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Two-variable zeta functions and their properties through covers

Master thesis

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Introduction

For a geometrically integral projective scheme X over a finite field $\mathbb{F}_q$, the (one-variable) zeta function $Z(X;t)$ is given by a power series counting effective 0-cycles on X. These zeta functions discuss geometrical problems in the flavor of arithmetic by considering effective 0-cycles on schemes. The central study of zeta functions is about the Weil conjecture and their cohomological interpretations.

In this thesis, we study properties of two-variable zeta functions of one-dimensional geometrically integral smooth projective $\mathbb{F}_q$-schemes. It turns out that parts of the Weil conjecture hold for two-variable zeta functions: the rationality and the functional equation. The analogue of the Riemann hypothesis for two-variable zeta functions is still unknown, but we shall prove that the numerator of a two-variable zeta function $Z(u,t)$ is irreducible in $\mathbb{C}(u)[t]$.

Weil cohomology theories give a good interpretation of the Weil conjecture of one-variable zeta functions. However, in general, such cohomology theories do not exist for two-variable zeta functions. It follows from the absolute irreducibility and the divisibility of numerators of zeta functions in covers, which we shall show in chapter two.

The last chapter focuses on two-variable zeta functions of specific curves. More precisely, we shall give an algorithm to compute two-variable zeta functions of hyperelliptic curves and some lower genus non-hyperelliptic curves, by using their one-variable zeta functions as well as the Riemann-Roch theorem. We also discuss some special curves, whose one-variable zeta functions have numerators that can be factored as products of numerators of zeta functions of lower genus curves. One important case is the curve with split Jacobian: the numerator of its one-variable zeta function can be represented by a product of polynomials, which are of form $1+at+qt^2$. These polynomials are numerators of zeta functions of elliptic curves that can be covered by the curve. Note all these product relations do not hold for two-variable zeta functions.

This thesis contributes to the study of divisors and divisor classes of a given curve, through zeta functions and covers. It generalizes the study of one-variable (Hasse-Weil) zeta functions.

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Contents

Chapter 1

Definition and absolute irreducibility

1.1 Definition of the two-variable zeta function

Our purpose in this section is to give the definition for the two-variable zeta function of one-dimensional geometrically integral smooth projective schemes over a finite field. It can be computed using the Riemann-Roch theorem. We begin with the definition of the one-variable zeta function. Let $\mathbb{F}_q$ be a finite field with $\#\mathbb{F}_q = q$ elements, and let k be an algebraic closure of $\mathbb{F}_q$.

Definition 1.1.1. Let X be a geometrically integral projective scheme over $\mathbb{F}_q$, let $n := \dim(X)$. Then we have a corresponding scheme $\bar{X} = X \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} k$ over k. For every $m \geq 1$, let $N_m = |\bar{X}(\mathbb{F}_{q^m})|$ be the number of points of $\bar{X}$ that are defined over $\mathbb{F}_{q^m}$. The one-variable zeta function of X is defined as

$$Z(X;t) = \exp\left(\sum_{m=1}^{\infty} N_m \frac{t^m}{m}\right).$$

Note that by the above definition, the one-variable zeta function is a power series with rational coefficients: $Z(X;t) \in \mathbb{Q}[[t]]$. It could be also given equivalently as the generating function counting effective 0-cycles on X, as stated in [12]. We have the following proposition:

Proposition 1.1.1. *For each geometrically integral projective scheme X over $\mathbb{F}_q$, we have*

$$Z(X;t) = \prod_{x \in X_{cl}} (1 - t^{\deg(x)})^{-1} = \sum_{\alpha} t^{\deg(\alpha)} = \sum_{n \geq 0} a_n t^n,$$

where X_{cl} is the set of closed points of X, $\deg(x) := [\mathbb{F}_q(x) : \mathbb{F}_q]$, with $\mathbb{F}_q(x)$ the residue field of the local ring $\mathscr{O}_{X,x}$, and the first sum is over all effective 0-cycles on X. In the second sum, a_n is the number of effective 0-cycles that have degree n. In particular, $Z(X;t) \in \mathbb{Z}[[t]]$.

It is well-known that the Weil conjecture [12] holds for the one-variable zeta function.

Proposition 1.1.2. ***Weil conjecture.***
Let $Z(X;t)$ be the one-variable zeta function of X as before, if X is moreover smooth, then we have:

(1) ***(Rationality.)*** *$Z(X;t)$ is a rational function, namely, $Z(X;t) \in \mathbb{Q}(t)$. More precisely, $Z(X;t)$ can be written as a finite product*

$$Z(X;t) = \prod_{i=0}^{2n} P_i(t)^{(-1)^{i+1}} = \frac{P_1(t)\dots P_{2n-1}(t)}{P_0(t)P_2(t)\dots P_{2n}(t)},$$

where $n = \dim(X)$ and $P_i(t) \in \mathbb{Z}[t]$.

(2) ***(Functional equation.)*** *The one-variable zeta function satisfies*

$$Z(X;(q^n t)^{-1}) = \pm q^{nE/2} t^E Z(X;t),$$

where E is the Euler characteristic of X.

(3) ***(Analogue of Riemann hypothesis.)*** *Write $Z(X;t)$ as a rational function in (1), then*

$$P_0(t) = 1-t, P_{2n}(t) = 1-q^n t,$$

$$P_i(t) = \prod_j (1-\alpha_{i,j}t), \textit{ for some complex numbers } \alpha_{i,j} \textit{ and } 1 \le i \le 2n-1.$$

These $\alpha_{i,j}$ satisfy $|\alpha_{i,j}| = q^{i/2}$ for all $1 \le i \le 2n-1$ and all j.

The special case for X of dimension one is the generating function counting effective divisors on X, which is discussed in [6]. If moreover X is smooth, the number of effective divisors which are linearly equivalent to a given effective divisor D is equal to $(q^{\ell(D)-1})/(q-1)$, where $\ell(D)$ is the dimension of the Riemann-Roch space $\mathscr{L}(D)$ of D. Hence, we have

$$Z(X;t) = \sum_{\deg(D)\ge 0} t^{\deg(D)} = \sum_{\deg(|D|)\ge 0} \frac{q^{\ell(D)}-1}{q-1} t^{\deg(D)},$$

where sums run over all divisors respective divisor classes with non-negative degrees. Based on this fact, Pellikaan [15] defines the two-variable zeta function $Z(X;t,u)$ of X as a generating function:

Definition 1.1.2. Let X be a geometrically integral smooth projective scheme over $\mathbb{F}_q$, of dimension one, define the two-variable zeta function $Z(X;u,t)$ as follows:

$$Z(X;u,t) = \sum_{n=0}^{\infty}\sum_{s=1}^{\infty} b_{n,s}\frac{u^s-1}{u-1}t^n,$$

where $b_{n,s}$ is the number of divisor classes of X of degree n and dimension s (note in this context, the dimension of a divisor D always means the dimension $\ell(D)$ of its associated Riemann-Roch space). Note $Z(X;u,t) \in \mathbb{Z}[[u,t]]$.

The class number of a smooth projective scheme is defined by $h := \#\mathrm{Pic}^n(X)$, for any $n \geq 0$.

Remark 1.1.3. Let X be a geometrically integral smooth projective scheme over $\mathbb{F}_q$.

(1) Let g be the genus of X, and let D be a divisor on X, then the dimension $\ell(D)$ of the divisor class that is linearly equivalent to D is equal to $n+1-g$ if $\deg(D) = n > 2g-2$, by the Riemann-Roch theorem. Thus, for any $n > 2g-2$, $b_{n,n+1-g} = h$ and $b_{n,s} = 0$ for all $s \geq 0$ with $s \neq n+1-g$.

(2) Let K be the canonical divisor and $|K|$ be the canonical divisor class, then the Riemann-Roch theorem shows that $\deg(K-D) = (2g-2) - \deg(D)$ and $\ell(K-D) = \ell(D) + g - 1 - \deg(D)$ for any divisor D on X. Hence, the operation $D \mapsto K-D$ is an involution on the set $\{D \in \mathrm{Pic}(X) \mid 0 \leq \deg(D) \leq 2g-2\}$, and $b_{n,s} = b_{2g-2-n,s-n-1+g}$.

Example 1.1.1. *(1) Let $X = \mathbb{P}^1$ be the projective line, we have $g(X) = 0$. The degree of the canonical divisor K is $2g-2 = -2$, hence, by the Riemann-Roch theorem, every effective divisor D on X has dimension $\ell(D) = \deg(D) + 1 = n+1$. Also, the class number of $\mathbb{P}^1$ is $h = 1$. Therefore, we have that the two-variable zeta function of $\mathbb{P}^1$ is*

$$Z(\mathbb{P}^1;u,t) = \sum_{n=0}^{\infty}\frac{u^{n+1}-1}{u-1}t^n = \sum_{n=0}^{\infty}(u^n+\cdots+u+1)t^n = \frac{1}{(1-t)(1-ut)}.$$

(2) Let E be an elliptic curve over $\mathbb{F}_q$ with N rational points. Since $g(E) = 1$, by Riemann-Roch, $\ell(D) \geq 1$, for any divisor D of degree 1. Divisor classes of degree one on X can be represented by an effective divisor, so they are represented by rational points on X, and they are mutually not linearly equivalent. Thus, there are N divisor classes of degree one, so N is the class number. Moreover, we have $g(E) = 1$, so $b_{n,n} = N$ and $s = n$ for all $n \geq 1$ by Remark 1.1.3.(1). Also, note that there is only one effective divisor class of degree zero. Hence, the two-variable zeta function of an elliptic curve with N rational points is

$$Z(E;u,t) = 1 + \sum_{n=1}^{\infty} N\frac{u^n-1}{u-1}t^n = \frac{1+(N-1-u)t+ut^2}{(1-t)(1-ut)}.$$

1.2 The Weil conjecture

Pellikaan [15] proved that parts of the Weil conjecture hold for two-variable zeta functions:

Proposition 1.2.1. *Let X be a geometrically integral smooth projective scheme over $\mathbb{F}_q$, of dimension one, and suppose the genus of X is g, then the two-variable zeta function of X satisfies:*

(1) ***Rationality.***

$$Z(X;u,t) = \frac{P(u,t)}{(1-t)(1-ut)},$$

where $P(u,t) \in \mathbb{Z}[u,t]$ is of degree $2g$ in t and degree g in u.

(2) ***Functional equation.***

$$Z(X;u,t) = u^{g-1}t^{2g-2}Z(X;u,(tu)^{-1}).$$

(3) If we write $P(u,t) = \sum_{i=0}^{2g} p_i(u)t^i$ with $p_i(u) \in \mathbb{Z}[u]$, then $p_0(u) = 1, p_{2g}(u) = u^g$, and

$$p_{2g-i}(u) = u^{g-i}p_i(u), \text{for all } 0 \leq i \leq 2g.$$

Furthermore, $P(u,1) = h$.

The analogue of Riemann hypothesis for two-variable zeta functions is still unknown, but Naumann [13] has proved the (absolute) irreducibility of the numerator $P(u,t)$. We shall prove it in a slightly different way later in the section 1.4.

Proof of Proposition 1.2.1. (1) The genus zero case follows from Example 1.1.1.(1), so we assume that $g \geq 1$, and hence, $2g-2 \geq 0$. In order to show that the rationality and the functional equation hold, we define two series:

$$F(u,t) = \sum_{\deg(D)>2g-2} u^{\ell(D)}t^{\deg(D)} - \sum_{D\in \mathrm{Pic}(X):\deg(D)\geq 0} t^{\deg(D)},$$

$$G(u,t) = \sum_{0\leq \deg(D)\leq 2g-2} u^{\ell(D)}t^{\deg(D)}.$$

Then we have

$$(u-1)Z(u,t) = F(u,t) + G(u,t).$$

By Remark 1.1.3, we have

$$F(u,t) = hu^{1-g}\sum_{n>2g-2}(ut)^n - h\sum_{n\geq 0} t^n = \frac{hu^g t^{2g-1}}{1-ut} - \frac{h}{1-t}, \tag{1.1}$$

where h is the class number. Now we consider the functional equation, note

$$u^{g-1}t^{2g-2}F(u,(tu)^{-1}) = \frac{ht^{-1}}{1-t^{-1}} - \frac{hu^{g-1}t^{2g-2}}{1-(ut)^{-1}} = F(u,t),$$

$$u^{g-1}t^{2g-2}G(u,(ut)^{-1}) = \sum_{0\le\deg(D)\le 2g-2} u^{\ell(D)+g-1-\deg(D)}t^{2g-2-\deg(D)}.$$

By Remark 1.1.3.(2), the operation $D \mapsto K-D$, with K the canonical divisor, gives an involution on the set $\{D \in \mathrm{Pic}(X) : 0 \le \deg(D) \le 2g-2\}$, so we have

$$u^{g-1}t^{2g-2}G(u,(ut)^{-1}) = \sum_{0\le\deg(D)\le 2g-2} u^{\ell(K-D)}t^{\deg(K-D)} = \sum_{0\le\deg(D')\le 2g-2} u^{\ell(D')}t^{\deg(D')}.$$

And hence it holds

$$u^{g-1}t^{2g-2}G(u,(ut)^{-1}) = G(u,t).$$

Therefore, we have

$$\begin{aligned}
Z(u,t) =& \frac{F(u,t)+G(u,t)}{u-1} \\
=& \frac{u^{g-1}t^{2g-2}(F(u,(ut)^{-1})+G(u,(ut)^{-1}))}{u-1} = u^{g-1}t^{2g-1}Z(u,(ut)^{-1}).
\end{aligned}$$

This finishes the proof of the functional equation of $Z(u,t)$.

