A Purity Theorem for Torsors

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Contents

Introduction

In this thesis, we first give an account of the Zariski-Nagata purity theorem, as it is stated in [2, X, §3]. Then, after introducing the setting, we establish a similar result in the context of fppf torsors under the action of a finite flat group scheme, as claimed in [4, Lemme 2]. In the last part, we take this as a starting point to study quotients by generically free actions and work out some examples.

The Zariski-Nagata purity theorem, known as "purity of the branch locus", is a result in Algebraic Geometry that, as stated in SGA2 by Alexander Grothendieck ([2, X, §3]), concerns finite étale coverings. More precisely, to any scheme S, one can associate a category $\mathrm{Et}(S)$, whose objects are morphisms of schemes $f\colon X \to S$ that are finite, flat and unramified, i.e. what we call a *finite étale covering.* If $S' \to S$ is any map of schemes, there is an induced functor $\mathrm{Et}(S) \to \mathrm{Et}(S')$ given by pull-back via $S' \to S$. In particular, if U is an open subscheme of a scheme S, the functor $\mathrm{Et}(S) \to \mathrm{Et}(U)$ induced by the inclusion map is simply the restriction of coverings $f\colon X \to S$ to $f|_{f^{-1}(U)}\colon f^{-1}(U) \to U$. Now, the purity theorem gives a sufficient condition for this functor to be an equivalence of categories:

Theorem 1.1 (Purity theorem for finite étale coverings)**.** Let S be a regular scheme, $U \subseteq S$ an open subscheme, $Z = S\backslash U$ its closed complement. Suppose that $\mathrm{codim}_S(Z) \geq 2$. Then, the pull-back functor:

$$\mathrm{Et}(S) \longrightarrow \mathrm{Et}(U)$$

is an equivalence of categories.

In particular, under the assumptions of the theorem, it is possible to extend any finite étale covering of U to the whole S.

This result is also the starting point for the study of "ramified coverings". In fact, it may happen that a finite morphism $f\colon X \to S$ is étale over a dense open subset $U \subseteq S$, but not necessarily everywhere on X; we say, in this case, that f is *generically étale.* If f is already flat, then the points where it is not étale are exactely the points where it ramifies, whence the name of *ramified covering.* In this case, one can define a closed subscheme B of S, consisting of the points over which f is ramified, called the *branch locus.* The purity theorem, then, has a strong consequence in terms of the shape of such locus, which is the reason for the name "purity of the branch locus":

Theorem 1.6 (Purity of the branch locus)**.** Let $f\colon X \to S$ be a finite flat morphism of schemes, with S regular. Suppose that f is generically étale. Then, its branch locus

B is either empty or pure of codimension 1 in S, i.e. $\operatorname{codim}_S(Z) = 1$ for all irreducible components Z of B.

In this thesis, we are mainly concerned with morphisms arising as quotients of schemes by the action of a finite flat group scheme. In particular, when the action is free and under suitable conditions ensuring that such quotients exist as schemes, the resulting morphism will be a *torsor*. The main property of torsors is that locally (with respect to a fixed Grothendieck topology, for us: fppf) they resemble the acting group scheme, so a torsor by the action of a finite flat group scheme is a finite flat morphism. Now, for group schemes over a field we have the following result by Cartier:

Theorem 2.13 (Cartier, affine case)**.** Let $G = \operatorname{Spec} A$ be an affine group scheme of finite type over a field k of characteristic zero. Then G is smooth over k.

In particular, over a field of characteristic zero, a finite group scheme is automatically étale, hence so is any torsor under the action of such a group. As a consequence, in this special case, we can apply the previous purity results to the context of our interest.

On the other hand, over a field k of positive characteristic p, there are important group schemes that are finite, but not étale. The basic examples are provided by the "infinitesimal group schemes", e.g. the group of p-th roots of unity $\mu_{p,k} = \operatorname{Spec} k[x]/(x^p - 1)$ and the group of p-th roots of zero $\alpha_{p,k} = \operatorname{Spec} k[x]/(x^p)$.

One may then wonder whether the consequences of the purity theorem extend, from the case of torsors by the action of a finite étale group scheme, to the more general case of all finite flat group schemes. Of course, to even ask such questions, we need to leave the context of finite étale coverings and concentrate on the property of being a torsor, under a fixed action of some group scheme. Thus, given a base scheme S and a group scheme G over S, we consider the category $\operatorname{Tors}(S, G)$ whose objects are G-torsors over S. Then, in complete analogy with the theory of finite étale coverings, a similar purity result holds for torsors by the action of a finite flat group scheme:

Theorem 3.1 (Purity theorem for fppf torsors)**.** Let S be a regular scheme, $U \subseteq S$ an open subscheme, $Z = S \setminus U$ its closed complement and suppose that $\operatorname{codim}_S(Z) \geq 2$. Let then $\pi\colon G \to S$ be a finite flat S-group scheme and denote by $\pi_U\colon G_U \to U$ its restriction to U. Then, the restriction functor:

$$\operatorname{Tors}(S, G) \longrightarrow \operatorname{Tors}(U, G_U)$$

is an equivalence of categories.

The next step is to bring, into the context of torsors, a similar notion to that of the branch locus, in order to study quotient morphisms $f\colon X \to S$ by some action $\rho\colon G \times_S X \to X$, such that f becomes a torsor for ρ after restricting to a dense open subset $U \subseteq S$. For this purpose, we need to turn back to the freeness condition on the action, which can be checked pointwise by means of the stabiliser. In general, if $f\colon X \to S$ is a finite flat morphism, which is invariant for the action $\rho\colon G \times_S X \to X$ of a finite flat group scheme G over S, we will be able to define a closed subscheme B^i of

S consisting of the points over which the G-action is not free: the *infinitesimal branch locus*. Our interest will be on the case of a *generically free* action, i.e. when ρ is free over a dense open subset of S, but not necessarily on all X. If we assume, moreover, that S is the quotient by the action concerned, then B^i really defines the locus of S over which f is not a torsor for ρ. In this situation, purity for fppf torsors has the following consequence, in analogy with the theory of finite étale coverings:

Theorem 3.7 (Purity of the infinitesimal branch locus)**.** Let $\pi\colon G \to S$ be a finite flat S-group scheme and $f\colon X \to S$ a finite flat S-scheme, with S regular. Let $\rho\colon G \times_S X \to X$ be a generically free action over S and suppose that S is the quotient scheme of ρ. Then, the infinitesimal branch locus $B^i \subseteq S$ is either empty or pure of codimension 1 in S, i.e. $\operatorname{codim}_S(Z) = 1$ for all irreducible components Z of B^i.

Finally, the infinitesimal branch locus B^i can be upgraded to an effective Weil divisor (as it can be done for the branch locus B). In other words, fixed an action of a finite flat group scheme G on another scheme X and assuming the quotient $f\colon X \to S$ exists and satisfies the assumptions of last theorem, we may attach a multiplicity to each irreducible component of $B^i \subseteq S$ in order to measure, over each component, "how much" f fails to be a torsor for the fixed action. As two basic examples show, this allows to recognise different behaviours for actions that give rise to the same quotient morphism, with the same infinitesimal branch locus, but different multiplicities of its components.

Overview of the contents

In the first chapter, we will introduce the category $\mathrm{Et}(S)$ of finite étale coverings of a scheme S and give a detailed proof of the Zariski-Nagata purity theorem in the local case; we will also explain how to derive the global statement from the local one. After that, we will prove purity of the branch locus using the previously established result (the local case will be enough) and quickly show how the purity theorem has an immediate consequence in terms of the étale fundamental group. We will finally see what happens weakening the main hypothesis of purity.

The second chapter mostly contains introductory material. We will first give an overview on group schemes, focusing on the affine case and explaining all the basic examples. A section is dedicated to the study of smoothness properties of group schemes over a field and the proof of the affine version of Cartier's theorem. We will close the chapter introducing fppf torsors, together with their main property, and recalling a fundamental result on the existence of quotients of schemes by the action of a group scheme; this will be the main source of torsors.

In the third chapter, we will give a detailed proof of the purity theorem for fppf torsors; this time, we will not go through a local statement, but we will be able to directly prove a global theorem, using a result in Commutative Algebra by Maurice Auslander. Following the same structure of the first chapter, we will then use the purity theorem for fppf torsors to prove purity of the infinitesimal branch locus, after defining the latter. Lastly, we will see how the main result of this chapter has an immediate consequence in terms of Nori's fundamental group scheme.

In the last chapter, we take the assumptions of purity of the infinitesimal branch locus as a starting point and study this situation in deeper detail. In particular, we will upgrade the infinitesimal branch locus to an effective Weil divisor, explaining a reasonable way to compute multiplicities. Finally, we will perform this computation in the fundamental examples.

Prerequisites and references

In order to make the exposition focused and compact, we exclude from this thesis the general definitions and facts in Algebraic Geometry that we reckon to be basic enough; therefore, we require from the reader a medium preparation in this subject. Of course, there is no univocal definition of what such a preparation should be, hence, for any incongruity between the level of this work and the knowledge of the reader, we refer to [11]; here one can find the basic definitions and facts about the theory of schemes very clearly explained. In particular, we rely on [11, Appendix C] for the permanence of properties of morphisms of schemes. Regarding étale morphisms, which play a fundamental role in the exposition, the main reference is [1, I]. As for the basics of Commutative Algebra, we forward the reader to [12] and, for more advanced topics, like the notion of *depth*, *projective* module and *projective dimension*, we refer to [13].

Throughout the discussion, we will use in our proofs some advanced tools, whose details would also fall outside the purpose of this thesis. However, the reader will find a good introduction to the theory of *cohomology with support* in [16, §1]. For *descent* theory, instead, we refer to [18, §4] and, for the particular case of *faithfully flat descent*, to [1, VIII-IX].

In this thesis, all rings are meant to be commutative with 1.

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Among these, I would first of all like to thank my advisor, Dr. Gabriel Zalamansky, who guided me through a beautiful path in Mathematics and taught me a way to explore myself this amazing world. Along this path, the contribution of Prof. Laurent Moret-Bailly has been crucial in providing a guideline for the proof of purity for torsors. In fact, let me add that a similar suggestion was given, in a letter dated September 20, 1961, by Jean-Pierre Serre to Alexander Grothendieck himself, in order to prove purity for finite étale coverings; Grothendieck, although, preferred a more conceptual proof, that is the one reproduced here (cf [23, pp 110-115]). My gratitude also goes to Prof. David Holmes, for posing interesting questions related to the first chapter of this work and eventually suggesting the content of §1.4. Given the structure of this thesis, it comes natural to ask the same questions within the setting of the third chapter; unfortunately, we do not know, at the present time, whether the result of proposition 1.9 in terms of finite étale coverings can be transported into the context of torsors.

I will also use this space to thank all the professors who contributed to my studies, thus making me able to develop this work. In this regard, special thanks go to Prof. Bas Edixhoven, for his precious advices during my time in Leiden, to Prof. Jochen Heinloth, for lighting my passion for Algebraic Geometry and to Prof. Jan Stienstra, for his picture of one of the deepest directions of this subject.

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Chapter 1

Purity for finite étale coverings

We begin our survey with the review of a known fact about finite étale coverings of a fixed scheme, namely "purity of the branch locus". The result was first established by Oscar Zariski and Masayoshi Nagata in [6] and [7]. Maurice Auslander gave an alternative, completely algebraic, proof in [5]. However, it was Alexander Grothendieck that turned this result into a statement about finite étale coverings, in SGA2. This also leads to an interpretation of the theorem in terms of the étale fundamental group of the base scheme.

In this chapter, we will illustrate Grothendieck's proof, following SGA2, precisely [2, X, §3], and using some results from the previous sections of the same chapter of SGA2. We will focus on the proof of the local version of the theorem. The main tools for such proof are cohomology with support, a bit of faithfully flat descent and, of course, a good amount of Commutative Algebra. In particular, we will make use of some outcomes of homological algebra, like the Auslander-Buchsbaum formula, and we will rely on Krull's principal ideal theorem ("Hauptidealsatz"). The reduction to a local problem follows from abstract arguments, located in [2, XIV, §1], which we will just summarise.

We will also formulate an outcome of the theorem, to which it owes the name of "purity of the branch locus", and an easy corollary about the consequences in terms of the étale fundamental group.

In the last section, we will examine the situation after removing the main hypothesis that makes purity work and see that a weaker statement still holds.

1.1 The result

Let S be a scheme. We denote by $\mathrm{Et}(S)$ the category of *finite étale coverings* of S. Objects of this category are S-schemes $f\colon X \to S$, such that f is finite étale.[1] A morphism between two objects $f\colon X \to S$, $g\colon Y \to S$ is a morphism of schemes $h\colon X \to Y$ over S, i.e. such that $g \circ h = f$.

[1] In some literature, e.g. [21], finite étale coverings are defined to be surjective; in this work, we follow Grothendieck's definition, which does not include surjectivity. However, we remark that the whole of this chapter can be carried on with the extra requirement of surjectivity without troubles.

Every morphism of schemes $S' \to S$ induces a pull-back functor $\mathrm{Et}(S) \to \mathrm{Et}(S')$, sending a finite étale covering $X \to S$ to $X \times_S S' \to S'$. This is well-defined, because the properties of being finite and étale are stable under base change.

There is a very useful point of view that one can adopt in this situation. Namely, recall that the category of affine schemes over S is anti-equivalent to the category of quasi-coherent $\mathcal{O}_S$-algebras, via the correspondence:

$$\begin{aligned} \text{Affine schemes over } S &\leftrightarrow \text{Quasi-coherent } \mathcal{O}_S\text{-algebras} \\ (f\colon T \to S) &\mapsto f_*\mathcal{O}_T \\ (\mathrm{Spec}(\mathscr{A}) \to S) &\leftarrow\!\shortmid \mathscr{A}, \end{aligned} \tag{1.1}$$

where $\mathrm{Spec}(\mathscr{A})$ denotes the relative *spectrum* of $\mathscr{A}$ (see [11, 12.2]). Under this anti-equivalence, fibered product over S of schemes corresponds to tensor product over $\mathcal{O}_S$ of algebras. Moreover, if $g\colon S' \to S$ is a map of schemes, pull-back of schemes corresponds to pull-back of sheaves, i.e., we have a commutative diagram:

$$\begin{array}{ccccc} T & & \text{Affine } S\text{-schemes} \longleftrightarrow \text{Quasi-coherent } \mathcal{O}_S\text{-algebras} & & \mathscr{A} \\ \downarrow & & \downarrow \qquad\qquad\qquad\qquad \downarrow & & \downarrow \\ T \times_S S' & & \text{Affine } S'\text{-schemes} \longleftrightarrow \text{Quasi-coherent } \mathcal{O}_{S'}\text{-algebras} & & g^*\mathscr{A}. \end{array}$$

Note that the property of being affine is stable under base-change and pull-back of quasi-coherent sheaves is again quasi-coherent ([11, 7.23]).

Now, recall that finite morphisms of schemes are in particular affine, hence, to every finite morphism $f\colon X \to S$ corresponds a quasi-coherent $\mathcal{O}_S$-algebra $f_*\mathcal{O}_X$, which is finite as an $\mathcal{O}_S$-module. If f is also flat and S is a locally Noetherian scheme (as it will be the case here), $f_*\mathcal{O}_X$ is also locally free as an $\mathcal{O}_S$-module (cf [11, 12.19]). Finally, quasi-coherent finite modules on a locally Noetherian scheme are coherent (cf [11, 7.45]), so that $f_*\mathcal{O}_X$ is a coherent $\mathcal{O}_S$-module. A finite étale covering $f\colon X \to S$ is in particular finite flat, hence, if S is locally Noetherian, the $\mathcal{O}_S$-algebra corresponding to f is a locally free coherent $\mathcal{O}_S$-module.

We denote by $\mathrm{L}(S)$ the category of locally free coherent $\mathcal{O}_S$-modules.

Theorem 1.1 (Purity theorem for finite étale coverings)**.** *Let S be a regular scheme, $U \subseteq S$ an open subscheme, $Z = S \setminus U$ its closed complement. Suppose that* $\mathrm{codim}_S(Z) \geq 2$. *Then, the pull-back functor:*

$$\mathrm{Et}(S) \longrightarrow \mathrm{Et}(U)$$

is an equivalence of categories.

Let us first shift to a local setting and explain later why this will lead to the purity theorem in the form just stated.

1.2 The local case

Let R be a Noetherian local ring, with maximal ideal $\mathfrak{m}$ and set $S := \operatorname{Spec}(R)$, $U := S \setminus \{\mathfrak{m}\}$. We say that R is *pure* if the pull-back functor $\operatorname{Et}(S) \to \operatorname{Et}(U)$ is an equivalence of categories. The local version of theorem 1.1 is the following.

Theorem 1.2. *A regular local ring $(R, \mathfrak{m})$ of dimension at least 2 is pure.*

Note that regular local rings are in particular Cohen-Macaulay rings (cf [14, 2.2.6]), i.e. their depth equals their dimension. The next two lemmas, therefore, apply to our case.

Lemma 1.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $S = \operatorname{Spec}(R)$, $U = S \setminus \{\mathfrak{m}\}$. If $\operatorname{depth} R \geq 2$, then the restriction functor:*

$$\mathrm{L}(S) \longrightarrow \mathrm{L}(U)$$

is fully faithful. In particular, the pull-back functor:

$$\operatorname{Et}(S) \longrightarrow \operatorname{Et}(U)$$

is fully faithful as well.

Proof. Let $\mathscr{A}$, $\mathscr{B}$ be two locally free coherent $\mathcal{O}_S$-modules and set $A := \mathscr{A}(S)$ and $B := \mathscr{B}(S)$. Because $S = \operatorname{Spec}(R)$ is affine, with R local, the fact that $\mathscr{A}$ and $\mathscr{B}$ are locally free amounts to saying that A and B are free R-modules. We need to show that the map:

$$\operatorname{Hom}_{\mathcal{O}_S}(\mathscr{A}, \mathscr{B}) \to \operatorname{Hom}_{\mathcal{O}_U}(\mathscr{A}|_U, \mathscr{B}|_U), \tag{1.2}$$

given by restriction of morphisms, is bijective; this is performed using the theory of cohomology with support. Now, since $\mathscr{A}$ and $\mathscr{B}$ are coherent on the affine scheme $S = \operatorname{Spec}(R)$, we have $\operatorname{Hom}_{\mathcal{O}_S}(\mathscr{A}, \mathscr{B}) \cong \operatorname{Hom}_R(A, B)$. Moreover, the $\mathcal{O}_S$-module $\mathscr{F} := \mathcal{H}om_{\mathcal{O}_S}(\mathscr{A}, \mathscr{B})$ is coherent too; note that $\operatorname{Hom}_{\mathcal{O}_S}(\mathscr{A}, \mathscr{B}) = \mathscr{F}(S)$ and $\operatorname{Hom}_{\mathcal{O}_U}(\mathscr{A}|_U, \mathscr{B}|_U) = \mathscr{F}(U)$.

By basic properties of the depth of modules, we have $\operatorname{depth} B = \operatorname{depth} R$ (because B is free), hence $\operatorname{depth} B \geq 2$ (by hypothesis), which implies that $\operatorname{depth} \operatorname{Hom}_R(A, B) \geq 2$ (cf [15, Tag 0AV5]). Now, the long exact sequence of cohomology with support ([16, 1.5.2]) yields an exact sequence of abelian groups:

$$\begin{gathered} 0 \to H^0_{\mathfrak{m}}(S, \mathscr{F}) \to H^0(S, \mathscr{F}) \to H^0(U, \mathscr{F}|_U) \to H^1_{\mathfrak{m}}(S, \mathscr{F}) \to \dots \\ \dots \to H^p_{\mathfrak{m}}(S, \mathscr{F}) \to H^p(S, \mathscr{F}) \to H^p(U, \mathscr{F}|_U) \to H^{p+1}_{\mathfrak{m}}(S, \mathscr{F}) \to \dots, \end{gathered} \tag{1.3}$$

where the H^i's are the classical cohomology groups and the $H^i_{\mathfrak{m}}$'s are the cohomology groups with support in $\{\mathfrak{m}\}$. However, since $\operatorname{depth} \mathscr{F}(S) = \operatorname{depth} \operatorname{Hom}_R(A, B) \geq 2$, we have $H^p_{\mathfrak{m}}(S, \mathscr{F}) = 0$ for $p = 0, 1$, by the characterisation of depth in terms of cohomology with support ([16, 1.7.1]; note that, by Nakayama lemma, we would have $\mathfrak{m}\mathscr{F}(S) = \mathscr{F}(S)$ only if $\mathscr{F}(S) = 0$, in which case there would be nothing to prove). But then, exactness of (1.3) implies that the restriction map $\mathscr{F}(S) = H^0(S, \mathscr{F}) \to H^0(U, \mathscr{F}|_U) = \mathscr{F}(U)$ is an isomorphism, which is exactely what we had to prove.

As for the second statement, we may use the interpretation of finite étale coverings of S in terms of locally free coherent $\mathcal{O}_S$-algebras outlined above. Thus, it suffices to prove that, for two locally free coherent $\mathcal{O}_S$-algebras $\mathscr{A}$ and $\mathscr{B}$, the map:

$$\mathrm{Hom}_{\mathcal{O}_S-\mathsf{Alg}}(\mathscr{A}, \mathscr{B}) \to \mathrm{Hom}_{\mathcal{O}_U-\mathsf{Alg}}(\mathscr{A}|_U, \mathscr{B}|_U),$$

given by restriction of morphisms, is bijective. However, homomorphisms of $\mathcal{O}_S$-algebras are in particular homomorphisms of $\mathcal{O}_S$-modules. Therefore, bijectivity of (1.2) proved above immediately implies injectivity of our map. Moreover, for $\varphi \in \mathrm{Hom}_{\mathcal{O}_U-\mathsf{Alg}}(\mathscr{A}|_U, \mathscr{B}|_U) \subseteq \mathrm{Hom}_{\mathcal{O}_U}(\mathscr{A}|_U, \mathscr{B}|_U)$, its preimage in $\mathrm{Hom}_{\mathcal{O}_S}(\mathscr{A}, \mathscr{B})$ is the push-forward $i_*\varphi$, where $i\colon U \to S$ denotes the inclusion map (because $(i_*\varphi)|_U = \varphi$). This is again a map of $\mathcal{O}_S$-algebras, proving thus surjectivity. □

Thanks to the lemma just proved, we have already gained fully faithfulness of the functor concerned in the theorem. The hardest part, however, is proving essential surjectivity. We plan to do this by induction on the dimension of R. For this purpose, we need some reduction statements.

Lemma 1.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, such that* $\operatorname{depth} R \geq 2$. *Then, if the completion $\hat{R}$ is pure, so is R.*

Proof. Let $S := \mathrm{Spec}(R)$, $U := S \setminus \{\mathfrak{m}\}$. We have to show that the pull-back functor $\mathrm{Et}(S) \to \mathrm{Et}(U)$ is an equivalence of categories. By lemma 1.3, we already know it is fully faithful, so we are left with proving essential surjectivity; this is done using faithfully flat descent. Set $S' := \mathrm{Spec}(\hat{R})$, with $g\colon S' \to S$ the canonical morphism. Since R is Noetherian, g is faithfully flat; being affine, g is also quasi-compact. Let then $U' := U \times_S S' = g^{-1}(U) = S' \setminus \{\hat{\mathfrak{m}}\}$, the last equality because $g^{-1}(\{\mathfrak{m}\}) = \mathrm{Spec}(\hat{R} \otimes_R R/\mathfrak{m}) = \mathrm{Spec}(\hat{R}/\hat{\mathfrak{m}}) = \{\hat{\mathfrak{m}}\}$ as topological spaces. We have a cartesian diagram:

$$\begin{tikzcd} U' \arrow[r, hook] \arrow[d, "g_U"] & S' \arrow[d, "g"] \\ U \arrow[r, hook] & S, \end{tikzcd}$$

where g_U is also faithfully flat and quasi-compact, since these properties are stable under base change.

Let $f_U\colon X_U \to U$ be a finite étale covering and consider its pullback $f'_U\colon X'_U = X_U \times_U U' \to U'$ via g_U. By hypothesis, there is a finite étale covering $f'\colon X' \to S'$ whose pull-back to U' is f'_U. Now, faithfully flat descent for étale morphisms ([1, IX, 4.1]) says that f', together with the obvious descent datum, yields an étale morphism $f\colon X \to S$, whose pull-back via g is f'. Since the property of being finite is stable under faithfully flat descent, f is actually a finite étale covering. Let $f|_U$ be the pull-back of f to U. By commutativity of the diagram above, the pull-back of $f|_U$ via g_U is also the pull-back of

f' to U', i.e. f'_U. Thus, $f|_U$ and f_U pull-back to the same thing via g_U, which means, by faithfully flat descent again, that they are the same covering. So f is a preimage of f_U under $\mathrm{Et}(S) \to \mathrm{Et}(U)$; this concludes the proof. □

The following lemma is proved making use of some results from [2, X, §1] and [2, X, §2], which relate the category of finite étale coverings of a scheme, with that of its *formal completion* along a fixed closed subset. The details of such results fall outside the purpose of this exposition. However, we forward the reader to [10, II, §9] for an introduction to formal completions of schemes.

