. By the universal property of the cofiltered limit, there must exist arrows $J(k) \to L$. This defines a unique arrow $J \to L$ such that for all $i \in I$ the following diagrams

commute

$$\begin{tikzcd} J \arrow[r] \arrow[dr] & L \arrow[d] \\ & D(i) \end{tikzcd}$$

□

Proposition 5.18. *Let $\mathcal{C}$ be a small category. Then the intermediate object of $\mathcal{C}$ is* $\mathrm{A}(\mathcal{C})$*.*

Proof. We still need to prove that for any category $\mathcal{D}$ with a filtered colimit preserving embedding $F\colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$ and a cofiltered limit preserving embedding $G\colon \mathrm{Pro}(\mathcal{C}) \to \mathcal{D}$ such that the following diagram commutes,

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathrm{Pro}(\mathcal{C}) \arrow[d] \arrow[ddr, bend left, "G"] & \\ \mathrm{Ind}(\mathcal{C}) \arrow[r] \arrow[drr, bend right, "F"'] & \mathrm{A}(\mathcal{C}) \arrow[dr, dotted] & \\ & & \mathcal{D} \end{tikzcd}$$

there exists a unique functor $U\colon \mathrm{A}(\mathcal{C}) \to \mathcal{D}$ that makes the diagram commute. We already know that there exists a unique functor $U'\colon \mathrm{Ind}(\mathcal{C}) \amalg_{\mathcal{C}} \mathrm{Pro}(\mathcal{C}) \to \mathcal{D}$ that makes the diagram commute, so we only need to say what the extra arrows in $E \subset \mathrm{A}(\mathcal{C})_1$ should be mapped to. Let $\big(\alpha\colon (C\colon I \to \mathcal{C}) \to (L\colon A \to \mathcal{C})\big) \in E$, so there exist arrows $C(i) \to L(a)$ in $\mathcal{C}$ for all $i \in I$ and for all $a \in A$ such that every diagram

$$\begin{tikzcd} C(i) \arrow[r] \arrow[d] & L(a) \arrow[d] \\ C(j) \arrow[r] & L(b) \end{tikzcd}$$

Note that C is an Ind-object, so it is the filtered colimit of objects $C(i)$ in $\mathcal{C}$ and these objects can be seen as Ind-objects via the embedding $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$. The arrow $C \to L$ gives us arrows $C(i) \to L$ from Ind-objects $C(i)$ to the Pro-object L in $\mathrm{Ind}(\mathcal{C}) \amalg_{\mathcal{C}} \mathrm{Pro}(\mathcal{C})$. Such an arrow $C(i) \to L$ is a family of arrows in $\mathcal{C}_1$. In this case the family of arrows consist of one exactly one arrow $C(i) \to L(a)$. Since F preserves filtered colimits $F(C)$ must be a filtered colimit of these Ind-objects $C(i)$. The images of the arrows $C(i) \to L$ under U' give us arrows $U'(C(i)) \to U'(L)$ in $\mathcal{D}$. Note that $F(C(i)) = U'(C(i))$. Because $F(C)$ is the filtered colimit of the $F(C(i))$, we can conclude that there exists a unique arrow $F(C) \to U'(L)$ satisfying the universal property of the filtered colimit $F(C)$. Then U must map α to this unique arrow.

If we take the composition of an extra arrow α with another arrow β, then indeed $U(\alpha \circ \beta) = U(\alpha) \circ U(\beta)$.

We have now shown that there exists a unique functor $U\colon \mathrm{A}(\mathcal{C}) \to \mathcal{D}$ that makes the diagram commute. □

Example 5.19. The intermediate object $\mathrm{Am}(\mathcal{C})$ of a category $\mathcal{C}$ is not in general complete (or cocomplete). We will give an example of an incomplete intermediate object. Let $\mathcal{C}$ be a category with two objects 1 and 2 and suppose that 1 and 2 do not have a product in $\mathcal{C}$. The diagram $D\colon \{1,2\} \to \mathcal{C}$, given by $D(1) = c_1$ and $D(2) = c_2$ is not filtered or co-filtered, so it is not an Ind-object or a Pro-object. Therefore the cofiltered limit of D is not an object in $\mathcal{C}$, nor an Ind-object or Pro-object, so it is not in $\mathrm{Am}(\mathcal{C})$.

Definition 5.20 ($\mathrm{Can}(\mathcal{C})$)**.** Let $\mathcal{C}$ be a small category. We define $\mathrm{Can}(\mathcal{C})$ as $\mathrm{Rex}\big(\mathrm{Am}(\mathcal{C})\big)$.

5.2.3 Characterisation of a canonical extension of categories $\mathcal{C}^\delta$

In this paragraph we will give a generalisation of the characterisation of canonical extensions of posets to categories. We will first give a generalisation of Definition 5.1.

Definition 5.21 (Open object)**.** Let $\mathcal{C}$ be a small category. Let $\mathcal{D}$ be a category of which $\mathcal{C}$ is a subcategory. Let $O \in \mathcal{D}_0$ be an object in this category. Then O is an open object, if it is the colimit in $\mathcal{D}$ of a filtered diagram $J \to \mathcal{C}$.

Dually we define a closed object as the limit of a cofiltered diagram. Note that every filtered diagram corresponds to an equivalence class in the Ind-completion, so the open objects correspond to the colimits in $\mathcal{D}$ of Ind-objects of $\mathcal{C}$.