(2) Regarding the rationality, it is clear that both $F(u,t)$ and $G(u,t)$ are rational functions, and the degree of $G(u,t)$ is $2g-2$ in t. Hence, $F(u,t)+G(u,t)$ is a rational function with denominator $(1-t)(1-ut)$, and the numerator $Q(u,t)$ of above sum is a polynomial of degree $2g$ in t. So we have the following:

$$(u-1)Z(u,t) = \frac{Q(u,t)}{(1-t)(1-ut)},$$

where $Q(u,t)$ is a polynomial in $\mathbb{Z}[u,t]$. By the definition of $F(u,t)$ and $G(u,t)$, we have $F(1,t)+G(1,t) = 0$, so $(u-1)$ is a factor of $F(u,t)+G(u,t)$ in $\mathbb{Q}[u,t]$, and hence a factor of $Q(u,t)$. Let $P(u,t) = Q(u,t)/(u-1)$, clearly, $P(u,t)$ is of degree $2g$ in t, then

$$Z(u,t) = \frac{P(u,t)}{(1-t)(1-ut)}.$$

(3) Suppose that $P(u,t) = \sum_{i=0}^{2g} p_i(u)t^i$ with $p_i(u) \in \mathbb{Z}[u]$, the functional equation implies that

$$P(u,t) = u^g t^{2g} P(u,(ut)^{-1}).$$

It then follows that $p_{2g-i}(u) = u^{g-i} p_i(u)$ for all $0 \leq i \leq 2g$. Now let $t = 0$, we have

$$F(u,0) + G(u,0) = -h + \sum_{D:\deg(D)=0} u^{\ell(D)}.$$

Since the only divisor class of degree zero and dimension one is 0, and other divisors of degree zero have dimension zero, we have $F(u,0)+G(u,0) = -h+u+(h-1) = u-1$. Hence, $Q(u,0) = u-1 = (u-1)P(u,0) = (u-1)p_0(u)$, so $p_0(u) = 1$, it immediately follows that $p_{2g}(u) = u^g p_0(u) = u^g$.
Regarding $P(u,1) = h$, substitute equality (1.1) in $F(u,t)+G(u,t) = (u-1)Z(u,t)$, we have

$$\begin{aligned}
P(u,t) =& Z(u,t)(1-t)(1-ut) = \frac{(1-t)(1-ut)}{(u-1)}(F(u,t)+G(u,t)) \\
=& \frac{(1-t)(1-ut)}{(u-1)} \left(\frac{hu^g t^{2g-1}}{1-ut} - \frac{h}{1-t} \right) + \frac{(1-t)(1-ut)}{(u-1)} G(u,t) \\
=& \frac{1}{(u-1)}(hu^g t^{2g-1}(1-t) - h(1-ut)) + \frac{(1-t)(1-ut)}{(u-1)} G(u,t).
\end{aligned}$$

Let $t = 1$, then it holds that $P(u,1) = -(-h) + 0 = h$.

□

1.3 Some properties of two-variable zeta functions

Let X be a one-dimensional geometrically integral smooth projective scheme over $\mathbb{F}_q$, of genus g, we define $Z(X;u,t)$ as before. We now consider the relation between $b_{n,s}$ and coefficients of the numerator $P(u,t)$ as a polynomial in t.
To start with, we consider the one-variable zeta function with $Z(t) = \sum_{i \geq 0} a_i t^i$, where a_i is the number of effective divisors of degree i on X. Thanks to Riemann-Roch, we have

$$a_i = \begin{cases} 0, & \text{for } \ i < 0. \\ h(q^{i+1-g} - 1)/(q-1), & \text{for } \ i > 2g-2. \end{cases}$$

For any $i > 2g$, we have

$$\begin{aligned}
a_i - (q+1)a_{i-1} + q a_{i-2} =& \frac{h(q^{i+1-g}-1)}{q-1} - \frac{h(q+1)(q^{i-g}-1)}{q-1} + \frac{hq(q^{i-1-g}-1)}{q-1} \\
=& \frac{hq^{i+1-g} - h - hqq^{i-g} - hq^{i-g} + hq + h + hqq^{i-1-g} - hq}{q-1} \\
=& 0.
\end{aligned}$$

Moreover, $a_i - (q+1)a_{i-1} + qa_{i-2} = 0$ holds trivially for all $i < 0$. Hence, it holds for any $i \notin \{0, 1, 2, \ldots, 2g\}$ that

$$a_i - (q+1)a_{i-1} + qa_{i-2} = 0. \tag{1.2}$$

Recall by the rational function of $Z(t)$, we have:

$$Z(t) = \frac{P(t)}{(1-t)(1-qt)}, \tag{1.3}$$

with $P(t) = p_0 + p_1 t + \cdots + p_{2g} t^{2g}$. In [5], Duursma stated without proof that there exists a relation between coefficients of $Z(t)$ and above $p_i(t)$s:

$$p_i = a_i - (q+1)a_{i-1} + qa_{i-2}.$$

The relation holds simply for the following reason. If we substitute it in $\frac{P(t)}{(1-t)(1-ut)}$ as a power series, then we have

$$\frac{P(t)}{(1-t)(1-ut)} = \left(\sum_{i=0}^{2g} p_i(u)t^i\right)\left(\sum_{i\geq 0} t^i\right)\left(\sum_{i\geq 0} (qt)^i\right) = \sum_{i\geq 0} a_i t^i = Z(u,t).$$

There is an analogous relation for $b_{n,s}$ and coefficients of $P(u,t)$ as a polynomial of t. With the Riemann-Roch theorem, one has that

$$b_{i,s} = \begin{cases} 0, & i < 0. \\ h, & i > 2g-2, \ s = i+1-g. \\ 0, & i > 2g-2, \ s \neq i+1-g. \end{cases}$$

Let $Z(X; u, t) = \frac{P(u,t)}{(1-t)(1-ut)}$ as a rational function, then these $p_i(u)$s are determined by $b_{i,s}$ for all i.

Proposition 1.3.1. *Let $P(u,t) = p_0(u) + p_1(u)t + \cdots + p_{2g}(u)t^{2g}$ with $p_i(u) \in \mathbb{Z}[u]$, then we have*

$$p_i(u) = \sum_{s\geq 1} b_{i,s}\frac{u^s - 1}{u-1} - (u+1)\left(\sum_{s\geq 1} b_{i-1,s}\frac{u^s-1}{u-1}\right) + u\left(\sum_{s\geq 1} b_{i-2,s}\frac{u^s-1}{u-1}\right). \tag{1.4}$$

Proof. The idea is the same as the above one-variable case. Note for any $s \geq 1$, we have $\frac{u^s-1}{u-1} = u^{s-1} + u^{s-2} + \cdots + u + 1$. Rewrite $Z(u,t)$ with $P(u,t) = p_0(u) + p_1(u)t + \cdots + p_{2g}(u)t^{2g}$, one has

$$Z(u,t) = \left(p_0(u) + p_1(u)t + \cdots + p_{2g}(u)t^{2g}\right)\left(\sum_{i\geq 0} t^i\right)\left(\sum_{i\geq 0} (ut)^i\right).$$

Substitute the relation in right-hand expansion, we get that it is identical with $Z(u,t)$ in sense of Definition 1.1.2. $\square$

Corollary 1.3.1. $P(u,1) = h$.

Proof. By Proposition 1.3.1, we have

$$
\begin{aligned}
P(u,1) =& \sum_{i=0}^{2g} p_i(u) \\
=& 1 + \left(\sum_{s\geq 1} b_{1,s} \frac{u^s-1}{u-1} - (u+1) \right) + \left(\sum_{s\geq 1} b_{2,s} \frac{u^s-1}{u-1} - (u+1) \left(\sum_{s\geq 1} b_{1,s} \frac{u^s-1}{u-1} \right) + u \right) \\
& + \cdots + \left(\sum_{s\geq 1} b_{2g,s} \frac{u^s-1}{u-1} - (u+1) \left(\sum_{s\geq 1} b_{2g-1,s} \frac{u^s-1}{u-1} \right) + u \left(\sum_{s\geq 1} b_{2g-2,s} \frac{u^s-1}{u-1} \right) \right) \\
=& \sum_{s\geq 1} b_{2g,s} \frac{u^s-1}{u-1} - u \sum_{s\geq 1} b_{2g-1,s} \frac{u^s-1}{u-1} \\
=& h \frac{u^{g+1}-1}{u-1} - uh \frac{u^g-1}{u-1} = h.
\end{aligned}
$$

□

Corollary 1.3.2. ***(Degree bound of** $p_i(u)$**.)*** *Let X as before, write its two-variable zeta functions as follows:*

$$
Z(X;u,t) = \frac{P(X;u,t)}{(1-t)(1-qt)} = \frac{\sum_{i=0}^{2g} p_i(u) t^i}{(1-t)(1-qt)}.
$$

Then we have that

$$
\deg_u(p_i(u)) \leq (i+1)/2, \ \textit{for all} \ \ 0 \leq i \leq 2g.
$$

Remark 1.3.1. Note that Naumann [13] obtained the bound $i/2+1$, so this corollary is an improvement of the result of Naumann.

Proof of Corollary 1.3.2. The proof is based on the following Clifford's theorem as well as Proposition 1.3.1. By Proposition 1.3.1, for $0 \leq i \leq 2g$, we have that

$$
p_i(u) = \sum_{s\geq 1} b_{i,s} \frac{u^s-1}{u-1} - (u+1) \left(\sum_{s\geq 1} b_{i-1,s} \frac{u^s-1}{u-1} \right) + u \left(\sum_{s\geq 1} b_{i-2,s} \frac{u^s-1}{u-1} \right).
$$

Thanks to Clifford's theorem (Theorem 1.3.2), we have that

$$
b_{i,s} = 0, \ \text{if} \ \ s > i/2+1.
$$

Hence, it holds

$$
\deg_u \left(\sum_{s\geq 1} b_{i,s} \frac{u^s-1}{u-1} \right) \leq i/2+1-1 = i/2,
$$

$$\deg_u\left((u+1)\left(\sum_{s\geq 1} b_{i-1,s}\frac{u^s-1}{u-1}\right)\right) \leq (i-1)/2+1-1+1 = (i+1)/2,$$

$$\deg_u\left(u\left(\sum_{s\geq 1} b_{i-2,s}\frac{u^s-1}{u-1}\right)\right) \leq (i-2)/2+1-1+1 = i/2.$$

It then follows that

$$\deg_u(p_i(u)) \leq \max(i/2, (i+1)/2, i/2) = (i+1)/2.$$

□

A divisor D is called special if $\ell(K-D) > 0$, Hartshorne [8] discusses the Clifford's theorem, which compares degrees with dimensions of effective special divisors:

Theorem 1.3.2. ***[Clifford]*** *Let D be an effective special divisor on one-dimensional geometrically integral smooth projective scheme X over an algebraically closed field k, then*

$$\ell(D) \leq \frac{1}{2}\deg(D)+1.$$

Moreover, equality holds if and only if either $D = 0$ or $D = K$ or X is hyperelliptic and D is a multiple of the unique g_2^1 on X. Note here g_2^1 means a divisor class of degree 2 and Riemann-Roch dimension 2.

Remark 1.3.3. (1) Clifford's theorem holds for schemes over an algebraic closure $k = \overline{\mathbb{F}_q}$ of $\mathbb{F}_q$, since $\ell(D)$ over k is the same as $\ell(D)$ over $\mathbb{F}_q$, it holds for schemes over $\mathbb{F}_q$.

(2) For any divisor D of X, let $n = \deg(D), s = \ell(D)$ be its degree and dimension, one has $s \leq n/2+1$, so $b_{n,s} = 0$ for all $s > n/2+1$.

1.4 Absolute irreducibility

Naumann [13] gives a criterion for irreducibility, by which we shall show the absolute irreducibility of the numerator of a two-variable zeta function. In fact, we have the following theorem:

Theorem 1.4.1. *Let X be geometrically integral smooth projective scheme X over $\mathbb{F}_q$ with $g \geq 1$, then the numerator $P(X;u,t)$ of a two-variable zeta function $Z(X;u,t)$ is irreducible in $\mathbb{C}(u)[t]$.*

Firstly, we give Naumann's criterion for irreducibility:

Lemma 1.4.1. *Let k be a field, let $F(x,y) \in k[x,y]$, and assume:*

(1) $F(x,y)$ is monic in y;

(2) the leading coefficient of $F(x,y)$ in x is irreducible in $k[y]$;

(3) there exist $\alpha, \beta \in k$ with $\beta \neq 0$ such that $F(x,\alpha) = \beta$.

