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Potential robustness measures for digraphs

Master thesis

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Contents

1 Introduction

Whenever a connection in a network fails, it could have grave consequences. It is important that even if these malfunctions occur, the network keeps on performing well. This ability is called robustness. The more a network is resistant to failures, the more robust it is. In this thesis, we will consider graphs as a representation of networks. While there has been a lot of research done for undirected graphs, there is little we know about directed graphs even though directed networks play a large roll in our world. Our aim is to find a mathematical definition and an approach to calculate the robustness of directed graphs such that it does not contradict with our sense of robustness. The most important property is that when an edge is added, the robustness should not decrease. The same holds if an edge weight is increased. In literature, it is also desired that a direct connection is equal or more robust than an indirect connection (triangle inequality). If there is a way to calculate the robustness of a network, it is possible to compare networks and to design networks that are more robust.

First we will show how to calculate robustness for undirected graphs. This will be done by using the concept of effective resistance between two nodes, which is explained in Section 2. In Section 3, the definition of the Laplacian of a graph will be given and some properties of the Laplacian will be shown. In Section 4, we will show how the robustness is calculated by using the Laplacian pseudoinverse. In the master thesis of W. Ellens [4], it is shown that the effective resistance function has some nice properties. That is, the total effective resistance of a graph strictly decreases (the graph is more robust) when an edge is added or weights are increased. Furthermore, the effective resistance function is a metric (distance function). The proof of these properties will not be given in this thesis. It can be found in Chapter 4.3 in [4]. Another way to find the Laplacian pseudoinverse will be shown in Section 5. This approach is obtained from [10]. In Section 6, an alternative expression of the effective resistance between two nodes which is obtained from [1] will be given.

We will then analyze directed graphs. In Section 7, it is shown how the method used in Section 4 fails for directed graphs. In Section 8, we analyze [10] and their definition of effective resistance for directed graphs. This is followed by Section 9 in which we propose and analyze a different, but similar definition. In Section 10, we will try to extend the expression found in Section 6 to directed graphs.

2 Effective resistance

A graph is denoted by $G = (V, E)$, where V the set of N vertices and E is the set of edges or arrows of G when G is undirected respectively directed. A weighted graph has weights on the edges. The weight of edge (i, j) is denoted with w_{ij}. We assume that all weights are positive. If no weight is written next to an edge, it is assumed that the edge weight is 1.

To determine the effective resistance, the graph is viewed as an electrical circuit, where an edge (i, j) corresponds to a resistor R_{ij}. For each pair of vertices the effective resistance between these vertices can be calculated with the well-known series and parallel manipulations. Two edges, corresponding to resistors with resistance R_1 and R_2, connected in series can be replaced by one edge with effective resistance $R_1 + R_2$. On the other hand, two resistors connected in parallel can be replaced by a resistor with resistance $\frac{1}{\frac{1}{R_1}+\frac{1}{R_2}}$. Notice that from these series and parallel manipulations, the effective resistance takes both the number of paths and their length into account, intuitively measuring the presence and quality of back-up possibilities.

Now using Kirchhoff's first law and Ohm's law, an expression for the effective resistance r_{ab} between vertices a and b is found. Kirchhoff's first law states that at any vertex in an electrical circuit, the sum of currents flowing into that vertex is equal to the sum of currents flowing out of that vertex. In other words, connect a voltage between vertices a and b and let I be the net current out of a and into b. Then the current y_{ij} between vertices i and j must satisfy

$$\sum_{j \in N(j)} y_{ij} = \begin{cases} I & \text{if } i = a, \\ -I & \text{if } i = b, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $N(i)$ is the set of first neighbors of vertex i. Now Ohm's law allows us to associate a potential v_i to any vertex i, such that for all edges (i, j) we have

$$y_{ij} R_{ij} = v_i - v_j. \tag{2}$$

Definition 2.1. *The effective resistance between vertex a and b is defined as*

$$r_{ab} = \frac{v_a - v_b}{I}.$$

Definition 2.2. *The total effective resistance for undirected graphs, also known as the Kirchhoff index K_f, is defined as the sum of the effective resistances over all pairs of vertices:*

$$K_f = \sum_{i=1}^{N} \sum_{j=i+1}^{N} r_{ij}.$$

Note that if one graph is more robust than another, the former's Kirchhoff index is smaller.

3 The (weighted) Laplacian L

For every graph G, directed or undirected, we can find the associated Laplacian matrix.

Definition 3.1. *The Laplacian L is given by*

$$L = D - A,$$

where D is the diagonal matrix of the node out-degrees and A is the adjacency matrix corresponding to G.

Definition 3.2. *For a graph with non-negative edge weights w_{ij}, the analogue of the adjacency matrix is the matrix of weights $W = (w_{ij})$. The weighted Laplacian, L^w, is given by*

$$L^w = S - W,$$

where S is the diagonal matrix of strengths, with $S_{ii} = \sum_{j=1}^{N} w_{ij}$.

Notice the Laplacian matrix is the weighted Laplacian matrix where the weights are either zero or one. For notational convenience we will write L for the weighted Laplacian.

We define the resistance of an edge R_{ij} as $\frac{1}{w_{ij}}$.

Theorem 3.1. *The effective resistance of an edge (i, j) is computed as follows*

$$r_{ij} = (\mathbf{e}_N^{(i)} - \mathbf{e}_N^{(j)})^\top Y (\mathbf{e}_N^{(i)} - \mathbf{e}_N^{(j)}) = Y_{ii} + Y_{jj} - Y_{ij} - Y_{ji}, \quad (3)$$

where $Y \in \mathbb{R}^{N \times N}$ acts as the inverse of the Laplacian matrix on $\mathbf{1}_N^\perp$ and as the zero map on sp$\{\mathbf{1}_N\}$.

Proof. This proof is similar to the proof of Theorem 4.1 in [4]. □

We will see in Section 4 that the Laplacian pseudoinverse satisfies these properties mentioned in Theorem 3.1.

3.1 Properties of the Laplacian

In this section some properties of the Laplacian are shown.

Lemma 3.1. *The (weighted) Laplacian corresponding to an undirected graph is symmetric, while the (weighted) Laplacian corresponding to a directed graph can be non-symmetric.*

Proof. For undirected graphs, we have

$$(i, j) \in E \Leftrightarrow (j, i) \in E$$

Thus, $L_{ij} = L_{ji}$ for all $i, j \in V$. On the otherhand, if G is a directed graph, it is possible that $(i, j) \in E$, while $(j, i) \notin E$ i.e. $L_{ij} = -w_{ij}$ and $L_{ji} = 0$. □

Whether or not the Laplacian is symmetric, the sum of each row is always equal to zero, thus $L\mathbf{1}_N = 0$ i.e. 0 is always an eigenvalue of L and $\mathbf{1}_N$ its corresponding eigenvector.

The following lemma shows that the eigenvalues of the Laplacian L of an undirected graph are all positive.

Lemma 3.2. *Let G be an undirected graph. The (weighted) Laplacian L of G is a positive semi-definite matrix.*

Proof. Taken from [4]. First convert G in a directed graph, by choosing an arbitrary direction for each edge in G. Now define an arc-vertex incidence matrix M, i.e. for an arc a and a vertex v:

$$M_{av} = \begin{cases} \sqrt{w_{ij}} & \text{if } a = (i,j) \text{ and } v = i, \\ -\sqrt{w_{ij}} & \text{if } a = (i,j) \text{ and } v = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then the Laplacian L satisfies $L = M^\top M$ and we have for all $z \in \mathbb{R}^N$:

$$\begin{aligned} z^\top L z &= z^\top M^\top M z \\ &= (Mz)^\top (Mz) \geq 0, \end{aligned}$$

where the last inequality holds, since it is the inner product of the vector Mz with itself. □

Lemma 3.3. *Let G be a directed graph. Then all eigenvalues of the Laplacian L of G have non-negative real part.*

Proof. It follows from Gerschgorin circles [8] that all eigenvalues of L have non-negative real part. □

As mentioned before, zero is always an eigenvalue of L. The algebraic multiplicity of zero is given in the next theorem. However, we introduce a definition first.

Definition 3.3. *A component C of a directed graph G is called a strongly connected component if for all vertices i, j of C, there is a directed path from i to j and a directed path from j to i. A directed path from $i \neq j$ to j is a sequence of arcs which connects a sequence of vertices starting from i and ending in j such that all arcs have the same direction. Assume that there is always a directed path from i to i for every $i \in G$. A strongly connected closed component C is a strongly connected component such that for all vertices j not belonging to C we have $w_{ij} = 0$ for all vertices i in C.*

Theorem 3.2. *Let G be an undirected graph. The algebraic multiplicity of the eigenvalue zero of the (weighted) Laplacian L of G is equal to the number of connected components of G. If G were a directed graph, then it is equal to the number of strongly connected closed components of G.*

Proof. The following proof is from [4].
Let G be an undirected graph and L its (weighted) Laplacian. For every connected component C, let $y^C \in \mathbb{R}^n$ be the vector with $y_i^C = 1$ if i is a vertex of C and $y_i^C = 0$ otherwise. Notice that y^C is an eigenvector of L with eigenvalue 0, since $Ly^C = 0$. Furthermore, the set of these eigenvectors is linearly independent. Thus, we will prove that all eigenvectors with eigenvalue zero are linear combinations of these eigenvectors.

Now suppose x is an eigenvector of L with eigenvalue zero, i.e. $Lx = \mathbf{0}$, then we have for all $i \in V$

$$x_i \sum_{j=1}^{n} w_{ij} = \sum_{j=1}^{n} x_j w_{ij}. \tag{4}$$

Now let C be an arbitrary component and i such that $x_i = \max_{j \in C} x_j$, then using (4) we find

$$x_i \sum_{j=1}^{n} w_{ij} = \sum_{j=1}^{n} x_j w_{ij} \leq \sum_{j=1}^{n} x_i w_{ij} = x_i \sum_{j=1}^{n} w_{ij}.$$

Thus $x_i = x_j$ for all neighbours j of i, since in that case we have $w_{ij} > 0$. Similarly, we see that $x_j = x_k$ for every neighbor k of j. By repeating this argument, we see that all eigenvectors x have $x_i = x_j$ when i and j are part of the same component. Therefore, all eigenvectors with eigenvalue zero are linear combinations of the vectors y^C.