Analogously to the definition of a completion of a poset, see Definition 2.15, we will call an embedding $\mathcal{C} \hookrightarrow \mathcal{D}$ a completion, if $\mathcal{D}$ is a complete and cocomplete category.

The following definition is a generalisation of the definition of denseness for posets, see Definition 5.2, which is why we will call it p-dense.

Definition 5.22 (P-dense completion)**.** Let $c\colon \mathcal{C} \to \mathcal{D}$ be a completion of categories. We call c p-dense, if every object $A \in \mathcal{D}_0$ is the colimit of all closed objects $C \in \mathcal{D}_0$ such that there exists an arrow $(C \to A) \in \mathcal{D}_1$ and it is the limit of all open objects $O \in \mathcal{D}_0$ such that there exists an arrow $(A \to O) \in \mathcal{D}_1$.

Note however that we could have chosen many different generalisations of denseness of posets to categories. The same goes for generalisations of compactness.

Definition 5.23 (Compact completion)**.** Let $c\colon \mathcal{C} \to \mathcal{D}$ be a completion of categories. We call c compact if it satisfies the following property: for any arrow from the limit in $\mathcal{D}$ of a cofiltered diagram $F\colon J \to \mathcal{C}$ to the colimit in $\mathcal{D}$ of a filtered diagram $I\colon J' \to \mathcal{C}$, there must exist $j \in J_0$ and $j' \in J'_0$ and an arrow $a\colon F(j) \to F(j')$ in $\mathcal{C}$.

Note that if $c\colon \mathcal{C} \to \mathcal{D}$ is a compact completion of categories, then we can form the following surjection

$$\bigsqcup_{j \in J, j' \in J'} \mathcal{C}(j, j') \to \mathcal{D}(\lim F\colon J \to \mathcal{C}, \operatorname{colim} I\colon J' \to \mathcal{C}).$$

Definition 5.24 (Canonical completion of a category)**.** Let $c\colon \mathcal{C} \to \mathcal{D}$ be a completion of categories. We call c canonical, if it is p-dense and compact. If c is canonical, then we will denote $\mathcal{D}$ by $\mathcal{C}^\delta$.

5.2.4 Link between $\mathcal{C}^\delta$ and $\mathrm{Can}(\mathcal{C})$?

In the above section we gave a generalisation of denseness and compactness of posets to categories, but of course one can generalise the same definition on posets to many different definitions on categories. We would like to find necessary and sufficient denseness and compactness properties of completions of categories such that $\mathcal{C}^\delta$ (with these new denseness and compactness properties) and $\mathrm{Can}(\mathcal{C})$ coincide.

To prove that $\mathcal{C}^\delta$ is equal to $\mathrm{Rex}(\mathrm{Am}(\mathcal{C}))$ it suffices by Theorem 4.18 to snow that $\mathcal{C}^\delta$ is reflexively complete and that there exists a snug embedding $\mathrm{Am}(\mathcal{C}) \to \mathcal{C}^\delta$.

Proposition 5.25. *Let $\mathcal{C}$ be a small category. There exists a filtered colimit preserving functor $U_{\mathrm{Ind}(\mathcal{C})}\colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}^\delta$ and a cofiltered limit preserving functor $U_{\mathrm{Pro}(\mathcal{C})}\colon \mathrm{Pro}(\mathcal{C}) \to \mathcal{C}^\delta$.*

Proof. By Theorem 3.20 the functor

$$\begin{aligned} U_{\mathrm{Ind}(\mathcal{C})}\colon \mathrm{Ind}(\mathcal{C}) &\to \mathcal{C}^\delta \\ (L \to \mathcal{C}) &\mapsto \mathrm{colim}(L \to \mathcal{C} \to C^\delta) \end{aligned}$$

is the unique filtered colimit preserving functor to make the following diagram commute

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "y"'] & & \mathcal{C}^\delta \\ & \mathrm{Ind}(\mathcal{C}) \arrow[ur, dotted] & \end{tikzcd}$$

Dually the following functor is a cofiltered limit preserving functor

$$\begin{aligned} U_{\mathrm{Pro}(\mathcal{C})}\colon \mathrm{Pro}(\mathcal{C}) &\to \mathcal{C}^\delta \\ (K \to \mathcal{C}) &\mapsto \lim(K \to \mathcal{C} \to C^\delta). \end{aligned}$$

$\square$

Corollary 5.26. *Let $\mathcal{C}$ be a small category. There exists a unique functor $U\colon \mathrm{Am}(\mathcal{C}) \to \mathcal{C}^\delta$ to make the following diagram commute*

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathrm{Pro}(\mathcal{C}) \arrow[d] \arrow[ddr, bend left] & \\ \mathrm{Ind}(\mathcal{C}) \arrow[r] \arrow[drr, bend right] & \mathrm{Am}(\mathcal{C}) \arrow[dr, dotted, "U"] & \\ & & \mathcal{C}^\delta \end{tikzcd}$$

This is by definition of the intermediate object.

We will prove that the intermediate object of a category is “meet- and join-dense” in the canonical extension.