Then $F(x,y)$ is irreducible in $k(x)[y]$.

In order to prove this criterion, we need Proposition 4.11 in [7] by Eisenbud:

Proposition 1.4.1. *Let $R \subset S$ be a ring extension, and suppose $f \in R[x]$ is a monic polynomial. Suppose we have the factorization $f = gh$ in $S[x]$, with g, h are monic, then the coefficients of g and h are integral over R.*

Proof of Lemma 1.4.1. We use contradiction to show the result. Assume that $F = fg$ in $k(x)[y]$ with $\deg(f)$ and $\deg(g)$ positive, and moreover, assume that f, g are monic. Thanks to Proposition 1.4.1, our assumption implies that coefficients of f and g are integral over $k[x]$. Moreover, $k[x]$ is integrally closed as it is a unique factorization domain, so we have that coefficients of f, g lie in $k[x]$, hence, $f, g \in k[x][y]$. So we may regard f, g as polynomials in $k[y][x]$. By condition (2), one of the leading coefficients of f and g is a unit in $k[y]$, so we may assume that the leading coefficient of f is in $k[y]^* = k^*$. In particular, $n := \deg_x f(x,y) = \deg_x f(x,\alpha)$. Note that $0 \neq \beta = f(x,\alpha)g(x,\alpha)$ by condition (3), so we get $n = 0$, thus $f \in k^*$, which contradicts our assumption. Therefore, $F(x,y)$ is irreducible in $k(x)[y]$. $\square$

Proof of Theorem 1.4.1. Let $F = \widetilde{P}(u,t) := t^{2g}P(u,t^{-1})$, it is a polynomial in $\mathbb{C}[u,t]$, if we show this polynomial F is irreducible in $\mathbb{C}(u)[t]$, then our target polynomial $P(u,t)$ would be also irreducible in $\mathbb{C}(u)[t]$, as we claimed before.
We apply the criterion to the polynomial F. The conditions (1) and (3) are easily verified by Proposition 1.2.1.(3), so it remains to verify the leading coefficient of F in u is irreducible in $\mathbb{C}[t]$. However, the leading coefficient of F can be evaluated by the following Proposition 1.4.2, which is enough to show the irreducibility of F, and hence, $P(u,t)$ is irreducible in $\mathbb{C}(u)[t]$. $\square$

Proposition 1.4.2. *Let $g \geq 1$, then $\widetilde{P}(u,t) = (1-t)u^g + O(u^{g-1})$, hence, the leading coefficient of $\widetilde{P}(u,t)$ in u is irreducible in $\mathbb{C}[t]$.*

Proof. Recall by Proposition 1.2.1.(3), if we write $P(u,t) = \sum_{i=0}^{2g} p_i(u)t^i$, then $p_0(u) = 1$, $p_{2g}(u) = u^g$, so we have

$$\widetilde{P}(u,t) = t^{2g}P(u,t^{-1}) = t^{2g} + p_1(u)t^{2g-1} + \cdots + p_g(u)t^g + up_{g-1}(u)t^{g-1} + \cdots + u^g t^0.$$

Since $\deg(P_i(u)) \leq (i+1)/2$, the leading term of $F = \widetilde{P}(u,t)$ is contained in $u^{g-1}p_1(u)t + u^g$, so it is enough to check the coefficient of $p_1(u)$. Thanks to Proposition 1.3.1, we have for $i = 1$,

$$p_1(u) = \left(b_{1,1} + b_{1,2}(u+1) + \sum_{s\geq 3} b_{1,s}\frac{u^s - 1}{u-1}\right) - (u+1).$$

By Remark 1.1.3.(2), we have $b_{1,s} = 0$ for all $s > 1/2+1$, so

$$p_1(u) = b_{1,1} - (u+1).$$

Therefore, $u^{g-1}p_1(u)t + u^g = (1-t)u^g + O(u^{g-1})$, so the leading coefficient of $\widetilde{P}(u,t)$ in u is irreducible in $\mathbb{C}[t]$.

□

Naumann has proved this proposition by using another method to compute $p_1(u)$ and $p_2(u)$ in [13]. However, Corollary 1.3.2 shows that computation of $p_2(u)$ is not necessary. Also, the proof above avoids Naumann's complicated computations.

Chapter 2

Cohomological interpretation of zeta functions

Weil cohomology theory gives a good interpretation of the Weil conjecture of one-variable zeta functions. We shall study zeta functions further by considering Weil cohomology theories in this chapter. On the one hand, such a cohomological interpretation for one-variable zeta functions shows the divisibility between one-variable zeta functions via finite, surjective and separable morphisms. On the other hand, we shall show that Weil cohomology theories do not exist for the two-variable zeta function, in the section 2.3.

2.1 Weil cohomology theories

We recall the Weil cohomology theory as in [4] and [12] before applying it to zeta functions. We shall work over a fixed algebraically closed field k, and all schemes are defined over k in this section.

Definition 2.1.1. A Weil cohomology theory H^* with coefficients in a field K of characteristic zero is given by the following data:

(D1) For every geometrically integral smooth projective scheme X over k, a graded commutative algebra $H^*(X)$ over K. The graded commutative algebra $H^*(X)$ is given by $H^*(X) = \bigoplus_{n\in\mathbb{Z}} H^n(X)$, whose grading is a direct sum decomposition of K-vector spaces. The multiplication $H^*(X) \times H^*(X) \to H^*(X)$, $(\alpha, \beta) \mapsto \alpha \cup \beta$ is called the cup product, which is K-bilinear, and graded commutative algebra means that $\alpha \cup \beta = (-1)^{\deg(\alpha)\deg(\beta)} \beta \cup \alpha$ for every homogeneous elements α and β.

(D2) For every morphism of geometrically integral smooth projective k-schemes $f : X \to Y$, a pull-back map $f^* : H^*(Y) \to H^*(X)$. Note id^* associated to the identity map on X is the identity map: $H^*(X) \to H^*(X)$. It is a K-algebra map preserving the grading and the cup product structure.

(D3) For every geometrically integral smooth projective scheme X over k, a K-linear trace map $\mathrm{Tr} : H^{2\dim(X)}(X) \to K$.

(D4) For every geometrically integral smooth projective scheme X over k, and for every closed geometrically integral subscheme $Z \subset X$ of codimension c, a cohomology class $cl(Z) \in H^{2c}(X)$.

The above data should satisfy the following axioms $(W1)-(W9)$.

(W1) Each $H^*(X)$ is a finite dimensional K-vector space, $H^i(X) \neq 0$ only if $0 \leq i \leq 2\dim(X)$, for any geometrically integral smooth projective k-scheme X.

(W2) Given morphisms $f : X \to Y$ and $g : Y \to Z$ of geometrically integral smooth projective schemes over k we have $(g \circ f)^* = f^* \circ g^*$ as maps $H^*(Z) \to H^*(X)$, and $\mathrm{id}^* : H^*(X) \to H^*(X)$ is the identity map, namely H^* is a contravariant functor.

(W3) **Künneth.** For any two geometrically integral smooth projective schemes X, Y over k, if $pr_X : X \times Y \to X$, $pr_Y : X \times Y \to Y$ are the canonical projections, then we have a K-algebra homomorphism

$$H^*(X) \otimes_K H^*(Y) \to H^*(X \times Y), \alpha \otimes \beta \mapsto pr_X^*(\alpha) \cup pr_Y^*(\beta),$$

which is an isomorphism.

(W4) **Poincaré duality.** For each geometrically integral smooth projective scheme X over k, the trace map $\mathrm{Tr} : H^{2\dim(X)}(X) \to K$ is an isomorphism, and for every i with $0 \leq i \leq 2\dim(X)$, the trace map and cup product induce a perfect pairing

$$H^i(X) \otimes_K H^{2\dim(X)-i}(X) \to K, \alpha \otimes \beta \mapsto \mathrm{Tr}_X(\alpha \cup \beta).$$

(W5) **Compatibility of Trace maps and products.** Given any two geometrically integral smooth projective schemes X, Y over k, the trace map

$$\mathrm{Tr}_{X \times Y} : H^{2\dim(X)+2\dim(Y)}(X \times Y) \to K$$

satisfies

$$\mathrm{Tr}_{X \times Y}(pr_X^*(\alpha) \cup pr_Y^*(\beta)) = \mathrm{Tr}_X(\alpha)\mathrm{Tr}_Y(\beta),$$

for every $\alpha \in H^{2\dim(X)}(X)$ and $\beta \in H^{2\dim(Y)}(Y)$.

(W6) **Exterior product of cohomology classes.** For every geometrically integral smooth projective schemes X, Y over k and geometrically irreducible closed subschemes $Z \subset X$, $W \subset Y$, we have $cl(Z \times W) = pr_X^*(cl(Z)) \cup pr_Y^*(cl(W))$.

(W7) **Cohomology classes and pull-back.** For every morphism $f : X \to Y$ of geometrically integral smooth projective schemes over k, and for every geometrically irreducible closed subscheme $Z \subset X$, we have for every $\alpha \in H^{2\dim(Z)}(Y)$,

$$\mathrm{Tr}_X(cl(Z) \cup f^*(\alpha)) = \deg(Z/f(Z)) \cdot \mathrm{Tr}_Y(cl(f(Z)) \cup \alpha).$$

(W8) **Pull-back of cohomology classes.** Suppose $f : X \to Y$ is a morphism of geometrically integral smooth projective schemes over k, and let $Z \subset Y$ be a geometrically irreducible closed subscheme. Assume that f is flat, and assume that all irreducible components $Z_1, \ldots, Z_r$ of $f^{-1}(Z)$ have pure dimension $\dim(Z) + \dim(X) - \dim(Y)$. Then if $[f^{-1}(Z)] = \sum_{i=1}^r n_i Z_i$, we have $f^*(cl(Z)) = \sum_{i=1}^r n_i cl(Z_i)$.

(W9) **Cohomology classes of a point.** Let $X = \mathrm{Spec}(k)$, then $cl(X) = 1$ and $\mathrm{Tr}_X(1) = 1$.

Definition 2.1.2. Push-forward. Let $f : X \to Y$ be a morphism of geometrically integral smooth projective schemes over k. For $\alpha \in H^i(X)$ with some i, we define the push-forward of α to be the unique $f_*(\alpha) \in H^{2\dim(Y)-2\dim(X)+i}(Y)$ such that

$$\mathrm{Tr}_Y(f_*(\alpha) \cup \beta) = \mathrm{Tr}_X(\alpha \cup f^*(\beta)),$$

for every $\beta \in H^{2\dim(X)-i}(Y)$.

Lemma 2.1.1. *$f_* : H^*(X) \to H^*(Y)$ is a group homomorphism.*

Proof. Let $\alpha, \beta \in H^i(X)$ for some i, and let $\gamma \in H^{2\dim(X)-i}(Y)$, we want to show that $f_*(\alpha+\beta) = f_*(\alpha) + f_*(\beta)$. By definition, it suffices to show that

$$\mathrm{Tr}_Y(f_*(\alpha+\beta) \cup \gamma) = \mathrm{Tr}_Y((f_*(\alpha) + f_*(\beta)) \cup \gamma).$$

However, this is straightforward by definition:

$$\mathrm{Tr}_Y(f_*(\alpha+\beta) \cup \gamma) = \mathrm{Tr}_X((\alpha+\beta) \cup f^*(\gamma)) = \mathrm{Tr}_X(\alpha \cup f^*(\gamma) + \beta \cup f^*(\gamma)),$$

$$\mathrm{Tr}_Y((f_*(\alpha) + f_*(\beta)) \cup \gamma) = \mathrm{Tr}_Y(f_*(\alpha) \cup \gamma + f_*(\beta) \cup \gamma) = \mathrm{Tr}_X(\alpha \cup f^*(\gamma) + \beta \cup f^*(\gamma)),$$

as we desired. Hence, f_* is a group homomorphism via addition.

