S.A. Postma

Pattern formation in mussel beds
Describing this phenomenon by the Cahn-Hilliard equation

Master’s thesis, defended on December 19, 2012

Thesis Advisors:
Prof. Dr. A. Doelman
Dr. V. Rottschäfer

Specialisation: Applied Mathematics

Mathematisch Instituut, Universiteit Leiden
Preface

After almost six and a half years my study in Mathematics has come to an end. In the final stage of my master’s at the University of Leiden I have done my research project in the field of Applied Mathematics. Over a year I have worked on the phenomenon of pattern formation in mussel banks. The ecologists Johan van de Koppel and Quan-Xing Liu, both located at the Royal Netherlands Institute for Sea Research in Yerseke, introduced this phenomenon to my supervisors. They both decided it would make a good subject for a master’s project, and it came to be my research.

It has been a long road, but I am very glad to have made it. For this I, first of all, would like to thank my parents, family and friends for supporting me at all time. A special thanks to Teun for his guidance and good advice through this entire process. I would also like to mention Johan van de Koppel and Quan-Xing Liu for their contribution and ideas to the project. It was fun to have visited you in the far south of the Netherlands. And finally, I would like to thank my both supervisors Arjen Doelman and Vivi Rottschäfer for their supervision in this project. I enjoyed working with them, and thank them for their support.

S.A.Postma
December 2012
# Contents

## Introduction

- **1 The equation**
  - 1.1 Structure ............................. 2
  - 1.2 The speed of the mussels ................. 3
  - 1.3 Further scaling .......................... 5

- **2 Analysis**
  - 2.1 The potential and the energy ............... 7
  - 2.2 System .................................... 9
  - 2.3 The Hamiltonian ............................ 10
  - 2.4 The setting $c_1 = 0$ ......................... 11
  - 2.5 The setting $c_1 \neq 0$ ....................... 13
    - 2.5.1 Perturbation theory ..................... 14
    - 2.5.2 Only two equilibria .................... 16
    - 2.5.3 Another example of three equilibria ........ 16
  - 2.6 Pattern formation .......................... 18

- **3 The movement speed**
  - 3.1 A non-positive speed ....................... 20
    - 3.1.1 One root .................................. 23
    - 3.1.2 Two roots ................................. 23
  - 3.2 A general speed ............................. 26
    - 3.2.1 A non-negative general speed ............. 27
    - 3.2.2 A negative general speed .................. 28
    - 3.2.3 Summary for a general speed ............... 29
    - 3.2.4 The parameters ........................... 29

- **4 Simulations**
  - 4.1 The code .................................. 31
  - 4.2 The equation and its parameter .............. 31
  - 4.3 The one-dimensional setting .................. 31
  - 4.4 The two-dimensional setting .................. 33

- **5 Conclusion**
  - 5.1 Further research ........................... 34

- **A Appendix**
  - A.1 Program code for finding the value of $c_1$ ...... 35
  - A.2 Program code for simulations of the Cahn-Hilliard equation ...... 36

- **Bibliography**
Introduction

For a long time mathematics has been used in scientifical fields. It has shown to be very useful in fields such as physics and astronomy. But since a few decades it has also shown its strength in the field of biology. Biological questions can be answered with the use of mathematical theories and models. For example for predicting the population size at a certain generation the size of the population from previous generation. Then a function can be composed for the population size at generation $t + 1$ which depends on the population size at generation $t$. With this function the population size can be studied over time and possible behaviour can be predicted. A clear example of one problem where both mathematics and biology are combined.

An area within the field of biology is ecology. Biological subjects can therefore also have an ecological character. The study of ecology treats a great amount of different subjects. From the behaviour of a small organism to a whole ecosystem. Both can be studied on a small time scale such as seconds to a few decades or centuries. Even though it is no very realistic to examine the behaviour of one small organism for two hundred years. When thinking of an ecosystem one can think of how certain species within this ecosystem organise themselves and how they interact with each other and their environment. An example is the phenomenon of spatial pattern formation within ecosystems. Several observations of this phenomenon have been observed. See Figure I, which is reproduced from [26].

![Figure I: A labyrinth pattern of bushy vegetation in Niger, regular maze patterns of shrubs and threes in West Siberia, and a spotted pattern of isolated three patches in Niger respectively.](image)

Self-organized spatial patterns

Spatial patterns can be of great importance in terms of early (warning) signals for potential catastrophic shifts. Also self-organized spatial patterns can be used to understand ecosystems better and give more information about the interactions within an ecosystem. Understanding how these spatial patterns, self-organized or not, are organized and come to be can be of high value for their protection and conservation. Because a great part of world’s nature is endangered, understanding these ecosystems is a good start [19].

The theory of spatial self-organization has been of great interest among theoretical ecologists in the past two decades [28]. Theoretical models have predicted that local interactions between organisms and the environment or between other organisms can cause the formation of large-scale spatial patterns. These patterns can take the form of regular spots, labyrinth structures, spiral waves, or patches [17].

For this thesis we will focus on the phenomenon of spatial pattern formation of a population mussels. These patterns are the result of self-organisation of the mussels. Different spatial pattern formation in young mussel beds have been observed, including in the Dutch Wadden See
Figure II: Spatial pattern formation in young mussel beds: (A) (nearly) uniformly covered, (B) stripped pattern and (C) rand patchy pattern.

Figure III: The (edible) blue mussel or common mussel, *Mytilus edulis* Linnaeus, 1758.

[16, 32]. See Figure II, which is reproduced from [16] and [18]. The animal we consider here is the (edible) blue mussel or common mussel, with the Latin name *Mytilus edulis* Linnaeus, 1758. A photograph is captured in Figure III.

Mussels communicate differently with each other on different distances. This is called scale-dependent feedback, which is suggested to be the cause of the striped patterns captured in Figure II(B) [15, 16, 32]. These patterns occur at various spatial scales [30]. The beds range in size from tens of meters to square kilometers. Within these beds mussels clump together in high-density patches alternated by sediments containing hardly any mussels. The great advantage of these clumps is the reduction of wave disturbance [14]. However, these aggregation of mussels can also form a disadvantage because the amount of food can become exhausted, also due to competition between the mussels, which reduces the growth of the mussels [16]. So the mussels find a balance between clinging together and getting enough nutrition which results in the observed patterns.
The Cahn-Hilliard equation as a model

In this thesis we model the local density or concentration of a population of mussels $M$. The change of this density is modeled by the particle flux, which indicates the rate of flow of a particle per unit area, using classical diffusion. For the particle flux the speed of the mussels and the long range or nonlocal interactions are taking into account. This eventually leads to the following equation:

$$\frac{\partial M}{\partial t} = \Delta [W'(M) - \kappa \Delta M].$$

(1)

Here $M(X,t) : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$, where $\Omega$ is a bounded open subset of $\mathbb{R}^n$. The function $W(M)$ is a symmetric double-well potential and $W'(M)$ its derivative, which will be denoted by the function $F(M)$, and $\kappa > 0$ is assumed constant. The contribution of the speed of the mussels is given by the derivative $W'(M)$ and the long range effects by $\kappa \Delta M$. This symmetry double-well potential $W(M)$ plays an important role in equation (1) in terms of the long term behaviour. The classical diffusion approach explains the $\Delta$-operator is in front of the equation.

The equation given in (1) is known as the Cahn-Hilliard equation. The equation was initially developed to describe the process of phase separation in a two component system, with $M$ representing the concentration of one of the two components. Concentration should here be understood as referring either to volume or mass fraction $[2, 3, 21]$. Phase separation or spinodal decomposition is the mechanism by which a mixture of two or more components separate into distinct regions or phases $[1]$. This process of phase separation generates patterns as seen in Figure IV. The patterns generated by the Cahn-Hilliard equation are the same type of patterns which are observed in mussel bed. Comparing Figures II and IV provides a clear example.

Outline

The aim of this thesis is to analyze the equation for the population mussels given in (1). For this we will first formulate the model using the classical diffusion for the particle flux in section 1. Then, in the next section, the model is analysed using gradient flow theory and phase plane analysis for a specific potential $W$. Also unstable situations are elaborated, even though the corresponding patterns will not be observed. In the third section the speed of the mussels is reviewed. We choose a different functions for the speed and, from a pure mathematical view, we also allow a negative speed. Finally in the last section we verify our results from the previous sections in both one and two dimensions using simulations. We end with a conclusion and summary together with further research suggestions.
1 The equation

We now derive an equation for the local density $M(X,t)$ of a population of particles, in our case mussels. In general particles can be cells, an amount of chemical, a number of animals, and so on. In principle we consider the $n$-dimensional case, but later on we will restrict us to the one-dimensional setting, where $n = 1$. This allows us to do the mathematical analysis for the model.

To model the change of concentration of mussels $M$, we use Fick’s first law of diffusion or Fickian diffusion, which is a classical approach to diffusion. This theory for diffusion uses the flux of the considered particles, the particle flux $J$, which indicates the rate of flow of a particle per unit area. The diffusion law says that the particle flux is proportional to the gradient of the concentration of the particles. In other words the particle flux moves from regions of high concentration to regions of low concentration - which indicates the minus sign - with a magnitude which is proportional to the concentration gradient - the spatial derivative. The magnitude is denoted by the diffusion coefficient $D$, which measures how efficiently the particles disperse from a high to a low centration [20, Ch.11]. Fick’s law states

$$J = -D\nabla M.$$  

(1.1)

We consider the case where the particle flux depends on two factors, namely the movement speed of the mussels and the effect of long range or nonlocal interactions. The movement speed depends on the concentration mussels $M$ and is denoted by $V(M)$. Furthermore, the particles turn with a tumbling rate which also depends on $M$ and is denoted by $\tau(M)$. In [29] the flow of particles in two boxes is modeled. From this model it is argued that if both $\tau(M)$ and $V(M)$ are $M$-dependent, one finds that the equilibrium density of particles is inversely proportional to $V(M)$ and that the diffusion coefficient for the velocity flux $D_v$ is equal to $[V(M)]^2/(n\tau)$. A generalization of this concept provides the related velocity flux

$$J_v = -\frac{V(M)}{n\tau(M)}\nabla V(M)M.$$  

(1.2)

This relation is valid under the condition $V|\nabla \tau| \ll \tau^2$ which states that the fractional change of the tumbling rate over the typical distance traveled between collisions must be small. Apart from the contribution of the movement speed to the flux, we also incorporate the effect of long range or nonlocal interactions as mentioned. The form of the particle flux though implies that diffusion is short range or local effect. This can be seen with the simple diffusion equation $M_t = D\nabla^2 M$, the one-dimensional heat equation. But when taking into account the long range or nonlocal effects it is shown in [20] that the first correction to the simple linear $\nabla M$ for the particle flux $J$ is a third order derivative term. Then the related long range or nonlocal interactions flux is given by

$$J_l = \kappa \nabla (\Delta M),$$  

(1.3)

where $\kappa$ is a measure of the long range effects and is in general smaller than diffusion coefficient for the velocity flux $D_v$, because we consider the particle flux $J_l$ to the stabilising the constant $\kappa > 0$ [20].

The general classical diffusion equation reads, using the equation for the diffusion (1.1),

$$\frac{\partial M}{\partial t} = -\nabla \cdot J = \nabla \cdot (D(M)\nabla M),$$

also see [20]. Because the population is studied on a relatively short time-scale at which no birth or death occurs, we assume there is no reproduction nor mortality in the population and therefore
no source term. Combining the above assumptions and using for the particle flux $J = J_v + J_l$ in (1.2) and (1.3) leads to the equation
\[
\frac{\partial M}{\partial t} = \nabla \cdot \left[ \frac{V(M)}{n\tau} \nabla (V(M) M) - \kappa \nabla (\Delta M) \right] = \nabla \cdot [f(M) \nabla M - \kappa \nabla (\Delta M)] ,
\] (1.4)
with the partial flux
\[
J = -f(M)\nabla M + \kappa \nabla (\Delta M),
\] (1.5)
and the function
\[
f(M) = \frac{V}{n\tau} \left( V + M \frac{dV}{dM} \right),
\] (1.6)
and where $V$ and $\tau$ are, in general, functions of $M$.

For simplicity, however, we consider here the tumbling rate $\tau$ to be independent of $M$. Moreover, we restrict the problem to one dimension where $n = 1$. Note that since $M$ is the population and $V(M)$ is the speed of the particles in the population, we know that $M \geq 0$ and $V(M) \geq 0$ for all $M \geq 0$.