If G were a directed graph, we let C be a strongly connected closed component and then the proof still holds. □

For example, notice in Example 8.1 there is only one strongly connected closed component, that is, $\{3\}$.

4 The Laplacian pseudoinverse L^+

As mentioned before there are multiple ways to calculate the effective resistance for undirected graphs. In [4] it is shown how one can calculate the effective resistance for undirected graphs using the Laplacian. Leaving the proofs aside, we will repeat some of the results.

Since the Laplacian has an eigenvalue zero, it is not invertible. However, if we restrict the linear transformation to the subspace orthogonal to the null space $\text{sp}\{\mathbf{1}_N\}$, the matrix can be inverted. Let L^+ be the matrix that corresponds to this inverse on $\mathbf{1}_N^\perp$ and to the zero map on $\text{sp}\{\mathbf{1}_N\}$. In other words:

Definition 4.1. *The Laplacian pseudoinverse L^+ is defined as the matrix satisfying*

$$L^+\mathbf{1}_N = \mathbf{0}$$

and for every $\mathbf{w}\perp\mathbf{1}_N$:

$$L^+\mathbf{w} = \mathbf{v} \Leftrightarrow L\mathbf{v} = \mathbf{w} \text{ and } \mathbf{v}\perp\mathbf{1_N}.$$

This definition is a specific case of the Moore-Penrose pseudoinverse for general $m \times n$ matrices provided that the graph is undirected. In Section 5.2, it is shown that it is unique.

One way to construct the Laplacian pseudoinverse is as follows. Since the Laplacian is symmetric, it has an orthonormal set of eigenvectors $v_1, \dots, v_N$. Let U be the matrix that has these eigenvectors as its columns, consequently $U^{-1} = U^\top$, and let D be the matrix with the corresponding eigenvalues on its diagonal and zero elsewhere. Notice that $\lambda_i = 0$ for some $i = 1, \dots, N$, because 0 is an eigenvalue of L. Without loss of generality, assume $\lambda_1 = 0$. Furthermore $\lambda_i \geq 0$ for all i, since L is a positive semi-definite matrix, see Lemma 3.2. Thus, D is of the form:

$$D = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_N \end{pmatrix}.$$

Then the Laplacian satisfies $L = UDU^{-1} = UDU^\top$. The pseudoinverse is then as follows

$$L^+ = UD^+U^{-1} = UD^+U^\top,$$

with D^+ :

$$D^+ = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & \frac{1}{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\lambda_N} \end{pmatrix}.$$

Furthermore, L^+ is similar to D^+, thus the eigenvalues of L^+ are $0, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_n}$. Also, L^+ is symmetric:

$$\begin{aligned}(L^+)^\top &= (UD^+U^\top)^\top = U(D^+)^\top U^\top \\ &= UD^+U^\top = L^+\end{aligned}$$

Lemma 4.1. *The Laplacian pseudoinverse L^+ satisfies*

$$L^+ = \int_0^\infty \left(e^{-Lt} - \frac{1}{N}\mathbf{1}_N \mathbf{1}_N^\top \right) dt.$$

Proof. Let $L = UDU^{-1}$ with U and D as above. Note the first column of U is $\left(\frac{1}{\sqrt{N}}, \ldots, \frac{1}{\sqrt{N}}\right)^\top$ and we have an orthonormal set of eigenvectors. Observe:

$$\begin{aligned}
e^{-Lt} &= \sum_{k=0}^{\infty} \frac{(-1)^k L^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k (UDU^{-1})^k t^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k U D^k U^{-1} t^k}{k!} = U \sum_{k=0}^{\infty} \frac{(-1)^k D^k t^k}{k!} U^{-1} \\
&= U \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & e^{-\lambda_2 t} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{-\lambda_N t} \end{pmatrix} U^{-1} \\
&= \begin{pmatrix} \frac{1}{N} & \cdots & \frac{1}{N} \\ \vdots & \ddots & \vdots \\ \frac{1}{N} & \cdots & \frac{1}{N} \end{pmatrix} + e^{-\lambda_2 t} v_2^\top v_2 + \ldots + e^{-\lambda_N t} v_N^\top v_N
\end{aligned}$$

Thus we find

$$\begin{aligned}
\int_0^\infty \left(e^{-Lt} - \frac{1}{N}\mathbf{1}_N \mathbf{1}_N^\top \right) dt &= \int_0^\infty \left(e^{-\lambda_2 t} v_2^\top v_2 + \ldots + e^{-\lambda_N t} v_N^\top v_N \right) dt \\
&= \frac{1}{\lambda_2} v_2^\top v_2 + \ldots + \frac{1}{\lambda_N} v_N^\top v_N \\
&= UD^+U^{-1} = L^+.
\end{aligned}$$

□

Since the Laplacian pseudoinverse is given by $L^+ = UD^+U^\top$, it acts as the inverse of the Laplacian matrix on $\mathbf{1}_N^\perp$ and as the zero map on $\mathrm{sp}\{\mathbf{1}_N\}$. This leads to the following corollary.

Corollary 4.1. *The effective resistance of an edge (i, j) is computed as follows*

$$r_{ij} = (\mathbf{e}_N^{(i)} - \mathbf{e}_N^{(j)})^\top L^+ (\mathbf{e}_N^{(i)} - \mathbf{e}_N^{(j)}) = L_{ii}^+ + L_{jj}^+ - 2L_{ij}^+.$$

Now the total effective resistance is given by the following theorem.

Theorem 4.1. *The Kirchhoff index K_f satisfies*

$$K_f = N \sum_{i=2}^{N} \frac{1}{\lambda_i^L}, \tag{5}$$

with λ_i^L's the eigenvalues of L.

Proof.

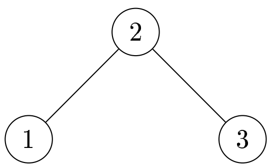
$$\begin{aligned} K_f &= \sum_{i=1}^{N}\sum_{j=i+1}^{N} r_{ij} = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} r_{ij} \\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left((L)^+_{ii} + (L)^+_{jj} - 2(L)^+_{ij}\right) \\ &= N\sum_{i=1}^{N}(L^+_{ii}) - \mathbf{1}_N^\top L^+ \mathbf{1}_N \\ &= N\,\mathrm{Tr}(L^+) = N\sum_{i=2}^{N}\frac{1}{\lambda_i^L}. \end{aligned}$$

□

4.1 L^+: Example undirected graph

Using the Laplacian pseudoinverse, the total effective resistance can be calculated. Observe a rather easy example.

Example 4.1. *Let G be an undirected graph with weights 1:*



with the Laplacian matrix $L = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$.

The eigenvalues of L are $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 3$. Since L is symmetric, it has an orthonormal set of eigenvectors, explicitly

$$\begin{aligned} v_1 &= \left(\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}\right), \\ v_2 &= \left(-\frac{1}{2}\sqrt{2}, 0, \frac{1}{2}\sqrt{2}\right), \\ v_3 &= \left(\frac{1}{6}\sqrt{6}, -\frac{1}{3}\sqrt{6}, \frac{1}{6}\sqrt{6}\right). \end{aligned}$$

Let U be the matrix that has these eigenvectors as its columns:

$$U = \begin{pmatrix} \frac{1}{3}\sqrt{3} & -\frac{1}{2}\sqrt{2} & \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{3} & 0 & -\frac{1}{3}\sqrt{6} \\ \frac{1}{3}\sqrt{3} & \frac{1}{2}\sqrt{2} & \frac{1}{6}\sqrt{6} \end{pmatrix}, U^{-1} = U^\top = \begin{pmatrix} \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} \\ -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ \frac{1}{6}\sqrt{6} & -\frac{1}{3}\sqrt{6} & \frac{1}{6}\sqrt{6} \end{pmatrix}.$$

Then let D be the matrix with the eigenvalues on the diagonal and zero elsewhere:

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Note $L = UDU^{-1} = UDU^\top$. Then D^+ is given by

$$D^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix},$$

and so the Laplacian pseudoinverse L^+ is

$$L^+ = UD^+U^{-1} = UD^+U^\top = \frac{1}{9}\begin{pmatrix} 5 & -1 & -4 \\ -1 & 2 & -1 \\ -4 & -1 & 5 \end{pmatrix}.$$

We have $L^+\mathbf{1} = 0$ and for every $\mathbf{w} \perp \mathbf{1}$:

$$L^+\mathbf{w} = \mathbf{v} \quad \text{such that } L\mathbf{v} = \mathbf{w} \text{ and } \mathbf{v} \perp \mathbf{1}.$$

For example take $\mathbf{w} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, then $\mathbf{v} = \frac{1}{9}\begin{pmatrix} 6 \\ -3 \\ -3 \end{pmatrix}$ and $L\mathbf{v} = \mathbf{w}$.

The total effective resistance of this graph is $K_f = 3 \cdot \left(1 + \frac{1}{3}\right) = 4$.

5 Different approach for the undirected graph

In this section, a different way to construct the Laplacian pseudoinverse is shown. This matrix, which we will refer to as X in this section, is thus identical to the Laplacian pseudoinverse L^+ for undirected graphs. However, in Section 8 the matrix X will be redefined such that it is unique and symmetric for directed graphs, while acting as the Laplacian pseudoinverse for undirected graphs. Effective resistance is then defined using the redefined X. Consequently, the square root of the effective resistance is a metric on the nodes of any connected directed graph. The following results are from [10].