Lemma 1.5. *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring, $t \in \mathfrak{m}$ a regular element (i.e. t not a zero-divisor). Suppose that, for all the prime ideals $\mathfrak{p}$ of R such that* $\dim R/\mathfrak{p} = 1$, *we have* $\operatorname{depth} R_\mathfrak{p} \geq 2$. *Then, if $R/(t)$ is pure, so is R.*

Proof. Set $S := \mathrm{Spec}(R)$, $Y := \mathrm{Spec}(R/(t))$, $j\colon Y \to S$ the closed immersion, $U := S \setminus \{\mathfrak{m}\}$, $Y_U := Y \times_S U = j^{-1}(U) = Y \setminus \{\bar{\mathfrak{m}}\}$, where $\bar{\mathfrak{m}}$ denotes the image of $\mathfrak{m}$ in $R/(t)$ (note that Y_U is a closed subscheme of U, as closed immersions are stable under base change). We have commutative diagrams:

$$\begin{tikzcd} Y_U \arrow[r] \arrow[d] & Y \arrow[d, "j"] \\ U \arrow[r] & S, \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Et}(Y_U) & \mathrm{Et}(Y) \arrow[l, "a"'] \\ \mathrm{Et}(U) \arrow[u, "b"] & \mathrm{Et}(S), \arrow[l, "c"] \arrow[u, "d"'] \end{tikzcd}$$

where the right one is induced by the left one and we have to prove that c is an equivalence of categories. Now, a is an equivalence by hypothesis. On the other hand, since R is complete, then it is also (t)-adically complete (cf [13, Ex. 8.2]). Hence, we may identify S with its formal completion along Y, so [2, X, 1.1] says that d is an equivalence of categories. To see that c is an equivalence, it suffices to show that b is fully faithful. Let $\hat{U}$ denote the formal completion of U along Y_U. Then, b factors as $\mathrm{Et}(U) \xrightarrow{b1} \mathrm{Et}(\hat{U}) \xrightarrow{b2} \mathrm{Et}(Y_U)$. By [2, X, 1.1], $b2$ is an equivalence and, by [2, X, 2.1(i), 2.3(i)], $b1$ is fully faithful (here is where the assumption in terms of $\operatorname{depth} R_\mathfrak{p}$ comes into play). Thus, b is fully faithful and this completes the proof. □

We can now proceed with the proof of the main result of this section.

Proof of theorem 1.2. Let $S := \mathrm{Spec}(R)$, $U := S \setminus \{\mathfrak{m}\}$ and denote by $i\colon U \to S$ the open immersion. We proceed by induction on the dimension of R.

Suppose $\dim R = 2$. Since R is a regular local ring, it is in particular Cohen-Macaulay (cf [14, 2.2.6]), hence $\operatorname{depth} R = \dim R = 2$. By lemma 1.3, the functor $\mathrm{Et}(S) \to \mathrm{Et}(U)$ is then fully faithful. As for essential surjectivity, let $f_U\colon X_U \to U$ be a finite étale covering and $\mathscr{A}_U = (f_U)_*\mathcal{O}_{X_U}$ the corresponding locally free coherent $\mathcal{O}_U$-algebra. We define the $\mathcal{O}_S$-algebra $\mathscr{A} := i_*\mathscr{A}_U$ and we set $A := \mathscr{A}(S)$. Then $\mathscr{A}$ corresponds to an affine morphism $f\colon X \to S$, with $X = \mathrm{Spec}(\mathscr{A}) = \mathrm{Spec}\, A$, which pulls-back to f_U via i. It suffices to show that f is actually a finite étale covering.

Now, the map i is quasi-compact and quasi-separated, as an open immersion of locally Noetherian schemes, hence $\mathscr{A}$ is a quasi-coherent $\mathcal{O}_S$-algebra and $\mathscr{A}|_U = i^*\mathscr{A} = \mathscr{A}_U$ (see [9, I, 9.4.2]); moreover, there exists a coherent $\mathcal{O}_S$-module that restricts to $\mathscr{A}_U$ on U (by [9, I, 9.4.3]). This allows us to apply a "finiteness criterion" for $\mathscr{A} = i_*\mathscr{A}_U$, namely [2, VIII, 2.3]: $\mathscr{A}$ is coherent if and only if $\operatorname{depth}(\mathscr{A}_U)_\mathfrak{p} \geq 1$ for every point $\mathfrak{p} \in U$ with $1 = \operatorname{codim}_{V(\mathfrak{p})}(\{\mathfrak{m}\}) = \dim R/\mathfrak{p}$. However, as we already observed, R is a Cohen-Macaulay ring, hence we have the formula (cf [14, 2.1.4]):

$$\dim R_\mathfrak{p} + \dim R/\mathfrak{p} = \dim R. \tag{1.4}$$

Therefore, $\dim R/\mathfrak{p} = 1$ if and only if $\dim R_\mathfrak{p} = \dim R - 1 = 2 - 1 = 1$. Furthermore, localisation preserves the property of being Cohen-Macaulay (cf [14, 2.1.3(b)]), so $\dim R_\mathfrak{p} = \operatorname{depth} R_\mathfrak{p}$. Finally, since $\mathscr{A}_U$ is locally free, we have: $\operatorname{depth}(\mathscr{A}_U)_\mathfrak{p} = \operatorname{depth} R_\mathfrak{p} = \dim R_\mathfrak{p}$; the condition of the finiteness criterion is thus satisfied. This means that f is a finite morphism.

The next claim is that the R-algebra A is free as an R-module. Since R is local, it suffices to show that A is a projective R-module (cf [13, 2.5]). We may do this proving that the projective dimension of A is zero, by means of the Auslander-Buchsbaum formula for finite modules of finite projective dimension over a Noetherian local ring (cf [13, 19.1] and note that, since R is regular, every R-module has finite projective dimension by [13, 19.2]):

$$\operatorname{pd} A + \operatorname{depth} A = \operatorname{depth} R,$$

where $\operatorname{pd} A$ denotes the projective dimension of A. Since $\operatorname{depth} R = \dim R = 2$, we have $\operatorname{pd} A = \operatorname{depth} R - \operatorname{depth} A = 2 - \operatorname{depth} A$, so it is sufficient to show that $\operatorname{depth} A \geq 2$. We will use again the theory of cohomology with support, in the opposite direction as before. We have an exact sequence of abelian groups (by [16, 1.5.2]):

$$0 \to H^0_\mathfrak{m}(S, \mathscr{A}) \to H^0(S, \mathscr{A}) \to H^0(U, \mathscr{A}_U) \to H^1_\mathfrak{m}(S, \mathscr{A}) \to H^1(S, \mathscr{A}).$$

Here, $H^0(S, \mathscr{A}) \to H^0(U, \mathscr{A}_U)$ is an isomorphism, by definition of $\mathscr{A}$ as the pushforward of $\mathscr{A}_U$, and $H^1(S, \mathscr{A}) = 0$, because $S = \operatorname{Spec}(R)$ is affine and $\mathscr{A}$ is quasi-coherent (cf [11, 12.32]). By exactness, then, we have $H^p_\mathfrak{m}(S, \mathscr{A}) = 0$ for $p = 0, 1$. Hence, by virtue of the characterisation of depth in terms of cohomology with support ([16, 1.7.1]), we have $\operatorname{depth} A = \operatorname{depth} \mathscr{A}(S) \geq 2$. This proves the claim.

As a consequence, $f\colon X \to S$ is a flat morphism. Recall that, then, we can check whether f is étale using the discriminant section (cf [1, I, 4.10]). More precisely, we have the trace map: $\mathscr{A} \to \mathcal{O}_S$, which is a homomorphism of $\mathcal{O}_S$-modules. Precomposing with the internal multiplication of $\mathscr{A}$, we get an $\mathcal{O}_S$-bilinear homomorphism $\mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{A} \to \mathcal{O}_S$. To this, we can associate a determinant Δ, which, in our case ($S = \operatorname{Spec} R$ affine, $A = \mathscr{A}(S)$ free over R) is really a section $\Delta \in \mathcal{O}_S(S) = R$ (the *discriminant section*). Then, f is étale if and only if Δ is a unit in R. However, we already know that the restriction of f to U, that is $f_U\colon X_U \to U$, is étale, hence Δ is a unit in all the local rings $\mathcal{O}_{S,s}$, for $s \in U$. In other words, $\Delta \in R \setminus \mathfrak{p}$ for all prime ideals $\mathfrak{p} \neq \mathfrak{m}$ of R. Suppose now, by contradiction, that Δ is not a unit in R, i.e., since R is local, $\Delta \in \mathfrak{m}$. Then,

$\dim R/(\Delta) = 0$, because the only prime ideal containing Δ is $\mathfrak{m}$. On the other hand, by Krull's principal ideal theorem (Δ is not a zero-divisor because R is regular, hence integral), we have $\dim R/(\Delta) = \dim R - 1 = 2 - 1 = 1$, which is a contradiction. Thus, Δ is a unit in R, i.e. f is étale, proving the base step of the induction.

Suppose now that $\dim R \geq 3$ and that the theorem holds for regular local rings of dimension strictly less than $\dim R$. The completion $\hat{R}$ is again a regular local ring of the same dimesion as R, hence we may assume R complete, by lemma 1.4. Choose an element $t \in \mathfrak{m} \setminus \mathfrak{m}^2$. Then, $R/(t)$ is a regular local ring of dimension $\dim R - 1$, by Krull's principal ideal theorem (regulariy follows from the fact that t can be completed to a minimal set of generators of $\mathfrak{m}$, which, as R is regular, counts $\dim R$ elements, so the maximal ideal of $R/(t)$ is generated by $\dim R - 1$ elements). By inductive hypothesis, $R/(t)$ is pure. We would like to conclude that R is pure using lemma 1.5. Let $\mathfrak{p}$ be a prime ideal of R such that $\dim R/\mathfrak{p} = 1$. By the formula (1.4), it follows that $\dim R_\mathfrak{p} = \dim R - \dim R/\mathfrak{p} = \dim R - 1 \geq 2$, because $\dim R \geq 3$. However, we already observed that $R_\mathfrak{p}$ is still Cohen-Macaulay, so $\operatorname{depth} R_\mathfrak{p} = \dim R_\mathfrak{p} \geq 2$. The hypothesis of lemma 1.5 is thus satisfied, hence R is pure. This concludes the proof. $\square$

From local to global

Let S be a scheme, $U \subseteq S$ an open subset and $Z = S \setminus U$ its complement. In [2, XIV, 1.2], it is defined the notion of *homotopic depth* of S with respect to Z, denoted by $\operatorname{prof\,hop}_Z S$. The main characterisation of the homotopic depth is given in [2, XIV, 1.4]: in particular, we have $\operatorname{prof\,hop}_Z S \geq 3$ if and only if the functor $\mathrm{Et}(S) \to \mathrm{Et}(U)$ is an equivalence of categories. Now, using theorem [2, XIV, 1.8], we can relate the homotopic depth of S with respect of Z, to the homotopic depth of the strict étale localisation $\bar{S}$ of S at the geometric points $\bar{z}$ of Z (cf [19, I, §4] for the definitions). In particular, in order to have $\operatorname{prof\,hop}_Z S \geq 3$, it suffices to have $\operatorname{prof\,hop}_{\bar{z}} \bar{S} \geq 3$ for all the geometric points $\bar{z}$ of Z.

Suppose now that S is regular and $\operatorname{codim}_Z(S) \geq 2$. Let $z \in Z$ and $\bar{z}$ be a geometric point on z. Then, $\mathcal{O}_{S,z}$ is a regular local ring of dimension at least 2. Now, the strict étale localisation $\bar{S}$ of S at $\bar{z}$ is the spectrum of the strict henselisation $\mathcal{O}_{S,z}^{sh}$ of $\mathcal{O}_{S,z}$ (see [19, I, §4] again), which is still a regular local ring of the same dimension (cf [15, Tag 06LK, Tag 06LN]). Hence, $\mathcal{O}_{S,z}^{sh}$ is pure by theorem 1.2, i.e. $\mathrm{Et}(\bar{S}) \to \mathrm{Et}(\bar{S} \setminus \{\, \bar{z} \,\})$ is an equivalence. By the characterisation above, then, $\operatorname{prof\,hop}_{\bar{z}} \bar{S} \geq 3$. Since this holds for all geometric points $\bar{z}$ of Z, we have $\operatorname{prof\,hop}_Z S \geq 3$, i.e. $\mathrm{Et}(S) \to \mathrm{Et}(U)$ is an equivalence of categories. We have thus proved theorem 1.1 using its local version 1.2.

1.3 Some consequences

Purity of the branch locus

The fact that the theorem proved above is known as "purity of the branch locus" is due to the statement that we will explain in this paragraph. In fact, theorem 1.2 allows to characterise, for certain morphisms, the locus on the base scheme over which they are *not* étale.

Let $f\colon X \to S$ be a finite flat morphism, with S locally Noetherian. We define the *ramification locus* $R \subseteq X$ of f to be the set of points where f is not étale; since being étale is an open condition, R is closed in X. Note that, as f is already flat, R is in fact the set of points where f is ramified, or, equivalently, the support of the sheaf of differentials $\Omega^1_{X/S}$ (cf [1, I, 3.1]). Next, we define the *branch locus* of f to be $B := f(R)$; since finite morphisms are closed, $B = f(R)$ is closed in S. We say that f is *generically étale* if there exists a dense open subset $U \subseteq S$ such that the restriction $f\colon f^{-1}(U) \to U$ is étale, or, equivalently, if $S \setminus B$ is dense in S.

Theorem 1.6. *Let $f\colon X \to S$ be a finite flat morphism of schemes, with S regular. Suppose that f is generically étale. Then, its branch locus B is either empty or pure of codimension* 1 *in S, i.e.* $\operatorname{codim}_S(Z) = 1$ *for all irreducible components Z of B.*

Proof. Set $U := S \setminus B$; by assumption, U is dense in S. Let now $Z \subseteq B$ be an irreducible component of B, $\eta \in Z$ its generic point, so that $\operatorname{codim}_S(Z) = \dim \mathcal{O}_{S,\eta}$. We proceed excluding both the possibilities that $\operatorname{codim}_S(Z) = 0$ and $\operatorname{codim}_S(Z) \geq 2$.

If $\operatorname{codim}_S(Z) = 0$, then Z is an irreducible component of S. However, being dense, U must intersects all the irreducible components of S (because the latter is locally Noetherian). Since $Z \subseteq B = S \setminus U$, this is impossible.

If $\operatorname{codim}_S(Z) \geq 2$, then $\mathcal{O}_{S,\eta}$ is pure by theorem 1.2. Let $S' := \operatorname{Spec} \mathcal{O}_{S,\eta}$ and let $f'\colon X' \to S'$ be the pull-back of f to S'; since the properties of being finite and flat are stable under base change, f' is still finite flat. Now, let U' denote the preimage of U in S' and note that $U' = S' \setminus \{\bar{\eta}\}$, where $\bar{\eta}$ is the maximal ideal of $\mathcal{O}_{S,\eta}$. Indeed, $\bar{\eta}$ maps to $\eta \in B = S \setminus U$ and, on the other hand, if a point $s \in S' \setminus \{\bar{\eta}\}$ mapped to B, it would contradict the maximality of $Z = \overline{\{\eta\}}$ as an irreducible subset of B. Observe, next, that the fibre of f' at a point of S' is the same as the fibre of f at the image of such point in S. Therefore, the restriction f'_U of f' to U' is étale, whereas f' is not étale over $\bar{\eta}$. However, by how we proved theorem 1.2, we see that the push-forward $i_* f'_U$ of f'_U to S' is étale (here $i\colon U' \to S'$ denotes the inclusion and with $i_* f'_U$ we mean the affine S'-scheme coresponding to the push-forward of the $\mathcal{O}_{U'}$-algebra of f'_U). We now claim that $f' = i_* f'_U$, which would contradict the fact that f' is not étale over $\bar{\eta}$. In fact, both f' and $i_* f'_U$ are finite flat, hence they correspond to locally free coherent $\mathcal{O}_{S'}$-algebras. Since f' and $i_* f'_U$ both pull-back to f'_U on U', the pull-backs of the corresponding algebras must coincide as well. But then, by lemma 1.3, these algebras are isomorphic, hence so are f' and $i_* f'_U$ (recall that $\operatorname{depth} \mathcal{O}_{S,\eta} = \dim \mathcal{O}_{S,\eta}$ because it is a regular local ring). This gives the desired contradiction and proves the theorem. $\square$

Remark 1.7. If we remove the assumption that f is generically étale, then the first step in the proof fails and it may well happen that the branch locus B contains an irreducible component of S. In particular, if S is connected, then it is irreducible by regularity and this would mean that $B = S$. As an example, let k be a field of positive characteristic

$p > 0$ and consider the Frobenius endomorphism F of $\mathbb{A}^1_k = \operatorname{Spec} k[x]$, defined by $x \mapsto x^p$. Then, F is ramified at all the points of $\mathbb{A}^1_k$, so we have $B = F(\mathbb{A}^1_k) = \mathbb{A}^1_k$.

On the other hand, over a field k of characteristic zero and with the additional assumption that X is regular over k, the map f is automatically generically étale, by generic smoothness (cf [11, Ex 10.40(b)]).

Also note that the assumption on the regularity of the base S is crucial. An example showing this can be found in [15, Tag 0BTE]. Given a field k of characteristic different from 2, consider the subring $k[x^2, xy, y^2]$ of $k[x, y]$. If we write $A := k[u, v, w]/(v^2 - uw)$, then the inclusion $k[x^2, xy, y^2] \hookrightarrow k[x, y]$ may be seen as the map:

$$A \to A[x, y]/(x^2 - u, xy - v, y^2 - w) \cong k[x, y].$$

Note that the scheme $S := \operatorname{Spec} A$ is not regular at the point $s = (u, v, w) \in S$. The corresponding morphism of schemes $f \colon \mathbb{A}^2_k = \operatorname{Spec} k[x, y] \to \operatorname{Spec} A = S$ is actually not even flat, but we can still consider the ramification locus $R = \operatorname{Supp} \Omega^1_{\mathbb{A}^2_k/S} = \{ \underline{0} \} \subseteq \mathbb{A}^2_k$. The branch locus, then, is $B = f(R) = \{ s \} \subseteq S$, which has codimension 2 in S; thus the theorem fails in this case. In fact, we may obtain a branch locus of any codimension $n > 1$, starting from the subring $k[t_i t_j | i, j \in \{ 1, \dots, n \}]$ of $k[t_1, \dots, t_n]$.

Purity in terms of the étale fundamental group

The purity theorem 1.1 has an immediate consequence in terms of étale fundamental groups. We will quickly recall the definitions concerned and a corollary will easily follow.

Let S be a scheme and $\bar{s} \colon \operatorname{Spec} \Omega \to S$ a geometric point. For every finite étale covering $X \to S$, we may consider the fibre $X_{\bar{s}} := X \times_S \operatorname{Spec} \Omega$. If $X \to Y$ is a morphism of finite étale coverings of S, we have an induced morphism $X_{\bar{s}} \to Y_{\bar{s}}$. This defines a functor:

$$\begin{aligned} Fib_{\bar{s}} \colon \operatorname{Et}(S) &\to \mathsf{Set} \\ (X \to S) &\mapsto X_{\bar{s}}, \end{aligned}$$

where $X_{\bar{s}}$ is considered as a set, forgetting the scheme structure and similarly for the induced morphisms. The *étale fundamental group* of S (with base point $\bar{s}$), denoted by $\pi_1(S, \bar{s})$, is the group of automorphisms of the functor $Fib_{\bar{s}}$, i.e. invertible natural transformations $Fib_{\bar{s}} \to Fib_{\bar{s}}$, with the group law given by composition.

Galois theory for schemes says that, when S is connected, $Fib_{\bar{s}}$ induces an equivalence of categories $\operatorname{Et}(S) \to \pi_1(S, \bar{s}){-}\mathsf{Set}$, where $\pi_1(S, \bar{s}){-}\mathsf{Set}$ is the category of finite sets with a continuous $\pi_1(S, \bar{s})$-action (cf [22, 1.11]). In fact, $\pi_1(S, \bar{s})$ turns out to be a profinite group (see [22, 1.8] for a definition), whose natural action on each $X_{\bar{s}}$ is continuous.

In the situation of theorem 1.1, fixed a geometric point $\bar{s} \colon \operatorname{Spec} \Omega \to U$, we have a commutative diagram:

$$\begin{tikzcd} \mathrm{Et}(S) \arrow[rr, "\sim"] \arrow[dr, "Fib_{\bar{s}}"'] & & \mathrm{Et}(U) \arrow[dl, "Fib_{\bar{s}}"] \\ & \mathsf{Set}, & \end{tikzcd}$$

which induces a map $\pi_1(U, \bar{s}) \to \pi_1(S, \bar{s})$. Since the top arrow is an equivalence of categories, we get the following corollary.

Corollary 1.8. *Let S be a regular scheme, $U \subseteq S$ an open subscheme, $Z = S \setminus U$ its closed complement and $\bar{s}\colon \operatorname{Spec}\Omega \to U$ a geometric point. Suppose that $\operatorname{codim}_S(Z) \geq 2$. Then, the map:*

$$\pi_1(U, \bar{s}) \to \pi_1(S, \bar{s})$$

is an isomorphism.

1.4 Lower codimension

One may ask whether the results of this chapter extend to a more general situation. In particular, if S is a regular scheme and $U \subseteq S$ an open subset, whose complement $Z = S \setminus U$ is of codimension 1 in S, what can we say about the restriction functor $\mathrm{Et}(S) \to \mathrm{Et}(U)$?

In fact, it turns out that such functor is fully faithful whenever U is dense in S (so $\operatorname{codim}_S(Z) \geq 1$ is enough). This property is due to the following more general proposition, which can be obtained as an application of Zariski's main theorem; we will refer to [17] for the formulations of Zariski's main theorem that are best suited for this purpose.

Proposition 1.9. *Let S be a locally Noetherian scheme, $U \subseteq S$ a dense open subset, X and Y two finite flat S-schemes; denote by $X_U = X \times_S U$ and $Y_U = Y \times_S U$ the pull-backs to U of X and Y respectively. Suppose that X is normal. Then, the restriction map:*

$$\operatorname{Hom}_{\mathsf{Sch}/S}(X, Y) \longrightarrow \operatorname{Hom}_{\mathsf{Sch}/U}(X_U, Y_U)$$

is a bijection.

In particular, if S is regular, the restriction functor:

$$\mathrm{Et}(S) \to \mathrm{Et}(U)$$

is fully faithful.

Proof. We will construct an inverse of the restriction map concerned, so let $f_U : X_U \to Y_U$ be a morphism over U. Consider the graph of f_U, i.e. the morphism:

$$\Gamma_{f_U} = (id_{X_U}, f_U)\colon X_U \to X_U \times_U Y_U.$$

Note that Γ_{f_U} is a section of the projection $X_U \times_U Y_U \to X_U$, which in turn is separated (because X is finite over S). Thus, Γ_{f_U} is a closed immersion (cf [11, 9.12]), i.e. it induces an isomorphism between X_U and a closed subscheme of $X_U \times_U Y_U$, which we denote again by Γ_{f_U}.

Now, $X_U \times_U Y_U \cong (X \times_S Y) \times_S U$ is an open subscheme of $X \times_S Y$ (namely the preimage of U via the structure map $X \times_S Y \to S$), hence we may see Γ_{f_U} as a subscheme of $X \times_S Y$. Moreover, X is normal by assumption, implying that X_U and hence Γ_{f_U} are normal too; in particular, Γ_{f_U} is reduced. Then, the inclusion: $\Gamma_{f_U} \hookrightarrow X \times_S Y$ factors through the closed subscheme $\overline{\Gamma_{f_U}}$ of $X \times_S Y$ given by the topological closure of Γ_{f_U} in $X \times_S Y$, with the reduced scheme structure (cf [11, 10.32]). By construction, we have a pull-back diagram:

$$\begin{tikzcd} \Gamma_{f_U} \arrow[r] \arrow[d, hook] & \overline{\Gamma_{f_U}} \arrow[d, hook] \\ X_U \times_U Y_U \arrow[r] & X \times_S Y, \end{tikzcd}$$

with Γ_{f_U} dense in $\overline{\Gamma_{f_U}}$.

Consider the composition $p \colon \overline{\Gamma_{f_U}} \hookrightarrow X \times_S Y \xrightarrow{pr_1} X$ and note that the pull-back of p to U, i.e. $p|_U \colon \Gamma_{f_U} \to X_U$ is the inverse of the isomorphism $X_U \xrightarrow{\sim} \Gamma_{f_U}$ induced by the graph, so it is an isomorphism too. Since Γ_{f_U} is dense in $\overline{\Gamma_{f_U}}$ and X_U is dense in X (because U is dense in X and the structure map $X \to S$ is finite flat, thus open), this means that p is birational. Furthermore, it is the composition of a closed immersion with a finite map, hence it is finite and, in particular, proper. We claim that p is in fact an isomorphism.

Now, being normal, X is a finite disjoint union of normal irreducible schemes (cf [15, Tag 033M]). Restricting p to each irreducible component of X, we may assume that $\overline{\Gamma_{f_U}}$ and X are irreducible (the preimage of an irreducible component of X will be an irreducible component of $\overline{\Gamma_{f_U}}$, as p is birational); since these are normal schemes, irreducibility implies that they are integral. Finally, p has connected fibers by Zariski's main theorem (cf [17, 1.1]), but, being it a finite morphism, such fibers are discrete, hence they consist of at most one point. On the other hand, finiteness also means that p is closed and, since its image contains X_U, it follows that p is surjective. This shows that, as a map of sets, p is bijective; a corollary to Stein factorization (cf [17, 3.9]) ensures that then p is an isomorphism, proving the claim.

