M.B.J. Zwaan

Rotations of Rigid Bodies

Master thesis

Thesis advisors:
Prof. S.J. Edixhoven

Date master exam:
July 21, 2017

Mathematisch Instituut, Universiteit Leiden

Contents

CHAPTER 1

Introduction

The subject of this thesis is the algebraic aspects of the rotation of a solid block in three dimensions.

The first topic of this thesis originates from a classical physical problem: What happens to a rotating solid object, under the assumption that there is no friction or gravitational pull affecting the object. In these conditions the object will not slow down or speed up as there is no force being applied to the object. This means that the center of gravity is moving at a constant rate, while the object rotates around its center of gravity. Intuitively you might think this is a simple problem as standard objects like a sphere will always keep rotating around the same axes at the same rate. However, for less regularly shaped objects this does not hold, for example if you were to spin a rectangular block around a non-main axes the rotational axis will change over time.

Mathematically we can describe a 3-dimensional object B by a density function $\rho : \mathbf{R}^3 \to \mathbf{R}$. We assume that the center of gravity of the object B is at $0 \in \mathbf{R}^3$. Then the position of the object can be described by a rotation, or element in $G := SO(\mathbf{R}^3) \subset M_3(\mathbf{R})$, here $SO(\mathbf{R}^3)$ is the rotation group and $M_3(\mathbf{R})$ is the set of 3 by 3 matrices, of the object in its starting position. Therefore we can represent the change in position of B over time t with a map $h : \mathbf{R} \to G$. The velocity at time t is then $h'(t)$, note that $G \subset M_3(\mathbf{R})$ and therefore we can calculate h' by considering h as a map from $\mathbf{R}$ to $\mathbf{R}^9$. We now have a map to the tangent bundle of G denoted T_G given by $(h, h') : \mathbf{R} \to T_G \subset G \times M_3(\mathbf{R})$. This map is important, because the basic principle of classical mechanics is that the position and velocity of an object at any time determine the path that it will take. In other words we have a vector field v on T_G which describes the course of (h, h'). The vector field v is uniquely determined by preservation of angular momentum, that is there is a unique vector field such that the angular momentum is preserved. The flow of a vector field is the function which describes all possible paths of the object along the vector field. More specifically the flow of v is a function $\Phi : T_G \times \mathbf{R} \to T_G$ such that $\dfrac{d\Phi(p,t)}{dt} = v(\Phi(p,t))$, here p describes a starting position on the vector field v and $\Phi(p, -)$ describes its path.

Rotating the reference system and rotating both the starting values and the vector field are equivalent. Therefore the vector field v is translation invariant under the left action of G, as a Lie group, on T_G. We can trivialize the tangent bundle with the manifold-isomorphism $\mathrm{Lie(G)} \times G \to T_G : (a, g) \mapsto (ga, g)$ by the left action of G, as G is a Lie group.

This means that we now have a new vector field v' on $\mathrm{Lie(G)} \times G$, which is simply the inverse image of the vector field v under the isomorphism. In this thesis we will show that the vector field v' is of the form $v'(a,g) = (v_1(a), v_2(a,g))$. This means that the vector field v' restricts to a vector field v_1 on $\mathrm{Lie}(G)$.

Here we see the difference in physical and mathematical terminology: what a physicist would describe as the rotating coordinate system a mathematician would consider a trivialization of the tangent bundle by left translation.

The angular momentum is constant in the non rotating reference system, however, in the rotating reference system it changes. We will show that the vector field which preserves angular momentum, in the non-rotating system, is given by

$$v'(a_1, a_2, a_3) = \left(a_2 a_3 \frac{I_2 - I_3}{I_1}, a_1 a_3 \frac{I_3 - I_1}{I_2}, a_1 a_2 \frac{I_1 - I_2}{I_3} \right). \tag{1}$$

Here I_1, I_2, I_3 are the principal moments of inertia. Formula (1) gives Euler's equation of motion, as found in chapter 6.29 A of [**1**], with torque equal to zero.

Subsequently we will move on to show that the vector field on $\mathrm{Lie}(G)$ retains both the kinetic energy and the norm squared of the angular momentum. The level surfaces of these quantities both form ellipsoids. The intersections of these level surfaces, when considered in $\mathbf{P}^3(\mathbf{C})$, are smooth algebraic curves of genus 1, which are elliptic curves for any choice of base point, as long as these intersections are non-degenerate. By non-degenerate we mean that the gradients of the two level surfaces are independent. We will see that the intersection will be non-degenerate if the three moments of inertia I_1, I_2, I_3 are pairwise different.

We will show that the vector field v_1 is translation invariant under the group law of any such elliptic curve E. In other words the group law of the flow of v_1 is algebraic, i.e. it can be derived from the group law of the elliptic curves. We will prove this by showing that the extension of the vector field v_1 has no poles in $\overline{E}$. Here $\overline{E}$ is the unique extension of E in $\mathbf{P}^3(\mathbf{C})$.

Next, we will show that the original 6-dimensional manifold, $\mathrm{Lie}(G) \times G$, has 4 preserved quantities, the kinetic energy and the angular momentum, the angular momentum representing 3 preserved quantities. The level surfaces of these quantities on the 6-dimensional manifold are circle-bundles over the elliptic curves in $\mathrm{Lie}(G)$. We then find that for any choice of zero on this circle-bundle Γ we can extend the group law of the corresponding elliptic curve in $\mathrm{Lie}(G)$.

The main result is then that we can find a group law of this type in such a way that the vector field v on the circle-bundles is translation invariant under this group law. This means that the group law of the flow of v coincides with that of the group. That is for any $p \in T_G, t_1, t_2 \in \mathbf{R}$ we have $\Phi(p, t_1 +_{\mathbf{R}} t_2) = \Phi(p, t_1) +_\Gamma \Phi(p, t_2)$.

CHAPTER 2

Algebraic description of a rigid rotating body

In this chapter we will consider the rotation of a rigid rotating body from an algebraic point of view. This problem is a physical one: how does a rigid body rotate when not under the influence of any force. This means that the center of mass of the object is moving at a constant speed ([**1**], Section 2.10) and so we are only interested in the orientation and let the reference system be the non-rotating reference system with origin the center of mass of the object. This means that the reference system is now unique up to rotation.

An analytic solution to this problem already exists, see [**1**] chapter 6.28 and [**3**], chapter 3; in this thesis, however, we are interested in an algebraic description of this problem, as this gives additional insight in the special functions occurring in the solutions of these equations.

We will begin by defining the problem in a more explicit way. Define $V := \mathbf{R}^3$ the real vector space with standard inner product, orientation and cross product. Let B denote the object, as we will see later we may assume that it is a rectangular block along the axes (after changing our reference system by a rotation). Let $\rho : V \to \mathbf{R}$ be the density function. We position B such that its center of mass is at the origin. We describe the rotation of the block over time by a map $h : \mathbf{R} \to SO(V)$. For ease of reading we define $G := SO(V)$.

For any point $x \in V$, and therefore in B, the path of x under h is given by $\mathbf{R} \to V : t \mapsto h(t)x$. Therefore the speed of x at time t is given by the function $\mathbf{R} \to V : t \mapsto \frac{dh(t)x}{dt} = h'(t)x$. As G is a Lie group and $h(t) \in G$ we have that $h'(t) \in T_G(h(t))$, here T_G denotes the tangentbundle of G and $T_G(h(t)$ the tangent space at $h(t)$. Therefore we have actually defined the following path $\mathbf{R} \to T_G : t \mapsto (h(t), h'(t))$.

In order to work with $\mathrm{Lie}(G) \times G$, where $\mathrm{Lie}(G)$ denotes the Lie algebra of G, instead of T_G, we consider the trivialization of the tangent bundle by left translation, which gives the following isomorphism of manifolds $\Psi : \mathrm{Lie}(G) \times G \to T_G : (a, g) \mapsto (ga, g)$. This isomorphism gives a representation of a corresponding rotating coordinate system.

Now we have introduced the basic objects which we will use in the first part of this thesis.

1. Vector field on $\mathrm{Lie}(G) \times G$

In this section we define the vector field v on $\mathrm{Lie}(G) \times G$ explicitly, taking a special interest in its projection onto $\mathrm{Lie}(G)$. In subsection 1.1 we determine the vector field v on $\mathrm{Lie}(G) \times G$ using the angular momentum and kinetic energy of the object. Then in the next subsection 1.3 we will show that there is a natural projection of this vector field v to $\mathrm{Lie}(G)$. Finally in subsection 1.2 we determine the formula that describes this vector field and confirm that it agrees with Euler's equation of motion.

1.1. Kinetic energy, angular momentum and the vector field.

In this subsection we will use the kinetic energy and angular momentum in combination with the explicit formulae for the path (h, h') of the object through $\mathrm{Lie}(G) \times G$ to define the vector field.

We know that within this system both the kinetic energy and the angular momentum are constant. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbf{R}^3$. The kinetic energy $\overline{K}$ is defined as

Definition 1.1. The kinetic energy function $\overline{K}$ is

$$\mathbf{R} \to \mathbf{R} : t \mapsto \int_{x \in V} \left(\frac{\rho(x)}{2} \right) \langle h'(t)x, h'(t)x \rangle d\mu$$

here μ is the Lebesgue measure.

This agrees with the definition of kinetic energy in [**4**] chapter 2.3

Denote $k(t) := h(t)^{-1}h'(t)$. Furthermore $k(t) = h^{-1}(t)h'(t) \in T_e(G) = \mathrm{Lie}(G)$. This equality lets us define K such that $K := \overline{K} \circ k$, which is the kinetic energy as function on $\mathrm{Lie}(G)$ instead of time.

Proposition 1.2. *On Lie*(G) *kinetic energy is given by* $K : \mathrm{Lie}(G) \to \mathbf{R} : a \mapsto \int_{x \in V} \left(\frac{\rho(x)}{2} \right) \langle ax, ax \rangle d\mu.$

Proof. Firstly

$$\overline{K} : \mathbf{R} \to \mathbf{R} : \int_{x \in V} \left(\frac{\rho(x)}{2} \right) \langle h'(t)x, h'(t)x \rangle d\mu =$$

$$\int_{x \in V} \left(\frac{\rho(x)}{2} \right) \langle h^{-1}(t)h'(t)x, h^{-1}(t)h'(t)x \rangle d\mu =$$

$$\int_{x \in V} \left(\frac{\rho(x)}{2} \right) \langle k(t)x, k(t)x \rangle d\mu.$$

We can replace $k(t)$ by $a \in \mathrm{Lie}(G)$. This yields

$$K : \mathrm{Lie}(G) \to \mathbf{R} : a \mapsto \int_{x \in V} \left(\frac{\rho(x)}{2} \right) \langle ax, ax \rangle d\mu.$$

$\square$

By composing K witht the projection $\mathrm{Lie}(G) \times G \to \mathrm{Lie}(G)$ we get a function on $\mathrm{Lie}(G) \times G$ we will call this function K as well. Note that the kinetic energy only depends on the momentum of an object and not on its position, therefore this definition of K on $\mathrm{Lie}(G) \times G$ is indeed the kinetic energy function on $\mathrm{Lie}(G) \times G$.

Now for the angular momentum. Let $\cdot \times \cdot$ denote the standard outer product on $\mathbf{R}^3$. We define the angular momentum $\overline{J}$ as follows.

DEFINITION 1.3. The angular momentum function $\overline{J}$ is

$$\mathbf{R} \to V : t \mapsto \int_{x \in V} \rho(x)(h(t)x) \times (h'(t)x) d\mu.$$

This definition is equivalent to that of [**8**] chapter 11.6.

