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Finite dimensional Riesz spaces and their automorphisms

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Contents

Introduction

In this thesis finite dimensional Riesz spaces will be studied. Riesz spaces are real vector spaces equipped with a partial order. Under this partial order the Riesz space must satisfy some axioms, including the axiom that it is a lattice.
In the first chapter the concept of a lattice and a Riesz space will be introduced and some examples will be given. Also some general definitions and properties will be given. Furthermore two partial orders on $\mathbb{R}^n$, the pointwise and the lexicographical order, will be introduced.
In the second chapter homomorphisms between Riesz spaces will be discussed. In chapter three we will use these homomorphisms to understand the structure of finite dimensional Riesz spaces. Here we will prove that each finite dimensional Riesz space E can be built up as the direct sum $\bigoplus_{j=1}^{n} I_j$ of a finite number of Riesz spaces, where every I_j can be constructed out of a lower dimensional Riesz space, using a variant of the lexicographical order. In the last chapter we will use these results to find the automorphism groups of finite dimensional Riesz spaces.

1 Lattices and Riesz spaces

Before studying Riesz spaces, we need to know the definition of a lattice.

Definition 1.1. A partially ordered set $(X, \leq)$ is called a lattice if for every $x, y \in X$, the set $\{x, y\}$ has an infimum and a supremum in X.

To make the notation simpler, we will write $x \vee y$ for $\sup\{x, y\}$ and $x \wedge y$ for $\inf\{x, y\}$.

Example 1.2. The space $(\mathbb{R}^2, \leq)$ equipped with the coordinatewise ordering (i.e. $(x_1, x_2) \leq (y_1, y_2)$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$) is a lattice. For each $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ the infimum and supremum are given by $(x_1, x_2) \wedge (y_1, y_2) = (\min\{x_1, y_1\}, \min\{x_2, y_2\})$ and $(x_1, x_2) \vee (y_1, y_2) = (\max\{x_1, y_1\}, \max\{x_2, y_2\})$.
However the closed unit disc $U = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$ equipped with the same coordinatewise ordering is not a lattice. To see this, consider $(1, 0), (0, 1) \in U$. We see that every upper bound of $\{(1, 0), (0, 1)\}$ must be greater than or equal to 1 in both its coordinates. Therefore U does not contain an upper bound of $\{(1, 0), (0, 1)\}$ and thus it does not contain its supremum.

Definition 1.3. A lattice $(X, \leq)$ is called distributive if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all $x, y, z \in X$.

Lemma 1.4. *A lattice $(X, \leq)$ is distributive if and only if*

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

for all $x, y, z \in X$.

Proof. Suppose that $(X, \leq)$ is distributive. Let $a := x \vee (y \wedge z)$, $b := (x \vee y) \wedge (x \vee z)$ and $w := x \vee y$. We have to prove that $a = b$. Now we find:

$$b = w \wedge (x \vee z) = (w \wedge x) \vee (w \wedge z) = ((x \vee y) \wedge x) \vee ((x \vee y) \wedge z) = x \vee (x \wedge y) \vee (x \wedge z) \vee (y \wedge z).$$

Notice that $x \wedge y \leq x$ and $x \wedge z \leq x$, so $x \vee (x \wedge y) \vee (x \wedge z) = x$. So we have indeed

$$a = x \vee (y \wedge z) = b.$$

Now suppose that $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in X$. Let $a := x \wedge (y \vee z)$, $b := (x \wedge y) \vee (x \wedge z)$ and $w := x \wedge z$. We have to prove that $a = b$. We find:

$$b = (x \wedge y) \vee w = (x \vee w) \wedge (y \vee w) = (x \vee (x \wedge z)) \wedge (y \vee (x \wedge z)) = x \wedge (x \vee z) \wedge (x \vee y) \wedge (y \vee z).$$

Notice that $x \vee z \geq x$ and $x \vee y \geq x$, so $x \wedge (x \vee z) \wedge (x \vee y) = x$. So we can conclude

$$a = x \wedge (y \vee z) = b.$$

□

The idea of lattices can be extended to partially ordered real vector spaces.

Definition 1.5. Let $(E, \leq)$ be a partially ordered real vector space. E is called an ordered vector space if for all $f, g, h \in E$ and for all $\lambda \in \mathbb{R}_{\geq 0}$ the following properties hold:

1. $f \leq g \Rightarrow f + h \leq g + h$;
2. $f \geq 0 \Rightarrow \lambda f \geq 0$;

If E is also a lattice with respect to its partial order, E is called a Riesz space.

We will write E^+ for the positive cone of a Riesz space E, which is defined to be the collection $\{f \in E : f \geq 0\}$. Now let us look at some examples.

Example 1.6. It is clear that the properties of an ordered vector space hold coordinatewise in $\mathbb{R}^n$, for $n \in \mathbb{Z}_{>0}$. Therefore $\mathbb{R}^n$ with coordinatewise order is an ordered vectorspace. Moreover it is a Riesz space, since the infimum of two elements is equal to the pointwise minimum and the supremum to the pointwise maximum.
Besides this coordinatewise ordering, we can also look at the lexicographical order. If we take $\mathbb{R}^2$, this order is defined in the following way: $(x_1, x_2) \leq (y_1, y_2)$ if and only if $x_1 < y_1$, or $x_1 = y_1$ and $x_2 \leq y_2$. In the same way we can order $\mathbb{R}^n$ for any $n \in \mathbb{Z}_{>0}$. Together with this order $\mathbb{R}^n$ becomes a totally ordered Riesz space.

Example 1.7. The vector space $(C[0,1], \leq)$ of all real continuous functions on the unit interval together with the pointwise order (i.e. $f \leq g$ if and only if $\forall x \in [0,1] : f(x) \leq g(x)$) is a Riesz space. For $f, g \in C[0,1]$ the supremum $f \vee g$ is given by the function $h : [0,1] \to \mathbb{R}$, $x \mapsto \max\{f(x), g(x)\}$. The infimum $f \wedge g$ is given in the same way.

Lemma 1.8. *Let $(E, \leq)$ be a Riesz space and let $f, g, h \in E$ and $\lambda \in \mathbb{R}_{\geq 0}$. Then the following equalities are valid:*

1. $(f \vee g) + h = (f + h) \vee (g + h)$;
2. $(f \wedge g) + h = (f + h) \wedge (g + h)$;

3. $(\lambda f \vee \lambda g) = \lambda(f \vee g)$;

4. $(\lambda f \wedge \lambda g) = \lambda(f \wedge g)$;

5. $(f \vee g) = -((-f) \wedge (-g))$;

6. $(f \wedge g) = -((-f) \vee (-g))$.

Proof. 1. From $f \leq f \vee g$ and $g \leq f \vee g$ it follows that $f + h \leq (f \vee g) + h$ and $g + h \leq (f \vee g) + h$. Therefore $(f \vee g) + h$ is an upper bound of $\{(f + h), (g + h)\}$. Now suppose that u is also an upper bound. Then it follows that $f \leq u - h$ and $g \leq u - h$, thus $(f \vee g) \leq u - h$ and $(f \vee g) + h \leq u$.

2. This proof is analogous to the proof of 1.

3. From $f \leq f \vee g$ and $g \leq f \vee g$ it follows that $\lambda f \leq \lambda(f \vee g)$ and $\lambda g \leq \lambda(f \vee g)$. Therefore $\lambda(f \vee g)$ is an upper bound of $\{\lambda f, \lambda g\}$. Now suppose that u is also an upper bound. Then it follows that $f \leq \lambda^{-1}u$ and $g \leq \lambda^{-1}u$, thus $f \vee g \leq \lambda^{-1}u$ and $\lambda(f \vee g) \leq u$.