Thanks to this, we can consider the composition: $f \colon X \xrightarrow{p^{-1}} X \times_S Y \xrightarrow{pr_2} Y$. By construction, the restriction of f to U is the map f_U. Conversely, if f_U is the restriction to U of a map $g \colon X \to Y$ over S, then we have that $\overline{\Gamma_{f_U}}$ coincides with the graph Γ_g of g and the resulting map f is again g. In other words, the construction above gives indeed an inverse of the restriction map: $\mathrm{Hom}_{\mathsf{Sch}/S}(X, Y) \longrightarrow \mathrm{Hom}_{\mathsf{Sch}/U}(X_U, Y_U)$.

As for the second statement, note that any finite étale covering of a normal scheme is itself normal (cf [15, Tag 025P]). If S is regular, then it is in particular normal and, for any two finite étale coverings $X, Y \in \mathrm{Et}(S)$, we may apply the previous result to the restriction map: $\mathrm{Hom}_{\mathsf{Sch}/S}(X, Y) \longrightarrow \mathrm{Hom}_{\mathsf{Sch}/U}(X_U, Y_U)$. □

On the other hand, there is no hope for the restriction functor $\mathrm{Et}(S) \to \mathrm{Et}(U)$ to be essentially surjective, when the codimension of the complement $Z = S \setminus U$ in S is 1.

First of all, extending finite étale coverings by push-forward, as we did in the proof of theorem 1.2, does not work any more. For instance, $\mathbb{Z}_p$ is a regular local ring of dimension 1 (i.e., a DVR), with maximal ideal $p\mathbb{Z}_p$. The punctured spectrum $\operatorname{Spec} \mathbb{Z}_p \setminus \{ p\mathbb{Z}_p \}$ consists only of the generic point $(0) \subseteq \mathbb{Z}_p$, so it is $\operatorname{Spec} \mathbb{Q}_p$. But the push-forward of $\mathbb{Q}_p$ itself via $\operatorname{Spec} \mathbb{Q}_p \to \operatorname{Spec} \mathbb{Z}_p$ is again $\mathbb{Q}_p$, which is not even finite over $\mathbb{Z}_p$.

More generally, taking for example $S = \mathbb{P}^1_{\mathbb{C}}$ and a ramified covering $f\colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$, the branch locus of f will be a non-empty closed subscheme $B \subseteq \mathbb{P}^1_{\mathbb{C}}$ of codimension 1 (a finite set of points). The restriction f_U of f to $U := \mathbb{P}^1_{\mathbb{C}} \setminus B$ is a finite étale covering of U. But any extension of f_U to $\mathbb{P}^1_{\mathbb{C}}$ must coincide with f itself, because the previous proposition applies. Since f is not étale, this means that f_U does not have a preimage in $\mathrm{Et}(S)$.

Remark 1.10. In the proposition above we saw that, for S a regular scheme and $U \subseteq S$ a dense open subset, the restriction functor $\mathrm{Et}(S) \to \mathrm{Et}(U)$ is fully faithful. An immediate consequence of this is that, for any geometric point $\bar{s}\colon \operatorname{Spec} \Omega \to U$, the induced map:

$$\pi_1(U, \bar{s}) \to \pi_1(S, \bar{s})$$

on the étale fundamental groups is surjective (cf [15, Tag 0BN6]).

Chapter 2

Group schemes and fppf torsors

The next aim of this thesis is to bring the results of the previous chapter to a setting of more specific interest, namely that of torsors by the action of a group scheme. In order to do so, we need to introduce the notions of "group scheme" and "torsor". We will focus, illustrating the classical examples, on affine group schemes (or, equivalently, Hopf algebras) and give a proof, in this case, of Cartier's theorem. The latter will allow us to distinguish different phenomena when working over a field of characteristic zero, rather than a field of positive characteristic. After that, we will define actions and torsors and illustrate the main property of the latter. We will conclude the chapter with a quick overview on quotients of schemes by a group scheme action and how they might yield a torsor.

In this chapter, it will be very useful to identify schemes at first with their functor of points and, eventually, to see them as sheaves in the categorical sense. This is a standard point of view in Algebraic Geometry and we forward any reader who may not be familiar with it to [18, §2].

2.1 Basic definitions and properties of group schemes

Definition 2.1. Let S be a scheme. A *group scheme over* S, or an S*-group scheme*, is an S-scheme $\pi\colon G \to S$, together with S-morphisms:

- $m\colon G \times_S G \to G$ (*multiplication*),
- $i\colon G \to G$ (*inverse*),
- $e\colon S \to G$ (*identity section*),

such that the following diagrams commute:

$$\begin{tikzcd} G \times_S G \times_S G \arrow[r, "m \times id_G"] \arrow[d, "id_G \times m"] & G \times_S G \arrow[d, "m"] \\ G \times_S G \arrow[r, "m"] & G, \end{tikzcd} \qquad \text{(associativity)}$$

$$\begin{tikzcd} S \times_S G \arrow[r, "e \times id_G"] \arrow[d, "\sim"] & G \times_S G \arrow[d, "m"] \\ G \arrow[r, "id_G"] & G, \end{tikzcd} \qquad \begin{tikzcd} G \times_S S \arrow[r, "id_G \times e"] \arrow[d, "\sim"] & G \times_S G \arrow[d, "m"] \\ G \arrow[r, "id_G"] & G, \end{tikzcd}$$

(neutral element)

$$\begin{tikzcd} G \arrow[r, "(i, id_G)"] \arrow[d, "\pi"] & G \times_S G \arrow[d, "m"] \\ S \arrow[r, "e"] & G, \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "(id_G, i)"] \arrow[d, "\pi"] & G \times_S G \arrow[d, "m"] \\ S \arrow[r, "e"] & G. \end{tikzcd} \qquad \text{(inverse)}$$

A group scheme G over S is *commutative* if, denoting by $s\colon G \times_S G \to G \times_S G$ the isomorphism switching the factors ($s = (pr_2, pr_1)$), we have $m = m \circ s\colon G \times_S G \to G$.

Given two group schemes G_1 and G_2 over S, with multiplication maps m_1 and m_2 respectively, a *homomorphism of S-group schemes* from G_1 to G_2 is a morphism of schemes $f\colon G_1 \to G_2$ over S, such that the following diagram commutes:

$$\begin{tikzcd} G_1 \times_S G_1 \arrow[r, "f \times f"] \arrow[d, "m_1"] & G_2 \times_S G_2 \arrow[d, "m_2"] \\ G_1 \arrow[r, "f"] & G_2. \end{tikzcd}$$

Remark 2.2. Let G be a scheme over a base scheme S. Identifying G with its functor of points: $\mathsf{Sch}/S \to \mathsf{Set}$, $T \mapsto \mathrm{Hom}_{\mathsf{Sch}/S}(T, G)$, via the Yoneda embedding, we see that the datum (m, i, e) verifying the above axioms is equivalent to giving a group structure to $G(T) = \mathrm{Hom}_{\mathsf{Sch}/S}(T, G)$ for all S-schemes T, functorial in T (in other words, we are saying that the functor of points of G lifts to a functor $\mathsf{Sch}/S \to \mathsf{Grp}$ to the category of groups, i.e. G is a *group object* in Sch/S).

Thus, G is commutative if and only if the group structure on $G(T)$ is commutative for all S-schemes T (i.e., the functor of points of G lifts to a functor $\mathsf{Sch}/S \to \mathsf{Ab}$ to the category of abelian groups).

Similarly, a homomorphism of S-group schemes $f\colon G_1 \to G_2$ is the same as group homomorphisms $f(T)\colon G_1(T) \to G_2(T)$ for all S-schemes T, natural in T.

Definition 2.3. Let G be a group scheme over a base scheme S. An *S-subgroup scheme* (respectively, *open S-subgroup scheme*, *closed S-subgroup scheme*) is an S-subscheme (respectively, open subscheme, closed subscheme) $H \overset{j}{\hookrightarrow} G$, such that:

- the identity section $e\colon S \to G$ factors through $H \overset{j}{\hookrightarrow} G$,
- the composition $H \xrightarrow{j} G \xrightarrow{i} G$ factors through $H \overset{j}{\hookrightarrow} G$,
- the composition $H \times_S H \xrightarrow{j \times j} G \times_S G \xrightarrow{m} G$ factors through $H \overset{j}{\hookrightarrow} G$;

in other words, H is an S-group scheme itself, with the same identity section as G and multiplication and inverse the restriction of those of G.

Equivalently, $H(T)$ is a subgroup of $G(T)$ for every S-scheme T.

Remark 2.4. If G is a group scheme over S and $f\colon S' \to S$ is a morphism of schemes, then the pull-back $G \times_S S'$ inherits the structure of an S'-group scheme. Indeed, the group structure is preserved by functoriality of the pull-back:

$$\begin{gathered} \mathsf{Sch}/S \to \mathsf{Sch}/S' \\ (\pi\colon T \to S) \mapsto (pr_2\colon T \times_S S' \to S') \end{gathered}$$

and the fact that $(G \times_S G) \times_S S' \cong (G \times_S S') \times_{S'} (G \times_S S')$.

Definition 2.5. Let $f\colon G \to G'$ be a morphism of S-group schemes and let $e'\colon S \to G'$ denote the identity section of G'. We define $\ker(f)$ to be the pull-back $G \times_{G'} S$:

$$\begin{tikzcd} \ker(f) \arrow[r] \arrow[d] & G \arrow[d, "f"] \\ S \arrow[r, "e'"] & G', \end{tikzcd}$$

i.e. the preimage of $e'(S) \subseteq G'$ by f.

Note that, since e' is a section of the structure map $G' \to S$, it is an immersion (respectively, a closed immersion if $G' \to S$ is separated), that is, S is a subscheme (respectively, a closed subscheme) of G' (cf [11, 9.12]). Because immersions are stable under base-change, we have that $\ker(f)$ is a subscheme of G.

For every S-scheme T, we have, by definition, $\ker(f)(T) = \ker(f(T)\colon G(T) \to G'(T))$, which is a subgroup of $G(T)$. This ensures that $\ker(f)$ is an S-subgroup scheme of G.

Translations. Let G be a group scheme over a base scheme S. As it is the case with abstract groups (i.e. groups in the usual sense), points of a group scheme define isomorphisms by multiplication. However, generalizing this idea to group schemes needs a little more effort, because we have to take into account "where" a point is defined (think of S the spectrum of a field and points with coordinates in a field extension). More precisely, if we view a point in the sense of the functor of points, i.e. as a map $T \to G$ from another S-scheme T, we may obviously have $T \neq S$. In this case, we shall pull-back G to the new base T, in order to have our point defined over the same base as the group scheme.

In details, given an S-scheme $T \to S$ and a point $g \in G(T)$, we can form the pull-back $G_T = G \times_S T$, which is now a group scheme over T, and we can pull-back the point g to $g_T = (g, id_T) \in G_T(T)$. Let m_T denote the multiplication of G_T. Then, the *right* *translation* and the *left translation* by g are defined respectively as:

$$t_g\colon G_T \xrightarrow{\sim} G_T \times_T T \xrightarrow{id_{G_T}\times g_T} G_T \times_T G_T \xrightarrow{m_T} G_T,$$

$$t'_g\colon G_T \xrightarrow{\sim} T \times_T G_T \xrightarrow{g_T\times id_{G_T}} G_T \times_T G_T \xrightarrow{m_T} G_T.$$

For a more familiar description, let $T' \to T$ be a T-scheme. Then, the maps $t_g(T')\colon G_T(T') \to G_T(T')$ and $t'_g(T')\colon G_T(T') \to G_T(T')$ are given respectively by $\gamma \mapsto \gamma g_T$ and $\gamma \mapsto g_T\gamma$, where $g_T \in G_T(T)$ is viewed as an element of $G_T(T')$ by precomposition with $T' \to T$ and the multiplication is the one of $G_T(T')$. Note that $t_g\colon G_T \to G_T$ (respectively $t'_g\colon G_T \to G_T$) is always an isomorphism, with inverse $t_{g^{-1}}$ (respectively $t'_{g^{-1}}$).

If here we take $T = G$ and $g = id_G \in G(G)$, we get the *universal right* (respectively *left*) *translation* $\tau\colon G \times_S G \to G \times_S G$, respectively $\tau'\colon G \times_S G \to G \times_S G$, which, as maps over S, are given on T-valued points (for an S-scheme T) by $(g_1, g_2) \mapsto (g_1g_2, g_2)$, respectively $(g_1, g_2) \mapsto (g_2g_1, g_2)$. These maps owe their name to the fact that every other translation is a pull-back of τ (respectively τ'), in the sense that for every S-scheme T and every point $g \in G(T)$ there are pull-back squares:

$$\begin{tikzcd} G_T = G\times_S T \arrow[r, "t_g"] \arrow[d, "id_G\times g"] & G_T \arrow[d, "id_G\times g"] \\ G\times_S G \arrow[r, "\tau"] & G\times_S G, \end{tikzcd} \qquad \begin{tikzcd} G_T \arrow[r, "t'_g"] \arrow[d, "id_G\times g"] & G_T \arrow[d, "id_G\times g"] \\ G\times_S G \arrow[r, "\tau'"] & G\times_S G. \end{tikzcd}$$

2.2 Affine group schemes

The anti-equivalence between the category of affine schemes and that of rings gives the possibility to interpret the above discussion in terms of Commutative Algebra, in the specific case of an affine scheme over an affine base; this allows for more insight into the topic. This situation, moreover, is the one of biggest interest for us, since we will mainly be concerned with finite group schemes over a field, or some pull-back of them.

Let $G = \operatorname{Spec} A$ be a scheme over $S = \operatorname{Spec} R$ with structure map $\pi\colon G \to S$, i.e. A an R-algebra via $\pi^\#\colon R \to A$. Then, the axioms of definition 2.1 are equivalent to giving homomorphisms of R-algebras:

- $m^\#\colon A \to A \otimes_R A$ (*co-multiplication*),
- $i^\#\colon A \to A$ (*co-inverse*),
- $e^\#\colon A \to R$ (*augmentation*),

making the dual diagrams of the ones in 2.1 commute, i.e.:

- $(m^\# \otimes id_A) \circ m^\# = (id_A \otimes m^\#) \circ m^\#$,
- $(e^\# \otimes id_A) \circ m^\# = j_1$ and $(id_A \otimes e^\#) \circ m^\# = j_2$, where $j_1 \colon A \to R \otimes_R A$ and $j_2 \colon A \to A \otimes_R R$ are the canonical isomorphisms,
- $\Delta^\# \circ (i^\# \otimes id_A) \circ m^\# = \pi^\# \circ e^\# = \Delta^\# \circ (id_A \otimes i^\#) \circ m^\#$, where $\Delta^\# \colon A \otimes_R A \to A$, $x \otimes y \mapsto xy$, is the multiplication map.

Commutativity of G can be expressed as $m^\# = s^\# \circ m^\#$, where $s^\# \colon A \otimes_R A \to A \otimes_R A$ is the map switching the factors ($s^\#(x \otimes y) = y \otimes x$).

A commutative R-algebra with unit A satisfying the above axioms is called a *Hopf algebra over* R (be aware that in the literature Hopf algebras need not be commutative algebras) and it is said to be *co-commutative* if the condition $m^\# = s^\# \circ m^\#$ holds.

A morphism f between two affine group schemes $G_1 = \operatorname{Spec} A_1$ and $G_2 = \operatorname{Spec} A_2$ over $\operatorname{Spec} R$ is then equivalent to a homomorphism of R-algebras $f^\# \colon A_2 \to A_1$ satisfying $(f^\# \otimes f^\#) \circ m_2^\# = m_1^\# \circ f^\#$, where $m_1^\#$ and $m_2^\#$ are the co-multiplications of A_1 and A_2 respectively; we say that $f^\#$ is a *homomorphism of Hopf algebras over* R.

Thus, the category of affine group schemes over $\operatorname{Spec} R$ is anti-equivalent to the category of Hopf algebras over R, with commutative group schemes corresponding to co-commutative Hopf algebras.

If A is a Hopf algebra over a ring R, the ideal $I := \ker e^\# \subseteq A$ is called the *augmentation ideal*. Here are the first properties of Hopf algebras.

Proposition 2.6. *Let $(A, m^\#, i^\#, e^\#)$ be a Hopf algebra over a ring R, $I = \ker e^\#$.*

(a) $A = I \oplus R \cdot 1$ as R-modules.

(b) For every $a \in I$, we have $m^\#(a) \equiv a \otimes 1 + 1 \otimes a \mod I \otimes I$.

(c) For every $a \in I$, we have $i^\#(a) \equiv -a \mod I^2$.

Proof. (a) Since $e^\#$ is an R-linear map, the composition $R \to A \xrightarrow{e^\#} R$ is the identity. Therefore, we have an exact sequence $0 \to I \to A \xrightarrow{e^\#} R \to 0$ which splits, as the structure map $R \to A$ gives a section of $e^\#$; the claim follows.

(b) Note that, by (a) and the fact that tensor product commutes with direct sum, we have $A \otimes_R A = (I \otimes I) \oplus (I \otimes R \cdot 1) \oplus (R \cdot 1 \otimes I) \oplus (R \cdot 1 \otimes R \cdot 1)$. Using now the neutral element axiom and the fact that $a \in I = \ker e^\#$, we find:

$$(e^\# \otimes id_A)(m^\#(a) - a \otimes 1 - 1 \otimes a) = a - 0 - a = 0,$$
$$(id_A \otimes e^\#)(m^\#(a) - a \otimes 1 - 1 \otimes a) = a - a - 0 = 0.$$

Hence $m^\#(a) - a \otimes 1 - 1 \otimes a \in \ker(e^\# \otimes id_A) \cap \ker(id_A \otimes e^\#)$. However, using the decomposition of $A \otimes_R A$ above and the fact that $e^\# \otimes id_A$ is injective on $R \cdot 1 \otimes I$ and $R \cdot 1 \otimes R \cdot 1$ (it actually induces the canonical isomorphisms $R \cdot 1 \otimes I \xrightarrow{\sim} I \subseteq A$ and $R \cdot 1 \otimes R \cdot 1 \xrightarrow{\sim} R \cdot 1 \subseteq A$), we see that $\ker(e^\# \otimes id_A) = (I \otimes I) \oplus (I \otimes R \cdot 1) = I \otimes A \subseteq A \otimes A$. Similarly, $\ker(id_A \otimes e^\#) = (I \otimes I) \oplus (R \cdot 1 \otimes I) = A \otimes I \subseteq A \otimes A$. Thus, $\ker(e^\# \otimes id_A) \cap \ker(id_A \otimes e^\#) = I \otimes I$, proving the claim.

(c) For $a \in I$, (b) implies that $m^{\#}(a) = a \otimes 1 + 1 \otimes a + x$, with $x \in I \otimes I$. By the inverse axiom, we get: $0 = e^{\#}(a) = (\Delta^{\#} \circ (i^{\#} \otimes id_A) \circ m^{\#})(a) = i^{\#}(a) + a + y$, where, if $x = \sum_i c_i \otimes d_i$, with $c_i, d_i \in I$, then $y = \sum_i i^{\#}(c_i) d_i \in I$. In order to conclude, we need to show that actually $y \in I^2$; for this, it is sufficient that $i^{\#}(c_i) \in I$ for all the c_i's. Let us then prove that in general, for $c \in I$, we have $i^{\#}(c) \in I$ again. The same computation as above shows that $0 = i^{\#}(c) + c + z$, with $z \in I$. Applying $e^{\#}$ to this equality we get $0 = e^{\#}(i^{\#}(c))$ (as both $c \in I$ and $z \in I$), i.e. $i^{\#}(c) \in \ker e^{\#} = I$. This concludes the proof. □

In the affine case, we can also give a very explicit description of the sheaf of differentials of a group scheme $G = \operatorname{Spec} A$ over a ring R, thanks to the following fact.

Proposition 2.7. *Let $(A, m^{\#}, i^{\#}, e^{\#})$ be a Hopf algebra over a ring R, $I = \ker e^{\#}$, $\Omega^1_{A/R}$ the A-module of R-differentials. Then, there is an isomorphism of A-modules:*

$$\Omega^1_{A/R} \cong (I/I^2) \otimes_R A,$$

where the right-hand side is an A-module by multiplication on the right factor.

Proof. Let $(m^{\#}, i_2)\colon A \otimes_R A \to A \otimes_R A$ be the map of R-algebras given by the two maps $m^{\#}\colon A \to A \otimes_R A$ and $i_2\colon A \to A \otimes_R A$, $i_2(b) = 1 \otimes b$; thus, $(m^{\#}, i_2)(a \otimes b) = m^{\#}(a) \cdot (1 \otimes b)$. If we consider $A \otimes_R A$ as an A-module via multiplication on the right factor, then $(m^{\#}, i_2)$ is an A-linear homomorphism. Note that, actually, $(m^{\#}, i_2)$ is the ring homomorphism corresponding to the universal right translation τ of the group scheme $\operatorname{Spec} A$ over $\operatorname{Spec} R$. In particular, it is an isomorphism. Now, using the neutral element axiom, we have a commutative diagram of A-linear homomorphisms:

$$\begin{tikzcd} A \otimes_R A \arrow[r, "{(m^{\#}, i_2)}"] \arrow[d, "\Delta^{\#}"] & A \otimes_R A \arrow[d, "e^{\#} \otimes id_A"] \\ A \arrow[r, "j_1"] & R \otimes_R A. \end{tikzcd}$$

Since the top and bottom maps are isomorphisms, this induces an isomorphism of sub-A-modules: $\ker \Delta^{\#} \cong \ker(e^{\#} \otimes id_A)$, where, as in the previous proof, $\ker(e^{\#} \otimes id_A) = I \otimes A$. Also recall that, with this A-module structure, we have $\Omega^1_{A/R} \cong (\ker \Delta^{\#})/(\ker \Delta^{\#})^2$. Therefore, $\Omega^1_{A/R} \cong (I \otimes A)/(I \otimes A)^2 \cong (I/I^2) \otimes A$, the last equality via the obvious isomorphism, which is easily A-linear. □

Remark 2.8. In the same setting as above, let $\pi\colon G = \operatorname{Spec} A \to \operatorname{Spec} R = S$ be the group scheme over R associated to the Hopf algebra A, with identity section $e\colon S \to G$. Then, by the proposition:

$$\begin{aligned} e^*\Omega^1_{G/S} = \Omega^1_{A/R} \otimes_{A,e^{\#}} R \cong ((I/I^2) \otimes_R A) \otimes_{A,e^{\#}} R \cong \\ \cong (I/I^2) \otimes_R (A \otimes_{A,e^{\#}} R) \cong (I/I^2) \otimes_R R \cong (I/I^2) \end{aligned}$$

as R-modules. But then:

$$\pi^* e^* \Omega^1_{G/S} = e^* \Omega^1_{G/S} \otimes_R A \cong (I/I^2) \otimes_R A \cong \Omega^1_{G/S}.$$

The outcome is that the sheaf of differentials of G over S is determined by its shape around the identity e, so we can expect that, under the right hypothesis, properties like smoothness need just to be checked around e (and this is indeed the case, as we will see later on).

Corollary 2.9. *Let $(A, m^\#, i^\#, e^\#)$ be a Hopf algebra over a field k, $I = \ker e^\#$. Then the A-module of k-differentials $\Omega^1_{A/k}$ is free.*

Proof. Since k is a field, I/I^2 is free over k. Thus, $\Omega^1_{A/k}$ is free over A by last proposition and the fact that tensor product commutes with direct sum. □

Definition 2.10. Let k be a field and $G = \operatorname{Spec} A$ a finite group scheme over k, meaning that the morphism $G \to \operatorname{Spec} k$ is finite, i.e. A is a finite k-algebra. Then, the *rank* of G is the dimension of A as a k-vector space.

Examples

Let us now list some basic examples of affine group schemes.

The additive group. Let R be a ring. The additive group scheme over R, denoted by $\mathbb{G}_{a,R}$, is given, as a scheme, by $\operatorname{Spec} R[x]$. For every scheme T over R, we have $\mathbb{G}_{a,R}(T) = \mathcal{O}_T(T)$, with its additive group structure. The group scheme structure is given, on the level of rings, by the following homomorphisms of R-algebras:

$$\begin{aligned} &m^\# \colon R[x] \to R[x] \otimes_R R[x], && x \mapsto x \otimes 1 + 1 \otimes x; \\ &i^\# \colon R[x] \to R[x], && x \mapsto -x; \\ &e^\# \colon R[x] \to R, && x \mapsto 0. \end{aligned}$$

The additive group scheme over $\mathbb{Z}$ is just denoted by $\mathbb{G}_a$. Note that, for every ring R, $\mathbb{G}_{a,R}$ can be obtained as the pull-back of $\mathbb{G}_a$, i.e. $\mathbb{G}_{a,R} = \mathbb{G}_a \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$.

The multiplicative group. Let R be a ring. The multiplicative group scheme over R, denoted by $\mathbb{G}_{m,R}$, is given, as a scheme, by $\operatorname{Spec} R[x, x^{-1}]$. For every scheme T over R, we have $\mathbb{G}_{m,R}(T) = \mathcal{O}_T(T)^\times$, with its multiplicative group structure. The group scheme structure is given, on the level of rings, by the following homomorphisms of R-algebras:

$$\begin{aligned} &m^\# \colon R[x, x^{-1}] \to R[x, x^{-1}] \otimes_R R[x, x^{-1}], && x \mapsto x \otimes x; \\ &i^\# \colon R[x, x^{-1}] \to R[x, x^{-1}], && x \mapsto x^{-1}; \\ &e^\# \colon R[x, x^{-1}] \to R, && x \mapsto 1. \end{aligned}$$

The multiplicative group scheme over $\mathbb{Z}$ is just denoted by $\mathbb{G}_m$. Note that, for every ring R, $\mathbb{G}_{m,R}$ can be obtained as the pull-back of $\mathbb{G}_m$, i.e. $\mathbb{G}_{m,R} = \mathbb{G}_m \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$.