Let $\Pi = I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top$, denote the orthogonal projection matrix onto the subspace of $\mathbb{R}^N$ perpendicular to $\mathbf{1}_N$. This subspace will be denoted with $\mathbf{1}_N^\perp$. Π is a symmetric matrix. Since the entries of the rows of L add up to 0 (i.e. the rows are perpendicular to 1), we have $L\mathbf{1}_N = \mathbf{0}$, $L\Pi = L$ and $\Pi L^\top = L^\top$ for any graph. Note, if the graph is balanced, i.e. the out-degree and in-degree are equal, which is true for undirected graphs, $\Pi L = \Pi L^\top = L^\top = L$ also hold.
Let $Q \in \mathbb{R}^{(N-1)\times N}$ be a matrix whose rows form an orthonormal basis for $\mathbf{1}_N^\perp$. Thus, we require $\left[\frac{1}{\sqrt{N}}\mathbf{1}_N \; Q^\top\right]$ to be an orthogonal matrix. It follows,

$$Q\mathbf{1}_N = 0, QQ^\top = I_{N-1} \text{ and } Q^\top Q = \Pi. \tag{6}$$

The last result holds since $\left[\frac{1}{\sqrt{N}}\mathbf{1}_N \; Q^\top\right]$ is an orthogonal matrix, thus

$$\left[\frac{1}{\sqrt{N}}\mathbf{1}_N \; Q^\top\right]\left[\frac{1}{\sqrt{N}}\mathbf{1}_N \; Q^\top\right]^\top = I_N$$

and by adding the $\frac{1}{\sqrt{N}}\mathbf{1}_N$ column to Q, we essentially add $\frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top$ to $Q^\top Q$.

X will be constructed such that X is equal on $\mathbf{1}_N^\perp$ to the inverse of L and 0 otherwise. Thus, it satisfies the definition of the Laplacian pseudoinverse in Section 4. This can be achieved by taking X as the unique generalized inverse of L.

5.1 Reduced Laplacian $\overline{L}$

We construct X by using the reduced Laplacian. This matrix acts as the Laplacian L on $\mathbf{1}_N^\perp$.

Let $b_1, \dots, b_{N-1}$ be an orthonormal basis for $\mathbf{1}_N^\perp$ and let $Q \in \mathbb{R}^{(N-1)\times N}$ be the matrix formed with these basis vectors as rows. Then for any $\mathbf{v} \in \mathbb{R}^N$, we can write $\mathbf{v} = \sum_{i=1}^{N-1} c_i b_i + c_N \mathbf{1}_N$, with $c_1, \dots, c_N \in \mathbb{R}$. By taking the innerproduct with any b_i on both sides, we find $b_i \cdot \mathbf{v} = c_i$. Thus, $\overline{\mathbf{v}} := Q\mathbf{v}$ is a coordinate vector (with respect to the chosen basis) of the orthogonal projection of $\mathbf{v}$ onto $\mathbf{1}_N^\perp$. For any $M \in \mathbb{R}^{N\times N}$, let $\overline{M} \in \mathbb{R}^{N-1\times N-1}$ be the linear transformation with respect to $b_1, \dots, b_{N-1}$ with $\overline{M} := QMQ^\top$. Then $\overline{M}$ acts as M on $\mathbf{1}_N^\perp$, i.e.

$$\overline{M}\overline{\mathbf{v}} = (QMQ^\top)Q\mathbf{v} = QM\Pi\mathbf{v} = QM\mathbf{v} = \overline{M\mathbf{v}} \text{ for any } \mathbf{v} \in \mathbf{1}_N^\perp.$$

Thus, on $\mathbf{1}_N^{\perp}$, the Laplacian matrix L is equivalent to

$$\overline{L} = QLQ^{\top}, \tag{7}$$

which we refer to as the reduced Laplacian. We see that $\overline{L}$ is symmetric if and only if the graph is balanced, since in that case we have $L = L^{\top}$:

$$\overline{L}^{\top} = (QLQ^{\top})^{\top} = QL^{\top}Q^{\top} = QLQ^{\top} = \overline{L}.$$

Theorem 5.1. *Let L be the Laplacian matrix, then $\overline{L}$ given in (7) has the same eigenvalues, except for a single zero eigenvalue, as L. Also, if v_α is an eigenvector with eigenvalue $\alpha \neq 0$ of L, then $\overline{v_\alpha}$ on $\mathbf{1}_N^{\perp}$ is an eigenvector with eigenvalue α of $\overline{L}$.*

Proof. This proof is partially taken from [9].

Define the matrix

$$V = \begin{bmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N^{\top} \\ Q \end{bmatrix}, \tag{8}$$

with Q satisfying (6). Then $VV^{\top} = I_N$ and $V^{\top}V = I_N$ hold, which means $V^{\top}$ is the inverse of V, i.e. $V^{\top} = V^{-1}$. Now we see that L is similar to $VLV^{\top}$ for the invertible matrix V, thus L and $VLV^{\top}$ have the same eigenvalues. Using the fact $L\mathbf{1}_N = 0$, it follows $VLV^{\top} = \begin{pmatrix} 0 & 0 \\ 0 & \overline{L} \end{pmatrix}$ with $\overline{L}$ given in (7) and we see that this is a block matrix. Thus, the eigenvalues of $VLV^{\top}$, which are the same as the eigenvalues of L, are the solutions of $\lambda \cdot \det(\lambda I_{N-1} - \overline{L}) = 0$, i.e. zero and the eigenvalues of $\overline{L}$. We conclude that $\overline{L}$ has the same eigenvalues as L except for a single zero eigenvalue.

Let v_α be an eigenvector of L with eigenvalue α. Then, we have

$$\begin{aligned} \overline{L}\overline{v_\alpha} &= QLQ^{\top}Qv_\alpha = QL\Pi v_\alpha \\ &= QLv_\alpha = Q\alpha v_\alpha = \alpha\overline{v_\alpha}. \end{aligned}$$

Thus, $\overline{v_\alpha}$ is an eigenvector with eigenvalue α of $\overline{L}$. □

Hence, it follows from Theorems 3.2 and 5.1 that for a connected graph, $\overline{L}$ is invertible since L only has one 0 eigenvalue.

5.2 Generalized inverse

Definition 5.1. *For $A \in \mathbb{R}^{m\times n}$, the generalized inverse of A is defined as a matrix $A^+ \in \mathbb{R}^{n\times m}$ satisfying the so-called Moore-Penrose criteria:*

1. $AA^+A = A$
2. $A^+AA^+ = A^+$
3. $(AA^+)^{\top} = AA^+$
4. $(A^+A)^{\top} = A^+A$

Theorem 5.2. *The generalized inverse of $A \in \mathbb{R}^{m\times n}$ is unique.*

Proof. Let $B, C \in R^{n\times m}$ both be a generalized inverse of $A \in \mathbb{R}^{m\times n}$, thus they satisfy the Moore-Penrose criteria. Then observe that

$$\begin{aligned} AB &= (AB)^\top = B^\top A^\top = B^\top (ACA)^\top = B^\top A^\top C^\top A^\top \\ &= (AB)^\top (AC)^\top = ABAC = AC. \end{aligned}$$

Analogously we find $BA = CA$. It follows,

$$B = BAB = BAC = CAC = C,$$

thus the generalized inverse of $A \in \mathbb{R}^{m\times n}$ is unique. □

As mentioned before, X will be constructed as the unique generalized inverse of L. Thus, it must satisfy the Moore-Penrose criteria. This can also be formulated differently.

Lemma 5.1. *If X satisfies the following:*

$$XL = LX = \Pi \quad \text{and } X\Pi = \Pi X = X, \tag{9}$$

then X satisfies the Moore-Penrose criteria.

Proof. Let X be a solution to (9). Since L is symmetric, we have:

1. $LXL = \Pi L = L$
2. $XLX = \Pi X = X$
3. $(LX)^\top = \Pi^\top = \Pi = LX$
4. $(XL)^\top = \Pi^\top = \Pi = XL$.

Thus it satisfies the Moore-Penrose criteria. □

5.3 Defining X

Now using the reduced Laplacian $\overline{L}$ we give an explicit construction for X:

$$X = Q^\top \overline{L}^{-1} Q. \tag{10}$$

Observe:

- $X\mathbf{1}_N = Q^\top \overline{L}^{-1} Q\mathbf{1}_N = 0$, since $Q\mathbf{1}_N = 0$,
- $\overline{X\mathbf{v}} = QX\mathbf{v} = QQ^\top \overline{L}^{-1} Q\mathbf{v} = \overline{L}^{-1} Q\mathbf{v} = \overline{L}^{-1}\overline{\mathbf{v}}$ for any $\mathbf{v} \in \mathbb{R}^N$.

Thus, X acts as the inverse of $\overline{L}$, which is equivalent to the inverse of L on $\mathbf{1}_N$ and X is 0 elsewhere.

Corollary 5.1. *Let $X = [x_{i,j}]$, then the effective resistance is given by*

$$r_{ij} = (\mathbf{e}_N^{(i)} - \mathbf{e}_N^{(j)})^\top X (\mathbf{e}_N^{(i)} - \mathbf{e}_N^{(j)}) = x_{i,i} + x_{j,j} - x_{i,j} - x_{j,i}. \tag{11}$$

Since $\overline{L}$ is symmetric for undirected graphs, so is $\overline{L}^{-1}$ and thus X is also symmetric:

$$X^\top = (Q^\top \overline{L}^{-1} Q)^\top = Q^\top (\overline{L}^{-1})^\top Q = Q^\top \overline{L}^{-1} Q = X$$

Consequently, we can write

$$r_{ij} = x_{i,i} + x_{j,j} - 2x_{i,j}.$$

Notice that we always have $r_{ij} = r_{ji}$, even if X is not symmetric.

Lemma 5.2. *X constructed as in (10) satisfies (9) when the graph is undirected and is thus the Moore-Penrose generalized inverse of L. Hence, it is unique.*

Proof. Using (6) and $L = L\Pi = \Pi L$, we find

$$XL = XL\Pi = Q^\top \overline{L}^{-1} Q L Q^\top Q = Q^\top \overline{L}^{-1} \overline{L} Q = Q^\top Q = \Pi$$

and analogously we find $LX = \Pi$. Furthermore,

$$X\Pi = Q^\top \overline{L}^{-1} Q \Pi = Q^\top \overline{L}^{-1} Q Q^\top Q = Q^\top \overline{L}^{-1} Q = X$$

and again in the same way $\Pi X = X$ also holds. □

Note $\overline{L}$ is not unique, since it depends on the choice of Q. However X is independent of the choice of Q as long it satisfies (6).