1.1 Structure

Equation (1.4) has the structure of the Cahn-Hilliard equation, which is also abbreviated by the CH-equation. The obvious CH-structure can be seen after writing the equation as
\[
\frac{\partial M}{\partial t} = \nabla^2 \left[ W'(M) - \kappa \nabla^2 M \right].
\] (1.7)

The derivative of the potential $W(M)$, $W'(M)$, is defined as the function $F(M) \equiv W'(M)$. And the derivative of the function $F(M)$, $F'(M)$, is then again defined as the function $f(M) \equiv F'(M)$. Therefore $W'(M)$ is given by $f(M)\nabla M$. For simplicity, the restriction to the case of one spatial dimension can be made but without boundaries, i.e. $x = x \in \mathbb{R}$ and $n = 1$. When setting one spatial dimension with boundaries $[-L, L]$, one can take the limit $L \to \infty$ which provides an unbounded one spatial dimension. The equation for the population mussels then reads
\[
\frac{\partial M}{\partial t} = \frac{\partial}{\partial x} [f(M) M_x - \kappa M_{xxx}] = \frac{\partial^2}{\partial x^2} [F(M) - \kappa M_{xx}].
\] (1.8)

In the most standard setting - sometimes referred to as the constant mobility - quartic polynomial case [21] - the potential $W$ is explicitly given by
\[
W(M) = \frac{1}{4} (M^2 - 1)^2 = \frac{1}{4} M^4 - \frac{1}{2} M^2 + \frac{1}{4},
\] (1.9)
so
\[
F(M) = M^3 - M.
\] (1.10)

Both $F$ and $W$ are represented in Figure 1.1. The CH-equation generates patterns of the type shown in Figure IV. The two states in these patterns are exactly the two minima $M_-$ and $M_+$ of the potential $W$. Due to the relation of $W$ with $F$ and $f$, these $M_\pm$ also correspond to the zeros of $F$ and the extrema of $f$ respectively. With the choice of (1.9), these preferred states are $M_- = -1$ and $M_+ = +1$. The spatial system switches between areas at which $M$ is at either one

\[\text{The constant $\kappa$ can be scaled out by rescaling the spatial variable $X$ by setting $\tilde{X} \equiv \frac{1}{\sqrt{\kappa}} X$ and the time variable $t$ by $\tilde{t} \equiv \frac{t}{\kappa}$.}\]
of the preferred states, indicated in the two different colors. The explanation for the preference of these two states will be done later on in section 2.

For the generation of CH-type patterns, it is not strictly necessary that $W$ has the explicit standard form given in (1.9). The condition needed is that there exist two preferred states $M_-$ and $M_+$, minima of the potential $W$, and that $M_-$ and $M_+$ are separated by a third unstable state $M_0$, a local maximum of the potential. In terms of its derivative this means that $F$ must have three zeros, $M_- < M_0 < M_+$. We assume $W \in C^4(\mathbb{R})$ and is qualitatively similar to the map $M \mapsto (M^2 - 1)^2$. To be more precise, it is usually assumed that there exist real numbers $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_5$ such that $W$ is strictly decreasing on $(-\infty, \alpha_1)$ and $(\alpha_3, \alpha_5)$ and strictly increasing on $(\alpha_1, \alpha_3)$ and on $(\alpha_5, \infty)$, and additionally concave up on $(-\infty, \alpha_2) \cup (\alpha_4, \infty)$, and concave down on $(\alpha_2, \alpha_4)$ [11, 12].

The potential $W$ and the function $f$ strongly correspond because $W'(M) = f(M)M_x$. Also because $f$ depends on the movement speed $V$, the choice for $V$ in terms of $M$ is crucial. Because the setting is one-dimensional the constant $n = 1$, and the function is

$$f(M) = \frac{V}{\tau} \left( V + M \frac{dV}{dM} \right).$$

(1.11)

Here $\tau > 0$ is assumed constant, so $V$ completely determines $f$. So the question is now: How do we model the movement speed $V$?

1.2 The speed of the mussels

In [15] data is obtained from experiments with the cluster size and the movement speed of the mussels. From these experiments it follows that the movement speed should be modeled in such a way it is minimal at a certain density value $M^*$. Then $V$ increases as $M$ moves away from $M^*$. To be more precise, we consider $V(M)$ with $V'(M) < 0$ for $M < M^*$ and $V'(M) > 0$ for $M > M^*$. For this choice, $f$ has one global minimum at $M^*$. But the position of this minimum with respect to the $M$-axis is controlled by the precise structure of $V$. The shape of $V$ found in the experiments is such that (1.8) could generate CH-type patterns, but this is not necessary and also depends on the precise structure of $V$. One of the most straightforward choices would be to take $V$ parabolic:

$$V(M) = aM^2 - bM + c,$$

(1.12)

where $a, b, c$ are constants that are need to be determined. This parabola has to have a minimum which implies that the parameter $a$ has to be positive. This minimum is located at $M^* = \frac{b}{2a}$.
with the value $V(M^*) = -\frac{b^2}{4c} + c$. The value $V(M^*)$ must be positive in order for the speed to be positive, which provides the condition $b^2 < 4ac$ for the parameter $b$. But since the minimum is located at $M^*$ and $a > 0$, the condition for $b$ is $0 < b < 2\sqrt{ac}$. At the intersection of the vertical axis at $M = 0$, the speed $V(0) = c$ and therefore also $c$ has to be positive. So all constants have to be positive together with an extra condition for $b$.

To be able to handle the movement speed better regarding the constants $a$, $b$ and $c$, equation (1.12) can be rescaled. The equation is first written as:

$$V(M) = aM^2 - bM + c = c\left(\frac{a}{c}M^2 - \frac{b}{c}M + 1\right).$$

If we now set the new variable $\tilde{M}$ as $\tilde{M} \equiv +\sqrt{\frac{c}{a}}M$ and define a new parameter $\gamma$ as

$$\gamma = \frac{b}{\sqrt{ac}} > 0.$$

Then the movement speed is given by

$$V(\tilde{M}) = c(\tilde{M}^2 - \gamma \tilde{M} + 1).$$

Substitution of (1.13) into the function (1.11) provides

$$f(\tilde{M}) = \frac{c^2}{\gamma}(\tilde{M}^2 - \gamma \tilde{M} + 1)(3\tilde{M}^2 - 2\gamma \tilde{M} + 1).$$

This last equation can be integrated to obtain the function $F(\tilde{M})$. When choosing (1.12) the function $f$ is a fourth and $F$ a fifth order function in $\tilde{M}$. This would imply that $W$ would be a sixth order function. But we have already argued that $W$ should be a function which is qualitatively similar to the function $(M^2 - 1)^2$, which is of fourth order. But when taking

$$V(\tilde{M}) = \sqrt{c(\tilde{M}^2 - \gamma \tilde{M} + 1)},$$

the functions become

$$f(\tilde{M}) = D(2\tilde{M}^2 - \frac{3}{2} \gamma \tilde{M} + 1),$$

$$(1.15)$$

$$F(\tilde{M}) = D(\frac{2}{3} \tilde{M}^3 - \frac{3}{4} \gamma \tilde{M}^2 + \tilde{M} - \tilde{H}),$$

$(1.16)$

in which $\tilde{H} \in \mathbb{R}$ is the integration constant and an irrelevant dummy variable. Now $W$ becomes a fourth order function. Therefore we will, for now, choose (1.14) as the movement speed $V$. In the next section we will also investigate the original movement speed given as (1.12). The diffusion coefficient $D$ is here introduced as $D \equiv \frac{\sqrt{c}}{\gamma}$. Hence, using equation (1.8) and the functions (1.15) and (1.16),

$$\tilde{M}_t = \frac{\partial}{\partial x} \left[ D \left( 2\tilde{M}^2 - \frac{3}{2} \gamma \tilde{M} + 1 \right) \tilde{M}_x - \kappa \tilde{M}_{xx} \right]$$

$$= -\frac{\partial^2}{\partial x^2} \left[ \kappa \tilde{M}_{xx} - D \left( \frac{2}{3} \tilde{M}^3 - \frac{3}{4} \gamma \tilde{M}^2 + \tilde{M} - \tilde{H} \right) \right].$$

Because the constant $\kappa$ for the long range interactions is much smaller than the coefficient $D$, the small parameter $\varepsilon$ can be set as $\varepsilon \equiv \frac{\varepsilon}{D} \ll 1$. The equation for $\tilde{M}$ becomes

$$\tilde{M}_t = -D \frac{\partial^2}{\partial x^2} \left[ \varepsilon \tilde{M}_{xx} + \left( \tilde{H} - \tilde{M} + \frac{3}{4} \gamma \tilde{M}^2 - \frac{2}{3} \tilde{M}^3 \right) \right].$$

$(1.17)$
1.3 Further scaling

To do further analysis for (1.17), it is useful to manipulate the equation by rescaling and introducing variables to bring the equation into the CH-form. Therefore we first introduce a new variable $\tilde{P}$ such that $M(x, t) = P(x, t) + \tilde{C}_1$, where $\tilde{C}_1$ is a constant which can be chosen in such a way that the quadratic term $\frac{3}{4}\gamma M^2$ in (1.17) can be eliminated.

$$
P_t = -D \frac{\partial^2}{\partial x^2} \left[ \varepsilon P_{xx} + \left( \tilde{H} - (P + \tilde{C}_1) + \frac{3}{4}\gamma (P + \tilde{C}_1)^2 - \frac{2}{3}(P + \tilde{C}_1)^3 \right) \right]
= -D \frac{\partial^2}{\partial x^2} \left[ \varepsilon P_{xx} + \tilde{H} - \tilde{C}_1 + \frac{3}{4}\gamma \tilde{C}_1^2 - \frac{2}{3}\tilde{C}_1^3 \right.
+ \left. \left( -1 + \frac{3}{2}\gamma \tilde{C}_1 - 2\tilde{C}_1^2 \right) P + \frac{1}{4}(3\gamma - 8\tilde{C}_1)P^2 - \frac{2}{3}P^3 \right].
$$

To eliminate the quadratic term, we need to set $3\gamma - 8\tilde{C}_1 = 0$ or $\tilde{C}_1 = \frac{3}{8}\gamma$. So $\tilde{M}(x, t) = P(x, t) + \frac{3}{8}\gamma$. Then

$$
P_t = -D \frac{\partial^2}{\partial x^2} \left[ \varepsilon P_{xx} + \tilde{H} + \frac{3}{128}\gamma (3\gamma^2 - 16) + \left( \frac{9}{32}\gamma^2 - 1 \right) P - \frac{2}{3}P^3 \right].
$$

When choosing $\tilde{H} = -\frac{3}{128}\gamma (3\gamma^2 - 16)$ there is no constant term in the cubic equation for $P$. The equation which remains is

$$
P_t = -D \frac{\partial^2}{\partial x^2} \left[ \varepsilon P_{xx} + \left( \frac{9}{32}\gamma^2 - 1 \right) P - \frac{2}{3}P^3 \right]. \quad (1.18)
$$

We want and need the condition $\frac{9}{32}\gamma^2 - 1 > 0$ in order for the potential $W$ to have the correct structure and the possible appearance of pattern formation. If we now introduce a new variable $Q$ such that $P(x, t) = \frac{1}{2}\sqrt{6}Q(x, t)$ then the cub term $\frac{2}{3}P^3$ in (1.18) is scaled to $P^3$. Then $Q$ is a solution of

$$
Q_t = D \frac{\partial^2}{\partial x^2} \left[ Q^3 - \beta Q - \varepsilon Q_{xx} \right]. \quad (1.19)
$$

Where the new parameter $\beta$ is introduced and defined as

$$
\beta := \frac{9}{32}\gamma^2 - 1,
$$

with the handy condition $\beta > 0$. The structure of (1.19) is the same as equation (1.8) with

$$
f(Q) = 3Q^2 - \beta \quad \text{and} \quad F(Q) = Q^3 - \beta Q. \quad (1.20)
$$

The corresponding potential $W$ can be obtained by integrating $F$ and setting the integration constant equal to $\frac{\beta^2}{4}$, so the potential is qualitatively similar to ($Q^2 - 1)^2$:

$$
W(Q) = \frac{1}{4}(Q^2 - \beta)^2 = \frac{1}{4}Q^4 - \frac{\beta}{2}Q^2 + \frac{\beta^2}{4}. \quad (1.21)
$$

These functions indeed all have the required structure such that (1.19) can exhibit pattern formation. The two minima of $W$ are now $Q_{\pm} = \pm\sqrt{\beta}$ and local maximum is $Q_0 = 0$ with
$W(Q_0) = \frac{\beta^2}{4}$. The main variable of (1.19) is $Q(x, t)$ while the variable for the population of mussels is $M(x, t)$. Therefore it is useful to keep in mind the variable $M$ is in terms of $Q$:

$$
M(x, t) = \sqrt{\frac{c}{a}} M(x, t) = \sqrt{\frac{c}{a}} \left( P(x, t) + \frac{3}{8} \gamma \right) = \sqrt{\frac{c}{a}} \left( \sqrt{\frac{3}{2}} Q(x, t) + \frac{3}{8} \gamma \right)
$$

$$
= \sqrt{\frac{c}{a}} \left( \sqrt{\frac{3}{2}} Q(x, t) + \sqrt{\frac{\beta + 1}{2}} \right) = \sqrt{\frac{c}{a}} \left( \sqrt{\frac{3}{2}} Q(x, t) + \frac{3b}{8\sqrt{ac}} \right).
$$

The condition for a positive population $M \geq 0$ implies for the variable $Q$ the condition $Q(x, t) \geq -\frac{1}{8} \sqrt{6} \gamma$. So the condition in terms of the original parameters is

$$
Q(x, t) \geq -\sqrt{\frac{\beta + 1}{2}} = -\frac{3b}{\sqrt{32ac}}.
$$
2 Analysis

The equation for the population mussels, after rescaling, is given in (1.19) and reads

$$\frac{\partial Q}{\partial t} = D \frac{\partial^2}{\partial x^2} \left[ Q^3 - \beta Q - \varepsilon Q_{xx} \right] = D \frac{\partial^2}{\partial x^2} \left[ F(Q) - \varepsilon Q_{xx} \right],$$

(2.1)

where

$$F(Q) = W'(Q),$$

or in terms of the other function

$$\frac{\partial Q}{\partial t} = D \frac{\partial}{\partial x} \left[ (3Q^2 - \beta)Q_x - \varepsilon Q_{xxx} \right] = D \frac{\partial}{\partial x} \left[ f(Q)Q_x - \varepsilon Q_{xxx} \right].$$

(2.2)

Note that our variable is now $Q$ due to the rescalings, but we still need to keep in mind that the variable $Q$ relates to the population mussels $M$. As mentioned before, the potential $W$ - which is given in (1.21) - plays an important role in the dynamics of the model. Therefore we will provide more inside onto this potential in this section, together with the analysis of the equation in (2.1).