4. This proof is analogous to the proof of 3.

5. From $f \leq f \vee g$ and $g \leq f \vee g$ it follows that $-f \geq -(f \vee g)$ and $-g \geq -(f \vee g)$. Therefore $-(f \vee g)$ is a lower bound of $\{-f, -g\}$. Now suppose that u is also a lower bound. Then it follows that $f \leq -u$ and $g \leq -u$, thus $(f \vee g) \leq -u$ and $-(f \vee g) \geq u$.

6. This proof is analogous to the proof of 5.

□

Definition 1.9. Let $(E, \leq)$ be a Riesz space. For any $f \in E$ we will use the following notation:

1. $f^+ := f \vee 0$;

2. $f^- := (-f) \vee 0$;

3. $|f| := f \vee (-f)$.

Definition 1.10. Let $(E, \leq)$ be a Riesz space and let $x, y \in E$. The order interval $[x, y]$ is the set

$$\{z \in E : x \leq z \leq y\}.$$

Lemma 1.11. *Let $(E, \leq)$ be a Riesz space and $f, g \in E$. Then the following equalities are valid:*

1. $f = f^+ - f^-$;

2. $|f| = f^+ + f^-$;

3. $(f \vee g) + (f \wedge g) = f + g$;

4. $(f \vee g) - (f \wedge g) = |f - g|$;
5. $f \vee g = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$;
6. $f \wedge g = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$.

Proof. 1. $f^+ - f = (f \vee 0) - f = 0 \vee (-f) = f^-$, so $f = f^+ - f^-$;

2. $|f| = f \vee (-f) = ((2f) \vee 0) - f = 2f^+ - (f^+ - f^-) = f^+ + f^-$;
3. $f \vee g = ((f - g) \vee 0) + g = (f - g)^+ + g$ and $f \wedge g = f + (0 \wedge (g - f)) = f - (0 \vee (f - g)) = f - (f - g)^+$, so $(f \vee g) + (f \wedge g) = f + g$;
4. Using $f \wedge g = f - (f - g)^+$ and $f \vee g = ((g - f) \vee 0) + f = (g - f)^+ + f$, we find: $(f \vee g) - (f \wedge g) = (g - f)^+ + (f - g)^+ = (f - g)^+ + (f - g)^- = |f - g|$;
5. This follows from adding 3 and 4;
6. This follows from subtracting 4 from 3.

□

Lemma 1.12 (Riesz decomposition property)**.** *Let $(E, \leq)$ be a Riesz space and let $g, f_1, f_2 \in E^+$ such that $g \leq f_1 + f_2$. Then there exist $g_1, g_2 \in E^+$ such that $g_1 \leq f_1$, $g_2 \leq f_2$ and $g = g_1 + g_2$.*

Proof. Let $g_1 = g \wedge f_1$ and $g_2 = g - g_1$. Now we have $g_1 \in E^+$ and $g_1 \leq f_1$. Furthermore $g_1 \leq g$, thus $g_2 = g - g_1 \geq 0$, which gives $g_2 \in E^+$. Also $g_1 + g_2 = g$, so we only have to prove that $g_2 \leq f_2$. We find:

$$g_2 = g - g_1 = g - g \wedge f_1 = g + (-g) \vee (-f_1) = 0 \vee (g - f_1) \leq 0 \vee f_2 = f_2.$$

□

Corollary 1.13. *Let $(E, \leq)$ be a Riesz space and let $x, y \in E^+$. Then*

$$[0, x + y] = [0, x] + [0, y].$$

Now we know the definition of a Riesz space, we can introduce the concept of Archimedean Riesz spaces.

Definition 1.14. The Riesz space $(E, \leq)$ is called Archimedean if for any $u, v \in E^+$ which satisfy $0 \leq v \leq n^{-1}u$ for all $n \in \mathbb{Z}_{>0}$, it follows that $v = 0$.

The difference between Archimedean and non-Archimedean Riesz spaces becomes clear if we compare the pointwise order to the lexicographical order on $\mathbb{R}^n$.

Lemma 1.15. *The vector space $\mathbb{R}^n$ is Archimedean under its pointwise order.*

Proof. Let $u = (u_1, ..., u_n), v = (v_1, ..., v_n) \in \mathbb{R}^n$ such that $0 \leq v \leq m^{-1}u$ for all $m \in \mathbb{Z}_{>0}$. Because $\mathbb{R}^n$ is ordered pointwise, it follows that $0 \leq v_i \leq m^{-1}u_i$ for all $i \in \{1, ..., n\}$ and for all $m \in \mathbb{Z}_{>0}$. Therefore $0 \leq v_i \leq \lim_{m\to\infty} m^{-1}u_i = 0$ for all $i \in \{1, ..., n\}$. Thus $v_i = 0$ for all $i \in \{1, ..., n\}$ and we can conclude that $v = 0$. Hence $\mathbb{R}^n$ is Archimedean under its pointwise order. □

Lemma 1.16. *The vector space $\mathbb{R}^n$, with $n \neq 1$, is non-Archimedean under its lexicographical order.*

Proof. Consider the element $(1, 1, ..., 1) \in \mathbb{R}^n$. For all $m \in \mathbb{Z}_{>0}$ the inequality $\frac{1}{m} > 0$ holds, so we get $\forall m \in \mathbb{Z}_{>0} : 0 \leq (0, 1, 1, ..., 1) \leq (\frac{1}{m}, \frac{1}{m}, ..., \frac{1}{m})$, while $(0, 1, 1, ..., 1) \neq (0, 0, ..., 0)$. This contradicts the definition of an Archimedean Riesz space, so $\mathbb{R}^n$ is non-Archimedean under its lexicographical order. □

In Riesz spaces ideals are frequently used.

Definition 1.17. Let $(E, \leq)$ be a Riesz space. A linear subspace I of E is called an ideal if for all $x \in I$ and $y \in E$ with $|y| \leq |x|$, it follows that $y \in I$.

For any Riesz space $(E, \leq)$, we write $\mathcal{I}(E)$ for the collection of ideals of E. Notice that $\{0\} \in \mathcal{I}(E)$ and $E \in \mathcal{I}(E)$.

Definition 1.18. An ideal $I \in \mathcal{I}(E)$ is called maximal if E is the only ideal which strictly contains I and I is called minimal if $\{0\}$ is the only ideal which is strictly contained in I. The intersection of all maximal ideals is called the radical R_E of E. We call E semi-simple if $R_E = \{0\}$ and simple if $\mathcal{I}(E) = \{\{0\}, E\}$. If E does not contain maximal ideals then $R_E = E$.

Example 1.19. Consider the Riesz space $(C[0, 1], \leq)$ defined in Example 1.7. The collection $\{f \in C[0, 1] | \forall x \in [0, \frac{1}{2}] : f(x) = 0\}$ is an ideal in $C[0, 1]$. For any $t \in [0, 1]$ the ideal I_t defined by $I_t = \{f \in C[0, 1] : f(t) = 0\}$ is a maximal ideal. From this it follows that $R_E = \{f \in C[0, 1] | \forall x \in [0, 1] : f(x) = 0)\} = \{0\}$. Hence $C[0, 1]$ is semi-simple.

Definition 1.20. Let $(E, \leq)$ be a Riesz space. A linear subspace V of E is called a Riesz subspace (sublattice) if for all $f, g \in V$ the elements $f \vee g$ and $f \wedge g$ are also elements of V (i.e. V is closed under taking suprema and infima).