The equality $k(t) = h^{-1}(t)h'(t)$ lets us define J such that $J := \overline{J} \circ k$, which is the kinetic energy as function on $\mathrm{Lie}(G)$ instead of time. as we did for K.

PROPOSITION 1.4. *On Lie$(G) \times G$ angular momentum is given by* $J : \mathrm{Lie}(G) \times G \to V : (a, g) \mapsto g \int_{x \in V} \rho(x)(x) \times (ax) d\mu$.

PROOF. Firstly

$$\overline{J} : \mathbf{R} \to V : t \mapsto \int_{x \in V} \rho(x)(h(t)x) \times (h'(t)x) d\mu =$$

$$h(t) \int_{x \in V} \rho(x)(h^{-1}(t)h(t)x) \times (h^{-1}(t)h'(t)x) d\mu =$$

$$h(t) \int_{x \in V} \rho(x)(x) \times (k(t)x) d\mu.$$

Replacing $k(t)$ by $a \in \mathrm{Lie}(G)$ yields

$$J : \mathrm{Lie}(G) \times G \to V : (a, g) \mapsto g \int_{x \in V} \rho(x) \cdot (x) \times (ax) d\mu.$$

□

By [**1**], Section 2.10 both the kinetic energy and angular momentum are constant in a system with no external system.

Note that the part inside of the integral in lemma 1.4 is independent of g, which leads us to the following definition:

DEFINITION 1.5. $j : \mathrm{Lie}(G) \to V : a \mapsto g^{-1} J((a, g))$

The rotation of the object over time is given by a vector field on $\mathrm{Lie}(G) \times G$. We call this vector field v. Its projection to $\mathrm{Lie}(G)$ we call v_1. The following lemma gives explicit descriptions of both v and v_1.

LEMMA 1.6. *The vector field v is given by* $v((a, g)) = (-j^{-1}(aj(a)), ga)$ *on* $\mathrm{Lie}(G) \times G$. *Its projection to* $\mathrm{Lie}(G)$ *is equal to* $v_1(a) = -j^{-1}(aj(a))$.

PROOF. The angular momentum is constant. Therefore

$$0 = J(k, h)' = (hj(k))' = h'j(k) + hj(k')$$

this gives

$$h'j(k) = -hj(k').$$

Multiplying both sides by h^{-1} gives

$$k' = -j^{-1}(kj(k)),$$

in corollary 1.12 we will see that j is invertible as long as $I_1 \neq 0, I_2 \neq 0$ and $I_3 \neq 0$. This shows that the angular momentum defines the a vector field on $\mathrm{Lie}(G)$ given by $v_1(a) = -j^{-1}(aj(a))$. The vector field v_1 can, by definition of $\mathrm{Lie}(G)$, be extended to a vector field v on $\mathrm{Lie}(G) \times G$, given by

$$v((a, g)) = (-j^{-1}(aj(a)), ga).$$

□

The proof of the previous lemma uses only the angular momentum. Therefore we have not yet shown that it preserves kinetic energy, which we will show later on. As a side note: it is unlikely that kinetic energy is not preserved as this would contradict the basic principles of classical mechanics.

We have now seen in lemma 1.6 that we can find a formula for the vector field v without using the kinetic energy. This means that we technically still have to check that the flow of the vector field leaves the kinetic energy invariant. We will prove this in subsection 1.2

We now have a formula for the vector field, $v((a, g)) = (-j^{-1}(aj(a)), ga)$. Lemma 1.6 also shows that v has a natural projection to $\mathrm{Lie}(G)$.

1.2. Vector field on the Lie algebra.

This subsection confirms that the vector field on the Lie algebra is simply a different form of Euler's equations of motion, as found in chapter 6.29 A of [**1**].

We start with an introduction in inertia tensors and a natural way of representing $\mathrm{Lie}(G)$ as $\mathbf{R}^3$.

Definition 1.7. The inertia tensor of a body $B \in \mathbf{R}^3$ with density function $\rho : V = \mathbf{R}^3 \to \mathbf{R}$ is the tensor represented by $I = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{23} & I_{33} \end{pmatrix}$, where $I_{ii} = \int_V \rho(\langle e_j, r\rangle^2 + \langle e_k, r\rangle^2 dV,\ I_{ij} = \int_V \langle e_i, r\rangle\langle e_j, r\rangle dV,$ for $i, j, k \in \{1, 2, 3\}$ different elements for given base choice.

Example 1.8. Consider the object $B \in V = \mathbf{R}^3$ with weights $\frac{m_1}{2}$ on $e_1, -e_1$, $\frac{m_2}{2}$ on $e_2, -e_2$ and $\frac{m_3}{2}$ on $e_3, -e_3$. Define

$$S := \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\}$$

then $I_{ij} = 0$ for every two different $i, j \in \{1, 2, 3\}$ since every $r \in S$ is perpendicular to two basis vectors. Furthermore $I_i = \sum_S \rho(r)(\langle e_j, r\rangle^2 + \langle e_k, r\rangle^2 = m_i + m_k$.

Lemma 1.9. *There is a frame of reference such that the matrix representing the inertia tensor as seen in definition 1.7 is a diagonal matrix.*

Proof. The matrix representing the inertia tensor is symmetric therefore it can be diagonalized by an orthogonal matrix, which proves this lemma. See also the proof on [**7**] page 360-361. $\square$

The elements on the diagonal of a diagonal matrix corresponding to a inertia tensor as in lemma 1.9 are called the principal moments of inertia. Note that they are not dependent on choice of diagonalization as they are the eigenvalues of any matrix which represents the inertia tensor.

Lemma 1.10. *We have an isomorphism of* $\mathbf{R}$ *vector spaces* $\psi : \mathrm{Lie}(G) \to \mathbf{R}^3$ *such that for all* $g \in G, a \in \mathrm{Lie}(G)$ *we have* $\psi(gag^{-1}) = g\psi(a)$*, furthermore for any* $r \in \mathbf{R}^3$ *we have* $ar = \psi(a) \times r$*.*

Proof. We note that $\mathrm{Lie}(G) = \left\{ \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} : a_1, a_2, a_3 \in \mathbf{R} \right\}$. Therefore $\psi : \mathrm{Lie}(G) \to \mathbf{R}^3 : \left\{ \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \mapsto (a_1, a_2, a_3)^T \right\}$ is an isomorphism of vector spaces. Furthermore we have $ar = (r_3a_2 - r_2a_3, r_1a_3 - r_3a_3, r_2a_1 - r_1a_2)^T = \psi(a) \times r$.

To prove that $\psi(gag^{-1}) = g\psi(a)$ we only need to show this holds for an orthononal basis $\{L_1 = \psi^{-1}(e_1), L_2 = \psi^{-1}(e_2), L_3 = \psi^{-1}(e_3)\}$. Now we get

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} =$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -c & -f & -i \\ b & e & h \end{pmatrix} = \begin{pmatrix} 0 & -fb + ce & -ib + ch \\ -ce + fb & 0 & -ie + fh \\ -ch + ib & -fh + ie & 0 \end{pmatrix}$$

Then by orthogonality of g we have:

$$\begin{pmatrix} 0 & -fb + ce & -ib + ch \\ -ce + fb & 0 & -ie + fh \\ -ch + ib & -fh + ie & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & -g & -d \\ g & 0 & -a \\ -d & -a & 0 \end{pmatrix} = \psi^{-1} \begin{pmatrix} a \\ d \\ g \end{pmatrix} = \psi^{-1}(ga)$$

$gag8^{-1}$ is a base change of a and therefore as ψ is an isomorphism of vector spaces the image will be the respective base change of $\psi(a)$, which is ga as desired. $\square$

Now we are ready to incorporate Euler's equations of motion.

Lemma 1.11. *The vector field* v_1 *satisfies Euler's equations of motion. Euler's equation of motion is given by*

$$v_1(a) = \left(a_2a_3\frac{I_2 - I_3}{I_1}, a_1a_3\frac{I_3 - I_1}{I_2}, a_1a_2\frac{I_1 - I_2}{I_3} \right),$$

where I_1, I_2, I_3 *are the principal tensors of inertia as in lemma 1.9.*

PROOF. By lemma 1.6 the conservation of angular momentum defines a vector field on Lie(G). We now wish to re-write this vector field in such a way that it can be compared to Euler's Equations of motion.

We take a to be the matrix $\begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$ or equivalently the vector $(a_1, a_2, a_3)^T$, via lemma 1.10.

By [**1**] chapter 6.28 D the moments of inertia I_1, I_2, I_3 completely determine the rotation of the body. That is any two bodies with identical moments of inertia will behave the same. We can therefore assume that the body has weight $\frac{m_1}{2}$ on $(\pm 1, 0, 0)$ and the same for $\frac{m_2}{2}, \frac{m_3}{2}$ on the second and third axes. By example 1.8 we then have $I_i = m_j + m_k$, for $i, j, k \in \{1, 2, 3\}$ different elements. The vector field on Lie(G) which corresponds to the angular momentum is given by $a' = j^{-1}(aj(a))$. At this point it is easy to find an explicit description of j:

$$j(a) = \int_{x \in V} \rho(x) x \times (ax) \cdot d\mu =$$

$$\frac{2m_1}{2}(0, a_2, a_3)^T + \frac{2m_2}{2}(a_1, 0, a_3)^T + \frac{2m_3}{2}(a_1, a_2, 0)^T = \tag{2}$$

$$((m_2 + m_3)a_1, (m_1 + m_3)a_2, (m_1 + m_2)a_3)^T = (I_1 a_1, I_2 a_2, I_3 a_3)^T$$

Therefore j is the linear transformation with matrix $\begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$. Hence $aj(a) = (a_2 a_3 (I_2 - I_3), a_1 a_3 (I_3 - I_1), a_1 a_2 (I_1 - I_2))^T$. Substituting this result into the equation for the vector field from lemma 1.6 gives

$$v_1(a) = j^{-1}((a_2 a_3 (I_2 - I_3), a_1 a_3 (I_3 - I_1), a_1 a_2 (I_1 - I_2))^T = \tag{3}$$

$$\left(a_2 a_3 \frac{I_2 - I_3}{I_1}, a_1 a_3 \frac{I_3 - I_1}{I_2}, a_1 a_2 \frac{I_1 - I_2}{I_3} \right)^T,$$

which agrees with Euler equations of motion.

□

COROLLARY 1.12. *For any object for which the principal moments of inertia are all non-zero the function j is invertible.*

PROOF. From the proof of lemma 1.11 we know that for a choice of axes such that the inertia tensor matrix is diagonal the function j is given by $\begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$, which is invertible as long as $I_1 \neq 0, I_2 \neq 0$ and $I_3 \neq 0$. □

In the proof of 1.11 we have seen that for a choice of axes such that the inertia tensor matrix is diagonal there is an explicit way of expressing j. This leads us to the following proposition.

PROPOSITION 1.13. *Let I_1, I_2, I_3 be principal moments of inertia for B and let $a \in Lie(G), g \in G$. Representing elements $a \in$ Lie(G) as elements of $\mathbf{R}^3$ using lemma 1.10 we get:*

- $j(a) = (I_1 a_1, I_2 a_2, I_3 a_3)$
- $\langle J(a,g), J(a,g)\rangle = \langle j(a), j(a)\rangle = I_1^2 a_2^2 + I_3^2 a_2^2 + I_3^2 a_3^2$
- $K(a) = \frac{1}{2}(I_1 a_2^2 + I_3 a_2^2 + I_3 a_3^2)$

Proof. Without loss of generality we again assume that the body has weights $\frac{m_1}{2}$ on $e_1, -e_1$, $\frac{m_2}{2}$ on $e_2, -e_2$ and $\frac{m_3}{2}$ on $e_3, -e_3$ We then have $I_i = m_j + m_k$, for $i, j, k \in \{1,2,3\}$ different elements.