The group of n-th roots of unity. Let R be a ring and n a positive integer. Then the scheme $\mu_{n,R} = \operatorname{Spec} R[x]/(x^n - 1)$ is a closed subgroup scheme of $\mathbb{G}_{m,R}$. Its functor of points associates, to every scheme T over R, the subgroup of $\mathbb{G}_{m,R}(T) = \mathcal{O}_T(T)^\times$ consisting of n-th roots of unity. Once again, we have $\mu_{n,R} = \mu_{n,\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} R$.

Note that $\mu_{n,R} = \ker[n]$, where $[n]\colon \mathbb{G}_{m,R} \to \mathbb{G}_{m,R}$ is the *n-th power map*, given, on the level of rings, by $[n]^\#\colon R[x, x^{-1}] \to R[x, x^{-1}]$, $x \mapsto x^n$.

If $R = k$ is a field, then $\mu_{n,k}$ is a finite group scheme of rank n. Interestingly enough, for every field k of characteristic p: $\mu_{p^n,k} = \operatorname{Spec} k[x]/(x^{p^n} - 1) = \operatorname{Spec} k[x]/(x-1)^{p^n}$ consists of just one point; nevertheless, it is not the trivial group scheme.

The group of p^n-th roots of zero. Let p be a prime number and k a field of characteristic p. Then the scheme $\alpha_{p^n,k} = \operatorname{Spec} k[x]/(x^{p^n})$ is a closed subgroup scheme of $\mathbb{G}_{a,k}$, finite of rank p^n. For every scheme T over k, we have $\alpha_{p^n,k}(T) = \{\, x \in \mathcal{O}_T(T) \mid x^{p^n} = 0 \,\}$, with the additive group structure (as a subgroup of $\mathbb{G}_{a,k}(T) = O_T(T)$). In this case, $\alpha_{p^n,k} = \alpha_{p^n} \times_{\operatorname{Spec}\mathbb{F}_p} \operatorname{Spec} k$, where $\alpha_{p^n} = \alpha_{p^n,\mathbb{F}_p}$.

Note that $\alpha_p = \ker F$, where $F\colon \mathbb{G}_{a,\mathbb{F}_p} \to \mathbb{G}_{a,\mathbb{F}_p}$ is the *Frobenius* endomorphism of $\mathbb{G}_{a,\mathbb{F}_p}$, given, on the level of rings, by $F^\#\colon \mathbb{F}_p[x] \to \mathbb{F}_p[x]$, $F^\#(x) = x^p$.

As in the previous example, $\alpha_{p^n,k}$ consists of only one point, but, as a group scheme, it is not trivial.

Constant group schemes. Let R be a ring and Γ an abstract finite group. Consider the scheme Γ_R given by the disjoint union of copies of $\operatorname{Spec} R$ indexed by Γ, i.e. $\Gamma_R = \coprod_{\gamma\in\Gamma}(\operatorname{Spec} R) = \operatorname{Spec}(\prod_{\gamma\in\Gamma} R)$. The group structure of Γ induces maps $m\colon \Gamma_R \times_{\operatorname{Spec} R} \Gamma_R \to \Gamma_R$, $i\colon \Gamma_R \to \Gamma_R$, $e\colon \operatorname{Spec} R \to \Gamma_R$, which just move (or choose, in the case of e) the components of Γ_R according to the group law of Γ, giving Γ_R the structure of a group scheme. On the rings, these maps are given by:

$$
\begin{aligned}
m^\#&\colon \prod_{\gamma\in\Gamma} R \to \Big(\prod_{\gamma'\in\Gamma} R\Big) \otimes_R \Big(\prod_{\gamma''\in\Gamma} R\Big), && t_\gamma \mapsto \sum_{\gamma',\gamma''\in\Gamma \mid \gamma'\gamma''=\gamma} t_{\gamma'} \otimes t_{\gamma''};\\
i^\#&\colon \prod_{\gamma\in\Gamma} R \to \prod_{\gamma\in\Gamma} R, && t_\gamma \mapsto t_{\gamma^{-1}};\\
e^\#&\colon \prod_{\gamma\in\Gamma} R \to R, && \text{projecting on the } 1_\Gamma\text{-factor;}
\end{aligned}
$$

where $t_\gamma = (0, \dots, 0, 1, 0, \dots, 0) \in \prod_{\gamma\in\Gamma} R$, with 1 in the γ-component. We have $\Gamma_R = \Gamma_\mathbb{Z} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} R$.

Note that, for a connected scheme T over R, we have $\Gamma_R(T) \cong \Gamma$ as groups (because any map $T \to \Gamma_R$ must factor through one of the components of Γ_R).

If $R = k$ is a field, then Γ_k is clearly a finite k-group scheme of rank the order of Γ.

Diagonalizable group schemes. Let R be a ring and Γ an abelian group. Then we can form the group algebra $R[\Gamma]$. As an R-module, $R[\Gamma]$ is the free R-module over Γ, hence elements are finite sums $\sum_{\gamma\in\Gamma} a_\gamma \cdot \gamma$, with $a_\gamma \in R$; multiplication is defined extending by R-linearity the rule $(1\cdot\gamma)(1\cdot\gamma') = 1\cdot\gamma\gamma'$. This has the structure of a Hopf algebra over R, given by the following maps:

$$\begin{aligned} &m^{\#}\colon R[\Gamma] \to R[\Gamma]\otimes_R R[\Gamma], && \gamma\mapsto\gamma\otimes\gamma;\\ &i^{\#}\colon R[\Gamma]\to R[\Gamma], && \gamma\mapsto\gamma^{-1};\\ &e^{\#}\colon R[\Gamma]\to R, && \gamma\mapsto 1.\end{aligned}$$

Therefore, the scheme $D_R(\Gamma) := \operatorname{Spec} R[\Gamma]$ is a group scheme over R. For every scheme T over R, we have $D_R(\Gamma)(T) \cong \operatorname{Hom}_{R\text{–}\mathsf{Alg}}(R[\Gamma], \mathcal{O}_T(T)) \cong \operatorname{Hom}_{\mathsf{Ab}}(\Gamma, \mathcal{O}_T(T)^\times)$.

Since $R[\Gamma]\cong\mathbb{Z}[\Gamma]\otimes_{\mathbb{Z}} R$, we have $D_R(\Gamma) = D_{\mathbb{Z}}(\Gamma)\times_{\operatorname{Spec}\mathbb{Z}}\operatorname{Spec} R$. Moreover, observe that $\mathbb{G}_{m,R}\cong D_R(\mathbb{Z})$ and, for every positive integer n, $\mu_{n,R}\cong D_R(\mathbb{Z}/n\mathbb{Z})$.

If $R = k$ is a field and Γ is a finite abelian group, then $D_k(\Gamma)$ is a finite k-group scheme of rank the order of Γ.

The general linear group. Let R be a ring and n a positive integer. The general linear group over R can be turned into a group scheme, as $\mathrm{GL}_{n,R} = \operatorname{Spec} R[x_{ij}, \det^{-1} | 1\le i,j\le n]$, where $\det\in R[x_{ij}|1\le i,j\le n]$ is the determinant polynomial. For every scheme T over R, $\mathrm{GL}_{n,R}(T)$ is the group of invertible $n\times n$ matrices with entries in $\mathcal{O}_T(T)$. The group scheme structure is given, on the level of rings, by the following homomorphisms of R-algebras (set $A := R[x_{ij}, \det^{-1}|1\le i,j\le n]$):

$$\begin{aligned} &m^{\#}\colon A\to A\otimes_R A, && x_{ij}\mapsto\sum_{k=1}^n x_{ik}\otimes x_{kj};\\ &i^{\#}\colon A\to A, && x_{ij}\mapsto y_{ij}\quad\text{(see below)};\\ &e^{\#}\colon A\to R, && x_{ij}\mapsto\begin{cases}1 & \text{if } i=j,\\ 0 & \text{if } i\neq j;\end{cases}\end{aligned}$$

here $y_{ij}\in A$ is the polynomial giving the (i,j)-entry of the inverse matrix.

Note that, for every ring R, the general linear group over R can be obtained as the pull-back of $\mathrm{GL}_{n,\mathbb{Z}}$, i.e. $\mathrm{GL}_{n,R} = \mathrm{GL}_{n,\mathbb{Z}}\times_{\operatorname{Spec}\mathbb{Z}}\operatorname{Spec} R$.

Generalisation

The preceding discussion has the following generalisation. Suppose that S is an arbitrary scheme and $\pi\colon G\to S$ is an affine S-group scheme, meaning that the map π is affine. Then, as we saw in the previous chapter, G may be viewed as an $\mathcal{O}_S$-algebra $\mathscr{A} = \pi_*\mathcal{O}_G$. The morphisms m, i, e constituting the group structure correspond to homomorphisms of $\mathcal{O}_S$-algebras:

$$m^{\#}\colon\mathscr{A}\to\mathscr{A}\otimes_{\mathcal{O}_S}\mathscr{A},\quad i^{\#}\colon\mathscr{A}\to\mathscr{A},\quad e^{\#}\colon\mathscr{A}\to\mathcal{O}_S$$

making the dual diagrams of those from definition 2.1 commute. This turns $\mathscr{A}$ into a sheaf of Hopf algebras over $\mathcal{O}_S$, in the sense that $\mathscr{A}(U)$ is a Hopf algebra over $\mathcal{O}_S(U)$ for all open subsets $U\subseteq S$.

This situation is achieved, for instance, starting from an affine group scheme $G = \operatorname{Spec} A$ over an affine base $\operatorname{Spec} R$, then taking any scheme S over R and considering the pull-back $G_S = G \times_{\operatorname{Spec} R} S$. Since the property of being affine is stable under base change, G_S is affine over S.

2.3 Group schemes over a field, smoothness

In this section, we fix the base scheme S to be the spectrum of an arbitrary field k and derive some consequences from this assumption, in terms of smoothness of group schemes over $S = \operatorname{Spec} k$.

Proposition 2.11. *Let G be a group scheme locally of finite type over a field k. Fix an algebraic closure $\bar{k}$ of k. For any field extension K of k, write $G_K = G \times_{\operatorname{Spec} k} \operatorname{Spec} K$ and let $e_K \colon \operatorname{Spec} K \to G_K$ denote the identity section of G_K. Then, the following are equivalent:*

(a1) G *is smooth over* k*;*

(b1) $G_{\bar{k}}$ *is regular;*

(c1) $G_{\bar{k}}$ *is reduced;*

(d1) G_K *is reduced for some perfect field* K *containing* k*;*

(a2) G *is smooth over* k *at* e*;*

(b2) $\mathcal{O}_{G_{\bar{k}}, e_{\bar{k}}} \cong \mathcal{O}_{G,e} \otimes_k \bar{k}$ *is regular;*

(c2) $\mathcal{O}_{G_{\bar{k}}, e_{\bar{k}}} \cong \mathcal{O}_{G,e} \otimes_k \bar{k}$ *is reduced;*

(d2) $\mathcal{O}_{G_K, e_K} \cong \mathcal{O}_{G,e} \otimes_k K$ *is reduced for some perfect field* K *containing* k*.*

Suppose, moreover, that $G = \operatorname{Spec} A$ is affine. Then, if K is a perfect field containing k, $\bar{A} := A \otimes_k K$ is the Hopf algebra corresponding to the group scheme G_K, $\mathfrak{N}$ its nilradical and $\bar{\mathfrak{m}}$ its augmentation ideal (since K is a field, $\bar{\mathfrak{m}} = \ker(e_K^\# \colon \bar{A} \to K)$ is a maximal ideal), the above properties are also equivalent to the following:

(e) $\operatorname{rk}_A \Omega^1_{A/k} = \dim G$*;*

(f) $\mathfrak{N} \subseteq \bar{\mathfrak{m}}^2$.

Proof. (a1)⇔(b1). This is a general fact in Algebraic Geometry, which can be found, e.g., in [11, 6.28].

(b1)⇔(c1). Clearly, if $G_{\bar{k}}$ is regular, then it is also reduced. Conversely, suppose that $G_{\bar{k}}$ is reduced. Then, the points where it is smooth over $\bar{k}$ form an open and dense subset (cf [11, 6.19]). In particular, there exists a closed point $p \in G_{\bar{k}}$ such that $G_{\bar{k}}$ is smooth at p (here, because $G_{\bar{k}}$ is locally of finite type over $\bar{k}$, then every non-empty locally closed subset contains a closed point, see [11, 3.35]). This implies that the local ring $\mathcal{O}_{G_{\bar{k}},p}$ is regular (smoothness implies regularity, cf [11, 6.26]). Now, since $\bar{k}$ is algebraically closed, p corresponds to a $\bar{k}$-valued point (that we still denote by p). But then, for evey other closed point $q \in G_{\bar{k}}$, corresponding to a $\bar{k}$-valued point $q \in G_{\bar{k}}(\operatorname{Spec} \bar{k})$, the translation $t_q \circ t_{p^{-1}} = t_{p^{-1}q} \colon G_{\bar{k}} \to G_{\bar{k}}$ induces an isomorphism between the stalks $\mathcal{O}_{G_{\bar{k}},p} \xrightarrow{\sim} \mathcal{O}_{G_{\bar{k}},q}$, so the local ring $\mathcal{O}_{G_{\bar{k}},q}$ is regular too. Thus, $G_{\bar{k}}$ is regular at all the closed points, but, $G_{\bar{k}}$ being locally of finite type over a field, this proves that it is a regular scheme (because the localization of a regular local ring at a prime ideal is again regular and, by [11, 3.35], every point specializes to a closed point).

(c1)⇔(d1). It is a fact in Algebraic Geometry that, for schemes over a field, both these notions are equivalent to G being geometrically reduced, i.e. $G \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ reduced for every field extension $k \subseteq k'$ (cf [11, 5.49] and [11, 5.54]).

(a2)⇔(b2). Recall that G is smooth at e if and only if all the local rings of the points $\bar{e} \in G_{\bar{k}}$ lying over e are regular (see [11, 6.28] again). However, since e is a k-valued point, its fibre in $G_{\bar{k}}$ consists of just one point with residue field $\bar{k}$, namely the $\bar{k}$-valued point defined by $e_{\bar{k}} \colon \operatorname{Spec} \bar{k} \to G_{\bar{k}}$ (indeed, such fibre is $\operatorname{Spec} k \times_{e,G} G \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k} \cong \operatorname{Spec} \bar{k}$). Thus, we conclude observing that the local ring $\mathcal{O}_{G_{\bar{k}},e_{\bar{k}}}$ is isomorphic to $\mathcal{O}_{G,e} \otimes_k \bar{k}$. To see this, let $\operatorname{Spec} A$ be an affine neighbourhood of e in G, let $A = k[x_1, \dots, x_n]/(f_1, \dots, f_r)$ be a presentation of A and let $\mathfrak{m} = (x_1 - \alpha_1, \dots, x_n - \alpha_n)$ be the maximal ideal of A defined by e. Then, the maximal ideal of $A \otimes_k \bar{k}$ defined by $e_{\bar{k}}$ corresponds to the maximal ideal $\bar{\mathfrak{m}} = (x_1 - \alpha_1, \dots, x_n - \alpha_n)$ generated by $\mathfrak{m}$ in $\bar{k}[x_1, \dots, x_n]/(f_1, \dots, f_r)$, under the identification:

$$A \otimes_k \bar{k} = k[x_1, \dots, x_n]/(f_1, \dots, f_r) \otimes_k \bar{k} \xrightarrow{\sim} \bar{k}[x_1, \dots, x_n]/(f_1, \dots, f_r).$$

This last identification, however, extends to an isomorphism:

$$(k[x_1, \dots, x_n]/(f_1, \dots, f_r))_{\mathfrak{m}} \otimes_k \bar{k} \xrightarrow{\sim} (\bar{k}[x_1, \dots, x_n]/(f_1, \dots, f_r))_{\bar{\mathfrak{m}}},$$

i.e. $\mathcal{O}_{G,e} \otimes_k \bar{k} \cong \mathcal{O}_{G_{\bar{k}},e_{\bar{k}}}$.

(b1)⇔(b2). We already observed in ((b1)⇔(c1)) that it suffices to check regularity at closed points and that the stalks at the closed points of $G_{\bar{k}}$ ($\mathcal{O}_{G_{\bar{k}},e_{\bar{k}}}$ is one of them) are all isomorphic via translations.

(c1)⇔(c2). Since the localization of a reduced ring is again reduced and $G_{\bar{k}}$ is locally of finite type over a field, it suffices to check reducedness at closed points (with the same argument as in ((b1)⇔(c1)) for regularity). But, once again, the stalks at the closed points of $G_{\bar{k}}$ are all isomorphic to $\mathcal{O}_{G_{\bar{k}},e_{\bar{k}}}$ via translations.

(c2)⇔(d2). As in ((c1)⇔(d1)) (consider the k-scheme $\operatorname{Spec} \mathcal{O}_{G,e}$). Note that $\mathcal{O}_{G_K,e_K} \cong \mathcal{O}_{G,e} \otimes_k K$, by the same argument as in ((a2)⇔(b2)).

(a1)⇔(e). Since G is of finite type over a field, smoothness is equivalent to $\Omega^1_{A/k}$ being locally free of rank $\dim G$ (cf [1, II, 5.5]). However, we already saw in corollary 2.9 that $\Omega^1_{A/k}$ is free, so the condition on the rank is sufficient for G to be smooth.

(f)⇒(d2). Let $G_{K,red} = \operatorname{Spec} \bar{A}/\mathfrak{N}$ be the reduced closed subscheme of G_K and note that $G_{K,red}$ is a subgroup scheme (one can see this using the universal property of the reduced closed subscheme of a scheme; the only non trivial thing to check is that $G_{K,red} \times_{\operatorname{Spec} K} G_{K,red}$ is again reduced, which boils down to the fact that K is perfect, see [11, 5.49]). Let $e_{K,red}$ denote its identity section and $\bar{\mathfrak{m}}_{red} = \bar{\mathfrak{m}}/\mathfrak{N}$ the maximal ideal that this defines in $\bar{A}/\mathfrak{N}$. Note that, since the prime ideals of $\bar{A}$ and those of $\bar{A}/\mathfrak{N}$ are in one-to-one, order preserving correspondence, $\bar{\mathfrak{m}}$ and $\bar{\mathfrak{m}}_{red}$ have the same height; in particular, $\dim \bar{A}_{\bar{\mathfrak{m}}} = \dim(\bar{A}/\mathfrak{N})_{\bar{\mathfrak{m}}_{red}}$. Now, since $G_{K,red}$ is reduced over a perfect field, then it is smooth at $e_{K,red}$ by ((d1)⇒(a2)), so $(\bar{A}/\mathfrak{N})_{\bar{\mathfrak{m}}_{red}} = \mathcal{O}_{G_{K,red},e_{K,red}}$ is regular. Hence, $\dim(\bar{A}/\mathfrak{N})_{\bar{\mathfrak{m}}_{red}} = \dim_K(\bar{\mathfrak{m}}_{red}/\bar{\mathfrak{m}}^2_{red})$. Furthermore, the hypothesis $\mathfrak{N} \subseteq \bar{\mathfrak{m}}^2$ implies that the surjection $\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2 \to (\bar{\mathfrak{m}}/\mathfrak{N})/(\bar{\mathfrak{m}}/\mathfrak{N})^2 = \bar{\mathfrak{m}}_{red}/\bar{\mathfrak{m}}^2_{red}$ is also injective, hence it is an isomorphism of K-vector spaces. Thus, $\dim_K(\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2) = \dim_K(\bar{\mathfrak{m}}_{red}/\bar{\mathfrak{m}}^2_{red})$. Altogether, we have:

$$\dim_K(\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2) = \dim_K(\bar{\mathfrak{m}}_{red}/\bar{\mathfrak{m}}^2_{red}) = \dim(\bar{A}/\mathfrak{N})_{\bar{\mathfrak{m}}_{red}} = \dim \bar{A}_{\bar{\mathfrak{m}}},$$

with $\bar{A}_{\bar{\mathfrak{m}}} = \mathcal{O}_{G_K,e_K}$, which is therefore regular, hence reduced.

(c1)⇒(f). Obvious, by the observation in ((c1)⇔(d1)). □

Remark 2.12. In the case of G an affine group scheme over a field k, we could prove ((a1)⇔(a2)) directly, using the criterion for smoothness in terms of the sheaf of differentials, as in ((a1)⇔(e)), together with the observation of remark 2.8.

Thanks to the proposition just shown, we can proceed to prove the main theorem of this section.

Theorem 2.13 (Cartier, affine case)**.** *Let $G = \operatorname{Spec} A$ be an affine group scheme of finite type over a field k of characteristic zero. Then G is smooth over k.*

Proof. Let $\mathfrak{m}$ denote the augmentation ideal of A and $\mathfrak{N}$ the nilradical of A. Since fields of characteristic zero are perfect, we only need to prove that $\mathfrak{N} \subseteq \mathfrak{m}^2$, by ((f)⇔(a1)) in the previous proposition. Let a be a nilpotent element of A; we distinguish two cases. If a maps to zero in $A_{\mathfrak{m}}$, then it maps to zero in $A_{\mathfrak{m}}/(\mathfrak{m}A_{\mathfrak{m}})^2 \cong A/\mathfrak{m}^2$, i.e. $a \in \mathfrak{m}^2$ and we are done. Suppose, otherwise, that a does not map to zero in $A_{\mathfrak{m}}$. The image of a in $A_{\mathfrak{m}}$ is, however, still nilpotent; let $n \geq 2$ be the smallest integer such that $a^n = 0$ in $A_{\mathfrak{m}}$, so that $a^{n-1} \neq 0$ in $A_{\mathfrak{m}}$. Then $\exists s \in A \setminus \mathfrak{m}$ such that $sa^n = 0$ in A; replacing a with sa, we have $a^n = 0$ in A, but still $a^{n-1} \neq 0$ in $A_{\mathfrak{m}}$ (note that if we prove that $sa \in \mathfrak{m}^2$, then sa maps to zero in $A/\mathfrak{m}^2$, but $A/\mathfrak{m}^2$ is a local ring with maximal ideal $\mathfrak{m}/\mathfrak{m}^2$, so the image of s in $A/\mathfrak{m}^2$ is a unit, implying that a is zero in $A/\mathfrak{m}^2$, i.e. $a \in \mathfrak{m}^2$). Since $a \in \mathfrak{m}$ (being nilpotent), by proposition 2.6(b) we have:

$$m^{\#}(a) = a \otimes 1 + 1 \otimes a + y,$$

with $y \in \mathfrak{m} \otimes \mathfrak{m}$, where $m^\#$ is the co-multiplication of A. Then:

$$0 = m^\#(a^n) = m^\#(a)^n = (a \otimes 1 + 1 \otimes a + y)^n = \sum_{i=0}^{n} \binom{n}{i} (a \otimes 1)^{n-i}(1 \otimes a + y)^i.$$

Here, the term with $i = 0$ vanishes, as $a^n = 0$; the term with $i = 1$ is $n(a^{n-1} \otimes 1)(1 \otimes a + y) = n(a^{n-1} \otimes a) + n(a^{n-1} \otimes 1)y$; the terms with $i \geq 2$ contain a factor $(1 \otimes a + y)^i \in A \otimes \mathfrak{m}^2$. It follows that $n(a^{n-1} \otimes a) + n(a^{n-1} \otimes 1)y \in A \otimes \mathfrak{m}^2$, hence:

$$n(a^{n-1} \otimes a) \in n(a^{n-1} \otimes 1)y + A \otimes \mathfrak{m}^2 \subseteq a^{n-1}\mathfrak{m} \otimes A + A \otimes \mathfrak{m}^2.$$

Projecting everything onto $A \otimes_k A/\mathfrak{m}^2$, we get:

$$n(a^{n-1} \otimes \bar{a}) \in a^{n-1}\mathfrak{m} \otimes A/\mathfrak{m}^2, \tag{2.1}$$

where $\bar{a}$ denotes the image of a in $A/\mathfrak{m}^2$; note that $n(a^{n-1} \otimes \bar{a}) = na^{n-1} \otimes \bar{a}$. Now, since k has characteristic zero, n is a unit in A, so (2.1) is equivalent to $a^{n-1} \otimes \bar{a} \in a^{n-1}\mathfrak{m} \otimes A/\mathfrak{m}^2$. This holds only if $a^{n-1} \otimes \bar{a}$ is zero in the quotient $(A \otimes A/\mathfrak{m}^2)/(a^{n-1}\mathfrak{m} \otimes A/\mathfrak{m}^2) \cong (A/a^{n-1}\mathfrak{m}) \otimes A/\mathfrak{m}^2$ (remember that we are working with k-vector spaces), which is true when either $a^{n-1} = 0 \in A/a^{n-1}\mathfrak{m}$ or when $\bar{a} = 0 \in A/\mathfrak{m}^2$. However, the first case is impossible, because if $a^{n-1} = a^{n-1}m$ for some $m \in \mathfrak{m}$, then $a^{n-1}(1 - m) = 0$, which implies $a^{n-1} = 0 \in A_\mathfrak{m}$ (as $(1 - m)$ is a unit in $A_\mathfrak{m}$), contradicting our assumptions. Therefore, $\bar{a}$ is zero in $A/\mathfrak{m}^2$, i.e. $a \in \mathfrak{m}^2$, as required. □

Remark 2.14. The theorem is actually true in bigger generality, namely for all group schemes locally of finite type over a field of characteristic zero (see [20] for a proof).

