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An analogue of Chern-Weil theory for the line bundle of weak Jacobi forms on a non-compact modular surface

Master Thesis

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Contents

Introduction

Let X be a smooth compact algebraic surface over $\mathbb{C}$. Chern-Weil theory implies that if $(L, ||\cdot||)$ is a line bundle on X equipped with a smooth metric, then we have

$$C \cdot C = \int_X c_1(L, ||\cdot||)^{\wedge 2},$$

where C is the equivalence class of Weil divisors associated to L and $c_1(L, ||\cdot||)$ is the first Chern form taken with respect to any connection on L compatible with $||\cdot||$. Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup which acts without fixed points on the upper half-plane $\mathcal{H}$ and let

$$p \colon E^0(\Gamma) \to Y(\Gamma)$$

be the universal elliptic curve over the modular curve $Y(\Gamma) = \Gamma \backslash \mathcal{H}$. We would like to use the aforementioned Chern-Weil theoretic result to compute the self-intersection of the theta divisor on $E^0(\Gamma)$ given by the image of the zero section, using a translation invariant metric $||\cdot||_{\mathrm{Pet}}$ on the line bundle of weak Jacobi forms $L_{4,4}(\Gamma)$ of weight 4 and index 4. But since $E^0(\Gamma)$ is not compact, Chern-Weil theory can not be applied directly. Following the same line of reasoning as in [5], we try to remedy this by extending $(L_{4,4}(\Gamma), ||\cdot||_{\mathrm{Pet}})$ to a metrized line bundle $(\overline{L_{4,4}(\Gamma)}, ||\cdot||)$ on the compactification

$$\overline{p} \colon E(\Gamma) \to X(\Gamma)$$

in the sense of Deligne-Rapoport. Let J be the divisor class associated to $L_{4,4}(\Gamma)$. It turns out not to be possible to extend $||\cdot||_{\mathrm{Pet}}$ smoothly. The closest thing we can get to a smooth extension of $||\cdot||_{\mathrm{Pet}}$ is a so-called *Mumford-Lear extension*. This extension picks up serious enough singularities along the singular locus of $E(\Gamma) - E^0(\Gamma)$ that the self-intersection of the associated divisor class $[J, ||\cdot||_{\mathrm{Pet}}]_{E(\Gamma)}$ is not equal to $\int_{E^0(\Gamma)} c_1(L_{4,4}(\Gamma), ||\cdot||_{\mathrm{Pet}})^{\wedge 2}$. The main result of this thesis, which is a generalization of [5, Theorem 5.2] to arbitrary congruence subgroups $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ which act without fixed points on $\mathcal{H}$, is the following.

Theorem. *For every smooth compactification E of $E^0(\Gamma)$, the Mumford-Lear extension of $L_{4,4}(\Gamma)$ to E exists. If we denote its associated divisor class by $[J, ||\cdot||_{\mathrm{Pet}}]_E$, then the following equality holds:*

$$\lim_{E \in \mathrm{CPT}(E^0(\Gamma))} [J, ||\cdot||_{\mathrm{Pet}}]_E^2 = \int_{E^0(\Gamma)} c_1(L_{4,4}(\Gamma), ||\cdot||_{\mathrm{Pet}})^{\wedge 2},$$

where $\mathrm{CPT}(E^0(\Gamma))$ denotes the directed set of isomorphism classes of smooth compactifications of $E^0(\Gamma)$.

Thus an analogue of Chern-Weil theory for the line bundle of weak Jacobi forms on $E^0(\Gamma)$ can be obtained by taking into account *all* smooth compactifications of $E^0(\Gamma)$.

We mention the following explicit results. If $\mathcal{C}(\Gamma)$ denotes the set of cusps of $X(\Gamma)$ and $h_{\mathfrak{c}}$ the period of cusp $\mathfrak{c}$, the self-intersection of $[J, ||\cdot||_{\mathrm{Pet}}]_{E(\Gamma)}$ is equal to

$$[J, ||\cdot||_{\mathrm{Pet}}]_{E(\Gamma)}^2 = \frac{64}{12} d_\Gamma + \sum_{\mathfrak{c} \in \mathcal{C}(\Gamma)} \frac{64}{12 h_{\mathfrak{c}}^2}$$

where d_Γ is the degree of the forgetful map $X(\Gamma) \to X(1)$. Furthermore, taking the limit of the self-intersection over all compactifications of $E^0(\Gamma)$, we obtain

$$\lim_{E \in \mathrm{CPT}(E^0(\Gamma))} [J, ||\cdot||_{\mathrm{Pet}}]_E^2 = \frac{64}{12} d_\Gamma.$$

Overview

Throughout this thesis, we fix a congruence subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ which acts without fixed points on $\mathcal{H}$.

The material in this paper is organized as follows. In the first chapter, we introduce the main objects of study in this thesis, the elliptic surface $E(\Gamma)$ and the line bundle $L_{k,m}(\Gamma)$ of weak Jacobi forms of weight k and index m, and provide some background on what purpose they serve in the general body of mathematics. This chapter fulfills mainly an expository role and does not contain many proofs.

In the second chapter, we introduce a set of tools to be used in the calculation of the Mumford-Lear extension of $|| \cdot ||_{\mathrm{Pet}}$ to the various smooth compactifications of $E^0(\Gamma)$. We also give an explicit description of the Petersson metric near the singular locus of $E(\Gamma) - E^0(\Gamma)$.

Finally, in the third chapter, the Mumford-Lear extension of $|| \cdot ||_{\mathrm{Pet}}$ to all smooth compactifications of $E^0(\Gamma)$ is computed and we show the associated system of divisors satisfies Chern-Weil theory and a Hilbert-Samuel type formula.

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1 Background

This chapter serves to put the calculations in the next two chapters in context. We begin by introducing the central object of study of this thesis, the elliptic surface $E(\Gamma)$ associated to a congruence subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$. As is well known (cf. [11, Proposition 2.1.1]), every $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ acts properly discontinuously on the upper half plane $\mathcal{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$, and this action extends to an action of $\Gamma^J := \Gamma \ltimes \mathbb{Z}^2$ on $\mathcal{H} \times \mathbb{C}$ given by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right).$$

If Γ acts without fixed points on $\mathcal{H}$, the quotient $E^0(\Gamma) = \Gamma^J \backslash \mathcal{H} \times \mathbb{C}$ comes equipped with the structure of non-compact complex manifold with the property that holomorphic functions on $E^0(\Gamma)$ correspond to Γ^J-periodic holomorphic functions on $\mathcal{H} \times \mathbb{C}$, and the purpose of section 1.1 is to describe the directed set

$$\mathrm{CPT}(E^0(\Gamma)) = \{\text{smooth compactifications of } E^0(\Gamma)\}/\cong$$

of isomorphism classes of complete smooth algebraic surfaces X over $\mathbb{C}$ which contain $E^0(\Gamma)$ as an open subset and which satisfy the property that $X - E^0(\Gamma)$ is a *normal crossings divisor*. We shall see that the minimal element of $\mathrm{CPT}(E^0(\Gamma))$ is $E(\Gamma)$, and all other elements of $\mathrm{CPT}(E^0(\Gamma))$ can be obtained from $E(\Gamma)$ by a finite series of blow-ups in points not contained in $E^0(\Gamma)$.

Next, we give a relation between $E(\Gamma)$ and certain moduli spaces of marked curves over a base scheme S. Here, by a *curve* over S (perhaps more aptly named a *family of curves*) we mean a 1-dimensional variety over S, i.e. a flat morphism $X \to S$ of finite type and of relative dimension 1 with geometrically connected fibers. If $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ acts without fixed points on $\mathcal{H}$ and is equal to one of

$$\Gamma(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \mod N, \quad b \equiv c \equiv 0 \mod N\right\}$$
$$\Gamma_1(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \mod N, \quad c \equiv 0 \mod N\right\}$$
$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \mod N\right\}$$

for some $N > 0$, then $E^0(\Gamma)$ can be interpreted as a moduli space of elliptic curves with extra structure. Here by an elliptic curve over S we mean a smooth proper curve over S whose geometric fibers are genus 1 curves, equipped with a section $O\colon S \to E$. A similar interpretation can be given to $E(\Gamma)$ in terms of moduli spaces of *semi-stable* curves, which we also introduce in section 1.2.

In section 1.3, we discuss the line bundle $L_{k,m}(\Gamma)$ of weak Jacobi forms of weight k and index m associated to Γ. For certain k and m, there is a particularly nice interpretation of $L_{k,m}(\Gamma)$ as the line bundle associated to a *theta divisor*. This gives a good motivation to study Jacobi forms and so we opt to develop the theory of theta divisors first.

Finally, in section 1.4 we introduce smooth metrics, Mumford-Lear extensions and the Petersson metric on the line bundle of weak Jacobi forms of weight k and index m.

1.1 The tower of compactifications of $E^0(\Gamma)$

In this section we introduce the tower of smooth compactifications of $E^0(\Gamma)$. As we shall see shortly, all smooth compactifications of $E^0(\Gamma)$ are (by definition) elliptic surfaces. Roughly speaking, an *elliptic surface* is a family of curves parametrized by a base curve C, almost all of which are smooth curves of genus 1.

Definition 1.1. Let k be a field. An *elliptic surface* E^0 over a curve C/k is a smooth 2-dimensional variety over k, equipped with a proper morphism $f\colon E^0 \to C$ with connected fibers such that the generic fiber $f^{-1}(\eta)$ of each irreducible component of C is a smooth genus 1 curve.

For the remainder of this section, to avoid working with the abstract machinery of étale coverings, we restrict our attention to varieties over $\mathbb{C}$. There is a functor from the category of schemes of finite type over $\mathbb{C}$ to the category of complex analytic spaces, which sends a nonsingular variety over $\mathbb{C}$ to a complex manifold (cf. [12, Appendix B]). If X is a non-singular variety over $\mathbb{C}$, we denote the image of X under this functor by X^{an}.

Generally speaking, if E^0 is an elliptic surface over a non-complete nonsingular curve C and $\overline{C}$ is the smooth completion of C, there is no way to define an elliptic surface $\pi\colon \widetilde{E^0} \to \overline{C}$ with the property that all fibers of π are smooth of genus 1. Instead we have to allow some of the fibers to be singular. As it turns out (cf. [2]) it is always possible to find an elliptic surface $E \to \overline{C}$ extending $E^0 \to C$ with the property that the complement $E - E^0$ is a *normal crossings divisor.*

Definition 1.2. Let X be a complex manifold and write $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. A *normal crossings divisor* D of X is a complex submanifold which admits for every $p \in D$ a coordinate system $z = (z_1, \ldots, z_n)\colon U \xrightarrow{\sim} \Delta^n$ with the property that $D \cap U$ is given by the equation $z_1 \ldots z_k = 0$. We say such a coordinate system z is *adapted to* D.

Now if X is a connected non-compact complex manifold, we say Y is a *smooth compactification* of X if it is a connected compact complex manifold which contains X as an open subset and which satisfies the property that $Y - X$ is a normal crossings divisor. Similarly, if Y is a complete non-singular variety over $\mathbb{C}$ containing X as a Zariski open subset, we say Y is a compactification of X if Y^{an} is a compactification of X^{an}. We call $D = Y - X$ the *boundary divisor* of the compactification Y/X.

Given an elliptic surface $E^0 \to C$, there is a compactification $E/\overline{C}$ of E^0, called the *relatively minimal model*, defined by the property that none of its fibers contain a -1-curve. Here, by a *-1-curve* we mean a rational curve on E^0 with self-intersection -1. Recall that by Castelnuovo's contractibility criterion, a curve on a surface can be blown down if and only if it is a -1-curve. Hence, since the Néron-Severi group of a surface is finitely generated and since a blow-down decreases the rank of the Néron-Severi group by one, a relatively minimal model of E^0 can be obtained by starting with an arbitrary compactification $\widetilde{E^0}/\overline{C}$ and blowing down a finite number of -1-curves. This yields the following result:

Theorem 1.1. *Let $p\colon E^0 \to C$ be an elliptic surface over a non-compact smooth curve C over $\mathbb{C}$ with completion $\overline{C}$. Consider the set $\mathrm{CPT}(E^0)$ of isomorphism classes of pairs (E, π), where*

- *$\pi\colon E \to \overline{C}$ is an elliptic surface extending the map $p\colon E^0 \to C$,*
- *$E - E^0$ is a normal crossings divisor of E,*

ordered by the relation that $E' \trianglelefteq E''$ if and only if E'' can be obtained from E' as a tower

$$E'' = E_n \to E_{n-1} \to \ldots \to E_0 = E',$$

where for all $k > 0$, E_k is obtained from E_{k-1} as a blow-up of E_{k-1} in a point $E_{k-1} - E^0$. Then $\mathrm{CPT}(E^0)$ is a directed set, with minimum given by the relatively minimal model of E^0.