For j we copy the proof from lemma 1.11:

$$j(a) = \int_{x \in V} \rho(x) x \times (ax) d\mu =$$

$$\frac{2m_1}{2}(0, a_2, a_3)^T + \frac{2m_2}{2}(a_1, 0, a_3)^T + \frac{2m_1}{2}(a_1, a_2, 0)^T =$$

$$((m_2 + m_3)a_1, (m_1 + m_3)a_2, (m_1 + m_2)a_3)^T = (I_1 a_1, I_2 a_2, I_3 a_3).$$

Furthermore

$$\langle J(a,g), J(a,g)\rangle = \langle gj(a), gj(a)\rangle = \langle j(a), j(a)\rangle =$$

$$\langle (I_1 a_2 + I_3 a_2 + I_3 a_3), (I_1 a_2 + I_3 a_2 + I_3 a_3)\rangle = I_1^2 a_2^2 + I_3^2 a_2^2 + I_3^2 a_3^2$$

gives the second equality.

Finally

$$\int_{x \in V} \left(\frac{\rho(x)}{2}\right) \langle ax, ax\rangle d\mu = \frac{1}{2} \sum_{i \in \{1,2,3\}} (m_j + m_k) a_i^2 = \frac{1}{2}(I_1 a_2^2 + I_3 a_2^2 + I_3 a_3^2)$$

gives the final equality. □

By lemma 1.11 and proposition 1.13 we have shown that the angular momentum and kinetic energy of an object only depend on its infinitesimal rotations and its principal moments of inertia. This in turn shows that our assumption of the object B being block shaped and of consistent density was without loss of generality.

Now we can verify that kinetic energy is indeed invariant under the vector field v.

Proposition 1.14. *The gradient of the level surfaces of the kinetic energy is perpendicular to the vector field. Here for any $b \in \mathrm{Lie}(G)$ the level surface of the kinetic energy at b is $\{a \in \mathrm{Lie}(G) : \frac{1}{2}(I_1 a_1^2 + I_2 a_2^2 + I_3 a_3^2) = K(b)\}$.*

Proof. The level surface of the kinetic energy function that passes through b is given by

$$\frac{1}{2}(I_1 a_1^2 + I_2 a_2^2 + I_3 a_3^2) = K(b).$$

Therefore the gradient at (a_1, a_2, a_3) is $(I_1 a_1, I_2 a_2, I_3 a_3)$. Evaluation the vector field v_1 at this point gives

$$v_1(a) = \left(a_2 a_3 \frac{I_2 - I_3}{I_1}, a_1 a_3 \frac{I_3 - I_1}{I_2}, a_1 a_2 \frac{I_1 - I_2}{I_3}\right).$$

Furthermore

$$\left\langle (I_1 a_1, I_2 a_2, I_3 a_3), \left(a_2 a_3 \frac{I_2 - I_3}{I_1}, a_1 a_3 \frac{I_3 - I_1}{I_2}, a_1 a_2 \frac{I_1 - I_2}{I_3}\right)\right\rangle =$$

$$a_1a_2a_3(I_2 - I_3 + I_3 - I_1 + I_1 - I_2) = 0$$

and therefore the gradient of the kinetic energy is perpendicular to the vector field. This shows that the kinetic energy is invariant under v_1. Furthermore, as the kinetic energy is independent of choice of $g \in G$, it is invariant under v.

□

In summary of this subsection the vector field is the same as the one dictated by Euler's equation of motion.

1.3. Projection of the vector field to an elliptic curve.

Here we will study the vector field as it appears on the elliptic curves.

Let $K_0 \in \mathbf{R}, J_0 \in V$ be initial values for kinetic energy and angular momentum respectively.

DEFINITION 1.15. Define $M(K_0, J_0) \subset \mathrm{Lie}(G) \times G$ such that K, J are identical to K_0, J_0 respectively, that is

$$M(K_0, J_0) = \{(a, g) \in \mathrm{Lie}(G) \times G : K(a) = K_0, J(a, g) = J_0\}.$$

DEFINITION 1.16. Define $E(K_0, J_0) := \{a \in \mathrm{Lie}(G) : K(a) = K_0, \langle j(a), j(a)\rangle = \langle J_0, J_0\rangle\} \subset \mathrm{Lie}(G)$. For $K_0 \neq 0$ and $J_0 \neq 0$

As long as the Principal moments of inertia are pairwise different $E(K_0, J_0)$ is the real part of an elliptic curve as seen at the end of section 2.1.

The following proposition shows that $E(K_0, J_0)$ is the projection of $M(K_0, J_0)$ onto $\mathrm{Lie}(G)$

PROPOSITION 1.17. *Let* $P : \mathrm{Lie}(G) \times G \to \mathrm{Lie}(G)$ *be the projection from* $\mathrm{Lie}(G) \times G$ *to* $\mathrm{Lie}(G)$*. Then the following diagram commutes.*

$$\begin{array}{ccc} M(K_0, J_0) & \xrightarrow[(K,J)]{} & \mathbf{R} \times V \\ {\scriptstyle P|_{M(K_0,J_0)}}\downarrow & & \downarrow{\scriptstyle (id, ||.||^2)} \\ E(K_0, J_0) & \xrightarrow{(K, ||j||^2)} & \mathbf{R} \times \mathbf{R}. \end{array}$$

PROOF. The functions K, J, j, from definitions 1.1, 1.3 and 1.5 respectively,give us the following diagram:

$$\begin{array}{ccc} M(K_0, J_0) & \xrightarrow[(K,J)]{} & \mathbf{R} \times V \\ & & \downarrow{\scriptstyle (id, ||.||^2)} \\ E(K_0, J_0) & \xrightarrow{(K, ||j||^2)} & \mathbf{R} \times \mathbf{R}. \end{array} \tag{4}$$

Furthermore the square of the angular momentum

$$\langle J(a, g), J(a, g)\rangle = \langle gj(a), gj(a)\rangle = \langle j(a), j(a)\rangle$$

is constant on $M(K_0, J_0)$ and only depends on a. Therefore $P(M(K_0, J_0)) = E(K_0, J_0)$. The commutativity then follows from the fact that $\forall (a, g) \in \mathrm{Lie}(G) \times G$ we have $(|| \cdot ||^2 \circ J)(a, g) = ||gj(a)||^2 = ||j(a)||^2$. □

The previous propositions gives the existence of a projection M to E. This leads to the following.

COROLLARY 1.18. *The vector field v_1 from lemma 1.6 lies along the curve $E(K_0, J_0)$.*

PROOF. The gradient of both the ellipsoids are perpendicular to the vector field:

This has been shown in 1.17 for the kinetic energy.

The level surface of the angular momentum is given by

$$(I_1^2 a_1^2 + I_2^2 a_2^2 + I_3^2 a_3^2) = \langle J_0, J_0 \rangle.$$

Therefore the gradient at (a_1, a_2, a_3) is $2(I_1^2 a_1, I_2^2 a_2, I_3^2 a_3)$. Evaluation the vector field v_1 at this point gives

$$v_1(a) = \left(a_2 a_3 \frac{I_2 - I_3}{I_1}, a_1 a_3 \frac{I_3 - I_1}{I_2}, a_1 a_2 \frac{I_1 - I_2}{I_3} \right).$$

Furthermore

$$\left\langle (I_1^2 a_1, I_2^2 a_2, I_3^2 a_3), \left(a_2 a_3 \frac{I_2 - I_3}{I_1}, a_1 a_3 \frac{I_3 - I_1}{I_2}, a_1 a_2 \frac{I_1 - I_2}{I_3} \right) \right\rangle =$$

$$a_1 a_2 a_3 (I_1 I_2 - I_1 I_3 + I_2 I_3 - I_1 I_2 - I_1 I_3 - I_2 I_3) = 0$$

This concludes the proof.

□

The set $E(K_0, J_0)$ is the intersection of two level surfaces, the level surfaces of j and K, in Lie(G). As both these level surfaces are ellipsoids this means that $E(K_0, J_0)$ is the affine real part of an elliptic curve, as long as it is non-singular and genus one. In particular this curve will be non-singular if and only iff the the principal moments of inertia of B are pairwise different and strictly greater than 0.

More importantly in this section we have shown that the vector field v flows along the intersection of two ellipsoids, which are level surfaces of the kinetic energy and the square of the norm of angular momentum for which we will give explicit formulas in the next section.

2. Group law of the elliptic curve

In this section we will show that the intersection of two level surfaces of kinetic energy and the square of the norm of the angular momentum on Lie(G) describes an elliptic curve, after choice of zero, and will study its group law in more detail. In the first subsection 2.1 we will discover that for any choice of zero the group law of the elliptic curve equals the group law of the flow of the vector field v. As a tangential topic, the second subsection 2.2, shows that the group law of the elliptic curve at infinity has some unique properties.

2.1. Translation invariant vector field.

In this section we show that the vector field is translation invariant under the group law of the elliptic curve and consequently that the group law of the flow coincides with that of the elliptic curve and as such is algebraic. This is a standard proof in the field of integrable systems [**2**]. More specifically we follow the notes of Prof. S.J. Edixhoven.

DEFINITION 2.1. Let G be a Lie group, L_g the left-translation on this group with differential dL_g. A vector field X on G is called translation invariant if $X \circ L_g = dL_g \circ X \quad \forall g \in G$.

Note that what is defined in 2.1 is usually called left invariant, but here we simply refer to it as invariant.

THEOREM 2.2. *Let $K_0 \in \mathbf{R}, J_0 \in \mathbf{R}^3$ be such that $E(K_0, J_0)$ is an elliptic curve. Then for any choice of $\mathbf{0}$ on the elliptic curve $E(K_0, J_0)$ the vector field v_1 is translation invariant under the group law of $(E(K_0, J_0), \mathbf{0})$.*

PROOF. We start by writing the equations of the ellipsoids that describe the fibers of the kinetic energy and length squared of the angular momentum. Using proposition 1.13 we get $\frac{1}{2}(I_1a_1^2 + I_2a_2^2 + I_3a_3^2) = K_0$ for the kinetic energy. Angular momentum gives $||J_0||^2 = I_1^2a_1^2 + I_2^2a_2^2 + I_3^2a_3^2$. Therefore

$$E(K_0, J_0) = \{a \in \mathbf{R}^3 : \frac{1}{2}(I_1a_1^2 + I_2a_2^2 + I_3a_3^2) = K_0, ||J_0||^2 = I_1^2a_1^2 + I_2^2a_2^2 + I_3^2a_3^2\}.$$

To find all the points on the elliptic curve we need to look at the complex points as well as most of the points on the elliptic curve are complex. Therefore we embed the ellipsoids into $\mathbf{P}^3(\mathbf{C})$, given by

$$\frac{1}{2}(I_1a_1^2 + I_2a_2^2 + I_3a_3^2) = K_0a_0^2,$$

$$||J_0||^2a_0^2 = I_1^2a_1^2 + I_2^2a_2^2 + I_3^2a_3^2.$$

We wish to find where these ellipsoids intersect the plane at infinity. For $a_0 = 0$ we get

$$\frac{1}{2}(I_1a_1^2 + I_2a_2^2 + I_3a_3^2) = 0,$$

$$I_1^2a_1^2 + I_2^2a_2^2 + I_3^2a_3^2 = 0.$$

Multiplying the first equation by I_1 and subtracting it from the second equation gives $I_1I_2(I_2 - I_1)a_2^2 + I_1I_3(I_3 - I_1)a_3^2 = 0$. Dividing both sides by $I_1I_2I_3$ gives

$$\frac{I_2 - I_1}{I_3}a_2^2 + \frac{I_3 - I_1}{I_2}a_3^2 = 0.$$

Repeating this process for I_2 and I_3 we get

$$\frac{I_1 - I_3}{I_2}a_1^2 + \frac{I_2 - I_3}{I_1}a_2^2 = 0,$$

$$\frac{I_3 - I_2}{I_1}a_3^2 + \frac{I_1 - I_2}{I_3}a_1^2 = 0.$$

Therefore the points at infinity

are

$$\left(0 : \pm\sqrt{\frac{I_3(I_3-I_2)}{I_1(I_2-I_1)}} : \pm i\sqrt{\frac{I_3(I_3-I_1)}{I_2(I_2-I_1)}} : 1\right), \tag{5}$$

where we assume that $I_3 > I_2 > I_1$ (if necessary we change the numbering of the coordinates). We know that, counting multiplicity, the ellipsoids have 4 common points at infinity, as on every other plane. Therefore as there are four unique points on the intersection of the ellipsoid, two pairs of conjugated points, the intersection is non-singular at infinity.