Theorem 5.3. *X is independent of the choice of Q.*

Proof. Let Q and Q' both satisfy (6). Define $P := Q'Q^\top$. P is orthogonal:

$$P^\top P = (Q'Q^\top)^\top Q'Q^\top = QQ'^\top Q'Q^\top = Q\Pi Q^\top = I_{N-1}$$

and analogously we have $PP^\top = I_{N-1}$, thus $P^\top$ is the inverse of P. Write

$$Q' = Q'\Pi = Q'Q^\top Q = PQ.$$

Hence,

$$\begin{aligned} X' &:= Q'^\top (Q'LQ'^\top)^{-1} Q' = Q^\top P^\top (PQLQ^\top P^\top)^{-1} PQ \\ &= Q^\top (QLQ^\top)^{-1} Q = X. \end{aligned}$$

Thus X is independent of the choice of Q. □

5.4 X: Total effective resistance

Theorem 5.4. *The Kirchhoff index satisfies*

$$K_f = N \sum_{i=2}^{N} \lambda_i^X$$

with λ_i^X's the eigenvalues of X.

Proof. Take the matrix V as in (8). Then X is similar to $VXV^\top$ for V, thus X and $VXV^\top$ have the same eigenvalues. Since $X = Q^\top \overline{L}^{-1} Q$, we can write $VXV^\top$ as the following: $VXV^\top = \begin{pmatrix} 0 & 0 \\ 0 & \overline{L}^{-1} \end{pmatrix}$. Thus, X has a single 0 eigenvalue and its remaining eigenvalues are the same as $\overline{L}^{-1}$. However, the eigenvalues of $\overline{L}^{-1}$ are one over the eigenvalues of $\overline{L}$ and those are the same as the eigenvalues of L, excluding the zero eigenvalue. This implies that X has a single 0 eigenvalue and its remaining eigenvalues are one over the eigenvalues of L. Then using (5) we have

$$K_f = N \sum_{i=2}^{N} \frac{1}{\lambda_i^L} = N \sum_{i=2}^{N} \lambda_i^X$$

with λ_i^L and λ_i^X the i'th eigenvalue of L respectively X. □

5.5 X: Example undirected graph

We will now show the calculation of X corresponding to Example 4.1. Take

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix}, \; Q^\top = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}.$$

Then we find

$$\overline{L} = QLQ^\top = \begin{pmatrix} \frac{5}{2} & -\frac{3}{\sqrt{12}} \\ -\frac{3}{\sqrt{12}} & \frac{9}{6} \end{pmatrix} \text{ and } \overline{L}^{-1} = \frac{1}{6} \begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 5 \end{pmatrix}.$$

X is given by $X = \frac{1}{9} \begin{pmatrix} 5 & -1 & -4 \\ -1 & 2 & -1 \\ -4 & -1 & 5 \end{pmatrix}$, which is the same as L^+ and we also have $K_f = 3 \cdot \left(1 + \frac{1}{3}\right) = 4$.

6 Computing effective resistance using determinants

As we have seen in Section 4, the effective resistance between node i and j is given by

$$r_{ij} = L^+_{ii} + L^+_{jj} - L^+_{ij} - L^+_{ji}.$$

We will show another expression for r_{ij} that is quite simple. This result holds for any weighted Laplacian of an undirected graph.

First define $L(i)$ to be the submatrix obtained by deleting the i-th row and i-th column of the Laplacian matrix L of a graph G. The submatrix $L(i,j)$ is obtained by deleting the i-th and j-th rows and i-th and j-th columns of L. The following theorem and proof are from [1].

Theorem 6.1. *Let $G = (V,E)$ be an undirected connected graph with $n \geq 3$ vertices, and $1 \leq i \neq j \leq n$. Let $L(i)$ and $L(i,j)$ be the above defined submatrices of the Laplacian matrix of G. Then the effective resistance between node i and j is given by*

$$r_{ij} = \frac{det\ L(i,j)}{det\ L(i)}. \tag{12}$$

Proof. Associate to each node v_i of the graph G a variable x_i and define the auxiliary function

$$f(x_1,\ldots,x_n) = \sum_{k<l,l\in N(k)} (x_k - x_l)^2 \tag{13}$$

where $N(k)$ denotes the set of first neighbors of v_k. The following relation is a result known in the theory of electrical networks [2]. For $i \neq j$,

$$r_{ij} = \max\left\{ \frac{1}{f(x_1,\ldots,x_n)} \middle| x_i = 1, x_j = 0, 0 \leq x_k \leq 1, k = 1,\ldots,n \right\}. \tag{14}$$

Then using (13) and (14), we derive for $k \neq i,j$:

$$\frac{\partial f}{\partial x_k} = 2 \sum_{l\in N(k)} (x_k - x_l) = 0. \tag{15}$$

Since the vertices of G are numbered arbitrarily, without loss of generality, we may consider the special case in which $i = n-1$ and $j = n$. Thus $x_{n-1} = 1$ and $x_n = 0$ may hold. Let $A = (a_{ij})$ be the adjacency matrix of G and denote the submatrix $L(n-1,n)$ with L_{n-2}, then we can write the Laplacian matrix as

$$L = \begin{pmatrix} L_{n-2} & B \\ B^\top & D \end{pmatrix},$$

where we have

$$B = \begin{pmatrix} -a_{1,n-1} & -a_{1,n} \\ \vdots & \vdots \\ -a_{n-2,n-1} & -a_{n-2,n} \end{pmatrix} \text{ and } D = \begin{pmatrix} \sum_{i=1}^N a_{n-1,i} & -a_{n-1,n} \\ -a_{n,n-1} & \sum_{i=1}^N a_{n,i} \end{pmatrix}.$$

Then from (15) the following can be derived for all $k \neq n-1, n$:

$$\begin{aligned}\sum_{l \in N(k)} (x_k - x_l) &= \sum_{l \in N(k) \setminus \{n-1,n\}} (x_k - x_l) + (x_k - x_{n-1})\mathbf{1}^{(\mathbf{k})}_{\mathbf{n-1}} + (x_k - x_n)\mathbf{1}^{(\mathbf{k})}_{\mathbf{n}} \\ &= 0, \end{aligned} \tag{16}$$

with

$$\mathbf{1}^{(\mathbf{k})}_{\mathbf{n-1}} = \begin{cases} 1 & \text{if } v_{n-1} \text{ is adjacent to } v_k, \\ 0 & \text{otherwise.} \end{cases}$$

and $\mathbf{1}^{(\mathbf{k})}_{\mathbf{n}}$ similar. Now since $x_{n-1} = 1$ and $x_n = 0$, we have

$$\sum_{l \in N(k) \setminus \{n,n-1\}} (x_k - x_l) + x_k \mathbf{1}^{(\mathbf{k})}_{\mathbf{n-1}} + x_k \mathbf{1}^{(\mathbf{k})}_{\mathbf{n}} = 1 \cdot \mathbf{1}^{(\mathbf{k})}_{\mathbf{n-1}}$$

Thus

$$L_{n-2}\mathbf{x} = \mathbf{b}, \tag{17}$$

where $\mathbf{x} = (x_1, \ldots, x_{n-2})^\top$ and $\mathbf{b} = (b_1, \ldots, b_{n-2})^\top$, with $b_k = 1$ if the vertices v_{n-1} and v_k are adjacent, otherwise $b_k = 0$, for all $k = 1, \ldots, n-2$.

Since $x_{n-1} = 1, x_n = 0$ holds and L is symmetric, the following relations hold:

$$\mathbf{x}^\top B(x_{n-1}, x_n)^\top = (x_{n-1}, x_n)B^\top \mathbf{x} = -\sum_{i=1}^{n-2} a_{n-1,i}x_i = -\sum_{k \in N(n-1)} x_k$$

and

$$(x_{n-1}, x_n)D(x_{n-1}, x_n)^\top = d_{n-1},$$

with d_{n-1} denoting the degree of the vertex v_{n-1}. Now rewrite (13) as follows

$$\begin{aligned} f(x_1, \ldots, x_n) &= (\mathbf{x}^\top, x_{n-1}, x_n)L(\mathbf{x}, x_{n-1}, x_n) \\ &= \mathbf{x}^\top L_{n-2}\mathbf{x} + (x_{n-1}, x_n)B^\top \mathbf{x} + \mathbf{x}^\top B(x_{n-1}, x_n)^\top \\ &\quad + (x_{n-1}, x_n)D(x_{n-1}, x_n)^\top \\ &= \mathbf{x}^\top \mathbf{b} - 2\sum_{k \in N(n-1)} x_k + d_{n-1} \\ &= d_{n-1} - \sum_{k \in N(n-1)} x_k, \end{aligned} \tag{18}$$

where the last equation holds because

$$\mathbf{x}^\top \mathbf{b} = \sum_{i=2}^{n-2} a_{n-1,i}x_i = \sum_{k \in N(n-1)} x_k.$$

Since G is an undirected, connected graph, its Laplacian matrix L is positive semi-definite. Thus, the submatrix L_{n-2} is a positive definite matrix, which means its inverse $(L_{n-2})^{-1}$ exists. Denote the (i,j)-th entry of $(L_{n-2})^{-1}$ with t_{ij}, then $\mathbf{x} = (L_{n-2})^{-1}\mathbf{b}$ implies

$$x_k = \sum_{l \in N(n-1)} t_{kl}. \tag{19}$$

We will use the following lemma.