Equip $Y(\Gamma) := \Gamma\backslash\mathcal{H}$ with the unique structure of Riemann surface satisfying the property that holomorphic functions $Y(\Gamma) \to \mathbb{C}$ correspond to Γ-periodic holomorphic functions $\mathcal{H} \to \mathbb{C}$. Associated to $E^0(\Gamma)$, we have a projection map

$$p\colon E^0(\Gamma) \to Y(\Gamma), \quad [\tau, z] \mapsto [\tau]$$

and a zero section

$$O\colon Y(\Gamma) \to E^0(\Gamma), \quad [\tau] \mapsto [\tau, 0],$$

turning $E^0(\Gamma)$ into an elliptic surface over $Y(\Gamma)$ in the strict sense that every fiber is an elliptic curve: for every $[\tau] \in Y(\Gamma)$, the fiber $p^{-1}[\tau]$ is equal to the complex torus $E_\tau := \mathbb{C}/(\mathbb{Z}\tau \oplus \mathbb{Z})$. The space $Y(\Gamma)$ can be compactified by adding so-called cusps, cf. [11, section 2.4], and we denote the resulting curve by $X(\Gamma)$. There is an interpretation of $X(\Gamma)$ and $E^0(\Gamma)$ as complex manifolds coming from smooth algebraic varieties over $\mathbb{C}$, and we define $E(\Gamma)/X(\Gamma)$ to be the relatively minimal model of $E^0(\Gamma)/Y(\Gamma)$.

1.2 Moduli spaces of elliptic curves

The purpose of this section is to discuss certain moduli spaces and various canonical maps that exist between them, so let us begin by recalling the definition of a moduli space. If $\mathcal{C}$ is a category, X is an object in $\mathcal{C}$ and $f\colon X \to Y$ is a morphism in $\mathcal{C}$, let $h_X = \mathrm{Hom}(-, X)$ be the functor which sends Y to the set of morphisms from Y to X and denote by $h_f = \mathrm{Hom}(-, f)$ the natural transformation given by post-composition with f.

Definition 1.3. Let S be a scheme and $\mathcal{F}\colon \mathrm{Sch}_S^{op} \to \mathrm{Sets}$ a contravariant functor from Sch_S to Sets. A pair (X, τ) consisting of a scheme over S and a natural transformation $\tau\colon \mathcal{F} \to h_X$ is said to be

1. A *fine moduli space* for $\mathcal{F}$ if τ is a natural isomorphism.
2. A *coarse moduli space* for $\mathcal{F}$ if τ satisfies the following properties:
   - For every $Y \in \mathrm{Sch}_S$ and every natural transformation $\sigma\colon \mathcal{F} \to h_Y$, there is a unique morphism $Y \to X$ such that $\sigma = h_f \circ \tau$.
   - For every one-point scheme $\mathrm{pt} = \mathrm{Spec}(k)$ over S with $k = k^{alg}$, the map $\tau_{\mathrm{pt}}\colon \mathcal{F}(\mathrm{pt}) \to X(\mathrm{pt})$ is a bijection.

Note that if X is a fine moduli space for $\mathcal{F}$, then there is a universal object over X associated to $\mathcal{F}$, given by $U_X = \tau_X^{-1}(\mathrm{id}_X)$. No such object can in general be associated to a coarse moduli space. Roughly speaking, the difference between coarse and fine moduli spaces is that fine moduli spaces parametrize all "families of geometric objects of a certain type", whereas coarse moduli spaces only parametrize "objects over points".

We refer to a functor $\mathcal{F}\colon \mathrm{Sch}_S^{op} \to \mathrm{Sets}$ as a *moduli problem* over S and to a fine moduli space (X, τ) for $\mathcal{F}$ as a *solution* of $\mathcal{F}$.

Given a morphism $X \to S$, a section $e\colon S \to X$ and a morphism $f\colon T \to S$, write $X_T = X \times_S T$ and $e_T\colon T \to X_T$ for the section corresponding to $(e \circ f, \mathrm{id}) \in X(T) \times_{S(T)} T(T)$.

Let g and n be positive integers. A natural moduli problem to consider is the moduli problem $\mathcal{M}_{g,n}$ of n-pointed genus g curves. This is the moduli problem which associates to $Y \in \mathrm{Sch}_S$ the set $\mathcal{M}_{g,n}(Y)$ of isomorphism classes of tuples $(C, P_1, \ldots, P_n)$ where

- C is a smooth, proper genus g curve over Y,
- $P_1, \ldots, P_n \in C(Y)$ are disjoint sections of $C \to Y$

and to a morphism $f : Y \to Y'$ the function

$$\mathcal{M}_{g,n}(Y') \to \mathcal{M}_{g,n}(Y) : (C, P_1, \ldots, P_n) \mapsto (C_Y, P_{1Y}, \ldots, P_{nY}).$$

Since the datum $(C, P_1, \ldots, P_n)$ may have non-trivial automorphisms (cf. [1, Theorem XII.2.5]), there is in general no solution to $\mathcal{M}_{g,n}$ in the category of schemes. This type of issue is common in the theory of moduli spaces. If we want to have a scheme that represents a family of objects with automorphisms, then either we need to settle with coarse moduli spaces, or we need to consider a different, *rigidified* moduli problem.

In the special case that $g = 1$, we are dealing with elliptic curves over a base scheme and the N-torsion can be used to eliminate automorphisms.

Definition 1.4. Let n and N be positive integers and let Γ be one of $\Gamma(N), \Gamma_0(N)$ or $\Gamma_1(N)$. Let (E, O) be an elliptic curve over a scheme S over $\mathrm{Spec}(\mathbb{Z}[1/N])$, with N-torsion subgroup scheme $E[N]$. Let $(\mathbb{Z}/N\mathbb{Z})_S$ be the scheme associated to the constant functor $\mathrm{Sch}_S \to \mathrm{Sets}$ with value $\mathbb{Z}/N\mathbb{Z}$. Then a *level structure on* (E, O) associated to Γ is:

- An isomorphism of group schemes $(\mathbb{Z}/N\mathbb{Z})_S^2 \to E[N]$ if $\Gamma = \Gamma(N)$.
- An embedding of group schemes $(\mathbb{Z}/N\mathbb{Z})_S \to E[N]$ if $\Gamma = \Gamma_1(N)$.
- A subgroup scheme of $E[N]$ étale locally isomorphic to $\mathbb{Z}/N\mathbb{Z}$ if $\Gamma = \Gamma_0(N)$.

The moduli problem of *elliptic curves with level structure and* $n - 1$ *marked points* is given on objects by the set $\mathcal{M}_{1,n}(\Gamma)(Y)$ of isomorphism classes of tuples $(E, O, P_2, \ldots, P_n, T)$, where

- E is a smooth, proper genus 1 curve over Y,
- $O, P_2, \ldots, P_n \in E(Y)$ are disjoint sections of $E \to Y$,
- T a level structure on (E, O) associated to Γ.

There is a one-to-one correspondence between $Y(\Gamma)$ and the set of isomorphism classes of elliptic curves over $\mathbb{C}$ with level structure associated to Γ (cf. [11, Theorem 1.5.1]). If N is such that Γ acts without fixed points on $\mathcal{H}$, then $\mathcal{M}_{1,1}(\Gamma)$ has a fine moduli space over $\mathrm{Spec}(\mathbb{Z}[1/N])$, and the analytification of its base change to $\mathbb{C}$ is isomorphic to $Y(\Gamma)$ (cf. [1, chapter XVI]). Even if Γ does not act without fixed points, the space $Y(\Gamma)$ is at least a coarse moduli space for $\mathcal{M}_{1,1}(\Gamma)$ over $\mathrm{Spec}(\mathbb{C})$ (cf. [1, chapter XII]). If $\Gamma = \Gamma(N)$, we use the abbreviation $\mathcal{M}_{1,n}(\Gamma(N)) = \mathcal{M}_{1,n}(N)$.

Note that for $n > 1$, there is a projection map $\pi_{n+1} : \mathcal{M}_{1,n+1}(\Gamma) \to \mathcal{M}_{1,n}(\Gamma)$, induced by the natural transformation which forgets the last point:

$$\pi_{n+1}(Y) : \mathcal{M}_{1,n+1}(\Gamma)(Y) \to \mathcal{M}_{1,n}(\Gamma)(Y), \quad (E, O, P_2, \ldots, P_{n+1}, T) \mapsto (E, O, P_2, \ldots, P_n, T)$$

The analytification of the base change of $\mathcal{M}_{1,2}(\Gamma)$ to $\mathbb{C}$ is isomorphic to $E^0(\Gamma) - O$ (cf. [1, chapter XVI]).

As mentioned in the previous section, the spaces $Y(\Gamma)$ and $E^0(\Gamma)$ are not compact, and neither are the $\mathcal{M}_{1,n}(\Gamma)$ with $n > 2$. If we were to compactify these spaces, we would like the result to be the solution to some interesting moduli problem. The way to do this is to add degenerated curves to the moduli problem in the form of *semi-stable* curves.

Definition 1.5. A *nodal curve* S is a curve over S with the property that every geometric fiber has only ordinary double points as singularities. We say a tuple $(C, P_1, \ldots, P_n)$ consisting of a proper nodal curve over S and an n-tuple of disjoint S-valued smooth points is *semi-stable* if the automorphism group of $(C_s, P_{1s}, \ldots, P_{ns})$ is reductive for all geometric points s of C.

In particular, we see that every semi-stable curve C of genus 1 over a field k is a union of rational and smooth genus 0 and 1 curves, satisfying the following properties (cf. [1, chapter X], [9], [15]):

- Every irreducible component of C that is smooth of genus 1 has at least 1 point that is marked or lies on another irreducible component.
- Every irreducible component of C that is rational has at least 2 points that are marked or lie on another irreducible component.

Now consider the moduli problem given on objects by the set of isomorphism classes of tuples $(E, O, P_2, \ldots, P_n, T)$ where

- (E, O, T) is a generalized elliptic curve with level structure in the sense of Deligne-Rapoport, cf. [10]
- $O, P_2, \ldots, P_n \in E^{sm}(Y)$ are disjoint sections of $E \to Y$ such that $(E, O, P_2, \ldots, P_n)$ is semi-stable,

If Γ acts without fixed points on $\mathcal{H}$ then each $\overline{\mathcal{M}}_{1,n}(\Gamma)$ admits a solution over $\mathrm{Spec}(\mathbb{Z}[1/N])$ and we have natural projection maps $\overline{\mathcal{M}}_{1,n}(\Gamma) \to \overline{\mathcal{M}}_{1,k}(\Gamma)$ for all $n > k$. The analytification of the base change of $\overline{\mathcal{M}}_{1,1}(\Gamma)$ to $\mathrm{Spec}(\mathbb{C})$ is isomorphic to $X(\Gamma)$, and the analytification of the base change of $\overline{\mathcal{M}}_{1,2}(\Gamma)$ to $\mathrm{Spec}(\mathbb{C})$ is isomorphic to $E(\Gamma) - O$ (cf. [1, chapter XVI]).

1.3 Jacobi forms and theta divisors

Let $E \to S$ be an elliptic curve over a scheme, with zero section $O\colon S \to E$. If $S = \mathrm{Spec}(k)$ for some field k, we have the following bijection between the k-valued points of E and the degree zero divisors of $E(k)$:

$$\varphi_O\colon E \to \mathrm{Div}^0(E), \quad P \mapsto [O] - [P].$$

Note the role of the zero divisor O in this construction. There is a generalization of φ_O to arbitrary schemes S, where different divisors (or rather, the line bundles associated to them) may take the place of O. This generalization makes use of the *Picard scheme of E/S*, which is the scheme representing the functor $\mathrm{Pic}_{E/S}\colon \mathrm{Sch}_S^{op} \to \mathrm{Sets}$ given by

$$\mathrm{Pic}_{E/S}(T) = \{(L, \alpha) : L \in \mathrm{Pic}(E_T),\ \alpha\colon \mathcal{O}_T \xrightarrow{\sim} O_T^* L\},$$

where the datum α, called a *rigidification of L along O_T*, serves a similar purpose as the level structure on elliptic curves we saw earlier, in that it is there to eliminate automorphisms. For each $T \in \mathrm{Sch}_S$, the set $\mathrm{Pic}_{E/S}(T)$ forms a group under the operation

$$(L, \alpha) \cdot (L', \alpha') = (L \otimes L', \alpha \otimes \alpha')$$

and this establishes a factorization of $\mathrm{Pic}_{E/S}$ through the forgetful functor $\mathrm{Grp} \to \mathrm{Sets}$. Thus $\mathrm{Pic}_{E/S}$ is in fact a group scheme. We refer to the connected component $\mathrm{Pic}^0_{E/S} \subset \mathrm{Pic}_{E/S}$ of the unit element as the *dual of* E and denote it by $E^\vee$. The k-valued points of $E^\vee$ are precisely the isomorphism classes of degree zero line bundles on E_k, and the correct generalization of the map φ_O above is a *polarization* $\varphi_L \colon E \to E^\vee$ associated to a line bundle $L \in \mathrm{Pic}(E)$.

Definition 1.6. Let L be a line bundle on E. Consider the *Mumford line bundle* on $E \times_S E$ given by

$$\Lambda(L) = m^* L \otimes p_1^* L^{\otimes -1} \otimes p_2^* L^{\otimes -1}.$$

This line bundle can be equipped with a rigidification, and we have a map φ_L corresponding to $\Lambda(L)$ via the universal property of $\mathrm{Pic}_{E/S}$. Fiberwise, ignoring rigidifications, it is given by $x \mapsto t_x^* L \otimes L^{\otimes -1}$. Note that the relative degree of $\Lambda(L)$ is zero. Hence φ_L factors through the map $E^\vee \to \mathrm{Pic}_{E/S}$. If φ_L is an isogeny $E \to E^\vee$ we call φ_L a *polarization*, and if φ_L is an isomorphism we call it a *principal polarization*.