The hypothesis on the characteristic of k, instead, is definitey necessary. Indeed, examples like $\mu_{p^n,k}$ and $\alpha_{p^n,k}$, with k a field of positive characteristic p, are finite over k, but still definitely not reduced.

Since later we will concentrate on group schemes that are finite over their base, let us observe that, in such situation, Cartier's theorem takes the form of the following corollary (recall that a finite and smooth morphism is étale, cf [1, II, 1.4]).

Corollary 2.15. *Let $G = \operatorname{Spec} A$ be a finite group scheme over a field k of characteristic zero. Then G is étale over k.*

2.4 Fppf torsors

The topics of this section are better explained using the abstract language of sheaves on a site. More precisely, as we did in remark 2.2, we may identify any scheme X over a base scheme S with its functor of points $\mathsf{Sch}/S \to \mathsf{Set}$, $T \mapsto \mathrm{Hom}_{\mathsf{Sch}/S}(T, X)$, via the Yoneda embedding. Now, if we equip the category Sch/S of schemes over S with a suitable Grothendieck topology (any *subcanonical* topology), the functor of points of any S-scheme is actually a sheaf with respect to such topology. Thus, the Yoneda embedding induces a fully faithful functor $\mathsf{Sch}/S \to \mathrm{Sh}(\mathsf{Sch}/S)$, to the category of sheaves of sets on Sch/S; we will use the same name for an object in Sch/S and the sheaf that it represents. What remark 2.2 says is that an S-group scheme is simply an S-scheme that gives rise, via this embedding, to a sheaf of groups.

In what follows, we will work in the bigger category $\mathrm{Sh}(\mathsf{Sch}/S)$, specifying what happens when the sheaves concerned are representable, i.e. they actually come from objects in Sch/S.

Fix a scheme S and consider the *fppf* topology on the category Sch/S of schemes over S: coverings of an S-scheme T are families of S-morphisms $\{ f_i : U_i \to T \}_i$ such that $\bigcup_i f_i(U_i) = T$ and each f_i is flat and locally of finite presentation. Equivalently, coverings are S-morphisms $f : U \to T$ that are faithfully flat and locally of finite presentation (take $U = \coprod_i U_i$ from above).

Definition 2.16. Let G be a sheaf of groups on Sch/S and X a sheaf of sets on Sch/S. An *action* of G on X is a morphism of sheaves $\rho : G \times X \to X$ such that, for all S-schemes T, the map of sets $\rho_T : (G \times X)(T) = G(T) \times X(T) \to X(T)$ is a $G(T)$-action on $X(T)$. We say that (X, ρ) (or just X, when the action is understood) is a *G-sheaf*.

A homomorphism between two G-sheaves (X, ρ) and (Y, σ) is a morphism of sheaves $f : X \to Y$ such that, for all S-schemes T, the map of sets $f_T : X(T) \to Y(T)$ is equivariant for the $G(T)$-actions. We say that f is *G-equivariant*.

A G-sheaf (X, ρ) is an *fppf G-torsor* over S (in some literature, e.g. [19], a *principal homogeneous space* for G) if:

(a) there exists a covering $U \to S$ such that $X(U) \neq \emptyset$;

(b) for every S-scheme T, the $G(T)$-action on $X(T)$ is simply transitive, i.e. free and transitive; equivalently, the *graph* morphism:

$$\Omega := (\rho, pr_2) : G \times X \to X \times X$$

is an isomorphism of sheaves.

Since we will only work with the fppf topology, we will just talk about *torsors*, without specifying "fppf". We denote by $\mathrm{Tors}(S, G)$ the category whose objects are G-torsors over S and whose morphisms are G-equivariant morphisms of sheaves.

Remark 2.17. Any morphism of schemes $g : S' \to S$ induces a functor $\mathsf{Sch}/S' \to \mathsf{Sch}/S$ (because every S'-scheme is also an S-scheme via composition with g) and hence a restriction functor $\mathrm{Sh}(\mathsf{Sch}/S) \to \mathrm{Sh}(\mathsf{Sch}/S')$, which consists just in precomposition with the previous functor. If X is a sheaf on Sch/S, we denote by $X|_{S'}$ the resulting sheaf on Sch/S'. Note that, for representable sheaves, this is just the usual pull-back via g; in other words, if X is a scheme, then $X|_{S'} = X \times_S S'$.

If G is a sheaf of groups on Sch/S, then $G|_{S'}$ is a sheaf of groups on Sch/S', generalising thus remark 2.4. Similarly, the restriction of a G-sheaf (X, ρ) will give rise to a $G|_{S'}$-sheaf $(X|_{S'}, \rho|_{S'})$ and the property of being a torsor is preserved. In particular, we have an induced functor $\mathrm{Tors}(S, G) \to \mathrm{Tors}(S', G|_{S'})$.

Remark 2.18. Note that a sheaf of groups G is itself a G-torsor, under the action given by multiplication, i.e., for any S-scheme T:

$$\begin{aligned} m_T \colon G(T) \times G(T) &\to G(T) \\ (h, g) &\mapsto h \cdot g. \end{aligned}$$

Indeed, we have $e \in G(S) \neq \emptyset$, where e denotes the identity of $G(S)$, and obviously G acts simply transitively on itself. Every G-torsor isomorphic (as G-sheaves) to G is called a *trivial* G-torsor.

Now, suppose that (X, ρ) is a G-torsor and let $U \to S$ be a covering such that $X(U) \neq \emptyset$. Then, we have that $(X|_U, \rho|_U)$ is isomorphic to $G|_U$ as $G|_U$-sheaves, i.e., $X|_U$ is a trivial $G|_U$-torsor. To justify this, choose a base point $x \in X(U)$. Then, for every U-scheme T, the map:

$$\begin{aligned} G|_U(T) = G(T) &\to X(T) = X|_U(T) \\ g &\mapsto g \cdot x|_T \end{aligned}$$

is an equivariant bijection, which is compatible with the restriction maps of $G|_U$ and $X|_U$. Therefore, it defines an isomorphism of $G|_U$-sheaves: $G|_U \cong X|_U$. Note that such an isomorphism is not canonical, but it depends on the choice of a base point.

Representable case. If $G \to S$ is a group scheme over S and $X \to S$ is an S-scheme, then, viewing them as sheaves, the definition above provides the notions of action of G on X over S and of G-equivariant morphisms; note that, by fully faithfulness, all morphisms of sheaves involving G and X are really morphisms of schemes over S (and be aware that the direct product as sheaves on Sch/S corresponds to the direct product of S-schemes, i.e. fibered product over S of schemes).

In this case, condition (a) for X to be a G-torsor over S means exactely that there exists a covering $U \to S$ that factors through $X \to S$. Now, if $X \to S$ is itself a covering, this is automatically verified, so only condition (b) has to be checked.

Affine case. Concentrating on a more specific situation, suppose that G is an affine S-group scheme and X an affine S-scheme. Then, by the correspondence between affine S-schemes and quasi-coherent $\mathcal{O}_S$-algebras, an action $\rho \colon G \times_S X \to X$ reflects into a homomorphism of $\mathcal{O}_S$-algebras:

$$\rho^\# \colon \mathscr{B} \to \mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{B},$$

where $\mathscr{A}$ and $\mathscr{B}$ are the algebras corresponding to G and X respectively. According to a previous observation (see the end of §2.2), $\mathscr{A}$ is a sheaf of Hopf algebras over $\mathcal{O}_S$. The fact that ρ is an action translates then into the commutativity of the following diagrams:

$$\begin{tikzcd} \mathscr{B} \arrow[r, "\rho^\#"] \arrow[d, "id_{\mathscr{B}}"'] & \mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{B} \arrow[d, "e^\# \otimes id_{\mathscr{B}}"] \\ \mathscr{B} \arrow[r, "\sim"] & \mathcal{O}_S \otimes_{\mathcal{O}_S} \mathscr{B} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{B} \arrow[r, "\rho^\#"] \arrow[d, "\rho^\#"'] & \mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{B} \arrow[d, "m^\# \otimes id_{\mathscr{B}}"] \\ \mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{B} \arrow[r, "id_{\mathscr{A}} \otimes \rho^\#"] & \mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{B}, \end{tikzcd}$$

where the maps $m^{\#}$ and $e^{\#}$ respect the notation introduced before for sheaves of Hopf algebras. We say, in this case, that $\rho^{\#}$ is a *co-action.* Similarly, we can define the notion of a *co-equivariant* homomorphism of $\mathcal{O}_S$-algebras with a co-action.

Let us now illustrate the main property of torsors, which, roughly, states that a torsor locally resembles the group acting on it. Recall that a property $\mathcal{P}$ of morphisms of schemes is *fppf local on the target* if it is stable under base change and, for any cartesian diagram:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

with g an fppf covering, f has $\mathcal{P}$ if and only if f' has $\mathcal{P}$. Properties that are stable under faithfully flat descent are clearly fppf local on the target.

Proposition 2.19. *Let $\pi\colon G \to S$ be an S-group scheme, $f\colon X \to S$ a G-torsor and $\mathcal{P}$ a property of morphisms of schemes which is fppf local on the target. Then, if π has $\mathcal{P}$, also f has $\mathcal{P}$. If, moreover, f is an fppf covering, then π has $\mathcal{P}$ if and only if f has $\mathcal{P}$.*

Proof. Consider first the commutative diagram, with left square cartesian:

$$\begin{tikzcd} G \arrow[d, "\pi"] & G \times_S X \arrow[l] \arrow[r, "\Omega"] \arrow[d] & X \times_S X \arrow[dl, "pr_2"] \\ S & X. \arrow[l, "f"] & \end{tikzcd} \tag{2.2}$$

If π has $\mathcal{P}$, then so does $G \times_S X \to X$. However, since X is a G-torsor, Ω is an isomorphism, so $pr_2\colon X \times_S X \to X$ has $\mathcal{P}$ too. Next, let $U \to S$ be a covering that factors through $f\colon X \to S$, using again that X is a G-torsor, and consider the commutative diagram with cartesian squares:

$$\begin{tikzcd} U \times_S X \arrow[r] \arrow[d] & X \times_S X \arrow[r, "pr_1"] \arrow[d, "pr_2"] & X \arrow[d, "f"] \\ U \arrow[r] & X \arrow[r, "f"] & S. \end{tikzcd} \tag{2.3}$$

From the fact that pr_2 has $\mathcal{P}$, we deduce that $U \times_S X \to U$ has $\mathcal{P}$ as well. But then, since $U \to S$ is an fppf covering, it follows that f has $\mathcal{P}$.

Suppose further that $f\colon X \to S$ is an fppf covering and assume conversely that f has $\mathcal{P}$. Then, by the rightmost square in diagram (2.3), we have that $pr_2\colon X \times_S X \to X$ has $\mathcal{P}$ and, by diagram (2.2), π has $\mathcal{P}$ too. $\square$

Quotients

One of the main sources of torsors in practice are quotients of a scheme by a free action of a group scheme. Unfortunately, the formation of quotients in the category of schemes is a pretty delicate matter. A possible approach is to work in the bigger category of sheaves on schemes and then investigate the conditions for the resulting sheaf to be represented by an actual scheme. Let us give a brief overview of the situation. The reader who shall be interested in a more thorough discussion may consult [3, V].

Fix a base scheme S and let G be an S-group scheme acting on an S-scheme X over S, say via $\rho\colon G \times_S X \to X$. We say that the action is *free* if, for every S-scheme T, the action of $G(T)$ on $X(T)$ is free, or, equivalently, if the graph morphism $\Omega\colon G \times_S X \to X \times_S X$ is a monomorphism of S-schemes.

Let $Y \to S$ be an S-scheme and $q\colon X \to Y$ a morphism over S. We say that Y is the *quotient scheme* of X by the action of G (or the *quotient scheme* of ρ) and that $q\colon X \to Y$ is the *quotient map* if $q \circ \rho = q \circ pr_2$ and, for any morphism of S-schemes $r\colon X \to Z$ such that $r \circ \rho = r \circ pr_2$ (i.e. r *invariant* for ρ), there exists a unique morphism $\bar{r}\colon Y \to Z$ such that $\bar{r} \circ q = r$:

$$\begin{tikzcd} G \times_S X \arrow[r, shift left, "\rho"] \arrow[r, shift right, "pr_2"'] & X \arrow[r, "r"] \arrow[d, "q"] & Z \\ & Y. \arrow[ur, dashed, "\bar{r}"] & \end{tikzcd} \tag{2.4}$$

Unfortunately, we cannot always find such an S-scheme Y, but note that, if it exists, then it is unique up to a unique isomorphism.

The situation, however, gets better if we switch to the bigger category $\mathrm{Sh}(\mathsf{Sch}/S)$. Indeed, define $G\backslash X \in \mathrm{Sh}(\mathsf{Sch}/S)$ to be the sheaf of sets associated to the presheaf $T \mapsto G(T)\backslash X(T)$, where $G(T)\backslash X(T)$ denotes the set-theoretical quotient of $X(T)$ by the action of $G(T)$. By construction, $G\backslash X$ comes together with a morphism of sheaves $\tilde{q}\colon X \to G\backslash X$, satisfying a similar universal property to (2.4), but in the category $\mathrm{Sh}(\mathsf{Sch}/S)$ of sheaves of sets on Sch/S, i.e. $\tilde{q} \circ \rho = \tilde{q} \circ pr_2$ and, for any morphism of sheaves $r\colon X \to Z$ such that $r \circ \rho = r \circ pr_2$, there exists a unique morphism $\bar{r}\colon G\backslash X \to Z$ such that $\bar{r} \circ \tilde{q} = r$.

Suppose now that $G\backslash X$ is representable by an S-scheme. Then, by fully faithfulness, $\tilde{q}$ is an S-morphism of schemes which satisfies the universal property (2.4) in the category of schemes over S. In particular, $G\backslash X$ is the quotient scheme of X by the action of G. In this case, we call $G\backslash X$ the *fppf quotient* of X by the action of G (or the *fppf quotient* of ρ) and $\tilde{q}\colon X \to G\backslash X$ the *quotient map*. To sum up, an fppf quotient is an S-scheme verifying the universal property (2.4) and which represents the sheaf $G\backslash X$.

Observe that, when the quotient scheme $q\colon X \to Y$ exists, then we can take Y to be our base scheme. More precisely, consider the pull-back Y-group scheme $G_Y = G \times_S Y$. Then, G_Y acts on X over Y, via:

$$\rho'\colon G_Y \times_Y X \xrightarrow{\sim} G \times_S X \xrightarrow{\rho} X.$$

This operation does not affect the freeness of the action. Indeed, we have a commutative diagram:

$$\begin{tikzcd} G_Y \times_Y X \arrow[r, "(\rho', id_X)"] \arrow[d, "\sim"] & X \times_Y X \arrow[d] \\ G \times_S X \arrow[r, "\Omega"] & X \times_S X, \end{tikzcd}$$

which shows that, if Ω is a monomorphism, then so is the graph morphism (ρ', id_X) of ρ'.

We have the following result ([3, V, 4.1]).

Theorem 2.20. *Let S be a locally Noetherian scheme, $\pi\colon G \to S$ a finite flat S-group scheme, $f\colon X \to S$ an affine S-scheme and let $\rho\colon G \times_S X \to X$ be an action of G on X over S. Denote by $\rho^{\#}\colon f_*\mathcal{O}_X \to \pi_*\mathcal{O}_G \otimes_{\mathcal{O}_S} f_*\mathcal{O}_X$ and $pr_2^{\#}\colon f_*\mathcal{O}_X \to \pi_*\mathcal{O}_G \otimes_{\mathcal{O}_S} f_*\mathcal{O}_X$ the morphisms of $\mathcal{O}_S$-algebras corresponding to $\rho\colon G \times_S X \to X$ and $pr_2\colon G \times_S X \to X$ respectively.*

Then, there exists a quotient scheme Y of X by the action of G. Moreover, Y is the affine S-scheme correponding to the $\mathcal{O}_S$-algebra $(f_*\mathcal{O}_X)^G$ defined, for every open subset $V \subseteq S$, by:

$$(f_*\mathcal{O}_X)^G(V) = \{\, a \in f_*\mathcal{O}_X(V) \mid \rho_V^{\#}(a) = pr_{2}{}_V^{\#}(a) \,\}\,.$$

In particular, if $U \subseteq S$ is an open subset, then $Y \times_S U$ is the quotient scheme of $X \times_S U$ by the action of $G \times_S U$.

Suppose, moreover, that ρ is a free action. Then, Y is the fppf quotient of X by the action of G, the quotient map $q\colon X \to Y$ is an fppf covering and X is a G_Y-torsor over Y. Furthermore, for any morphism of schemes $S' \to S$, we have that $Y \times_S S'$ is the fppf quotient of $X \times_S S'$ by the pull-back action of $G \times_S S'$.

Chapter 3

Purity for fppf torsors

Thanks to the work done in the previous chapter, we can now specify a precise setting where we aim at solving similar problems to those addressed in the first chapter.

Consider the following situation. S is a regular scheme and G is a finite flat S-group scheme acting on a finite flat S-scheme X, in such a way that S is the quotient scheme (we saw in the last section of the previous chapter how to reduce to this setting). Suppose, moreover, that the action of G on X is "generically free", meaning that there exists a dense open subset $U \subseteq S$ such that, after restriction to U, the action of $G_U = G \times_S U$ on $X_U = X \times_S U$ is free. Then, as a consequence of theorem 2.20, X_U is an fppf G_U-torsor over U. Here, it comes natural to ask whether some purity result, in the same fashion as the statements of chapter 1, allows to derive some consequences about the whole quotient map $X \to S$. For instance, is it possible, as we did in theorem 1.6 for finite étale coverings, to determine a locus on S such that X is not a G-torsor over it and give such locus some precise geometric shape?

As we learned in the first chapter, a strictly related question is the following. Suppose we have an fppf G_U-torsor X_U over an open subset U of S. Is it, then, possible to extend it to a G-torsor X over S, under suitable hypothesis? In other words, is there a sufficient condition for the pull-back functor $\mathrm{Tors}(S, G) \to \mathrm{Tors}(U, G_U)$ to be essentially surjective?

Now, if G is a finite étale group scheme over S, then, by proposition 2.19 and the fact that the properties of being finite and étale are fppf local on the target, every G-torsor is finite étale over S as well; similarly, G_U will be finite étale over U and so will be every G_U-torsor over U. In this case, the question is settled by the results on finite étale coverings. Indeed, if the codimension of $S \setminus U$ in S is at least 2, we may first extend torsors as finite étale coverings (as in theorem 1.1) and then it will not be hard to check that we get a torsor again.

Still, it is possible to extend this result to a wider class of group schemes, namely that of finite flat group schemes over their base. As a first remark, note that if a finite flat S-group scheme G is defined over a field k of characteristic zero, meaning that it is the pull-back to S of some finite group scheme over k, then it is automatically étale over S by corollary 2.15 of Cartier's theorem (and the fact that being étale is stable under base change); in this case, we get back to the previous situation. On the other hand, over a field k of positive characteristic, we have seen examples of finite group schemes that are not étale, namely the "infinitesimal group schemes" $\mu_{p^n,k}$ and $\alpha_{p^n,k}$, where p is the characteristic of k. These are, in fact, the main examples addressed by the results of this chapter. In this case, it is not possible to rely on purity for finite étale coverings, but a new proof has to be established.

In what follows, we will prove an analogue of theorem 1.1 for the category of torsors under the action of a finite flat group scheme over a regular scheme, as claimed by Laurent Moret-Bailly in [4, Lemme 2]. Besides making, once again, use of the theory of cohomology with support, the proof relies on a result in Commutative Algebra, that Maurice Auslander established in order to prove purity of the branch locus in [5]. An alternative proof of essential surjectivity, due to Madhav V. Nori, can be found in [8, II, Proposition 7]; Nori's proof, however, restricts to finite group schemes over a field and relies on a decomposition result for such group schemes.

After that, we will give a positive answer to the first question, defining the "infinitesimal branch locus" and proving an analogue of theorem 1.6 in the setting of torsors. We will also derive an easy corollary regarding the consequences in terms of Nori's fundamental group scheme; this corollary was in fact Nori's aim in the proposition mentioned above.

3.1 The result

Let us begin stating the main theorem of this section; recall that we consider torsors with respect to the fppf topology.

Theorem 3.1 (Purity theorem for fppf torsors)**.** *Let S be a regular scheme, $U \subseteq S$ an open subscheme, $Z = S \setminus U$ its closed complement and suppose that* $\operatorname{codim}_S(Z) \geq 2$*. Let then $\pi\colon G \to S$ be a finite flat S-group scheme and denote by $\pi_U\colon G_U \to U$ its restriction to U. Then, the restriction functor:*

$$\operatorname{Tors}(S, G) \longrightarrow \operatorname{Tors}(U, G_U)$$

is an equivalence of categories.

Remark 3.2. Under the assumptions of the theorem (actually whenever G is an affine S-group scheme, for any scheme S), we have the nice fact, as we find in [19, III, 4.3(a)], that any G-torsor $X \in \operatorname{Tors}(S, G)$ is represented by an S-scheme. The argument is actually pretty simple and uses the theory of faithfully flat descent. By remark 2.18, there is a covering $T \to S$ such that $X|_T \cong G|_T$, where $G|_T$ is represented by the affine T-scheme $G_T = G \times_S T$. Therefore, $X|_T$ is also represented by an affine T-scheme, corresponding to a quasi-coherent $\mathcal{O}_T$-algebra $\mathscr{B}_T$. Together with the obvious descent data, $\mathscr{B}_T$ yields a quasicoherent $\mathcal{O}_S$-algebra $\mathscr{B}$, whose pull-back to T is $\mathscr{B}_T$, by faithfully flat descent of quasi-coherent modules ([1, VIII, 1.1]). But then, $\mathscr{B}$ corresponds to an affine S-scheme X', whose pull-back to T is $X|_T$. Since X' and X coincide on the covering $T \to S$, the scheme X' must represent the sheaf X.

Another immediate fact is that any G-torsor $X \in \mathrm{Tors}(S, G)$, which we can now take as a scheme, is also finite flat over S. This follows from 2.19 and the fact that the property of being finite flat is fppf local on the target.

Tools for the proof.

Contrary to the strategy used in the first chapter, we will prove the theorem directly, without going through a local statement. In order to do so, however, we need to upgrade our tools in cohomology with support. We will use the following criterion from SGA2 ([2, III, 3.5]).

Proposition 3.3. *Let S be a locally Noetherian scheme, $U \subseteq S$ an open subscheme, $Z = S \setminus U$ its closed complement. For every coherent $\mathcal{O}_S$-module $\mathscr{F}$, the following conditions are equivalent:*

(a) for every $z \in Z$, $\operatorname{depth} \mathscr{F}_z \geq 2$ (as an $\mathcal{O}_{S,z}$-module);

(b) for every open subset $V \subseteq S$, the restriction map:

$$\mathscr{F}(V) \longrightarrow \mathscr{F}(V \cap U)$$

is bijective.

The key to prove the theorem is the following result in Commutative Algebra, due to Maurice Auslander ([5, Theorem 1.3]).

Theorem 3.4. *If R is a regular local ring and M is a reflexive R-module such that $\operatorname{End}_R(M)$ is isomorphic to a direct sum of copies of M, then M is R-free.*

Recall that an R-module M is *reflexive* if $M^{\vee\vee} \cong M$, where $M^\vee = \operatorname{Hom}_R(M, R)$ is the dual module. Free modules of finite rank are clearly reflexive. In general, we have the following criterion ([14, 1.4.1(b)]).

Proposition 3.5. *Let R be a Noetherian ring and M a finite R-module. Then M is reflexive if and only if:*

(a) $M_\mathfrak{p}$ is a reflexive $R_\mathfrak{p}$-module for all prime ideals $\mathfrak{p} \subseteq R$ with $\operatorname{depth} R_\mathfrak{p} \leq 1$ and

(b) $\operatorname{depth} M_\mathfrak{p} \geq 2$ for all prime ideals $\mathfrak{p} \subseteq R$ with $\operatorname{depth} R_\mathfrak{p} \geq 2$.

Now that we gathered all the necessary material, we may proceed proving the central result of this chapter.

Proof of theorem 3.1.

Let $\mathscr{A} := \pi_* \mathcal{O}_G$ be the $\mathcal{O}_S$-algebra corresponding to G and note that, because G is finite flat over S, which in turn is locally Noetherian, $\mathscr{A}$ is coherent and locally free as an $\mathcal{O}_S$-module (see the observations at the beginning of the first chapter). Since $G_U = G \times_S U = \pi^{-1}(U)$, the same holds for $\mathscr{A}_U := \pi_{U*} \mathcal{O}_{G_U} = \mathscr{A}|_U$ as an $\mathcal{O}_U$-module.

Essential surjectivity. Let X_U be a G_U-torsor over U; by remark 3.2, this is a finite flat U-scheme $f_U\colon X_U \to U$. Denote by $\mathscr{B}_U := (f_U)_*\mathcal{O}_{X_U}$ the corresponding locally free coherent $\mathcal{O}_U$-algebra. We have to find a G-torsor X over S, whose pull-back to U coincides with X_U.