Lemma 6.1. *The following equation holds:*

$$det\ L(i,j) = det\ L(j,i).$$

Proof. We have

$$L(i,j) = (L(j,i))^\top \text{and}$$
$$\det L(i,j) = \det(L(i,j))^\top.$$

Thus, it follows

$$\det L(i,j) = \det L(j,i).$$

□

Let $L_{n-2}(k,l)$ be the submatrix obtained by removing from L_{n-2} the k-th row and the l-th row. Then using Cramer's rule and Lemma 6.1 we obtain

$$t_{kl} = (-1)^{k+l}\frac{\det L_{n-2}(l,k)}{\det L_{n-2}} = (-1)^{k+l}\frac{\det L_{n-2}(k,l)}{\det L_{n-2}}. \tag{20}$$

Next we can rewrite (18) using (19) and (20) into

$$f(x_1,\dots,x_n) = d_{n-1} - \sum_{k\in N(n-1)}\sum_{l\in N(n-1)}(-1)^{k+l}\frac{\det L_{n-2}(k,l)}{\det L_{n-2}}. \tag{21}$$

Now look at $\det L(n)$ by expanding it with respect to its last column, which is the $(n-1)$-th column of L:

$$\det L(n) = d_{n-1}\det L_{n-2} - \sum_{k\in N(n-1)}(-1)^{k+n-1}\det L(n;k,n-1), \tag{22}$$

where $L(n;k,n-1)$ is the submatrix obtained by removing the k-th row and the $(n-1)$-th column from $L(n)$. Furthermore, we expand $\det L(n;k,n-1)$ with respect to its last row,

$$\det L(n;k,n-1) = \sum_{l\in N(n-1)}(-1)^{l+n-1}\det L_{n-2}(k,l).$$

Substituting this in (22) yields

$$\begin{aligned}\det L(n) &= d_{n-1}\det L_{n-2} - \sum_{k\in\Gamma(n-1)}(-1)^{k+n-1}\sum_{l\in N(n-1)}(-1)^{l+n-1}\det L_{n-2}(k,l)\\ &= d_{n-1}\det L_{n-2} - \sum_{k\in N(n-1)}\sum_{l\in N(n-1)}(-1)^{k+l}\det L_{n-2}(k,l), \end{aligned}\tag{23}$$

which results in the following expression when substituted back in (21)

$$f(x_1,\dots,x_n) = \frac{\det L(n)}{\det L_{n-2}}.$$

Finally, substituting this in (14), gives us (12). □

According to Kirchhoff's theorem, the number of spanning trees is equal to any cofactor of the Laplacian matrix of G. Thus

$$\det\, L(i) = t(G),$$

where $t(G)$ is the number of spanning trees of G. Now, according to [7], $\det\, L(i,j)$ is equal to the number of trees of G', where G' is the graph where vertex v_i and vertex v_j is contracted to one vertex v. Thus,

$$\det L(i,j) = t(G')$$

and we can write

$$r_{ij} = \frac{t(G')}{t(G)}.$$

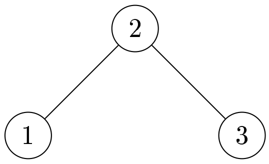
This can be interpreted as follows. The effective resistance between vertex i and j is equal to the number of trees of G containing the edge (i,j) divided by the total number of spanning trees of G. This can be seen as how much the edge (i,j) contributes to the number of spanning trees of G.

Remark: Since any cofactor of the Laplacian is equal to the number of spanning trees, we have $\det L(i) = \det L(j)$ for every $i, j \in V$. Using Lemma 6.1, we can write

$$r_{ij} = \frac{\det\, L(i,j)}{\det\, L(i)} = \frac{\det\, (L(j,i))^\top}{\det\, L(j)} = \frac{\det\, L(j,i)}{\det\, L(j)} = r_{ji}.$$

6.1 Example undirected graph using determinants

Recall Example 4.1:



with the Laplacian matrix $L = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$.

Using the corresponding Laplacian pseudoinverse of this graph

$$\frac{1}{9}\begin{pmatrix} 5 & -1 & -4 \\ -1 & 2 & -1 \\ -4 & -1 & 5 \end{pmatrix},$$

yields

$$\begin{aligned} r_{12} &= L^+_{11} + L^+_{22} - 2L^+_{12} = \frac{5}{9} + \frac{2}{9} + \frac{2}{9} = 1, \\ r_{13} &= L^+_{11} + L^+_{33} - 2L^+_{13} = \frac{5}{9} + \frac{5}{9} + \frac{8}{9} = 2, \\ r_{23} &= L^+_{22} + L^+_{33} - 2L^+_{23} = \frac{2}{9} + \frac{5}{9} + \frac{2}{9} = 1. \end{aligned}$$

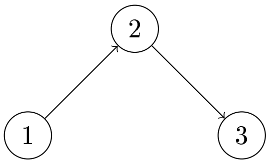
These are the same as using (12):

$$r_{12} = \frac{\det L(1,2)}{\det L(1)} = 1,$$
$$r_{13} = \frac{\det L(1,3)}{\det L(1)} = 2,$$
$$r_{23} = \frac{\det L(2,3)}{\det L(2)} = 1.$$

7 L^+: Directed graph

It is not always possible to compute L^+ for directed graphs as in Section 4 for undirected graphs. The matrix U such that $L = UDU^{-1}$ can not always be constructed.

Example 7.1. *Consider*



$$\textit{with } L = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of L are $\lambda_1 = 0$ with multiplicity two and $\lambda_2 = 1$ with multiplicity one. The eigenvectors are

$$v_1 = \left(\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}\right) \text{ and } v_2 = (1, 0, 0)\,.$$

This means that L is not diagonalizable; we can't construct U such that $L = UDU^{-1}$. Furthermore, even if we were to restrict the graphs, in [6] it has been argued that the robustness if defined through L^+ could decline when an edge is added, which contradicts with our concept of robustness. Thus, this approach is not suitable for directed graphs.

8 X: Directed graph

Here as well, for directed graphs we can't always compute X in the same manner as in Section 5. From Theorem 5.1 it follows that in the case of a directed graph, the algebraic multiplicity of the eigenvalue zero is equal to the number of strongly connected closed components, which is not always equal to one. Therefore, the inverse of $\overline{L}$ does not necessarily need to exist.

Therefore in [10], the authors propose a different way to construct X when the graph is directed. This approach will lead to a matrix X that is unique and symmetric. Furthermore, it is equal to the Laplacian pseudoinverse when calculated for undirected graphs and in Section 8.2 it is shown that the square root of the effective resistance is a metric on the nodes of any connected directed graph. The symmetry of X will be used to prove this. If we would take X as the Moore-Penrose inverse neither the effective resistance nor the square root of it is a metric.

In section Conclusion and discussion, we will discuss the effective resistance function satisfying symmetry.

The approach used in [10] considers the following Lyapunov equation:

$$A^*\Sigma + \Sigma A + B = 0, \tag{24}$$

with A, B square matrices, A^* the conjugate transpose of A and Σ unknown.

Theorem 8.1. *If A has only eigenvalues with a negative real part, then*

$$\Sigma = \int_0^\infty e^{A^*t} B e^{At} dt \tag{25}$$

is the unique solution to (24)*.*

Proof. This proof is taken from [3].
Let Σ be given as in (25). Then we have

$$A^*\Sigma + \Sigma A = \int_0^\infty \frac{d}{dt}\left[e^{A^*t} B e^{At}\right] dt = e^{A^*t} B e^{At}\Big|_0^\infty = -B,$$

which means that (25) is a solution for (24). To prove the uniqueness of (25), we define a linear map $\Gamma : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$, with $\Gamma(\Sigma) = A^*\Sigma + \Sigma A$. Now for all $B \in \mathbb{C}^{n\times n}$, the equation $\Gamma(\Sigma) = B$ has a solution. Thus, the dimension of the image of Γ is n^2. Note that the dimension of the domain of Γ is also n^2. Therefore, $\ker(\Gamma) = 0$ holds. We conclude $\Gamma(\Sigma) = B$ has a unique solution for each B. □

Lemma 8.1. *Furthermore, if B is positive definite, then Σ is also positive definite.*

Proof. Let $z \in \mathbb{C}^n$ and z^* the conjugate transpose of z, then observe

$$z^*\Sigma z = z^* \int_0^\infty e^{A^*t} B e^{At} dt z = \int_0^\infty z^* e^{A^*t} B e^{At} z dt \tag{26}$$

$$= \int_0^\infty (e^{At}z)^* B \left(e^{At}z\right) dt > 0. \tag{27}$$

The last inequality follows since B is positive definite. □

Now by taking $A = -\overline{L}^\top$ and $B = I_{N-1}$, (24) can be rewritten to

$$\overline{L}\Sigma + \Sigma\overline{L}^\top = I_{N-1}. \tag{28}$$

It follows from Lemma 8.1 that Σ is positive definite. We will see that Σ is also symmetric.

Lemma 8.2. *Σ satisfying* (28) *is symmetric.*

Proof. Let Σ satisfy (28). Then from Theorem 8.1 it follows that

$$\Sigma = \int_0^\infty e^{-\overline{L}t}e^{-\overline{L}^\top t}dt.$$

Observe the following:

$$\Sigma^\top = \int_0^\infty \left(e^{-\overline{L}t}e^{-\overline{L}^\top t}\right)^\top dt = \int_0^\infty e^{-\overline{L}t}e^{-\overline{L}^\top t}dt = \Sigma.$$

Thus, Σ is symmetric. □

Construct X as

$$X = 2Q^\top \Sigma Q, \tag{29}$$

with Q satisfying (6) and Σ the unique solution to the Lyapunov equation (28). Note that since Σ is symmetric, X is also symmetric for any graph. Now if $\overline{L}$ is symmetric, we have

$$\Sigma = \frac{1}{2}\overline{L}^{-1}. \tag{30}$$

Corollary 8.1. *If L is symmetric, X in* (29) *is equal to the Laplacian pseudoinverse L^+.*

Proof. Straightforward. □

Corollary 8.2. *Σ satisfies*

$$\Sigma = \int_0^\infty Qe^{-Lt}e^{-L^\top t}Q^\top dt.$$

Proof. Let Σ satisfies (25) with $A = -\overline{L}^\top$ and $B = I_{N-1}$. Then we can write the following:

$$
\begin{aligned}
\Sigma &= \int_0^\infty e^{-\overline{L}t} e^{-\overline{L}^\top t} dt \\
&= \int_0^\infty \sum_{k=0}^\infty \frac{(-1)^k (QLQ^\top)^k t^k}{k!} \sum_{k=0}^\infty \frac{(-1)^k (QL^\top Q^\top)^k t^k}{k!} dt \\
&= \int_0^\infty Q \sum_{k=0}^\infty \frac{(-1)^k L^k t^k}{k!} Q^\top Q \sum_{k=0}^\infty \frac{(-1)^k (L^\top)^k t^k}{k!} Q^\top dt \\
&= \int_0^\infty Q \sum_{k=0}^\infty \frac{(-1)^k L^k t^k}{k!} \Pi \sum_{k=0}^\infty \frac{(-1)^k (L^\top)^k t^k}{k!} Q^\top dt \\
&= \int_0^\infty Q e^{-Lt} e^{-L^\top t} Q^\top - \frac{1}{N} Q \sum_{k=0}^\infty \frac{(-1)^k L^k t^k}{k!} \mathbf{1}_N \mathbf{1}_N^\top e^{-L^\top t} Q^\top dt \\
&= \int_0^\infty Q e^{-Lt} e^{-L^\top t} Q^\top - \frac{1}{N} Q \mathbf{1}_N \mathbf{1}_N^\top e^{-L^\top t} Q^\top dt \\
&= \int_0^\infty Q e^{-Lt} e^{-L^\top t} Q^\top dt.
\end{aligned}
$$

□

The following lemma shows us that what kind of eigenvalues X has.