Notice that the requirement for V to be closed under taking infima follows from the requirements for it to be a linear subspace and to be closed under taking suprema. To see this, suppose that $f, g \in V$. Using Lemma 1.8, we find:

$$f \wedge g = -((-f) \vee (-g)) \in V.$$

Lemma 1.21. *Let $(E, \leq)$ be a Riesz space. Every ideal in E is a Riesz subspace.*

Proof. Let $I \in \mathcal{I}(E)$ and $f, g \in I$. From Lemma 1.11 it follows that

$$f \vee g = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|.$$

By definition I is a linear subspace, so $f+g \in I$ and $f-g \in I$. Furthermore we have $||f-g|| = |f-g| \leq |f-g|$, so $|f-g| \in I$. Hence $f \vee g \in I$ and I is a Riesz subspace. □

Definition 1.22. Two elements f and g in a Riesz space $(E, \leq)$ are called disjoint if $|f| \wedge |g| = 0$ (notation $f \perp g$). For any non-empty subset $D \subseteq E$, the disjoint complement $D^{\perp}$ is defined by: $D^{\perp} = \{f \in E : \forall g \in D : f \perp g\}$. Notice that $D^{\perp}$ is an ideal in E [3, Theorem 8.4]. Two non-empty subsets D_1 and D_2 of E are called disjoint if $d_1 \perp d_2$ for all $d_1 \in D_1$ and $d_2 \in D_2$ (notation $D_1 \perp D_2$).

Lemma 1.23. *Let $(E, \leq)$ be a Riesz space and let $I, J \in \mathcal{I}(E)$. Then:*

1. $I + J \in \mathcal{I}(E)$;
2. $I \cap J \in \mathcal{I}(E)$;
3. $I \cap I^{\perp} = \{0\}$.

Proof. 1. Let $f \in I + J$ and let $|g| \leq |f|$. We have to prove that $g \in I + J$. Because $f \in I + J$, we can write $f = f_1 + f_2$, where $f_1 \in I$, $f_2 \in J$. Now we find

$$g^{+} \leq |g| \leq |f| \leq |f_1| + |f_2|.$$

Now it follows from Lemma 1.12 that there exist $g_1, g_2 \in E^{+}$ such that $g_1 \leq |f_1|$, $g_2 \leq |f_2|$ and $g^{+} = g_1 + g_2$. Thus $g_1 \in I$, $g_2 \in J$ and $g^{+} = g_1 + g_2 \in I + J$. In the same way we find $g^{-} \in I + J$, thus $g \in I + J$.

2. As intersection of linear subspaces $I \cap J$ is a linear subspace of E. Now suppose that $x \in I \cap J$ and $y \in E$ such that $|y| \leq |x|$. Then $y \in I$ and $I \in J$, so $y \in I \cap J$.

3. Notice that $0 \in I$ and $0 \in I^{\perp}$, so $0 \in I \cap I^{\perp}$. Now suppose that $x \in I \cap I^{\perp}$. Then $|x| \wedge |x| = 0$, so $x = 0$.

□

From this lemma it follows that the algebraic sum $I + I^{\perp}$ is in fact a direct sum $I \bigoplus I^{\perp}$. So any $f \in I \bigoplus I^{\perp}$ can be writen uniquely as $f = g + h$, with $g \in I$ and $h \in I^{\perp}$. We can even say more:

Lemma 1.24. *Let $(E, \leq)$ be a Riesz space and let $I \in \mathcal{I}(E)$. Let $f, g \in I \bigoplus I^{\perp}$, having the unique decompositions $f = f_1 + f_2$ and $g = g_1 + g_2$. If $f \leq g$, then $f_1 \leq g_1$ and $f_2 \leq g_2$.*

Proof. First we want to prove that $(I \bigoplus I^{\perp})^{+} = I^{+} \bigoplus (I^{\perp})^{+}$. We see that $I^{+} \bigoplus (I^{\perp})^{+} \subseteq (I \bigoplus I^{\perp})^{+}$. Now suppose that $h \in (I \bigoplus I^{\perp})^{+}$. Then there exists an $h_1 \in I$ and an $h_2 \in I^{\perp}$ such that $h = h_1 + h_2$ and we find $h = |h| = |h_1 + h_2| \leq |h_1| + |h_2|$. Now the Riesz decomposition property tells us that $h = h' + h''$, where $0 \leq h' \leq |h_1|$ and $0 \leq h'' \leq |h_2|$. So $h' \in I^{+}$ and $h'' \in (I^{\perp})^{+}$.
Now suppose that $f, g \in I \bigoplus I^{\perp}$, such that $f \leq g$. Then we have $g - f = (g_1 + g_2) - (f_1 + f_2) = (g_1 - f_1) + (g_2 - f_2) \geq 0$, so $g - f \in (I \bigoplus I^{\perp})^{+}$, $g_1 - f_1 \in I^{+}$ and $g_2 - f_2 \in (I^{\perp})^{+}$. Hence $g_1 - f_1 \geq 0$ and $g_2 - f_2 \geq 0$. □

In particular we are interested in projection bands.

Definition 1.25. Let $(E, \leq)$ be a Riesz space. An element $I \in \mathcal{I}(E)$ is called a projection band if $I \bigoplus I^{\perp} = E$.

Lemma 1.26. *Let $(E, \leq)$ be a Riesz space and $D \subseteq E$. The smallest ideal $I \in \mathcal{I}(E)$ such that $D \subseteq I$ is given by the set*

$$D' = \{g \in E : |g| \leq |\lambda_1 f_1| + ... + |\lambda_n f_n| : \lambda_1, ..., \lambda_n \in \mathbb{R},\ f_1, ..., f_n \in D\}.$$

This ideal is called the ideal generated by D.

Proof. We first have to show that D' is indeed an ideal. From the definition of D' it follows that $0 \in D'$. Now suppose that $a, b \in D'$ and $\lambda \in \mathbb{R}$. Then we have

$$|a| \leq |\lambda_1 f_1| + ... + |\lambda_n f_n|$$

and

$$|b| \leq |\mu_1 g_1| + ...|\mu_m g_m|,$$

where $\lambda_1, ..., \lambda_n, \mu_1, ..., \mu_m \in \mathbb{R}$ and $f_1, ..., f_n, g_1, ..., g_m \in D$. Now we find:

$$|a + b| \leq |a| + |b| \leq |\lambda_1 f_1| + ... + |\lambda_n f_n| + |\mu_1 g_1| + ...|\mu_m g_m|$$

and

$$|\lambda a| = |\lambda||a| \leq |\lambda|(|\lambda_1 f_1| + ... + |\lambda_n f_n|) = |\lambda\lambda_1 f_1| + ... + |\lambda\lambda_n f_n|.$$

Thus D' is a linear subspace of E. Now it follows from the definition of D' that it is indeed an ideal.
Now let $J \in \mathcal{I}(E)$ such that $D \subseteq J$. Because J is a linear subspace of E, we find that $\lambda f \in J$ for all $\lambda \in \mathbb{R}$ and for all $f \in D$. Because J is an ideal, $|\lambda f| \in J$ as well. Now we find $|\lambda_1 f_1| + ... + |\lambda_n f_n| \in J$ for all $\lambda_1, ..., \lambda_n \in \mathbb{R}$ and $f_1, ..., f_n \in D$. From this it follows that $D' \subseteq J$. □

Lemma 1.27. *Let $(E, \leq)$ be a Riesz space. Under set inclusions $\mathcal{I}(E)$ becomes a distributive lattice. Its lattice operations are given by $I \vee J = I + J$ and $I \wedge J = I \cap J$.*