□

Proposition 2.1.1. *Let X be a geometrically integral smooth projective scheme over k, then we have*

i) $\dim_K(H^0(X)) = 1$.

ii) $cl(X) = 1 \in H^0(X)$.

iii) $Tr_X(\alpha \cup 1) = Tr_X(\alpha)$*, moreover,* $\alpha \cup 1 = \alpha$*, for all* $\alpha \in H^{2\dim(X)}(X)$*.*

Proof. i) We apply Poincaré duality for $i = 0$, then we get a perfect pairing

$$H^0(X) \times H^{2\dim(X)}(X) \to K, \ \ (\alpha, \beta) \mapsto \mathrm{Tr}_X(\alpha \cup \beta).$$

Since the trace map $\mathrm{Tr}_X : H^{2\dim(X)}(X) \to K$ is an isomorphism by Poincaré duality, we have that $\dim_K(H^0(X)) = 1$.

ii) Now consider the morphism $g : X \to \mathrm{Spec}(k)$, by $(W9)$, we have $cl(\mathrm{Spec}(k)) = 1$ and $\mathrm{Tr}_{\mathrm{Spec}(k)}(1) = 1$. Then applying $(W8)$, we have $g^*(cl(\mathrm{Spec}(k))) = cl(X)$, so $cl(X) = g^*(1) = 1 \in H^0(X)$.

iii) Apply $Y = \mathrm{Spec}(k)$ and $\beta = 1$ in $(W5)$, then for all $\alpha \in H^{2\dim(X)}(X)$, we have

$$\mathrm{Tr}_X(\alpha \cup 1) = \mathrm{Tr}_{X \times \mathrm{Spec}(k)}(pr_X^*(\alpha) \cup pr_Y^*(1)) = \mathrm{Tr}_X(\alpha) \cdot \mathrm{Tr}_Y(1) = \mathrm{Tr}_X(\alpha).$$

Recall the trace map $\mathrm{Tr}_X : H^{2\dim(X)}(X) \to K$ is an isomorphism by Poincaré duality, we have that $\alpha \cup 1 = \alpha$.

□

Proposition 2.1.2. *Let* $f : X \to Y$ *be a finite, surjective morphism between geometrically integral smooth projective schemes over k with same dimension d.*

i) $Tr_X(f^*(\alpha)) = \deg(f) \cdot Tr_Y(\alpha)$*, for every* $\alpha \in H^{2\dim(Y)}(Y)$*.*

ii) $f_*(1) = \deg(f) \cdot 1$*.*

Proof. i) Recall the pull-back of f is given by $f^* : H^*(Y) \to H^*(X)$, which is a K-algebra map preserving the grading. Consider $X \subset X$ as a closed subscheme of codimension 0, then there is a cohomology class $cl(X) \in H^0(X)$. Note f is flat by our assumption. Regard $Y \subset Y$ as a subscheme, then $f^{-1}(Y) = X$, so $(W8)$ implies that

$$f^*(cl(Y)) = cl(X) \in H^0(X).$$

For every $\alpha \in H^{2d}(Y)$, $f^*(\alpha) \in H^{2d}(X)$, by $(W7)$, we have

$$\mathrm{Tr}_X(cl(X) \cup f^*(\alpha)) = \deg(f)\mathrm{Tr}_Y(cl(f(X)) \cup \alpha).$$

By previous proposition, we know $cl(X) = 1 \in H^0(X)$ and $cl(Y) = 1 \in H^0(Y)$. Hence,

$$\mathrm{Tr}_X(cl(X) \cup f^*(\alpha)) = \mathrm{Tr}_X(1 \cup f^*(\alpha)) = \mathrm{Tr}_X(f^*(\alpha)),$$

$$\deg(f)\mathrm{Tr}_Y(cl(f(X)) \cup \alpha) = \deg(f)\mathrm{Tr}_Y(1 \cup \alpha) = \deg(f)\mathrm{Tr}_Y(\alpha).$$

Therefore, we have $\mathrm{Tr}_X(f^*(\alpha)) = \deg(f) \cdot \mathrm{Tr}_Y(\alpha)$, for every $\alpha \in H^{2\dim(Y)}(Y)$.

ii) By the definition of push-forward of f, for $w = 2\dim(X) - 2\dim(Y) = 0$, we have that the following diagram commutes:

$$\begin{array}{ccc} H^{2\dim(Y)}(X) \times H^w(X) & \xrightarrow{\mathrm{Tr}_X} & K \\ f^*\big\uparrow \qquad\qquad \big\downarrow f_* & & \big\| \\ H^{2\dim(Y)}(Y) \times H^0(Y) & \xrightarrow[\mathrm{Tr}_Y]{} & K \end{array}$$

Let $1 \in H^w(X)$ and $\alpha \in H^{2\dim(Y)}(Y)$, then we have

$$(f^*(\alpha), 1) \mapsto \mathrm{Tr}_X(f^*(\alpha) \cup 1),$$

$$(\alpha, f_*(1)) \mapsto \mathrm{Tr}_Y(\alpha \cup f_*(1)).$$

Hence, $\mathrm{Tr}_X(f^*(\alpha) \cup 1) = \mathrm{Tr}_Y(\alpha \cup f_*(1))$. Proposition 2.1.1.(iii) shows that

$$\mathrm{Tr}_Y(\alpha \cup \deg(f) \cdot 1) = \deg(f)\mathrm{Tr}_Y(\alpha \cup 1) = \deg(f)\mathrm{Tr}_Y(\alpha).$$

By part (i) and Proposition 2.1.1.(iii), we have that

$$\deg(f)\mathrm{Tr}_Y(\alpha) = \mathrm{Tr}_Y(f^*(\alpha)) = \mathrm{Tr}_X(f^*(\alpha) \cup 1).$$

Therefore, by definition, $f_*(1) = \deg(f) \cdot 1$.

□

Lemma 2.1.2. *Projection formula.* *Let $f : X \to Y$ be a finite and surjective morphism of geometrically integral smooth projective schemes over k with same dimension d. It then follows that there is a K-algebra pull-back map $f^* : H^*(Y) \to H^*(X)$. Let $f_* : H^*(X) \to H^*(Y)$ be the push-forward map. For any $\alpha \in H^*(X)$ and $\beta \in H^*(Y)$, we have $f_*(\alpha \cup f^*(\beta)) = f_*(\alpha) \cup \beta$.*

Proof. The definition of push-forward implies the following diagram commutes:

$$\begin{array}{ccc} H^{2\dim(X)-i-j}(X) \times (H^i(X) \times H^j(X)) & \xrightarrow{\mathrm{Tr}_X} & K \\ f^*\big\uparrow \qquad\qquad \big\downarrow f_* \qquad f^*\big\uparrow & & \big\| \\ H^{2\dim(X)-i-j}(Y) \times (H^i(Y) \times H^j(Y)) & \xrightarrow[\mathrm{Tr}_Y]{} & K \end{array}$$

Let $\alpha \in H^i(X)$ and $\beta \in H^j(Y)$, then $\alpha \cup f^*(\beta) \in H^{i+j}(X)$. It suffices to show that

$$\mathrm{Tr}_X(f^*(\gamma) \cup (\alpha \cup f^*(\beta))) = \mathrm{Tr}_Y(\gamma \cup (f_*(\alpha) \cup \beta)),$$

for all $\gamma \in H^{2\dim(X)-i-j}$. However, by definition, it is straightforward that

$$LHS = v\mathrm{Tr}_X(f^*(\gamma \cup \beta) \cup \alpha) \stackrel{def}{=} v\mathrm{Tr}_Y((\gamma \cup \beta) \cup f_*(\alpha)) = RHS,$$

where $v = (-1)^{\deg(\alpha)\deg(f^*(\beta))} = (-1)^{\deg(f^*(\alpha))\deg(\beta)}$.

□

Proposition 2.1.3. *Let $f : X \to Y$ be a finite and surjective morphism of geometrically integral smooth projective schemes over k with same dimension d. Let f^* and f_* be the pull-back and the push-forward respectively as before, then $f_*f^*(\alpha) = \deg(f)\alpha$ for every $\alpha \in H^*(Y)$.*

Proof. Recall by Proposition 2.1.2.(ii), it holds that $f_*(1) = \deg(f) \cdot 1$. Apply Projection formula for $1 \in H^*(X)$ and $\alpha \in H^*(Y)$, we have

$$f_*(1 \cup f^*(\alpha)) = f_*(1) \cup \alpha = \deg(f) \cdot 1 \cup \alpha = 1 \cup \deg(f)\alpha.$$

By Proposition 2.1.1.(iii), we have that

$$f_*f^*(\alpha) = f_*(1 \cup f^*(\alpha)) = 1 \cup \deg(f)\alpha = \deg(f)\alpha.$$

□

Proposition 2.1.4. *Let $f : X \to Y$ be a finite and surjective morphism of geometrically integral smooth projective schemes over k with same dimension d. Recall that char$(K) = 0$, so $f^* : H^*(Y) \to H^*(X)$ is injective.*

Proof. Assume that $f^*(\alpha) = 0$ for some $\alpha \in H^*(Y)$, then by Lemma 2.1.1, $f_*(f^*(\alpha)) = 0$ as f_* is a group homomorphism. Moreover, by Proposition 2.1.3, we have that $0 = f_*(f^*(\alpha)) = \deg(f) \cdot \alpha$. Since $\mathrm{char}(K) = 0$, $0 = \deg(f) \cdot \alpha$ implies that $\alpha = 0$ in $H^*(Y)$. Therefore, $f^* : H^*(Y) \to H^*(X)$ is injective. □

2.2 Divisibility of zeta functions of curves in a cover

In this section, we shall prove the divisibility of one-variable zeta functions in a cover, by the Weil conjecture and the Weil cohomology theory.
We firstly introduce the construction of Frobenius morphism by Hartshorne in [8, p. 301,302]: let X be a geometrically integral smooth projective scheme over $\mathbb{F}_q$, let k be an algebraic closure of $\mathbb{F}_q$, and let $\bar{X} = X \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathrm{Spec}\,k$ be the corresponding scheme over Spec k.

Definition 2.2.1. (Frobenius morphism.) Let X, $\bar{X}$ as above, we define the Frobenius morphism $\mathrm{Fr}_X : X \to X$ as follows: for any open subset $U \subset X$, $\mathscr{O}_X(U)$ is a $\mathbb{F}_q$-algebra of sheaves, so it admits a Frobenius endomorphism $F : \mathscr{O}_X(U) \to \mathscr{O}_X(U)$ given by the q-th power map. For any open subset $V \subset U$, the restriction of F on V is the Frobenius morphism on V, then gluing those Frobenius morphisms we get an endomorphism of X. This endomorphism is defined as Frobenius morphism on X. As Mustata stated in [12], we have an induced endomorphism of $\bar{X}$ given by

$$\mathrm{Fr} := \mathrm{Fr}_{\bar{X}} = \mathrm{Fr}_X \times \mathrm{Id} : \bar{X} \to \bar{X}.$$

By above definition, we have a commutative diagram:

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\mathrm{Fr}} & \bar{X} \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \mathrm{Spec}\ k & \xrightarrow{\mathrm{Fr}} & \mathrm{Spec}\ k, \end{array}$$

where $\pi : \bar{X} \to \mathrm{Spec}\ k$ is the corresponding morphism of $\bar{X}$ as a k-scheme. Then the Frobenius morphism on $\bar{X}$ is a k-linear Frobenius morphism if we regard $\bar{X}$ to be a k-scheme with structure morphism $\mathrm{Fr} \circ \pi$.
With Frobenius morphism given above, Weil cohomology theories explain the Weil conjecture, as Theorem 4.11 in [12], in the following way:

Proposition 2.2.1. *Let X be a geometrically integral smooth projective scheme over $\mathbb{F}_q$, with dimension n, then*

$$Z(X;t) = \frac{P_1(t) \cdot P_3(t) \dots P_{2n-1}(t)}{P_0(t) \cdot P_2(t) \dots P_{2n}(t)},$$

where for every i with $0 \le i \le 2n$ we have $P_i(t) = \det(1 - tFr^*; H^i(\bar{X}))$. Note Fr^* is the morphism induced by $Fr : \bar{X} \to \bar{X}$, and $H^*(\bar{X})$ is the l-adic cohomology over $K = \mathbb{Q}_l$.