Lemma 8.3. *X satisfying (29) has a single 0 eigenvalue and its remaining eigenvalues are twice those of Σ.*

Proof. Take $V = \begin{bmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N & Q^\top \end{bmatrix}$ with Q as in Section 5. Then V is an orthogonal matrix. X is similar to $V^\top X V = \begin{bmatrix} 0 & \mathbf{0}^\top \\ \mathbf{0} & 2\Sigma \end{bmatrix}$. Hence, X has a single 0 eigenvalue and its remaining eigenvalues are twice those of Σ. □

Now in [10] they define the effective resistance between nodes in the graph as the following.

Definition 8.1. *Let G be a connected graph with N nodes and Laplacian matrix L. Then the effective resistance between nodes i and j in G is defined as*

$$r_{ij} = (\mathbf{e}_N^{(i)} - \mathbf{e}_N^{(j)})^\top X (\mathbf{e}_N^{(i)} - \mathbf{e}_N^{(j)}) = x_{i,i} + x_{j,j} - 2x_{i,j}, \tag{31}$$

where

$$X = 2Q^\top \Sigma Q, \quad \overline{L}\Sigma + \Sigma\overline{L}^\top = I_{N-1}, \quad \overline{L} = QLQ^\top, \tag{32}$$

and Q satisfying (6).

Remark: This expression for the effective resistance comes from (3). However X defined as in (32) does not satisfy all the properties mentioned in Theorem 3.1. X acts as the zero map on $\mathrm{sp}\{\mathbf{1}_N\}$, but it does not act as the inverse of the Laplacian matrix on $\mathbf{1}_N^\perp$; $XL = \Pi$ does not hold.

By taking (31) as the definition for the effective resistance between two nodes, the definition of the Kirchhoff index for undirected graphs can be generalized by summing over all distinct effective resistances of pairs of vertices of a graph, directed or undirected.

Definition 8.2. *The total effective resistance, K_f, of a graph is the sum over all distinct effective resistances of pairs of vertices:*

$$K_f = \sum_{i=1}^{N} \sum_{j=i+1}^{N} r_{ij} \tag{33}$$

Theorem 8.2. *The Kirchhoff index satisfies*

$$K_f = N \sum_{i=1}^{N} \lambda_i^X,$$

with λ_i^X's the eigenvalues of the matrix X satisfying (32)*.*

Proof. Similarly to the proof of Theorem 4.1, the Kirchhoff index can be written as

$$\begin{aligned} K_f &= \sum_{i=1}^{N} \sum_{j=i+1}^{N} r_{ij} = N \sum_{i=1}^{N} (x_{i,i}) - \mathbf{1}_N^\top X \mathbf{1}_N \\ &= N \sum_{i=1}^{N} (x_{i,i}) - \mathbf{1}_N^\top (2Q^\top \Sigma Q) \mathbf{1}_N = N \sum_{i=1}^{N} x_{i,i} \\ &= N \operatorname{Tr}(X) = N \sum_{i=1}^{N} \lambda_i^X, \end{aligned}$$

with λ_i^X's the eigenvalues of X. □

Then from Lemma 8.3 it follows

$$K_f = 2N \sum_{i=1}^{N-1} \lambda_i^\Sigma,$$

with λ_i^Σ's the eigenvalues of Σ.

8.1 Effective resistance for the directed graph is well-defined

To confirm that the concept of effective resistance for directed graphs given in Definition 8.1 is indeed well-defined, observe the following two lemmas from [10].

Lemma 8.4. *The value of the effective resistance between two nodes in a connected directed graph is independent of the choice of Q.*

Proof. Let Q an Q' be two matrices satisfying (6) and let the effective resistance between nodes i and j be r_{ij}, r'_{ij}, computed using Q and Q' respectively. Furthermore, define $W = Q'Q^\top$. Then we have $WQ = Q'Q^\top Q = Q'\Pi = Q'$ and

$WW^\top = Q'Q^\top(Q'Q^\top)^\top = Q'Q^\top QQ'^\top = Q'\Pi Q'^\top = Q'Q'^\top = I_{N-1}$. $W^\top W$ is computed analogously. Now using (7) and the above we can write

$$\overline{L}' = Q'LQ'^\top = WQL(WQ)^\top = WQLQ^\top W^\top = W\overline{L}W^\top.$$

Substituting this expression in (28) for $\overline{L}'$, we see $\Sigma = W^\top\Sigma' W$. And thus we find $X = 2Q^\top\Sigma Q = 2Q^\top W^\top\Sigma' WQ = 2Q'^\top\Sigma' Q' = X'$, hence $r_{j,k} = r'_{j,k}$. □

From this lemma it follows that the effective resistance is independent of the choice of Q as long as it satisfies (6). The effective resistance is also independent of the labelling of the nodes, which will be proven in the following lemma.

Lemma 8.5. *The value of the effective resistance between two nodes in a connected directed graph is independent of the labelling of the nodes.*

Proof. Let L and L' be two Laplacian matrices associated with the same graph, but with different labellings of the nodes. Then L' can be found from L by permuting its rows and columns. That is, there exists a matrix P such that $L' = PLP^\top$. Note that since P is a permutation matrix, P has exactly one 1 in every row and column, and every other entry is 0. Consequently, we have $P^{-1} = P^\top, P\mathbf{1}_N = \mathbf{1}_N$ and $\mathbf{1}_N^\top P = \mathbf{1}_N^\top$. Furthermore, we have

$$\overline{P}^\top = (QPQ^\top)^\top = QP^\top Q^\top = QP^{-1}Q^\top = \overline{P}^{-1},$$

$$\begin{aligned}\overline{P}Q &= QPQ^\top Q = QP\Pi = QP(I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top)\\ &= QP - QP\frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top = QP\end{aligned}$$

and analogously $Q^\top\overline{P} = PQ^\top$. Now using (7) and the properties above, it follows

$$\overline{L}' = QL'Q^\top = QPLP^\top Q^\top = \overline{P}QLQ^\top\overline{P}^\top = \overline{P}\overline{L}\overline{P}^\top.$$

Then substituting this expression in (28) for $\overline{L}'$ yields $\Sigma = \overline{P}^\top\Sigma'\overline{P}$. Hence,

$$\begin{aligned}X &= 2Q^\top\Sigma Q = 2Q^\top\overline{P}^\top\Sigma'\overline{P}Q = 2(\overline{P}Q)^\top\Sigma' QP\\ &= 2P^\top Q^\top\Sigma' QP = P^\top X'P.\end{aligned}$$

Thus, if P permutes node i to node k and node j to node l, $r'_{kl} = r_{ij}$ holds. □

8.2 The square root of the effective resistance is a metric

The following theorem is taken from [10].

Theorem 8.3. *The square root of the effective resistance is a metric on the nodes of any connected directed graph. That is, we have for all nodes i, j, k*

$$\begin{aligned}r_{ij} &\geq 0,\\ r_{ij} = 0 &\Leftrightarrow i = j,\\ r_{ij} &= r_{ji}, \quad \textit{and}\\ \sqrt{r_{ik}} + \sqrt{r_{kj}} &\geq \sqrt{r_{ij}}.\end{aligned}$$

Furthermore, the effective resistance itself is not a metric since it fails to satisfy the triangle inequality.

Proof. The effective resistance between two nodes can be computed as

$$r_{ij} = (\mathbf{e}_N^{(i)} - \mathbf{e}_N^{(j)})^\top X (\mathbf{e}_N^{(i)} - \mathbf{e}_N^{(j)}), \tag{34}$$

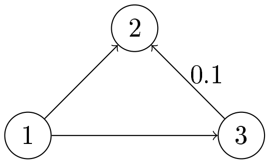
where $X = 2Q^\top \Sigma Q$ and Σ is a positive definite matrix. From Lemma 8.3 we know that X has a single 0 eigenvalue and its remaining eigenvalues are twice those of Σ. Furthermore, $X\mathbf{1}_N = 0$ holds, thus X is positive semi-definite with null space given by the span of $\mathbf{1}_N$.

Now since X is symmetric and positive semi-definite, there exists an matrix $P \in \mathbb{R}^{N\times N}$ such that $X = P^\top P$ (by Cholesky decomposition [5]). Then the effective resistance can be written as follows

$$r_{ij} = \left\| P(\mathbf{e}_N^{(i)} - \mathbf{e}_N^{(j)}) \right\|^2,$$

therefore $\sqrt{r_{ij}} = \left\| P(\mathbf{e}_N^{(i)} - \mathbf{e}_N^{(j)}) \right\|$. Now by associating every node i of G with a point $p_i := P\mathbf{e}_N^{(i)} \in \mathbb{R}^N$, it follows that $\sqrt{r_{ij}}$ is equal to the Euclidean distance in $\mathbb{R}^N$ between p_i and p_j. Since $\mathbf{e}_N^{(i)} - \mathbf{e}_N^{(j)}$ is perpendicular to $\mathbf{1}_N$ for $i \neq j$, $\mathbf{e}_N^{(i)} - \mathbf{e}_N^{(j)}$ is not in the null space of P and so we have $p_i \neq p_j$ for $i \neq j$. Hence, the square root of the effective resistance is a metric on the nodes of the graph.