Proposition 5.27. *Let $\mathcal{C}$ be a small category and let $A \in \mathcal{C}^\delta_0$. Then A is a cofiltered limit in $\mathcal{C}^\delta$ of a diagram $D\colon J \to \mathrm{Ind}(\mathcal{C})$. and a colimit in $\mathcal{C}^\delta$ of a diagram $D'\colon J' \to \mathrm{Pro}(\mathcal{C})$.*

Proof. We will prove that every object $A \in \mathcal{C}_0^\delta$ is a colimit in $\mathcal{C}^\delta$ of a diagram $D \colon J \to \mathrm{Ind}(\mathcal{C})$, the other statement can be proved analogously. Since $\mathcal{C}^\delta$ is p-dense, every object $A \in \mathcal{C}_0^\delta$ is the limit of all open objects $C \in \mathcal{C}_0^\delta$ such that there exists an arrow $(A \to C) \in \mathcal{C}_1^\delta$. By definition of an open object, this means that every object $A \in \mathcal{C}_0^\delta$ is the limit of all colimits C in $\mathcal{C}^\delta$ of Ind-objects such that there exists an arrow $(C \to A) \in \mathcal{C}_1^\delta$. $\square$

In particular, every $A \in \mathcal{C}_0^\delta$ is a colimit in $\mathcal{C}^\delta$ of a diagram $D \colon J \to \mathrm{Am}(\mathcal{C})$ and a limit in $\mathcal{C}^\delta$ of a diagram $D' \colon J' \to \mathrm{Am}(\mathcal{C})$.

Remark 5.28. Remark that by Proposition 4.12 every object $F \in \mathrm{Rex}\big(\mathrm{Am}(\mathcal{C})\big)$ is a colimit in $\mathrm{Rex}\big(\mathrm{Am}(\mathcal{C})\big)$ of a diagram $D \colon J \to \mathrm{Am}(\mathcal{C})$ and a limit in $\mathrm{Rex}\big(\mathrm{Am}(\mathcal{C})\big)$ of a diagram $D' \colon J' \to \mathrm{Am}(\mathcal{C})$ as well.

We would like to find necessary and sufficient denseness and compactness conditions on $\mathcal{C}^\delta$ such that $\mathcal{C}^\delta$ is reflexively complete and such that the functor $U \colon \mathrm{Am}(\mathcal{C}) \to \mathcal{C}^\delta$ is a dense and codense embedding.

The following is a more strict definition of a compact $c \colon \mathcal{C} \to \mathcal{D}$ completion of categories, than the one we gave in Definition 5.23.

Definition 5.29 (S-compact completion)**.** Let $c \colon \mathcal{C} \to \mathcal{D}$ be a completion of categories. We call c s-compact if it satisfies the following property: for any arrow from the limit of a cofiltered diagram $F \colon J \to \mathcal{D}$ to the colimit of a filtered diagram $I \colon J' \to \mathcal{D}$, up to equivalence there must exist a unique arrow $F(j) \to F(j')$ in $\mathcal{D}$ and $j \in \mathcal{J}_0$ and $j' \in \mathcal{J}'_0$. Here the equivalence relation $\sim$ is generated by relations of the form: $\big(F(j) \to F(j'), F(l) \to F(j')\big) \in \sim$ if there exists an arrow $F(l) \to F(j)$ for $l, j \in J_0$ and $\big(F(j) \to F(j'), F(j) \to F(k)\big) \in \sim$ if there exists an arrow $F(j') \to F(k)$ for $j', k \in J_0$.

If $c \colon \mathcal{C} \to \mathcal{D}$ is an s-compact completion of categories, then we can form the following bijection

$$\bigsqcup_{j \in J, j' \in J'} \mathcal{C}(j, j')/\sim \to \mathcal{D}(\lim F \colon J \to \mathcal{C}, \operatorname{colim} I \colon J' \to \mathcal{C}).$$

Proposition 5.30. *If the completion $\mathcal{C} \to \mathcal{C}^\delta$ is s-compact, then the functor $U \colon \mathrm{Am}(\mathcal{C}) \to \mathcal{C}^\delta$ in Corollary 5.26 is an embedding.*

Proof. Let P, P' be Pro-objects and I, I' Ind-objects.

By s-compactness $\mathcal{C}^\delta(U(P), U(I)) \cong \mathrm{Am}(P, I)$.

Note that since the following diagram commutes

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathrm{Pro}(\mathcal{C}) \arrow[d] \arrow[ddr, bend left, "U_{\mathrm{Pro}(\mathcal{C})}"] & \\
\mathrm{Ind}(\mathcal{C}) \arrow[r] \arrow[drr, bend right, "U_{\mathrm{Ind}(\mathcal{C})}"'] & \mathrm{Am}(\mathcal{C}) \arrow[dr, dotted, "U"] & \\
 & & \mathcal{C}^\delta
\end{tikzcd}$$

and since $U_{\mathrm{Ind}(\mathcal{C})} \colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}^\delta$ and $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Am}(\mathcal{C})$ preserve filtered colimits and $U_{\mathrm{Pro}(\mathcal{C})} \colon \mathrm{Pro}(\mathcal{C}) \to \mathcal{C}^\delta$ and $\mathrm{Pro}(\mathcal{C}) \to \mathrm{Am}(\mathcal{C})$ preserve cofiltered limits, U preserves filtered colimits of Ind-objects and cofiltered limits of Pro-objects.

The universal properties of the filtered colimit I and the universal properties of the cofiltered limit P determine $\mathcal{C}^\delta(U(I), U(P)) \cong \mathrm{Am}(I, P)$.