Theorem 2.2.2. *Let $f : X \to Y$ be a finite, surjective morphism of geometrically integral smooth projective schemes over $\mathbb{F}_q$, note that X and Y have same dimension, which we denote as n. Then the numerator of $Z(Y;t)$ divides the numerator of $Z(X;t)$ in $\mathbb{Z}[t]$.*

Proof. Let $\bar{X} = X \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathrm{Spec}\ k$ and $\bar{Y} = Y \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathrm{Spec}\ k$ with k be an algebraic closure of $\mathbb{F}_q$. Let $H^*(\bar{X})$ and $H^*(\bar{Y})$ be the l-adic cohomology over $K = \mathbb{Q}_l$ on $\bar{X}$ and $\bar{Y}$ respectively. Since X, Y are geometrically integral smooth projective schemes over $\mathbb{F}_q$, by Proposition 2.2.1, we have

$$Z(X;t) = \frac{P_1(X;t) \cdot P_3(X;t) \dots P_{2n-1}(X;t)}{P_0(X;t) \cdot P_2(X;t) \dots P_{2n}(X;t)},$$

$$Z(Y;t) = \frac{P_1(Y;t) \cdot P_3(Y;t) \dots P_{2n-1}(Y;t)}{P_0(Y;t) \cdot P_2(Y;t) \dots P_{2n}(Y;t)},$$

where for each i with $0 \le i \le 2n$,

$$P_i(X;t) = \det(1 - tFr^*; H^i(\bar{X})), \;\; P_i(Y;t) = \det(1 - tFr^*; H^i(\bar{Y})).$$

The l-adic cohomology is a Weil cohomology theory. By our assumption, the map $f^* : H^*(\bar{Y}) \to H^*(\bar{X})$ induced by f satisfies the conditions in Proposition 2.1.4, so we have $f^* : H^*(\bar{Y}) \to H^*(\bar{X})$ is injective. Thus, for any $i \in \mathbb{Z}$, the map $f^* : H^i(\bar{Y}) \to H^i(\bar{X})$

is also injective.
Recall that $H^i(\bar{X}), H^i(\bar{Y})$ are K-vector spaces with K characteristic zero field, we may regard $H^i(\bar{Y})$ as a subspace of $H^i(\bar{X})$ via f^*. The restriction on $H^i(\bar{Y})$ of morphism Fr^* is the morphism on $H^i(\bar{Y})$ induced by $Fr: \bar{Y} \to \bar{Y}$. Hence, we have $P_i(Y;t)$ divides $P_i(X;t)$ in $K[t]$, for each i with $0 \leq i \leq 2n$. Furthermore, by the Weil conjecture, we have $P_i(X;t), P_i(Y;t) \in \mathbb{Z}[t]$ with $P_i(X;0) = P_i(Y;0) = 1$, so we may assume that $P_i(Y;t) = c_n t^n + \cdots + c_1 t + 1$ with $c_1, \ldots, c_n \in \mathbb{Z}$ and $P_i(X;t)/P_i(Y;t) = d_m t^m + \cdots + d_1 t + d_0$ with $d_0, \ldots, d_n \in K$. Then we have

$$\begin{aligned} P_i(X;t) =& (d_m t^m + \cdots + d_1 t + d_0)(c_n t^n + \cdots + c_1 t + 1) \\ =& c_n d_m t^{m+n} + \cdots + (d_0 c_2 + d_2 + d_1 c_1) t^2 + (d_0 c_1 + d_1) t + d_0. \end{aligned}$$

As $P_i(X;t) \in \mathbb{Z}[t]$ with $P_i(X;0) = 1$, we get $d_0 = 1$. Inductively, we have all $d_i \in \mathbb{Z}$ for $1 \leq i \leq m$, since $c_i \in \mathbb{Z}$ for all $1 \leq i \leq n$. Therefore the numerator of $Z(Y;t)$ divides the numerator of $Z(X;t)$ in $\mathbb{Z}[t]$, as we desired. □

Corollary 2.2.1. *Let $f: X \to Y$ as in Theorem 2.2.2, and suppose that* $\dim(X) = \dim(Y) = 1$*, then the class number of Y divides the class number of X: $h_Y \mid h_X$.*

Proof. The one-variable zeta functions of X and Y are just their two-variable zeta functions substituting $u = q$. Proposition 1.2.1.(3) shows that $P(u,1) = h$ for any u, it then follows that $P(X;1) = P(X;q,1) = h_X$ and $P(Y;1) = P(Y;q,1) = h_Y$. By Theorem 2.2.2, $P(Y;1) = h_Y$ divides $P(X;1) = h_X$. □

2.3 Non-existence of the Weil cohomology theory

Unlike one-variable zeta functions, there is no suitable Weil cohomology theory for two-variable zeta functions. Let k be an algebraic closure of $\mathbb{F}_q$.

Theorem 2.3.1. *There is no Weil cohomology theory $H^*(-)$ over a field K containing $\mathbb{Q}[u]$ such that: for any one-dimensional geometrically integral smooth projective scheme Y over $\mathbb{F}_q$, the two-variable zeta function of Y can be written as*

$$Z(Y;u,t) = \prod_{i=0}^{2} P_i(u,t)^{(-1)^{i+1}},$$

with $P_i(u,t) = \det(1 - tFr^*; H^i(\bar{Y}))$ for $i = 0,1,2$, where Fr^* is the morphism induced by $Fr: \bar{Y} \to \bar{Y}$ on Weil cohomology $H^*(\bar{Y})$ with $\bar{Y} = Y \times_{Spec\, \mathbb{F}_q} Spec\, k$.

Proof. We shall show that the existence of Weil cohomology theories will admit a contradiction. Let Y be a one-dimensional geometrically integral smooth projective scheme over $\mathbb{F}_q$, assume that $g(Y) \geq 1$, let k be an algebraic closure of $\mathbb{F}_q$, and let

$\bar{Y} = Y \times_{\operatorname{Spec}\mathbb{F}_q} \operatorname{Spec} k$ be the corresponding scheme over k. Let $Z(Y;u,t)$ be the two-variable zeta function of Y. Assume that there exists a suitable Weil cohomology theory for the two-variable zeta function $Z(Y;u,t)$, note that the coefficients of such Weil cohomology theory would be in some characteristic zero field K that contains $\mathbb{Q}[u]$. Then the numerator of $Z(Y;u,t)$, which is denoted as $P(Y;u,t)$, would be

$$P(Y;u,t) = \det(1 - tFr^*; H^i(\bar{Y})),$$

where the map Fr^* is induced by the Frobenius morphism $Fr_{\bar{Y}}$.
We now consider a finite, surjective and separable morphism $f : X \to Y$ with $\deg(f) > 1$, where X is some suitable one-dimensional geometrically integral smooth projective scheme over $\mathbb{F}_q$. Such morphism exists, because we can always find a finite separable field extension of the function field of Y, where $\mathbb{F}_q$ is algebraically closed, then this field extension admits a one-dimensional geometrically integral smooth projective scheme X for the cover. Similarly, we could define $\bar{X}$ over k, and we have $Z(X;u,t)$, $P(X;u,t)$ of Y, then $P(X;u,t) = \det(1 - tFr^* : H^1(\bar{X}))$.
However, by Proposition 2.1.4, $f^* : H^*(\bar{Y}) \to H^*(\bar{X})$ is injective. As $H^i(\bar{X})$ and $H^i(\bar{Y})$ are K-vector spaces, f^* induces a K-subspace (of $H^i(\bar{X})$) structure on $H^i(\bar{Y})$, hence, the Frobenius morphism Fr_X^* on $H^i(\bar{X})$ leaves fixed on the subspace $H^i(\bar{Y})$, so we have $P(Y;u,t)$ divides $P(X;u,t)$ in $K[t]$. If X and Y have different genus, then $P(Y;u,t)$ and $P(X;u,t)$ have different degrees in t, but $P(Y;u,t) \mid P(X;u,t)$ contradicts to the fact that $P(X;u,t)$ is absolutely irreducible.
It is enough to show that there is no Weil cohomology theory suitable for all two-variable zeta functions of one-dimensional geometrically integral smooth projective $\mathbb{F}_q$-schemes. □

Remark 2.3.2. With the same notation as above,

(1) If $g(Y) = 0$, then $P(Y;u,t)$ is just constant.

(2) Note that in the case that $g_Y = g_X \neq 0$, for a cover $f : X \to Y$, it does not admit any contradiction in the proof, as both $P(X;u,t)$ and $P(Y;u,t)$ have same degree in t. It can only happen if X and Y are elliptic curves and $f : X \to Y$ is an isogeny: by the Riemann-Hurwitz theorem, one has that

$$2g(X) - 2 - \deg(f) \cdot (2g(Y) - 2) = \deg R, \text{ where } R \text{ is the ramification divisor of} f.$$

Since $\deg R \geq 0$, if $g(X) = g(Y)$, we have $g(X) = g(Y) = 1$ or $\deg(f) = 1$. The morphism that we are interested in has degree > 1, so it suffices to consider the case that X and Y are elliptic curves. In fact, the Weil cohomology theory does not exist for elliptic curves either, which is the following proposition.

Proposition 2.3.1. *Let X an elliptic curve, thus of genus one, then there does not exist any Weil cohomology theories $H^*(X)$ such that the two-variable zeta function $Z(X;u,t)$ can be represented by*

$$Z(X;u,t) = \prod_{i=0}^{2} P_i(u,t)^{(-1)^{i+1}},$$

with $P_i(u,t) = \det(1 - tFr^*; H^i(\bar{X}))$ for all $i = 0,1,2$, where Fr^* is the morphism induced by $Fr : \bar{X} \to \bar{X}$ on Weil cohomology $H^*(\bar{X})$ with $\bar{X} = X \times_{Spec\,\mathbb{F}_q} Spec\,k$.

In order to talk about Weil cohomology theories on elliptic curves, we need Lefschetz fixed-point formula, which is given by Hartshorne in [8]:

Theorem 2.3.3. ***(Lefschetz fixed-point formula.)*** *Let X be smooth and proper over k with k an algebraic closure of $\mathbb{F}_q$, let $f : X \to Y$ be a morphism with isolated fixed points, and assume that the action of $1 - df$ on Ω_X^1 is injective for each specific point $x \in X$. The last condition says that the fixed point has "multiplicity 1". Let $L(f,X)$ be the number of fixed points of f. Then*

$$L(f,X) = \sum (-1)^i Tr(f^*; H^i(X,K))$$

where f^* is the induced map on the cohomology of X, and K is a characteristic zero field containing $\mathbb{Q}[u]$.

Proof of Proposition 2.3.1. Let X be a one-dimensional geometrically integral smooth projective scheme over a finite field $\mathbb{F}_q$, suppose it has genus one. Thanks to the Hasse-Weil bound ([17, Theorem 5.2.3.]), X always contains a rational point over $\mathbb{F}_q$. Further, by Theorem 2 in [16], this rational point can be translated to be the point at infinity ∞, so X is an elliptic curve. Let $\bar{X} = X \times_{\mathrm{Spec}\,\mathbb{F}_q} k$ be the corresponding scheme over an algebraic closure k of $\mathbb{F}_q$. We define the Frobenius morphism as in Definition 2.2.1, then it is a k-linear Frobenius morphism. As $\bar{X}$ is given by equations with coefficients in $\mathbb{F}_q$, $\mathrm{Fr}(P)$ is also a point of $\bar{X}$ for $P \in \bar{X}$. In general, P is a fixed point of Fr^m if and only if it has coordinates in the field $\mathbb{F}_{q^m}$. With the notation in Definition 1.1.1 for one-variable zeta function, we have

$$N_m = \#\{\text{fixed points of } \mathrm{Fr}^m\} = L(\mathrm{Fr}^m, \bar{X}).$$

By Lefschetz fixed-point formula, if there exists a suitable Weil cohomology theory over K for $Z(X;u,t)$, then

$$N_m = \sum_{i=0}^{2} (-1)^i \mathrm{Tr}(\mathrm{Fr}^{m*}; H^i(\bar{X},K)), \tag{2.1}$$

where K is a characteristic zero field containing $\mathbb{Q}[u]$. Recall by Example 1.1.1.(2), we have that the two-variable zeta function for an elliptic curve over $\mathbb{F}_q$ is:

$$Z(X;u,t) = \frac{1 + (N_1 - 1 - u)t + ut^2}{(1-t)(1-ut)}.$$

The zeta function can be represented as

$$Z(X;u,t) = \prod_{i=0}^{2} \det(1 - t\mathrm{Fr}^*; H^i(\bar{X}))^{(-1)^{i+1}}.$$

The two expansions of $Z(X;u,t)$ are identical if $H^0(\bar{X}) = K$ with $\mathrm{Fr}^* = id$, and $H^2(\bar{X}) \cong K$ with $\mathrm{Fr}^* = u \cdot id$, and $H^i(\bar{X}) = 0$ for $i > 2$. Now equation (2.1) implies that

$$u + 1 - N_1 = \mathrm{Tr}(\mathrm{Fr}^*; H^1(\bar{X})).$$

However, (2.1) does not hold for $m \geq 2$. We discuss the case $m = 2$: we want to show that $u^2 + 1 - N_2 \neq \mathrm{Tr}((\mathrm{Fr}^2)^*; H^i(\bar{X}))$. As $H^i(\bar{X})$ are K-vector space for each i, $(\mathrm{Fr}^2)^*$ is a linear map, then its trace is

$$\mathrm{Tr}((\mathrm{Fr}^2)^*) = (\mathrm{Tr}(\mathrm{Fr}^*))^2 - 2\det(\mathrm{Fr}^*).$$

By the theory of elliptic curves, we know that $N_m = q^m + 1 - (\alpha^m + \beta^m)$ for some $\alpha, \beta \in \mathbb{C}$, so it follows

$$\mathrm{Tr}(\mathrm{Fr}^*; H^i(\bar{X})) = u + 1 - N_1 = u - q + (\alpha + \beta).$$

By cohomological interpretations, the following holds:

$$P(u,t) = 1 + (N_1 - 1 - u)t + ut^2 = \det(1 - t\mathrm{Fr}^*; H^1(\bar{X})),$$

hence, $\det(\mathrm{Fr}^*; H^1(\bar{X})) = u$. It follows that

$$\begin{aligned}
\mathrm{Tr}(\mathrm{Fr}^{2*}) =& (\mathrm{Tr}(\mathrm{Fr}^*))^2 - 2\det(\mathrm{Fr}^*) \\
=& (u - q + \alpha + \beta)^2 - 2u \\
=& u^2 + 2(\alpha + \beta - q - 1)u + ((\alpha + \beta) - q)^2.
\end{aligned}$$

On the other hand,

$$u^2 + 1 - N_2 = u^2 + 1 - (q^2 + 1 - \alpha^2 - \beta^2) = u^2 - (q^2 + \alpha^2 + \beta^2).$$

Since $-N_1 = \alpha + \beta - q - 1 \neq 0$, $\mathrm{Tr}(\mathrm{Fr}^{2*})$ always has linear items, so $u^2 + 1 - N_2 \neq \mathrm{Tr}(\mathrm{Fr}^{2*})$, hence (2.1) does not hold for $m = 2$. Therefore, the Weil cohomology theory does not exist for two-variable zeta functions of elliptic curves.