Step I: extending the space. Let $i\colon U \to S$ denote the inclusion map and consider the $\mathcal{O}_S$-algebra $\mathscr{B} := i_*\mathscr{B}_U$. The map i is quasi-compact and quasi-separated, as an open immersion of locally Noetherian schemes, hence, as we already saw in the proof of theorem 1.2 (using [9, I, 9.4.2]), $\mathscr{B}$ is a quasi-coherent $\mathcal{O}_S$-module and $\mathscr{B}|_U = i^*\mathscr{B} = \mathscr{B}_U$. Thus, $\mathscr{B}$ defines an S-scheme $f\colon X \to S$, with $X = \operatorname{Spec}(\mathscr{B})$, whose pull-back to U is $f_U\colon X_U \to U$. To see that $\mathscr{B}$ is a coherent $\mathcal{O}_S$-module, i.e. that f is finite, we may apply the same finiteness criterion as in the proof of theorem 1.2 (namely [2, VIII, 2.3]): it suffices to check that $\operatorname{depth}(\mathscr{B}_U)_s \geq 1$ for all $s \in U$ with $c(s) := \operatorname{codim}_{\overline{\{s\}}}(Z \cap \overline{\{s\}}) = 1$. Now, $\mathscr{B}_U$ is locally free, so $\operatorname{depth}(\mathscr{B}_U)_s = \operatorname{depth}\mathcal{O}_{S,s} = \dim\mathcal{O}_{S,s}$ for all $s \in U$, the second equality because S is regular by assumption, hence $\mathcal{O}_{S,s}$ is a regular local ring and therefore Cohen-Macaulay. But $\dim\mathcal{O}_{S,s} < 1$ only if s is a generic point of S, in which case we must have $c(s) \geq 2$, as $\operatorname{codim}_S(Z) \geq 2$ by hypothesis. The criterion is thus satisfied and f is finite.

It now suffices to show that X is a G-torsor over S, whose G-action pulls-back to the G_U-action on X_U via i. Of course, the first thing to do is to provide X with such an action.

Step II: extending the action. By our analysis of the affine case in §2.4, the action of G_U on X_U, which we denote by $\rho_U\colon G_U \times_U X_U \to X_U$, corresponds to a map of $\mathcal{O}_U$-algebras:

$$\rho_U^\#\colon \mathscr{B}_U \to \mathscr{A}_U \otimes_{\mathcal{O}_U} \mathscr{B}_U,$$

satisfying the axioms of a co-action. Applying the push-forward functor i_*, we get a homomorphism of $\mathcal{O}_S$-algebras:

$$i_*\rho_U^\#\colon \mathscr{B} \to i_*(\mathscr{A}_U \otimes_{\mathcal{O}_U} \mathscr{B}_U).$$

However, since $\mathscr{A}_U = \mathscr{A}|_U = i^*\mathscr{A}$ and $\mathscr{A}$ is a locally free coherent $\mathcal{O}_S$-module, there is a natural isomorphism:

$$i_*(\mathscr{A}_U \otimes_{\mathcal{O}_U} \mathscr{B}_U) \xrightarrow{\sim} \mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{B}$$

(the "Projection formula", cf [10, II, Ex. 5.1(d)] and note that this respects the structures of $\mathcal{O}_S$-algebra). Thus, we obtain a homomorphism:

$$\rho^\#\colon \mathscr{B} \to \mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{B}.$$

By functoriality of i_* and naturality of the isomorphism above, $\rho^\#$ is again a co-action. Moreover, up to the natural identification $i^*(\mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{B}) \cong i^*\mathscr{A} \otimes_{\mathcal{O}_U} i^*\mathscr{B} = \mathscr{A}_U \otimes_{\mathcal{O}_U} \mathscr{B}_U$ (cf [11, 7.10]), we have $i^*\rho^\# = \rho_U^\#$. Then, $\rho^\#$ corresponds to an action $\rho\colon G \times_S X \to X$ of G on X (over S), whose pull-back via i is ρ_U.

The aim is now to show that ρ makes X a G-torsor over S. The strategy is to prove, first, that $\mathscr{B}$ is a locally free $\mathcal{O}_S$-module. Since S is locally Noetherian, it suffices to show that the stalks $\mathscr{B}_s$ are free $\mathcal{O}_{S,s}$-modules, for all $s \in S$ (cf [10, II, Ex. 5.7(b)]). Here is where Auslander's theorem 3.4 comes into play: the idea is to compare the $\mathcal{O}_S$-module $\mathcal{E}\mathrm{nd}_{\mathcal{O}_S}(\mathscr{B})$ with $\mathcal{H}\mathrm{om}_{\mathcal{O}_S}(\mathscr{A}, \mathscr{B})$ and then look at the stalks, as we already know that $\mathscr{A}$ is a locally free $\mathcal{O}_S$-module. Let us spell out the details.

Step III: local freeness. Denote by $\Omega = (\rho, pr_2)\colon G \times_S X \to X \times_S X$ the graph morphism of the action ρ. Note that, since the pull-back of ρ to U is ρ_U, then the pull-back of Ω to U is the graph Ω_U of ρ_U. Now, because $\mathscr{A}$ and $\mathscr{B}$ are coherent $\mathcal{O}_S$-modules, the $\mathcal{O}_S$-modules $\mathcal{H}\mathrm{om}_{\mathcal{O}_S}(\mathscr{A}, \mathscr{B})$ and $\mathcal{E}\mathrm{nd}_{\mathcal{O}_S}(\mathscr{B})$ are coherent too. Let us define a map:

$$\Phi\colon \mathcal{H}\mathrm{om}_{\mathcal{O}_S}(\mathscr{A}, \mathscr{B}) \longrightarrow \mathcal{E}\mathrm{nd}_{\mathcal{O}_S}(\mathscr{B})$$

in the following way. Suppose $V \subseteq S$ is an open subset and let $v \in \mathcal{H}\mathrm{om}_{\mathcal{O}_S}(\mathscr{A}, \mathscr{B})(V) = \mathrm{Hom}_{\mathcal{O}_V}(\mathscr{A}|_V, \mathscr{B}|_V)$. Then, we define $\Phi_V(v) \in \mathcal{E}\mathrm{nd}_{\mathcal{O}_S}(\mathscr{B})(V) = \mathrm{End}_{\mathcal{O}_V}(\mathscr{B}|_V)$ to be the homomorphism of $\mathcal{O}_V$-modules:

$$\Phi_V(v)\colon \mathscr{B}|_V \xrightarrow{\rho^{\#}|_V} \mathscr{A}|_V \otimes_{\mathcal{O}_V} \mathscr{B}|_V \xrightarrow{(v, id_{\mathscr{B}|_V})} \mathscr{B}|_V,$$

where $(v, id_{\mathscr{B}|_V})$ is the map given by the universal property of tensor product.

Now, since X_U is a G_U-torsor, then the graph $\Omega_U\colon G_U \times_U X_U \to X_U \times_U X_U$ is an isomorphism. This allows us to find an inverse:

$$\Phi|_U{}^{-1}\colon \mathcal{E}\mathrm{nd}_{\mathcal{O}_S}(\mathscr{B})|_U = \mathcal{E}\mathrm{nd}_{\mathcal{O}_U}(\mathscr{B}_U) \longrightarrow \mathcal{H}\mathrm{om}_{\mathcal{O}_U}(\mathscr{A}_U, \mathscr{B}_U) = \mathcal{H}\mathrm{om}_{\mathcal{O}_S}(\mathscr{A}, \mathscr{B})|_U$$

as follows. For $V \subseteq U$ open and $v \in \mathcal{E}\mathrm{nd}_{\mathcal{O}_U}(\mathscr{B}_U)(V) = \mathrm{End}_{\mathcal{O}_V}(\mathscr{B}|_V)$, set $(\Phi|_U{}^{-1})_V(v) \in \mathcal{H}\mathrm{om}_{\mathcal{O}_U}(\mathscr{A}_U, \mathscr{B}_U)(V) = \mathrm{Hom}_{\mathcal{O}_V}(\mathscr{A}|_V, \mathscr{B}|_V)$ to be the homomorphism of $\mathcal{O}_V$-modules:

$$(\Phi|_U{}^{-1})_V(v)\colon \mathscr{A}|_V \xrightarrow{j_1} \mathscr{A}|_V \otimes_{\mathcal{O}_V} \mathscr{B}|_V \xrightarrow{(\Omega_U^{-1}|_V)^{\#}} \mathscr{B}|_V \otimes_{\mathcal{O}_V} \mathscr{B}|_V \xrightarrow{(v, id_{\mathscr{B}|_V})} \mathscr{B}|_V,$$

where j_1 is the canonical homomorphism.

We have a commutative diagram, for every open subset $V \subseteq S$:

$$\begin{tikzcd} \mathcal{H}\mathrm{om}_{\mathcal{O}_S}(\mathscr{A}, \mathscr{B})(V) \arrow[r, "\Phi_V"] \arrow[d, "res"] & \mathcal{E}\mathrm{nd}_{\mathcal{O}_S}(\mathscr{B})(V) \arrow[d, "res"] \\ \mathcal{H}\mathrm{om}_{\mathcal{O}_S}(\mathscr{A}, \mathscr{B})(V \cap U) \arrow[r, "\Phi_{V \cap U}"] & \mathcal{E}\mathrm{nd}_{\mathcal{O}_S}(\mathscr{B})(V \cap U), \end{tikzcd} \tag{3.1}$$

where the bottom arrow is an isomorphism, because $\Phi|_U$ is invertible. We want to show, with the help of cohomology with support, that the vertical arrows are bijective too.

By definition of $\mathscr{B}$ and the fact that $\mathscr{B}|_U = \mathscr{B}_U$, we have that $\mathscr{B}(V) = \mathscr{B}_U(V \cap U) = \mathscr{B}(V \cap U)$ for every open subsets $V \subseteq S$. Hence, by proposition 3.3, depth $\mathscr{B}_z \geq 2$ for all $z \in Z$. From basic properties of the depth, it follows that depth $\mathrm{Hom}_{\mathcal{O}_{S,z}}(\mathscr{A}_z, \mathscr{B}_z) \geq 2$

and $\operatorname{depth} \operatorname{End}_{\mathcal{O}_{S,z}}(\mathscr{B}_z) \geq 2$ as well, for all $z \in Z$ (cf [15, Tag 0AV5]). However, since $\mathscr{A}$ and $\mathscr{B}$ are of finite presentation (because they are coherent and S is locally Noetherian), we have $\mathcal{H}om_{\mathcal{O}_S}(\mathscr{A}, \mathscr{B})_s \cong \operatorname{Hom}_{\mathcal{O}_{S,s}}(\mathscr{A}_s, \mathscr{B}_s)$ and $\mathcal{E}nd_{\mathcal{O}_S}(\mathscr{B})_s \cong \operatorname{End}_{\mathcal{O}_{S,s}}(\mathscr{B}_s)$ for all $s \in S$ (cf [11, 7.27(1)]). Therefore, $\operatorname{depth} \mathcal{H}om_{\mathcal{O}_S}(\mathscr{A}, \mathscr{B})_z \geq 2$ and $\operatorname{depth} \mathcal{E}nd_{\mathcal{O}_S}(\mathscr{B})_z \geq 2$ for all $z \in Z$. Again by proposition 3.3, then, the restriction maps $\mathcal{H}om_{\mathcal{O}_S}(\mathscr{A}, \mathscr{B})(V) \to \mathcal{H}om_{\mathcal{O}_S}(\mathscr{A}, \mathscr{B})(V \cap U)$ and $\mathcal{E}nd_{\mathcal{O}_S}(\mathscr{B})(V) \to \mathcal{E}nd_{\mathcal{O}_S}(\mathscr{B})(V \cap U)$, i.e. the vertical arrows in the diagram above, are bijective for every open subset $V \subseteq S$.

As a consequence, the top arrow Φ_V in the diagram (3.1) is an isomorphism for every $V \subseteq S$ open, i.e. $\Phi\colon \mathcal{H}om_{\mathcal{O}_S}(\mathscr{A}, \mathscr{B}) \to \mathcal{E}nd_{\mathcal{O}_S}(\mathscr{B})$ is an isomorphism of $\mathcal{O}_S$-modules. Taking stalks, we have that for every point $s \in S$:

$$\operatorname{End}_{\mathcal{O}_{S,s}}(\mathscr{B}_s) \cong \mathcal{E}nd_{\mathcal{O}_S}(\mathscr{B})_s \cong \mathcal{H}om_{\mathcal{O}_S}(\mathscr{A}, \mathscr{B})_s \cong \operatorname{Hom}_{\mathcal{O}_{S,s}}(\mathscr{A}_s, \mathscr{B}_s). \tag{3.2}$$

Here, since $\mathscr{A}$ is locally free, we have that $\mathscr{A}_s$ is a free $\mathcal{O}_{S,s}$-module of finite rank, say $r(s)$. But then:

$$\operatorname{Hom}_{\mathcal{O}_{S,s}}(\mathscr{A}_s, \mathscr{B}_s) \cong \operatorname{Hom}_{\mathcal{O}_{S,s}}(\mathcal{O}_{S,s}^{\oplus r(s)}, \mathscr{B}_s) \cong \operatorname{Hom}_{\mathcal{O}_{S,s}}(\mathcal{O}_{S,s}, \mathscr{B}_s)^{\oplus r(s)} \cong \mathscr{B}_s^{\oplus r(s)}. \tag{3.3}$$

Pulling together the two equations, we get that $\operatorname{End}_{\mathcal{O}_{S,s}}(\mathscr{B}_s)$ is isomorphic to a direct sum of copies of $\mathscr{B}_s$. Recalling that $\mathcal{O}_{S,s}$ is a regular local ring, as S is a regular scheme, it suffices to see that $\mathscr{B}_s$ is a reflexive $\mathcal{O}_{S,s}$-module, in order to apply theorem 3.4 and conclude that it is free. For this purpose, we may use criterion 3.5 for reflexivity.

Let $s \in S$ and note that, if $\mathfrak{p} \subseteq \mathcal{O}_{S,s}$ is a prime ideal, then $(\mathcal{O}_{S,s})_{\mathfrak{p}}$ is the stalk $\mathcal{O}_{S,t}$ at a point $t \in S$ whose preimage in $\operatorname{Spec} \mathcal{O}_{S,s}$ is $\mathfrak{p}$. In order to show that $\mathscr{B}_s$ is a reflexive $\mathcal{O}_{S,s}$-module for all $s \in S$, then, criterion 3.5 says that we only need to check that $\mathscr{B}_s$ is reflexive for all $s \in S$ with $\operatorname{depth} \mathcal{O}_{S,s} \leq 1$ and that $\operatorname{depth} \mathscr{B}_s \geq 2$ for all $s \in S$ with $\operatorname{depth} \mathcal{O}_{S,s} \geq 2$. Also recall that $\operatorname{depth} \mathcal{O}_{S,s} = \dim \mathcal{O}_{S,s}$ for any $s \in S$, by regularity. Now, if $\dim \mathcal{O}_{S,s} = \operatorname{depth} \mathcal{O}_{S,s} \leq 1$, then, since $\operatorname{codim}_S(Z) \geq 2$, we must have $s \in S \setminus Z = U$, so $\mathscr{B}_s = (\mathscr{B}_U)_s$ is $\mathcal{O}_{S,s}$-free of finite rank (as $\mathscr{B}_U$ is finite locally free) and therefore reflexive. On the other hand, if $\operatorname{depth} \mathcal{O}_{S,s} \geq 2$, we already saw that $\operatorname{depth} \mathscr{B}_s \geq 2$ for $s \in Z$, while, for $s \in U$, we have $\operatorname{depth} \mathscr{B}_s = \operatorname{depth}(\mathscr{B}_U)_s = \operatorname{depth} \mathcal{O}_{S,s} \geq 2$, the second equality because $(\mathscr{B}_U)_s$ is $\mathcal{O}_{S,s}$-free. This shows that, for all $s \in S$, $\mathscr{B}_s$ is a reflexive $\mathcal{O}_{S,s}$-module, which, together with equations (3.2) and (3.3), implies, by theorem 3.4, that $\mathscr{B}_s$ is a free $\mathcal{O}_{S,s}$-module at all the points $s \in S$; hence $\mathscr{B}$ is a locally free $\mathcal{O}_S$-module.

We can finally prove that X is indeed a G-torsor over S and this will conclude the proof of essentially surjectivity.

Step IV: we have a torsor. Let us first show that $f\colon X \to S$ is an fppf covering. We already observed that f is finite (because $\mathscr{B}$ is coherent), hence it is a fortiori of finite type and, since S is locally Noetherian, it is of finite presentation. Furthermore, we now know that $\mathscr{B}$ is locally free as an $\mathcal{O}_S$-module, which means that f is flat. As for surjectivity, we need some more articulate argument. First, we observe that $f_U\colon X_U \to U$ is surjective. Indeed, the structure map $\pi_U\colon G_U \to U$ is surjective, as it has a section $e_U : U \to G_U$ (the identity section). Since surjectivity is fppf local on the target, then, $f_U : X_U \to U$ is surjective too by proposition 2.19. Let now $s \in S$ and choose an affine open neighbourhood $V = \operatorname{Spec} R \subseteq S$ of s. By construction, we have $f^{-1}(V) = \operatorname{Spec} \mathscr{B}(V)$, with $\mathscr{B}(V)$ a finite, hence integral R-algebra. Recall that integral extensions of rings yield surjective maps on spectra, hence, unless $\mathscr{B}(V) = 0$, we can find a preimage of s in $f^{-1}(V) \subseteq X$. Suppose, by contradiction, that $\mathscr{B}(V) = 0$. Since $\operatorname{codim}_S(Z) \geq 2 > 0$, U is dense in S, hence $U \cap V \neq \emptyset$. Then, we can find a non-empty standard affine open subset $W = \operatorname{Spec} R_a \subseteq U \cap V \subseteq V = \operatorname{Spec} R$ $(a \in R)$. But then, since $\mathscr{B}$ is coherent, $\mathscr{B}(W) = \mathscr{B}(V)_a = 0$. On the other hand, $W \subseteq U$, so $\mathscr{B}(W) = \mathscr{B}_U(W)$ and this cannot be zero, because f_U is surjective and $\operatorname{Spec} \mathscr{B}_U(W) = f_U^{-1}(W)$. This contradiction shows that $\mathscr{B}(V) \neq 0$, hence we find a preimage of s, hence f is surjective. Thus, f is faithfully flat and of finite presentation, i.e. an fppf covering.

Lastly, we have to prove that the graph $\Omega : G \times_S X \to X \times_S X$ is an isomorphism. Equivalently, we may show that $\Omega^{\#} : \mathscr{B} \otimes_{\mathcal{O}_S} \mathscr{B} \to \mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{B}$ is an isomorphism of $\mathcal{O}_S$-algebras. As $f_U : X_U \to U$ is a torsor, we know that Ω_U is an isomorphism, hence so is $\Omega_U^{\#}$. Like before, we have a commutative diagram, for every open subset $V \subseteq S$:

$$\begin{tikzcd}
(\mathscr{B} \otimes_{\mathcal{O}_S} \mathscr{B})(V) \arrow[r, "(\Omega^{\#})_V"] \arrow[d, "res"] & (\mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{B})(V) \arrow[d, "res"] \\
(\mathscr{B} \otimes_{\mathcal{O}_S} \mathscr{B})(V \cap U) \arrow[r, "(\Omega^{\#})_{V \cap U}"] & (\mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{B})(V \cap U),
\end{tikzcd} \tag{3.4}$$

where the bottom arrow is an isomorphism, as $(\Omega^{\#})_{V \cap U} = (\Omega_U^{\#})_{V \cap U}$. However, both $\mathscr{A}$ and $\mathscr{B}$ are locally free $\mathcal{O}_S$-modules, hence, for every point $z \in Z$, we have that $(\mathscr{B} \otimes_{\mathcal{O}_S} \mathscr{B})_z = \mathscr{B}_z \otimes_{\mathcal{O}_{S,z}} \mathscr{B}_z$ is a free $\mathcal{O}_{S,z}$-module and similarly for $(\mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{B})_z$. Therefore, $\operatorname{depth}(\mathscr{B} \otimes_{\mathcal{O}_S} \mathscr{B})_z = \operatorname{depth}(\mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{B})_z = \operatorname{depth} \mathcal{O}_{S,z} = \dim \mathcal{O}_{S,z} \geq 2$, because $\operatorname{codim}_S(Z) \geq 2$ (and $\mathcal{O}_{S,z}$ is a regular local ring). It follows from proposition 3.3 that, for all open subset $V \subseteq S$, the vertical arrows in the diagram are bijective, hence the top arrow is an isomorphism. Thus, as required, $\Omega^{\#} : \mathscr{B} \otimes_{\mathcal{O}_S} \mathscr{B} \to \mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{B}$ is an isomorphism of $\mathcal{O}_S$-algebras.

Fully faithfulness. Let $p : X \to S$, $q : Y \to S$ be two G-torsors and let $\mathscr{F} := p_* \mathcal{O}_X$, $\mathscr{G} := q_* \mathcal{O}_Y$. As already observed, $\mathscr{F}$ and $\mathscr{G}$ are locally free and coherent as $\mathcal{O}_S$-modules, hence so is the $\mathcal{O}_S$-module $\mathcal{H}om_{\mathcal{O}_S}(\mathscr{F}, \mathscr{G})$. By locally freeness of $\mathscr{G}$, we have $\operatorname{depth} \mathscr{G}_z = \operatorname{depth} \mathcal{O}_{S,z} = \dim \mathcal{O}_{S,z} \geq 2$ and hence $\operatorname{depth}(\mathcal{H}om_{\mathcal{O}_S}(\mathscr{F}, \mathscr{G}))_z = \operatorname{depth} \operatorname{Hom}_{\mathcal{O}_{S,z}}(\mathscr{F}_z, \mathscr{G}_z) \geq 2$ for all $z \in Z$ (using again [15, Tag 0AV5]). By proposition 3.3, then, that the restriction map:

$$\operatorname{Hom}_{\mathcal{O}_S}(\mathscr{F}, \mathscr{G}) = \mathcal{H}om_{\mathcal{O}_S}(\mathscr{F}, \mathscr{G})(S) \to \mathcal{H}om_{\mathcal{O}_S}(\mathscr{F}, \mathscr{G})(U) = \operatorname{Hom}_{\mathcal{O}_U}(\mathscr{F}|_U, \mathscr{G}|_U)$$

is bijective. Morphisms of torsors, looked on the corresponding algebras, are in particular morphisms of $\mathcal{O}_S$-modules. Therefore, this bijection implies that our functor is faithful.

As for fullness, working again on the level of $\mathcal{O}_S$-algebras, a co-equivariant morphism $\psi : \mathscr{F}|_U \to \mathscr{G}|_U$ seen as $\psi \in \operatorname{Hom}_{\mathcal{O}_U}(\mathscr{F}|_U, \mathscr{G}|_U)$, has preimage $i_* \psi \in \operatorname{Hom}_{\mathcal{O}_S}(\mathscr{F}, \mathscr{G})$

under the bijection above. But, by functoriality of i_*, it is immediate that $i_*\psi$ is again co-equivariant for the co-actions of $\mathscr{A}$. This proves that our functor is full and, together with the previous steps, concludes the proof of theorem 3.1. $\square$

3.2 Some consequences

Purity of the infinitesimal branch locus

As the analogy with the theory of finite étale coverings suggests, we expect the result just proved to yield a similar consequence to that of theorem 1.6. In other words, fixed an action ρ on a scheme X and given an invariant morphism $f\colon X \to S$, we aim at having a good description of the locus on S over which f is *not* a torsor for ρ, under reasonable conditions on f. However, it is not immediately clear, from the definition of torsor, what this locus should be; one way to identify it, is to detect the points where the action is not free, looking at their stabiliser.

Therefore, we will first introduce the notion of stabiliser and this will allow us to define a suitable locus. Similarly to the ramification locus being the support of the sheaf of differentials, we can give an algebraic characterisation in this case too. After that, we will be able to prove a theorem in the same fashion as purity of the branch locus.

Let $\pi\colon G \to S$ be a finite flat group scheme, with S locally Noetherian, and let $f\colon X \to S$ be a finite flat morphism. Suppose, moreover, that $\rho\colon G \times_S X \to X$ is a G-action on X over S and denote by $\Omega = (\rho, pr_2)\colon G \times_S X \to X \times_S X$ the graph morphism.

We define the *stabiliser* St_ρ of ρ to be the fibered product of the graph Ω and the diagonal $\Delta\colon X \to X \times_S X$, so we have a cartesian diagram:

$$\begin{tikzcd} \mathrm{St}_\rho \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\ G \times_S X \arrow[r, "\Omega"] & X \times_S X. \end{tikzcd} \tag{3.5}$$

St_ρ is naturally an X-scheme and the projection $\mathrm{St}_\rho \to G \times_S X$ is, by construction, a morphism over X. Note that, since f is finite and hence separated, the diagonal Δ is a closed immersion. But then, the projection $\mathrm{St}_\rho \to G \times_S X$ is a closed immersion too (as this property is stable under base change), so that St_ρ is a closed X-subscheme of $G \times_S X$.