We now wish to see whether the vector field has poles or zeroes on the elliptic curve. From the equation for the vector field in lemma 1.11 it is clear that on the finite part this vector field has no poles. We now wish to find out what happens at infinity. We first write the general tangent vector in homogeneous coordinates. We can create a tangent vector at a point $(1 : a_1 : a_2 : a_3)$ by multiplying the derivative gained from Euler's equation of motion by an element ϵ such that $\epsilon^2 = 0$. This gives us

$$\left(1 : a_1 + \epsilon\frac{I_2 - I_3}{I_1}a_2a_3 : a_2 + \epsilon\frac{I_3 - I_1}{I_2}a_3a_1 : a_3 + \epsilon\frac{I_1 - I_2}{I_3}a_1a_2.\right) \tag{6}$$

The last coordinate is $a_3 + \epsilon\frac{I_1-I_2}{I_3}a_1a_2 = a_3\left(1+\epsilon\frac{I_1-I_2}{I_3}\frac{a_1a_2}{a_3}\right)$ the inverse is $\frac{1}{a_3}\left(1-\epsilon\frac{I_1-I_2}{I_3}\frac{a_1a_2}{a_3}\right) = \left(\frac{1}{a_3}-\epsilon\frac{I_1-I_2}{I_3}\frac{a_1a_2}{a_3^2}\right)$, as $(1+\epsilon)(1-\epsilon) = 1-\epsilon^2 = 1$ and therefore $(1+\epsilon)^{-1} = (1-\epsilon)$. We multiply the entries of (6) by this inverse to get

$$\begin{pmatrix} \frac{1}{a_3} - \epsilon\frac{I_1-I_2}{I_3}\frac{a_1a_2}{a_3^2} : \\ \frac{a_1}{a_3} + \epsilon\left(\frac{I_2-I_3}{I_1}a_2 - \frac{I_1-I_2}{I_3}\frac{a_1^2a_2}{a_3^2}\right) : \\ \frac{a_2}{a_3} + \epsilon\left(\frac{I_3-I_1}{I_2}a_1 - \frac{I_1-I_2}{I_3}\frac{a_1a_2^2}{a_3^2}\right) : \\ 1 \end{pmatrix}$$

We switch to the affine plane where the last coordinate is one and define $b_0 = \frac{1}{a_3}$, $b_1 = \frac{a_1}{a_3}$ and $b_2 = \frac{a_2}{a_3}$ we have the vector

$$\begin{pmatrix} b_0 + \epsilon\frac{I_1-I_2}{I_3}b_1b_2 \\ b_1 + \epsilon\left(\frac{I_2-I_3}{I_1}\frac{b_2}{b_0} - \frac{I_1-I_2}{I_3}\frac{b_1^2b_2}{b_0}\right) \\ b_2 + \epsilon\left(\frac{I_3-I_1}{I_3}\frac{b_1}{b_0} - \frac{I_1-I_2}{I_3}\frac{b_1b_2^2}{b_0}\right) \end{pmatrix}.$$

The first coordinate clearly does not have a pole. We re-write the epsilon part of the second and third coordinates to

$$\frac{b_2}{b_0}\left(\frac{I_2-I_3}{I_1} - \frac{I_1-I_2}{I_3}b_1^2\right),$$

$$\frac{b_1}{b_0}\left(\frac{I_3 - I_1}{I_2} - \frac{I_1 - I_2}{I_3}b_2^2\right).$$

The points at infinity are those where $b_0 = 0$. Therefore, as the points at infinity of the intersection are of multiplicity 1, both $b_1/b_0, b_2/b_0$ have a simple pole at infinity along the curve, because the points at infinity have multiplicity 1 as there are four of them shown in equation (5). Furthermore $\left(\frac{I_2 - I_3}{I_1} - \frac{I_1 - I_2}{I_3}b_1^2\right) = \left(\frac{I_3 - I_1}{I_3} - \frac{I_1 - I_2}{I_3}b_2^2\right) = 0$ at infinity, simply substituting in the points at infinity from equation (5) will show this. In other words they have a zero at infinity of multiplicity at least one. Therefore the products

$$\frac{b_2}{b_0}\left(\frac{I_2 - I_3}{I_1} - \frac{I_1 - I_2}{I_3}b_1^2\right),$$

$$\frac{b_1}{b_0}\left(\frac{I_3 - I_1}{I_3} - \frac{I_1 - I_2}{I_3}b_2^2\right)$$

do not have a pole at $b_0 = 0$ as b_1/b_0 and b_2/b_0 have a zero here of degree at most one. Hence the vector field does not have a pole at infinity.

This shows that $E(J_0, K_0)$ is of genus 1, as a non-trivial vector field bestaat without poles exists on it. This means that as it is a non-singular projective curve it is indeed an Elliptic curve.

Thus, the vector field has no poles on the elliptic curve. We can trivialize the tangent bundle of the elliptic curve by using left translation with the group law. After trivialization the vector field is a global regular function. Any global regular function on a genus one curve such as $E(K_0, J_0)$ must be constant. As such the vector field is translation invariant under the group law of the elliptic curve for any choice of 0.

□

Theorem 2.2 implies that the group law of the flow of the vector field coincides with that of the elliptic curves. Hence, the group law of the flow of the vector field v is algebraic as the group laws of the elliptic curves are algebraic. This is interesting as the flow itself is not algebraic.

2.2. Points at infinity.

In this subsection we will show that the difference between the points at infinity of the elliptic curves is of order 2. The points at infinity of an elliptic curve are in general like any other points on the elliptic curve and therefore it is a special property that their differences are all of order 2. This fact has no further bearing on the rest of this thesis aside from the fact that it highlights some other special properties of the elliptic curves.

We will need two preliminary lemmas

LEMMA 2.3. *A morphism of elliptic-curves is either surjective or constant.*

PROOF. Theorem 5.12 [**9**]. □

LEMMA 2.4. *the automorphism group of an elliptic curve is the semi-direct product of the homomorphisms and translations of the curve when seen as a group.*

PROOF. This follows from [**5**] III.3.4, III.3.7. □

EXAMPLE 2.5. As an example of lemma 2.4 we get

$$\mathrm{Aut}(E(K_0,J_0)) = \mathrm{Aut}(E(K_0,J_0),0) \rtimes (E(K_0,J_0),0)$$

for any choice of zero.

THEOREM 2.6. *Let $K_0 \in R, J_0 \in R^3$ be such that $E(K_0,J_0)$ is an elliptic curve. Then for any choice of $\mathbf{0}$ on the elliptic curve the differences of the points at infinity are of order* 2.

PROOF. First we note that the maps

$$\begin{aligned} s_0 &: (x_0 : x_1 : x_2 : x_3) \mapsto (-x_0 : x_1 : x_2 : x_3) \\ s_1 &: (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : -x_1 : x_2 : x_3) \\ s_2 &: (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1 : -x_2 : x_3) \\ s_3 &: (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1 : x_2 : -x_3) \end{aligned}$$

are automorphisms of the $\mathbf{P}^3(\mathbf{C})$. As $E(K_0,J_0)$ is mapped onto itself it holds that $s_0,s_1,s_2,s_3 \in \mathrm{Aut}(E(K_0,J_0))$. Furthermore any three of these will generate the same group of automorphisms, as the concatenation of three different ones will be equal to the last one. Moreover s_0,s_1,s_2,s_3 all have order 2.

Recall from lemma 2.4 that the automorphism group of an elliptic curve is the semi-direct product of the homomorphisms and translations of the curve when seen as a group i.e. $\mathrm{Aut}(E(K_0,J_0)) = (E(K_0,J_0),\mathbf{0}) \rtimes \mathrm{Aut}(E(K_0,J_0),\mathbf{0})$ for any choice of zero.

We now want to show that there is only one endomorphism of $(E(K_0,J_0),\mathbf{0})$ of order 2. Consider the endomorphism ring of an elliptic curve $(E(K_0,J_0),\mathbf{0})$. We note that $\forall f \in \mathrm{End}(E(K_0,J_0),\mathbf{0})$ we have that $f \neq 0 \Rightarrow f$ surjective by lemma 2.3. Therefore for any two non-zero elements $f,g \in \mathrm{End}(E(K_0,J_0),\mathbf{0})$ both f,g are surjective. Hence $fg := f \circ g$ is a surjective map and as such fg is not zero. This proves that $\mathrm{End}(E(K_0,J_0),\mathbf{0})$ is a domain. Now we note that for any endomorphism $f \in \mathrm{End}(E(K_0,J_0),\mathbf{0})$ such that $f^2 = id$ then we have $f^2 - id = 0$ so $(f-id)(f+id) = 0$ (here we use that $id, -id \in Z(\mathrm{End}(E(K_0,J_0),\mathbf{0}))$ which implies $f = -id$ or $f = id$ as $\mathrm{End}(E(K_0,J_0),\mathbf{0})$ is a domain. Therefore $-id$ is the only element of order 2 in $\mathrm{End}(E(K_0,J_0),\mathbf{0})$, as id is of order 1.

Now, by Proposition III.3.7 from [**5**], for every complex elliptic curve there is a $\alpha \in \mathbf{C}$ such that $(E(K_0,J_0),0)$ is isomorphic to $\mathbf{C}/(\mathbf{Z}+\alpha\mathbf{Z})$, as groups. We note that $\mathbf{C}/(\mathbf{Z}+\alpha\mathbf{Z})$ has order 2 elements $\overline{\alpha/2}, \overline{1/2}, \overline{\alpha/2+1/2}$, because the order 2 elements are precisely those a such that $a \neq 0 \in \mathbf{C}/(\mathbf{Z}+\alpha\mathbf{Z})$ and $2a \in (\mathbf{Z}+\alpha\mathbf{Z})$.