Now a *theta divisor* on E is a divisor Θ on E with the property that the map φ_L with $L = \mathcal{O}_E(\Theta)$ is a principal polarization. Viewing $E^0(\Gamma)$ as a family of elliptic curves over $Y(\Gamma)$, one can show that the polarization associated to the image of the zero section $Y(\Gamma) \to E^0(\Gamma)$ is given fiberwise by the map which sends $P \in E_\tau$ to $\mathcal{O}_{E_\tau}(O - P) \in \mathrm{Pic}^0(E_\tau)$ for all $\tau \in \mathcal{H}$. Thus O is a theta divisor on $E^0(\Gamma)$.

We describe the global sections of the line bundle associated to O. Because $E^0(\Gamma)$ is defined as the quotient of a contractible space by a discontinuous group action, there is a fairly explicit description of $\mathrm{Pic}(E^0(\Gamma))$ in terms of *systems of multipliers*.

Proposition 1.2. *Let X and Y be complex manifolds. Suppose $f \colon Y \to X$ is a Galois cover with automorphism group Ψ and suppose the Picard group of Y is trivial. Then there is a one-to-one correspondence between*

- *Families of invertible holomorphic functions $(e_\psi)_{\psi\in\Psi} \in \prod_{\psi\in\Psi} H^0(Y, \mathcal{O}_Y^*)$ satisfying the property that*
$$e_{\psi\psi'}(z) = e_\psi(\psi' \cdot z) e_{\psi'}(z)$$
for all $\psi, \psi' \in \Psi$ and $z \in Y$.
- *Pairs (L, α), consisting of a line bundle $L \in Pic(X)$ and a trivialization $\alpha \colon f^* L \xrightarrow{\sim} \mathcal{O}_Y$.*

If $(e_\psi)_{\psi\in\Psi}$ is a system of multipliers, the vector space of global sections of the associated line bundle L is given by

$$H^0(X, L) = \{\theta(z) \in H^0(Y, \mathcal{O}_Y) : \theta(\psi \cdot z) = e_\psi(z)\theta(z) \text{ for all } \psi \in \Psi\}.$$

For details we refer the reader to [3, page 105]. Now to calculate the system of multipliers associated to O, we write down a holomorphic function $f \colon \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ with $\mathrm{div}(f) = O$ and determine the quotient $e_\gamma(z) = f(\gamma \cdot z)/f(z)$. A common choice for such a function is the *Riemann theta function*, which is defined by the convergent power series

$$\theta_{1,1}(\tau, z) = \sum_{n\in\mathbb{Z}} \exp\left(\pi i \tau \left(n + \frac{1}{2}\right)^2 + 2\pi i \left(z + \frac{1}{2}\right)\left(n + \frac{1}{2}\right)\right).$$

Associated to the Riemann theta function we have the following system of multipliers:

$$c'(\psi; \tau, z) = \chi(\gamma) \cdot (c\tau + d)^{1/2} \exp\left(-\pi i \left(\lambda^2 \tau + 2\lambda z - \frac{c(z + \lambda\tau + \mu)^2}{c\tau + d}\right)\right),$$

where $\psi = (\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (\lambda, \mu)) \in \Gamma^J$ and $\chi\colon \Gamma \to \mathbb{C}^*$ is a character which assigns to each $\gamma \in \Gamma$ an eighth root of unity. We define the line bundle $L_{k,m}(\Gamma)$ by a similar system of multipliers,

$$c(\psi; \tau, z) = (c\tau + d)^k \exp\left(-2\pi i m \left(\lambda^2\tau + 2\lambda z - \frac{c(z + \lambda\tau + \mu)^2}{c\tau + d}\right)\right),$$

and call a global section of $L_{k,m}(\Gamma)$ a *weak Jacobi forms of weight k and index m*. Thus $\theta_{1,1}^8$ is a weak Jacobi form of weight 4 and index 4, and the line bundle associated to $8O$ is isomorphic to $L_{4,4}(\Gamma)$.

We end this section with a brief discussion of non-weak Jacobi forms. Consider the action of Γ^J on $L_{4,4}(\Gamma)$ given by

$$(\psi \cdot f)(\tau, z) = f(\psi \cdot (\tau, z)).$$

Note that weak Jacobi forms are invariant under $\psi = (\pm \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right), (0, n))$ for all $m, n \in \mathbb{Z}_{>0}$ with $\psi \in \Gamma^J$. More generally, since $((\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), (0, 1)) \in \Gamma^J$ for all congruence subgroups $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$, if $\mathfrak{c} = \alpha \cdot \infty$ is a cusp of $X(\Gamma)$ and

$$h_{\mathfrak{c}} = \min\{m \in \mathbb{Z}_{>0} : ((\pm \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma\}$$

is its period, then the function $f(\tau, z) \cdot c(\alpha^{-1}; \tau, z)$ is $h_{\mathfrak{c}}\mathbb{Z}$-periodic in its first and $\mathbb{Z}$-periodic in its second argument. Therefore, the function admits a Fourier expansion

$$f(\tau, z) \cdot c(\alpha^{-1}; \tau, z) = \sum_{n \in \mathbb{Z}, r \in \mathbb{Z}} a_{\mathfrak{c}}(n, r) \exp(2n\pi i\tau/h_{\mathfrak{c}}) \exp(2\pi i z).$$

We say f is a *Jacobi form of weight k and index m* if it is a weak Jacobi form with the property that $a_{\mathfrak{c}}(n, r) = 0$ for all pairs $(n, r) \in \mathbb{Z}^2$ with $n < 0$ or $4mn - h_{\mathfrak{c}}r^2 < 0$. If in addition this equality holds for all (n, r) with $n = 0$ or $4mn - h_{\mathfrak{c}}r^2 = 0$, we say f is a *Jacobi cusp form*. The Jacobi (cusp) forms of weight k and index m form a $\mathbb{C}$-vector space, which we denote by $J_{k,m}(\Gamma)$ ($J_{k,m}^{cusp}(\Gamma)$), and as we will see in chapter 3, the quantity $\dim J_{4\ell,4\ell}(\Gamma)$ grows like

$$C \cdot \ell^2/2! + o(\ell^2),$$

as $\ell \to \infty$, where C is the limit of the self-intersection of the Mumford-Lear extension of $L_{4,4}(\Gamma)$ to all compactifications of $E^0(\Gamma)$.

1.4 Metrics and Mumford-Lear extensions

Suppose X is a complex manifold equipped with a holomorphic line bundle $\mathcal{L}$. Let us begin by recalling the definition of a smooth metric on $\mathcal{L}$ and its associated first Chern form $c_1(\mathcal{L}, ||\cdot||)$.

Definition 1.7. Let X be a complex manifold equipped with a holomorphic line bundle $\mathcal{L}$. A *smooth metric* on $\mathcal{L}$ is a map $||\cdot||$ which associates to every local section $s \in \mathcal{L}(U)$ a function $||s||\colon U \to \mathbb{R}_{\geq 0}$ in such a manner that the following two conditions hold:

- For all $f \in \mathcal{O}_X(U)$, we have $||f \cdot s|| = |f| \cdot ||s||$.
- For every section $s \in \mathcal{L}(U)$ generating $\mathcal{L}(U)$, the function $||s||^2$ is smooth and positive valued.

To a smooth metric $||\cdot||$, we associate a $(1,1)$-form $c_1(\mathcal{L}, ||\cdot||)$, called the *first Chern form*, by choosing a connection ∇ on $\mathcal{L}$ compatible with $||\cdot||$ and taking $c_1(\mathcal{L}, ||\cdot||) = \frac{1}{2\pi i}\operatorname{Tr}(k_\nabla)$, where k_∇ is the curvature form associated to ∇. This Chern form can be shown to be equal to

$$c_1(\mathcal{L}, ||\cdot||) = \frac{1}{2\pi i}\partial\overline{\partial}\log||s||^2,$$

where s is any locally generating section of $\mathcal{L}$. Given smooth metrics $||\cdot||_1$ and $||\cdot||_2$ on line bundles $\mathcal{L}$ and $\mathcal{M}$, the tensor product $\mathcal{L}\otimes\mathcal{M}$ comes equipped with a smooth metric $||\cdot||$ determined by the formula $||s\otimes t|| = ||s||_1\cdot||t||_2$. The first Chern form is additive on tensor products:

$$c_1(\mathcal{L}\otimes\mathcal{M}, ||\cdot||) = c_1(\mathcal{L}, ||\cdot||_1) + c_1(\mathcal{M}, ||\cdot||_2).$$

If $f\colon Y\to X$ is a morphism of complex manifolds, the pull-back line bundle $f^*\mathcal{L}$ comes equipped with a smooth metric uniquely determined by the property that $||f^*s|| = ||s||\circ f$ for all $s\in\mathcal{L}(U)$, and the first Chern form is compatible with this pullback: $c_1(f^*\mathcal{L}, ||\cdot||) = f^*c_1(\mathcal{L}, ||\cdot||)$.

We have the following relation between the intersection pairing on line bundles on a complete smooth algebraic surface over $\mathbb{C}$ and integrals involving Chern forms of smooth metrics.

Proposition 1.3. *Let X^{an} be the complex manifold associated to a complete smooth algebraic surface X over $\mathbb{C}$. Suppose $||\cdot||_1$ and $||\cdot||_2$ are smooth metrics on line bundles $\mathcal{L}$ and $\mathcal{M}$ on X^{an}, and C and D are the equivalence classes of Weil divisors associated to $\mathcal{L}$ and $\mathcal{M}$, respectively. Then*

$$C\cdot D = \int_X c_1(\mathcal{L}, ||\cdot||_1)\wedge c_1(\mathcal{M}, ||\cdot||_2).$$

Obviously it is of interest to extend this result as much as possible to metrized line bundles on non-compact algebraic surfaces. Let X be a non-compact algebraic surface over $\mathbb{C}$ equipped with a smoothly metrized line bundle $(\mathcal{L}, ||\cdot||)$, let Y be a smooth compactification of X and write $\mathrm{Pic}_{\mathcal{L}}(Y)$ for the set of line bundles $\overline{\mathcal{L}}\in\mathrm{Pic}(Y)$ such that $\overline{\mathcal{L}}|_X = \mathcal{L}$. We can try to find a smoothly metrized line bundle $(\overline{\mathcal{L}}, ||\cdot||')$ on Y with the property that $\overline{\mathcal{L}}|_X$ is isometric to $\mathcal{L}$. This is not always possible, but sometimes a singular metric on an extension $\overline{\mathcal{L}}\in\mathrm{Pic}_{\mathcal{L}}(Y)$ can be given which has mild enough singularities along $Y-X$ that it can be used to determine intersection numbers. In particular this is the case if the extension can be equipped with a so-called *pre-log metric* (cf. [17]).

Definition 1.8. Let Y be a smooth compactification of a non-compact complex manifold X and let $D = Y - X$ be its boundary divisor. We say a holomorphic function f on X has *log-log growth* along D if for every coordinate system $z = (z_1,\dots,z_n)$ adapted to D such that D is locally given by the equation $z_1\dots z_k = 0$ there is a positive integer M and a positive real number C such that the following inequality holds:

$$|f(z_1,\dots,z_n)| < C\cdot\prod_{i=1}^{k}\log(\log(1/|z_i|))^M.$$

Let $j\colon X\to Y$ be the inclusion map. The Dolbeault algebra of *log-log forms* is the sub-algebra of $j_*\mathcal{E}_X$ which is closed under the operators $\wedge, \partial$ and $\overline{\partial}$ and which is generated by the holomorphic functions with log-log growth and the differential forms

$$\frac{dz_i}{z_i\log(1/|z_i|)},\quad \frac{d\overline{z}_i}{\overline{z}_i\log(1/|z_i|)}\quad \text{for } i = 1,\dots,k,$$
$$dz_i,\quad d\overline{z}_i\quad \text{for } i > k.$$

We say a differential form η is *pre-log-log* if it is a log-log form with the property that $\partial\eta$, $\overline{\partial}\eta$ and $\partial\overline{\partial}\eta$ are also log-log forms.

Intuitively speaking, a function is pre-log-log if it has log-log growth and its "derivatives" also have log-log growth. We note that by [8, Proposition 7.4], the property of being a pre-log-log form is respected by pull-backs.

Definition 1.9. Let X^{an} be the complex manifold associated to a complete smooth algebraic variety X over $\mathbb{C}$ of dimension n, equipped with a holomorphic line bundle $\mathcal{L}$. Suppose $D \subset X^{an}$ is a normal crossings divisor on X^{an}. Then a *pre-log metric along* D is a smooth metric $||\cdot||$ on $X - D$ satisfying the following properties:

- It has logarithmic growth along D: that is, for every $x \in X$, there is an adapted coordinate system $z = (z_1, \ldots, z_n)$, a positive integer M and a locally generating section s of $\mathcal{L}$ defined on z such that

$$C \cdot \prod_{i=1}^{k} \log(1/|z_i|)^{-M} < ||s(z_1, \ldots, z_n)|| < C' \cdot \prod_{i=1}^{k} \log(1/|z_i|)^{M}$$

  for all $(z_1, \ldots, z_n) \in X - D$ and some positive real numbers C, C'.

- For every rational section s of L, the function $\log ||s||$ is a pre-log-log form along $D - \operatorname{div}(s)$ on $X - \operatorname{div}(s)$.