Proof. According to Lemma 1.23 $\mathcal{I}(E)$ is closed under taking sums and intersections.
Let $I, J \in \mathcal{I}(E)$. We see that $I, J \subseteq I + J$, so $I + J$ is an upper bound. Suppose that there exists a $K \in \mathcal{I}(E)$, such that $I, J \subseteq K$. Each $x \in I + J$ can be written as $x = x_1 + x_2$, where $x_1 \in I$ and $x_2 \in J$. This means that $x_1, x_2 \in K$, thus $x \in K$. Hence $I + J \subseteq K$ and $I \vee J = I + J$.
We see that $I \cap J$ is a lower bound of I and J. Now suppose that there exists a $K \in \mathcal{I}(E)$ such that $K \subseteq I, J$. Then we also find $K \subseteq I \cap J$, thus $I \wedge J = I \cap J$.
To prove that $\mathcal{I}(E)$ is distributive, we have to show that $(I + J) \cap K = (I \cap K) + (J \cap K)$ for all $I, J, K \in \mathcal{I}(E)$. Suppose that $z \in (I \cap K) + (J \cap K)$. Then we can write $z = z_1 + z_2$, with $z_1 \in I \cap K$ and $z_2 \in J \cap K$. This gives $z \in I + J$ and $z \in K$, so $z \in (I + J) \cap K$.
Now suppose that $z \in (I + J) \cap K$. Then we can write $z = x + y$, with $x \in I$ and $y \in J$. This gives $|z| \leq |x| + |y|$, so $|z| \in [0, |x| + |y|]$. Using Corollary 1.13 we find $|z| \in [0, |x|] + [0, |y|]$, so we can write $|z| = u + v$ with $0 \leq u \leq |x|$ and $0 \leq v \leq |y|$. Thus $u \in I$ and $v \in J$.
Because $z \in K$, we find $|z| \in K$. Furthermore we have $|u| = u \leq |z|$ and $|v| = v \leq |z|$ which gives $u, v \in K$ and thus $u \in I \cap K$ and $v \in J \cap K$. From this it follows that $|z| \in (I \cap K) + (J \cap K)$, so $z \in (I \cap K) + (J \cap K)$. □

2 Lattice homomorphisms

If we have two Riesz spaces we can define a homomorphism between them.

Definition 2.1. Let E and F be Riesz spaces. A linear map $T : E \to F$ is called a lattice homomorphism if $T(f \vee g) = T(f) \vee T(g)$ and $T(f \wedge g) = T(f) \wedge T(g)$ for all $f, g \in E$. It is called a lattice isomorphism if T is also bijective. In that case T^{-1} is a lattice homomorphism as well.

Notice that the requirement for T to preserve finite infima follows from the requirement for it to be linear and to preserve finite suprema. To see this, let $f, g \in E$. We find:

$$T(f \wedge g) = T(-[(-f) \vee (-g)]) = -T((-f) \vee (-g)) = -[T(-f) \vee T(-g)] =$$

$$-[(-T(f)) \vee (-T(g))] = T(f) \wedge T(g).$$

Lemma 2.2. *Let $T : E \to F$ be a lattice homomorphism. Then:*

1. *For all $f, g \in E$ the inequality $f \geq g$ implies $T(f) \geq T(g)$;*
2. *$T(|f|) = |T(f)|$.*

Proof. 1. If $f \geq g$, then $f = f \vee g$. Now we find

$$T(f) = T(f \vee g) = T(f) \vee T(g) \geq T(g).$$

2. For $f \in E$ we find

$$T(|f|) = T(f \vee (-f)) = T(f) \vee T(-f) = T(f) \vee (-T(f)) = |T(f)|.$$

□

From this lemma it follows that a Riesz homomorphism $T : E \to F$ is a positive map. That means that it maps E^+ into F^+.

Lemma 2.3. *Let E, F be Riesz spaces. A positive and bijective linear map $T : E \to F$ is a lattice isomorphism if T^{-1} is also positive.*

Proof. Let $f, g \in E$. Since T is positive and $f \vee g \geq f$ and $f \vee g \geq g$, we find

$$T(f \vee g) \geq T(f) \vee T(g).$$

Since T^{-1} is also positive and $T(f) \vee T(g) \geq T(f)$ and $T(f) \vee T(g) \geq T(g)$, we find

$$T^{-1}(T(f) \vee T(g)) \geq f \vee g$$

and thus

$$T(f) \vee T(g) \geq T(f \vee g).$$

Hence $T(f \vee g) = T(f) \vee T(g)$ for all $f, g \in E$ and T is a lattice homomorphism. □

Now let E be a Riesz space and let $I \in \mathcal{I}(E)$. Then we can define the canonical map $q : E \to E/I$, $f \mapsto f + I$.

Lemma 2.4. *If E/I is ordered by $q(f) \leq q(g)$ if and only if there exist an $f_1 \in f + I$ and a $g_1 \in g + I$ such that $f_1 \leq g_1$, then E/I becomes a Riesz space and $q : E \to E/I$ a lattice homomorphism.*

Proof. Because q is a linear map, property 1 and 2 of Definition 1.5 hold. Furthermore we see that the order on E/I is reflexive and transitive. Now suppose that there exists an $f \in E$ such that $q(f) \leq 0$ and $q(f) \geq 0$. Then there exist $f_1, f_2 \in E$ such that $0 \leq f_1 \in f + I$ and $0 \leq f_2 \in -f + I$, so $f_1 + f_2 \in I$. Now the inequality $|f_1| \leq |f_1 + f_2|$ gives $f_1 \in I$, so $q(f) = 0$. Thus the order on E/I is antisymmetric as well.
Now let $f, g \in E$. From the inequalities $f \vee g \geq f$ and $f \vee g \geq g$, it follows that $q(f \vee g) \geq q(f)$ and $q(f \vee g) \geq q(g)$, so $q(f \vee g)$ is an upper bound of $\{q(f), q(g)\}$. Now let $h \in E$ be such that $q(h) \geq q(f)$ and $q(h) \geq q(g)$. Then there exist $h_1, h_2 \in h + I$, such that $h_1 \geq f$ and $h_2 \geq g$. From this it follows that $h_1 - h_2 \in I$ and, because I is a Riesz subspace of E, we find $|h_1 - h_2| \in I$. So $k := h_2 + |h_1 - h_2| \in h + I$, where $k \geq h_2 + h_1 - h_2 = h_1 \geq f$ and $k \geq h_2 \geq g$. This gives us $k \geq f \vee g$, so $q(h) = q(k) \geq q(f \vee g)$. Thus we can conclude that $q(f \vee g) = q(f) \vee q(g)$. Hence E/I is a Riesz space and q is a lattice homomorphism. □

Lemma 2.5. *Let E be a Riesz space and $I \in \mathcal{I}(E)$. Let $q : E \to E/I$ be its canonical map. The map $J \mapsto q(J)$ is a surjective homomorphism $\mathcal{I}(E) \to \mathcal{I}(E/I)$. Its restriction to the Riesz subspace of ideals of E which contain I is an isomorphism.*

Proof. For any $J \in \mathcal{I}(E)$, we have that J is a linear subspace of E, so $q(J)$ is a linear subspace of E/I. Now let $x \in J$ and $y \in E$ such that $|q(y)| \leq |q(x)|$. Then we have $q(y)^+ = q(y) \vee 0 = q(y \vee 0) = q(y^+) \leq q(|y|) \leq q(|x|)$ and in the same way $q(y)^- = q(y^-) \leq q(|x|)$. Now we find $q(y^+) = q(|x|) \wedge q(y^+) = q(|x| \wedge y^+)$ and $q(y^-) = q(|x|) \wedge q(y^-) = q(|x| \wedge y^-)$. Furthermore $|x| \wedge y^+ \in J$ and $|x| \wedge y^- \in J$, because J is an ideal. From this it follows that $|x| \wedge y^+ - |x| \wedge y^- \in J$. Therefore $q(y) = q(|x| \wedge y^+ - |x| \wedge y^-) \in q(J)$. Hence $q(J) \in \mathcal{I}(E/I)$, so the map is well defined.
Let $J, K \in \mathcal{I}(E)$. Now we find $q(J+K) = J+K+I = J+I+K+I = q(J)+q(K)$ and, using the fact that $\mathcal{I}(E)$ is distributive, $q(J \cap K) = (J \cap K) + I = (J + I) \cap (K + I) = q(J) \cap q(K)$. Thus the map is a lattice homomorphism.
Now let $H \in \mathcal{I}(E/I)$. Because H is a linear subspace of E/I, $q^{-1}(H)$ is a linear subspace of E. Suppose that $x \in q^{-1}(H)$ and $y \in E$ are such that