As we have seen in the previous section the elliptic curve $E(K_0,J_0)$ only has 4 points at infinity. Now choose the zero of $E(K_0,J_0)$ to be one of the 4 elements at infinity, call it a. We see that the subgroup of $\mathrm{Aut}(E(K_0,J_0))$ generated by s_0,s_1,s_2 only has elements of order 1 and 2. Now we note that s_0 is not the identity and maps 0 to 0, so it must be $-id$. Therefore the elements invariant under this map are of order 1 or 2. This means that all elements at infinity apart from a are of order 2 for this choice of zero. Therefore the (non-trivial) differences of any of these elements are of order 2. Therefore this holds for any choice of zero.

Finally we note that $(E(K_0, J_0), 0)$ only has 4 elements of order at most 2. All of the differences of two of these elements will therefore have order 2 for any choice of 0 for $E(K_0, J_0)$.

$\square$

REMARK 2.7. Analogously to theorem 2.6 we could prove that the difference of any two points on the same standard planes, those being

$$\{(0 : x_1 : x_2 : x_3)\}, \{(x_0 : 0 : x_2 : x_3)\}, \{(x_0 : x_1 : 0 : x_3)\}, \{(x_0 : x_1 : x_2 : 0)\},$$

will be of order 2.

The fact that these differences all have order 2 is unusual, just like the fact that the vector field v is translation invariant. We can, however, use this proof for any automorphism $s \in \mathrm{Aut}(E)$ of order 2 which stabilizes 4 elements, for any elliptic curve E.

3. The group law on $\mathrm{Lie}(G) \times G$

In this section we will prove that for any K_0, J_0 such that $E(K_0, J_0)$ is an elliptic curve there is a unique algebraic group law on $M(K_0, J_0)$ for any choice of zero such that the flow is translation invariant with respect to this group law. We have previously, subsection 2.1, shown that we have such a group law on $E(K_0, J_0)$. We now wish to extend this group law to $M(K_0, J_0)$. We will achieve this by showing that $M(K_0, J_0) \to E(K_0, J_0)$ is a circle bundle. After complexification and projectivization this circle bundle becomes a $\mathbf{C}^*$ bundle of degree 0. We will then see that we have a unique group law on the fibers of this map, for any choice of zero, under which the vector field is translation invariant. This group law can then be combined with the group law of $E(K_0, J_0)$ to form a group law on $M(K_0, J_0)$ as a whole.

3.1. The group law.

In this subsection we will focus on extending $M(K_0, J_0)$ and $E(K_0, J_0)$ over a complex projective version of $\mathrm{Lie}(G) \times G$. In particular we will expand $M(K_0, J_0)$ into a $\mathbf{C}^*$-bundle Γ over the extention of $E(K_0, J_0)$. This will help us then describe the group law on this Γ with some nice properties that will help us in combining it with the vector field in the following subsection.

DEFINITION 3.1. For any manifold N a circle bundle is a manifold $\mathcal{N}$ with a projection $p : \mathcal{N} \to N$ such that for every $n \in N$ there is an open subset $U \subset N$ such that $p^{-1}(U) \cong U \times S^1$.

In algebraic geometry over $\mathbf{R}$ we have the analogous notion, and with S^1 replaced by $\mathbf{C}^*$ we even have the analogous notion of $\mathbf{C}^*$-bundle.

PROPOSITION 3.2. *The manifold $M(K_0, J_0)$, as defined in 1.15, is a circle bundle over $E(K_0, J_0)$, as defined in 1.16.*

PROOF. Take $(a, g) \in M(K_0, J_0)$ we note that $a \in E(K_0, j_0)$ and $J_0 = gj(a)$. Furthermore $j(a) = (I_1 a_1, I_2 a_2, I_3 a_3)^T$ by definition has the same norm as J_0. Therefore $g \in G = SO(V)$ satisfies $g(I_1 a_1, I_2 a_2, I_3 a_3)^T = J_0$.

For any $\gamma \in SO(V)$ such that $\gamma(J_0) = J_0$ we have that $\gamma g(a) = \gamma(J_0) = J_0$. Furthermore $\{\gamma \in SO(V) : \gamma(J_0) = J_0\}$ is isomorphic to $SO(\mathbf{R}^2)$ as group. We can make an isomorphism by choosing a orthonormal basis $\mathcal{B}$ in V with $J_0 \in \mathcal{B}$ then $SO(\mathbf{R})$ is isomorphic to $SO(\mathrm{Span}(B \setminus \{J_o\}))$. Furthermore as $SO(\mathrm{Span}(B \setminus \{J_o\})$ does not depend on the choice of basis vectors down to orientation. Therefore if we choose the orientation to be positive around J_0 we have a natural isomorphism from $SO(\mathbf{R})$ to the stabilizer of J_0 denoted $\mathrm{Stab}((J_0))$.

Finally $SO(\mathbf{R}^2)$ can be seen as S^1 with added group law. Therefore the fiber of a point of $E(K_0, J_0)$ under the projection $M(K_0, J_0) \to E(K_0, J_0)$ is isomorphic to S^1 and therefore $M(K_0, J_0)$ is a circle bundle over an elliptic curve. □

We now wish to construct a rational section of the projection $M(K_0, J_0) \to E(K_0, J_0)$ by considering the projection of two bigger objects. This section will be central to extending $M(K_0, J_0)$ to a $\mathbf{C}^*$ bundle over an elliptic curve.

Without loss of generality we fix the $J_0 \in \mathbf{R}^3$ to be e_1.

Definition 3.3. Define

- $E(J_0) := \{r \in \mathbf{R}^3 : ||r|| = ||J_0||\} = \{r \in \mathbf{R}^3 : ||r||^2 = 1\}$
- $M'(J_0) = \{(g, r) \in G \times \mathrm{E}(J_0) : gJ_0 = r\}$
- $M(J_0) = \{(g, r) \in G \times \mathrm{E}(J_0) : gJ_0 = r\}$.

Now we have a projection $\pi : M'(J_0) \to E(J_0)$. Analogous to proposition 3.2 $M'(J_0)$ is a circle bundle over $E(J_0)$. Furthermore $M(J_0)$ is isomorphic to $M'(J_0)$ as variety over $E(J_0)$ by $\iota : M'(J_0) \to M(J_0) : (g, r) \mapsto (g^{-1}, r)$, this means that finding a section of $M(J_0)$ with divisor D is equivalent to finding a section of $M'(J_0)$ with the same dvisor D.

If we complexify this circle bundle we get a $\mathbf{C}^*$ bundle instead by the following proposition.

Proposition 3.4. *The complexification of* $S^1(\mathbf{R})$ *is* $\mathbf{C}^*$.

Proof. A real circle is given by $\{(a, b) \in \mathbf{R} : a^2 + b^2 = 1\}$. Its complexification is therefore given by $\{(a, b) \in \mathbf{C} : a^2 + b^2 = 1\}$. In $\mathbf{C}$, however, we can rewrite this to $\{(a, b) \in \mathbf{C} : (a + bi)(a - bi) = 1\}$. For any such a, b the element $a + bi \in \mathbf{C}^*$ furthermore for any $\alpha \in \mathbf{C}^*$ there is a $a, b \in \mathbf{C}$ such that $a + bi = \alpha$, $(a + bi)(a - bi) = 1$. Therefore $\{(a, b) \in \mathbf{C} : (a + bi)(a - bi) = 1\} = \mathbf{C}^*$ □

The degree of a $\mathbf{C}^*$-bundle can be conveniently determined over complex projective curves. This degree is defined as the degree of the divisor of any rational section of the projection map; this does not depend on the choice of rational section because the degree of the divisor of any rational function is zero. Therefore we now wish to construct an algebraic section of this map.

Definition 3.5. A rational section of a projection of schemes $\mathcal{N} \to N$ is a rational map $s : U \to \mathcal{N}$, for a dense open subset U of N, such that $p \circ s = id_U$.

Example 3.6. Now we will give two examples of rational sections P and Q.

To make a rational section of $p : M(J_0) \to E(J_0)$ we need to find a way to make a rotation matrix that maps a vector e_1 to $v = (a, b, c)^T \in \mathbf{R}^3$ such that $||v||^2 = 1$. To make such a matrix we can find two reflections one which maps v to e_1 and vice-versa and one that maps e_1 to itself and then composing them.

To do this is it easier to change the coordinate system from the canonical basis to $(e_1, v, v \times e_1)$. In this basis

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the reflection that maps e_1 to v and vice versa. Furthermore

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

is the reflection in the plane spanned by e_1, v. By composing these maps we get the desired matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Now we rewrite this matrix in terms of the canonical basis. We get

$$P := \begin{pmatrix} 1 & a & 0 \\ 0 & b & c \\ 0 & c & -b \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & a & 0 \\ 0 & b & c \\ 0 & c & -b \end{pmatrix}^{-1} = \tag{7}$$

$$\begin{pmatrix} a & -\dfrac{(a^2-1)b}{b^2+c^2} & -\dfrac{(a^2-1)c}{b^2+c^2} \\ b & -\dfrac{ab^2+c^2}{b^2+c^2} & -\dfrac{(a-1)bc}{b^2+c^2} \\ c & -\dfrac{(a-1)bc}{b^2+c^2} & -\dfrac{ac^2+b^2}{b^2+c^2} \end{pmatrix}.$$

This matrix is defined if and only if $b^2 + c^2 \neq 0$.

We can make a different second section by constructing a matrix that maps switches v and e_2 and then composing this with a matrix that swaps e_2 to e_1. To make a matrix that swaps v and e_2 we follow the same principles as when making the rotation matrix that for v and e_1.

If we switch the basis to $\{e_2, v, v \times e_2\}$ the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the reflection that maps e_2 to v and vice versa. Furthermore

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

is the reflection in the plane spanned by e_2, v. By composing these maps we get the desired matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Now we rewrite this matrix in terms of the canonical basis and compose with

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

to rotate e_2 to e_1. We get

$$Q := \begin{pmatrix} 0 & a & c \\ 1 & b & 0 \\ 0 & c & -a \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & a & c \\ 1 & b & 0 \\ 0 & c & -a \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} =$$

$$\begin{pmatrix} a & -\dfrac{a^2b+c^2}{a^2+c^2} & \dfrac{(ab-a)c}{a^2+c^2} \\ b & -\dfrac{ab^2-a}{a^2+c^2} & \dfrac{\left(b^2-1\right)c}{a^2+c^2} \\ c & -\dfrac{(ab-a)c}{a^2+c^2} & \dfrac{bc^2+a^2}{a^2+c^2} \end{pmatrix}.$$

Note that when we replace the real numbers a, b, c in the definitions of these section by complex numbers, we get a section of the complexification $\overline{p} : \overline{M(J_0)} \to \overline{E(J_0)}$

DEFINITION 3.7. Let P, Q be the rational sections of $\overline{p} : \overline{M(J_0)} \to \overline{E(J_0)}$ given by:

1

$$P := \begin{pmatrix} a & -\dfrac{\left(a^2-1\right)b}{b^2+c^2} & -\dfrac{\left(a^2-1\right)c}{b^2+c^2} \\ b & -\dfrac{ab^2+c^2}{b^2+c^2} & -\dfrac{(a-1)bc}{b^2+c^2} \\ c & -\dfrac{(a-1)bc}{b^2+c^2} & -\dfrac{ac^2+b^2}{b^2+c^2} \end{pmatrix}.$$

2

$$Q := \begin{pmatrix} a & -\dfrac{a^2b+c^2}{a^2+c^2} & \dfrac{(ab-a)c}{a^2+c^2} \\ b & -\dfrac{ab^2-a}{a^2+c^2} & \dfrac{\left(b^2-1\right)c}{a^2+c^2} \\ c & -\dfrac{(ab-a)c}{a^2+c^2} & \dfrac{bc^2+a^2}{a^2+c^2} \end{pmatrix}.$$

The following expressions will be relevant in the remainder of this section

$$Q^{-1}P = Q^TP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{-a^2b^2+abc^2-ac^2-bc^2}{(a^2+c^2)(b^2+c^2)} & \dfrac{-a^2bc-ab^2c+abc-c^3}{(a^2+c^2)(b^2+c^2)} \\ 0 & \dfrac{a^2bc+ab^2c-abc+c^3}{(a^2+c^2)(b^2+c^2)} & \dfrac{-a^2b^2+abc^2-ac^2-bc^2}{(a^2+c^2)(b^2+c^2)} \end{pmatrix}.$$

DEFINITION 3.8. Let $\overline{E(K_0, J_0)}$ denote the complexification of the elliptic curve $E(K_0, J_0)$.