Since I is an Ind-object it is a cofiltered diagram of objects in $\mathcal{C}$ which we will call C_i, we have

$$\mathcal{C}^\delta(U(I), U(I')) \cong \mathcal{C}^\delta(U(\operatorname{colim} C_i), U(I'))$$

Because we can view the C_i as Ind-objects and because U preserves filtered colimits of Ind-objects we have

$$\mathcal{C}^\delta(U(\operatorname{colim} C_i), U(I')) \cong \mathcal{C}^\delta(\operatorname{colim} U(C_i), U(I')).$$

We can take this colimit out to obtain

$$\mathcal{C}^\delta(\operatorname{colim} U(C_i), U(I')) \cong \operatorname{colim} \mathcal{C}^\delta(U(C_i), U(I')).$$

Because we can view the C_i as Pro-objects we have

$$\operatorname{colim} \mathcal{C}^\delta(U(C_i), U(I')) \cong \operatorname{colim} \mathrm{Am}(\mathcal{C})(C_i, I').$$

Because filtered colimits of objects in $\mathcal{C}$ exist in $\mathrm{Am}(\mathcal{C})$ we have

$$\operatorname{colim} \mathrm{Am}(\mathcal{C})(C_i, I') \cong \mathrm{Am}(\mathcal{C})(\operatorname{colim} C_i, I')$$

which is again isomorphic to $\mathrm{Am}(\mathcal{C})(I, I')$.

Analogously we can write $U(P')$ as a limit of objects C_j in $\mathcal{C}$ to obtain

$$\begin{aligned}\mathcal{C}^\delta(U(P), U(P')) &\cong \mathcal{C}^\delta(U(P), \lim U(C_j))\\ &\cong \mathrm{Am}(P, P').\end{aligned}$$

□

One could hope that p-denseness of $\mathcal{C}^\delta$ implies denseness of the functor $U\colon \mathrm{Am}(\mathcal{C}) \to \mathcal{C}^\delta$. To prove denseness of the functor U one needs to show that

$$[\mathrm{Am}(\mathcal{C})^{\mathrm{op}}, \mathrm{Sets}]\big(\mathcal{C}^\delta(U(-), C), \mathcal{C}^\delta(U(-), C') \cong \mathcal{C}^\delta(C, C').$$

Indeed the fact that by p-denseness of $\mathcal{C}^\delta$ any object $C \in \mathcal{C}^\delta$ can be written as a limit of Ind-objects and as a colimit of Pro-objects does give us room to play. However we expect that a stricter definition of denseness is needed.

6 Properties of the canonical extension of categories

In this chapter we will investigate which properties of the canonical extension of posets can be generalised to categories. First we will see in Paragraph 6.1 that like canonical extensions of posets, canonical extensions of categories do not seem to be functorial. In Paragraph 6.2 that like in the poset case, taking the canonical extension of categories commutes with taking the opposite. Lastly we investigate in Section 6.3 whether the canonical extension of categories commutes with taking the product, but we do not find a definite answer.

6.1 Functoriality

6.1.1 Functoriality of the intermediate object

Proposition 6.1. *The construction of the intermediate object* $\mathrm{Am}(\mathcal{C})$ *out of a small category* $\mathcal{C}$ *is functorial.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be small categories and $F\colon \mathcal{A} \to \mathcal{B}$ a functor. Define $\mathrm{Am}(F)\colon \mathrm{Am}(\mathcal{A}) \to \mathrm{Am}(\mathcal{B})$ as the functor that maps a diagram $D\colon J \to \mathcal{A}$ to the composition $F \circ D$. Firstly we show that $\mathrm{Am}(F)$ indeed is a functor for all functors $(F\colon \mathcal{A} \to \mathcal{B}) \in \mathrm{Cat}$. Note that if D is (co)filtered, then $F \circ D$ is (co)filtered, so $\mathrm{Am}(F)(D) \in \mathcal{B}$ and $\mathrm{Am}(F)$ maps Ind-objects to Ind-objects and Pro-objects to Pro-objects.

Suppose we are given two diagrams $D\colon J \to \mathcal{A}$ and $D'\colon J' \to \mathcal{A}$ and suppose there exists an arrow $\alpha\colon D \to D'$ between them in $\mathrm{Am}(\mathcal{A})$. The arrow $\alpha\colon D \to D'$ is a family of arrows $D(j) \to D'(j')$ in $\mathcal{A}$. We can map the arrows $D(j) \to D'(j')$ to arrows $F(D(j)) \to F(D'(j'))$ to obtain an arrow $F(\alpha)\colon F(D) \to F(D')$ in $\mathrm{Am}(\mathcal{B})$.

Secondly we show that this makes Am into a functor.

Let $F\colon \mathcal{A} \to \mathcal{A}$ be the identity. Then the functor $D \mapsto F \circ D$ is the identity.

Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ be functors. Let $D\colon J \to \mathcal{A}$ be a diagram. Then

$$\begin{aligned}\mathrm{Am}(G) \circ \mathrm{Am}(F)(D) &= \mathrm{Am}(G)(F \circ D)\\ &= G \circ (F \circ D)\\ &= (G \circ F) \circ D\\ &= \mathrm{Am}(G \circ F)(D).\end{aligned}$$

$\square$

6.1.2 Functoriality of the reflexive completion?

We can map a functor $F\colon \mathcal{A} \to \mathcal{B}$ to the map $\mathrm{Rex}(F)\colon \mathrm{Rex}(\mathcal{A}) \to \mathrm{Rex}(\mathcal{B})$ determined by mapping representable functors $\mathcal{A}(-,a)$ to $\mathcal{B}(-,F(a))$. This looks like a likely candidate for a functor that maps small categories to their reflexive completion.