Let $E(\Gamma)$ be the elliptic surface associated to Γ and let D^{sing} be the singular locus of $D = E(\Gamma) - E^0(\Gamma)$. Write $y = \Im(z)$ and $\eta = \Im(\tau)$. Then the line bundle $L_{k,m}(\Gamma)$ of weak Jacobi forms can be equipped with the following so-called *Petersson metric*:

$$||f(\tau, z)||^2_{\mathrm{Pet}} = |f(\tau, z)|^2 \exp(-4\pi m y^2/\eta)\eta^k$$

The content of the next proposition is that $||\cdot||_{\mathrm{Pet}}$ is *translation invariant*, meaning that we have $||\psi \cdot f||_{\mathrm{Pet}} = ||f||_{\mathrm{Pet}}$ for all local sections f of $L_{k,m}(\Gamma)$ and all $\psi \in \Gamma^J$.

Proposition 1.4. *Let* $\psi = (\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (\lambda, \mu)) \in \Gamma^J$ *and let* U *be an open subset of* $E^0(\Gamma)$. *For every* $f \in H^0(U, L_{k,m}(\Gamma))$ *and* $(\tau, z) \in \mathcal{H} \times \mathbb{C}$ *we have*

$$\left|\left| f\left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right)\right|\right|^2_{Pet} = ||f(\tau, z)||^2_{Pet}.$$

Proof. Tedious but straightforward computation. □

As we will see in later chapters, the Petersson metric on the line bundle of weak Jacobi forms can be extended to a pre-log metric on $E(\Gamma) - D^{sing}$ along $D - D^{sing}$, but not to a pre-log metric on the entirety of $E(\Gamma)$. This phenomenon fits in the framework of *Mumford-Lear extensions*, which was introduced in [5] and which we recall here.

Definition 1.10. Let Y be a smooth compactification of a smooth variety X over $\mathbb{C}$ and let $(\mathcal{L}, ||\cdot||)$ be a smoothly metrized line bundle on X. Let D be the boundary divisor of Y/X. A *Mumford-Lear extension* of $(\mathcal{L}, ||\cdot||)$ is a triple $(\overline{\mathcal{L}}, ||\cdot||', e)$ consisting of a line bundle $\overline{\mathcal{L}} \in \operatorname{Pic}(Y)$, a positive integer e and a metric $||\cdot||'$ on $\overline{\mathcal{L}}|_X$ such that $\overline{\mathcal{L}}|_X$ is isometric to $\mathcal{L}^{\otimes e}$ and such that $\overline{\mathcal{L}}|_X$ is pre-log along $D - S$ for some algebraic subset $S \subset Y$ of codimension at least 2.

Remark 1.11. Mumford-Lear extensions are so named because they are a common generalization of the extensions defined by Mumford (cf. [17]) and Lear (cf. [16]): a Mumford extension of a norm $||\cdot||$ on $\mathcal{L}$ to a line bundle $\overline{\mathcal{L}}$ on Y is pre-log along the entirety of D, whereas a Lear extension of $||\cdot||$ to $\overline{\mathcal{L}}$ is one that extends continuously, but only to a subset $D - S$ with S of codimension at least 2 in Y.

Associated to a Mumford-Lear extension $(\overline{\mathcal{L}}, ||\cdot||', e)$ we have the $\mathbb{Q}$-line bundle $[\mathcal{L}, ||\cdot||]_Y = \frac{1}{e}\overline{\mathcal{L}}$. Note that since the Picard group is not influenced by what happens on codimension two subsets, this line bundle is uniquely determined by $(\mathcal{L}, ||\cdot||)$ (cf. [5, Proposition 3.9]). Note furthermore that Mumford-Lear extensions are compatible with tensor products. If C is the equivalence class of Weil divisors associated to $\mathcal{L}$, we write $[C, ||\cdot||]_Y$ for the divisor class associated to $[\mathcal{L}, ||\cdot||]_Y$ and call it the Mumford-Lear extension of C to Y. If it is clear from the context what metric is being used, we drop $||\cdot||$ from the notation.

2 Tools

In this chapter we introduce the tools necessary to compute the Mumford-Lear extension of $L_{4,4}(\Gamma)$ to the various compactifications of $E^0(\Gamma)$. We begin by discussing two techniques for calculating Mumford-Lear extensions that are somewhat similar in flavor, the method of test curves and the method of locally generating sections. The former allows us to reduce the calculation of the Mumford-Lear extension of a divisor to the case where the underlying manifold is Δ, while the latter provides a method to reason about Mumford-Lear extensions entirely by way of locally trivializing sections.

In section 2.2, we introduce the *Deligne pairing* of line bundles, which is a relative version of the usual intersection pairing of divisors on algebraic surfaces. We collect some identities satisfied by this pairing, most notably an adjunction formula and a relation between Deligne pairings of degree zero divisors on one hand and the Poincaré bundle on the Jacobian variety associated to a family of nodal curves on the other.

In section 2.3 we describe the singularities of the Petersson metric in greater detail. As mentioned in the previous chapter, the smooth metric $||\cdot||_{\mathrm{Pet}}$ can not be extended to a pre-log metric along the entirety of $D = E(\Gamma) - E^0(\Gamma)$. The degree to which the existence of such an extension fails can be quantified locally on a coordinate system (W, u, v) around a point $p \in D^{sing}$ in the following manner. There is, associated to p, a rational number $q \in \mathbb{Q}_{>0}$ such that the metric $||\cdot||$ given by the formula

$$\log||\cdot||^2 = \log||\cdot||^2_{\mathrm{Pet}} + q \cdot \frac{\log|u|\log|v|}{\log|u| + \log|v|}.$$

on $E^0(\Gamma) \cap W$ does not only admit a Mumford-Lear extension, but in fact a Mumford extension along the entirety of $D \cap W$. This property is the key ingredient in calculating the Mumford-Lear extension of $L_{4,4}(\Gamma)$ to all compactifications of $E^0(\Gamma)$.

2.1 Method of test curves and locally generating sections

Let $(\mathcal{L}, ||\cdot||)$ be a metrized line bundle on a smooth algebraic variety X over $\mathbb{C}$ with associated divisor class C and let Y be a smooth compactification of X. Write $V = \{D_1, \ldots, D_n\}$ for the set of irreducible components of $D = Y - X$. Suppose a divisor C' on Y is given with the property that $\mathcal{O}_Y(C')|_X \cong \mathcal{L}$. Suppose furthermore that the Mumford-Lear extension of $\mathcal{L}$ to Y exists. Since Y is smooth, $\mathrm{Pic}_{\mathcal{L}}(Y)$ is a $\mathbb{Z}^V$-torsor, so $[C]_Y$ is necessarily of the form $[C]_Y = C' + \sum_{i=1}^n a_i D_i$ for certain $a_i \in \mathbb{Q}$. The method of test curves allows us to determine these coefficients a_i in the following manner.

Proposition 2.1. *Suppose we are given for each $i = 1, \ldots, n$ a curve $\overline{f_i}\colon \Delta \to Y$ which intersects D_i transversally in a point $p_i \in D_i - S$ and which restricts to a curve $f_i\colon \Delta^* \to X$. Then for each $i = 1, \ldots, n$, the Mumford-Lear extension of $f_i^*\mathcal{L}$ to Δ exists, and we have $[f_i^* C]_\Delta = \overline{f_i}^* C' + a_i \cdot [0]$, where $[0]$ is the divisor on Δ corresponding to the closed submanifold $\{0\}$.*

Proof. Since pull-backs and Mumford-Lear extensions are compatible with tensor products, it suffices to prove the proposition in the case that $a_i \in \mathbb{Z}$. Then $[\mathcal{L}]_Y$ comes equipped with a smooth metric on $[\mathcal{L}]_Y|_X$ which is pre-log along $D - S$. Evidently this implies that the pull-back metric on $\overline{f_i}^*[\mathcal{L}]_Y$ has logarithmic growth along $\{0\}$, so since pre-log-log forms are preserved under pullbacks, we deduce that $[f_i^* C]_\Delta = \overline{f_i}^* [C]_Y$. Now we calculate:

$$[f_i^* C]_\Delta = \overline{f_i}^* [C]_Y = \overline{f_i}^* \left(C' + \sum_{j=1}^n a_j D_j \right) = \overline{f_i}^* C' + a_i \cdot [0],$$

where in the last step we use that $f_i^* D_i = [0]$ and $f_i^* D_j$ is the empty divisor for all $i \neq j$. □

Thus to find a_i it suffices to compute the component of $[0]$ for $[f_i^* C]_\Delta$. In view of Proposition 2.1, we say a curve f_i is a *test curve for the i-th component of D* if it satisfies the properties in the statement of Proposition 2.1.

Next we describe the method of locally generating sections. Evidently the property of being a Mumford extension of $(\mathcal{L}, ||\cdot||)$ is a local one, so it suffices to characterize existence of Mumford extensions on compactifications of the form $(\Delta^*)^k \times \Delta^{n-k} \subset \Delta^n$. Then a generating section of an extension of $\mathcal{L}$ can be used to characterize existence of a Mumford extension in the following manner.

Proposition 2.2. *Let C' be a divisor on Δ^n with the property that $\mathcal{O}_{\Delta^n}(C')$ restricts to $\mathcal{L}$ on $(\Delta^*)^k \times \Delta^{n-k}$ and let s be a generating section of $\mathcal{O}_{\Delta^n}(C')$. Consider for each $i = 1, \ldots, k$ the divisor D_i given by the equation $z_i = 0$ and note that $D_1, \ldots, D_k$ are the irreducible components of $D = \Delta^n - (\Delta^*)^k \times \Delta^{n-k}$. Let $a_1, \ldots, a_k \in \mathbb{Q}_{>0}$. The following statements are equivalent:*

- *The function $\log ||s|| - \sum_{i=1}^k a_i \log |z_i|$ on $\Delta^n - D$ is a pre-log-log form along D*
- *The $\mathbb{Q}$-divisor $C' + \sum_{i=1}^k a_i \cdot D_i$ is the underlying $\mathbb{Q}$-divisor of a Mumford extension of $\mathcal{L}$ to Δ^n.*

Proof. Again it suffices to prove this statement under the assumption that $a_1, \ldots, a_k \in \mathbb{Z}$. Then we simply note that $s \cdot z_1^{-a_1} \ldots z_k^{-a_k}$ is a generating section of $\mathcal{O}_{\Delta^n}(C' + \sum_{i=1}^k a_i \cdot D_i)$, so all information about the growth of $\log ||\cdot||$ along D can be deduced from information about the growth of $\log ||s|| - \sum_{i=1}^k a_i \log |z_i|$. □

2.2 Deligne pairing

Throughout this section, by a *nodal curve over* $\mathbb{C}$ we mean a complete algebraic curve over $\mathbb{C}$ that has only ordinary double points as singularities.

Definition 2.1. Let $p\colon X \to S$ be a family of nodal curves over a complex manifold S, that is, a flat proper holomorphic map $p\colon X \to S$ with fibers that are nodal curves over $\mathbb{C}$, and let $\mathcal{L}$ and $\mathcal{M}$ be two line bundles on X. The *Deligne pairing* $\langle \mathcal{L}, \mathcal{M} \rangle$ is defined as follows:

- If S is a point, then X is a nodal curve. Consider the vector space V of pairs (ℓ, m) of rational sections of $\mathcal{L}, \mathcal{M}$ with the property that $\operatorname{div}(\ell)$ and $\operatorname{div}(m)$ have disjoint support and ℓ, m are non-zero on the components of X and regular on the nodes of X. Then $\langle \mathcal{L}, \mathcal{M} \rangle$ is the quotient of V by the relations
$$(f\ell, m) \sim f(\operatorname{div}(m))(\ell, m)$$
$$(\ell, gm) \sim g(\operatorname{div}(\ell))(\ell, m),$$

where f, g are rational functions on X and where for a divisor $D = \sum_{P \in X} n_P P$ on X we define

$$f(D) = \prod_{P \in X} f(P)^{n_P}$$

- If S is not a point, we define $\langle \mathcal{L}, \mathcal{M} \rangle(U)$ to be the set of tuples $(u_s)_{s \in U} \in \prod_{s \in U} \langle \mathcal{L}_s, \mathcal{M}_s \rangle$ with the property that for all $x \in U$ there are an open $V \ni x$ and rational sections ℓ, m on $\pi^{-1}(V)$ with $u_t = \langle \ell_t, m_t \rangle$ for all $t \in V$.

For a proof that $\langle \mathcal{L}, \mathcal{M} \rangle$ is really a line bundle, we refer the reader to [1, section XIII.5].

From the "pointwise" construction of $\langle \mathcal{L}, \mathcal{M} \rangle$ it is clear that the Deligne pairing is compatible with base change. Another property that is immediately apparent is the "bilinearity" and "symmetry" of the Deligne pairing: if $\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}_1$ and $\mathcal{M}_2$ are line bundles on X, we have natural isomorphisms

$$\langle \mathcal{L}_1, \mathcal{M}_1 \rangle \otimes \langle \mathcal{L}_1, \mathcal{M}_2 \rangle \otimes \langle \mathcal{L}_2, \mathcal{M}_1 \rangle \otimes \langle \mathcal{L}_2, \mathcal{M}_2 \rangle \to \langle \mathcal{L}_1 \otimes \mathcal{L}_2, \mathcal{M}_1 \otimes \mathcal{M}_2 \rangle$$

given by $\langle \ell_1, m_1 \rangle \otimes \langle \ell_1, m_2 \rangle \otimes \langle \ell_2, m_1 \rangle \otimes \langle \ell_2, m_2 \rangle \mapsto \langle \ell_1 \otimes \ell_2, m_1 \otimes m_2 \rangle$ and

$$\langle \mathcal{L}, \mathcal{M} \rangle \to \langle \mathcal{M}, \mathcal{L} \rangle$$

given by $\langle \ell, m \rangle \mapsto \langle m, \ell \rangle$.