Let $\mathcal{E}(K_0, J_0)$ be the projectivization of $\overline{E(K_0, J_0)}$.

We define Γ to be the $\mathbf{C}^*$-bundle on $\mathcal{E}(K_0, J_0)$ whose projection to $\overline{E(K_0, J_0)}$ is $\overline{M(K_0, J_0)}$ and whose rational section P has order 0 at all 4 points at infinity.

LEMMA 3.9. *The $\mathbf{C}^*$-bundle Γ has degree 0. For any choice of η on Γ, we have a group law on Γ, which is compatible with the group law of $\mathcal{E}(K_0, J_0)$ and the action of $\mathbf{C}^*$, such that η is the unit element.*

PROOF. Using the complexification we can now determine the zeroes and poles of P from definition 3.7 by calculating the zeroes and poles of $Q^{-1}P$ when considered as element of $\mathbf{C}(a, b, c)/(a^2 + b^2 + c^2 - 1)$, the function field corresponding to the complex variety describing the surface $a^2 + b^2 + c^2 = 1$. In this setting we can finally talk about the divisor of the rational section P, which will lead us to the degree of the $\mathbf{C}^*$-bundle. Furthermore the projective closure of of this surface in $\mathbf{P}^3(\mathbf{C})$, given by $a^2 + b^2 + c^2 = d^2$, is isomorphic to $\mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$, as we will see below.

We define

$$\alpha = \frac{-a^2b^2 + abc^2 - ac^2 - bc^2}{(a^2 + c^2)(b^2 + c^2)}, \beta = \frac{a^2bc + ab^2c - abc + c^3}{(a^2 + c^2)(b^2 + c^2)}.$$

Then

$$\alpha + i\beta = \frac{-a^2b^2 + abc^2 - ac^2 - bc^2}{(a^2 + c^2)(b^2 + c^2)} + i\frac{a^2bc + ab^2c - abc + c^3}{(a^2 + c^2)(b^2 + c^2)}$$

is a function in $\mathbf{C}(a, b, c)/(a^2 + b^2 + c^2 - 1)$. For ease of use we will refer to the numerators and denominators of $\alpha, \beta, \alpha + i\beta, \alpha - i\beta$ as they appear here as the numerators and denominators of $\alpha, \beta, \alpha + i\beta, \alpha - i\beta$, even though these are rational functions and this way of writing them is not unique. We can do this because to calculate the multiplicity of the poles and zeroes this representation will do. The denominators of both $\alpha + i\beta$ and $\alpha - i\beta$ are

$$(a^2 + c^2)(b^2 + c^2) = (a + ic)(a - ic)(b + ic)(b - ic).$$

By using the fact that $Q^{-1}P$ is orthogonal we see

$$(\alpha + i\beta)(\alpha - i\beta) = \alpha^2 + \beta^2 = 1.$$

Therefore the numerator of neither $\alpha + i\beta$ or $\alpha - i\beta$ has a zero where the denominators, both equal to

$$(a^2 + c^2)(b^2 + c^2) = (a + ic)(a - ic)(b + ic)(b - ic),$$

are not zero either.

Note that for a different choice of Q we would not have had the factor $a^2 + c^2$ and as we are interested in the divisor of the rational section P and not of Q the zeroes of the denominators due to $(a^2 + c^2)$ are irrelevant. The factor $b^2 + c^2$ is zero when $b = ic$ or $b = -ic$. On $E(J_0)$ this also means that $a^2 + b^2 + c^2 = 1$ so $a^2 + 0 = 1$ which gives $a = \pm 1$. Therefore there are 4 lines in $E(J_0)$ where $b^2 + c^2$ is zero given by $a = \pm 1, b = \pm ic$. Define $L_{1,\pm}$ to be the lines given by $a = 1, b = \pm ic$ and $L_{-1,\pm}$ to be the lines given by $a = -1, b = \pm ic$ respectively. Now we will show that the valuation of $b^2 + c^2$ at any of these lines is 1. To do this we need to show that $b^2 + c^2$ splits into four lines over the variety $\mathbf{C}[a, b, c]/(a^2 + b^2 + c^2)$. Note that $b^2 + c^2 = (b + ic)(b - ic)$ therefore we can consider $b + ic, b - ic$ separately instead. We consider the ideal generated by $(b + ic), (a^2 + b^2 + c^2 - 1)$ over $\mathbf{C}[a, b, c]$. We substitute $b = -ic$ in $a^2 + b^2 + c^2 = 1$ this gives $a^2 - 1 = 0$ over $\mathbf{C}[a, b, c]/(b - ic) \cong \mathbf{C}[a, c]$.

Then $\mathbf{C}[a,c]/(a^2-1) \cong \mathbf{C}[a,c]/((a+1)\times(a-1)) \cong \mathbf{C}[c]\times\mathbf{C}[c]$. This means that on the variety $b+ic=0$ gives two lines $L_{\pm 1,+}$ and as such $b+ic$ and so b^2+c^2 have valuation 1 at these lines. The analogue holds for $b-ic=0$.

Now to determine the zeroes and poles of $\alpha+i\beta$ all we have to do is calculate the numerators of both $\alpha+i\beta$ and $\alpha-i\beta$ on these lines. That is to say we substitute $c=\pm ib$ into the "numerators" of $\alpha\pm i\beta$ as seen in the table below.

TABLE 1. Numerators of $\alpha,\beta,\alpha\pm i\beta$.

	$a=1,b=ic$	$a=1,b=-ic$
α	$c^2+ic^3-c^2-ic^3=0$	$c^2-ic^3-c^2+ic^2=0$
β	$ic^2-c^3-ic^2+c^3=0$	$ic^2-c^3+ic^2+c^3=0$
$\alpha+i\beta$	0	0
$\alpha-i\beta$	0	0
	$a=-1,b=ic$	$a=-1,b=-ic$
α	$c^2-ic^3+c^2-ic^3=2(c^2-ic^3)$	$c^2i+ic^3+c^2+ic^3=2(c^2+ic^3)$
β	$ic^2+c^3+ic^2+c^3=2(ic^2+c^3)$	$-ic^2+c^3+-ic^2+c^3=2(-ic^2+c^3)$
$\alpha+i\beta$	0	$4(c^2+ic^3)$
$\alpha-i\beta$	$4(c^2-ic^3)$	0

From the table 1 we first conclude that both numerators of $\alpha+i\beta$ and $\alpha-i\beta$ have zeroes of degree at least one at the lines $a=1,b=\pm ic$. Recall that $(\alpha+i\beta)(\alpha-i\beta)=1$, hence, their valuations are additive inverses of each other on any co-dimension one sub-variety. We get $V_{L_{1,\pm}}(\alpha+i\beta)+V_{L_{1,\pm}}(\alpha-i\beta)=0$ and $V_{L_{1,\pm}}(\alpha+i\beta)\geq 0$ and $V_{L_{1,\pm}}(\alpha-i\beta)\geq 0$, as the numerators have a zero of degree at least zero and the denominators have a zero of degree one, which gives $V_{L_{1,\pm}}=0$. Here $V_L(f)$ denotes the valuation of f at the subvarieties L.

Secondly we determine that $\alpha-i\beta$ has a pole of order 1 at $a=-1,b=ic$ and $\alpha+i\beta$ has a pole of order 1 at $a=-1,b=-ic$. Thus $V_{L_{-1,\pm}}(\alpha+i\beta)=\pm 1$.

Now we need to see what happens at infinity. The variety at infinity is given by $a^2+b^2+c^2=0$, which is irreducible over $\mathbf{C}$. Then as $(\alpha+i\beta)=(\alpha-i\beta)^{-1}=\overline{(\alpha-i\beta)}$ we have that, for D the divisor on the variety given by $C[a,b,c]/(a^2+b^2+c^2)$, the equality $D(\alpha-i\beta)=-D(\alpha+i\beta)=\overline{D(\alpha+i\beta)}$ holds. Because $a^2+b^2+c^2$ only has real coefficients $\overline{V(a^2+b^2+c^2)}=V(a^2+b^2+c^2))$ this means that the valuation at $V(a^2+b^2+c^2)$ of $\alpha+i\beta$ is zero as the valuations at $\overline{V(a^2+b^2+c^2)},V(a^2+b^2+c^2))$ are additive inverses.

This means that the divisor of P as rational section to the $\mathbf{C}^*$-bundle is

$$+L_{-1,+}-L_{-1,-}.$$

We consider the map

$$\gamma:\mathbf{P}^1(\mathbf{C})\times\mathbf{P}^1(\mathbf{C})\to\overline{E(J_0)}:$$

$$((x_1:x_2),(y_1:y_2))\mapsto(x_2y_2+x_1y_1:x_2y_1+x_1y_2:i(x_2y_1-x_1y_2):x_2y_2-x_1y_1),$$

here $\overline{E(J_0)}$ denotes the completion of $E(J_0)$ in $\mathbf{P}^3(\mathbf{C})$ i.e. as sets

$$\overline{E(J_0)}=\{(u:v:w:x)\in\mathbf{P}^3(\mathbf{C}):u^2+v^2+w^2=x^2\}.$$

The map γ is an isomorphism of projective varieties as it is the Segre embedding composed with a bijective linear map.

We now wish to calculate the inverse images of $L_{-1,\pm}$ under this map.

The line that corresponds to $L_{-1,+}$ in $\overline{E(J_0)}$ is given by $u = -x, v = iw$. The first condition, $a = -d$, implies that $x_2y_2 + x_1y_2 = -x_2y_2 + x_1y_2$ which is equivalent to $x_2y_2 = 0$. The second condition implies $x_2y_1 + x_1y_2 = -x_2y_1 + x_1y_2$ which is equivalent to $x_2y_1 = 0$ because y_1 and y_2 cannot simultaneously be zero we see that these are the points such that $x_2 = 0$ i.e. $L_1 = (1:0) \times \mathbf{P}^1(\mathbf{C}) \subset \mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$.

Doing the same for the line that corresponds to $L_{-1,-}$ in $\overline{E}(J_0)$ given by $u = -x, v = -iw$. We find that $a = -d$ implies $x_2y_2 = 0$ as before. Furthermore $v = -iw$ necessitates $x_1y_2 = 0$. Therefore the inverse image of $L_{-1,-}$ is $L_2 := \mathbf{P}^1(\mathbf{C}) \times (1:0)$.

The inverse image under γ of any elliptic curve on $\overline{E(J_0)}$ has bi-degree $(2,2)$. Therefore the inverse image of the completion of $E(K_0, J_0)$ for any $K_0 \in \mathbf{R}_{>0}$ has bi-degree $(2,2)$ in $\mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$ as well. As the bi-degree of the divisor is $(1,-1)$ the degree of the intersection with the elliptic curve is $2 \cdot 1 + 2 \cdot -1 = 0$. We have therefore shown that the circle bundle has degree zero over $\overline{E(K_0, J_0)}$.