In order to see that this definition agrees with the usual notion of stabiliser, we observe that, for every X-scheme $x\colon T \to X$, i.e. $x \in X(T)$, we have:

$$St_\rho(T) = \{\, g \in G(T) \mid g \cdot x = x \,\}\,.$$

As this formula defines a subgroup of $G(T) = \mathrm{Hom}_{\mathsf{Sch}/S}(T, G) = \mathrm{Hom}_{\mathsf{Sch}/X}(T, G \times_S X) = (G \times_S X)(T)$, we also see that St_ρ is a closed X-subgroup scheme of $G \times_S X$. From this description, it is clear that the action ρ is free if and only if St_ρ is the trivial X-group scheme, i.e. the structure morphism $\mathrm{St}_\rho \to X$ is an isomorphism.

Note, finally, that St_ρ is a finite scheme over X. Indeed, $\pi\colon G \to S$ is finite, hence so is $G \times_S X \to X$, hence so is the composition $\mathrm{St}_\rho \hookrightarrow G \times_S X \to X$, since the first map is a closed immersion.

The fact that St_ρ is a scheme over X means that, for every point $x \in X$, we can define the *stabiliser of ρ at x*, which we denote by $\mathrm{St}_\rho(x)$, as the fibre of $\mathrm{St}_\rho \to X$ at x, i.e. the pull-back:

$$\begin{tikzcd} \mathrm{St}_\rho(x) \arrow[r] \arrow[d] & \mathrm{St}_\rho \arrow[d] \\ \operatorname{Spec} k(x) \arrow[r] & X, \end{tikzcd}$$

where $k(x)$ is the residue field at x.

We can now define the *infinitesimal ramification locus* $R^i \subseteq X$ of ρ to be the set of points $x \in X$ such that $\mathrm{St}_\rho(x)$ is not trivial, i.e. the structure morphism $\mathrm{St}_\rho(x) \to \operatorname{Spec} k(x)$ is not an isomorphism.

Remark 3.6. We can also give an algebraic description of R^i. As we already observed, St_ρ is finite, hence affine over X; let $\mathscr{H}$ be the corresponding quasi-coherent $\mathcal{O}_X$-algebra. Since X is locally Noetherian (because it is finite over S), $\mathscr{H}$ is coherent as an $\mathcal{O}_X$-module by finiteness of $\mathrm{St}_\rho \to X$ (cf [11, 7.45]). As St_ρ is a subgroup scheme of $G \times_S X$, it is equipped with an identity section $e_X\colon X \to \mathrm{St}_\rho$, which is the factorisation of the identity section of $G \times_S X$ through the inclusion $\mathrm{St}_\rho \hookrightarrow G \times_S X$. Let $e_X^\#\colon \mathscr{H} \to \mathcal{O}_X$ be the corresponding morphism of $\mathcal{O}_X$-algebras and denote by $\mathscr{I}$ the kernel sheaf; since S is locally Noetherian and $\mathscr{H}$ is coherent, then $\mathscr{I}$ is a coherent $\mathcal{O}_X$-module. In analogy with the theory of Hopf algebras, we call $\mathscr{I}$ the *augmentation ideal*. Note that $e_X^\#$ is surjective, because, being a morphism of $\mathcal{O}_X$-algebras, its composition after the structure map $\mathcal{O}_X \to \mathscr{H}$ must give the identity on $\mathcal{O}_X$. Thus, we have an exact sequence of $\mathcal{O}_X$-modules:

$$0 \to \mathscr{I} \to \mathscr{H} \xrightarrow{e_X^\#} \mathcal{O}_X \to 0$$

which splits, as the structure map $\mathcal{O}_X \to \mathscr{H}$ gives a section of $e_X^\#$. Therefore, generalising proposition 2.6(a), we have $\mathscr{H} \cong \mathscr{I} \oplus \mathcal{O}_X$ as $\mathcal{O}_X$-modules.

Let now $x \in X$ be a point of X and denote by $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ its maximal ideal and by $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ its residue field. The pull-back of $\mathscr{H}$ to $\operatorname{Spec} k(x)$ is $\mathscr{H}_x \otimes_{\mathcal{O}_{X,x}} k(x) = \mathscr{H}_x/\mathfrak{m}_x\mathscr{H}_x$, hence, by the correspondence explained at the beginning of the first chapter, we have: $\mathrm{St}_\rho(x) = \operatorname{Spec} \mathscr{H}_x/\mathfrak{m}_x\mathscr{H}_x$. However, by the decomposition of $\mathscr{H}$ found above:

$$\begin{aligned} \mathscr{H}_x/\mathfrak{m}_x\mathscr{H}_x = \mathscr{H}_x \otimes_{\mathcal{O}_{X,x}} k(x) &\cong (\mathscr{I}_x \oplus \mathcal{O}_{X,x}) \otimes_{\mathcal{O}_{X,x}} k(x) \cong \\ &\cong (\mathscr{I}_x \otimes_{\mathcal{O}_{X,x}} k(x)) \oplus (\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{X,x}} k(x)) \cong \mathscr{I}_x/\mathfrak{m}_x\mathscr{I}_x \oplus k(x). \end{aligned}$$

Thus, the structure map $\mathrm{St}_\rho(x) \to \operatorname{Spec} k(x)$, which corresponds to $k(x) \to \mathscr{H}_x/\mathfrak{m}_x\mathscr{H}_x \cong \mathscr{I}_x/\mathfrak{m}_x\mathscr{I}_x \oplus k(x)$ on the algebras, is an isomorphism if and only if $\mathscr{I}_x/\mathfrak{m}_x\mathscr{I}_x = 0$. Since $\mathscr{I}$ is coherent, this is equivalent to $\mathscr{I}_x = 0$, by Nakayama lemma.

We found: $R^i = \operatorname{Supp}\mathscr{I}$, i.e., the infinitesimal ramification locus coincides with the support of the augmentation ideal; in particular, since $\mathscr{I}$ is coherent, R^i is a closed subset of X (cf [11, 7.31]).

Let us define the *infinitesimal branch locus* of ρ to be $B^i := f(R^i) \subseteq S$. Since finite morphisms are closed, $B^i = f(R^i)$ is closed in S. Note that, if $U := S \setminus B^i$ denotes the complement of B^i in S, which is an open subscheme, then the restricted action ρ_U of $G_U = G \times_S U$ on $X_U = X \times_S U$ is free. Indeed, the stabiliser of ρ_U is $\mathrm{St}_{\rho_U} = \mathrm{St}_\rho \times_S U$ and corresponds to the $\mathcal{O}_{X_U}$-algebra $\mathscr{H}|_{X_U}$. But, as $X_U = f^{-1}(U)$, we have $\mathscr{I}_x = 0$ at all the points $x \in X_U$, so that $e_X^{\#}\big|_{X_U} : \mathscr{H}|_{X_U} \to \mathcal{O}_{X_U}$ is an isomorphism at all the stalks and hence an isomorphism of $\mathcal{O}_{X_U}$-algebras. This means that St_{ρ_U} is the trivial X_U-group scheme, i.e. that ρ_U is a free action.

We say that ρ is *generically free* if there exists a dense open subset $V \subseteq S$ such that the restricted action ρ_V of $G \times_S V$ on $X \times_S V$ is free, or, equivalently, if $U = S \setminus B^i$ is dense in S.

Theorem 3.7. *Let $\pi\colon G \to S$ be a finite flat S-group scheme and $f\colon X \to S$ a finite flat S-scheme, with S regular. Let $\rho\colon G \times_S X \to X$ be a generically free action over S and suppose that S is the quotient scheme of ρ, with quotient map f. Then, the infinitesimal branch locus $B^i \subseteq S$ is either empty or pure of codimension 1 in S, i.e.* $\operatorname{codim}_S(Z) = 1$ *for all irreducible components Z of B^i.*

Remark 3.8. Note that, given an action $\rho\colon G \times_S X \to X$ over S of a finite flat S-group scheme $\pi\colon G \to S$ on an affine S-scheme $f\colon X \to S$, theorem 2.20 guarantees the existence of the quotient scheme. Then, we saw before the same theorem how to reduce to the setting of an action over such quotient, as we assume here.

The proof of the theorem is similar to that of theorem 1.6.

Proof of the theorem. Set $U := S \setminus B^i$, which is dense in S by assumption, and let $f_U\colon X_U \to U$ be the pull-back of f to U. Note that f_U is still a finite flat morphism, as these properties are stable under base change. Moreover, the preceding discussion shows that the action ρ_U of $G_U := G \times_S U$ on X_U is free. However, since S is the quotient scheme of ρ, theorem 2.20 implies that U is the quotient scheme of ρ_U and, as ρ_U is free, U is also the fppf quotient of ρ_U and f_U is a torsor for the G_U-action.

Let now $Z \subseteq B^i$ be an irreducible component of B^i, $\eta \in Z$ its generic point, so that $\operatorname{codim}_S(Z) = \dim \mathcal{O}_{S,\eta}$. We proceed excluding both the possibilities that $\operatorname{codim}_S(Z) = 0$ and $\operatorname{codim}_S(Z) \geq 2$.

If $\operatorname{codim}_S(Z) = 0$, then Z is an irreducible component of S. However, being dense, U must intersects all the irreducible components of S (because the latter is locally Noetherian). Since $Z \subseteq B^i = S \setminus U$, this is impossible.

Suppose, on the other hand, that $\operatorname{codim}_S(Z) \geq 2$. Let $S' := \operatorname{Spec} \mathcal{O}_{S,\eta}$ and let $f'\colon X' \to S'$ be the pull-back of f to S'; since the properties of being finite and flat are stable under base change, f' is still finite flat. Now, let $U' = U \times_S S'$ denote the preimage of U in S' and note that $U' = S' \setminus \{\bar\eta\}$, where $\bar\eta$ is the maximal ideal of $\mathcal{O}_{S,\eta}$. Indeed, $\bar\eta$ maps to $\eta \in B^i = S \setminus U$ and, on the other hand, if a point $s \in S' \setminus \{\bar\eta\}$ mapped to B^i, it would contradict the maximality of $Z = \overline{\{\eta\}}$ as an irreducible subset of B^i. Consider finally the pull-back $f'_U \colon X'_U \to U'$ of f' to U' and note that this is a torsor under the action of $G'_U := G_U \times_U U'$. Indeed, this map can also be obtained as the pull-back of f_U to U', with f_U a G_U-torsor; in other words, we have a commutative cube with cartesian faces:

$$\begin{tikzcd}[row sep=small, column sep=small]
& X_U \arrow[rr] \arrow[dd, "f_U" near start] & & X \arrow[dd, "f"] \\
X'_U \arrow[ur] \arrow[rr, crossing over] \arrow[dd, "f'_U"] & & X' \arrow[ur] & \\
& U \arrow[rr] & & S \\
U' \arrow[ur] \arrow[rr] & & S'. \arrow[ur] \arrow[from=uu, crossing over, "f'" near start] &
\end{tikzcd}$$

However, as $\operatorname{codim}_{S'}(\{\bar\eta\}) = \dim \mathcal{O}_{S,\eta} = \operatorname{codim}_S(Z) \geq 2$, with $\{\bar\eta\} = S' \setminus U'$, theorem 3.1 gives an equivalence of categories:

$$\operatorname{Tors}(S', G') \longrightarrow \operatorname{Tors}(U', G'_U),$$

where $G' = G \times_S S'$ and clearly G'_U is the same as $G' \times_{S'} U'$. Thus, f'_U can be extended to a G'-torsor over S'. We claim that such extension coincides with f'.

Let $\mathscr{A}'$ and $\mathscr{B}'$ be the $\mathcal{O}_{S'}$-algebras corresponding respectively to $G' \to S'$ and $f' \colon X' \to S'$ and denote by $\rho'^{\#} \colon \mathscr{B}' \to \mathscr{A}' \otimes_{\mathcal{O}_{S'}} \mathscr{B}'$ the co-action of $\mathscr{A}'$ on $\mathscr{B}'$ corresponding to the G'-action ρ' on X'. Let then $\mathscr{A}'_U$ and $\mathscr{B}'_U$ be the $\mathcal{O}_{U'}$-algebras corresponding respectively to $G'_U \to U'$ and $f'_U \colon X'_U \to U'$; by construction, these are respectively the restrictions of $\mathscr{A}'$ and $\mathscr{B}'$ to U'. Similarly, the G'_U-action on X'_U corresponds, on the algebras, to the restriction $\rho'^{\#}_U \colon \mathscr{B}'_U \to \mathscr{A}'_U \otimes_{\mathcal{O}_{U'}} \mathscr{B}'_U$ of $\rho'^{\#}$ to U'. Let now $i \colon U' \to S'$ denote the inclusion. Then, by how we proved theorem 3.1, the extension of f'_U to a G'-torsor over S' is given by the affine S'-scheme corresponding to the push-forward algebra $i_*\mathscr{B}'_U$ with the push-forward action $i_*\rho'^{\#}_U$ (up to some natural identifications). However, both $i_*\mathscr{B}'_U$ and $\mathscr{B}'$ pull-back to $\mathscr{B}'_U$ via i, as well as both $i_*\rho'^{\#}_U$ and $\rho'^{\#}$ restrict to $\rho'^{\#}_U$ on U'. Recall, though, that the restriction functor:

$$\mathrm{L}(S') \longrightarrow \mathrm{L}(U')$$

on locally free coherent modules is fully faithful by lemma 1.3, since $2 \leq \dim \mathcal{O}_{S,\eta} = \operatorname{depth} \mathcal{O}_{S,\eta}$, the last equality by regularity. Since all our algebras are locally free and coherent modules, by finite flatness and because S is locally Noetherian, this means that $i_*\mathscr{B}'_U$ is isomorphic to $\mathscr{B}'$ and the actions $i_*\rho'^{\#}_U$ and $\rho'^{\#}$ correspond to each other via this isomorphism. Therefore, $f' \colon X' \to S'$ coincides in a G'-equivariant way to the extension of f'_U to S', proving the claim.

As a consequence, $f'\colon X' \to S'$ is a G'-torsor, but this is impossible. Indeed, since $\eta \in B^i$, there exists a point $x \in X$ in the fibre of η such that $\mathrm{St}_\rho(x)$ is not trivial. Now, the fibre of f' at $\bar\eta$ is the same as that of f at η, hence x corresponds to a point $x' \in X'$ over $\bar\eta$ and the stabiliser $\mathrm{St}_{\rho'}(x')$ of ρ' at x' coincides with $\mathrm{St}_\rho(x)$, hence it is not trivial either. This prevents ρ' from being a free action and hence $f'\colon X' \to S'$ from being a torsor. Such a contradiction excludes the possibility that $\mathrm{codim}_S(Z) \geq 2$ and proves the theorem. □

Remark 3.9. The proof just seen shows that, under the hypotheses of the theorem, the pull-back f_U of f to U is a G_U-torsor. Moreover, U is the biggest open subset of S such that this holds, in the sense that if $V \subseteq S$ is an open subset such that the pull-back f_V of f to V is a torsor, then $V \subseteq U$. In other words, B^i really defines the locus on S over which f is not a torsor for ρ.

Also note that, still in the situation of the theorem, the degree of f, as a map $\deg(f)\colon S \to \mathbb{N}$, is the same as the degree $\deg(\pi)$ of π. Indeed, let $s \in S$ and choose an affine open neighbourhood $V = \operatorname{Spec} R \subseteq S$ of s such that both $\mathscr{A}|_V$ and $\mathscr{B}|_V$ are free $\mathcal{O}_V$-modules, say of rank n and r respectively (here $\mathscr{A} = \pi_*\mathcal{O}_G$ and $\mathscr{B} = f_*\mathcal{O}_X$). We have $n = \deg(\pi)(s)$ and $r = \deg(f)(s)$. Now, as $U = S \setminus B^i$ is dense in S, we must have $U \cap V \neq \emptyset$, hence we can find a non-empty standard affine open subset $W = \operatorname{Spec} R_a \subseteq U \cap V \subseteq V = \operatorname{Spec} R$ ($a \in R$). Since $\mathscr{A}$ and $\mathscr{B}$ are coherent, $\mathscr{A}|_W$ and $\mathscr{B}|_W$ are again free $\mathcal{O}_W$-modules of rank n and r respectively. However, $W \subseteq U$ and, as we observed above, f_U is a G_U-torsor over U. Thus, the graph morphism induces an isomorphism of $\mathcal{O}_W$-modules:

$$\Omega^\#\Big|_W \colon \mathscr{B}|_W \otimes_{\mathcal{O}_W} \mathscr{B}|_W \longrightarrow \mathscr{A}|_W \otimes_{\mathcal{O}_W} \mathscr{B}|_W.$$

Here, on the left we have a free $\mathcal{O}_W$-module of rank r^2 and on the right a free $\mathcal{O}_W$-module of rank rn. Note that $r = 0$ would contradict the fact that $f\colon X \to S$ is a quotient, because, by the description of the latter in theorem 2.20, $\mathcal{O}_S$ is a subsheaf of $\mathscr{B}$. Therefore, $n = r$ as required.

Remark 3.10. As for theorem 1.6, the assumption on the regularity of the quotient S is very important. In fact, this can be seen with the same example as in remark 1.7, but now over a base field k of characteristic 2. In this case, the inclusion $k[x^2, xy, y^2] \hookrightarrow k[x, y]$ corresponds to a map of schemes, say $f\colon \mathbb{A}^2_k \to S$, that is everywhere ramified. Nevertheless, we may view S as the quotient scheme of $\mathbb{A}^2_k$ by the multiplicative action of $\mu_{2,k} = \operatorname{Spec} k[t]/(t^2 - 1)$ and study the quotient map f with the tools of this section. The action, which we denote by ρ, is given on rings by:

$$\begin{aligned}\rho^\#\colon k[x, y] &\to k[x, y, t]/(t^2 - 1) \cong k[t]/(t^2 - 1) \otimes_k k[x, y]\\ x &\mapsto tx,\\ y &\mapsto ty;\end{aligned}$$

the fact that S is indeed the quotient scheme is a consequence of the explicit description in theorem 2.20. As in remark 1.7, note that S is not regular at the point $s = (x^2, xy, y^2) \in S$

and f is not flat. The stabiliser St_ρ is however still defined and a direct computation shows that the infinitesimal ramification locus is $R^i = \{\,\underline{0}\,\} \subseteq \mathbb{A}^2_k$ (see §4.2 for an example of this kind of computations). But then, the infinitesimal branch locus is $B^i = f(R^i) = \{\,s\,\} \subseteq S$, which has codimension 2 in S, hence the theorem fails in this case. Again, we may find an infinitesimal ramification locus of any codimension $n > 1$, working on $\mathbb{A}^n_k$ with the same multiplicative action of $\mu_{2,k}$.

Purity in terms of Nori's fundamental group scheme

The analogy with the theory of finite étale coverings also extends to fundamental groups. As in the previous part, we have to find suitable substitutes for the objects involved. In this context, the purity theorem for fppf torsors has an immediate consequence in terms of *Nori's fundamental group scheme*. Let us quickly recall its construction, following [8, II], and a corollary to theorem 3.1 will easily follow.

First, however, let us remark that Nori's construction is carried on only for schemes over a field. In this case, as we explained at the beginning of the chapter, the content of the last pages is of new interest, compared to the first chapter, only when the characteristic of the base field is not zero (due to corollary 2.15 to Cartier's theorem). Thus, we will restict to the case of a field of positive characteristic.

Fix a field k of positive characteristic. Let S be a k-scheme and $s_0\colon \operatorname{Spec} k \to S$ a k-rational point. Consider the category $\mathrm{Tors}(S, s_0)$ whose objects are triples (G, X, x_0), where G is a finite k-group scheme acting over k on an S-scheme X, such that X is a G_S-torsor over S (here $G_S = G \times_{\operatorname{Spec} k} S$) and $x_0\colon \operatorname{Spec} k \to X$ is a k-rational point over s_0. A morphism $(G, X, x_0) \to (G', X', x'_0)$ in $\mathrm{Tors}(S, s_0)$ is a couple (g, f), where $g\colon G \to G'$ is a homomorphism of k-group schemes and $f\colon X \to X'$ is an S-morphism of schemes with $f \circ x_0 = x'_0$, such that f intertwines the actions of G and G' on X and X' respectively, i.e. the following diagram:

$$\begin{tikzcd} G \times_{\operatorname{Spec} k} X \arrow[r, "\rho"] \arrow[d, "g\times f"] & X \arrow[d, "f"] \\ G' \times_{\operatorname{Spec} k} X' \arrow[r, "\rho'"] & X' \end{tikzcd}$$

commutes (here ρ and ρ' are respectively the G-action on X and the G'-action on X').

In [8, II, Proposition 2], Nori proves that if S is reduced and connected, then $\mathrm{Tors}(S, s_0)$ has fiber products. More precisely, if $(g_1, f_1)\colon (G_1, X_1, x_1) \to (G, X, x_0)$ and $(g_2, f_2)\colon (G_2, X_2, x_2) \to (G, X, x_0)$ are two morphisms in $\mathrm{Tors}(S, s_0)$, then:

$$(G_1 \times_G G_2, X_1 \times_X X_2, (x_1, x_2))$$

is again an object of $\mathrm{Tors}(S, s_0)$. This makes $\mathrm{Tors}(S, s_0)$ a projective system and we can define *Nori's fundamental group scheme* of S (with base point s_0) to be the projective limit:

$$\pi_1^N(S, s_0) := \varprojlim_{(G,X,x_0)\in \mathrm{Tors}(S,s_0)} G.$$

Then, $\pi_1^N(S, s_0)$ is a k-group scheme satisfying the property that, for any finite k-group scheme G, there exists a bijection between the set of homomorphisms of k-group schemes $\pi_1^N(S, s_0) \to G$ and that of isomorphism classes of G_S-torsors X over S with a fixed k-rational point $x_0 \colon \operatorname{Spec} k \to X$ over s_0 (here two G_S-torsors X, X' over S with fixed k-rational points x_0 and x'_0 respectively are isomorphic if there exists a G_S-equivariant isomorphism of schemes $f \colon X \to X'$ over S with $f \circ x_0 = x'_0$).

In the situation of theorem 3.1, we have equivalences of categories:

$$\mathrm{Tors}(S, G_S) \longrightarrow \mathrm{Tors}(U, G_U)$$

for any finite group scheme G over k (note that $G_S = G \times_{\operatorname{Spec} k} S$ is finite flat over S, as a pull-back of G). Fixing a base point $s_0 \colon \operatorname{Spec} k \to U$, these induce an equivalence:

$$\begin{aligned} \mathrm{Tors}(S, s_0) &\longrightarrow \mathrm{Tors}(U, s_0) \\ (G, X, x_0) &\mapsto (G, X_U, x_0). \end{aligned} \tag{3.6}$$

To be precise, fully faithfulness of the last functor is not automatic, because in $\mathrm{Tors}(S, s_0)$ we allow morphisms between triples featuring different group schemes. However, the same argument that we used to prove fully faithfulness in theorem 3.1 applies here as well. Now, when S is reduced and connected, the equivalence (3.6) implies that $\pi_1^N(S, s_0) = \pi_1^N(U, s_0)$ by construction. Thus, recalling that regular schemes are in particular reduced, we have the following corollary.

Corollary 3.11. *Let k be a field of positive characteristic, S a regular connected k-scheme, $U \subseteq S$ an open subscheme, $Z = S \setminus U$ its closed complement and $s_0 \colon \operatorname{Spec} k \to U$ a k-rational point. Suppose that $\operatorname{codim}_S(Z) \geq 2$. Then, we have an equality of Nori's fundamental group schemes:*

$$\pi_1^N(S, s_0) = \pi_1^N(U, s_0).$$

Chapter 4

"Infinitesimal" ramification theory

In this final part of our work, we would like to study in deeper detail the infinitesimal branch locus B^i introduced at the end of last chapter. In particular, consider the situation of theorem 3.7: we have a finite flat morphism $f\colon X \to S$, which is the quotient map, in the category of schemes, by a generically free action over S of a finite flat S-group scheme $\pi\colon G \to S$. If S is regular, the theorem shows that B^i is a union of irreducible closed subsets of codimension 1 in S. In fact, we can upgrade B^i to an effective Weil divisor, i.e., we can attach suitable multiplicities to the single irreducible components.

Indeed, in remark 3.9 we saw that $\deg(f) = \deg(\pi)$ as maps $S \to \mathbb{N}$, hence the graph morphism Ω gives rise, on the algebras, to a morphism of locally free $\mathcal{O}_S$-modules of the same rank. This allows to locally define determinant sections, which detect whether Ω is an isomorphism. The collection of these local sections is an effective Cartier divisor, whose support turns out, as expected, to be B^i; looking at the orders of vanishing at each irreducible component, we get an effective Weil divisor.

Note that this is in complete analogy with the characterisation of finite étale coverings in terms of the discriminant section, which we used in the proof of theorem 1.2. In fact, we could have employed a similar technique in StepIV of the proof of theorem 3.1, using a determinant section as just explained.

After giving the details of the construction of the divisor introduced above, we will compute it in some basic examples.

4.1 The infinitesimal branch divisor

Let us assume the hypotheses of theorem 3.7: S a regular scheme, $\pi\colon G \to S$ a finite flat S-group scheme, $f\colon X \to S$ a finite flat S-scheme and $\rho\colon G \times_S X \to X$ a generically free action over S, such that S is the quotient scheme.