$|y| \leq |x|$. Then we have, using Lemma 2.2, $q(|y|) = |y|+I = |q(y)| = |y+I|$ and $q(|x|) = |x| + I = |q(x)| = |x + I|$ and, using $|y| \leq |x|$, we find $|q(y)| \leq |q(x)|$. Thus $y \in q^{-1}(H)$ and $q^{-1}(H) \in \mathcal{I}(E)$. Hence q is surjective.
Now suppose that $J \in \mathcal{I}(E)$ such that $I \subseteq J$. Then we find $q^{-1}(q(J)) = J+I = J$, thus the restriction to the Riesz subspace of ideals of E which contain I is an isomorphism. □

Lemma 2.6. *Let $(E, \leq)$ be a Riesz space. The Riesz space E/R_E is semi-simple and Archimedean.*

Proof. Suppose that E does not contain maximal ideals. Then $R_E = E$, so $E/R_E = \{0\}$ and E is indeed semi-simple and Archimedean.
If E does contain maximal ideals, Lemma 2.5 tells us that the collection of ideals of E which contain R_E is isomorphic to $\mathcal{I}(E/R_E)$ (using the canonical map $q : E \to E/R_E$). So the set of maximal ideals of E is isomorphic to the set of maximal ideals of E/R_E and the radical of E is mapped onto the radical of E/R_E. Hence E/R_E has radical $\{0\}$ and is semi-simple.
Let $u, v \in (E/R_E)^+$ such that $0 \leq nv \leq u$ for all $n \in \mathbb{Z}_{>0}$. Suppose that I is a maximal ideal in E/R_E and suppose that $u \in I$. Then $v \in I$ as well. Now suppose that $u \notin I$ and consider the ideal J generated by $I \cup \{v\}$. If $u \notin J$, J is strictly contained in E. But I is a maximal ideal and therefore $J = I$ and $v \in I$.
If $u \in J$ there exist an $x \in E^+$ and an $n \in \mathbb{Z}_{>0}$ such that $u \leq x + nv$. Now we find:

$$0 \leq u - 2nv \leq x - nv.$$

This gives $nv \leq x$ and therefore $|nv| \leq x$. Thus $nv \in I$ and $v \in I$.
We can conclude that v is contained in the radical of E/R_E and must be zero. Hence E is Archimedean. □

Now suppose that $(E, \leq)$ is a semi-simple Riesz space. Then $E/R_E = E$, so E is itself Archimedean.

Lemma 2.7. *Let $(E, \leq) \neq \{0\}$ be a Riesz space. The following claims are equivalent:*

1. $\dim(E) = 1$;
2. *E is isomorphic to $\mathbb{R}$;*
3. *E is simple;*
4. *E is totally ordered and Archimedean.*

Proof. $1 \Rightarrow 2$: Let $f \in E$ such that $f \neq 0$. Then $f^- > 0$ or $f^+ > 0$. Call a part that is unequal to zero g. Because E is one-dimensional, g is a basis of E. Now the mapping $\lambda g \mapsto \lambda$ is an isomorphism $E \to \mathbb{R}$.
$2 \Rightarrow 1$: $\mathbb{R}$ is one-dimensional, so E is one-dimensional.

$2 \Rightarrow 3$: $\mathbb{R}$ is simple, so E is also simple.
$3 \Rightarrow 4$: Let $f \in E$ and suppose that $f^+ > 0$ and $f^- > 0$. Then the ideals generated by f^+ and f^- are both non zero and different from each other, which contradicts the assumption that E is simple. Hence $f^+ = 0$ or $f^- = 0$ and therefore $f \geq 0$ or $f \leq 0$ for each $f \in E$. Furthermore E is Archimedean according to Lemma 2.6.
$4 \Rightarrow 2$: Let $e \in E$ such that $e > 0$ and define the sets $A_x := \{\lambda \in \mathbb{R} : x \leq \lambda e\}$ and $B_x := \{\lambda \in \mathbb{R} : x \geq \lambda e\}$ for each $x \in E$. Because E is totally ordered, $\mathbb{R} = A_x \cup B_x$.
Suppose that $A_x = \emptyset$. Then we find $\forall \lambda \in \mathbb{R} : x > \lambda e$ and in particular $\forall n \in \mathbb{Z}_{>0} : ne \leq x$. Now follows from the Archimedean property the contradiction $e \leq 0$. Hence $A_x \neq \emptyset$ and in a similar way we find $B_x \neq \emptyset$.
Now suppose that there exists a $\lambda \in \overline{A_x}$ such that $\lambda \notin A_x$. Let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence in A_x such that $\lim_{n \to \infty} \lambda_n = \lambda$. Because $\mathbb{R} = A_x \cup B_x$ and $\lambda \notin A_x$, we see that $\lambda \in B_x$ and $x \neq \lambda e$, thus $x > \lambda e$.
Now we find $\lambda e < x \leq \lambda_n e$ for all $n \in \mathbb{N}$, thus $0 < x - \lambda e \leq (\lambda_n - \lambda)e$ for all $n \in \mathbb{N}$. This contradicts the assumption that E is Archimedean, because $\lambda_n - \lambda \to 0$ for $n \to \infty$. Hence A_x is closed in $\mathbb{R}$. In the same way we find that B_x is closed in $\mathbb{R}$.
Because $\mathbb{R}$ is connected and A_x and B_x are non empty closed subsets of $\mathbb{R}$, $A_x \cap B_x \neq \emptyset$. On the other hand $A_x \cap B_x$ can only contain the element $\lambda_x \in \mathbb{R}$ which satisfies $x = \lambda_x e$. So the map $x \mapsto \lambda_x$ becomes an isomorphism $E \to \mathbb{R}$. □

Corollary 2.8. *Let $(E, \leq)$ be a Riesz space. The ideal $I \in \mathcal{I}(E)$ is minimal if and only if it is isomorphic to $\mathbb{R}$ and maximal if and only if E/I is isomorphic to $\mathbb{R}$.*

Proof. If I is a minimal ideal of E, it is simple and therefore isomorphic to $\mathbb{R}$. Now suppose that $I \cong \mathbb{R}$. Then I is simple, thus minimal.
If I is a maximal ideal of E, Lemma 2.5 tells us that E/I is simple and thus that E/I is isomorphic to $\mathbb{R}$. If $E/I \cong \mathbb{R}$, then $\operatorname{codim}(I) = 1$, so I must be maximal. □

Lemma 2.9. *Let $(E, \leq)$ be an Archimedean Riesz space and let I be a minimal ideal of E. Then I is a projection band and $I^\perp$ is a maximal ideal.*

Proof. Suppose that I is a minimal ideal of E. Then Corollary 2.8 gives $I \cong \mathbb{R}$. Because E is Archimedean, [1, Theorem 26.4.iii] gives that I is a projection band. Because I has dimension 1, the dimension of $I^\perp$ must be one lower than the dimension of E. Hence $I^\perp$ is a maximal ideal. □

If we have a Riesz space M we can always extend it to a Riesz space of one dimension higher by looking at the lexicographic union of $\mathbb{R}$ with M.

Definition 2.10. Let M be a Riesz space. The lexicographic union of $\mathbb{R}$ with M (notation $\mathbb{R} \circ M$) is the Riesz space $\mathbb{R} \times M$ ordered by $(a, b) \geq 0$ if and only if $a > 0$ or $a = 0$ and $b \in M^+$.