Let $P \colon S \to X$ be a section of p such that the image of P is contained in the largest open subset $U \subset X$ where $p \colon X \to S$ is smooth. Then there is a natural isomorphism between $\langle \mathcal{L}, \mathcal{O}(P) \rangle$ and $P^*\mathcal{L}$, cf. [1, XIII.5.18]. Thus we have the following "adjunction formula" involving the relative dualizing sheaf $\omega_{X/S}$:

Lemma 2.3. *There is a natural isomorphism* $\langle \mathcal{O}(P), \mathcal{O}(P) \rangle \to P^*\omega_{X/S}^{\otimes -1}$.

Proof. By the usual adjunction formula, we have $P^*(\mathcal{O}(P) \otimes \omega_X) \cong \omega_S$. Note that as p is smooth on U, we have $\omega_{U/S} = \omega_U \otimes p^*\omega_S^{\otimes -1}$. Hence, since $P^*p^*\omega_S = \omega_S$ and since the image of P is contained in U, we have

$$P^*\mathcal{O}(P) = P^*(\omega_X^{\otimes -1} \otimes p^*\omega_S) = P^*(\omega_U^{\otimes -1} \otimes p^*\omega_S) = P^*\omega_{U/S}^{\otimes -1} = P^*\omega_{X/S}^{\otimes -1}.$$

Now the result follows by applying the above natural isomorphism to $\mathcal{L} = \mathcal{O}(P)$. □

We note that if X is a family of nodal curves over a compact Riemann surface S, then the Deligne pairing generalizes the usual intersection pairing in the sense that $\deg_S \langle \mathcal{L}, \mathcal{M} \rangle = C \cdot D$, where C and D are the divisor classes associated to $\mathcal{L}$ and $\mathcal{M}$.

Next we give a relation between the Deligne pairing of two degree zero divisors and the *Poincaré bundle on the Jacobian variety* associated to a family of smooth curves $p \colon X \to S$. To set this up, we need the fact that an analogue of the Picard scheme from section 1.3 exists in the category of complex analytic spaces. Once this is known, we infer by universality of $\mathrm{Pic}_{X/S}$ that the space $X \times \mathrm{Pic}_{X/S}$ comes equipped with a line bundle $\mathcal{P}$, called the *Poincaré bundle*. Then on the Jacobian variety $J(X/S)$ of X, which is by definition the connected component of the identity element of $\mathrm{Pic}_{X/S}$, we get a line bundle on $X \times \mathrm{Pic}_{X/S}$ by pulling back $\mathcal{P}$ along the inclusion map $X \times J(X/S) \to X \times \mathrm{Pic}_{X/S}$. By abuse of notation, we write $\mathcal{P}$ for this line bundle and also call it the Poincaré bundle.

Proposition 2.4. *Suppose E is a family of elliptic curves over a complex manifold S and suppose $\mathcal{L}$ and $\mathcal{M}$ are line bundles on E with the property that for all $s \in S$, $\mathcal{L}_s$ and $\mathcal{M}_s$ are degree zero line bundles on E_s. Consider the map $\nu \colon S \to J(E/S) \times_S J(E/S)$ associated to*

$$(\mathcal{L}, \mathcal{M}) \in Pic(E/S) \times_S Pic(E/S)$$

and let $\lambda \colon E^\vee \to E^{\vee\vee} \cong E$ be the dual of the principal polarization associated to $L = \mathcal{O}_E(O)$. Then there is a natural isomorphism

$$\langle \mathcal{L}, \mathcal{M} \rangle^{\otimes -1} \xrightarrow{\sim} ((\lambda, \mathrm{id}) \circ \nu)^* \mathcal{P}$$

We refer the reader to [13, Corollary 4.2] for a proof of this proposition.

We end this section by noting that if $\mathcal{L}$ and $\mathcal{M}$ are equipped with smooth metrics, the Deligne pairing $\langle \mathcal{L}, \mathcal{M} \rangle$ comes equipped with a smooth metric as well. A smooth metric can also be defined on the Poincaré bundle associated to $X \to S$ and the natural isomorphisms introduced in this section become isometries when the Deligne pairings and Poincaré bundles are equipped with their natural smooth metrics.

2.3 The Petersson metric near singular points of $E(\Gamma) - E^0(\Gamma)$

To introduce coordinates on $E(\Gamma)$, one has no choice but to give an explicit construction of it. In [4], this is done using the theory of *toric varieties*. We characterize the resulting space in the following proposition.

Proposition 2.5. *Let $\mathfrak{c}$ be a cusp of $X(\Gamma)$. Then the fiber $D = \overline{p}^{-1}\{\mathfrak{c}\}$ is an h-gon, where h is the period of $\mathfrak{c}$. Moreover, there is a numbering $C_0, \ldots, C_{h-1}$ of the irreducible components of D such that for each $\nu \in \mathbb{Z}/h\mathbb{Z}$ there is a coordinate system*

$$(W_{\mathfrak{c},\nu}, u_\nu, v_\nu)$$

centered around $p_\nu = D_\nu \cap D_{\nu+1}$ such that $W_{\mathfrak{c},\nu} \cap C_\nu$ is given by the equation $u_\nu = 0$, $W_{c,\nu} \cap C_{\nu+1}$ is given by the equation $v_\nu = 0$, and the following relations between (u_ν, v_ν) and the natural coordinate system (τ, z) on $W_{\mathfrak{c},\nu} \cap E^0(\Gamma)$ hold:

$$u_\nu v_\nu = \exp(2\pi i \tau / h), \quad u_\nu^{\nu+1} v_\nu^{\nu} = \exp(2\pi i z).$$

Here, by an *h-gon* on a smooth algebraic surface X we mean the following. If $h > 1$, it is a curve isomorphic to the one obtained by taking h copies $\{C_\nu\}_{\nu \in \mathbb{Z}/h\mathbb{Z}}$ of $\mathbb{P}^1$ and gluing ∞ to 0 for each $\nu \in \mathbb{Z}/h\mathbb{Z}$, embedded in X in such a manner that $C_\nu^2 = -2$ and $C_\nu \cdot C_{\nu+1} = 1$ for all $\nu \in \mathbb{Z}/h\mathbb{Z}$. If $h = 1$, it is a curve isomorphic to the projective closure of $\mathrm{Spec}(\mathbb{C}[x, y]/(y^2 - x^2(x + 1)))$.

With respect to the coordinate system (u_ν, v_ν), we have the following description of $|| \cdot ||_{\mathrm{Pet}}$:

Lemma 2.6. *Let f be a local section of $L_{4,4}(\Gamma)$. Around $p_\nu = C_\nu \cap C_{\nu+1}$, the metric $|| \cdot ||_{\mathrm{Pet}}$ satisfies the following formula:*

$$\begin{aligned} \log ||f(\tau, z)||^2_{Pet}|_{W_\nu} &= \log(|f(\tau, z)|^2)|_{W_\nu} \\ &+ \frac{8}{h}\left((\nu+1)^2 \log|u_\nu| + \nu^2 \log|v_\nu| - \frac{\log|u_\nu| \log|v_\nu|}{\log|u_\nu| + \log|v_\nu|}\right) \\ &+ 4 \log\left(-\frac{2h_c}{4\pi}(\log|u| + \log|v|)\right). \end{aligned}$$

Proof. The proof is identical to that of [5, Lemma 2.10], noting that $\log|u_\nu| = \frac{1}{2}\log(u_\nu \overline{u}_\nu)$ and $\log|v_\nu| = \frac{1}{2}\log(v_\nu \overline{v}_\nu)$. □

Now we have enough material to describe the singularities of $||\cdot||_{\mathrm{Pet}}$ near the singular points of the boundary divisor of $E(\Gamma)/E^0(\Gamma)$.

Proposition 2.7. *Let $p \in E(\Gamma)$ be a singular point of the boundary divisor of $E(\Gamma)/E^0(\Gamma)$, above a cusp of period h. Then there is a coordinate chart (W, u, v) around p such that the metric $||\cdot||$ determined by the formula*

$$\log||\cdot||^2 = \log||\cdot||^2_{\mathrm{Pet}} + \frac{8}{h}\frac{\log|u|\log|v|}{\log|u| + \log|v|}$$

admits a Mumford extension along $\{uv = 0\}$.

Proof. Assume without loss of generality that $p = C_\nu \cap C_{\nu+1}$ and take $u = u_\nu$, $v = v_\nu$. By the method of locally generating sections, it suffices to check for a local generator s of $L_{4,4}(\Gamma)$ that there are $a, b \in \mathbb{Q}$ such that $\log||s|| - a\log|u| - b\log|v|$ is pre-log-log along $\{uv = 0\}$. We shall do this for $\theta_{1,1}^8$. Note that by Proposition 2.5, we can write $\theta_{1,1}$ as

$$\theta_{1,1}(\tau, z) = \sum_{n\in\mathbb{Z}} \exp(\pi(n+1)/2) u^{h/2(n+1/2)^2 + (\nu+1)(n+1/2)} v^{h/2(n+1/2)^2 + \nu(n+1/2)}.$$

Let c and d be the smallest powers of u respectively v occurring in the above power series. Then

$$\log\left(\log\left|\frac{\theta_{1,1}(\tau,z)^8}{u^{8c}v^{8d}}\right|\right) + 4\log\left(-\frac{2h_c}{4\pi}(\log|u| + \log|v|)\right)$$

is pre-log-log along $\{uv = 0\}$. Hence taking $a = c + \frac{m}{h}(\nu+1)^2$ and $b = d + \frac{m}{h}\nu^2$ we see $\log||\theta_{1,1}^8|| - a\log|u| - b\log|v|$ is pre-log-log along $\{uv = 0\}$, as was to be shown. □

3 Calculations

This chapter contains the main calculations of this thesis. We first determine the Mumford-Lear extension of $L_{4,4}(\Gamma)$ to $E(\Gamma)$. Then we compute the Mumford-Lear extension of $L_{4,4}(\Gamma)$ to all smooth compactifications of $E^0(\Gamma)$ and show the associated system of divisors satisfies Chern-Weil theory and a Hilbert-Samuel type formula.

3.1 Mumford-Lear extension of $L_{4,4}(\Gamma)$ to $E(\Gamma)$

Let $\mathcal{C}(\Gamma) = X(\Gamma) - Y(\Gamma)$ be the set of cusps of $X(\Gamma)$ and consider the space

$$U(\Gamma) = E^0(\Gamma) \times_{Y(\Gamma)} E^0(\Gamma),$$

viewed as a family of elliptic curves over $E^0(\Gamma)$ via the projection on the first coordinate. Denote by $O, P\colon E^0(\Gamma) \to U(\Gamma)$ the zero and diagonal section, respectively. Recall that $L_{4,4}(\Gamma)$ can be identified with $\mathcal{O}_{E^0(\Gamma)}(8O)$. Let

$$\mathcal{Q} = (\mathrm{id} \times \varphi_O)^* \mathcal{P}$$

be the pullback of the Poincaré bundle $\mathcal{P}$ on $E^0(\Gamma) \times_{Y(\Gamma)} E^0(\Gamma)^\vee$ along the principal polarization associated to O. Consider the *Hodge bundle* $\lambda_{Y(\Gamma)} = O^* \omega_{E^0(\Gamma)/Y(\Gamma)}$ on $Y(\Gamma)$. This bundle comes equipped with a smooth metric $||\cdot||_{\mathrm{Pet}}$, with respect to which we have the following theorem:

Theorem 3.1. *Equip $P^*\mathcal{Q}$ with the pullback of the natural metric $||\cdot||_{can}$ on $\mathcal{P}$ along $(\mathrm{id} \times \varphi_O) \circ P$. Then there is an isometry*

$$(\mathcal{O}_{E^0(\Gamma)}(8O), ||\cdot||_{\mathrm{Pet}}) \xrightarrow{\sim} (P^*\mathcal{Q}, ||\cdot||_{can})^{\otimes 4} \otimes (p^*\lambda_{Y(\Gamma)}, ||\cdot||_{\mathrm{Pet}})^{\otimes 4}.$$

Thus to find a Mumford-Lear extension of $L_{4,4}(\Gamma)$ it suffices to find Mumford-Lear extensions of $P^*\mathcal{Q}$ and $p^*\lambda_{Y(\Gamma)}$. Mumford's original paper (cf. [17]) implies that $\lambda_{Y(\Gamma)}$ has a Mumford extension with underlying line bundle $\lambda_{X(\Gamma)} = O^*\omega_{E(\Gamma)/X(\Gamma)}$. As for $P^*\mathcal{Q}$, the existence of the Lear extension follows from Lear's thesis (cf. [16]), and we will determine it using the method of test curves.