However this map does not have to be well-defined. Consider the following diagram

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, hook] \arrow[d, "F"'] & {[\mathcal{A}^{\mathrm{op}}, \mathrm{Sets}]} \arrow[r, shift right, "R_{\mathcal{A}}"'] \arrow[d] & {[\mathcal{A}, \mathrm{Sets}]^{\mathrm{op}}} \arrow[l, shift right, "L_{\mathcal{A}}"'] \arrow[d] & \mathcal{A} \arrow[l, hook'] \arrow[d, "F"] \\
\mathcal{B} \arrow[r, hook] & {[\mathcal{B}^{\mathrm{op}}, \mathrm{Sets}]} \arrow[r, shift right, "R_{\mathcal{B}}"'] & {[\mathcal{B}, \mathrm{Sets}]^{\mathrm{op}}} \arrow[l, shift right, "L_{\mathcal{B}}"'] & \mathcal{B} \arrow[l, hook']
\end{tikzcd}
$$

Here the functor $[\mathcal{A}^{\mathrm{op}}, \mathrm{Sets}] \to [\mathcal{B}^{\mathrm{op}}, \mathrm{Sets}]$ is determined by mapping colimits of representable functors $\mathcal{A}(-, a)$ to colimits of representable functors $\mathcal{B}(-, F(a))$ and the functor $[\mathcal{A}, \mathrm{Sets}]^{\mathrm{op}} \to [\mathcal{B}, \mathrm{Sets}]^{\mathrm{op}}$ is determined by mapping limits of representable functors $\mathcal{A}(-, a)$ to limits of representable functors $\mathcal{B}(-, F(a))$ However, since right adjoints do not necessarily preserve colimits (and left adjoints do not necessarily preserve limits), the above diagram does not have to commute.

A lattice is a poset in which every two elements have a unique join and meet. Taking the canonical extension of posets is not functorial in general either (even the canonical extension of lattices is not in general functorial), see [2] page 1944.

6.2 Taking the opposite commutes with taking the canonical extension

Note that a poset is complete if and only if it is cocomplete. The dual notion of denseness for posets is again denseness. The same goes for compactness. We can conclude from this that taking the opposite of a poset commutes with taking the canonical extension.

6.2.1 Taking the opposite of the intermediate object

Proposition 6.2. *Let $\mathcal{C}$ be a small category. Then* $\mathrm{Am}(\mathcal{C})^{op} \cong \mathrm{Am}(\mathcal{C}^{op})$.

Proof. Let $\mathcal{D}$ be a small category. First note that the following diagram commutes

$$
\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[r] \arrow[d] & \mathrm{Pro}(\mathcal{C}^{\mathrm{op}}) \arrow[d] \arrow[ddr, bend left] & \\
\mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \arrow[r] \arrow[dr, bend right] & \mathrm{Am}(\mathcal{C}^{\mathrm{op}}) \arrow[dr, dotted] & \\
& & \mathcal{D}
\end{tikzcd}
$$

if and only if the following diagram does

$$
\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathrm{Pro}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \arrow[d] \arrow[ddr, bend left] & \\
\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \arrow[r] \arrow[dr, bend right] & \mathrm{Am}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \arrow[dr, dotted] & \\
& & \mathcal{D}^{\mathrm{op}}
\end{tikzcd}
$$

Secondly, recall that $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \cong \mathrm{Pro}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \cong \mathrm{Ind}(\mathcal{C})$, so the above diagram is the same as the diagram below

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathrm{Ind}(\mathcal{C}) \arrow[d] \arrow[dr, bend left] & \\
\mathrm{Pro}(\mathcal{C}) \arrow[r] \arrow[rr, bend right] & \mathrm{Am}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \arrow[r, dotted] & \mathcal{D}^{\mathrm{op}}
\end{tikzcd}$$

Suppose that $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \to \mathcal{D}$ a colimit preserving functor and $\mathrm{Pro}(\mathcal{C}^{\mathrm{op}}) \to \mathcal{D}$ a limit preserving functor. Then the functor $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is a limit preserving functor and $\mathrm{Pro}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is a colimit preserving functor. We can conclude that $\mathrm{Am}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ satisfies the universal property of $\mathrm{Am}(\mathcal{C})$. Hence $\mathrm{Am}(\mathcal{C}) \cong \mathrm{Am}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$. □

6.2.2 Taking the opposite of the reflexive completion

Proposition 6.3. *Let $\mathcal{C}$ be a small category. Then* $\mathrm{Rex}(\mathcal{C}) \cong \mathrm{Rex}(\mathcal{C}^{\,op})^{\,op}$.

Proof. See the talk [13]. Let $\mathcal{C}$ be a category then the following is an adjoint pair

$$\begin{aligned} L\colon [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}] &\to [\mathcal{C}, \mathrm{Sets}]^{\mathrm{op}} \\ P &\mapsto := \big(c \mapsto [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}](P, \mathcal{C}(-,c))\big) \end{aligned}$$

and functor

$$\begin{aligned} R\colon [\mathcal{C}, \mathrm{Sets}]^{\mathrm{op}} &\to [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}] \\ Q &\mapsto \hat{Q} := \big(c \mapsto [\mathcal{C}, \mathrm{Sets}](Q, \mathcal{C}(c,-))\big). \end{aligned}$$

Now consider the adjoint pair corresponding to the category $\mathcal{C}^{\mathrm{op}}$

$$\begin{aligned} L'\colon [\mathcal{C}, \mathrm{Sets}] &\to [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]^{\mathrm{op}} \\ P &\mapsto := \big(c \mapsto [\mathcal{C}, \mathrm{Sets}](P, \mathcal{C}(c,-))\big) \end{aligned}$$

and functor

$$\begin{aligned} R'\colon [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}]^{\mathrm{op}} &\to [\mathcal{C}, \mathrm{Sets}] \\ Q &\mapsto \hat{Q} := \big(c \mapsto [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}](Q, \mathcal{C}(-,c))\big). \end{aligned}$$

From this adjoint pair we obtain $\mathrm{Rex}(\mathcal{C}^{\mathrm{op}})$ by taking the presheaves P of $\mathcal{C}^{\mathrm{op}}$ such that $R'(L'(P)) \cong P$.