2.1 The potential and the energy

The spatial setting of the CH-equation in (2.1) is in one dimension without boundaries. This was done for simplicity. But we can also take a one-dimensional spatial bounded domain $\Sigma = [-L, L]$ for $L \to \infty$. These boundary are subject to boundary conditions, the Neumann and the zero mass flux boundary conditions [8, 21]:

$$\frac{\partial Q}{\partial x} \bigg|_{x=\pm L} = f(Q)Q_x - \varepsilon Q_{xxx} \bigg|_{x=\pm L} = 0.$$

Typically, the domain $\Omega$ is assumed to be a bounded domain with a sufficiently smooth boundary $\partial \Omega$. These boundary conditions are equivalent to

$$\frac{\partial Q}{\partial x} \bigg|_{x=\pm L} = \frac{\partial^3 Q}{\partial x^3} \bigg|_{x=\pm L} = 0.$$

So the CH-equation in one dimension, subject to these boundary conditions, is given by

$$\begin{cases} 
\frac{\partial Q}{\partial t} = D \frac{\partial^2}{\partial x^2} \left( W'(Q) - \varepsilon Q_{xx} \right) & \text{for } (x, t) \in \Omega \times \mathbb{R}^+, \\
Q_x = Q_{xxx} = 0 & \text{for } (x, t) \in \partial \Omega \times \mathbb{R}^+, 
\end{cases}$$

(2.3)

and the initial conditions

$$Q(x, 0) = q_0(x) \quad \text{for } x \in \Omega.$$

The change of the total mass is

$$\frac{d}{dt} \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} Q(x, t) \, dx = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \frac{\partial}{\partial t} Q(x, t) \, dx = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \left( W'(Q) - \varepsilon Q_{xx} \right) \, dx = 0.$$

So the total mass remains for all $t > 0$, due to the boundary conditions,

$$\lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} Q(x, t) \, dx = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} Q(x, 0) \, dx = q_0(x).$$
The amount of mussels in the population remains constant over time, so no mussels are added or removed nor is there any reproduction or mortality [8].

For understanding the importance of the potential $W$ we need the corresponding energy. This Ginzburg-Landau energy acts as a Lyapunov functional for (2.3) when $t > 0$, with the requirement that the energy is bounded [8, 9, 11, 21, 22, 23].

**Definition 2.1 (Energy).** The Ginzburg-Landau free energy is defined as

$$E(Q) = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \left( W(Q) + \frac{1}{2} Q_x^2 \right) \, dx.$$  

(2.4)

This energy can give more inside onto the behaviour of our system and therefore the population mussels, besides the conservation of mass.

**Lemma 2.1 (Decay of energy).** The energy $E$ decays in time under the dynamics of $M$.

**Proof.** The derivative of $E$ in (2.4) with respect to $t$:

$$\frac{dE}{dt} = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \frac{\partial}{\partial t} \left( W(Q) + \frac{1}{2} Q_x^2 \right) \, dx$$

$$= \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} (W'(Q)Q_t + Q_x Q_{xx}) \, dx$$

$$= \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} W'(Q)Q_t \, dx + \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} Q_x \frac{\partial}{\partial x} Q_x \, dx. \quad (2.5)$$

The second integral in (2.5) can be expanded using partial integration:

$$\lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} Q_x \frac{d}{dx} Q_x \, dx = \lim_{L \to \infty} \frac{1}{2L} \left\{ [Q_x Q_t]_{x=-L}^{x=L} - \int_{-L}^{L} Q_{xx} Q_t \, dx \right\}$$

Due to the Neumann boundary condition the term $[Q_x Q_t]_{x=-L}^{x=L} = 0$. Then (2.5) becomes

$$\frac{dE}{dt} = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} (W'(Q) - Q_{xx}) Q_t \, dx,$$

where the expression for $Q_t$ can be substituted using (2.3).

$$\frac{dE}{dt} = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} (W'(Q) - Q_{xx}) \left[ \frac{\partial^2}{\partial x^2} (W'(Q) - Q_{xx}) \right] \, dx.$$  

This equation can also be expanded using partial integration:

$$\frac{dE}{dt} = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \left( W'(Q) - Q_{xx} \right) \left[ \frac{\partial^2}{\partial x^2} (W'(Q) - Q_{xx}) \right] \, dx$$

$$= \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \left[ \frac{\partial}{\partial x} (W'(Q) - Q_{xx}) \right]_{-L}^{L} - \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \left[ \frac{\partial}{\partial x} (W'(Q) - Q_{xx}) \right]^2 \, dx$$

$$= - \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \left[ \frac{\partial}{\partial x} (W'(Q) - Q_{xx}) \right]^2 \, dx \leq 0. \quad \square$$
This provides us more information. First of all the total mass is conserved and the energy of the system decays under the dynamics of \( Q \). In order to study the CH-equation (2.3) further, we consider \( Q \) to be only spatial dependent. So the variable \( Q(x,t) = q(x) \). Then the critical points of \( E \) can be found which could be potential minima of interest due to the decay of energy.

Equation (2.3) can be integrated twice with respect to \( x \).

\[
\frac{d^2}{dx^2} (W'(q) - \varepsilon q_{xx}) = 0
\]

\[
W'(q) - \varepsilon q_{xx} = c_1 + c_2 x,
\]

where both \( c_1, c_2 \in \mathbb{R} \) are integration constants. The constant term \( c_2 x \) tends to infinity if \( x \) tends to infinity. To compensate this effect the left-hand side, \( F(q) - \varepsilon q_{xx} \), should also tend to infinity for \( x \to \infty \). Because both \( F(q) \) and \( q_{xx} \) depend on the variable \( x \) this implies that the condition \( q(x) \to \infty \) if \( x \to \infty \) should hold. But because the variable \( q \) relates to the population mussels and this population cannot become unbounded and blow up, our interest lies in bounded solution \( q(x) \) for all \( x \). To cancel this effect the constant term \( c_2 x \equiv 0 \). Notice that the parameter \( \varepsilon \) can be scaled out by rescaling the spatial variable \( x \) as \( \tilde{x} = \frac{1}{\sqrt{\varepsilon}} x \). Dropping the tildes for convenience provides then

\[
W'(q) - q_{xx} = c_1. \tag{2.6}
\]

In section 1.3 we ended with explicit expressions for the functions \( f \) and \( F \) and potential \( W \) given in (1.20) and (1.21). There \( W'(q) = F(q) = q^3 - \beta q \) together with the condition \( \beta > 0 \), which can be used in equation (2.6). In Figure 2.1 the function \( F(q) \) is represented. The two extrema and the three roots are indicated to point out the structure of the function. Because of the condition \( \beta > 0 \) the qualitative structure of the function is the same\(^2\).

### 2.2 System

The one-dimensional equation for interest (2.6) can be written as a one-dimensional system:

\[
\begin{cases}
q_x = r \\
r_x = F(q) - c_1.
\end{cases} \tag{2.7}
\]

The equilibria of system (2.7) can be computed by setting both equations \( q_x \) and \( r_x \) to 0. These equilibria, which we denote by \( (\tilde{q}, \tilde{r}) \), satisfy \( r = 0 \) and \( F(q) = c_1 \). Our interest therefore lies in

\(^2\)For this reason the parameter \( \beta = 1 \) in the coming figures.
the intersection of $F$ with the constant $c_1$. This constant $c_1$ can be varied to find (a) different (number of) equilibria. To be able to identify the character of these equilibria, we use the Jacobian matrix corresponding to (2.7):

$$
J(q,r) = \begin{pmatrix} 0 & 1 \\ 3\tilde{q}^2 - \beta & 0 \end{pmatrix}.
$$

(2.8)

The eigenvalues of the Jacobian (2.8) are $\lambda_\pm = \pm \sqrt{3\tilde{q}^2 - \beta} = \pm \sqrt{f(\tilde{q})}$. So $f$ determines the characters of the equilibria. Namely if $f > 0$ the equilibrium is a saddle point, and if $f < 0$ it is a center.

It is still convenient to keep in mind the relation of the variable $q$ to the population mussels. Recall the requirement $q \geq -\sqrt{\frac{\beta + 1}{3}}$. This determines the domain of interest because it provides a lower bound. Because the considered system is given in (2.7), this requirement can be put into place. So we investigate how $-\sqrt{\frac{\beta + 1}{3}}$ varies in the phase plane with respect to $\beta$. The requirement is negative, therefore it is important to keep the three negative points indicated in Figure 2.1, $-\frac{2\sqrt{3}}{\sqrt{3}}, -\sqrt{\beta}$ and $-\sqrt{\frac{\beta}{3}}$, in mind. In Figure 2.2 the relation between the requirement and the three points is made clear in terms of $\beta$. To clarify this relation we take for example $\beta = \frac{5}{12}$. Then the requirement $-\sqrt{\frac{\beta + 1}{3}}$ lies, for this value of $\beta$, between the points $-\frac{2\sqrt{3}}{\sqrt{3}}$ and $-\sqrt{\beta}$. This is useful when working with system (2.7) where $F$ is given and the constant $c_1$ still to be chosen.

2.3 The Hamiltonian

Equation (2.6) can again be integrated with respect to $x$:

$$
W(q) - \frac{1}{2}q_x^2 = c_1 q - H.
$$

(2.9)

Here $H$ is again an integration constant, but is also equal to the Hamiltonian of the system. This Hamiltonian $H$ can be expressed in terms of $q$ explicitly using (1.21):

$$
H = H(q,q_x) = \frac{1}{2}q_x^2 - W(q) + c_1 q = \frac{1}{2}q_x^2 - \frac{1}{4}q^4 + \frac{\beta}{2}q^2 - \frac{\beta^2}{4} + c_1 q.
$$

(2.10)

Using (2.9) and (2.10), the level curves of the system in the phase plane $(q,q_x)$ are given by

$$
q_x = \pm \sqrt{2H + 2W(q) - 2c_1 q} = \pm \sqrt{2H + \frac{1}{2}q^4 - \beta q^2 + \frac{\beta^2}{2} - 2c_1 q}.
$$

(2.11)
Figure 2.3: The level curves of system (2.7) for $c_1 = 0$, together with the corresponding energy $E$.

Also $W$ given in (2.9) can be used for the energy given in (2.4). Substituting this $W$ into the energy $E$ in (2.4) provides the critical points of $E$. This energy will only depend on the spatial variable $x$ and not $t$, but also on the two constants $c_1$ and $H$. For this reason the energy is here denoted by $E(c_1, H)$:

$$E(c_1, H) = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \left( q_x^2 + c_1 q - H \right) dx.$$  \hspace{1cm} (2.12)

2.4 The setting $c_1 = 0$

What the minima of the energy $E$, given in (2.12), are in terms of $c_1$ and $H$ is still unknown. But because the system in (2.7) also depends on $c_1$ we could investigate this constant first. A logical first choice is to examine the setting where $c_1 = 0$. What are the dynamics in the phase plane? For this we use the Hamiltonian in (2.10), the level curves in (2.11) and the energy $E$ in (2.12). For $c_1 = 0$ the equilibria are explicitly known:

$$\left( \tilde{q}, \tilde{r} \right) = (0, 0), (\pm \sqrt{\beta}, 0).$$

Evaluating these three points into the function $f$ provides $-\beta + 3\tilde{q}^2|_{\tilde{q}=0} = -\beta < 0$ and $-\beta + 3\tilde{q}^2|_{\tilde{q}=\pm \sqrt{\beta}} = 2\beta > 0$. Hence the character of the origin $(0, 0)$ is a center, and the other two equilibria $(\pm \sqrt{\beta}, 0)$ are both saddle points. The value of $H$ at these equilibria is

$$H(0, 0) = -\frac{\beta^2}{4}, \quad H(\pm \sqrt{\beta}, 0) = 0.$$ 

Because both $c_1 = 0$ and $q_x = 0$ for the equilibria, the value of $H$ and the energy $E$ have opposite signs. So $E(0, 0) = \frac{\beta^2}{4}$ and $E(\pm \sqrt{\beta}, 0) = 0$. Due to the decaying energy in time the two non-trivial equilibria are preferred by the system instead of the origin. This is a key observation. In Figure 2.3 the level curves are plotted in the phase plane $(q, q_x)$. The level curves for $H = 0$ at $(\pm \sqrt{\beta}, 0)$ are indicated in red and for $H = -\frac{\beta^2}{4}$ at the origin in green. For more inside another level curve is plotted at $(\pm \frac{1}{2}\sqrt{\beta}, 0)$ with $H = -\frac{9}{64}\beta^2$, indicated in blue. When investigating the
Hamiltonian further, it also provides that above the upper and below the lower red line $H < 0$, and left of the left-hand and right of the right-hand green line $H < -\frac{\beta^2}{4}$. Which implies that the energy will be at a minimum 0 at the two equilibria $(\pm\sqrt{\beta}, 0)$.