Note that by Proposition 2.4, we have $P^*\mathcal{Q} \cong \langle P - O, P - O\rangle^{\otimes -1}$. Hence, it suffices to determine the Mumford-Lear extension of $\langle P - O, P - O\rangle^{\otimes -1}$. Let $\mathfrak{c} \in \mathcal{C}(\Gamma)$ be a cusp of $X(\Gamma)$ and let $h = h_{\mathfrak{c}}$ be its period. By Proposition 2.5, we know that the fiber above $\mathfrak{c}$ is an h-gon D, say with irreducible components

$$V = \{C_{\mathfrak{c},0}, \ldots, C_{\mathfrak{c},h-1}\},$$

with the zero section O passing through $C_{\mathfrak{c},0}$ and then ordered cyclically. Relabel the irreducible components by $C_i = C_{\mathfrak{c},i}$ for the moment and let $\overline{f}_i\colon \Delta \to E(\Gamma)$ be a test curve for the i-th component of D which does not intersect the zero section. Then by compatibility of the Deligne pairing and pullbacks, we have

$$f_i^*\langle P - O, P - O\rangle^{\otimes -1} = \langle P_{\Delta^*} - O_{\Delta^*}, P_{\Delta^*} - O_{\Delta^*}\rangle^{\otimes -1},$$

where O_{Δ^*} is the zero section of the fibration $p_{\Delta^*}\colon U(\Gamma)_{\Delta^*} \to \Delta^*$ and P_{Δ^*} is a section of p_{Δ^*} which is disjoint from O_{Δ^*} and which intersects the h-gon above 0 in the i-th component. Let $\widetilde{U(\Gamma)}_{\Delta^*}/\Delta$ be a compactification and desingularization of U_{Δ^*}/Δ^* and let

$$O_\Delta, P_\Delta\colon \Delta \to \widetilde{U(\Gamma)}_{\Delta^*}$$

be the unique extensions of O_{Δ^*} and P_{Δ^*} to Δ. Since $P_{\Delta^*} - O_{\Delta^*}$ is of relative degree 0, we have that by [13, Theorem 2.2], the Mumford-Lear extension of $\langle P_{\Delta^*} - O_{\Delta^*}, P_{\Delta^*} - O_{\Delta^*}\rangle^{\otimes -1}$ to Δ is equal to

$$[\langle P_{\Delta^*} - O_{\Delta^*}, P_{\Delta^*} - O_{\Delta^*}\rangle^{\otimes -1}]_\Delta = \langle P_\Delta - O_\Delta + \Phi_i, P_\Delta - O_\Delta + \Phi_i\rangle^{\otimes -1},$$

where Φ_i is a $\mathbb{Q}$-divisor with support contained in D_Δ, uniquely determined by the properties that the coefficient for $C_{0\Delta}$ is zero and $P_\Delta - O_\Delta + \Phi_i$ has zero intersection with $C_{\nu\Delta}$ for all $\nu \in \mathbb{Z}/h\mathbb{Z}$. Applying this second property to Φ_i to get the equality $\langle P_\Delta - O_\Delta + \Phi_i, \Phi_i\rangle = 0$, we see the Mumford-Lear extension can also be expressed as follows:

$$\begin{aligned}[\langle P_{\Delta^*} - O_{\Delta^*}, P_{\Delta^*} - O_{\Delta^*}\rangle^{\otimes -1}]_\Delta &= \langle P_\Delta - O_\Delta + \Phi_i, P_\Delta - O_\Delta + \Phi_i\rangle^{\otimes -1} \\ &= \langle P_\Delta - O_\Delta + \Phi_i, P_\Delta - O_\Delta - \Phi_i\rangle^{\otimes -1} \\ &= \langle P_\Delta - O_\Delta, P_\Delta - O_\Delta\rangle^{\otimes -1} \otimes \langle \Phi_i, \Phi_i\rangle \\ &= \langle P_\Delta - O_\Delta, P_\Delta - O_\Delta\rangle^{\otimes -1} \otimes \mathcal{O}_\Delta(\Phi_i^2 \cdot [0]),\end{aligned}$$

where we use in the last equality that $\langle \Phi_i, \Phi_i\rangle$ and $\mathcal{O}_\Delta(\Phi_i^2 \cdot [0])$ are both uniquely characterized by the fact that they are supported on $[0]$ and have degree Φ_i^2. Thus the coefficient for C_i is Φ_i^2. The following lemma gives an explicit description of Φ_i^2.

Lemma 3.2. *There is exactly one $\mathbb{Q}$-divisor $\Phi_i = \sum_{\nu\in\mathbb{Z}/h\mathbb{Z}} v_\nu C_{\nu\Delta}$ with $v_0 = 0$ satisfying the property that $P_\Delta - O_\Delta + \Phi_i$ has zero intersection with $C_{\nu\Delta}$ for all $\nu \in \mathbb{Z}/h\mathbb{Z}$. Its coefficients $v_\nu \in \mathbb{Q}$ are given by*

$$v_\nu = \begin{cases} \nu \cdot T_0 & \text{if } \nu < i \\ \nu \cdot T_0 + i - \nu & \text{else} \end{cases},$$

where $T_0 = \frac{h-i}{h}$. The self-intersection of Φ_i is given by

$$\Phi_i^2 = -\frac{i(h-i)}{h}.$$

Proof. Relabel the irreducible components of D_Δ as $A_\nu = C_{\nu\Delta}$ for the moment. If Φ_i exists, it should satisfy the following relations:

$$\begin{aligned} 0 &= \langle P_\Delta - O_\Delta + \Phi_i, A_i\rangle = 1 + \langle \Phi_i, A_i\rangle \\ 0 &= \langle P_\Delta - O_\Delta + \Phi_i, A_0\rangle = -1 + \langle \Phi_i, A_0\rangle \\ 0 &= \langle P_\Delta - O_\Delta + \Phi_i, A_\nu\rangle = \langle \Phi, A_\nu\rangle \text{ for all } \nu \neq i \end{aligned}$$

Thus, since $A_\nu^2 = -2$ and $A_\nu \cdot A_{\nu+1} = 1$ for all $\nu \in \mathbb{Z}/h\mathbb{Z}$, we see the vector $v = (v_0, \ldots, v_{h-1})$ is a solution of the linear system $Mv = b$, where $b = e_0 - e_i$ and

$$M = \begin{pmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & 1 & -2 \end{pmatrix}$$

Note that M is of rank $h-1$, with kernel given by $\mathbb{Q} \cdot (1, 1, \ldots, 1)^\top$. Hence the solution set of $Mv = b$ is of the form $u + \mathbb{Q} \cdot (1, 1, \ldots, 1)^\top$ for some vector $u \in \mathbb{Q}^h$. This implies that there is exactly one $v = (v_0, \dots, v_{h-1})$ with $v_0 = 0$ and $Mv = b$. Its coefficients are determined by the following recurrence relations:

$$v_\nu = \begin{cases} 2v_{\nu-1} - v_{\nu-2} & \text{if } \nu \neq i+1; \\ 2v_i - v_{i-1} - 1 & \text{otherwise.} \end{cases}$$

A small calculation shows that these relations are satisfied by the v_ν in the statement of this lemma, so we infer by unicity of Φ_i that the coefficients of Φ_i are as claimed.

Next we calculate the self-intersection of Φ_i. Note that

$$v_{\nu+1} - v_\nu = \begin{cases} T_0 & \text{if } \nu < i \\ T_0 - 1 & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} \Phi_i^2 &= -2\sum_{\nu=0}^{h-1} v_\nu^2 + 2\sum_{\nu=0}^{h-1} v_\nu v_{\nu+1} = 2\sum_{\nu=0}^{h-1}(v_{\nu+1} - v_\nu)v_\nu \\ &= T_0^2 \cdot i(i-1) + (T_0 - 1)\cdot((T_0 - 1)\cdot(h(h-1) - i(i-1) + 2(i(h-i))) \\ &= -\frac{i(h-i)}{h}, \end{aligned}$$

as was to be shown. □

Now consider for each $\mathfrak{c} \in \mathcal{C}(\Gamma)$ the $\mathbb{Q}$-divisor

$$V_\mathfrak{c} = \sum_{i \in \mathbb{Z}/h_\mathfrak{c}\mathbb{Z}} -\frac{i \cdot (h_\mathfrak{c} - i)}{h_\mathfrak{c}} C_{\mathfrak{c},i}.$$

The previous discussion implies that $[P^*\mathcal{Q}, ||\cdot||_{\text{can}}]_{E(\Gamma)} = \langle P - O, P - O\rangle^{\otimes -1} \otimes \bigotimes_{\mathfrak{c}\in\mathcal{C}(\Gamma)} \mathcal{O}_{E(\Gamma)}(V_\mathfrak{c})$. Note that $\langle P, O\rangle = P^*\mathcal{O}_{U(\Gamma)}(O) = \mathcal{O}_{E(\Gamma)}(O)$. Hence, since $\langle P, P\rangle^{\otimes -1}$ and $\langle O, O\rangle^{\otimes -1}$ are isomorphic to $\lambda_{Y(\Gamma)}$ by Lemma 2.2, we find using bilinearity of the Deligne pairing that

$$[P^*\mathcal{Q}, ||\cdot||_{\text{can}}]_{E(\Gamma)} = \overline{p}^*\lambda_{X(\Gamma)}^{\otimes 2} \otimes \mathcal{O}_{E(\Gamma)}(2O) \otimes \bigotimes_{\mathfrak{c}\in\mathcal{C}(\Gamma)} \mathcal{O}_{E(\Gamma)}(V_\mathfrak{c}).$$

Thus we arrive at the following result.

Theorem 3.3. *Let F be the divisor class associated to $\overline{p}^*\lambda_{X(\Gamma)}$ and let J be the divisor class associated to $L_{4,4}(\Gamma)$. Then $[J, ||\cdot||_{\text{Pet}}]_{E(\Gamma)} = 12F + 8O + 4\sum_{\mathfrak{c}\in\mathcal{C}(\Gamma)} V_\mathfrak{c}$.*

To compute the self-intersection of $[J, ||\cdot||_{\text{Pet}}]_{E(\Gamma)}$, we need the following auxiliary results.

Lemma 3.4. *For each $\mathfrak{c} \in \mathcal{C}(\Gamma)$, the self-intersection of $V_\mathfrak{c}$ is given by*

$$V_\mathfrak{c}^2 = -\frac{h_\mathfrak{c}^2 - 1}{3h_\mathfrak{c}}$$

Proof. Fix a cusp $\mathfrak{c} \in \mathcal{C}(\Gamma)$ and relabel for the moment the period of $\mathfrak{c}$ as $h = h_\mathfrak{c}$ and the irreducible components of $D = D_\mathfrak{c}$ as $C_i = C_{\mathfrak{c},i}$ for all $i \in \mathbb{Z}/h\mathbb{Z}$. Then

$$\begin{aligned} V_\mathfrak{c}^2 &= -2\sum_{i=0}^{h} \frac{i^2(h-i)^2}{h^2} + 2\sum_{i=0}^{h} \frac{i(i+1)(h-i)(h-i-1)}{h^2} \\ &= \frac{h^2(h+1)^2 - \frac{1}{3}(3h-1)h(h+1)(2h+1) + h^2(h+1)(h-1)}{h^2} \\ &= -\frac{h^2-1}{3h} \end{aligned}$$

□

Lemma 3.5. *Let d_Γ be the degree of the forgetful map $\pi\colon X(\Gamma) \to X(1)$. Then we have the following equality:*

$$d_\Gamma = \sum_{\mathfrak{c}\in\mathcal{C}(\Gamma)} h_\mathfrak{c}.$$

Proof. Choose $N \gg 0$ such that $\Gamma(N) \leq \Gamma$ and let $\pi_1\colon X(N) \to X(1)$ and $\pi_\Gamma\colon X(N) \to X(\Gamma)$ be the natural projection maps. Note that since $\Gamma(N)$ is normal in $\mathrm{SL}_2(\mathbb{Z})$, it is also normal in Γ. By [11, page 67] and the fact that $\Gamma(N)$ is a normal subgroup of Γ, the ramification index $e_\mathfrak{c}$ of any cusp $T \in \mathcal{C}(\Gamma(N))$ above $\mathfrak{c} \in \mathcal{C}(\Gamma)$ is equal to $h/h_\mathfrak{c}$, where h is the period of T. As all cusps in $X(N)$ have period N (cf. [11, page 101]), we deduce that $h_\mathfrak{c} = N/e_\mathfrak{c}$.

Now consider the *divisor of cusps* $\delta_0 := \pi^*[\infty]$ of $X(\Gamma)$. The pull-back of a point is supported on the fiber above that point, so we may write $\delta_0 = \sum_{\mathfrak{c}\in\mathcal{C}(\Gamma)} n_\mathfrak{c} \cdot \mathfrak{c}$ for certain $n_\mathfrak{c} \in \mathbb{Z}$. As $\pi_1 = \pi \circ \pi_\Gamma$, we deduce that $\pi_\Gamma^*\delta_0 = \pi_1^*[\infty]$. Therefore, since each cusp in $X(N)$ has ramification index N above $[\infty]$, we find

$$\sum_{\mathfrak{c}\in\mathcal{C}(\Gamma)} \sum_{T\in\pi_\Gamma^{-1}(\mathfrak{c})} n_\mathfrak{c} e_\mathfrak{c} \cdot T = \pi_\Gamma^* \sum_{\mathfrak{c}\in\mathcal{C}(\Gamma)} n_\mathfrak{c} \cdot \mathfrak{c} = \sum_{T\in\mathcal{C}(\Gamma(N))} N \cdot T.$$

We deduce that $\delta_0 = \sum_{\mathfrak{c}\in\mathcal{C}(\Gamma)} h_\mathfrak{c} \cdot \mathfrak{c}$, so indeed

$$\sum_{\mathfrak{c}\in\mathcal{C}(\Gamma)} h_\mathfrak{c} = \deg \sum_{\mathfrak{c}\in\mathcal{C}(\Gamma)} h_\mathfrak{c} \cdot \mathfrak{c} = \deg \delta_0 = \deg \pi \cdot \deg[\infty] = d_\Gamma \cdot 1 = d_\Gamma,$$

as was to be shown. □

We end this section with a calculation of the self-intersection of $[J, ||\cdot||_{\mathrm{Pet}}]_{E(\Gamma)}$.