To show that in general the effective resistance is not a metric, consider the following graph:



with $L = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & -0.1 & 0.1 \end{pmatrix}$.

We have

$$\overline{L} = \begin{pmatrix} \frac{3}{2} & \frac{\sqrt{3}}{2} \\ \frac{7\sqrt{3}}{15} & \frac{3}{5} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \frac{131}{84} & \frac{-103\sqrt{3}}{84} \\ \frac{-103\sqrt{3}}{84} & \frac{133}{36} \end{pmatrix}$$

and

$$X = \frac{1}{189} \begin{pmatrix} 64 & -62 & -2 \\ -62 & 991 & -929 \\ -2 & -929 & 931 \end{pmatrix}.$$

In this case, we find $r_{23} = 20$, $r_{21} = \frac{131}{21} \approx 6.24$ and $r_{13} = \frac{37}{7} \approx 5.29$. Thus, $r_{23} > r_{21} + r_{13}$, and the triangle inequality does not hold. $\square$

As said earlier, neither the effective resistance nor the square root of it is a metric if we were to take X as the Moore-Penrose inverse. Consider the same graph as above. Keep in mind that this is not the same as $Q^\top \overline{L}^{-1} Q$, since this is only the Moore-Penrose inverse for undirected graphs. Then calculating the Moore-Penrose inverse [8], we find

$$X = \frac{1}{6}\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{1}{6} & 0 & -5 \\ -\frac{1}{6} & 0 & 5 \end{pmatrix}$$

and thus this yields

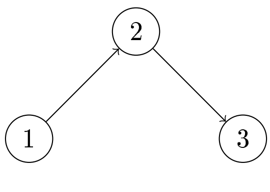
$$\begin{aligned} r_{21} &= \tfrac{1}{2}, & \sqrt{r_{21}} &\approx 0.71, \\ r_{13} &= 5\tfrac{1}{2}, & \sqrt{r_{13}} &\approx 2.35, \\ r_{23} &= 10, & \sqrt{r_{23}} &\approx 3.16. \end{aligned}$$

Indeed, we have $r_{23} > r_{21} + r_{13}$ and $\sqrt{r_{23}} > \sqrt{r_{21}} + \sqrt{r_{13}}$. Therefore, neither is a metric.

8.3 X: Examples directed graph

To research if this approach satisfies our concept of robustness, we will look at some examples of directed graphs and observe how adding an edge or increasing an edge weight affects the total effective resistance.

Example 8.1. *Let G be the following*



with the Laplacian matrix $L = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$.

We have

$$\overline{L} = QLQ^{\top} = \begin{pmatrix} \frac{3}{2} & -\frac{3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} & \frac{1}{2} \end{pmatrix}$$

and solving (28) for Σ yields

$$\Sigma = \begin{pmatrix} \frac{17}{35} & \frac{16}{35\sqrt{3}} \\ \frac{16}{35\sqrt{3}} & \frac{27}{35} \end{pmatrix}.$$

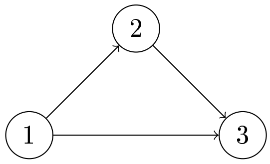
Thus we find

$$\begin{aligned} X &= 2Q^{\top}\Sigma Q \\ &= \frac{2}{105}\begin{pmatrix} 55 & -12 & -43 \\ -12 & 23 & -11 \\ -43 & -11 & 54 \end{pmatrix}. \end{aligned}$$

Using Theorem 8.2, the total effective resistance of this graph is $K_f = \frac{264}{35} \approx 7.54$.

Now by adding an edge to G, for example the directed edge $(1, 3)$, we construct G' and observe how this affects the total effective resistance.

Example 8.2. *The graph G':*



with the Laplacian matrix $L' = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$.

We find

$$\overline{L'} = \begin{pmatrix} 2 & 0 \\ \frac{2}{\sqrt{12}} & 1 \end{pmatrix} \quad \text{and} \Sigma' = \begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{36} \\ -\frac{\sqrt{3}}{36} & \frac{19}{36} \end{pmatrix}.$$

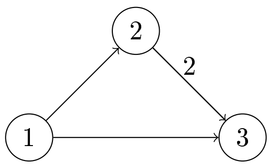
Thus, this yields

$$X' = \frac{4}{108} \begin{pmatrix} 10 & -2 & -8 \\ -2 & 13 & -11 \\ -8 & -11 & 19 \end{pmatrix},$$

which means $K'_f = 3\,\mathrm{Tr}(X') = \frac{32}{9} \approx 3.56$. We see that by adding an edge the total effective resistance has decreased. This agrees with our intuition.

Now we examine what kind of effect increasing an edge weight will have on the total effective resistance.

Example 8.3. *Let G'' be G' except $w_{23} = 2$:*



The associated Laplacian and reduced Laplacian are

$$L'' = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \overline{L}'' = \begin{pmatrix} \frac{5}{2} & -\frac{3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} & \frac{3}{2} \end{pmatrix}.$$

Thus yields

$$\Sigma = \begin{pmatrix} \frac{7}{32} & \frac{\sqrt{3}}{32} \\ \frac{\sqrt{3}}{32} & \frac{31}{96} \end{pmatrix} \text{ and } X'' = \frac{1}{72} \begin{pmatrix} 28 & -8 & -20 \\ -8 & 19 & -11 \\ -20 & -11 & 31 \end{pmatrix}.$$

The total effective resistance of G'' is $K''_f = 3\,\mathrm{Tr}(X'') = \frac{13}{4} = 3.25$. Hence, by increasing an edge weight the total effective resistance has decreased. Again, this agrees with our concept of robustness.

Even though these examples point to a decrease in total effective resistance when an edge is added or when an edge weight is increased, we have not succeeded into proving that this is always the case. This is still open for further research.

8.3.1 Example: Corresponding Laplacian of X

While researching what the effects are of adding an edge to a graph on the total effective resistance, we found the following remark.

Notice that in Example 8.2 we have found $X' = \frac{4}{108}\begin{pmatrix} 10 & -2 & -8 \\ -2 & 13 & -11 \\ -8 & -11 & 19 \end{pmatrix}$, which is a symmetric matrix with row sums equal to zero. Now if we let $L^+_{X'} := X'$ be a Laplacian pseudoinverse, we can find the Laplacian $L_{X'}$ and the corresponding undirected graph $G_{X'}$. Consequently, the total effective resistance of G' is equal to that of $G_{X'}$.

Thus, let

$$L^+_{X'} := \frac{4}{108}\begin{pmatrix} 10 & -2 & -8 \\ -2 & 13 & -11 \\ -8 & -11 & 19 \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = 0, \lambda_2 = \frac{1}{9}(7+\sqrt{7})$ and $\lambda_3 = \frac{1}{9}(7-\sqrt{7})$ and the set of orthonormal eigenvectors is

$$\begin{aligned} v_1 &= \left(\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}\right), \\ v_2 &= \frac{1}{\sqrt{28-10\sqrt{7}}}(-3+\sqrt{7}, 2-\sqrt{7}, 1), \\ v_3 &= \frac{1}{\sqrt{28+10\sqrt{7}}}(-3-\sqrt{7}, 2+\sqrt{7}, 1). \end{aligned}$$

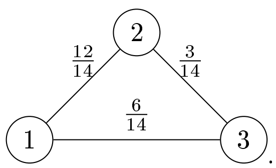
From Section 4, we can write $L_{X'} = UDU^\top$ with

$$U = \begin{pmatrix} \frac{1}{3}\sqrt{3} & \frac{-3+\sqrt{7}}{\sqrt{28-10\sqrt{7}}} & \frac{-3-\sqrt{7}}{\sqrt{28+10\sqrt{7}}} \\ \frac{1}{3}\sqrt{3} & \frac{2-\sqrt{7}}{\sqrt{28-10\sqrt{7}}} & \frac{2+\sqrt{7}}{\sqrt{28+10\sqrt{7}}} \\ \frac{1}{3}\sqrt{3} & \frac{1}{\sqrt{28-10\sqrt{7}}} & \frac{1}{\sqrt{28+10\sqrt{7}}} \end{pmatrix}, \; D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{9}{7+\sqrt{7}} & 0 \\ 0 & 0 & \frac{9}{7-\sqrt{7}} \end{pmatrix}$$

and the Laplacian matrix is given by

$$L_{X'} = \frac{1}{14}\begin{pmatrix} 18 & -12 & -6 \\ -12 & 15 & -3 \\ -6 & -3 & 9 \end{pmatrix}$$

with the corresponding undirected graph $G_{X'}$ as follows:

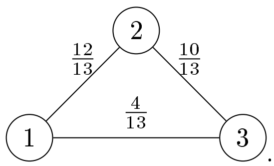


We also do this for Example 8.3 in which we have increased the edgeweight of

$(2,3)$ where $X'' = \frac{1}{72}\begin{pmatrix} 28 & -8 & -20 \\ -8 & 19 & -11 \\ -20 & -11 & 31 \end{pmatrix}$. This yields

$$L_{X''} = \frac{1}{13}\begin{pmatrix} 16 & -12 & -4 \\ -12 & 22 & -10 \\ -4 & -10 & 14 \end{pmatrix}$$

with the corresponding undirected graph $G_{X''}$



Remark: If we compare $G_{X"}$ and $G_{X'}$, we see that not all edge weights are increased. Although w_{12} and w_{23} have increased, w_{13} has decreased. Thus, a decrease in total effective resistance does not lead to an increase on all the edge weights.

9 Different definition for X

In Section 8, X is defined as $X = 2Q^\top \Sigma Q$. If the graph is undirected we have $\Sigma = \frac{1}{2}\overline{L}^{-1}$, thus $X = Q^\top \overline{L}^{-1} Q$. This is the same as the definition of X for undirected graphs (see Section 5). We can also define the matrix X differently such that this also holds. The flaw of the following definition, however, is that X is not unique; it depends on the choice of Q.