Taking the opposite of $\mathrm{Rex}(\mathcal{C}^{\mathrm{op}})$ comes down to reversing the arrows between the reflexive presheaves, which is the same as taking the reflexive completion that corresponds to the following adjoint pair

$$\begin{aligned} L'^{\mathrm{op}}\colon [\mathcal{C}, \mathrm{Sets}]^{\mathrm{op}} &\to [\mathcal{C}^{\mathrm{op}}, \mathrm{Sets}] \\ P &\mapsto := \big(c \mapsto [\mathcal{C}, \mathrm{Sets}](P, \mathcal{C}(c,-))\big) \end{aligned}$$

and to R' to obtain the functor

$$\begin{aligned} R'^{\,\mathrm{op}} \colon [\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}] &\to [\mathcal{C}, \mathrm{Sets}]^{\,\mathrm{op}} \\ Q &\mapsto \hat{Q} := \big(c \mapsto [\mathcal{C}^{\,\mathrm{op}}, \mathrm{Sets}](Q, \mathcal{C}(-, c))\big). \end{aligned}$$

We find that $L'^{\,\mathrm{op}} = R$ and that $R'^{\,\mathrm{op}} = L$. Hence the presheaves P of $\mathcal{C}$ such that $L'^{\,\mathrm{op}}(R'^{\,\mathrm{op}}(P)) \cong P$ are exactly the same as the presheaves Q such that $R(L(Q)) \cong Q$. We can conclude that $\mathrm{Rex}(\mathcal{C}) \cong (\mathrm{Rex}(\mathcal{C}^{\,\mathrm{op}})^{\,\mathrm{op}}$. □

6.3 Does taking a product commute with taking the canonical extension?

6.3.1 Taking a product of the intermediate object

Proposition 6.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories. Then $\mathrm{Ind}(\mathcal{A} \times \mathcal{B}) \cong \mathrm{Ind}(\mathcal{A}) \times \mathrm{Ind}(\mathcal{B})$ and $\mathrm{Pro}(\mathcal{A} \times \mathcal{B}) \cong \mathrm{Pro}(\mathcal{A}) \times \mathrm{Pro}(\mathcal{B})$.*

Proof. Consider the following functors

$$\begin{aligned} S_{\mathcal{A}} \colon \mathrm{Ind}(\mathcal{A} \times \mathcal{B}) &\to \mathrm{Ind}(\mathcal{A}) \\ (D \colon J \to \mathcal{A} \times \mathcal{B}) &\mapsto P_{\mathcal{A}} \circ D \end{aligned}$$

and

$$\begin{aligned} S_{\mathcal{B}} \colon \mathrm{Ind}(\mathcal{A} \times \mathcal{B}) &\to \mathrm{Ind}(\mathcal{B}) \\ (D \colon J \to \mathcal{A} \times \mathcal{B}) &\mapsto P_{\mathcal{B}} \circ D. \end{aligned}$$

Let T be a category and $F_{\mathcal{A}} \colon T \to \mathrm{Ind}(\mathcal{A})$ and $F_{\mathcal{B}} \colon T \to \mathrm{Ind}(\mathcal{B})$ be functors. Consider the following diagram

$$\begin{tikzcd} & T \arrow[d, dotted] \arrow[ddl, bend right, "F_{\mathcal{A}}"'] \arrow[ddr, bend left, "F_{\mathcal{B}}"] & \\ & \mathrm{Ind}(\mathcal{A} \times \mathcal{B}) \arrow[dl, "P_{\mathcal{A}}"] \arrow[dr, "P_{\mathcal{B}}"'] & \\ \mathrm{Ind}(\mathcal{A}) & & \mathrm{Ind}(\mathcal{B}) \end{tikzcd}$$

Define the functor U as follows

$$\begin{aligned} U \colon T &\to \mathrm{Ind}(\mathcal{A} \times \mathcal{B}) \\ t &\mapsto (F_{\mathcal{A}}, F_{\mathcal{B}}) \colon J \times I \to \mathcal{A} \times \mathcal{B}. \end{aligned}$$

Then U is the unique functor to make the diagram commute. □

Proposition 6.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories. Then $\mathrm{Am}(\mathcal{A} \times \mathcal{B}) \cong \mathrm{Am}(\mathcal{A}) \times \mathrm{Am}(\mathcal{B})$.*

6.3.2 Taking a product of the reflexive completion

Taking the opposite commutes with taking (co)limits. In particular, for $\mathcal{A}$ and $\mathcal{B}$ categories, we have $[(\mathcal{A}+\mathcal{B})^{\text{op}}, \text{Sets}] \cong [\mathcal{A}^{\text{op}}+\mathcal{B}^{\text{op}}, \text{Sets}]$ and $[(\mathcal{A}\times\mathcal{B})^{\text{op}}, \text{Sets}] \cong [\mathcal{A}^{\text{op}}\times\mathcal{B}^{\text{op}}, \text{Sets}]$.