$\square$

Chapter 3

Examples

In this chapter, we use the word "a curve" to mean a one-dimensional geometrically integral smooth projective scheme over a finite field. We shall give some specific examples for two-variable zeta functions of curves. Pellikaan [15] stated that the two-variable zeta function of a hyperelliptic curve is determined by its one-variable zeta function, we will discuss it with detailed proofs in the first part. For non-hyperelliptic curves, we consider the cases of genus three and genus four, by comparing coefficients and the class number. In fact, two-variable zeta functions of such curves are also determined by their one-variable zeta functions, but in a way differing from hyperelliptic curves. In the rest of this chapter, we give the zeta function of some specific curves.

3.1 Hyperelliptic curves

Definition 3.1.1. Let X be a curve over $\mathbb{F}_q$, X is called hyperelliptic if its genus $g \geq 2$ and there exists a finite morphism $\varphi : X \to \mathbb{P}^1$ of degree 2.

Hyperelliptic curves have many important properties, we represent three properties here:

(1) [9, Chapter 1.53]. If X is a curve of genus $g = 2$, then X is hyperelliptic.

(2) [17, Lemma 6.2.2.(a)]. For a curve X of genus ≥ 2, it is hyperelliptic if and only if it has exactly one divisor class of degree 2 and dimension 2.

(3) [9, Chapter 1.22]. The morphism $\varphi : X \to \mathbb{P}^1$ in the definition is unique up to automorphisms of X and $\mathbb{P}^1$, and the inverse images of rational points on the projective line define $q+1$ effective divisors of degree 2, which are called hyperelliptic divisors of the curve.

For further information, [2] and [3] discuss effective divisors of hyperelliptic curves.

Let X be a hyperelliptic curve of genus g, and let $\varphi : X \to \mathbb{P}^1$ be the unique morphism (up to automorphisms) as before. Then X allows a unique involution (conjugation), the hyperelliptic involution, which we denote by σ. Fixed points of σ are called hyperelliptic points (or ramification points). Note divisors of form $P+\sigma(P)$ are hyperelliptic divisors of X in above property (3).

Lemma 3.1.1. *Let X as above, let H be a hyperelliptic divisor on X, then every effective divisor D of X is equivalent to $T+r\cdot H$, where T does not contain any hyperelliptic pairs or H in its support.*

Proof. Let D be an effective divisor of X, by separating conjugate pairs from the support of D, we can write $D \sim T + r(P+\sigma(P)) \sim T + r\cdot H$, where T does not contain any conjugated pairs in its support.

$\square$

Lemma 3.1.2. *Let X and $D \sim T + r\cdot H$ as above, let $\deg(T) = d$, then*

$$\ell(D) = \begin{cases} r+1 & if \;\; d+r \leq g-1, \\ d+2r+1-g & if \;\; d+r > g-1. \end{cases}$$

If $\deg(T) \leq g$, then $\ell(T) = 1$ and T is unique.

Proof. Since $\ell(D)$ does not change when we extend the ground field $\mathbb{F}_q$, so we shall work on an algebraic closure k of $\mathbb{F}_q$. By Lemma 3.1.1, we can write $D \sim T + r\cdot H \sim P_1 + \cdots + P_d + r\cdot H$ with $P_i \neq \sigma(P_j)$ for all $i \neq j$ and $i,j \in \{1,\ldots,d\}$. Let $D' = P_1 + \cdots + P_d + P_{d+1} + \sigma(P_{d+1}) + \cdots + P_{d+r} + \sigma(P_{d+r}) \sim D$, and let $D'' = P_1 + \sigma(P_1) + \cdots + P_d + \sigma(P_d) + P_{d+1} + \sigma(P_{d+1}) + \cdots + P_{d+r} + \sigma(P_{d+r})$. Let f_i be the rational functions such that $\mathrm{div}(f_i) = H - P_i - \sigma(P_i)$ for each $i = 1,2,\ldots,d+r$.
If $d+r \leq g-1$, then $\deg(D'') \leq 2g-2$, so $0 \leq \deg(K-D'') \leq 2g-2$. Note that $\ell(K) \geq 1$, since $g \geq 2$, so we can represent the canonical divisor class by an effective divisor, so $f_1 f_2 \ldots f_r \in \mathscr{L}(K-D'')$, and hence $\ell(K-D'') \geq 1$, thus, D'' is a special divisor. By Clifford's theorem, one has that $\ell(D'') \leq \deg(D'')/2+1 = d+r+1$. Also, $< 1, f_1, \ldots, f_{d+r} > \subset \mathscr{L}(D'')$, so we have $< 1, f_1, \ldots, f_{d+r} > = \mathscr{L}(D'')$ as a k-vector space. Notice that $\mathscr{L}(D') \subset \mathscr{L}(D'')$ and $\mathscr{L}(D') \cap < f_1, \ldots, f_d > = \{0\}$, so $\ell(D') \leq r+1$. On the other hand, $\mathscr{L}(rH) \subset \mathscr{L}(D')$, so $\ell(D') \geq \ell(rH)$. Since $\mathscr{L}(rH) \supset < 1, f_{d+1}^{-1}, \ldots, f_{d+r}^{-1} >$, we have $\ell(rH) \geq r+1$, so $\ell(D') = r+1$. Hence, $\ell(D) = \ell(D') = r+1$ if $d+r \leq g-1$.
If $d+r > g-1$, then $\deg(D'') > 2g-2$, by Riemann-Roch, $\ell(D'') = 2d+2r+1-g$, thus, we have $\mathscr{L}(D') \subset \mathscr{L}(D'') = < 1, f_1, \ldots, f_{d+r}, h_1, \ldots, h_{d+r-g} >$ for some non-constant rational functions $h_1, \ldots, h_{d+r-g}$. Similar as before, $\mathscr{L}(D') \cap < f_1, \ldots, f_d > = \{0\}$, so $\ell(D) = \ell(D') \leq d+2r+1-g$. By the Riemann-Roch theorem, $\ell(D) \geq d+2r+1-g$, hence, $\ell(D) = d+2r+1-g$.
In the case that P_i are not distinct, suppose that $P_{i_1} = P_{i_2} = \cdots = P_{i_u}$, then the proof can

be continued with $f_{i_2} = f_{i_1}^2, \dots, f_{i_r} = f_{i_1}^u$.
For the case of $\deg(T) \le g$, we can apply previous result for $D = T$, then one has $\ell(T) = 1$. Let $D \sim T + rH$, suppose T is not unique, namely, there exists T' such that $D \sim T' + rH$, then $T' \sim T$, so $T - T' \sim \operatorname{div}(f)$ for some rational function f. However, $T \sim T' + \operatorname{div}(f) \ge 0$ implies that $f \in \mathscr{L}(T') \cong k$, hence, $\operatorname{div}(f) = 0$, it immediately follows that $T = T'$. This shows the uniqueness of T, and hence, completes the proof.

□

Proposition 3.1.1. *Let X be a hyperelliptic curve of genus g with two-variable zeta function $Z(u,t)$, then $Z(u,t)$ is determined by its one-variable zeta function:*

$$b_{n,s} = \begin{cases} a_{n-2s+2} - (q+1)a_{n-2s} + qa_{n-2s-2} & if\ \ 0 \le n \le 2g-2, s \ge 1, \\ h & if\ \ n > 2g-2, s = n+1-g, \\ 0 & otherwise. \end{cases}$$

Proof. Let

$$Z(u,t) = \sum_{n \ge 0} \sum_{s \ge 1} b_{n,s} \frac{u^s - 1}{u - 1} t^n, \ \ Z(t) = \sum_{n \ge 0} a_n t^n = \sum_{n \ge 0} \sum_{s \ge 1} a_{n,s} t^n$$

be the two-variable zeta function and the one-variable zeta function of X, respectively. Note $a_{n,s} = b_{n,s} \frac{q^s - 1}{q-1}$.
Suppose $0 \le n \le g$, let D be a divisor of degree n and dimension s. By Lemma 3.1.1, we can write $D \sim T + rH$, where T does not contain any conjugated pairs in its support and H is some hyperelliptic divisor. Note $\deg(T) + 2r = n \le g$, by Lemma 3.1.2, $s = \ell(D) = r + 1$, so D is equivalent to $T + (s-1)H$, where $\deg(T) = n - 2s + 2$. Moreover, T is unique, so one has

$$b_{n,s} \frac{q^s - 1}{q - 1} = a_{n,s} = a_{n-2s+2,1} \frac{q^s - 1}{q - 1}.$$

It follows that $b_{n,s} = a_{n-2s+2,1}$. Also, $a_{n-2s+2,1} = b_{n-2s+2,1}$, so $b_{n,s} = a_{n-2s+2,1} = b_{n-2s+2,1}$.
Now we prove the formula $b_{n,1} = a_n - (q+1)a_{n-2} + qa_{n-4}$ for $0 \le n \le g$ by induction. If $n = 0$, then $b_{0,1} = a_0$, which is clear. Assume that for any $m < n \le g$, the formula holds. Recall that

$$a_n = \sum_{s \ge 1} a_{n,s} = \sum_{s \ge 1} b_{n-2s+2,1} \frac{q^s - 1}{q - 1}.$$

By induction, we have

$$\begin{aligned} a_n =& b_{n,1} + \sum_{s \ge 2} (a_{n-2s+2} - (q+1)a_{n-2s} + qa_{n-2s-2}) \frac{q^s - 1}{q - 1} \\ =& b_{n,1} + \sum_{s \ge 2} (a_{n-2s+2} - (q+1)a_{n-2s} + qa_{n-2s-2}) (q^{s-1} + q^{s-2} + \cdots + q + 1) \\ =& b_{n,1} + (q+1)a_{n-2} - qa_{n-4}. \end{aligned}$$

Hence, the formula holds for $b_{n,1}$. Moreover, $b_{n,s} = b_{n-2s+2,1}$, so we have that

$$b_{n,s} = a_{n-2s+2} - (q+1)a_{n-2s} + qa_{n-2s-2}.$$

If $g \le n \le 2g-2$, by Remark 1.1.3.(2), we have

$$b_{n,s} = b_{2g-2-n,s-n-1+g}.$$

Hence, $b_{n,s} = a_{n-2s+2} - (q+1)a_{n-2s} + qa_{n-2s-2}$ as well.
Regarding the case that $n > 2g-2$, we have that $b_{n,s} = h$ if $s = n+1-g$ and zero otherwise, thanks to the Riemann-Roch theorem.

□

3.2 Non-hyperelliptic curves of low genus

3.2.1 Non-hyperelliptic curves of genus 3

Let X be a non-hyperelliptic curve of genus 3, namely, let X be a one-dimensional geometrically integral smooth projective scheme over $\mathbb{F}_q$ of genus 3, and it does not have any divisor classes of degree 2 and dimension 2.

Theorem 3.2.1. *For any curve X as above, let $Z(u,t) = \sum_{n\ge0}\sum_{s\ge1} b_{n,s}\frac{u^s-1}{u-1}t^n$ and $Z(t) = \sum_{n\ge0} a_n t^n$ be the two-variable and the one-variable zeta functions of X, as in Definition 1.1.2 and Proposition 1.1.1, respectively. Then $Z(u,t)$ is determined by $Z(t)$ as follows:*

$$b_{4,3} = b_{0,1} = a_0 = 1, b_{1,1} = b_{3,2} = a_1, b_{2,1} = a_2, b_{3,1} = a_3-(q+1)a_1, b_{4,2} = a_3 - a_1q - 1,$$

$b_{n,n-2} = h = a_3 - a_1 q$ for all $n \ge 5$, and $b_{n,s} = 0$ for all other $n \ge 0$ and $s \ge 1$.