Recall the two minima $M_-$ and $M_+$ of the potential $W$ described in section 1.1. For the potential which is now used and is given in 1.9 these two minima are $-\sqrt{\beta}$ and $\sqrt{\beta}$. And the unstable state is $M_0 = 0$. These three points correspond to the three equilibria. From the system together with its level curves given in Figure 2.3 it follows two heteroclinic connections exist at the two non-trivial equilibria. The explicit expression for these two heteroclinic connections is

$$q_h^+(x) = \pm\sqrt{\beta} \tanh \left( \sqrt{\frac{\beta}{2}} x \right),$$

with initial condition $q_h(0) = 0$. Here $q_h^+(x)$ is the connection from $-\sqrt{\beta}$ to $\sqrt{\beta}$ and vice versa for $q_h^-(x)$. In Figure 2.4 the two functions are plotted, together with the two stable states.

In time the energy of the system decays which follows from Lemma 2.1. We have already seen that the minimum of $E$, for $c_1 = 0$, is 0. This is the case in two situations, namely the two equilibria $(\pm\sqrt{\beta}, 0)$. For the heteroclinic connections we clarify the situation with Lemma 2.2.

**Lemma 2.2** (The energy at the two heteroclinic connections). The energy $E$ at the heteroclinic connections, which exist for $c_1 = 0$, is 0 on the one-dimensional spatial domain $[-L, L]$ for $L \to \infty$.

**Proof.** We consider the energy given in (2.4). Choose $\varepsilon > 0$, then there exists a $K$ such that $0 \leq q^*_L < \varepsilon$ and $0 \leq W(q) < \frac{1}{2}\varepsilon$ for all $|x| \geq K$. Then

$$0 \leq E(q^*_L(x)) = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \left( W(q^*_L(x)) + \frac{1}{2} (q^*_L(x))^2 \right) dx$$

$$< \lim_{L \to \infty} \frac{1}{2L} \left[ \int_{-L}^{-K} \left( \frac{1}{2}\varepsilon^2 + \frac{1}{2}\varepsilon \right) dx + \int_{K}^{L} \left( \frac{1}{2}\varepsilon^2 + \frac{1}{2}\varepsilon \right) dx \right]$$

$$+ \lim_{L \to \infty} \frac{1}{2L} \int_{-K}^{K} \left[ W(q^*_L(x)) + \frac{1}{2} (q^*_L(x))^2 \right] dx$$

At the interval $[-K, K]$ the energy of the heteroclinic orbits is bounded. Hence

$$0 \leq E(q^*_L(x)) < \lim_{L \to \infty} \frac{1}{2L} [(-K + L)\varepsilon + (L - K)\varepsilon] + 0 = \lim_{L \to \infty} \frac{1}{L} (L - K)\varepsilon = \varepsilon.$$
So if only one heteroclinic connection exists on the entire unbounded domain, the energy is 0. Because this is the minimal amount of energy this connection is an expected dynamic of the system. Note that it is also necessary to have these connections due to the mass conservation; the population mussels stays constant. If the system has multiple heteroclinic connections, then in time one would expect the amount of connections to decrease because multiple connections do not have an energy 0. This process is called coarsening, and is very well-known for the CH-equation [7, 9], see Figure 2.5 for illustration. In theory one would expect to have only one heteroclinic connection for \( t \to \infty \) on the entire domain. Due to the mass conservation it is also not possible for the system to enter one of the two stable states \( M_\pm \) and remain there. So the stable states are the two minima \( M_\pm \) and the two heteroclinic connections \( q_0^\pm(x) \). These four states are exactly the ones shown in Figure IV and 2.5. In Section 4 this process will also be discussed.

In [12] it is shown that there are precisely three types of bounded non-constant stationary solutions of the CH-equation on \( \mathbb{R} \). These types are periodic solutions, homoclinic solutions referred to as (pulse-type) reversal solutions and heteroclinic solutions referred to as (monotonic) transition waves. The spectrum of the linear operator obtained upon linearization is studied for each of these solutions and only for the heteroclinic solutions linear stability exists. For our analysis we also consider the CH-equation in \( \mathbb{R} \). From this article we know now that the transition waves or heteroclinic connections have linear stability and so these are the solutions we are interested in. Note that the potential \( W \) must have the double-well form described in section 1.1. In [12] the spectral analysis for stationary solutions is done on \( \mathbb{R}^d \) for \( d \geq 2 \).

**Remark.** Recall again the requirement for a positive population, \( q \geq -\sqrt{\beta + \frac{1}{\beta}} \). Because the heteroclinic connections both also have \( E = 0 \) these are also stable states. But therefore it is necessary that the equilibrium \((-\sqrt{\beta}, 0)\) is part of the domain which implies \(-\sqrt{\beta + \frac{1}{\beta}} \leq -\sqrt{\beta} \). In terms of the parameter \( \beta \) this implies \( 0 < \beta \leq \frac{1}{2} \). Which is equivalent to \( \frac{\sqrt{32}}{3} < \gamma \leq \frac{4}{\sqrt{3}} \) for the parameter \( \gamma \), where \( \frac{\sqrt{32}}{3} \approx 1.8856 \) and \( \frac{4}{\sqrt{3}} \approx 2.3094 \). In Figure 2.6 the setting is sketched, where the range of the domain is represented in brown.

### 2.5 The setting \( c_1 \neq 0 \)

We have just reasoned in the previous section that the setting with \( E = 0 \), for \( c_1 = 0 \) and \( H = 0 \), provides stable states. But what happens if \( c_1 \neq 0 \)? To begin to understand this setting, we first assume \( c_1 \) is small.
2.5.1 Perturbation theory

For this analysis we use perturbation theory where $c_1$ is seen as a small perturbation of $c_1 = 0$ and $c_1$ is $O(\delta)$ for $0 < \delta \ll 1$, see Figure 2.7 for illustration.

The trivial equilibrium For $c_1 = O(\delta)$, the trivial equilibrium $(\tilde{q}, \tilde{r}) = (0, 0)$ shifts. For $q \approx 0$, the equation $r_x = q^3 - \beta q - c_1$ from the system becomes $r_x \approx -\beta q - c_1$ because $|q^3| \ll |\beta q|, |c_1|$. Then approximately $q = -\frac{c_1}{\beta}$ with

$$q = -\frac{c_1}{\beta} \quad \left\{ \begin{array}{ll} < 0 & \text{if } c_1 > 0 \\ > 0 & \text{if } c_1 < 0. \end{array} \right.$$ 

The equilibrium $(0, 0)$ for $c_1 = 0$ changes to leading order to $(-\frac{c_1}{\beta}, 0)$ for $c_1 \neq 0$. In other words, for $c_1 > 0$ small the trivial equilibrium shifts to the left and the $q$-coordinate becomes negative, and for $c_1 < 0$ small the equilibrium shifts to the right and the $q$-coordinate becomes positive.

At the trivial equilibrium $H(0, 0) = 0$ for $c_1 = 0$. For $c_1 \neq 0$, the Hamiltonian at $q_x = 0$ is $H(q, 0) = -\frac{1}{2}q^4 + \frac{\beta}{2}q^2 - \frac{\beta^2}{4} + c_1 q$. For $q \approx 0$, to leading order the equation becomes $H(q, 0) \approx \frac{\beta}{2}q^2 - \frac{\beta^2}{4} + c_1 q$, because $|\frac{1}{2}q^4| \ll |\frac{1}{2}q^2|, |\frac{\beta^2}{4}|, |c_1 q|$. Then

$$H(-\frac{c_1}{\beta}, 0) = -\frac{\beta^2}{4} - \frac{c_1^2}{2\beta},$$
and
\[ E(\frac{\beta^2}{4} + \frac{c_1^2}{\beta}, c_1) = \frac{\beta^2}{4} + \frac{c_1^2}{2\beta} + c_1 \cdot \frac{\beta^2}{4} = \frac{\beta^2}{4} - \frac{c_1^2}{2\beta}. \]

So the value of \( H \) at the perturbed trivial equilibrium becomes even smaller than \(-\frac{\beta^2}{4}\), the value of \( H \) at the trivial equilibrium. The corresponding energy \( E \) is also smaller though, but because \( c_1 \) is \( O(\delta) \) the energy will still be larger than zero. Therefore this equilibrium is not preferred by the system.

**The two non-trivial equilibria** We will now focus be on the change of the two non-trivial equilibria \((\tilde{q}, \tilde{r}) = (\pm \sqrt{\beta}, 0)\) when \( c_1 \neq 0 \). Also here we use perturbation theory to describe the changes of the two equilibria and the value of \( H \) and \( E \). For this we define \( \delta \equiv c_1 \) as the small parameter, where \( \delta \) can be both positive and negative.

For the positive equilibrium \((\sqrt{\beta}, 0)\) we set \( q = \sqrt{\beta} + d_1 \delta + d_2 \delta^2 \), with \( d_1 \) and \( d_2 \) constants which are to be determined. Substitution into the equation for \( r_x \) in (2.7) provides
\[ r_x = \delta(-1 + 2\beta d_1) + \delta^2(2\beta d_2 + 3\sqrt{\beta} d_1^2) + O(\delta^3) = 0. \]

By setting the different powers of \( \delta \) to 0, the constants \( d_1 \) and \( d_2 \) can be determined. From the equation of \( O(\delta) \) the constant \( d_1 \) and \( d_2 \) from \( O(\delta^2) \),
\[ d_1 = \frac{1}{2\beta}, \quad d_2 = -\frac{3\sqrt{\beta}}{8\beta^3}. \]

Hence
\[ q = \sqrt{\beta} + \frac{1}{2\beta} c_1 - \frac{3\sqrt{\beta}}{8\beta^3} c_1^2 + O(c_1^3) \quad \begin{cases} > \sqrt{\beta} & \text{if } c_1 > 0 \\ < \sqrt{\beta} & \text{if } c_1 < 0. \end{cases} \]

So the equilibrium \((\sqrt{\beta}, 0)\) for \( c_1 = 0 \) shifts to leading order to \((\sqrt{\beta} + \frac{1}{2\beta} c_1 - \frac{3\sqrt{\beta}}{8\beta^3} c_1^2, 0)\) for \( c_1 \neq 0 \) small. In other words, for \( c_1 > 0 \) small the positive equilibrium shifts to the right and becomes larger than \( \sqrt{\beta} \), and for \( c_1 < 0 \) small the equilibrium shifts to the left and becomes smaller than \( \sqrt{\beta} \).

For the negative equilibrium \((-\sqrt{\beta}, 0)\), we use the same method by setting \( q = -\sqrt{\beta} + e_1 \delta + e_2 \delta^2 \) with here \( e_1 \) and \( e_2 \) the constants to be determined. Then
\[ r_x = \delta(-1 + 2\beta e_1) + \delta^2(2\beta e_2 - 3\sqrt{\beta} e_1^2) + O(\delta^3) = 0, \]

and
\[ e_1 = \frac{1}{2\beta}, \quad e_2 = \frac{3\sqrt{\beta}}{8\beta^3}. \]

Hence
\[ q = -\sqrt{\beta} + \frac{1}{2\beta} c_1 + \frac{3\sqrt{\beta}}{8\beta^3} c_1^2 + O(c_1^3) \quad \begin{cases} > -\sqrt{\beta} & \text{if } c_1 > 0 \\ < -\sqrt{\beta} & \text{if } c_1 < 0. \end{cases} \]

So the equilibrium \((-\sqrt{\beta}, 0)\) for \( c_1 = 0 \) shifts to leading order to \((-\sqrt{\beta} + \frac{1}{2\beta} c_1 + \frac{3\sqrt{\beta}}{8\beta^3} c_1^2, 0)\) for \( c_1 \neq 0 \) small. In other words, for \( c_1 > 0 \) small the negative equilibrium shifts to the right and becomes larger than \(-\sqrt{\beta} \), and for \( c_1 < 0 \) small the equilibrium shifts to the left and becomes smaller than \(-\sqrt{\beta} \). A summary of the change of the three equilibria for \( c_1 \neq 0 \) small is given in Figure 2.7.
either one or two equilibria, depending on the value of $c_1$. But when keeping Figure (2.1) in mind, one can observe it is also possible to have three equilibria for the considered system for a whole range of values for $H$. In Figure 2.9 this situation is represented in respectively the two considered values of $c_1$.