We note that Γ is the $\mathbf{C}^*$ bundle corresponding to the divisor $+L_{-1,+} - L_{-1,-}$ over $\overline{E(K_0, J_0)}$ and let p be the projection from Γ to $\mathcal{E}(K_0, J_0)$. By the above Γ can be extended to a line bundle, by adding zero, of degree zero. For any element $e \in \mathcal{E}(K_0, J_0)$ and any element $\eta \in \Gamma$ which lies over this point with respect to the rational section then we have a group law on Γ for which η is the identity element by proposition I.9.2 of [**6**]. Furthermore this group law is compatible with that of $(E(K_0, J_0), e)$ under the projection and the action of C^* on the fibers of the projection map to $\mathcal{E}(K_0, J_0)$ of e.

$\square$

The isomorphism from the first part of 3.9 respects the projection in the sense that p' in conjunction with the isomorphism and $p : M(K_0, J_0) \to E(K_0, J_0)$ are identical on the complexification of $M(K_0, J_0)$.

Lemma 3.9 shows that there exist a group law on Γ. We now show that this group law is unique for given choice of unit element η.

Lemma 3.10. *Let E be an elliptic curve over $\mathbf{C}$. Then let $\mathcal{G}$ be a $\mathbf{C}^*$-bundle over E, with action of $\mathbf{C}^*$ on the fibers of the projection $p : \mathcal{G} \to E$. Then for every choice of unit element $\eta \in \mathcal{G}$ there is not more than one group law $\cdot$ on $\mathcal{G}$ for which p is a group homomorphism and is compatible with the structure of the $\mathbf{C}^*$-bundle.*

Proof. Assume $\mathcal{G}$ has 2 group laws $\cdot_1, \cdot_2$ with the same unit element $\eta \in \mathcal{G}$. Furthermore assume both group laws are compatible with the $\mathbf{C}^*$-bundle structure and for which p is a group homomorphism. Note that $\mathbf{C}^* \to \mathcal{G}$ is the kernel of p as group homomorphism. Furthermore for all $\lambda \in \mathbf{C}^*$ and $g \in \mathcal{G}$ we have $\lambda \cdot_2 g = \lambda g = \lambda \cdot_1 g$, where λg denotes the action of $\mathbf{C}^*$ on $\mathcal{G}$. Now for any $g \in \mathcal{G}$ consider the map $f_g : \mathcal{G} \to \mathcal{G} : x \mapsto g^{-1} \cdot_2 (g \cdot_1 x)$, where g^{-1} denotes the inverse of g with respect to $\cdot_2$. These maps are automorphisms of G as complex algebraic variety. We see that $f_g(x) \in p^{-1}p(x)$ as $p(g^{-1} \cdot_2 (g \cdot_1 x)) = p(x)$. Furthermore as $p(g^{-1} \cdot_2 (g \cdot_1 x)) = p(x)$ the isomorphism f_g is actually an element of $\mathrm{Aut}_{\mathbf{C}^*-\mathrm{bundle}/E}(\mathcal{G}) = \mathbf{C}^*$. For any $g \in \mathcal{G}$ we have $g^{-1} \cdot_2 (g \cdot_1 \eta)) = g^{-1} \cdot_2 g$ because η is the unit element for $\cdot_1$ and $g^{-1} \cdot_2 g = \eta$ because η is the unit element for $\cdot_2$. Therefore $f_g(\eta) = \eta$ and the only element of $\mathrm{Aut}_{\mathbf{C}^*-\mathrm{bundle}/E}(\mathcal{G})$ with this property is id. Therefore f_g is $id \in \mathrm{Aut}_{\mathbf{C}^*-\mathrm{bundle}/E}(\mathcal{G})$. This implies that the two group laws are identical. □

Therefore by 3.10 for the group law on Γ is unique up to choice of η.

COROLLARY 3.11. *For real $\eta \in \Gamma$ the group law restricts to the circle bundle over $E(K_0, J_0)$.*

PROOF. If η is real then it is invariant under complex conjugation and therefore the group law is invariant under complex conjugation as well, as such the group law restricts to the circle bundle over $E(K_0, J_0)$ □

We have now constructed an extension of $M(K_0, J_0) \to E(K_0, J_0)$ in the form of $\Gamma \to \mathcal{E}(K_0, J_0)$, where Γ is a $\mathbf{C}^*$-bundle over the complex projective elliptic curve $\mathcal{E}(K_0, J_0)$. Furthermore we have shown that there is exactly one group law on Γ for any choice of unit element η in Γ and we have shown that for $\eta \in \mathcal{E}(K_0, J_0)$ the group law of the elliptic curve with that base point is the projection of this group law on Γ.

3.2. The vector field on Γ.

In this subsection we will show that the vector field v extends to a vector field on Γ. We will then proceed to show that any vector field on Γ, which is invariant under $\mathbf{C}^*$ and the group law of $\mathcal{E}(K_0, J_0)$, is translation invariant under the group law of Γ.

Let $\overline{E(K_0, J_0)}$ be the complexification of $E(K_0, J_0)$ and let $\Gamma' \subset \Gamma$ be the inverse image of $\overline{E(K_0, J_0)}$ under the map $\Gamma \to \mathcal{E}(K_0, J_0)$.

THEOREM 3.12. *The vector field v, as defined in 1.6, extends to Γ.*

We will prove this theorem at the end of this section after proving a preliminary lemma. Furthermore we will for the remainder of this subsection often refer to $\Gamma \setminus \Gamma'$ as the part of Γ at infinity.

On $\mathrm{Lie}(G) \times G$ the vector field v is given by $(a, g)' = (-j^{-1}(a \cdot j(a)), g \cdot a)$ as in lemma 1.6. We then rewrite this as a vector field on $\mathbf{R}^3 \times G$ via the isomorphism $j : \mathrm{Lie}(G) \to \mathbf{R}^3$, which gives the vector field

$$(u, g)' = (-j^{-1}(u) \cdot u, g \cdot j^{-1}(u)). \tag{8}$$

For any starting point $(a, g) \in \mathrm{Lie}(G) \times G$ and image $(u, g) \in \mathbf{R}^3 \times G$ the image of its level surface determined by the length of the angular momentum is given by $S = \{w \in \mathbf{R}^3 : ||w|| = ||u||\}$.

At this point we define a section of the projection $\Gamma \to \mathcal{E}(K_0, J_0)$. Define $P(w) := (w, p(w))$ wheret $p(w)$ is defined as the element of G obtained by first reflecting in the plane perpendicular to $w - gu$ (that is, the plane spanned by $w + gu$ and $w \times gu$), and then in the plane perpendicular to $w \times gu$. This rotation maps $w \times gu$ to $gu \times w$, w to gu and vice versa. It is therefore the same as the rotation around $w + gu$ by π. Note that this is analogous to the definition of P from example . Let S' be S without the zeroes of $b^2 + c^2$, this way S' is a subset of the domain of the section P defined in equation (7). Now let $M := \{(w, h) : w \in S, hw = gu\}$.

Then let $p : S' \to M$ be such that $P(w) = (w, p(w)) \in M$. Note that, as before, we have a S^1 action, now as rotation group around the axis gu: for γ in G such that $\gamma{\cdot}gu = gu$ and (w, h) in M we have $(w, \gamma{\cdot}h)$ in M.

To eventually determine the poles of v on Γ we wish to show that the extension of v from $M(K_0, J_0)$ to Γ is equal to a vector field on Γ with no poles at infinity. For $w \in S$ we have $(w, p(w)) \in M$, therefore we have that $P_*(-j^{-1}(w) \cdot w)$ and $v((w, p(w)))$ in $T_M(w, p(w))$. These vector fields on M have the same first coordinate, namely w', because $w' = -j^{-1}(w) \cdot w$ and P followed by the projection from M to S' is the identity on S'. Let α be a non-trivial vector field on Γ given by the $\mathbf{C}^*$ action on Γ. Then there exists a unique f such that

$$v((w, p(w))) - P_*(-j^{-1}(w) \cdot w) = f(w)\alpha(w, p(w), \tag{9}$$

as on both sides of the equation we have vector fields which are zero along $P(S')$.

We can now introduce the preliminary lemma.

Lemma 3.13. *The rational function f such that*

$$v((w, p(w))) - P_*(-j^{-1}(w) \cdot w) = f(w)\alpha(w, p(w)) \tag{10}$$

has no poles on $\Gamma \setminus \Gamma'$.

Proof. First we elaborate on the terms of (10).

Starting off with the second term of $\alpha(w, p(w))$. The function α represents the infinitesimal rotation around gu as $\mathbf{C}^*$ is the stabilizer of gu. The infinitesimal rotation around gu is given by $id + \epsilon \cdot \frac{1}{||gu||} gu \times -$. Then to obtain the second term of $\alpha(w, p(w))$ we need only multiply by $p(w)$ on the left. This gives us $(\alpha(w, p(w)) = p(w) + \frac{\epsilon}{||u||} \times ((gu) \times -)$. The second term of this equation, dividing out the ϵ, is

$$\alpha(w, p(w)) = \frac{1}{||u||}(gu) \times p(w) \cdot - = \frac{1}{||u||} p(w)(p(w)h^{-1}gu) \times - = p(w)(\frac{1}{||u||}(w) \times -).$$

Now we consider $P_*(-j(w), w)$. When we consider the infinitesimal movement along v at w the second term of $P_*(-j(w), w)$ is $p(w - \epsilon j^{-1}(w)w)$. The reflection in a line $\mathbf{R}a$ in V is $y \mapsto -y + \frac{\langle y, a\rangle}{\langle a, a\rangle} a$. Now with $a = w + gu + \epsilon j^{1-}(w)w$ we get

$$y \mapsto -y + \frac{2\langle y, w + gu\rangle + 2\epsilon\langle y, j^{-1}(w)w\rangle}{\langle w + gu, w + gu\rangle + 2\epsilon\langle j^{-1}(w)w, gu\rangle} \cdot (w + gu + \epsilon j^{-1}(w)w) =$$

$$-y + \frac{2\langle y, w + gu\rangle}{\langle w + gu, w + gu\rangle}(w + gu)$$

$$+ \frac{\epsilon}{\langle w + gu, w + gu\rangle^2}(\langle w + gu, w + gu\rangle(2\langle y, w + gu\rangle \cdot j^{-1}(w)w \tag{11}$$

$$+2\langle y, j^{-1}(w)w\rangle \cdot (w + gu)) - 2\langle y, w + gu\rangle \cdot 2\langle j^{-1}(w)w, gu\rangle \cdot (w + gu))$$

Finally we expand the second term of $v((w, p(w)))$, which is $p(w) \cdot j^{-1}(w)$, this follows directly from the equation of the vector field given in equation 8. When we write this in terms an infinitesimal change of a point for the sake of homogeneity we get $p(w) + \epsilon p(w)j^{-1}(w) = p(w)(id + \epsilon j^{-1}(w)$. Now (10) becomes

$$p(w) \cdot (id + \epsilon j^{-1}(w)) - p(w - \epsilon j^{-1}(w)w) = f(w) \cdot \epsilon \frac{(gu) \times (p(w)-)}{||u||}.$$

Multiplying the left and right side by $p(w)$, using $p(w)^2 = id$ and $p(w)gu = w$, and dividing out by ϵ we get

$$j^{-1}(w) - p(j^{-1}(w)w) = f(w)\frac{w \times -}{||v||} \tag{12}$$

Note that when substituting in $y = (w + gu) \times j^{-1}(w)w$ into (11) we get zero as $(w + gu) \times j^{-1}(w)w$ is perpendicular to both $(w + gu)$ and $j^{-1}(w)w$. Then for $y = (w + gu) \times j^{-1}(w)w$ (12) becomes

$$j^{-1}(w) \cdot ((w + gu) \times (j^{-1}(w)w) = \frac{f(w)}{||u||}(-||w||^2 j^{-1}(w)w + w \times (gu \times (j^{-1}(w)w).$$

This gives

$$f(w) = \frac{||u||j^{-1}(w) \cdot ((w + gu) \times (j^{-1}(w)w)}{(-||w||^2 j^{-1}(w)w + w \times (gu \times (j^{-1}(w)w)}. \tag{13}$$

As $||u|| = ||w||$ we can assume that $||w|| = 1$ without loss of generality.