Proposition 6.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories. Then $[(\mathcal{A}+\mathcal{B})^{op}, \text{Sets}] \cong [\mathcal{A}^{op}, \text{Sets}]\times[\mathcal{B}^{op}, \text{Sets}]$ and $[\mathcal{A}+\mathcal{B}, \text{Sets}]^{op} \cong [\mathcal{A}, \text{Sets}]^{op}\times[\mathcal{B}, \text{Sets}]^{op}$.*

Proof. Let $P_\mathcal{A}$ and $P_\mathcal{B}$ be the projections of $\mathcal{A}^{\text{op}}+\mathcal{B}^{\text{op}}$ on $\mathcal{A}^{\text{op}}$ and on $\mathcal{B}^{\text{op}}$, respectively. Define

$$\begin{aligned} R_\mathcal{A} \colon [\mathcal{A}^{\text{op}}+\mathcal{B}^{\text{op}}, \text{Sets}] &\to [\mathcal{A}^{\text{op}}, \text{Sets}] \\ P &\mapsto P_\mathcal{A} \circ P \end{aligned}$$

and

$$\begin{aligned} R_\mathcal{B} \colon [\mathcal{A}^{\text{op}}+\mathcal{B}^{\text{op}}, \text{Sets}] &\to [\mathcal{B}^{\text{op}}, \text{Sets}] \\ P &\mapsto P_\mathcal{B} \circ P. \end{aligned}$$

Let T be a category and let $F_\mathcal{A} \colon T \to [\mathcal{A}^{\text{op}}, \text{Sets}]$ and $F_\mathcal{B} \colon T \to [\mathcal{B}^{\text{op}}, \text{Sets}]$ be functors. Then the unique functor to make the following diagram

$$\begin{tikzcd}
 & T \arrow[d, dotted] \arrow[ddl, bend right, "F_\mathcal{A}"'] \arrow[ddr, bend left, "F_\mathcal{B}"] & \\
 & {[\mathcal{A}^{\text{op}}+\mathcal{B}^{\text{op}}, \text{Sets}]} \arrow[dl, "Ref_\mathcal{A}"] \arrow[dr, "Ref_\mathcal{B}"'] & \\
{[\mathcal{A}^{\text{op}}, \text{Sets}]} & & {[\mathcal{B}^{\text{op}}, \text{Sets}]}
\end{tikzcd}$$

commute is

$$\begin{aligned} F \colon T &\to [\mathcal{A}^{\text{op}}+\mathcal{B}^{\text{op}}, \text{Sets}] \\ t &\mapsto F_\mathcal{A}(t) + F_\mathcal{B}(t). \end{aligned}$$

□

Consider the following diagram

$$\begin{tikzcd}
 & {[\mathcal{A}^{\text{op}}, \text{Sets}] \times [\mathcal{B}^{\text{op}}, \text{Sets}]} \arrow[dl, "Ref_\mathcal{A}"'] \arrow[dr, "Ref_\mathcal{B}"] \arrow[ddd, dotted, shift left] & \\
{[\mathcal{A}^{\text{op}}, \text{Sets}]} \arrow[d, shift left, "L_\mathcal{A}"] & & {[\mathcal{B}^{\text{op}}, \text{Sets}]} \arrow[d, shift left, "L_\mathcal{B}"] \\
{[\mathcal{A}, \text{Sets}]^{\text{op}}} \arrow[u, shift left, "R_\mathcal{A}"] & & {[\mathcal{B}, \text{Sets}]^{\text{op}}} \arrow[u, shift left, "R_\mathcal{B}"] \\
 & {[\mathcal{A}, \text{Sets}]^{\text{op}} \times [\mathcal{B}, \text{Sets}]^{\text{op}}} \arrow[ul, "S_\mathcal{A}"] \arrow[ur, "S_\mathcal{B}"'] \arrow[uuu, dotted, shift left] &
\end{tikzcd}$$

Proposition 6.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories. Let $L \dashv R$ be the adjoint pair of the reflexive completion* $\mathrm{Rex}(\mathcal{A}+\mathcal{B})$*, $L_{\mathcal{A}} \dashv R_{\mathcal{A}}$ the adjoint pair corresponding to* $\mathrm{Rex}(\mathcal{A})$ *and $L_{\mathcal{B}} \dashv R_{\mathcal{B}}$ the adjoint pair corresponding to* $\mathrm{Rex}(\mathcal{B})$*. Then the left adjoint L is equal to the following functor*

$$\begin{aligned}[\mathcal{A}^{\,op} + \mathcal{B}^{\,op}, \mathrm{Sets}] &\to [\mathcal{A}+\mathcal{B}, \mathrm{Sets}]^{\,op} \\ P &\mapsto L_{\mathcal{A}}(T_{\mathcal{A}}(P)) + L_{\mathcal{B}}(T_{\mathcal{A}}(P))\end{aligned}$$

and analogously the right adjoint R is equal to

$$\begin{aligned}[\mathcal{A}^{\,op} + \mathcal{B}^{\,op}, \mathrm{Sets}] &\to [\mathcal{A}+\mathcal{B}, \mathrm{Sets}]^{\,op} \\ P &\mapsto R_{\mathcal{A}}(S_{\mathcal{A}}(P)) + R_{\mathcal{B}}(S_{\mathcal{B}}(P)).\end{aligned}$$