Lemma 2.11. *Let $(E, \leq)$ be a Riesz space. Suppose that E strictly contains an ideal M which contains every ideal which is strictly contained in E. Then E contains no projection bands other than $\{0\}$ and E and is isomorphic to $\mathbb{R} \circ M$.*

Proof. Let I be a projection band of E and suppose that $\{0\} \neq I \neq E$. Then $\{0\} \neq I^\perp \neq E$ as well and by assumption we find $I \subseteq M$ and $I^\perp \subseteq M$, thus $E \subseteq M$, which is a contradiction. Hence E contains no projection bands other than $\{0\}$ and E.
By assumption M is a maximal ideal, so it must be of a dimension one lower than the dimension of E (Corollary 2.8). Thus we can write E as $L(x) + M$, where $L(x)$ is the linear span of an element $x \in E^+ \backslash M$. Now let $z = \lambda x + y \in E$, where $y \in M$ and suppose that $\lambda > 0$. Then $z \notin M$, so for the ideal I which is generated by z we have $I \not\subseteq M$ and thus $I = E$. So we can write $E = J \bigoplus K$, where J is the ideal generated by z^+ and K is the ideal generated by z^-. We just proved that $J = \{0\}$ or $K = \{0\}$.
Now consider the canonical map $q : E \to E/M$. We find $q(z^+) \geq q(z) = q(\lambda x + y) = q(\lambda x) = \lambda q(x) > 0$. From this it follows that $J \neq \{0\}$ and therefore $K = \{0\}$. So $z^- = 0$ and $z > 0$. Hence we have $z > 0$ if $\lambda > 0$ and therefore E is isomorphic to $\mathbb{R} \circ M$. $\square$

Definition 2.12. Let $(E, \leq)$ be a Riesz space. A subset $D \subseteq E^+$ is called a disjoint system if $0 \notin D$ and if $u \wedge v = 0$ for all $u, v \in D$ with $u \neq v$.

Lemma 2.13. *Let $(E, \leq)$ be a Riesz space and let $I \in \mathcal{I}(E)$. Let q be the canonical map $E \to E/I$. For each finite disjoint system $\{y_i : i = 1, ..., n\}$ in E/I there exists a disjoint system $\{x_i : i = 1, ..., n\}$ in E, such that $y_i = q(x_i)$ for all i.*

Proof. We will use induction on n. For $n = 1$ the claim is true. Now suppose that the claim is true for all $n < N$, with $N > 1$. Consider the disjoint system $\{y_i : i = 1, ..., N\}$ in E/I. By induction hypothesis there exists a disjoint system $\{x_i : i = 1, ..., N-1\}$ in E such that $y_i = q(x_i)$ for all i satisfying $1 \leq i \leq N-1$. Let $x_N \in E^+$ such that $q(x_N) = y_N$ and define:

$$x'_i := x_i - x_i \wedge x_N \quad \text{for } i = 1, ..., N-1;$$
$$x'_N := x_N - x_N \wedge (x_1 + ... + x_{N-1}).$$

Now we find for each i satisfying $1 \leq i \leq N - 1$:

$$q(x'_i) = q(x_i) - q(x_i \wedge x_N) = q(x_i) - y_i \wedge y_N = q(x_i) = y_i$$

and

$$q(x'_N) = q(x_N) - q(x_N \wedge (x_1 + ... + x_{N-1})) = q(x_N) = y_N.$$

Thus the system $\{x'_i : i = 1, ..., N\}$ is a disjoint system in E which satisfies the requirement. $\square$

3 Structure of finite dimensional Riesz spaces

From now on we will concentrate on finite dimensional Riesz spaces.

Theorem 3.1. *Let $(E, \leq)$ be a Riesz space of finite dimension $n \geq 1$ en let r be the dimension of its radical. Then E is isomorphic to the sum of $k = n - r$ mutually disjoint ideals I_j (for $j = 1, ..., k$), such that each of these ideals contains a unique maximal ideal M_j and is therefore isomorphic to $\mathbb{R} \circ M_j$. This decomposition of E is unique, except for a permutation of the indices.*

Proof. If $\dim(E) = n$ and $\dim(R_E) = r$, then $\dim(E/R_E) = k = n - r$ and according to Lemma 2.6 E/R_E is Archimedean. From Corollary 2.8 and Lemma 2.9 it follows that every minimal ideal is a projection band of dimension one in E/R_E. So inductively we find that E/R_E is the sum of k mutually disjoint minimal ideals. Now we can make a disjoint system $\{y_j : j = 1, ..., k\}$ in E/R_E by taking an element out of each disjoint minimal ideal.
Using Lemma 2.13 we find a disjoint system $\{x_j : j = 1, ..., k\}$ in E such that $q(x_j) = y_j$ for $1 \leq j \leq k$ (where q denotes the canonical map $E \to E/R_E$). Since $\{y_j : j = 1, ..., k\}$ is a basis of E/R_E, we can write $E = L + R_E$, where L is the linear span of $\{x_j : j = 1, ..., k\}$.
Now consider the ideal $I(L)$ which is generated by L and suppose that $I(L) \neq E$. Because $E = L + R_E$, $R_E \not\subseteq I(L)$, so $I(L)$ is not a maximal ideal in E. Thus there exists a maximal ideal J in E such that $I(L) \subseteq J$. However J is strictly contained in E, so $R_E \not\subseteq J$, which contradicts the definition of R_E. Thus $I(L) = E$.
Denote the ideal generated by x_j by I_j. Because $\{x_j : j = 1, ..., k\}$ is a disjoint system in E, the ideals I_j are mutually disjoint, so E is the sum of k mutually disjoint ideals.
We claim that $M_j := I_j \cap R_E$ is the unique maximal ideal in each I_j. Notice that M_j is an ideal in E and therefore an ideal in I_j, since it is an intersection of ideals in E. Notice that $x_j \in I_j$, but $x_j \notin M_j$, because $x_j \notin R_E$. Therefore $\dim(I_j/M_j) > 0$. Furthermore we can write

$$E = \bigoplus_{j=1}^{k} I_j$$

and

$$R_E = R_E \cap E = R_E \cap (\bigoplus_{j=1}^{k} I_j) = \bigoplus_{j=1}^{k} (R_E \cap I_j) = \bigoplus_{j=1}^{k} M_j.$$

Now we find

$$E/R_E \cong \bigoplus_{j=1}^{k} (I_j/M_j),$$

which gives us

$$k = \dim(E/R_E) = \sum_{j=1}^{k} \dim(I_j/M_j).$$

Because $\dim(I_j/M_j) > 0$, we find $\dim(I_j/M_j) = 1$ for all j with $0 \leq j \leq k$. Hence M_j is a maximal ideal in I_j.
Take a maximal ideal K in I_j. Then $K' := K + \sum_{i \neq j} I_i$ is an ideal in E, since it is a sum of ideals. Furthermore $\dim(K') = n - 1$, so K' is maximal in E and $R_E \subseteq K'$. Now we find $K = K' \cap I_j \supseteq R_E \cap I_j = M_j$, so $M_j = K$. Thus M_j is indeed the unique maximal ideal in each I_j.
Suppose that I_j strictly contains an ideal K such that $K \not\subseteq M_j$. Then K is either a maximal ideal, or it is contained in a maximal ideal. Both are impossible, since M_j is the unique maximal ideal of I_j. Thus M_j contains each ideal which is strictly contained in I_j. Now it follows from Lemma 2.11 that $I_j \cong \mathbb{R} \circ M_j$.
All that remains to be proven is the unicity of the decomposition. Suppose that $E = J \bigoplus J^\perp$, where J is isomorphic to $\mathbb{R} \circ N$, with N a unique maximal ideal in J. For each j satisfying $1 \leq j \leq k$ we have $I_j = (J \bigoplus J^\perp) \cap I_j = J \cap I_j \bigoplus J^\perp \cap I_j$, because of the distributivity of $\mathcal{I}(E)$. Now it follows from Lemma 2.11 that I_j contains only the projection bands $\{0\}$ and I_j, thus $J \cap I_j = \{0\}$ or $J \cap I_j = I_j$. However $J \cap I_j = \{0\}$ cannot hold for all j, so there must be an m satisfying $1 \leq m \leq k$ such that $J \cap I_m = I_m$. Now suppose that $J \neq I_m$. Then I_m is a non trivial projection band in J, which is a contradiction according to Lemma 2.11. So the decomposition is indeed unique. □