Recall that

$$j^{-1}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 & -a_3/I_3 & a_2/I_2 \\ a_3/I_3 & 0 & -a_1/I_1 \\ -a_2/I_2 & a_1/I_1 & 0 \end{pmatrix}.$$

Now for $w = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $gu = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ equation(13) becomes:

$$f(w) = \frac{\big(b_3a_3 + a_3^2\big)I_1I_2 + \big(a_2b_2 + b_2^2\big)I_1I_3 + \big(a_1b_1 + b_1^2\big)I_2I_3}{(a_1b_1 + a_2b_2 + a_3b_3 + 1)I_1I_2I_3}.$$

Now we need to see whether f has poles at infinity. Applying j to the points describes by (5) we get $(a_0 : a_1 : a_2 : a_3) = \left(0 : \pm\sqrt{\dfrac{I_1I_3(I_3 - I_2)}{I_2 - I_1}} : \pm i\sqrt{\dfrac{I_2I_3(I_3 - I_1}{I_2 - I_1}} : I_3\right)$. We verify that these are indeed the correct points by caluclating the kinetic energy and angular momentum which are

$$a_1^2/I_1 + a_2^2/I_2 + a_3^2/I_3 = \frac{I_3(I_3 - I_2) - I_3(I_3 - I_1 + I_3(I_2 - I_1)}{I_2 - I_1} = 0$$

and

$$a_1^2 + a_2^2 + a_3^2 = \frac{I_1I_3(I_2 - I_3) - I_2I_3(I_3 - I_1) + I_3^2(I_2 - I_1)}{I_2 - I_1} = 0$$

respectively.

Now on to $\mathbf{P}^3$. Let the vector (a_1, a_2, a_3) in $\mathbf{R}^3$ correspond to the vector $(1 : a_1 : a_2 : a_3) = 1/a_3 : a_1/a_3 : a_2/a_3 : 1$ in $\mathbf{P}^3$. Now let $c_1 = 1/a_3, c_2 = a_1/a_3, c_3 = a_2/a_3$ then

$$f(w) = \frac{I_1I_2 \cdot \frac{1}{c_1}(\frac{1}{c_1} + b_3) + I_3I_1\frac{c_3}{c_1}(\frac{c_3}{c_1} + b_2) + I_2I_3\frac{c_2}{c_1}(\frac{c_2}{c_1} + b_1)}{I_1I_2I_3(1 + b_1\frac{c_2}{c_1} + b_2\frac{c_3}{c_1} + z\frac{1}{c_1})} =$$

$$\frac{I_1I_2(1+c_1b_3)+I_3I_1c_3(c_3+c_1b_2)+I_2I_3c_2(c_2+c_1b_1)}{I_1I_2I_3(c_1+b_1c_2+b_2c_3+b_3)\cdot c_1}.$$

The variable c_1 has a simple zero at ∞: $\left(0, \pm\sqrt{\frac{I_1(I_3-I_2}{I_3(I_2-I_1)}}, \pm i\sqrt{\frac{I_2(I_3-I_1)}{I_3(I_2-I_1)}}\right)$.
Now the numerator f in ∞ is given by:

$$I_1I_2 - I_3I_1\frac{I_2(I_3-I_1)}{I_3(I_2-I_1)} + I_2I_3\frac{I_1(I_3-I_2}{I_3(I_2-I_1)} = 0$$

.

Now we take a closer look at the denominator. It is a regular function on $\mathbf{R}^3$ equal to $I_1I_2I_3\left(\pm b_1\sqrt{\frac{I_1(I_3-I_2)}{I_3(I_2-I_1)}} \pm b_2 i\sqrt{\frac{I_2(I_3-I_1)}{I_3(I_2-I_1)}} + b_3\right)$. There are only 16 values of gv such that this equation is zero. Therefore we can easily choose starting values such that f has no poles on ∞. Both $P_*(-j^{-1}(w)\cdot w)$ and $f(w)(0, \alpha(w, p(w))$ do not have poles at infinity.

$\square$

We are now ready to prove theorem 3.12.

Proof. By lemma 3.13 we have

$$v((w, p(w))) - P_*(-j^{-1}(w)\cdot w) = f(w)(0, \alpha(w, p(w)),$$

where f is a rational function without poles on $\Gamma \setminus \Gamma'$. Therefore

$$v((w, p(w))) = f(w)(0, \alpha(w, p(w)) + P_*(-j^{-1}(w)\cdot w).$$

Furthermore as both $(0, \alpha(w, p(w)), P_*(-j^{-1}(w)\cdot w)$ have no poles on $\Gamma\setminus\Gamma'$, neither does $v((w, p(w)))$. $\square$

Now the only thing left to show is that the vector field v on $\mathrm{Lie}(G)\times G$ is invariant under the group law of Γ. Note that v is a vector field on Γ which is C^*-invariant, as it is G invariant. Let v_1 be the projection of v on $\mathcal{E}(K_0, J_0)$. Note that v_1 is translation-invariant under the group action $\mathcal{E}(K_0, J_0)$ as it has no poles. Define w to be the translation invariant vector field on Γ such that $w(\eta) = v(\eta)$ and let $\overline{w}$ be the projection of w to $\mathcal{E}(K_0, J_0)$. We wish to show that $v = w$ as this is equivalent to v being translation invariant under the group action of Γ. We define $u = v - w$. We get $u(\eta) = v(\eta) - w(\eta) = 0$.

We now introduce a lemma that lets us equate u to a global section of the trivial line bundle.

Lemma 3.14. *Let V be a complex variety. Let B be a $\mathbf{C}^*$-bundle over V, with the projection $p : B \to V$. Then we have an isomorphism between the complex line bundle L of $\mathbf{C}^*$-invariant vector fields on the fiber of $B \to V$ and the trivial line bundle on V.*

Proof. For all $b \in B$ we have an isomorphism of varieties $o_b : \mathbf{C}^* \to p^{-1}p(b) : z \mapsto z\cdot b$. Therefore for any $\mathbf{C}^*$ invariant vector field on $\mathbf{C}^*$ we get a $\mathbf{C}^*$ invariant vector field of $p^{-1}p(b)$. We wish to show that this vector field is independent of the choice of $b' \in p^{-1}p(b)$. For any $\lambda \in \mathbf{C}^*$ consider the function $\lambda\cdot : \mathbf{C}^* \to \mathbf{C}^*$ defined by multiplication by λ. Then on the tangent spaces we get $a \in T_1(\mathbf{C}^*) = \mathbf{C}$ gets mapped to $\lambda\cdot a \in T_\lambda(\mathbf{C}^*) = \mathbf{C}$, which gives the same vector field as the vector field is translation invariant. Now for any $b' \in p^{-1}p(b)$, $\lambda \in \mathbf{C}^*$ such that $\lambda b = b'$ the following diagram commutes

$$\begin{tikzcd} \mathbf{C}^* \arrow[r, "o_b"'] \arrow[d, "\lambda^{-1}"'] & p^{-1}p(b) \arrow[d, "id"] \\ \mathbf{C}^* \arrow[r, "o'_b"] & p^{-1}p(b), \end{tikzcd}$$

as $z \mapsto a \cdot b' = z(\lambda b) = \lambda z b = \lambda o_b(z)$. Now consider a translation invariant vector field v on $\mathbf{C}^*$. It is mapped onto itself by λ^{-1} that means that $o_b(v) = id(o_b(v)) = o_{b'}(v)$. Therefore the vector field is indeed independent of the choice of $b' \in p^{-1}p(b)$.

This means that we have a natural isomorphism from translation invariant vector fields on $\mathbf{C}^*$ to translation invariant vector fields on B in the direction of the fibers of p. Therefore as there is a natural isomorphism between the translation invariant vector fields on $\mathbf{C}^*$ and elements of $T_1(\mathbf{C}^*) = \mathbf{C}$ we get an isomorphism $V \times \mathbf{C} \to L$. $\square$

Using 3.14 we can consider u to be a global section of the trivial line bundle. Therefore as $u(\eta) = 0$ we get $u = 0$ this implies that $v = w$ as desired and as such v is translation invariant under the group action of Γ. Whats more by 3.11 for the right choice of unit $\eta \in \Gamma$ this group law will be real.

The existence of this algebraic group law is remarkable as the flow of the vector field is not algebraic. In general polynomial vector fields on $\mathrm{Lie}(G) \times G$ will not have this property.

4. Summary and Conclusion

This thesis describes the motion of a rigid rotating body. Such a description is already known from literature e.g. [**3**] and [**1**]. As in the literature we have described the defining properties of a rotating body, such as the path, kinetic energy and angular momentum, and we have shown that the orientation of such an object and its derivative, the angular velocity, can be seen as an element of the product of the Lie group $SO(\mathbf{R}^3)$ and its Lie algebra. It is known (see the references above) that the trajectory of the angular velocity, in the rotating reference frame of the object, hence in the Lie algebra, called the flow, moves along an elliptic curve, and that (which is really miraculous) the flow is translation invariant under the group law of this elliptic curve for any choice of zero.

The main result of this thesis concerns the flow on the product of the Lie group and its Lie algebra. This flow moves along an algebraic circle bundle with the elliptic curve as its base. Furthermore we can define an algebraic group law on this circle bundle which projects to the group law on the elliptic curve and agrees with the left action of the circle on the circle bundle, such that the flow is translation invariant under this group law. We also consider this as a miracle, because we have only a very long and computational proof for it.

As a result we now have an addition operation on the fibres of the energy momentum map from the state space of the object to $\mathbf{R}^4$, for which the flow is translation invariant. Furthermore this group law can be expressed exclusively in polynomial functions.

Bibliography

[1] V. Arnold. *Mathematical Methods of Classical Mechanics*. Springer, 1989.

[2] M. Audin. *Spinning Tops*. The press syndicate of the univesity of cambridge, 1999.

[3] R. Cushman and L. Bates. *Global aspects of Classical Intregrable Systems*. Birkhauser Basel, 1997.

[4] G. Fowles and G. Cassiday. *Analytical mechanics*. Thomson, 2005.

[5] J. Milne. *Elliptic Curves*. BookSurge Publishers, 2008.

[6] J. S. Milne. Abelian varieties (v2.00), 2008. Available at www.jmilne.org/math/.

[7] C. Norman. *Undergraduate Algebra*. Oxford university press, 1986.

[8] K. Riley, M. Hobson, and S. Bence. *Mathematical Methods for Physics and Engineering*. Cambridge university press, 2010.

[9] A. Sutherland. Lecture notes 5. `http://math.mit.edu/classes/18.783/2013/LectureNotes5.pdf`, 2013.

Index