Theorem 3.6. *Let d_Γ be the degree of the forgetful map $\pi\colon X(\Gamma) \to X(1)$. Then the self-intersection of $[J, ||\cdot||_{\mathrm{Pet}}]_{E(\Gamma)}$ is given by*

$$[J, ||\cdot||_{\mathrm{Pet}}]_{E(\Gamma)}^2 = \frac{64}{12} d_\Gamma + \sum_{\mathfrak{c}\in\mathcal{C}(\Gamma)} \frac{64}{12 h_\mathfrak{c}}.$$

Proof. Note that $O \cdot V_\mathfrak{c} = 0$ for all $\mathfrak{c} \in \mathcal{C}(\Gamma)$ since O intersects $D_\mathfrak{c}$ only in $C_{\mathfrak{c},0}$ and since the coefficient for $C_{\mathfrak{c},0}$ is zero. The self-intersection of F is also zero since F can be written as a sum of fibers, which have zero self-intersection since every fiber is algebraically equivalent to a fiber which doesn't intersect it. In a similar way we see that F has zero intersection with $V_{\mathfrak{c}}$ for all $\mathfrak{c} \in \mathcal{C}(\Gamma)$. Thus we obtain the following formula for $[J, ||\cdot||_{\mathrm{Pet}}]^2_{E(\Gamma)}$:

$$[J, ||\cdot||_{\mathrm{Pet}}]^2_{E(\Gamma)} = 192 \cdot F \cdot O + 64 \cdot O^2 - 64 \sum_{\mathfrak{c} \in \mathcal{C}(\Gamma)} \frac{h_{\mathfrak{c}}^2 - 1}{12 h_{\mathfrak{c}}}$$

By the adjunction formula we have $-O \cdot O = F \cdot O = \frac{d_\Gamma}{12}$ (cf. [6, Proposition 3.2]). Substituting this in the above formula and using that $d_\Gamma = \sum_{\mathfrak{c} \in \mathcal{C}(\Gamma)} h_{\mathfrak{c}}$ we obtain

$$\begin{aligned}[J, ||\cdot||_{\mathrm{Pet}}]^2_{E(\Gamma)} &= \left(\frac{196}{12} - \frac{64}{12} - \frac{64}{12}\right) d_\Gamma + \sum_{\mathfrak{c} \in \mathcal{C}(\Gamma)} \frac{64}{12 h_{\mathfrak{c}}} \\ &= \frac{64}{12} d_\Gamma + \sum_{\mathfrak{c} \in \mathcal{C}(\Gamma)} \frac{64}{12 h_{\mathfrak{c}}},\end{aligned}$$

as was to be shown. □

3.2 Comparison to Chern-Weil theory

Let $\mathrm{CPT}'(E^0(\Gamma))$ be the subset of $\mathrm{CPT}(E^0(\Gamma))$ consisting of the isomorphism classes of compactifications $Y/E^0(\Gamma)$ that are obtained from $E(\Gamma)$ by a finite number of blow-ups in *singular* points of the boundary divisor. By Lemma 3.7 below, to compute Mumford-Lear extensions to arbitrary compactifications of $E^0(\Gamma)$, it suffices to compute $[L_{4,4}(\Gamma), ||\cdot||_{\mathrm{Pet}}]_Y$ for all $Y \in \mathrm{CPT}'(E^0(\Gamma))$.

Lemma 3.7. *Let Y be a smooth compactification of a variety X over $\mathbb{C}$ with boundary divisor D and suppose $(\mathcal{L}, ||\cdot||)$ is a smoothly metrized line bundle on X. Suppose furthermore that $(\overline{\mathcal{L}}, ||\cdot||, e)$ is a Mumford-Lear extension of $\mathcal{L}$ to Y and let $S \subset D$ be the smallest subset for which*

$$(\mathcal{L}|_{X-S}, ||\cdot||)$$

admits a Mumford extension along $D - S$. Let $\pi\colon Y' \to Y$ be the compactification of X obtained by blowing up Y in a point $p \in D - S$. Then the Mumford-Lear extension of $\mathcal{L}$ to Y' exists and is equal to

$$(\pi^*\overline{\mathcal{L}}, \pi^*||\cdot||, e)$$

Proof. Just note that $\pi^*||\cdot||$ is pre-log along $\pi^{-1}(D - S)$. □

Corollary 3.8. *If E^0/C is an elliptic surface equipped with a metrized line bundle $(\mathcal{L}, ||\cdot||)$, the Mumford-Lear extensions of $\mathcal{L}$ to a smooth compactification Y' of E^0 can be computed in the following manner: if Y is the greatest element of $\mathrm{CPT}'(E^0)$ with $Y \trianglelefteq Y'$ and the Mumford-Lear extension of $\mathcal{L}$ to Y exists, then so does the Mumford-Lear extension of $\mathcal{L}$ to Y' and we have*

$$[\mathcal{L}, ||\cdot||]_Y = \pi^*[\mathcal{L}, ||\cdot||]_{Y'},$$

where $\pi\colon Y' \to Y$ is the projection map induced by the chain of blow-ups turning Y into Y'.

If we blow up a singular point p of the boundary divisor D of $Y/E^0(\Gamma)$, the exceptional divisor E intersects the strict transform of D in two points. Thus whenever we blow up a singular point, we get two singular points back, which again can be blown up to get four points and so on. This yields the following result.

Lemma 3.9. *Take for each* $\mathfrak{c} \in \mathcal{C}(\Gamma)$ *and* $i = 0, \ldots, h_{\mathfrak{c}} - 1$ *a copy* $T_{\mathfrak{c},i}$ *of the full binary tree. Then* $\mathrm{CPT}'(E^0(\Gamma))$ *is order-isomorphic to*

$$\prod_{\mathfrak{c}\in\mathcal{C}(\Gamma)} \prod_{i=0}^{h_{\mathfrak{c}}-1} T_{\mathfrak{c},i}$$

equipped with the product order.

Let Y be a compactification of $E^0(\Gamma)$. We say a singular point $p \in Y - E^0(\Gamma)$ has type (n, m) and multiplicity μ if there is an adapted coordinate system (V, u, v) of p with the property that the norm given by the formula

$$\log || \cdot ||^2 = \log || \cdot ||^2_{\mathrm{Pet}} + \frac{2\mu}{nm} \frac{\log |u| \log |v|}{n \log |u| + m \log |v|}$$

admits a Mumford extension along $D \cap V$. Note that by Proposition 2.12, a singular point p in the fiber above a cusp $\mathfrak{c} \in \mathcal{C}(\Gamma)$ has type $(1, 1)$ and multiplicity $4/h_{\mathfrak{c}}$. As we will see in Theorem 3.11 below, if we blow up a singular point of type (n, m) and multiplicity μ we obtain two new points of multiplicity μ, one of type $(n + m, m)$ and the other of type $(n, n + m)$. Before we can prove this statement we need to study the expression on the right hand side of the above equation in more detail.

Proposition 3.10. *Let* (n, m) *be a pair of coprime positive integers and let* D *be the boundary divisor of the compactification* Δ^2 *of* $(\Delta^*)^2$*. Let* $f_{n,m}$ *be the function* $(\Delta^*)^2 \to \mathbb{C}$ *defined by the formula*

$$f_{n,m}(u, v) = \frac{2}{nm} \frac{\log |u| \log |v|}{n \log |u| + m \log |v|}.$$

Then $f_{n,m}$ *is a pre-log-log function along* $D - \{(0, 0)\}$*. Moreover, if* π *is the function* $\Delta^2 \to \Delta^2$ *given by* $(s, t) \mapsto (st, t)$ *then*

$$f_{n,m}(\pi(s, t)) = \frac{2}{nm(n + m)} \log |t| + f_{n,n+m}(s, t).$$

Similarly, if π' *is the function* $\Delta^2 \to \Delta^2$ *given by* $(s, t) \mapsto (s, st)$*, then*

$$f_{n,m}(\pi'(s, t)) = \frac{2}{nm(n + m)} \log |s| + f_{n+m,n}(s, t).$$

Proof. For a proof that $f_{n,m}$ is pre-log-log along $D - \{(0, 0)\}$, we refer the reader to [5, Proposition 4.1.(i)]. As for the second statement, this follows from the following calculation:

$$\begin{aligned} f_{n,m}(st, t) &= \frac{2}{nm(n + m)} \frac{\log |t|(n \log |s| + (n + m) \log |t|) + m \log |s| \log |t|}{n \log |s| + (n + m) \log |t|} \\ &= \frac{2}{nm(n + m)} \log |t| + f_{n,m}(s, t) \end{aligned}$$

The third statement is proved in an identical manner. □

Let $\mathrm{Bl}_{(0,0)}\Delta^2 \subset \mathbb{P}^1 \times \Delta^2$ be the blow-up of Δ^2 in $(0, 0)$. Note that the map π from the previous proposition is the composition of the projection map $\mathrm{Bl}_{(0,0)}\Delta^2 \to \Delta^2$ with the coordinate map $\Delta^2 \to \mathrm{Bl}_{(0,0)}\Delta^2$

$$(s, t) \mapsto ((s : 1), st, t),$$

and that the exceptional divisor E of $\psi\colon \mathrm{Bl}_{(0,0)}\Delta^2 \to \Delta^2$ is given on this chart by the equation $t = 0$. Similarly, with respect to the coordinate map $(s,t) \mapsto ((1:t), s, st)$ around p_2 the projection map ψ is given by π' and the exceptional divisor E by the equation $t = 0$.

Theorem 3.11. *Let Y be a compactification of $E^0(\Gamma)$ with boundary divisor D and singular locus S. Suppose $p \in D$ is a singular point of type (n,m) and multiplicity μ. Consider the blow-up $\psi\colon Y' \to Y$ of Y in the point p. Then the singular locus of Y' is equal to $S - \{p\} \cup \{p_1, p_2\}$, where p_1 and p_2 are both of multiplicity μ, p_1 is of type $(n, n+m)$ and p_2 is of type $(n+m, n)$. Moreover, if the Mumford-Lear extension of $L_{4,4}(\Gamma)$ to Y exists then so does the Mumford-Lear extension of $L_{4,4}(\Gamma)$ to Y', and we have*

$$[J, ||\cdot||_{\mathrm{Pet}}]_{Y'} = \pi^*[J, ||\cdot||_{\mathrm{Pet}}]_Y - \frac{\mu}{nm(n+m)}E,$$

where E is the exceptional divisor of the blow-up $\pi\colon Y' \to Y$.

Proof. Let g be a generating section of $L_{4,4}|_V$, where (V, u, v) is a coordinate system for Y around p satisfying the property that the norm given by

$$\log ||\cdot||^2 = \log ||\cdot||^2_{\mathrm{Pet}} + \mu \cdot f_{n,m}(u,v)$$

admits a Mumford extension along $D \cap V$. Denote by $D_1 = \{u = 0\}$ and $D_2 = \{v = 0\}$ the irreducible components of $D \cap V$. Then by Proposition 2.2, there are $c, d \in \mathbb{Q}_{>0}$ such that

$$\log ||g|| - c \cdot \log |u| - d \cdot \log |v|$$

is a pre-log-log function along $D \cap V$. The points of intersection p_1, p_2 of the strict transform of D with E correspond to the points $((0:1),0,0), ((1:0),0,0)$ on $\mathrm{Bl}_{(0,0)}\Delta^2$ and in the coordinate chart (s,t) around p_1, the pull-back of $||\cdot||$ along ψ is given by

$$\log ||\pi^* g|| = \log ||\pi^* g||_{\mathrm{Pet}} + \frac{\mu}{nm(n+m)} \log |t| + \tfrac{1}{2}\mu \cdot f_{n,n+m}(s,t).$$

Thus $\log ||\pi^* g||_{\mathrm{Pet}} - c \log |s| + \frac{\mu}{nm(n+m)} \log |t| + \frac{1}{2}\mu \cdot f_{n,n+m}(s,t)$ is pre-log-log along $D \cap V$, so p_1 is a point of type $(n, n+m)$ and multiplicity μ. In an entirely analogous manner, using that ψ is given by π' on the coordinate chart around p_2, we see that p_2 is a point of type $(n+m, m)$ and multiplicity μ.

Now since $f_{k,\ell}(u,v)$ is pre-log-log along $D^{sm} \cap V$ for all pairs of coprime integers (k, ℓ), we deduce that the Mumford-Lear extension of $L_{4,4}(\Gamma)$ to Y' exists. As the pullback of D_2 respectively D_1 along ψ is given on the coordinate around p_1 respectively p_2 by the equation $s = 0$, we see the coefficients for the irreducible components of $\psi^{-1}(D)$ and E of this Mumford-Lear extension coincide with those in the statement of this theorem, so we are done. □

Now we have enough material to calculate the limit of the self-intersection of the Mumford-Lear extension of $L_{4,4}(\Gamma)$ to all compactifications of $E^0(\Gamma)$.