Proof. Note $g = 3$. It is straightforward for all $n > 2g-2 = 4$, thanks to the Riemann-Roch theorem. For $0 \le n \le 4$, recall by Remark 1.1.3.(2), for $0 \le i \le 2g-2 = 4$, $b_{4-i,s} = b_{i,s+i-2}$, so it suffices to consider $0 \le n \le 2$. By definition, $b_{0,1} = a_0 = 1$, and $b_{0,s} = 0$ for all $s \ge 2$. As $a_n = \sum_{s\ge1} b_{n,s}\frac{q^s-1}{q-1}$, we have $a_1 = b_{1,1}$. Moreover, Remark 1.1.3.(2) shows that $b_{n,s} = 0$ for $0 \le n < 2s-2$. Thus in our case, we have $b_{1,s} = 0$ for all $s \ge 2$, and $b_{2,s} = 0$ for all $s \ge 3$. Now we can apply Remark 1.1.3.(2), which implies that

$$b_{4,3} = b_{0,1} = a_0 = 1, b_{4,2} = b_{0,0} = h - b_{0,1} = h-1, b_{3,2} = b_{1,1} = a_1, b_{3,1} = b_{1,0} = h - a_1.$$

Regarding the divisors with degree two, by the property (2) of hyperelliptic curves, X does not have any divisor classes that has dimension 2 and degree 2, so $b_{2,2} = 0$. Hence, $a_2 = \sum_{s\ge0} b_{2,s}\frac{q^s-1}{q-1} = b_{2,1}$.
Furthermore, since $a_3 = \sum_{s\ge0} b_{3,s}\frac{q^s-1}{q-1} = h - a_1 + a_1(q+1)$, we have $h = a_3 - a_1 q$. This completes our proof. □

Example 3.2.1. *([15]). Let X be the curve over $\mathbb{F}_2$ defined by $y^4+y=x^3+1$. Note the projective closure $\bar{X}$ of X in $\mathbb{P}^2$ is given by $y^4+yz^3=x^3z+z^4$. The infinity point of $\bar{X}$ is* $(1:0:0)$*. Reduce to the affine open $x=1$, we find that $\bar{X}$ is smooth at* $(1:0:0)$*, so $\bar{X}$ is a smooth curve in $\mathbb{P}^2$, and hence, it is a complete intersection in $\mathbb{P}^2$. By Hartshorne [8, p. 348], a complete intersection is not hyperelliptic, so X is not hyperelliptic. Also, since it is smooth, we get $g(X)=3$ by the genus formula.*
The one-variable zeta function of X is

$$Z(t)=\frac{1+4t^2+8t^4+8t^6}{(1-t)(1-2t)}.$$

The two-variable zeta function of X is

$$Z(u,t)=\frac{1+(2-u)t+(8-2u)t^2+(10-5u)t^3+(8-2u)ut^4+(2-u)u^2t^5+u^3t^6}{(1-t)(1-ut)}.$$

It is easy to get

$$a_0=1,a_1=3,a_2=11,a_3=27,h=21,$$

$$b_{0,1}=1,b_{1,1}=3,b_{2,1}=11,b_{3,1}=18,b_{3,2}=3.$$

This satisfies Theorem 3.2.1.

3.2.2 Non-hyperelliptic curves of genus 4

Let X be a non-hyperelliptic curve of genus 4 over a characteristic $\neq 2$ field. Let $|K|$ be the canonical divisor class of X. Then by [10, Chapter 7, Lemma 4.8], one has $|K|$ is very ample if and only if X is not hyperelliptic, so in our case, $|K|$ is very ample, hence, the class of a canonical divisor $|K|$ induces an embedding $\phi : X \hookrightarrow \mathbb{P}^3$, which is called the canonical embedding. With this setting, the invertible sheaf $\mathscr{L}(K)$ associated to $|K|$ is just $\phi^*\mathscr{O}_{\mathbb{P}^3}(1)$. Suppose that there exists a divisor D of degree 3 and dimension 2 in X, we define $\overline{\varphi(D)}$ as in [1] by

$$\overline{\varphi(D)}=\bigcap_{\varphi(D)\subset H} H,\ \text{ where } H \text{ runs over all hyperplanes containing } \phi(D) \text{ in } \mathbb{P}^3.$$

Note we have

$$\ell(K-D)=g-1-\dim(\overline{\phi(D)}),$$

further by the Riemann-Roch theorem, one has

$$\ell(D)-1=\deg(D)-1-\dim(\overline{\phi(D)}).$$

Hence, $\dim(\overline{\phi(D)})=1$, thus, the linear span in $\mathbb{P}^3$ corresponding to D is a line, which we denote L.
As Hartshorne shows in [8], the canonical embedding of X in $\mathbb{P}^3$ is a curve of degree 6,

and X is contained in a unique irreducible quadric surface Q, in fact, X is the complete intersection of Q with an irreducible cubic surface C. Note L lies in Q.
Depending on whether Q is smooth or not, the number of divisor classes of degree three and dimension two is either one or two. Since Q is a ruled surface, divisor classes of degree 3 and dimension 2 on X correspond to L cut out by the line of a ruling of Q. If Q is not smooth, then Q is a cone, there is only one such ruling, so $b_{3,2} = 1$. If Q is smooth, then $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, it is doubly ruled, divisor classes of degree 3 and dimension 2 correspond to the intersections between L and the vertical ruling line and the horizontal ruling line of Q, and hence, there are two such divisor classes, so $b_{3,2} = 2$.
Noet $D \mapsto K - D$ is an involution on $\mathrm{Pic}^3(X) := \{|D| \in \mathrm{Pic}(X) | \deg(D) = 3\}$, for any $|D| \in \mathrm{Pic}^3(X)$ with $\ell(D) = 2$, it is clear that $K - D$ is also a divisor of degree 3 and dimension 2, since by the Riemann-Roch theorem, $\ell(D) - \ell(K - D) = 3 + 1 - 4 = 0$.

Proposition 3.2.1. *([1, p. 206].) If X is over an algebraically closed field, then*

$$b_{3,2} = \begin{cases} 1 & Q \text{ is not smooth}, \\ 2 & Q \text{ is smooth}. \end{cases}$$

Moreover, as divisors do not change if we restrict to a finite field $\mathbb{F}_q$, so it also holds for X over $\mathbb{F}_q$.

Theorem 3.2.2. *$Z(X; u, t) = \sum_{n \geq 0} \sum_{s \geq 1} b_{n,s} \frac{u^s - 1}{u - 1} t^n$ is determined by $Z(X; t) = \sum_{n \geq 0} a_n t^n$ as follows:*

$$b_{6,4} = b_{0,1} = a_0 = 1, b_{6,3} = h - 1, b_{1,1} = b_{5,3} = a_1, b_{5,2} = h - a_1, b_{2,1} = b_{4,2} = a_2,$$

$$b_{4,1} = h - a_2,\ b_{3,1} + b_{3,2}(q + 1) = a_3,\ b_{3,2} = 1 \ \text{ or } \ 2.$$

Moreover, $b_{n,n-3} = h = a_4 - a_2 q$ for all $n \geq 7$, and $b_{n,s} = 0$ for all other $n \geq 0$ and $s \geq 1$.

Proof. The proof is similar as in Theorem 3.2.1. Since $g = 4$, we have $2g - 2 = 6$, by the Riemann-Roch theorem and Remark 1.1.3.(2), it suffices to consider the case $n = 0, 1, 2, 3$. It is straightforward that $b_{0,1} = a_0 = 1$ by definition. Furthermore, Remark 1.3.3.(2) shows that $b_{n,s} = 0$ for all $0 \leq n < 2s - 2$, so $b_{1,1} = a_1$ and $b_{1,s} = 0$ for all $s \geq 2$. If $n = 2$, then $b_{2,2} = 0$ since X is non-hyperelliptic, so $b_{2,1} = a_2$ and $b_{2,s} = 0$ for all $s \geq 3$. If $n = 3$, then we have $b_{3,1} + (q + 1) b_{3,2} = a_3$. Proposition 3.2.1 shows that $b_{3,2}$ is either 1 or 2, so that $b_{3,1} = a_3 - (q + 1)$ or $a_3 - 2(q + 1)$ respectively, depending on whether Q is smooth or not, where X embeds into the complete intersection of a quadric Q and a cubic in $\mathbb{P}^3$. The cases that $n = 4, 5, 6$ follow from Remark 1.1.3.(2). If $n \geq 7 = 2g - 1$, then Riemann-Roch implies that $b_{n,n-3} = h$ and $b_{n,s} = 0$ for all $s \neq n - 3$. Furthermore, we have

$$a_4 = b_{4,1} + b_{4,2}(q + 1) = h - a_2 + a_2(q + 1),$$

so we have $h = a_2 + a_4 - a_2(q + 1) = a_4 - a_2 q$, as we desired. □

3.3 Zeta functions over extensions of $\mathbb{F}_q$

Let X be a geometrically integral smooth projective scheme over a finite field $\mathbb{F}_q$. Let $\mathbb{F}_{q^r}$ be a finite field extension of $\mathbb{F}_q$, and let $X' = X \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathrm{Spec}\,\mathbb{F}_{q^r}$ be the corresponding scheme over $\mathbb{F}_{q^r}$. Let $Z(X';t)$ and $Z(X,t)$ be one-variable zeta functions of X' and X respectively, it is well-known that the following holds:

Proposition 3.3.1. *([12, Prop. 2.14])*

$$Z(X';t^r) = \prod_{i=1}^{r} Z(X;\xi^i t),$$

where ξ is a r-th primitive root of unity.

Proof. Let $N'_m := |\bar{X}'(\mathbb{F}_{q^{rm}})|$ and $N_m := |\bar{X}(\mathbb{F}_{q^m})|$, clearly, $N'_m = N_{rm}$. By the definition of one-variable zeta function, we just need to show

$$\sum_{m=1}^{\infty} \frac{N_{mr}}{m} t^{mr} = \sum_{i=1}^{r} \sum_{\ell=1}^{\infty} \frac{N_\ell}{\ell} \xi^{i\ell} t^\ell.$$

Since ξ is a r-th primitive root of unity, we have

$$\sum_{i=1}^{r} \xi^{i\ell} = \begin{cases} 0, & \text{if } r \text{ does not divide } \ell, \\ r, & \text{otherwise.} \end{cases}$$

It follows that

$$\sum_{i=1}^{r} \sum_{\ell=1}^{\infty} \frac{N_\ell}{\ell} \xi^{i\ell} t^\ell = \sum_{\ell=1}^{\infty} \sum_{i=1}^{\ell} \frac{N_\ell}{\ell} \xi^{i\ell} t^\ell = \sum_{\ell=1, r|\ell}^{\infty} \frac{N_\ell}{\ell/r} t^\ell = \sum_{m=1}^{\infty} \frac{N_{mr}}{m} t^{mr}.$$

□

In the case that $\dim(X) = 1$, or X is a curve, the one-variable zeta function of X is just substituting $u = q$ in its two-variable zeta function. However, it is a different situation for the two-variable case, as it is absolutely irreducible. We shall give a specific example from [15].

Example 3.3.1. *Let X be the curve over $\mathbb{F}_2$ that is given by $y^2 + y = x^3 + 1$, its one-variable zeta function is*

$$Z(X/\mathbb{F}_2;t) = \frac{1+2t^2}{(1-2t)(1-t)},$$

so one has
$$Z(X/\mathbb{F}_2;u,t) = \frac{1+(2-u)t+ut^2}{(1-ut)(1-t)}.$$
We now consider a simple case: $r=2$*. By Proposition 3.3.1, it follows that*
$$Z(X/\mathbb{F}_{2^2};t^2) = Z(X/\mathbb{F}_2;t)\cdot Z(X/\mathbb{F}_2;-t) = \frac{(1+2t^2)^2}{(1-4t^2)(1-t^2)} = \frac{1+4t^2+4t^4}{(1-4t^2)(1-t^2)}.$$
Hence, we have
$$Z(X/\mathbb{F}_{2^2};t) = \frac{1+4t+4t^2}{(1-4t)(1-t)}, Z(X/\mathbb{F}_{2^2};u,t) = \frac{1+(8-u)t+ut^2}{(1-ut)(1-t)}.$$
It implies that the relation in Proposition 3.3.1 can not be extended to the two variable case.

3.4 On curves with split Jacobians

The divisibility of one-variable zeta functions shows that one can identify, in some sense, the one-variable zeta function of a curve by its covers to some lower genus curves. In fact, for some special curves, which are known as curves with split Jacobian, their one-variable zeta functions can be given by covers to elliptic curves.

Let C be a smooth curve defined over a field k, with Jacobian $J(C)$. We call C a curve with split Jacobian, or $J(C)$ is split, if $J(C)$ is isogenous (over k) to a product of elliptic curves ([18]). As the Jacobian $J(E)$ of an elliptic curve E is isomorphic to E ([11]), for a smooth curve C of genus g with split Jacobian $J(C)$, we have that $J(C)$ is isogenous to
$$\bigoplus_{i=1}^{g} E_i \cong \bigoplus_{i=1}^{g} J(E_i), \text{ where } E_i \text{ are elliptic curves.}$$

Proposition 3.4.1. *Let C be a curve over $\mathbb{F}_q$ of genus g with split Jacobian, say $J(C) = \bigoplus_{i=0}^{g} J(E_i)$. Then the numerator $P(C;t)$ of the one-variable zeta function of C satisfies*
$$P(C;t) = \prod_{i=1}^{g} P(E_i;t), \textit{ where } P(E_i;t) \textit{ is the numerator of the zeta function of } E_i.$$

Remark 3.4.1. (1) The covers in Proposition 3.4.1 are unique up to isogeny, and hence, the product in Proposition 3.4.1 is unique up to isogeny and numbering.