Proof. Let $P \in [\mathcal{A}^{\,\mathrm{op}} + \mathcal{B}^{\,\mathrm{op}}, \mathrm{Sets}]$. Note that $[\mathcal{A}^{\,\mathrm{op}} + \mathcal{B}^{\,\mathrm{op}}, \mathrm{Sets}]\big(P, \mathcal{A}(-,a)\big) \cong [\mathcal{A}^{\,\mathrm{op}}, \mathrm{Sets}]\big(T_{\mathcal{A}}(P), \mathcal{A}(-,a)\big)$ for all $a \in \mathcal{A}$. The image of P under the functor L is defined as

$$\begin{aligned}\mathcal{A}+\mathcal{B} &\to \mathrm{Sets} \\ a &\mapsto [\mathcal{A}^{\,\mathrm{op}} + \mathcal{B}^{\,\mathrm{op}}, \mathrm{Sets}]\big(P, \mathcal{A}(-,a)\big) \\ b &\mapsto [\mathcal{A}^{\,\mathrm{op}} + \mathcal{B}^{\,\mathrm{op}}, \mathrm{Sets}]\big(P, \mathcal{A}(-,b)\big)\end{aligned}$$

which is equal to

$$\begin{aligned}\mathcal{A}+\mathcal{B} &\to \mathrm{Sets} \\ a &\mapsto [\mathcal{A}^{\,\mathrm{op}}, \mathrm{Sets}]\big(T_{\mathcal{A}}(P), \mathcal{A}(-,a)\big) \\ b &\mapsto [\mathcal{B}^{\,\mathrm{op}}, \mathrm{Sets}]\big(T_{\mathcal{B}}(P), \mathcal{B}(-,b)\big).\end{aligned}$$

Hence the left adjoint L is equal to the following functor

$$\begin{aligned}[\mathcal{A}^{\,\mathrm{op}} + \mathcal{B}^{\,\mathrm{op}}, \mathrm{Sets}] &\to [\mathcal{A}+\mathcal{B}, \mathrm{Sets}]^{\,\mathrm{op}} \\ P &\mapsto L_{\mathcal{A}}(T_{\mathcal{A}}(P)) + L_{\mathcal{B}}(T_{\mathcal{A}}(P)).\end{aligned}$$

$\square$

Corollary 6.8. *The adjoint pair $L \dashv R$ is the same as the pair $L_{\mathcal{A}} \times L_{\mathcal{B}} \dashv L_{\mathcal{A}} \times L_{\mathcal{B}}$.*

6.3.3 Taking the product of the canonical extension

It could be that the adjoint pair that corresponds to $\mathrm{Rex}(\mathcal{A}\times\mathcal{B})$ corresponds to $L_{\mathcal{A}} \times L_{\mathcal{B}} \dashv L_{\mathcal{A}} \times L_{\mathcal{B}}$, which means that $\mathrm{Rex}(\mathcal{A}\times\mathcal{B})$ is the same as $\mathrm{Rex}(\mathcal{A}) \times \mathrm{Rex}(\mathcal{B})$ To prove this one can not just use properties of the category of presheaves, like we did to prove that $L \dashv R$ is the same as the pair $L_{\mathcal{A}} \times L_{\mathcal{B}} \dashv L_{\mathcal{A}} \times L_{\mathcal{B}}$. One would have to use properties of the reflexive completion, perhaps the fact that every reflexive presheaf is both a colimit and a limit of representable functors could help.

In the poset case, it is true that taking the product commutes with taking the Dedekind-MacNeille completion. Let P and Q be posets. We claim that $A \in \mathrm{DM}(P)$ and $B \in \mathrm{DM}(Q)$ if and only if $A \times B \in \mathrm{DM}(P \times Q)$. Note that $(A\times B)^u = A^u \times B^u$ and that $(A\times B)^l = A^l \times B^l$, so $((A\times B)^u)^l = (A^u)^l \times (B^u)^l$. Hence $((A\times B)^u)^l = A \times B$ if and only if $(A^u)^l = A$ and $(B^u)^l = B$.

If it is the case that $\mathrm{Rex}(\mathcal{A}\times\mathcal{B})$ is the same as $\mathrm{Rex}(\mathcal{A}) \times \mathrm{Rex}(\mathcal{B})$, then $\mathrm{Rex}(\mathrm{Am}(\mathcal{A}\times\mathcal{B})) \cong \mathrm{Rex}(\mathrm{Am}(\mathcal{A}) \times \mathrm{Am}(\mathcal{B})) \cong \mathrm{Rex}(\mathrm{Am}(\mathcal{A})) \times \mathrm{Rex}(\mathrm{Am}(\mathcal{B}))$.

7 Ideas for future research

An idea for future research would be to finish what we started in Paragraph 5.2.4.

It would be interesting to know what denseness and compactness properties $\mathrm{Can}(\mathcal{C})$ has. In Paragraph 5.2.3 we gave generalisations of the notion of denseness and compactness of canonical extensions of posets, but of course one can generalise the same definition on posets to many different definitions on categories. We would like to have necessary and sufficient conditions, that look like denseness and compactness, such that a completion of $\mathcal{C}$ with these conditions coincides with $\mathrm{Can}(\mathcal{C})$.

Once one has fixed generalisations of denseness and compactness, one could try to prove analogously to the proof of unicity and existence of canonical extensions of posets given in Paragraph 5.1.2, that a completion with those denseness and compactness properties satisfies the universal property of $\mathrm{Rex}(\mathrm{Am}(\mathcal{C}))$. This would mean proving that a completion $\mathcal{D}$ of $\mathcal{C}$ with those denseness and compactness properties is reflexively complete and has a snug embedding of $\mathrm{Am}(\mathcal{C})$ into $\mathcal{D}$. Developing more theory about reflexive completions would make it easier to find such a proof.

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