And to leading order:

$$E(\pm \sqrt{c_1}, c_1, 0) = E(-\sqrt{c_1}, c_1, 0) = c_1^2 > 0.$$  

So indeed also here the energy is not at a minimum.

### 2.5.2 Only two equilibria

In the previous sections we assumed system (2.7) had three equilibria, so either $c_1 = 0$ or $c_1$ small enough. But when keeping Figure (2.1) in mind, one can observe it is also possible to have either one or two equilibria, depending on the value of $c_1$. Only for two values of $c_1$ there are two equilibria, namely $c_1 = \pm \frac{2\beta^2}{3\sqrt{3}}$.

Suppose $c_1 = \frac{2\beta^2}{3\sqrt{3}}$. Then the two equilibria of the system are ($-\sqrt{\frac{\beta}{3}}, 0$) and ($\frac{2\sqrt{\beta}}{\sqrt{3}}, 0$). And analogue, suppose $c_1 = -\frac{2\beta^2}{3\sqrt{3}}$, then the two equilibria are ($-\frac{2\sqrt{\beta}}{\sqrt{3}}, 0$) and ($\frac{2\sqrt{\beta}}{\sqrt{3}}, 0$). The value of $H$ is

$$H(\pm \sqrt{\beta}, 0) = -\frac{\beta^2}{3}, \quad H(\pm \frac{2\sqrt{\beta}}{\sqrt{3}}, 0) = \frac{5\beta^2}{12},$$

together with the corresponding energy

$$E(\frac{\beta^2}{3}, \pm \frac{2\beta^2}{3\sqrt{3}}) = \frac{\beta^2}{9}, \quad E(-\frac{5}{12} \beta^2, \pm \frac{2\beta^2}{3\sqrt{3}}) = \frac{\beta^2}{36}.$$  

In Figure 2.9 this situation is represented in respectively the two considered values of $c_1$.

### 2.5.3 Another example of three equilibria

It is possible to have three equilibria for the considered system for a whole range of values for $c_1$, namely $-\frac{2\beta^2}{3\sqrt{3}} < c_1 < \frac{2\beta^2}{3\sqrt{3}}$. To give another example we choose $c_1 = \pm \frac{2\beta^2}{3\sqrt{3}}$.

The three equilibria of the system, for $c_1 = \frac{2\beta^2}{3\sqrt{3}}$, are ($\tilde{q}, \tilde{r}$) = ($q_1, 0$), ($q_2, 0$), ($q_3, 0$) with

$$q_1 = \frac{-\sqrt{3} \sin(\frac{\pi}{\sqrt{3}})}{\sqrt{3}} \beta \approx -0.8846 \sqrt{\beta}, \quad q_2 = \frac{\sqrt{3} \sin(\frac{\pi}{\sqrt{3}})}{\sqrt{3}} \beta \approx -0.2005 \sqrt{\beta}, \quad q_3 = \frac{2\cos(\frac{\pi}{\sqrt{3}})}{\sqrt{3}} \beta \approx 1.0851 \sqrt{\beta}.$$
Figure 2.9: The cases $c_1 = \frac{2\beta^2}{3\sqrt{3}}$ respectively $c_1 = -\frac{2\beta^2}{3\sqrt{3}}$ together with the function $F$ and the three different level curves in the corresponding phase planes of system (2.7).

The value of $H$ at these equilibria is

$$H(q_1, 0) = \frac{\beta^2}{36} \left[ \cos \left( \frac{4\pi}{9} \right) - 3 - 3\sqrt{3} \sin \left( \frac{\pi}{9} \right) + 3\sqrt{3} \sin \left( \frac{2\pi}{9} \right) - 4\cos \left( \frac{\pi}{9} \right) - 2\cos \left( \frac{2\pi}{9} \right) \right]$$

$$\approx -0.1821\beta^2,$$

$$H(q_2, 0) = \frac{\beta^2}{36} \left[ \cos \left( \frac{4\pi}{9} \right) - 3 + 3\sqrt{3} \sin \left( \frac{\pi}{9} \right) - 3\sqrt{3} \sin \left( \frac{2\pi}{9} \right) - 4\cos \left( \frac{\pi}{9} \right) - 2\cos \left( \frac{2\pi}{9} \right) \right]$$

$$\approx -0.2689\beta^2,$$

$$H(q_3, 0) = \frac{\beta^2}{36} \left[ -2\cos \left( \frac{4\pi}{9} \right) + 4\cos \left( \frac{2\pi}{9} \right) - 3 + 8\cos \left( \frac{\pi}{9} \right) \right]$$

$$\approx 0.2010\beta^2,$$

together with the corresponding energy:

$$E(H(q_1, 0), \frac{1}{3\sqrt{3}}\beta^2) = \frac{\beta^2}{36} \left[ -\sqrt{3} \sin \left( \frac{4\pi}{9} \right) - \cos \left( \frac{4\pi}{9} \right) + 3 + 2\cos \left( \frac{2\pi}{9} \right) - 2\sqrt{3} \sin \left( \frac{2\pi}{9} \right) \right]$$

$$\approx 0.0118\beta^2,$$

$$E(H(q_2, 0), \frac{1}{3\sqrt{3}}\beta^2) = \frac{\beta^2}{36} \left[ \sqrt{3} \sin \left( \frac{4\pi}{9} \right) - \cos \left( \frac{4\pi}{9} \right) + 3 + 2\cos \left( \frac{2\pi}{9} \right) + 2\sqrt{3} \sin \left( \frac{2\pi}{9} \right) \right]$$

$$\approx 0.2303\beta^2,$$

$$E(H(q_3, 0), \frac{1}{3\sqrt{3}}\beta^2) = \frac{\beta^2}{36} \left[ 2\cos \left( \frac{4\pi}{9} \right) - 4\cos \left( \frac{2\pi}{9} \right) + 3 \right]$$

$$\approx 0.0079\beta^2.$$

The three equilibria of the system, for $c_1 = \frac{-1}{3\sqrt{3}}\beta^2$, are $(\tilde{q}, \tilde{r}) = (q_1, 0), (q_2, 0), (q_3, 0)$ with

$$q_1 = 2\cos \left( \frac{2\pi}{9} \right) \sqrt{3} / \sqrt{3} \approx 0.8848\sqrt{3},$$

$$q_2 = \frac{\sqrt{3} \sin \left( \frac{2\pi}{9} \right) - \cos \left( \frac{2\pi}{9} \right)}{\sqrt{3}} \sqrt{3} \approx 0.2005\sqrt{3},$$

$$q_3 = \frac{-\sqrt{3} \sin \left( \frac{2\pi}{9} \right) + \cos \left( \frac{2\pi}{9} \right)}{\sqrt{3}} \sqrt{3} \approx -1.0851\sqrt{3}.$$
to the bounded orbits, preferred in terms of bounded dynamics. The setting where the system has three equilibria is, due solutions of the system are unbounded. The variable

The setting where system (2.7) has either one or two equilibria is not preferred, because then the variable "c"

2.6 Pattern formation

In Figure 2.10 this situation is represented in respectively the two considered values of "c1" and the corresponding energy:

The value of "H" at these equilibria is

\[
H(q_1, 0) = \frac{\beta^2}{36} \left[ 2 \cos \left( \frac{\pi}{9} \right) + 4 \cos \left( \frac{4\pi}{9} \right) - 3 - 8 \cos \left( \frac{2\pi}{9} \right) \right] \\
\approx -0.1821\beta^2
\]

\[
H(q_2, 0) = \frac{\beta^2}{36} \left[ 4 \cos \left( \frac{2\pi}{9} \right) - 3 - \cos \left( \frac{\pi}{9} \right) - 3\sqrt{3} \sin \left( \frac{2\pi}{9} \right) - 3\sqrt{3} \sin \left( \frac{4\pi}{9} \right) - 2 \cos \left( \frac{4\pi}{9} \right) \right] \\
\approx -0.2689\beta^2
\]

\[
H(q_3, 0) = \frac{\beta^2}{36} \left[ 4 \cos \left( \frac{2\pi}{9} \right) - 3 - \cos \left( \frac{\pi}{9} \right) + 3\sqrt{3} \sin \left( \frac{2\pi}{9} \right) + 3\sqrt{3} \sin \left( \frac{4\pi}{9} \right) - 2 \cos \left( \frac{4\pi}{9} \right) \right] \\
\approx 0.2010\beta^2
\]

together with the corresponding energy:

\[
E(H(q_1, 0), -\frac{\beta^2}{3\sqrt{3}}) = \frac{\beta^2}{36} \left[ -2 \cos \left( \frac{\pi}{9} \right) - 4 \cos \left( \frac{4\pi}{9} \right) + 3 \right] \\
\approx 0.0118\beta^2
\]

\[
E(H(q_2, 0), -\frac{\beta^2}{3\sqrt{3}}) = \frac{\beta^2}{36} \left[ \sqrt{3} \sin \left( \frac{\pi}{9} \right) + \cos \left( \frac{\pi}{9} \right) + 3 + 2 \cos \left( \frac{4\pi}{9} \right) + 2\sqrt{3} \sin \left( \frac{4\pi}{9} \right) \right] \\
\approx 0.2303\beta^2
\]

\[
E(H(q_3, 0), -\frac{\beta^2}{3\sqrt{3}}) = \frac{\beta^2}{36} \left[ -\sqrt{3} \sin \left( \frac{\pi}{9} \right) + \cos \left( \frac{\pi}{9} \right) + 3 + 2 \cos \left( \frac{4\pi}{9} \right) - 2\sqrt{3} \sin \left( \frac{4\pi}{9} \right) \right] \\
\approx 0.0079\beta^2
\]

In Figure 2.10 this situation is represented in respectively the two considered values of "c1".

2.6 Pattern formation

The setting where system (2.7) has either one or two equilibria is not preferred, because then the solutions of the system are unbounded. The variable "q" corresponds to the population mussels and therefore cannot be unbounded. The setting where the system has three equilibria is, due to the bounded orbits, preferred in terms of bounded dynamics.

The three equilibria which exist for \(0 < c_1 < \frac{2\beta^2}{3\sqrt{3}}\) or \(\frac{2\beta^2}{3\sqrt{3}} < c_1 < 0\) are denoted by \((q_1, 0)\), \((q_2, 0)\) and \((q_3, 0)\) respectively as in Figure 2.10. As argued before by using \(f\) both \((q_1, 0)\) and
$(q_3, 0)$ are saddle points and $(q_2, 0)$ is a center. If one would evaluate these equilibria into the potential $W$ given in (1.21), then $W(q_1), W(q_2), W(q_3) > W(\pm \sqrt{\beta}) = 0$. So

$$E(H(q_1, 0), c_1), E(H(q_2, 0), c_1), E(H(q_3, 0), c_1) > \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} (W(\pm \sqrt{\beta}) + 0) \ dx = 0 = E(0, 0).$$

(2.13)

As argued before the setting where the system exhibits transition waves or heteroclinic solutions is stable due to the minimum energy. Even if the system does have the three equilibrium and therefore bounded solutions but no heteroclinic solutions, the energy of these points is not minimum as shown in (2.13). This is also shown in [12], and we will also use this in latter sections for different functions of $f$. 
3 The movement speed

Recall the equation we have been studying for the population mussels, in (1.8):

\[
\frac{\partial M}{\partial t} = \frac{\partial}{\partial x} [f(M)M_x - \kappa M_{xxx}] = \frac{\partial^2}{\partial x^2} [F(M) - \kappa M_{xx}].
\]  

(3.1)

For the speed of the mussels, the movement speed, the function \( V(\tilde{M}) = \sqrt{c(\tilde{M}^2 - \gamma \tilde{M} + 1)} \) was chosen, where \( \tilde{M} \) was the rescaled variable with \( \tilde{M} = \sqrt{\tau} M \). The main reason for this choice was the order of the functions \( f \) and \( F \), together with the potential \( W \). This potential should be a fourth order function which is qualitatively similar to the function \( (M^2 - 1)^2 \). And with this choice of \( V(\tilde{M}) \) this was accomplished. Also with this movement speed we have shown that it is possible to generate CH-type patterns. But we also stated that one of the most straightforward choices was to take \( V \) to be parabolic as

\[
V(\tilde{M}) = c(\tilde{M}^2 - \gamma \tilde{M} + 1),
\]  

(3.2)

where \( c \) is a positive constant and \( \gamma \) a positive parameter. In equation (1.13) this function is also mentioned. With this movement speed the function \( f \) is given by

\[
f(\tilde{M}) = \frac{V}{\tau} \left( V + \tilde{M} \frac{dV}{d\tilde{M}} \right)
\]

\[
= \frac{c}{\tau} \left( \frac{c(\tilde{M}^2 - \gamma \tilde{M} + 1)}{V} (3\tilde{M}^2 - 2\gamma \tilde{M} + 1) \right),
\]  