Theorem 3.12. *The limit of the self-intersection $[L_{4,4}(\Gamma), ||\cdot||_{\mathrm{Pet}}]^2_Y$ over all $Y \in \mathrm{CPT}(E^0(\Gamma))$ exists and is equal to $\frac{64}{12}d_\Gamma$.*

Proof. Since the pull-back along the projection map associated to the blow-up in a non-singular point does not change the value of the self-intersection, we may as well calculate

$$\lim_{Y \in \mathrm{CPT}'(E^0(\Gamma))} [J, ||\cdot||_{\mathrm{Pet}}]^2_Y$$

instead. For each $\mathfrak{c} \in \mathcal{C}(\Gamma)$ and $i \in \{0, \ldots, h_{\mathfrak{c}} - 1\}$, there is a one-to-one correspondence between the elements of $T_{\mathfrak{c},i}$ and the set $\mathcal{N}$ of pairs of coprime integers, defined recursively by the following conditions:

- The element $(1,1)$ maps to the blow-up of $E(\Gamma)$ in the i-th singular point above $\mathfrak{c}$.
- If (n,m) maps to a compactification Y, then $(n, n+m)$ maps to the blow-up of Y in the unique point of type $(n, n+m)$ and $(n+m, n)$ maps to the blow-up in the unique point of type $(n+m, m)$.

Recall that if $\pi\colon Y \to X$ is the blow-up of a smooth algebraic surface in a point p with exceptional divisor E then $\mathrm{Cl}(X) \oplus \mathbb{Z}$ is isomorphic to $\mathrm{Cl}(Y)$ via the map $(D,n) \mapsto \pi^*D + nE$. Hence if $Y' \to Y$ is the blow-up of a compactification $Y/E^0(\Gamma)$ in a point p of multiplicity μ and type (n,m) then

$$[J, ||\cdot||_{\mathrm{Pet}}]^2_{Y'} = [J, ||\cdot||_{\mathrm{Pet}}]^2_{Y} - \frac{\mu^2}{n^2m^2(n+m)^2}.$$

Thus a point of multiplicity μ and type (n,m) contributes $-\frac{\mu^2}{n^2m^2(n+m)^2}$ to the limit of self-intersections, so if the limit exists, it is equal to

$$\begin{aligned}\lim_{Y \in \mathrm{CPT}'(E^0(\Gamma))} [J, ||\cdot||_{\mathrm{Pet}}]^2_{Y} &= [J, ||\cdot||_{\mathrm{Pet}}]^2_{E(\Gamma)} - \sum_{\mathfrak{c}\in\mathcal{C}(\Gamma)} h_{\mathfrak{c}} \sum_{(n,m)\in\mathcal{N}} \frac{4^2}{h_{\mathfrak{c}}^2 n^2m^2(n+m)^2} \\ &= \frac{64}{12} d_\Gamma + \sum_{\mathfrak{c}\in\mathcal{C}(\Gamma)} \frac{64}{12h_{\mathfrak{c}}} \left(1 - \sum_{(n,m)\in\mathcal{N}} \frac{3}{n^2m^2(n+m)^2}\right).\end{aligned}$$

As in the proof of [5, Theorem 4.11], we deduce from the results of [18] that the sum $\sum_{(n,m)\in\mathcal{N}} \frac{1}{n^2m^2(n+m)^2}$ converges to $1/3$. Substituting this value in the above equation we obtain the desired result. □

Next up is the promised analogue of Chern-Weil theory for the system of divisors associated to the tower of compactifications of $E^0(\Gamma)$.

Theorem 3.13. *The integral* $\int_{E^0(\Gamma)} c_1(L_{4,4}(\Gamma), ||\cdot||_{\mathrm{Pet}})^{\wedge 2}$ *exists and is equal to* $\frac{64}{12} d_\Gamma$.

Proof. Let $([J, ||\cdot||_{\mathrm{Pet}}]_{E(\Gamma)}, 1, ||\cdot||')$ be a Mumford extension of the norm $||\cdot||$ on $L_{4,4}(\Gamma)$ which is given locally around a double point p above a cusp of period h by the formula

$$\log ||\cdot||^2 = \log ||\cdot||^2_{\mathrm{Pet}} + \frac{8}{h} \frac{\log|u| \log|v|}{\log|u| + \log|v|}.$$

Write $\omega = c_1(L_{4,4}(\Gamma), ||\cdot||)$, $\omega' = c_1(L_{4,4}(\Gamma), ||\cdot||')$ and

$$f = \log ||s||^2 - \log ||s||'^2$$

for some meromorphic section s of $L_{4,4}(\Gamma)$. Then $\omega = \omega' + \frac{1}{2\pi i}\partial\overline{\partial} f$, so writing out the wedge product we see the following equality holds:

$$\int_{E^0(\Gamma)} \omega^{\wedge 2} = \int_{E^0(\Gamma)} \omega'^{\wedge 2} - \int_{E^0(\Gamma)} d\left(\frac{1}{(2\pi i)^2} \partial f \wedge \partial\overline{\partial} f\right).$$

Since an analogue of Chern-Weil theory holds for pre-log metrics and since pre-log-log forms have no residue (cf. [17]), we can rewrite the above integral as

$$[J, ||\cdot||_{\mathrm{Pet}}]^2_{E^0(\Gamma)} - \sum_{\mathfrak{c}\in\mathcal{C}(\Gamma)} \sum_{\nu=0}^{h_{\mathfrak{c}}-1} \lim_{\varepsilon\to 0} \int_{\partial V_{\mathfrak{c},\nu,\varepsilon}} \partial f \wedge \partial\overline{\partial} f$$

where $V_{\mathfrak{c},\nu,\varepsilon} = \{(u,v) \in W_{\mathfrak{c},\nu} : |u| \leq \varepsilon, |v| \leq \varepsilon\}$. As in the proof of [5, Theorem 5.2], we compute

$$\lim_{\varepsilon\to 0} \int_{\partial V_{\mathfrak{c},\nu,\varepsilon}} \partial f \wedge \partial\overline{\partial} f = \frac{64}{12 h_{\mathfrak{c}}^2}.$$

Putting this all together, we find

$$\begin{aligned} \int_{E^0(\Gamma)} \omega^{\wedge 2} &= [J, ||\cdot||_{\mathrm{Pet}}]^2_{E(\Gamma)} - \sum_{\mathfrak{c}\in\mathcal{C}(\Gamma)} h_{\mathfrak{c}} \cdot \frac{64}{12 h_{\mathfrak{c}}^2} \\ &= \frac{64}{12} d_\Gamma + \sum_{\mathfrak{c}\in\mathcal{C}(\Gamma)} \frac{64}{12 h_{\mathfrak{c}}} - \sum_{\mathfrak{c}\in\mathcal{C}(\Gamma)} \frac{64}{12 h_{\mathfrak{c}}} = \frac{64}{12} d_\Gamma, \end{aligned}$$

as was to be shown. □

We end this section by showing that the system of divisors associated to the tower of compactifications of $E^0(\Gamma)$ satisfies a Hilbert-Samuel type formula, generalizing [5, Theorem 5.2] to arbitrary subgroups $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ which act without fixed points on $\mathcal{H}$.

Proposition 3.14. *The following equality holds:*

$$\lim_{X\in\mathrm{CPT}(E^0(\Gamma))} [J, ||\cdot||_{\mathrm{Pet}}]^2_X = \lim_{\ell\to\infty} \frac{\dim J_{4\ell,4\ell}(\Gamma)}{\ell^2/2!}.$$

Proof. By [6, Proposition 3.8], the following equality holds for all $\ell \gg 0$ with $h_{\mathfrak{c}} | \ell$ for all $\mathfrak{c} \in \mathcal{C}(\Gamma)$:

$$\dim J_{4\ell,4\ell}(\Gamma) = \frac{16}{6} d_\Gamma \cdot \ell^2 - d_\Gamma \cdot \ell - \sum_{\mathfrak{c}\in\mathcal{C}(\Gamma)} \left(\frac{h_{\mathfrak{c}}}{4} Q\left(\frac{16\ell}{h_{\mathfrak{c}}}\right) + \frac{h_{\mathfrak{c}}}{2} \sum_{\substack{\Delta | 16\ell/h_{\mathfrak{c}}, \Delta<0 \\ 16\ell/(h_{\mathfrak{c}}\Delta) \text{ squarefree}}} H(\Delta) \right),$$

where $Q(n)$ denotes the largest integer whose square divides n and $H(\Delta)$ is the Hurwitz class number. As in the proof of [5, Proposition 2.6] and [5, Remark 2.7], we deduce

$$\dim J_{4\ell,4\ell}(\Gamma) = \frac{32}{12} d_\Gamma \cdot \ell^2 + o(\ell^2).$$

Now we calculate:

$$\lim_{X\in\mathrm{CPT}(E^0(\Gamma))} [J, ||\cdot||_{\mathrm{Pet}}]^2_X = \frac{64}{12} d_\Gamma = \lim_{\ell\to\infty} \frac{\frac{32}{12} d_\Gamma \cdot \ell^2 + o(\ell^2)}{\ell^2/2!} = \lim_{\ell\to\infty} \frac{\dim J_{4\ell,4\ell}(\Gamma)}{\ell^2/2!}.$$

□

4 Concluding remarks

We have succeeded in generalizing [5, Theorem 5.1] and [5, Theorem 5.2] to all congruence subgroups $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ which act without fixed points on $\mathcal{H}$. The next question is whether these results can be generalized to arbitrary families of abelian varieties that come equipped with a theta divisor[1]. It goes beyond the scope of this thesis to even formulate an analogue of [5, Theorem 5.2] for families of abelian varieties. But we will mention the following higher-dimensional analogues of the two main ingredients enabling the calculation of $[J, ||\cdot||_{\mathrm{Pet}}]_E$ for all $E \in \mathrm{CPT}(E^0(\Gamma))$:

(i) Suppose $p\colon \mathcal{A} \to Y$ is a family of abelian varieties and $\lambda\colon \mathcal{A} \to \mathcal{A}^\vee$ is the principal polarization induced by a theta divisor Θ on $\mathcal{A}$. Write $P = (\mathrm{id}, \lambda)$ and let $\mathcal{P}$ be the Poincaré bundle on $\mathcal{A} \times_Y \mathcal{A}^\vee$. Let $\lambda_Y = \det O^*\Omega^1_{\mathcal{A}/Y}$ be the determinant of the Hodge bundle of $\mathcal{A}/Y$. Then with respect to a translation invariant metric $||\cdot||_{\mathrm{Pet}}$ on $\mathcal{O}_{\mathcal{A}}(2\Theta)$ and certain natural metrics $||\cdot||_{\mathrm{Pet}}$ on λ_Y and $||\cdot||_{\mathrm{can}}$ on $\mathcal{P}$, we have an isometry

$$(\mathcal{O}_{\mathcal{A}}(2\Theta), ||\cdot||_{\mathrm{Pet}}) \xrightarrow{\sim} P^*(\mathcal{P}, ||\cdot||_{\mathrm{can}}) \otimes p^*(\lambda_Y, ||\cdot||_{\mathrm{Pet}})$$

Thus, if $\overline{p}\colon \overline{\mathcal{A}} \to \overline{Y}$ is an extension of p, we can again use the original results of Mumford and Lear to deduce $\mathcal{O}_{\mathcal{A}}(2\Theta)$ has a Mumford-Lear extension to $\overline{\mathcal{A}}$.

(ii) In [7], it is proved that if $\mathcal{A}/Y$ is the Jacobian of a family of curves over Y satisfying certain properties, then the singularities of the Lear extension of $P^*\mathcal{P}$ to $\overline{\mathcal{A}}$ can be characterized on a coordinate neighborhood $(U, z_1, \ldots, z_n)$ adapted to $D = \overline{\mathcal{A}} - \mathcal{A}$ in the following manner. If D is given locally by the equation $z_1 \ldots z_k = 0$ then there exists a positive integer d, a homogeneous polynomial $Q \in \mathbb{Z}[x_1, \ldots, x_k]$ of degree d with no zeroes on $\mathbb{R}^k_{>0}$ and for each locally generating section s of $P^*\mathcal{P}$, a homogeneous polynomial $P_s \in \mathbb{Z}[x_1, \ldots, x_k]$ of degree $d+1$ such that for $f_s = P_s/Q$ we have that

$$-\log ||s|| - f_s(-\log |z_1|, \ldots, -\log |z_k|)$$

is bounded on $U \cap \mathcal{A}$ and extends continuously over $U - D^{sing}$. Although no analogue of relatively minimal models exists for higher-dimensional varieties, it may again be fruitful to study the behavior under blow-ups of rational functions that arise in this manner.

We also note that $[J, ||\cdot||_{\mathrm{Pet}}]^2_{E(\Gamma)}$ can be expressed entirely in terms of invariants related to the action of Γ on $\mathcal{H}$. This suggests that it may be possible to extend the results of this thesis to elliptic surfaces over curves that are defined as the quotient of some space under a properly discontinuous group action.

[1] Dual abelian varieties and theta divisors on abelian varieties are defined in exactly the same way as dual elliptic curves and theta divisors on elliptic curves.

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