Denote by $\mathscr{A} := \pi_*\mathcal{O}_G$ and $\mathscr{B} := f_*\mathcal{O}_X$ the locally free coherent $\mathcal{O}_S$-algebras corresponding respectively to π and f. Let $V = \operatorname{Spec} R \subseteq S$ be an affine open subset such that both $\mathscr{A}|_V$ and $\mathscr{B}|_V$ are free $\mathcal{O}_V$-modules; by remark 3.9 they have the same rank, say n. Choose isomorphisms $\varphi\colon \mathscr{A}|_V \xrightarrow{\sim} \widetilde{R^n}$ and $\psi\colon \mathscr{B}|_V \xrightarrow{\sim} \widetilde{R^n}$, i.e. *local trivialisations* on V. Then, the graph morphism induces an endomorphism of $\widetilde{R^{n^2}} \cong \widetilde{R^n} \otimes_{\mathcal{O}_V} \widetilde{R^n}$:

$$\begin{tikzcd}
\mathscr{B}|_V \otimes_{\mathcal{O}_V} \mathscr{B}|_V \arrow[r, "\Omega^\#|_V"] \arrow[d, "\psi\otimes\psi"] & \mathscr{A}|_V \otimes_{\mathcal{O}_V} \mathscr{B}|_V \arrow[d, "\varphi\otimes\psi"] \\
\widetilde{R^{n^2}} \cong \widetilde{R^n} \otimes_{\mathcal{O}_V} \widetilde{R^n} \arrow[r] & \widetilde{R^n} \otimes_{\mathcal{O}_V} \widetilde{R^n} \cong \widetilde{R^{n^2}},
\end{tikzcd}$$

which corresponds to an endomorphism of the free R-module R^{n^2}; let $\delta_V \in R = \mathcal{O}_S(V)$ be its determinant, so $\Omega^\#\big|_V$ is an isomorphism if and only if $\delta_V \in R^\times$. Note that, with a different choice of trivialisations $\varphi' \colon \mathscr{A}|_V \to \widetilde{R^n}$ and $\psi' \colon \mathscr{B}|_V \to \widetilde{R^n}$, the resulting determinant δ'_V differs from δ_V by a unit of R (to be precise, we have $\delta'_V \cdot \det((\psi' \otimes \psi') \circ (\psi \otimes \psi)^{-1}) = \det((\varphi' \otimes \psi') \circ (\varphi \otimes \psi)^{-1}) \cdot \delta_V)$.

Let now $S = \bigcup_i V_i$ be an affine open cover of S, such that on each $V_i = \operatorname{Spec} R_i$ we have isomorphisms $\varphi_i \colon \mathscr{A}|_{V_i} \to \widetilde{R_i^{n_i}}$ and $\psi_i \colon \mathscr{B}|_{V_i} \to \widetilde{R_i^{n_i}}$. Consider then the collection $(V_i, \delta_{V_i})_i$, where each δ_{V_i} is constructed as above, and observe that, whenever $V_i \cap V_j \neq \emptyset$, $\delta_{V_i}|_{V_i \cap V_j}$ and $\delta_{V_j}\big|_{V_i \cap V_j}$ differ by a unit $\alpha \in \mathcal{O}_S(V_i \cap V_j)^\times$. Indeed, we can find an open affine covering $V_i \cap V_j = \bigcup_k W_k$, with each $W_k = \operatorname{Spec}(R_i)_{a_k} \cong \operatorname{Spec}(R_j)_{b_k}$ ($a_k \in R_i$ and $b_k \in R_j$). Then, for every k, $\delta_{V_i}|_{W_k}$ and $\delta_{V_j}\big|_{W_k}$ are two determinants of $\Omega^\#\big|_{W_k}$ computed on (possibly) different trivialisations, hence, by the argument above, they differ by a unit $\alpha_k \in \mathcal{O}_S(W_k)^\times$. By construction, the α_k's coincide on the intersections of the W_k's (as they only depend on the two different trivialisations on V_i and V_j, see the explicit dependence relation above), hence they yield a section $\alpha \in \mathcal{O}_S(V_i \cap V_j)^\times$ such that $\delta_{V_i}|_{V_i \cap V_j} \cdot (\delta_{V_j}\big|_{V_i \cap V_j})^{-1} = \alpha$.

As a consequence, $(V_i, \delta_{V_i})_i$ defines an effective Cartier divisor on S. Moreover, the same argument as above shows that, with a different choice of cover $S = \bigcup_j V'_j$ and trivialisations, the resulting collection $(V'_j, \delta'_{V_j})_j$ defines the same Cartier divisor. Thus, we may just denote it by δ.

Remark 4.1. Consider now the support of δ, i.e. $\operatorname{Supp} \delta = \{\, s \in S \mid \delta_s \in \mathcal{O}_{S,s} \setminus \mathcal{O}_{S,s}^\times \,\}$, where δ_s is the image in $\mathcal{O}_{S,s}$ of δ_{V_i} for any i such that $s \in V_i$. Then, we have $\operatorname{Supp} \delta = B^i$.

Indeed, suppose that $s \in S \setminus B^i =: U$. In remark 3.9 we saw that the pull-back f_U of f to U is a G_U-torsor, hence $\Omega^\#\big|_U$ is an isomorphism. For i such that $s \in V_i = \operatorname{Spec} R_i$, choose a standard affine open neighbourhood $W = \operatorname{Spec}(R_i)_a \subseteq V_i \cap U \subseteq V_i = \operatorname{Spec} R_i$ of s ($a \in R$). Then, $\delta_{V_i}|_W$ is a unit in $\mathcal{O}_S(W)$, because $W \subseteq U$ and hence $\Omega^\#\big|_W$ is an isomorphism. It follows that $\delta_s \in \mathcal{O}_{S,s}^\times$, i.e. $s \in S \setminus \operatorname{Supp} \delta$.

Conversely, suppose that $s \in S \setminus \operatorname{Supp} \delta$. For i such that $s \in V_i = \operatorname{Spec} R_i$, the image of δ_{V_i} in $\mathcal{O}_{S,s}$ is a unit. Then, we may find a standard affine open neighbourhood $W = \operatorname{Spec}(R_i)_a \subseteq \operatorname{Spec} R_i = V$ of s ($a \in R$), such that $\delta_{V_i}|_W \in \mathcal{O}_S(W)^\times$. This means that $\Omega^\#\big|_W$ is an isomorphism. Now let $X_W := f^{-1}(W) = X \times_S W$. For any point $x \in X$

mapping to $s \in W$ via f, the morphism $\operatorname{Spec} k(x) \to X$, where $k(x)$ is the residue field at x, factors through $X_W \to X$ and we can obtain the stabiliser $\mathrm{St}_\rho(x)$ at x as the pull-back:

$$\begin{tikzcd}
\mathrm{St}_\rho(x) \arrow[r] \arrow[d] & \mathrm{St}_{\rho_W} \arrow[r] \arrow[d] & G_W \times_W X_W \arrow[d, "\Omega|_W"] \\
\operatorname{Spec} k(x) \arrow[r] & X_W \arrow[r, "\Delta"] & X_W \times_W X_W,
\end{tikzcd}$$

where $G_W = \pi^{-1}(W) = G \times_S W$ and St_{ρ_W} is the stabiliser of the restricted action $\rho_W \colon G_W \times_W X_W \to X_W$. Here, $\Omega|_W$ is an isomorphism, hence so is $\mathrm{St}_\rho(x) \to \operatorname{Spec} k(x)$, so x does not belong to the infinitesimal ramification locus R^i. Since this holds for all $x \in X$ in the fibre of s and we defined $B^i = f(R^i)$, it follows that $s \in S \setminus B^i$.

Recall that to each effective Cartier divisor we may associate an effective Weil divisor. Let S^1 denote the set of irreducible closed subschemes of codimension 1 in S. For $C \in S^1$ and $\eta \in C$ its generic point, we have $\dim \mathcal{O}_{S,\eta} = 1$, hence, since S is regular, $\mathcal{O}_{S,\eta}$ is a discrete valuation ring. We set $\mathrm{ord}_C(\delta) := v_\eta(\delta_\eta)$, where $v_\eta \colon \mathcal{O}_{S,\eta} \setminus \{ 0 \} \to \mathbb{N}$ is the discrete valuation of $\mathcal{O}_{S,\eta}$. Then, the effective Weil divisor associated to δ is $\sum_{C \in S^1} \mathrm{ord}_C(\delta)[C]$. Note that $\mathrm{ord}_C(\delta) > 0$ if and only if $\eta \in \operatorname{Supp} \delta = B^i$ (η being the generic point of C), which in turn is equivalent to $C \subseteq B^i$, since B^i is closed. Thus, the summands appearing in the effective Weil divisor associated to δ are exactely the irreducible components of B^i. This allows for some small abuse of notation, upgrading the definition of B^i from the previous chapter to:

$$B^i := \sum_{C \in S^1} \mathrm{ord}_C(\delta)[C],$$

where δ is the Cartier divisor constructed above and given locally by the determinant of the graph morphism $\Omega^\#$. We will call this the *infinitesimal branch divisor*.

4.2 Some examples

We will now compute the inifinitesimal branch divisor in two fundamental examples, after checking that they verify the conditions of theorem 3.7. We will see how this new tool provides means to discern between the two examples, which could not be distinguished by the sole quotient map.

The $\mu_{p,k}$-action on the affine line

Let k be a field of positive characteristic p. The group of p-th roots of unity $\mu_{p,k} = \operatorname{Spec} k[x]/(x^p - 1)$ over k is a finite k-group scheme of rank p, which is obviously flat, but not étale, as $k[x]/(x^p - 1) = k[x]/(x-1)^p$ is not a reduced ring. We let $\mu_{p,k}$ act on the affine line $X := \mathbb{A}^1_k = \operatorname{Spec} k[y]$ over k by multiplication, i.e., for any k-scheme T:

$$\begin{aligned} \rho(T) \colon \mu_{p,k}(T) \times \mathbb{A}^1_k(T) &\to \mathbb{A}^1_k(T) \\ (\nu, \beta) &\mapsto \nu \cdot \beta, \end{aligned}$$

where $\mu_{p,k}(T) = \{\, \nu \in \mathcal{O}_T(T)^\times \mid \nu^p = 1 \,\}$ and $\mathbb{A}^1_k(T) = \mathcal{O}_T(T)$. On the algebras, the action is given by:[1]

$$\begin{aligned} \rho^\#\colon k[y] &\to k[x,y]/(x^p-1) \cong k[x]/(x^p-1) \otimes_k k[y] \\ y &\mapsto xy. \end{aligned}$$

By theorem 2.20, the quotient scheme of ρ is given by the k-algebra:

$$\begin{aligned} k[y]^{\mu_{p,k}} &= \{\, p(y) \in k[y] \mid \rho^\#(p(y)) = p(y) \,\} = \\ &= \{\, p(y) \in k[y] \mid p(xy) = p(y) \,\} = k[y^p] \cong k[t] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad y^p \;\leftarrow\!\shortmid\; t. \end{aligned}$$

Hence, the quotient scheme is still an affine line $S := \operatorname{Spec} k[t] \cong \mathbb{A}^1_k$ and the quotient map $f\colon X \to S$ is given by the inclusion: $k[t] \to k[y]$, mapping $t \mapsto y^p$. As we explained before theorem 2.20, we may take S to be our base scheme:

$$G := \mu_{p,k} \times_{\operatorname{Spec} k} S = \operatorname{Spec} k[x]/(x^p-1) \otimes_k k[t] \cong \operatorname{Spec} k[x,t]/(x^p-1)$$

is a finite flat S-scheme of degree p and we still denote by $\rho\colon G \times_S X \to X$ the action over S. Note that $S \cong \mathbb{A}^1_k$ is a regular scheme and $k[y]$ is a free $k[y^p]$-module of rank p, so that $f\colon X \to S$ is finite flat of degree p.

Let us now check that the action is generically free. We have:

$$G \times_S X = \operatorname{Spec} k[x,t]/(x^p-1) \otimes_{k[t]} k[y] \cong \operatorname{Spec} k[x,y]/(x^p-1)$$

and, writing two copies of X as $X = \operatorname{Spec} k[y_1]$ and $X = \operatorname{Spec} k[y_2]$:

$$X \times_S X = \operatorname{Spec} k[y_1] \otimes_{k[t]} k[y_2] \cong \operatorname{Spec} k[y_1,y_2]/(y_1^p - y_2^p).$$

The graph morphism $\Omega\colon G \times_S X \to X \times_S X$ is, on the algebras:

$$\begin{aligned} \Omega^\#\colon k[y_1,y_2]/(y_1^p-y_2^p) &\to k[x,y]/(x^p-1) \\ y_1 &\mapsto xy, \\ y_2 &\mapsto y. \end{aligned}$$

On the other hand, writing $X = \mathbb{A}^1_k = \operatorname{Spec} k[z]$, we can describe the diagonal map $\Delta\colon X \to X \times_S X$ as:

$$\begin{aligned} \Delta^\#\colon k[y_1,y_2]/(y_1^p-y_2^p) &\to k[z] \\ y_1 &\mapsto z, \\ y_2 &\mapsto z. \end{aligned}$$

[1] We will admit the same notation for a variable in a polynomial ring and its image in any quotient.

Thus, the pull-back diagram (3.5) giving the stabiliser reflects, on the algebras, into:

$$\begin{tikzcd} k[y_1,y_2]/(y_1^p-y_2^p) \arrow[r, "\Omega^\#"] \arrow[d, "\Delta^\#"] & k[x,y]/(x^p-1) \arrow[d] & y \arrow[d, mapsto] \\ k[z] \arrow[r] & k[x,z]/(x^p-1, z(x-1)) & z, \end{tikzcd}$$

so the stabiliser of ρ is:

$$\mathrm{St}_\rho = \operatorname{Spec} k[x,z]/(x^p-1, z(x-1)) \longrightarrow \operatorname{Spec} k[z] = X.$$

Computing the fibers for $\mathfrak{p} \in X = \operatorname{Spec} k[z]$, we find: $\mathrm{St}_\rho(\mathfrak{p}) \cong k(\mathfrak{p})$ for all the points $\mathfrak{p} \in X \setminus \{ (z) \} = \mathbb{A}^1_k \setminus \{ \underline{0} \}$, where $k(\mathfrak{p})$ is the residue field at $\mathfrak{p}$. On the other hand, $\mathrm{St}_\rho((z)) \cong \operatorname{Spec} k[x]/(x^p-1) \cong \mu_{p,k}$. Therefore, the infinitesimal ramification locus is $R^i = \{ \underline{0} \}$ and the infinitesimal branch locus is $B^i = f(R^i) = \{ (t) \} \subseteq \operatorname{Spec} k[t] = S$. Since $U := S \setminus B^i$ is dense in S, the action is indeed generically free. As we could predict by theorem 3.7, the unique irreducible component of B^i has codimension 1 in S.

We will now proceed to the computation of the infinitesimal branch divisor B^i. As guaranteed by the preceding discussion, $\deg(f) = p$ equals the degree of $G \to S$, hence $\Omega^\#$ is a homomorphism between free $k[t]$-modules of the same rank p^2. We have the $k[t]$-bases $\{ y_i^i y_2^j \}_{0\le i,j\le p-1}$ for $k[y_1,y_2]/(y_1^p - y_2^p)$ (with $t \mapsto y_1^p = y_2^p$) and $\{ x^i y^j \}_{0\le i,j\le p-1}$ for $k[x,y]/(x^p-1)$ (with $t \mapsto y^p$). By the formula for $\Omega^\#$ above, we see that, for $0 \le i,j \le p-1$:

$$\Omega^\# \colon y_1^i y_2^j \mapsto x^i y^{i+j} = \begin{cases} 1 \cdot x^i y^{i+j} & (0 \le i+j \le p-1), \\ t \cdot x^i y^{i+j-p} & (p \le i+j \le 2p-2). \end{cases}$$

Thus, ordering the first basis as:

$$1, y_2, \dots, y_2^{p-1}, \quad y_1, y_1y_2, \dots, y_1y_2^{p-1}, \quad \dots, \quad y_1^{p-1}, y_1^{p-1}y_2, \dots, y_1^{p-1}y_2^{p-1}$$

and the second basis as:

$$1, y, \dots, y^{p-1}, \quad x, xy, \dots, xy^{p-1}, \quad \dots, \quad x^{p-1}, x^{p-1}y, \dots, x^{p-1}y^{p-1},$$

the matrix of $\Omega^\#$ as a $k[t]$-linear homomorphism is a block diagonal matrix:

$$A = \begin{pmatrix} A_0 & 0 & \dots & 0 \\ 0 & A_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & A_{p-1} \end{pmatrix}, \quad \text{with:} \quad A_i = \begin{pmatrix} 0 & t \cdot \mathbb{I}_i \\ \mathbb{I}_{p-i} & 0 \end{pmatrix}, \quad \text{for} \quad 0 \le i \le p-1,$$

where $\mathbb{I}_n$ denotes the identity $n \times n$ matrix, for $n \in \mathbb{N}$. Up to a factor (-1), which is anyway a unit, the determinant of A is:

$$\delta = \prod_{i=0}^{p-1} \det A_i = \prod_{i=0}^{p-1} t^i = t^{\frac{p(p-1)}{2}} \in k[t].$$

Irreducible closed subschemes of codimension 1 in S correspond to closed points of S, i.e. maximal ideals of $k[t]$. The only maximal ideal of $k[t]$ containing δ is (t) and we have $\mathrm{ord}_{(t)}(\delta) = p(p-1)/2$ (in particular, $\operatorname{Supp}\delta = B^i$ as a set, confirming remark 4.1). Thus, identifying $S \cong \mathbb{A}^1_k$, the inifinitesimal branch divisor is:

$$B^i = \frac{p(p-1)}{2} \cdot [\{\, \underline{0} \,\}].$$

The $\alpha_{p,k}$-action on the affine line

Let again k be a field of positive characteristic p. The group of p-th roots of zero $\alpha_{p,k} = \operatorname{Spec} k[x]/(x^p)$ over k is a finite k-group scheme of rank p, which, like $\mu_{p,k}$, is not étale over k. Consider the action ρ of $\alpha_{p,k}$ on the affine line $X := \mathbb{A}^1_k$ given, for any k-scheme T, by:

$$\begin{aligned} \rho(T)\colon \alpha_{p,k}(T) \times \mathbb{A}^1_k(T) &\to \mathbb{A}^1_k(T) \\ (\nu, \beta) &\mapsto \frac{\beta}{1+\nu\beta}, \end{aligned}$$

where $\alpha_{p,k}(T) = \{\, \nu \in \mathcal{O}_T(T) \mid \nu^p = 0 \,\}$ and $\mathbb{A}^1_k(T) = \mathcal{O}_T(T)$. Note that, for $\nu, \beta \in \mathcal{O}_T(T)$ with $\nu^p = 0$, we have that $(1+\nu\beta)$ is a unit in $\mathcal{O}_T(T)$, with inverse $(1+\nu\beta)^{p-1} = (1 - \nu\beta + \cdots + \nu^{p-1}\beta^{p-1})$. On the algebras:

$$\begin{aligned} \rho^\#\colon k[y] &\to k[x,y]/(x^p) \cong k[x]/(x^p) \otimes_k k[y] \\ y &\mapsto \frac{y}{1+xy}. \end{aligned}$$

Similarly to the previous example, we have:

$$\begin{aligned} k[y]^{\alpha_{p,k}} &= \{\, p(y) \in k[y] \mid \rho^\#(p(y)) = p(y) \,\} = \\ &= \{\, p(y) \in k[y] \mid p\Big(\frac{y}{1+xy}\Big) = p(y) \,\} = k[y^p] \cong k[t] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad y^p \;\leftarrow\!\shortmid\; t. \end{aligned}$$

Indeed, for $n \in \mathbb{N}$, the term $\rho^\#(y^n) = y^n/(1+xy)^n$ contains a summand $-nxy^{n+1}$, which is not zero unless $p|n$ and, for different n's these monomials are all linearly independent over k. Using theorem 2.20, we find that the quotient scheme of ρ is the affine line $S = \operatorname{Spec} k[t]$, with quotient map $f\colon X \to S$ given by the inclusion: $k[t] \to k[y]$, $t \mapsto y^p$. We set $G := \alpha_{p,k} \times_{\operatorname{Spec} k} S = \operatorname{Spec} k[x,t]/(x^p)$ and write again $\rho\colon G \times_S X \to X$ for the action over S. As before, $G \to S$ and $f\colon X \to S$ are finite flat of degree p and S is regular.

Let us now compute the infinitesimal branch locus, in order to check that ρ is generically free. We have:

$$G \times_S X = \operatorname{Spec} k[x,t]/(x^p) \otimes_{k[t]} k[y] \cong \operatorname{Spec} k[x,y]/(x^p),$$

while $X \times_S X \cong \operatorname{Spec} k[y_1, y_2]/(y_1^p - y_2^p)$ as in the previous example. The graph morphism $\Omega\colon G \times_S X \to X \times_S X$ and the diagonal map $\Delta\colon X \to X \times_S X$ are given, on the algebras, by:

$$\begin{aligned} \Omega^{\#}\colon k[y_1, y_2]/(y_1^p - y_2^p) &\to k[x, y]/(x^p) \\ y_1 &\mapsto \frac{y}{1 + xy}, \\ y_2 &\mapsto y \end{aligned}$$

and:

$$\begin{aligned} \Delta^{\#}\colon k[y_1, y_2]/(y_1^p - y_2^p) &\to k[z] \\ y_1 &\mapsto z, \\ y_2 &\mapsto z, \end{aligned}$$

if the source of Δ is $X = \operatorname{Spec} k[z]$. We can then compute the algebra corresponding to the stabiliser with the dual diagram of (3.5):

$$\begin{tikzcd} k[y_1, y_2]/(y_1^p - y_2^p) \arrow[r, "\Omega^{\#}"] \arrow[d, "\Delta^{\#}"] & k[x, y]/(x^p) \arrow[d] & y \arrow[d, mapsto] \\ k[z] \arrow[r] & k[x, z]/(x^p, xz^2) & z, \end{tikzcd}$$

so: $\mathrm{St}_\rho = \operatorname{Spec} k[x, z]/(x^p, xz^2) \longrightarrow \operatorname{Spec} k[z] = X$. The fibers for $\mathfrak{p} \in X = \operatorname{Spec} k[z]$ are $\mathrm{St}_\rho(\mathfrak{p}) \cong k(\mathfrak{p})$ if $\mathfrak{p} \in X \setminus \{ (z) \} = \mathbb{A}_k^1 \setminus \{ \underline{0} \}$, where $k(\mathfrak{p})$ is the residue field at $\mathfrak{p}$, whereas $\mathrm{St}_\rho((z)) \cong \operatorname{Spec} k[x]/(x^p) \cong \alpha_{p,k}$. Exactely as for the multiplicative action of $\mu_{p,k}$, thus, the infinitesimal ramification locus is $R^i = \{ \underline{0} \}$, the infinitesimal branch locus is $B^i = f(R^i) = \{ (t) \} \subseteq \operatorname{Spec} k[t] = S$ and the action is therefore generically free.

Finally, let us determine the infinitesimal branch divisor B^i. In order to compute the determinant of the graph morphism $\Omega^{\#}$, we will use the same $k[t]$-basis of $k[y_1, y_2]/(y_1^p - y_2^p)$ (with $t \mapsto y_1^p = y_2^p$) as in the previous example, although this time it is convenient to choose a different order:

$$1, y_2, \dots, y_2^{p-1}, \quad y_1^{p-1}, y_1^{p-1} y_2, \dots, y_1^{p-1} y_2^{p-1}, \quad \dots, \quad y_1, y_1 y_2, \dots, y_1 y_2^{p-1}.$$

For the $k[t]$-module $k[x, y]/(x^p)$ (with $t \mapsto y^p$), instead, we have the basis:

$$1, y, \dots, y^{p-1}, \quad x, xy, \dots, xy^{p-1}, \quad \dots, \quad x^{p-1}, x^{p-1} y, \dots, x^{p-1} y^{p-1}.$$

Now, for $0 \le j \le p - 1$ and $1 \le i \le p - 1$, the graph $\Omega^{\#}$ has the following description on the basis elements:

$$\begin{aligned} y_2^j &\mapsto y^j = 1 \cdot y^j, \\ y_1^{p-i} y_2^j &\mapsto \frac{y^{p-i}}{(1 + xy)^{p-i}} y^j = y^{j+p-i}(1 + xy)^i &&= x^i y^{j+p} + \text{terms of lower degree in } x \\ & &&= t \cdot x^i y^j + \text{terms of lower degree in } x. \end{aligned}$$

Thus, the matrix of $\Omega^{\#}$ as a $k[t]$-linear homomorphism is an upper triangular matrix, whose diagonal features the number 1 in the first p entries and t in the other $(p-1)p$ entries:

$$A = \begin{pmatrix} \mathbb{I}_p & * & \dots & * \\ 0 & t & * & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & t \end{pmatrix}.$$

In particular:

$$\delta = \det A = t^{p(p-1)} \in k[t].$$

Here, $\operatorname{ord}_{(t)}(\delta) = p(p-1)$, whereas $\delta_s \in \mathcal{O}_{S,s}^{\times}$ for all closed points $s \in S \setminus \{ (t) \}$. Therefore, identifying $S \cong \mathbb{A}_k^1$, the infinitesimal branch divisor is:

$$B^i = p(p-1) \cdot [\{ \underline{0} \}].$$

Remark 4.2. Note that the two actions studied in these examples gave rise to the same quotient and the same infinitesimal branch locus, as a set. On the other hand, the infinitesimal branch divisor is $\frac{p(p-1)}{2} \cdot [\{ \underline{0} \}]$ for the $\mu_{p,k}$-action and $p(p-1) \cdot [\{ \underline{0} \}]$ for the $\alpha_{p,k}$-action. The upgrade of B^i to an effective Weil divisor has thus been substancial, in order to distinguish a different behaviour in each situation.

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