(3.3)

and is indeed not a second but fourth order function. The constant \( D \) is positive and the function \( \tilde{V} \) represent the speed of the mussels. For now we assume this speed is positive, because the mussels cannot move with a negative speed. The constant \( \frac{c^2}{\kappa^2} > 0 \) in (3.3) and also the constant \( \kappa > 0 \) can both be scaled out of (3.1) by rescaling the spatial variable as \( \tilde{x} = \frac{\sqrt{\kappa}}{c} x \) and the time variable as \( \tilde{t} = \frac{\kappa}{c^2} t \). So \( \tilde{M}(\tilde{x}, \tilde{t}) = \tilde{M}(\sqrt{\frac{\kappa}{c^2}} \tilde{x}, \frac{\kappa}{c^2} \tilde{t}) \). \(^3\) Then the function \( f \) becomes

\[
f(M) = \left( M^2 - \gamma M + 1 \right) \left( 3M^2 - 2\gamma M + 1 \right).
\]  

(3.4)

Note the bars have been dropped for convenience. To be complete, the function \( F \):

\[
F(M) = \frac{3}{5} M^5 - \frac{5}{4} \gamma M^4 + \frac{2}{3} (\gamma^2 + 2) M^3 - \frac{3}{2} \gamma M^2 + M.
\]

To investigate equation (3.1) further an analogue procedure as described in section 2 is done. We consider \( \tilde{M} \) to be only spatial dependent, \( M(x) = M(x) \). Then the one-dimensional system of interest becomes

\[
\begin{align*}
M_x &= N \\
N_x &= F(M) - c_1.
\end{align*}
\]  

(3.5)

Notice the similarity with the system given in 2.7, with the only difference being the function \( F \). As mentioned before, to be able to find the equilibria of this system we need to investigate \( F \), because the equilibria are the points where both \( M_x = 0 \) and \( F(M) = c_1 \). In order to have multiple equilibrium points there is also a requirement for the constant \( c_1 \). In Figure 3.1 a sketch is given of the situation for both \( F \) and \( c_1 \).

\(^3\)An equivalent procedure has been mentioned in section 1.1.
Because $F$ is a fifth order function and $f$ has a nice structure we will investigate $f$ to learn more about the structure of the function $F$. We can set $f = 0$ to find the extrema of $F$. Then it is sufficient to search for roots of $V$ and $g$. First the roots of $V$:

$$M_V^\pm = \frac{1}{2} \gamma \pm \frac{1}{2} \sqrt{\gamma^2 - 4}.$$  (3.6)

The claim $V$ has to be positive also implies a claim for the positive parameter $\gamma$, namely $0 < \gamma < 2$. $V$ can also have one root for $\gamma = 2$ and two roots for $\gamma > 2$, for a non-positive movement speed. These three cases are represented in Figure 3.2. We will consider the latter two cases later on.

Due to the positive movement speed the roots of $f$ are given by the roots of the function $g(M) = 3M^2 - 2\gamma M + 1$, which are

$$M_\pm^g = \frac{1}{3} \gamma \pm \frac{1}{3} \sqrt{\gamma^2 - 3}.$$  (3.7)

To be able to generate patterns $F$ has to have multiple extrema and therefore $f$ multiple roots. So if $V > 0$, then $g$ has to have multiple roots. Another requirement for the parameter $\gamma$ is $\gamma > \sqrt{3} \approx 1.7321$. Because the rescalings of the spatial variable are only positive constants multiples both roots $M_\pm^g$ are positive for these values of $\gamma$. Which is in agreement with a positive population mussels.

Suppose now that both $c_1$ and $\gamma$ are chosen such that there are three equilibria, which we denote by $M_1, M_2$ respectively $M_3$, with $0 < M_1 < M_2 < M_3$. Evaluating these three points in $f$ provides $f(M_1), f(M_2) > 0$ and $f(M_3) < 0$ due to the slope of $F$ at these points. In Figure 3.3 the situation is represented. Therefore the character of these points is also determined.

---

Figure 3.1: The structure of the function $F$ for $V > 0$ with the parameter $\gamma = 1.8$.

Figure 3.2: The three cases for the movement speed $V$ with respectively zero, one and two roots.
equilibria $M_1$ and $M_3$ are saddle points and $M_2$ a center. The exact values of these points is not needed, only the sign of $f$. The explanation for the use of $f$ for this purpose is already given in section 2.2.

**Example** To be able to explain this setting for $V > 0$ in terms of the potential $W$ we give an example. Because of the requirement $\sqrt{3} < \gamma < 2$, we choose $\gamma = 1.8$. Then the function $F$ and potential $W$ are given by

$$F(M) = \frac{3}{5}M^5 - \frac{9}{4}M^4 + \frac{262}{75}M^3 - \frac{27}{10}M^2 + M$$

$$W(M) = \frac{1}{10}M^6 - \frac{9}{20}M^5 + \frac{185}{75}M^4 - \frac{9}{10}M^3 + \frac{1}{2}M^2.$$  

The roots of $g(M)$ together with their value of $F$:

$$M^g_+ = \frac{3}{5} + \frac{1}{10}\sqrt{6} \approx 0.7633, \quad M^g_- = \frac{3}{5} - \frac{1}{10}\sqrt{6} \approx 0.1354$$

$$F(M^g_+) = \frac{36211}{225000} - \frac{428}{225000}\sqrt{6} \approx 0.1354, \quad F(M^g_-) = \frac{36211}{225000} + \frac{428}{225000}\sqrt{6} \approx 0.4367.$$

In section 2.4 we have argued that the CH-equation tends to form heteroclinic connections. For system (3.7) this will be no different. In order for these connections to exist the potential $W$ must have the same shape as in Figure 1.1 where the two minima $F(M_-) = F(M_+)$. Because the two saddle points already exists when the system has three equilibria, we are searching for the value of $c_1$ in the $n_x$-equation of (3.7) such that the heteroclinic connections exist. The constant $c_1$ controls the amount of equilibria of the system and also the position of these points. And therefore also the existence of the wanted connections. In order for $W$ to have this standard form the two integrals over the area between the function $F$ and constant $c_1$ between the two saddle points needs to be equal. Finding this value of $c_1$ is done by a self-written program in Matlab, which is attached in Appendix A.1. Hence

$$c_1 \approx 0.137669786947617.$$  

In Figure 3.4 this situation for $F$, $c_1$ and $W$ is represented combined with the phase plane of the system for $c_1 = 0.136, 0.140$ and the found value $c_1 \approx 0.1377$. The heteroclinic connections are indicated in green. The two values $c_1 = 0.136, 0.140$ are given as illustration for the setting.

### 3.1 A non-positive speed

Now we consider the two cases where the speed of the mussels can also become non-positive. These situation will not be realistic in terms of the population mussels, but only from a mathematical point of view.
3.1.1 One root

Suppose \( V \) has one root. So the movement speed can become 0, but for the rest of the domain \( V \) is strictly positive. This is the case if \( \gamma = 2 \). The single root is then located at \( M^V_{\pm} = \frac{1}{2} \gamma = 1 \), and the roots of \( g \) are located at \( M^g_{-} = \frac{1}{3} \) and \( M^g_{+} = 1 \). So the root \( M = 1 \) has an algebraic multiplicity of 2. For \( \gamma = 2 \), the function \( f \) is given by

\[
 F(M) = \frac{3}{5} M^5 - \frac{5}{2} M^4 + 4M^3 - 3M^2 + M,
\]

with \( F(\frac{1}{3}) = \frac{97}{310} \approx 0.1198 \) and \( F(1) = \frac{1}{10} \). Because \( F \) still has one maximum and one minimum the standard form of \( W \) is not different then for \( \sqrt{3} < \gamma < 2 \), the situation where \( V \) is positive. The value of \( c_1 \) which is found in Matlab is \( c_1 = \frac{17}{160} = 0.10625 \). In Figure 3.5 this situation for \( F \), \( c_1 \) and \( W \) is represented combined with the phase plane of the system for \( c_1 = 0.103, 0.110 \) and the found value \( c_1 = 0.10625 \).

3.1.2 Two roots

Suppose now \( V \) has a double root, so both \( M^V_{\pm} \) given in (3.6) are real-valued. The requirement being \( \gamma > 2 \). Because \( f \) now has four roots this implies \( F \) has four extrema. For a suitable choice of \( c_1 \) system (3.5) can have five equilibria. To investigate the structure of \( F \) with its four extrema, \( M^V_{\pm} \) and \( M^g_{\pm} \) in (3.6) and (3.7), we use the double derivative of \( F \):

\[
 F''(M) = f''(M) = 12M^3 - 15\gamma M^2 + 4(2 + \gamma^2)M - 3\gamma.
\]

At the roots of \( g \):

\[
 f'(M^g_{-}) = -\frac{2}{9} \gamma (\gamma^2 - 3) + \frac{8}{9} (\gamma^2 - 6) \sqrt{\gamma^2 - 3} \quad < \quad 0 \quad \text{for} \quad \gamma > \sqrt{3}
\]

\[
 f'(M^g_{+}) = -\frac{2}{9} \gamma (\gamma^2 - 3) - \frac{8}{9} (\gamma^2 - 6) \sqrt{\gamma^2 - 3} \quad < \quad 0 \quad \text{for} \quad \gamma > 2.
\]
Figure 3.5: For $\gamma = 2$, the function $F$ and $c_1$ as needed for the standard form of the potential $W$. Together with the corresponding phase plane for three values of $c_1$ including $c_1 = \frac{17}{160} = 0.10625$.

And also at the roots of $V$:

$$f'(M_V^-) = \frac{1}{2}(\gamma^2 - 4)(\gamma - \sqrt{\gamma^2 - 4}) > 0 \quad \text{for} \quad \gamma > 2$$

$$f'(M_V^+) = \frac{1}{2}(\gamma^2 - 4)(\gamma + \sqrt{\gamma^2 - 4}) > 0 \quad \text{for} \quad \gamma > 2.$$

So both roots of $g, M_V^\pm$, are maxima of $F$ for $\gamma > 2$, and both roots of $V, M_V^\pm$, are minima of $F$ for $\gamma > 2$.

The minimum $F(M_V^-)$ is always smaller than the other minimum $F(M_V^+)$ for $\gamma > 2$. But for the two maxima $F(M_V^+)$ and $F(M_V^-)$ this is not the case. Namely for $\sqrt{3} < \gamma < \frac{1}{3}\sqrt{42}$ $F(M_V^+) > F(M_V^-)$ but if $\gamma > \frac{1}{3}\sqrt{42} \approx 2.1602$ then $F(M_V^+) < F(M_V^-)$. For $\gamma = \frac{1}{3}\sqrt{42}$ both maxima are equal, $F(M_V^+) = F(M_V^-) = \frac{1}{60}\sqrt{42} \approx 0.1080$. From this a requirement in terms of $c_1$ can be composed for the existence of five equilibria:

$$F(M_V^+) < c_1 < \min\{F(M_V^+), F(M_V^-)\},$$

or more explicitly

$$F(M_V^+) < c_1 < \begin{cases} F(M_V^+) & \text{for } 2 < \gamma \leq \frac{1}{3}\sqrt{42} \\ F(M_V^-) & \text{for } \gamma \geq \frac{1}{3}\sqrt{42} \end{cases}.$$

**Example** For example we take two values of $\gamma$ as illustration, namely $\gamma = 2.1$ and $\gamma = 2.4$.

Then for $\gamma = 2.1$ the function $F$ is given by

$$F(M) = \frac{3}{5}M^5 - \frac{21}{8}M^4 + \frac{641}{150}M^3 - \frac{63}{20}M^2 + M.$$
with the roots

\[
\begin{align*}
M_g^b &= \frac{7}{10} - \frac{1}{20} \sqrt{141} \approx 0.3041 & F(M_g^b) &\approx 0.1121 \\
M_V^b &= \frac{4}{5} - \frac{1}{15} \sqrt{69} \approx 0.7298 & F(M_V^b) &\approx 0.0927 \\
M_g^b &= \frac{7}{10} + \frac{1}{20} \sqrt{141} \approx 1.0958 & F(M_g^b) &\approx 0.0994 \\
M_V^b &= \frac{4}{5} + \frac{1}{15} \sqrt{69} \approx 1.3702 & F(M_V^b) &\approx 0.0945.
\end{align*}
\]

To give an idea of this situation we choose three values of \(c_1\) and provide the function \(F\) and the corresponding phase planes for the three values in Figure 3.6. These values of \(c_1\) are 0.095, 0.097, 0.099.

![Phase planes diagram](image)

Figure 3.6: The function \(F\) for \(\gamma = 2.1\) with respectively the phase plane for the values \(c_1 = 0.095, 0.097, 0.099\).

We also see here the existence of the two types of bounded non-constant stationary solutions, the periodic solutions and reversal waves. In theory it should also be possible to construct transition waves. But due to the dependence of the function \(F\) on \(\gamma\) and \(c_1\) a clear example in terms of \(\gamma\) and \(c_1\) is not so obvious. But in Figure 3.7 the situation is sketched for the potential \(W\) and corresponding function \(F\) such that there exist six heteroclinic connections. The literature used for the standard form of \(W\), [12], does not consider the setting where the potential \(W\) has a different form, e.g. when \(W\) has five extrema.