Definition 9.1. *Let*

$$X = Q^\top B^{-1} Q, \tag{35}$$

with Q satisfying (6), *$\overline{L}$ satisfying* (7) *and such that B is a positive definite matrix satisfying $B^2 = \overline{LL^\top}$.*

Remark: $\overline{LL^\top} = QLQ^\top QL^\top Q^\top = QL\Pi L^\top Q^\top = QLL^\top Q^\top = \overline{LL^\top}$

Notice that $\overline{LL^\top}$ is a positive definite matrix. Consequently, there exists a positive definite matrix B such that $B^2 = \overline{LL^\top}$ holds. And since $\overline{LL^\top}$ is also a symmetric matrix, we hence have that X is a symmetric positive definite matrix. Now for an undirected graph, the Laplacian is symmetric which means $\overline{L}$ is also symmetric. Then X defined in (35) will be the same as in (10). Furthermore, the total resistance is calculated in the same way.

Theorem 9.1. *The total resistance K_f still satisfies*

$$K_f = N \sum_{i=1}^{N} \lambda_i^X,$$

with λ_i^X the i'th eigenvalue of X.

Proof. Like earlier proofs, we can write the following:

$$\begin{aligned} K_f &= \sum_{i=1}^{N} \sum_{j=i+1}^{N} r_{ij} = N \sum_{i=1}^{N} (x_{i,i}) - \mathbf{1}_N^\top X \mathbf{1}_N \\ &= N \sum_{i=1}^{N} (x_{i,i}) - \mathbf{1}_N^\top (Q^\top B^{-1} Q) \mathbf{1}_N = N \sum_{i=1}^{N} x_{i,i} \\ &= N \operatorname{Tr}(X) = N \sum_{i=1}^{N} \lambda_i^X, \end{aligned}$$

□

Another obvious expression for X which is the same as (10) for undirected graphs is

$$X = Q^\top (QBQ^\top)^{-1} Q,$$

where B satisfies $B^2 = LL^\top$. However, the inverse of $QBQ^\top$ does not always have to exist, since $LL^\top$ is not necessarily invertible on $\mathbf{1}_N^\perp$.

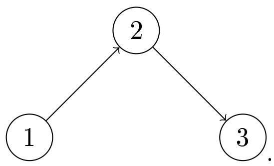
9.1 Examples corresponding to the different definition of X

Applying the former definition on the examples in Section 8.3 with

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix},$$

yields the following.

First, recall Example 8.1 with G:



The corresponding positive definite matrix B is

$$B = \begin{pmatrix} \frac{\sqrt{11}}{2} & 0 \\ 0 & \frac{1}{3}\sqrt{3} \end{pmatrix}.$$

Thus, this yields

$$X = Q^\top B^{-1} Q = \frac{1}{11} \begin{pmatrix} \sqrt{11} + \frac{11}{6}\sqrt{3} & -\sqrt{11} + \frac{11}{6}\sqrt{3} & -\frac{22}{6}\sqrt{3} \\ -\sqrt{11} + \frac{11}{6}\sqrt{3} & \sqrt{11} + \frac{11}{6}\sqrt{3} & -\frac{22}{6}\sqrt{3} \\ -\frac{22}{6}\sqrt{3} & -\frac{22}{6}\sqrt{3} & \frac{44}{6}\sqrt{3} \end{pmatrix}.$$

The total effective resistance of G is $K_f \approx 7.01$. By adding the edge $(1,3)$, we find

$$X' = \frac{1}{42} \begin{pmatrix} 3\sqrt{21} & -\sqrt{21} & -2\sqrt{21} \\ -\sqrt{21} & 5\sqrt{21} & -4\sqrt{21} \\ -2\sqrt{21} & -4\sqrt{21} & 6\sqrt{21} \end{pmatrix},$$

with $K'_f \approx 4.58$ which has decreased. Lastly, we increase the edge weight of $(2,3)$ by one. Then we find

$$X'' = \frac{1}{312} \begin{pmatrix} 16\sqrt{39} & -4\sqrt{39} & -12\sqrt{39} \\ -4\sqrt{39} & 14\sqrt{39} & -10\sqrt{39} \\ -12\sqrt{39} & -10\sqrt{39} & 22\sqrt{39} \end{pmatrix},$$

with $K''_f \approx 3.12$ which has also decreased. This corresponds with our concept of robustness. However, as mentioned before, X depends on Q. By taking

$$Q' = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

we would find for G:

$$X^{Q'} = \frac{1}{11} \begin{pmatrix} \frac{\sqrt{11}}{3} + \frac{11\sqrt{3}}{2} & -\frac{\sqrt{11}}{3} + \frac{11\sqrt{3}}{2} & -\frac{2\sqrt{11}}{3} \\ -\frac{\sqrt{11}}{3} + \frac{11\sqrt{3}}{2} & \frac{\sqrt{11}}{3} + \frac{11\sqrt{3}}{2} & -\frac{2\sqrt{11}}{3} \\ -\frac{2\sqrt{11}}{3} & -\frac{2\sqrt{11}}{3} & \frac{4\sqrt{11}}{3} \end{pmatrix}$$

with $K_f^{Q'} \approx 2.34$, therefore $K_f^{Q'} \neq K_f$.

10 Determinants: Directed graph

In this section we will try to extend the expression found in Section 6 to directed graphs. Consider Example 8.1, with

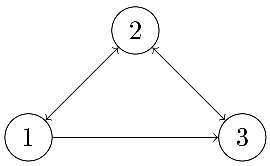
$$L = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Calculating the effective resistance r_{23} as (12), we would find

$$r_{23} = \frac{\det\ L(2,3)}{\det\ L(2)} = \frac{1}{0},$$

which is not defined. This problem is solved by only looking at strongly connected directed graphs. Keep in mind that the derivation of the expression does not hold anymore. Nevertheless, we would still like to consider this expression for the directed graph.

Example 10.1. *Consider the following strongly connected graph*



with the Laplacian matrix $L = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$

Then using (12), we find $r_{12} = 1$ while $r_{21} = \frac{1}{2}$ because $\det\ L(1) \neq \det\ L(2)$. The determinants are unequal because $\det\ L(i)$ is the amount of rooted trees with the orientation towards the root i.

Even though the effective resistance between vertex a and b is not equal to the effective resistance between vertex b and a, it does not necessary imply that it is "incorrect". It might even make more sence that it is not symmetric, since the graph is directed.

Interpret in the above example the nodes as cities and an arc (i, j) as an one way traffic from j to i. Note that this is the opposite of how edges are normally viewed in literature. Now removing the edge $(1, 3)$ would not affect the people traveling from city 1 to city 2. However, it would affect the people traveling from city 2 to city 1 since they have lost the alternative route via 3. We would find $r'_{12} = 1 = r_{12}$ and $r'_{21} = 1 > r_{21}$.

Nevertheless, this approach contradicts with our concept of robustness. If the arc $(3, 1)$ is added to the graph in Example 10.1, we have $r_{ij} = \frac{2}{3}$ for every $i, j \in V$, $i \neq j$. Notice that r_{21} has increased, even though an edge is added. Therefore, this method is not suitable.

11 Conclusion and discussion

The aim of this thesis is to find a way to measure robustness for directed graphs such that it does not contradict with our concept of robustness. If an edge were to be added or an edge weight were to be increased, the graph should not become less robust.

We started by looking at undirected graphs, a discussion of which it is the topic of [4]. We showed some properties of the Laplacian and using these, a method to calculate the total effective resistance is discussed. Two different approaches to determine the Moore-Penrose inverse of the Laplacian matrix were shown. Furthermore, we found another expression for the effective resistance between two different nodes.

We continued by looking at directed graphs and researched how former methods could be extended. It was shown that it is not always possible to calculate the Laplacian pseudoinverse using the matrix U which has the eigenvectors of the Laplacian as its columns, since there are not always N distinct eigenvectors. Also, in [6], it is shown that this method does not satisfy our concept of robustness. Using this method, it is possible for the robustness of a graph to decrease when an edge is added.

Next, we studied the extension of effective resistance to directed graphs based on [10]. The authors proposed a different definition for the matrix X which is used in their calculation of the effective resistance. This matrix is unique and the same as the Moore-Penrose inverse for undirected graphs. However, we have some critique on this approach.

- The matrix is symmetric and it is not clear what the underlying motivation is other than using it in the proof of Theorem 8.3.
- X does not satisfy all the properties mentioned in Theorem 3.1. The authors simply define the effective resistance with this matrix X and do not argue or motivate this definition.
- The effective resistance function defined this way satisfies symmetry which we believe is counterintuitive. If there are paths from node i to node j but none from j to i, we would still have $r_{ij} = r_{ji}$. It would be expected that they are unequal. Consider the following. Remove an edge of a path from i to j. This has an effect on r_{ij} even though it has no effect on any path from j to i, thus r_{ji} should not change.
- It is proven that the square root of the effective resistance function is a metric on the nodes. Unfortunatly, the effective resistance function it self is not a metric.
- Most importantly, the authors have not mentioned, let alone proven, that by adding an edge or by increasing an edge weight, the robustness does not decrease. Examples in Section 8.3 indicate that this is the case, but unfortunately we were not able to prove that this is always true.

Thus, if there exists networks in which the effective resistance should be symmetric, then this approach might be useful. Whether adding an edge or increasing an edge weight always lead to a decrease (or at least not an increase) in total effective resistance is still open for further research.

Lastly, we researched if it was possible to extend the determinants expression of the effective resistance to directed graphs. We know that the derivation of the expression did not hold anymore, nevertheless we studied it anyway. By using the same expression, it was not defined for arbitrary directed graphs, thus we considered only strongly connected graphs. Interestingly, using this approach the effective resistance function did not satisfy symmetry, which we find more intuitive. Unfortunately, it is possible for the effective resistance to increase while having an edge added.

Thus, the only definition which we have found that is slightly promising is the definition of effective resistance using the matrix X, see Section 8. For further research, it would be interesting to find a definition of robustness such that the intuitive asymmetry of directed connections is also taken into account. This can be followed by comparing networks and researching designs which are more robust. Perhaps, it is even possible to implement some of these robust designs, especially in networks where failures are likely to occur and which are important to our highly networked world.

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