Now suppose that $(E, \leq)$ is a Riesz space of finite dimension $n \geq 1$. Then we can write E as the direct sum of k mutually disjoint ideals I_j according to Theorem 3.1. Furthermore we can write each of these ideals as a lexicographical union $\mathbb{R} \circ M_j$. Because each M_j is itself a Riesz space of a lower dimension than the dimension of E, we can continue this process inductively. In this way the structure of any finite dimensional Riesz space can be understood.

Corollary 3.2. *Let $(E, \leq)$ be a finite dimensional Riesz space. The set $\mathcal{B}(E)$ of all projection bands in E is isomorphic to $\mathcal{I}(E/R_E)$.*

Proof. According to Theorem 3.1 we can write E uniquely as the direct sum $\bigoplus_{j=1}^k I_j$ of mutually disjoint ideals. So each projection band in E is of the form $\bigoplus_{j \in H} I_j$, with $H \subseteq \{1, ..., k\}$. Furthermore E/R_E is the direct sum of k mutually disjoint minimal ideals which are given by $q(I_j)$ (where q denotes the canonical map $E \to E/R_E$). Thus each ideal in E/R_E can be writen as $\bigoplus_{j \in H} q(I_j)$, so the map $B \mapsto q(B)$ is an isomorphism $\mathcal{B}(E) \to \mathcal{I}(E/R_E)$. □

Corollary 3.3. *Let $(E, \leq) \neq \{0\}$ be a finite dimensional Riesz space. If E contains no projection bands other than $\{0\}$ and E, it is isomorphic to $\mathbb{R} \circ M$. Here M denotes the unique maximal ideal of E.*

Proof. From Corollary 3.2 it follows that E/R_E is simple, so $\dim(E/R_E) = 1$ according to Lemma 2.7. Therefore R_E must be a maximal ideal in E and hence a unique maximal ideal in E. Now it follows from Lemma 2.11 that E is isomorphic to $\mathbb{R} \circ R_E$. □

Theorem 3.4. *Let $(E, \leq)$ be a Riesz space of finite dimension $n \geq 1$. The following claims are equivalent:*

1. *E is Archimedean;*

2. *E is semi-simple;*

3. *E is isomorphic to $\mathbb{R}^n$ under its pointwise order.*

Proof. $1 \Rightarrow 2$: Suppose that $R_E \neq \{0\}$. As intersection of ideals R_E is itself an ideal and therefore contains a minimal ideal $I \neq \{0\}$. Now Lemma 2.9 tells us that $I^\perp$ is maximal, while $R_E \not\subset I^\perp$. From this contradiction it follows that $R_E = \{0\}$.
$2 \Rightarrow 3$: From Theorem 3.1 it follows that E is isomorphic to the direct sum of n disjoint ideals of the form $\mathbb{R} \circ \{0\}$. So E must be isomorphic to $\mathbb{R}^n$ under its pointwise order.
$3 \Rightarrow 1$: Lemma 1.15 tells us that $\mathbb{R}^n$ under its pointwise order is Archimedean, so E is Archimedean. □

Theorem 3.5. *Let $(E, \leq)$ be a totally ordered Riesz space of finite dimension $n \geq 1$. Then E is isomorphic to $\mathbb{R}^n$ under its lexicographical order.*

Proof. We will use induction with respect to the dimension of E. Suppose $\dim(E) = 1$. Then Lemma 2.7 tells us that E is isomorphic to $\mathbb{R}$.
Now suppose that the claim holds for all $n < N$, for $N \in \mathbb{Z}_{>1}$. Let E be a totally ordered Riesz space such that $\dim(E) = N$. Suppose that there exists a projection band I, such that $I \neq \{0\} \neq I^\perp$. Then there exist a $u, v \in E$ such that $u \neq 0 \neq v$, but $|u| \wedge |v| = 0$. This contradicts the assumption that E is totally ordered, thus E contains only the projection bands $\{0\}$, E. Now Corollary 3.3 tells us that E is isomorphic to $R \circ M$, where M is totally ordered and has dimension $N - 1$. From the induction hypothesis it follows that M is isomorphic to $\mathbb{R}^{N-1}$ under its lexicographical order, so E is isomorphic to $\mathbb{R}^N$ under its lexicographical order. □

4 Automorphism groups of finite dimensional Riesz spaces

According to Theorem 3.1 a finite dimensional Riesz space $(E, \leq)$ is isomorphic to the direct sum of a finite number of mutually disjoint ideals. This decomposition of E is unique, except for a permutation of the indices. Now we can write this decomposition of E in particular as $\bigoplus_{i=1}^{n} K_i$, where K_i is the direct sum $\bigoplus_{j=1}^{m_i} I_{ij} \cong \bigoplus_{j=1}^{m_i} I$ of all mutually disjoint ideals in the decomposition of E which are contained in one isomorphism class. Because of the unicity of this decomposition, $\sigma(K_i) = K_i$ for all $\sigma \in \mathrm{Aut}(E)$. Therefore the automorphism group $\mathrm{Aut}(E)$ becomes isomorphic to the Cartesian product of the automorphism groups $\mathrm{Aut}(K_i)$. Now let us look at these automorphism groups.

Theorem 4.1. *Let $(E, \leq)$ be a Riesz space and let $K = \bigoplus_{j=1}^{n} I_j \cong \bigoplus_{j=1}^{n} I$ be the direct sum of all mutually disjoint ideals in the decomposition of E which are contained in one isomorphism class. Then $Aut(K)$ is isomorphic to $\prod_{j=1}^{n} Aut(I) \rtimes S_n$, where $\prod_{j=1}^{n} Aut(I)$ denotes the Cartesian product of the automorphism group of I and S_n denotes the collection of all permutations of n-tuples.*

Proof. Let $\sigma \in \mathrm{Aut}(K)$. From the uniqueness of Theorem 3.1 it follows that

$$\sigma(I_j) = I_{\pi_\sigma(j)}$$

for every $j \in \{1, ..., n\}$, with $\pi_\sigma \in S_n$.
Now let $P_j : K \to I$ be the projection onto the j-th coordinate and $\phi_j : I \to K$ be the embedding at the j-th coordinate. Then we find for each $x \in K$:

$$x = \sum_{j=1}^{n} \phi_j P_j x. \tag{1}$$

Now define the map

$$\tau_\sigma : K \to K,\, x \mapsto \sum_{j=1}^{n} \phi_{\pi_\sigma(j)} P_j x$$

which permutes the coordinates of an element in the same way as σ permutes the disjoint ideals of K. Then we get for any l satisfying $1 \leq l \leq n$ and for $x \in K$:

$$P_l \tau_\sigma x = \sum_{j=1}^{n} P_l \phi_{\pi_\sigma(j)} P_j x = P_{\pi_\sigma^{-1}(l)} x. \tag{2}$$