For \(\gamma = 2.4\) the function \(F\) is given by

\[
F(M) = \frac{3}{5} M^5 - 3 M^4 + \frac{388}{75} M^3 - \frac{18}{5} M^2 + M,
\]

with the roots

\[
\begin{align*}
M_g^B &= \frac{4}{5} - \frac{1}{15} \sqrt{69} \approx 0.2462 & F(M_g^B) &\approx 0.0947 \\
M_V^B &= \frac{4}{5} - \frac{1}{15} \sqrt{11} \approx 0.5367 & F(M_V^B) &\approx 0.0773 \\
M_g^B &= \frac{4}{5} + \frac{1}{15} \sqrt{69} \approx 1.3538 & F(M_g^B) &\approx 0.2433 \\
M_V^B &= \frac{4}{5} + \frac{1}{15} \sqrt{11} \approx 1.8633 & F(M_V^B) &\approx 0.1458.
\end{align*}
\]
Figure 3.7: The potential \( W \) and function \( F \) which provide six heteroclinic connections.

Figure 3.8: The function \( F \) for \( \gamma = 2.4 \) with respectively the phase plane for the values \( c_1 = 0.08, 0.12, 0.2 \).

Again three values of \( c_1 \) are chosen, \( c_1 = 0.08, 0.12, 0.2 \). In Figure 3.8 the situation is represented.

Because for \( \gamma = 2.4 \) it is possible to construct three equilibria the same setting and arguments hold as in section 2 and in the beginning of section 3. For two values of \( c_1 \) the heteroclinic connections exist. From the last plot of Figure 3.8 it follows that one value \( c_1 \approx 0.2 \) and the other value \( c_1 \approx 0.09 \).

### 3.2 A general speed

As explained in section 1.2 a logical choice for \( V \), after rescaling, was

\[
V(\dot{M}) = c(\dot{M}^2 - \gamma \dot{M} + 1),
\]

(3.10)
also given in (3.2) and used in the beginning of section 3. In sections 1 and 2 we have also used the square root of (3.10) given in (1.14). Therefore we introduce a movement speed $V_k$ as

$$V_k(M) = \left[ c(M^2 - \gamma M + 1) \right]^k,$$  \hspace{1cm} (3.11)

where we assume $V > 0$, as before, and $k$ is a positive parameter. So the two previous considered cases are $V_{\frac{1}{2}}$ and $V_1$ with $k = \frac{1}{2}$ and $k = 1$ in (3.11). We would like to investigate this movement speed for general $k$. Therefore we need the general function $f_k$ in one spatial dimension:

$$f_k(M) = \frac{V_k}{\tau} \left( V_k + M \frac{dV_k}{dM} \right)$$

$$= \frac{1}{\tau} \left[ c(M^2 - \gamma M + 1) \right]^k \left( \left[ c(M^2 - \gamma M + 1) \right]^k + M \cdot k \left[ c(M^2 - \gamma M + 1) \right]^{k-1} \cdot c(2M - \gamma) \right)$$

$$= \frac{c}{\tau} \left[ c(M^2 - \gamma M + 1) \right]^{2k-1} \left[ (2k + 1)M^2 - \gamma(k + 1)M + 1 \right].$$  \hspace{1cm} (3.12)

Also here the tildes are dropped. The constant term $D = \frac{c}{\tau}$ we have encountered before. The movement speed $V$ has here a power $2k - 1$, which we denote by $V_{2k-1}$. Finally we have the function $g_k$. By rescaling as before the constant $\frac{c}{\tau}$ can be scaled out, and will not be considered here. So

$$f_k(M) = \left[ (M^2 - \gamma M + 1) \right]^{2k-1} \left[ (2k + 1)M^2 - \gamma(k + 1)M + 1 \right].$$  \hspace{1cm} (3.13)

A small remark for the parameter $k$ is in place here. We assume $k > 0$, but there is a difference for $0 < k < \frac{1}{2}$ and $k > \frac{1}{2}$. For $0 < k < \frac{1}{2}$ the exponent of $V$ is $-1 < 2k - 1 < 0$. Because $V$ is a positive parabola, the only difference is that for $0 < k < \frac{1}{2}$ the movement speed $V$ is a concave down and for $k > \frac{1}{2}$ concave up function. For $k = \frac{1}{2}$ the function $V_{2k-1}$ is equal to 1.

The roots of $f_k(M)$, in (3.13), are given by the roots of $V_{2k-1}$ and $g_k$ denoted by respectively $M^V_\pm$ and $g^{g_k}_\pm$.

$$M^V_\pm = \frac{1}{2} \gamma \pm \frac{1}{2} \sqrt{\gamma^2 - 4},$$  \hspace{1cm} (3.14)

$$g^{g_k}_\pm = \frac{\gamma(k + 1) \pm \sqrt{\gamma^2(k + 1)^2 - 4(2k + 1)}}{2(2k + 1)}.$$  \hspace{1cm} (3.15)

The roots of $V_{2k-1}$ do not depend on the parameter $k$ and are therefore the same as the roots of $V$ which are described in the beginning of section 3 in (3.10)

### 3.2.1 A non-negative general speed

If we assumed $V > 0$, the requirement for the parameter is $0 < \gamma < 2$. In order for $g_k$ to have two real roots another requirement follows

$$\gamma^2 > \frac{4(2k + 1)}{(k + 1)^2}.$$  \hspace{1cm} (3.16)

So for $V > 0$ and general $k > 0$, the requirement for pattern formation, in terms of the parameter, is

$$\frac{2 \sqrt{2k + 1}}{k + 1} < \gamma < 2.$$  \hspace{1cm} (3.17)
Where the lower bound is explained due to the function $g_k$ having two real roots and the upper bound due to $V > 0$. The case $k = \frac{1}{2}$ is special due to the non-existence of the upper bound 2, because the movement speed $V_k = 1$ for $k = \frac{1}{2}$. This case had an upper bound due to the existence of the homoclinic orbits of $\gamma \leq \frac{1}{\sqrt{3}}$. This upper bound was found in section 2.4. Note that the inequality $2\sqrt{\frac{k+1}{k+1}} < 2$ holds for $k > 0$ (and $-\frac{1}{2} \leq k < 0$).

The general function (3.13) can be checked for the two cases $k = \frac{1}{2}$ and $k = 1$:

$$k = \frac{1}{2}: \quad f_\frac{1}{2}(\tilde{M}) = \frac{2}{3}(\tilde{M}^2 - \frac{2}{3}\tilde{M} + 1) = Dg_{\frac{1}{2}}.$$

$$k = 1: \quad f_1(\tilde{M}) = \frac{2}{3}c(\tilde{M}^2 - \gamma\tilde{M} + 1)(3\tilde{M}^2 - 2\gamma\tilde{M} + 1) = DV_1g_1.$$

And indeed these match with the functions $f$ found in (1.15) and (3.3). For $k = 1$ the roots of $g_k$ are indeed given by $M_{\pm}^g = \frac{1}{2}\gamma \pm \frac{1}{3}\sqrt{\gamma^2 - 3}$ and the lower bound by $\gamma^2 > 3$. This is in agreement with the range of interest $\sqrt{3} < \gamma < 2$ found in Section 2. The roots of $g_{\frac{1}{2}} = 2\tilde{M}^2 - \frac{3}{2}\gamma\tilde{M} + 1$ are $M_{\pm} = \frac{3}{8}\gamma \pm \frac{1}{2}\sqrt{9\gamma^2 - 32}$. To have two real roots for $g_{\frac{1}{2}}$ the requirement is $\gamma > \frac{32}{9}$ (or $\beta > 0$), which is in agreement with the requirement found in section 1.3.

If we now assume $V \geq 0$, viewed from a non-ecological point of view. Then this situation is not different from the setting described in section 3.1.1. Because $V$ has only one root for $\gamma_0 = 0$ at $M = 1$. The roots of $g_k$, for $\gamma_0$, are

$$M_{\pm}^g = \frac{2(k+1) \pm \sqrt{4(k+1)^2 - 4(2k+1)}}{2(2k+1)} = \frac{k+1 \pm k}{2k+1}.$$

So the positive root $M_{+}^g$ does not depend on $k$ and is always equal to 1, $M_{+}^g = 1$. So this root has a multiplicity of 2. The negative root does depend on $k$ and is $M_{-}^g = \frac{k-1}{2k+1}$. To give an idea of the behaviour of this negative root we can take the limit of this root for $k$:

$$\lim_{k \to 0} \frac{1}{2k+1} = 1, \quad \lim_{k \to \infty} \frac{1}{2k+1} = 0.$$

So its value varies between 0 and 1. Because also here for $\gamma_0 = 2$, the function $f_k$ has two roots this setting is not different from the setting where $V > 0$ in terms of the pattern formation.

### 3.2.2 A negative general speed

The requirement $k > 0$, for the parameter $k$, is not sufficient anymore due to the possibility that $V$ can become negative. In order for the function $f_k$ to be real-valued the power $2k - 1$ of $V_{2k-1}$ should be a positive integer. Therefore the requirement here holds $2k - 1 \in \mathbb{Z}_{>0}$, which implies

$$k \in \left\{ \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \right\}.$$

So for general speed $V$, where $V$ has two roots, the general function $f_k$ has four roots given in (3.14) and (3.15). Both roots $M_{-}^g$ have a multiplicity of 1 and $M_{+}^g$ both a multiplicity of $2k - 1$. For a fixed value of the parameter $\gamma$ the position of these four roots with respect to each other does not depend on $k$. Although the roots of $g_k$ depend on $k$:

$$\lim_{k \to \infty} M_{-}^g = 0, \quad \lim_{k \to 0} M_{-}^g = M_{+}^V, \quad \lim_{k \to \infty} M_{+}^g = \frac{1}{2}\gamma, \quad \lim_{k \to 0} M_{+}^g = M_{+}^V,$$

the roots are ordered as

$$M_{-}^g < M_{+}^V < M_{+}^g < M_{+}^V.$$

Note that this setting is not different than the setting described in section 3.1.2 if $k > 1$, where the system has five equilibria.
3.2.3 Summary for a general speed

So the number of roots of $f_k$, and therefore the number of extrema of $F_k$, depends on the parameter $\gamma$:

$$
\begin{align*}
0 < \gamma < \frac{2\sqrt{2k+1}}{k+1} & \quad \text{0 roots,} \\
\gamma = \frac{2\sqrt{2k+1}}{k+1} & \quad \text{1 root,} \\
\frac{2\sqrt{2k+1}}{k+1} < \gamma \leq 2 & \quad \text{2 roots,} \\
\gamma > 2 & \quad \text{4 roots.}
\end{align*}
$$

Notice that this setting for a general $k$ is a generalization of the setting described in the beginning of section 3 and 3.1. Therefore we will only give some examples for different values of the parameter $\gamma$ and $k$ together with the functions $f$ and $F$. First to illustrate the behaviour of the $k$-depend bound we choose three values of $k$, namely $k = 2, 3, 4$. In Table 3.1 the $k$-dependent bound is given for these three values of $k$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\frac{2\sqrt{2k+1}}{k+1}$</th>
<th>$\frac{2\sqrt{k}}{3} \approx 1.4907$</th>
<th>$\frac{2\sqrt{4}}{4} \approx 1.3229$</th>
<th>$\frac{6}{5} = 1.2$</th>
</tr>
</thead>
</table>

Table 3.1: The $k$-dependent bound for three values of $k$.

Now we choose three values of the parameter $\gamma$, namely $\gamma = 1.6, 2, 2.4$ to illustrate the functions $f_k$ and $F_k$. These functions are illustrated in Figure 3.9 for each $\gamma$ and $k$. Because the behaviour of these functions is known also the behaviour of the considered system (3.5). Because if $0 < k < \frac{1}{2}$ or $k > \frac{1}{2}$ the system exhibits pattern formation for $\frac{2\sqrt{2k+1}}{k+1} < \gamma \leq 2$ and if $k = \frac{1}{2}$ for $\frac{\sqrt{32}}{3} = \frac{2\sqrt{2k+1}}{k+1} < \gamma < \frac{4}{\sqrt{3}}$, for the general function $f_k$ given in (3.13).

3.2.4 The parameters

Besides the parameter $\gamma$ we have also encountered the parameters $\beta$ and the constants $a, b, c$ for the movement speed given in (1.12). To be able to relate the two parameters $\beta$ and $\gamma$ we give examples, also in terms of the constants. Recall $\beta = \frac{3}{2\gamma^2} - 1$, so $\gamma = \frac{\sqrt{32}}{3} \beta + 1$ and $\gamma = \frac{b}{\sqrt{ac}}$. We take again $\gamma = 1.6, 2, 2.4$ to represent the three different cases as before. For these values of $\gamma$ we give values of the constants $a, b, c$ where $b = 1$ and $c = 25$ are fixed and $a$ varies. These are respectively the three cases for the movement speed $V$ as represented in Figure 3.2.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\beta$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.6</td>
<td>$\frac{3}{8}$</td>
<td>1</td>
<td>$c = 25$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{3}$</td>
<td>1</td>
<td>$c = 25$</td>
<td></td>
</tr>
<tr>
<td>2.4</td>
<td>$\frac{3}{11}$</td>
<td>1</td>
<td>$c = 25$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.2: The two parameters $\beta$ and $\gamma$ for the three different cases of the movement speed together with an example of the constants $a, b, c$. 