(2) An elliptic curve is a curve of genus 1 with a rational point ∞, note that any curves of genus 1 over a finite field always have a rational point ∞ (see the proof of Proposition 2.3.1), so curves of genus 1 over a finite field are elliptic curves.

Proof of Proposition 3.4.1. For the Jacobian of the curve C, there is an induced morphism on the first cohomology group $H^1(J(C))$ from the Frobenius morphism on C, we denote this morphism as the Frobenius morphism on $J(C)$, $\mathrm{Fr} : H^1(J(C)) \to H^1(J(C))$. Note the numerator of $Z(C;t)$ is related to the characteristic polynomial $L(t)$ of the Frobenius morphism on $H^1(J(C))$ ([9]), by the following equation:

$$P(t) = t^{2g} L(1/t).$$

By Honda-Tate theorem (Theorem 3.4.2), elliptic curves up to isogeny correspond bijectively to eigenvalues of the Frobenius morphism, hence, elliptic curves in an isogeny class have same characteristic polynomial of Fr. In particular, it means that

$$P(C;t) = P\left(\bigoplus_{i=1}^{g} E_i;t\right) = \prod_{i=1}^{g} P(E_i;t).$$

□

An abelian variety is called simple if it is not isogenous to a product of abelian varieties of lower dimension.

Theorem 3.4.2. ***(Honda-Tate theorem.)****([11] and [14])*
Let A be a simple abelian variety over $\mathbb{F}_q$, and let Fr_A be the Frobenius morphism on A. Note as A is simple, $Fr_A \in End(A)$, which is a division algebra, and $\mathbb{Q}[Fr_A] \subset End(A)$ is a number field. Define a Weil q-integer to be an algebraic integer such that, for any embedding $\sigma : \mathbb{Q}[Fr_A] \hookrightarrow \mathbb{C}$, $|\sigma(Fr_A)| = q^{1/2}$. Let $W(q)$ be the set of Weil q-integers in $\mathbb{C}$. The map $A \mapsto Fr_A$ defines a bijection

$$\{\textit{simple abelian varieties}/\mathbb{F}_q\}/(\textit{isogeny}) \to W(q)/(\textit{conjugacy}).$$

Corollary 3.4.1. *Let C, E_i be the same as in Proposition 3.4.1, suppose $Z(E_i;t) = (1+a_it+qt^2)(1-t)^{-1}(1-qt)^{-1}$. Then we have $b_{1,1} = \sum_{i=1}^{g} a_i + q + 1$.*

Proof. Let $Z(C;u,t) = (\sum_{n=0}^{2g} p_n(u)t^n)(1-t)^{-1}(1-ut)^{-1}$. Since $P(C;t) = \prod_{i=1}^{g} P(E_i;t)$, we have that $p_1(q) = \sum_{i=1}^{g} a_i$. Recall in the proof of Proposition 1.4.2, we have that $p_1(u) = b_{1,1} - (u+1)$, so $p_1(q) = b_{1,1} - (q+1)$, hence, $b_{1,1} = \sum_{i=1}^{g} a_i + q + 1$.

□

3.5 A curve without split Jacobian

We consider the curves over $\mathbb{F}_2$:

$$X : y^4 + y = x^3 + 1,\ Y : y^2 + y = x^3 + 1.$$

Then X is an Artin-Schreier cover of Y by the map $(x,y) \mapsto (x, y^2+y)$. Note the genus of X and Y are $g_X = 3$ and $g_Y = 1$, respectively. The one-variable zeta functions of X and Y are, respectively,

$$Z(X;t) = \frac{1+4t^2+8t^4+8t^6}{(1-t)(1-2t)}, \ Z(Y;t) = \frac{1+2t^2}{(1-t)(1-2t)}.$$

Moreover, by Theorem 2.2.2, the numerator of its one-variable zeta function can be written as a product with two polynomials:

$$1+4t^2+8t^4+8t^6 = (1+2t^2)(1+2t^2+4t^4).$$

Y is an elliptic curve with 3 rational points, and X is a non-hyperelliptic curve, so their two-variable zeta functions are, respectively,

$$Z(Y;u,t) = \frac{1+(2-u)t+ut^2}{(1-t)(1-ut)},$$

$$Z(X;u,t) = \frac{1+(2-u)t+(8-2u)t^2+(10-5u)t^3+(8-2u)ut^4+(2-u)u^2t^5+u^3t^6}{(1-t)(1-ut)}.$$

Note there is a cover $X \to Y'$ defined by the map $(x,y) \mapsto (x, y^4+y^2)$ with Y' given by $y^2+y = x^6+1$. This curve Y' has genus 2, and hence, it has zeta functions (see proof below):

$$Z(Y',t) = \frac{1+2t^2+4t^4}{(1-t)(1-2t)}, \ Z(Y';u,t) = \frac{1+(2-u)t+(4-u)t^2+u(2-u)t^3+u^2t^4}{(1-t)(1-ut)}.$$

Clearly, the numerator of $Z(X;t)$ is the product of numerators of $Z(Y;t)$ and $Z(Y';t)$. However, the absolute irreducibility of two-variable zeta functions shows that there is no analogous product relations in the two-variable situation.

Remark 3.5.1. $P(Y';t)$ is not a product of numerators of zeta functions of two elliptic curves. In particular, the Jacobian of Y' is not split.

Proof. Assume the contrary that $P(Y';t)$ is a product of two numerators of zeta functions of some elliptic curves over $\mathbb{F}_2$, say E_1, E_2. Let $P(E_1;t) = 1+at+2t^2, P(E_2;t) = 1+bt+2t^2$ be numerators of zeta functions of E_1, E_2. Then we have

$$1+2t^2+4t^2 = (1+at+2t^2)(1+bt+2t^2) = 1+(a+b)t+(ab+4)t^2+(2a+2b)t^3+4t^4.$$

Hence,

$$a+b = 0, \ ab+4 = 2.$$

However, by Weil conjecture, we know that $a,b \in \mathbb{Z}$, so there is no such a,b satisfying our assumption. Therefore, $P(Y';t)$ is not a product of numerators of zeta functions of elliptic curves. $\square$

Proof of the zeta functions of Y'. The one-variable zeta function can be verified by taking logarithm derivative of the zeta function of Y' in the sense of Definition 1.1.1:

$$\frac{d}{dt}\log\left(\exp\left(\sum_{m=1}^{\infty} N_m \frac{t^m}{m}\right)\right) = \sum_{m=1}^{\infty} N_m t^{m-1}.$$

Moreover, the polynomial $(1+2t^2+4t^4)(1-t)^{-1}(1-2t)^{-1}$ has logarithm derivative

$$\begin{aligned}
&(16t^5-36t^4+16t^3-6t^2+3)(1-2t)^{-1}(1-t)^{-1}(1+2t^2+4t^4)^{-1}\\
=&(16t^5-36t^4+16t^3-6t^2+3)\left(\sum_{n\geq 0}(2t)^n\right)\left(\sum_{n\geq 0}(t)^n\right)\left(\sum_{n\geq 0}(-2t^2-4t^4)^n\right)\\
=&3+9t+9t^2+O(t^3).
\end{aligned}$$

In $\mathbb{F}_2$, one can easily get that $N_1 = 3$.
In $\mathbb{F}_4$, all points satisfy $u^4-u=(u^2-u)(u^2+u+1)=0$. If $y\in\mathbb{F}_2$, namely $y^2-y=0$, then $y^2+y=0$, so $x^6+1=0$, hence, $x\in\mathbb{F}_4^*$, there are 6 points with $y\in\mathbb{F}_2$. Moreover, if $y\in\mathbb{F}_4\setminus\mathbb{F}_2$, then $y^2+y+1=0$, so $y^2+y=-1$, then $y^2+y=x^6+1$ implies $x^6=0$, hence, $x=0$. There are 2 points with $y\in\mathbb{F}_4\setminus\mathbb{F}_2$. Therefore, together with the point at infinity, we have $N_2=9$.
In $\mathbb{F}_8$, if $(x,y)\in(\mathbb{F}_2)^2$, then $(1,0),(1,1)$ lie in the curve. Note $\mathbb{F}_8=\{0,1,\alpha,\alpha^2,\ldots,\alpha^6\}$, with $\alpha^7=1$ and $\alpha^3+\alpha+1=0$. Hence, we have

$$\alpha^3=\alpha+1, \alpha^6=\alpha^2+1, \alpha^4=\alpha^2+\alpha.$$

It follows that there are 6 points lie on the curve with $(x,y)\notin(\mathbb{F}_2)^2$:

$$(\alpha,\alpha^4),(\alpha,\alpha^5),(\alpha^2,\alpha),(\alpha^2,\alpha^3),(\alpha^4,\alpha^2),(\alpha^4,\alpha^6).$$

Together with the point at infinity, we have $N_3=9$.
For any $m\geq 1$, one can check that N_m is identical with the coefficients of the logarithm derivative of $(1+2t^2+4t^4)(1-t)^{-1}(1-2t)^{-1}$, thus, $(1+2t^2+4t^4)(1-t)^{-1}(1-2t)^{-1}$ is the one-variable zeta function of Y', as we claimed.
Now we compute the two-variable zeta function by Proposition 3.1.1. Note

$$Z(Y';t)=\frac{1+2t^2+4t^4}{(1-t)(1-2t)}=\sum_{n=0}^{\infty}a_n t^n.$$

Compare coefficients in both sides, we have that

$$a_0=1, a_1=3, a_2=9, a_3=21.$$

Proposition 3.1.1 shows that

$$b_{0,1}=1, b_{1,1}=3, b_{2,1}=6, b_{2,2}=1.$$

The numerator of $Z(Y';u,t)$ can be computed by Proposition 1.3.1 and the functional equation,

$$p_0(u) = 1, p_1(u) = 2-u, p_2(u) = -u+4, p_3(u) = u(2-u), p_4(u) = u^2.$$

Hence, the two-variable zeta function of Y' is

$$Z(Y';u,t) = \frac{1+(2-u)t+(4-u)t^2+u(2-u)t^3+u^2t^4}{(1-t)(1-ut)}.$$

$\square$

Bibliography

[1] Arbarello, E., Cornalba, M., Griffiths, P., and Harris, J. D. (2013). *Geometry of algebraic curves*, volume 1. Springer Science & Business Media.

[2] Cantor, D. G. (1987). Computing in the jacobian of a hyperelliptic curve. *Mathematics of computation*, 48(177):95–101.

[3] de Boer, M. A. (1998). The generalized hamming weights of some hyperelliptic codes. *Journal of Pure and Applied Algebra*, 123(1-3):153–163.

[4] de Jong, A. J. (2007). Weil cohomology theories. Note available at `http://math.columbia.edu/~dejong/seminar/note_on_weil_cohomology.pdf`.

[5] Duursma, I. M. (2004). Combinatorics of the two-variable zeta function. In *Finite fields and applications*, pages 109–136. Springer.

[6] Edixhoven, B., Holmes, D., and Taelman, L. (2016). Algebraic geometry. Note available at `http://pub.math.leidenuniv.nl/~holmesdst/teaching/2016-2017/Mastermath_AG/AG_notes.pdf`.

[7] Eisenbud, D. (2013). *Commutative Algebra: with a view toward algebraic geometry*, volume 150. Springer Science & Business Media.

[8] Hartshorne, R. (2013). *Algebraic geometry*, volume 52. Springer Science & Business Media.

[9] Hurt, N. E. (2013). *Many rational points: coding theory and algebraic geometry*, volume 564. Springer Science & Business Media.

[10] Liu, Q. (2002). *Algebraic geometry and arithmetic curves*, volume 6. Oxford University Press on Demand.

[11] Milne, J. S. (1986). Abelian varieties. In *Arithmetic geometry*, pages 103–150. Springer.

[12] Mustaţa, M. (2011). Zeta functions in algebraic geometry. Note available at http://www-personal.umich.edu/~mmustata/zeta_book.pdf.

[13] Naumann, N. (2003). On the irreducibility of the two variable zeta-function for curves over finite fields. *Comptes Rendus Mathématique*, 336(4):289–292.

[14] Oort, F. (2007). Abelian varieties over finite fields. Note available at http://www.math.nyu.edu/~tschinke/books/finite-fields/final/05_oort.pdf.

[15] Pellikaan, R. (1996). On special divisors and the two variable zeta function of algebraic curves over finite fields. *Arithmetic, Geometry and Coding theory, Walter de Gruyter, Berlin*, pages 175–184.

[16] Siksek, S. (2007). Genus 1 preliminaries. Note available at http://www.math.leidenuniv.nl/~evertse/siksek-genus1.pdf.

[17] Stichtenoth, H. (2009). *Algebraic function fields and codes*, volume 254. Springer Science & Business Media.

[18] Yamauchi, T. (2008). On curves with split jacobians. *Communications in Algebra*, 36(4):1419–1425.