Now we find, using (1), for any $x \in K$:

$$\sigma x = \sum_{j=1}^{n} \sigma \phi_j P_j x = \sum_{j=1}^{n} \phi_{\pi_\sigma(j)} P_{\pi_\sigma(j)} \sigma \phi_j P_j x.$$

By substituting $j = \pi_\sigma^{-1}(l)$, this equation becomes:

$$\sigma x = \sum_{l=1}^{n} \phi_l P_l \sigma \phi_{\pi_\sigma^{-1}(l)} P_{\pi_\sigma^{-1}(l)} x.$$

Using (2) this equation finally becomes:

$$\sigma x = \sum_{l=1}^{n} \phi_l P_l \sigma \phi_{\pi_\sigma^{-1}(l)} P_l \tau_\sigma x.$$

Therefore every $\sigma \in \mathrm{Aut}(E)$ is responsible for a permutation of the coordinates of an element, followed by a pointwise automorphism of $\mathrm{Aut}(E)$. Furthermore $\prod_{j=1}^{n} \mathrm{Aut}(I) \cap S_n = \{id\}$, where id denotes the identity map. Therefore any $\sigma \in \mathrm{Aut}(E)$ can be written uniquely as the product of a $\tau \in \prod_{j=1}^{n} \mathrm{Aut}(I)$ and a $\pi \in S_n$. Hence $\mathrm{Aut}(E)$ can be embedded in $\prod_{j=1}^{n} \mathrm{Aut}(I) \times S_n$.
If we take $\pi \in S_n$ and $\tau \in \prod_{j=1}^{n} \mathrm{Aut}(I)$, we see that $\pi\tau\pi^{-1} \in \prod_{j=1}^{n} \mathrm{Aut}(I)$. Thus there exists a homomorphism $S_n \to \mathrm{Aut}(\prod_{j=1}^{n} \mathrm{Aut}(I))$, $\pi \mapsto (\sigma_\pi : \tau \mapsto \pi\tau\pi^{-1})$. Therefore

$$\mathrm{Aut}(K) \cong \prod_{j=1}^{n} \mathrm{Aut}(I) \rtimes S_n.$$

□

In order to fully understand the structure of $\mathrm{Aut}(K)$, we now need to know the structure of $\mathrm{Aut}(I)$. According to Theorem 3.1 the ideal I is of the form $\mathbb{R} \circ M$, where M denotes the unique maximal ideal of I. Now the following lemma will explain the stucture of $\mathrm{Aut}(I)$.

Lemma 4.2. *Let $(I, \leq)$ be a finite dimensional Riesz space of the form $\mathbb{R} \circ M$ (where M denotes the unique maximal ideal of I). Its automorphism group is isomorphic to*

$$\left\{ \left(\begin{array}{c|c} \lambda & 0 \cdots 0 \\ \hline a_2 & \\ \vdots & \sigma \\ a_n & \end{array} \right) : \lambda \in \mathbb{R}_{>0},\ a_2, ..., a_n \in \mathbb{R},\ \sigma \in Aut(M) \right\}.$$

Here σ is given with respect to the basis $\{v_2, ..., v_n\}$ of M and the matrices of the studied automorphism group are given with respect to the basis $\{(1,0), (0,v_2), ..., (0,v_n)\}$ of $\mathbb{R} \circ M$.

Proof. Let ψ be such an automorphism. Because M is the unique maximal ideal of I, $\psi M = M$. Therefore the matrix representing ψ must be of the form

$$\left(\begin{array}{c|c} \lambda & 0 \cdots 0 \\ \hline a_2 & \\ \vdots & \sigma \\ a_n & \end{array} \right),$$

where $a_2, ..., a_n \in \mathbb{R}$ and $\sigma \in \mathrm{Aut}(M)$. Furthermore $\psi(a,0)^T > 0$ for all $a > 0$, because ψ must be a positive map. From this it follows that $\lambda \in \mathbb{R}_{>0}$.
Now we have to show that all the matrices in the prescribed collection are indeed automorphisms. Suppose that ψ is a map of the form

$$\psi = \left(\begin{array}{c|c} \lambda & 0 \cdots 0 \\ \hline a_2 & \\ \vdots & \sigma \\ a_n & \end{array}\right),$$

where $\lambda \in \mathbb{R}_{>0}$, $a_2, ..., a_n \in \mathbb{R}$ and $\sigma \in \mathrm{Aut}(M)$. We see that ψ is a positive bijective linear map, since $\lambda \in \mathbb{R}_{>0}$ and $\sigma \in \mathrm{Aut}(M)$. Furthermore we see that ψ^{-1} is given by

$$\left(\begin{array}{c|c} \frac{1}{\lambda} & 0 \cdots 0 \\ \hline b_2 & \\ \vdots & \sigma^{-1} \\ b_n & \end{array}\right),$$

where $b_2, ..., b_n \in \mathbb{R}$. Because $\lambda > 0$, $\frac{1}{\lambda} > 0$ as well. Furthermore $\sigma^{-1} \in \mathrm{Aut}(M)$, because $\sigma \in \mathrm{Aut}(M)$. Therefore we can conclude that ψ^{-1} is also a positive map. Now Lemma 2.3 tells us that ψ is an automorphism.

□

The maximal ideal M of I used in Lemma 4.2 is itself a Riesz space. Therefore we can find its automorphism group by using Theorem 4.1 and the preceding remarks. Thus given a finite dimensional Riesz space $(E, \leq)$ we can always inductively find its automorphism group, making use of Theorem 4.1 and Lemma 4.2. In the Archimedean and totally ordered case we can give the automorphism group in more detail.

Theorem 4.3. *Let $(E, \leq)$ be an Archimedean Riesz space having finite dimension $n \geq 1$. Its automorphism group is isomorphic to*

$$\left\{ \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} : \lambda_1, ..., \lambda_n \in \mathbb{R}_{>0} \right\} \rtimes S_n.$$

Proof. According to Theorem 3.1 and 3.4, E is isomorphic to the sum of n mutually disjoint ideals of the form $\mathbb{R} \circ \{0\}$. Thus the decomposition of E only consists of one isomorphism class. Now it follows from Theorem 4.1 that the automorphism group $\mathrm{Aut}(E)$ is isomorphic to

$$\prod_{j=1}^{n} \mathrm{Aut}(\mathbb{R}) \rtimes S_n.$$

Furthermore an automorphism of $\mathbb{R}$ can only be a scalar multiplication with a positive scalar. Therefore $\prod_{j=1}^{n} \mathrm{Aut}(\mathbb{R})$ must be of the form

$$\left\{ \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} : \lambda_1, ..., \lambda_n \in \mathbb{R}_{>0} \right\}.$$

□

Theorem 4.4. *Let $(E, \leq)$ be a totally ordered Riesz space of finite dimension $n \geq 1$. Its automorphism group is isomorphic to*

$$\left\{ \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ a_{21} & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ a_{n1} & \cdots & a_{nn-1} & \lambda_n \end{pmatrix} : \lambda_1, ..., \lambda_n \in \mathbb{R}_{>0},\ a_{21}, ..., a_{nn-1} \in \mathbb{R} \right\}.$$

Proof. According to Theorem 3.5, E is isomorphic to $\mathbb{R} \circ M$, where M is totally ordered and has dimension $n-1$. Now it follows from Lemma 4.2 that $\mathrm{Aut}(E)$ must be isomorphic to

$$\left\{ \left(\begin{array}{c|c} \lambda & 0 \cdots 0 \\ \hline a_2 & \\ \vdots & \sigma \\ a_n & \end{array} \right) : \lambda \in \mathbb{R}_{>0},\ a_2, ..., a_n \in \mathbb{R},\ \sigma \in \mathrm{Aut}(M) \right\}.$$

Because M is a totally ordered Riesz space of dimension $n-1$, we can continue this proces inductively. This gives indeed the prescribed automorphism group.

□

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