29
Figure 3.9: The functions $f$ and $F$ both for $\gamma = 1.6, 2, 2.4$ and for each value of $\gamma$ the values $k = 2, 3, 4$. 
4 Simulations

To be able to verify the analysis of the Cahn-Hilliard equation given in (1.8), we use simulations for the function $f$ and different values of our parameter $\gamma$. We choose only one function of $f$ because we expect the same dynamics in time for different functions. These simulations are provided to give a nice comparison of different dynamics for different parameter values.

The program which is used is written in Python and written by Quan-Xing Liu\(^4\). The code of the program is attached in Appendix A.2 but adjusted for the use of this thesis.

The whole setting for the analysis done in the previous sections is one-dimensional. But to give an impression of the equation in two dimensions we will use the one-dimensional results in terms of the parameters and apply them to the two-dimensional setting.

4.1 The code

The program uses a grid of $60 \times 0.5$ in one dimension and $60 \times 60$ in two dimensions. The spatial steps, $dx$ and $dy$, are both set to 0.5 and also the time steps are set $dt = 0.5$. After 8 time steps, $t = 4$, an image is made of the simulated equation and saved. These are the images which are used in this section. As initial conditions we use noise that has a Gaussian distribution with a certain mean $\mu$ and variance $\sigma = 0.04$. By adjusting the mean the total amount of mussels changes and different structures and patterns of the CH-equation arise.

4.2 The equation and its parameter

For the movement speed we choose:

$$V(M) = c(M^2 - \gamma M + 1),$$

given in (3.2) which provides the function $f(M) = (M^2 - \gamma M + 1)(3M^2 - 2\gamma M + 1)$ given in (3.4). For $\sqrt{3} < \gamma < 2$ we expect the population mussels to exhibit pattern formation. For $0 < \gamma < \sqrt{3}$ we expect to see no patterns at all. And for $\gamma > 2$ we do expect patterns which are not interesting from an ecological point of view. Because we consider these three cases in terms of the parameter we choose one value of $\gamma$ for each case: $\gamma = 1.8$, $1.18$, $2.4$. Recall that for both $\gamma = 1.8$, $2.4$ we did explicit calculations in section 2.

4.3 The one-dimensional setting

The most interesting value is $\gamma = 1.8$. As mentioned before we expect to observe pattern formation. For $\gamma = 1.8$, the three equilibria of the CH-equation are located at $M_1 \approx 0.3373$, $M_1 \approx 0.5976$ and $M_3 \approx 0.8989$. So the two preferred states are $M_1$ and $M_3$. Due to the position of these points we first choose the mean $\mu = 0.6$. This means that $M$ is approximately 0.6 at every grid point. For the one-dimensional grid this implies that the total amount of $M$ is circa 72, and for the two-dimensional grid circa 8.640. Because the mean is located in the middle of the two equilibria $M_1$ and $M_3$ these two states will be preferred by the system. In Figure 4.1 simulations of the equation are given for $\gamma = 1.8$ and $\mu = 0.6$.

We have also chosen $\mu = 0.2, 0.45, 0.75, 1$ respectively to point out the different dynamics. For $\mu = 0.2$ there will not be enough mussels in the entire system to reach one of the two preferable states $M_1$ and $M_3$, and vice versa for $\mu = 1$ there are too many. Therefore a homogeneous state is

\(^{4}\text{Quan-Xing is currently doing his PhD project leaded by Prof. dr. Johan van de Koppel at NIOZ-Yerseke (Department of Spatial Ecology) and works on a project on ecosystem functioning of spatial patterning in mussel beds. For more information: http://johanvandekoppel.nl/}.\)
shown. For \( \mu = 0.45 \) there will be more mussels in state \( M_1 \) than \( M_3 \) and vice versa for \( \mu = 0.75 \). Because it will take the system quite some time the final states for these values of \( \mu \) will not show. In Figure 4.2 simulations of the equation are given for \( \gamma = 1.8 \) and \( \mu = 0.2, 0.45, 0.75, 1 \) respectively.

For \( \gamma = 1.4 \) no patterns arise at all. We choose \( \mu = 0.4 \) and quite fast this homogeneous state is exhibit. In Figure 4.3 simulations of the equation are given for \( \gamma = 1.4 \) and \( \mu = 0.4 \) respectively.

For \( \gamma = 2.4 \) there is a special situation. The system can choose between four stable states, indicated in Figure 3.8 as the four extrema of \( F \). These states are located at \( M \approx 0.2462, 0.5367, 1.3538, 1.8633 \). For this reason we choose \( \mu = 0.4, 1, 1.5 \). Then for \( \mu = 0.4 \) the first two states \( 0.2462, 0.5367 \) are preferred and for \( \mu = 1 \) both \( 1.3538, 1.8633 \). These values are also calculated in section 3.1.2. The situation where \( \mu = 1 \) can be explained analogue as the situation where \( \gamma = 1.8 \) with \( \mu = 0.45, 0.75 \). For these values the system has too little and respectively too much mussels in order to exhibit any of the preferable states, and will exhibit the homogeneous state at the mean \( M = 1 \). In Figure 4.4 simulations of the equation are given for \( \gamma = 2.4 \) and \( \mu = 0.4, 1, 1.5 \) respectively.
4.4 The two-dimensional setting

The most interesting setting is not changed. Still $\gamma = 1.8$ with $\mu = 0.6$ as in the one-dimensional setting. In Figure 4.5 simulations of the equation are given for $\gamma = 1.8$ and $\mu = 0.6$ in two dimensions. The preferable states are the same. So the colors in the simulations are the same as in Figure 4.1. The pattern formation is very clear to see and are equivalent to patterns observed in Figures IV and 2.5.
5 Conclusion

In this thesis we aimed at describing the phenomenon of pattern formation in mussel beds. To explain the occurrence of these patterns we modeled the local density of a population mussels. Where the rate of change of the population depends on the speed and the long range or nonlocal interactions of the mussels. Taking into account these two factors the Cahn-Hilliard equation, given in (1.7), arises.

In section 1 we furthermore investigate the function which describes the speed of the animals, the movement speed. A choice is made for this function which depends on the parameter $\gamma$. After scaling the equation is given in the beginning of section 2 in (2.1), together with the requirement for a positive population and the parameter $\beta$. Also the concept of the potential $W$ and the energy is introduced and elaborated to provide a handle for the rest of the analysis in terms of $\beta$. In the third section the movement speed of the mussels is reviewed and the different choices for the movement speed are made. A more general function is considered and also the possibility for a negative speed.

To be able to verify the results found in section 2 and 3 the considered system (1.7) is simulated in section 4. Different values of the parameter $\gamma$ are considered together with different initial conditions.

5.1 Further research

A logical research suggestion would be now to investigate the two-dimensional setting. All the analysis done in this thesis is in one dimension. From the simulations one could think that the same requirements hold for the parameters.

When deriving the equation for the local density in section 1 we assumed that particle flux $J$ depends on two factors. Namely the movement speed of the mussels and the effect of long range or nonlocal interactions. We could also incorporate other factors instead of these two or consider these factors in another way.

A final suggestion would be to consider another the movement speed $V$. Perhaps a more general function where requirements for $V$ can be obtained in terms of its derivative.
Appendix

A.1 Program code for finding the value of $c_1$

```
clc; clear all;
format long

% define variables
gamma = 2; % from sqrt(3) < gamma <= 2
epsilon = 10^-18;
maxiter = 100;

G = @(M)0.6*M.^5-1.25*gamma*M.^4+2/3*(gamma^2+2)*M.^3-1.5*gamma*M.^2+M;
Mup = 1/3*gamma - 1/3*sqrt(gamma^2-3);
M0 = 1/3*gamma;
Mlow = 1/3*gamma + 1/3*sqrt(gamma^2-3);
cup = G(Mup) - 0.001;
clow = G(Mlow) + 0.001;

% define upper function
Fup = @(M)0.6*M.^5-1.25*gamma*M.^4+2/3*(gamma^2+2)*M.^3-1.5*gamma*M.^2+M - cup;
z1up = fzero(Fup,0);
z2up = fzero(Fup,M0);
z3up = fzero(Fup,2);
intup = quad(Fup,z1up,z3up);

% define lower function
Flow = @(M)0.6*M.^5-1.25*gamma*M.^4+2/3*(gamma^2+2)*M.^3-1.5*gamma*M.^2+M - clow;
z1low = fzero(Flow,0);
z2low = fzero(Flow,M0);
z3low = fzero(Flow,2);
intlow = quad(Flow,z1low,z3low);

intF = 1;
iter = 0;
while ((abs(intF)>epsilon) & (iter<maxiter))
    rc = (intlow-intup)/(clow-cup); % linear interpolation
    c = (rc*cup-intup)/rc % computing the new constant
    F = @(M)0.6*M.^5-1.25*gamma*M.^4+2/3*(gamma^2+2)*M.^3-1.5*gamma*M.^2+M - c;
z1 = fzero(F,Mup - 1/3*sqrt(gamma^2-3));
z2 = fzero(F,M0);
z3 = fzero(F,Mlow + 1/3*sqrt(gamma^2-3));
intF = quad(F,z1,z3);
if intF > 0
    clow = c;
    intlow = intF;
else
```
cup = c;
intup = intF;
end
iter = iter+1
end
c

%plot of the function F and the found constant c1
Fzero = @(M)c;
fplot(G,[0,1.5],’b’)
hold on
fplot(Fzero,[0,2],’k’)
axis([0 1.6 0.096 0.1248])
legend(’The function F’,’The constant c_1’,’Location’,’NorthWest’) hold off

A.2 Program code for simulations of the Cahn-Hilliard equation

from fipy import *
import scipy.integrate as spi
import numpy as np
import pylab as pl
import time, os
from matplotlib import pylab
import matplotlib
import matplotlib.pyplot as plt

params = {'backend': 'ps',
          'font.size': 8,
          'ines.markersize': 25,
          'axes.labelsize': 8,
          'text.fontsize': 8,
          'legend.fontsize': 8,
          'xtick.labelsize': 20,
          'ytick.labelsize': 20,
          'xtick.major.size': 8,
          'ytick.major.size': 8,
          'xtick.minor.size': 4,
          'ytick.minor.size': 4,
          'font.family': 'sans-serif',
          }

matplotlib.rcParams.update(params)
matplotlib.rcParams[‘xtick.direction’] = ‘in’
matplotlib.rcParams[‘ytick.direction’] = ‘in’
colors = (’b’, ’g’, ’r’, ’c’, ’m’, ’y’, ’k’)

fig = plt.figure(figsize=(6,6))
fig.add_subplot(111)
plt.subplots_adjust(hspace=0.25)

plt.subplots_adjust(wspace=0.25)
ax1 = plt.subplot((111))
plt.axis('off')

## ====================begin the main code===================
t = time.time()
Lx=60
Ly=0.5
dx = 0.5
dy = 0.5
dt = 0.5
elapsed = 0.0
duration = 2000
count_t = 0
store_n = 0

# parameter of model
kappa = 0.05
gamma = 1.86

nx = np.fix(Lx/dx)
ny = np.fix(Ly/dy)
mesh = PeriodicGrid2D(nx=nx, ny=ny, dx=dx, dy=dy)
M = CellVariable(name="", mesh=mesh,hasOld=True)
M.setValue(GaussianNoiseVariable(mesh=mesh,mean=0.35,variance=0.04))

titleplot = '$M(X,t)$ for $\gamma$ = ' + str(gamma)
viewer = Matplotlib2DViewer(vars=(M,),datamin=0,datamax=1.4,axes=ax1,
title=titleplot,cmap=pylab.cm.jet, colorbar='True')

## Here the variable M the function f are created, together
## with the CH-equation
Mf = M.getArithmeticFaceValue()
funcf = (Mf*Mf-gamma*Mf+1)*(3.*Mf*Mf-2.*gamma*Mf+1)
eq = (TransientTerm()== DiffusionTerm(coeff=funcf)
- ImplicitDiffusionTerm(coeff=(Een, kappa)))

foldnname='simulatie1D6'
if os.path.exists(foldnname)==True:
    pass
else:
    os.mkdir(foldnname)

while elapsed < duration:
    print "time=%03.3f %% | dt=%3.3G" % (elapsed, dt)
elapsed += dt
count_t +=1

37
M.updateOld()
eq.solve(var=M, dt=dt)
if count_t % 4 == 0:
    print "time=%03.3f %%% | dt=%3.3G" % (elapsed, dt)
    store_n +=1
viewer.plot(filename=foldname+/frame%03d.png %store_n)
References


