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Methods for determining the effective resistance

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Introduction

We deal with networks on a daily basis, such as traffic network, telecommunication, water pipes and the internet. When a network fails to perform, it could just be a little frustrating, but at times, it comes with great consequences. It is thus important that a network is able to perform well, when there are some failures. This ability is called the robustness of the network.

In order to compare networks a measure for robustness is needed. In [6], many different measures have been studied and it was concluded, that the measure related to the electrical effective resistance between two points in the network satisfies the following two desirable properties for a measure of robustness. Firstly, the effective resistance is a distance [9]. Thus it satisfies the triangle inequality. Hence a network with direct connections is more robust than one with indirect connections. Secondly, when adding (removing) an edge, the robustness decreases (increases).

This led us to study the effective resistance in more detail. It turns out that the effective resistance can be determined in many different ways. Or rather, there are many relations between the effective resistance and different branches of mathematics.

In this thesis we present these known results alongside with another method to calculate the effective resistance that we have not yet seen in literature during our research. Later on, we found that it has been mentioned in [7]. We will also prove many well-known relations by using this method. These proofs are generally shorter and give better insight to the relations. It also shows that one does not need to include more physics than the basic rules of electrical circuits in order to obtain relations into the different branches of mathematics. Lastly we give an explicit formula of the increase or decrease of the effective resistances in a network when an edge is deleted or added respectively. With this formula we hope to understand how to modify our network in order to make it more robust.

Overview

Each sections of this thesis discusses relationships between the effective resistance and a specific theory. In Section 1, we focus on the physics behind the effective resistance. We start by providing the electrical laws and a definition of the effective resistance in Section 1.1. In Section 1.2 we present the known techniques (series law, parallel law and Y-Δ transform) to determine the effective resistances. The shortcomings of these techniques are discussed as well. Lastly, in Section 1.3, we give the relation between the effective resistance and the energy dissipation along the network. This also leads to the Rayleigh Monotonicity Law.

Section 2 is a compilation of correlations between the effective resistance and the Laplacian L, a matrix that characterises a graph. We introduce the Laplacian and its properties in Section 2.1. In Sections 2.2, 2.3 and 2.4, we discuss some generalized inverses of L and point out their correspondence to the effective resistance. Finally, we relate the determinants of submatrices of L to the effective resistances, with our own proof, in Section 2.5. The latter relation leads to a connection between effective resistances and the number of spanning trees, which we present in Section 3, alongside some minor applications of it.

We provide results from random walk theory related to the effective resistance in Section 4, giving our own proofs. Here we see how the commute time between two vertices and the escape probability at a vertex can be computed using the effective resistance.

Lastly, we procure and analyse a formula for the increase or decrease of the effective resistances in a network when an edge is deleted or added respectively in Section 5.

Acknowledgements

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I would like to thank Marcel de Jeu as well. Even though he did not play a major part in this thesis, he was the one who took the effort to convince me to continue studying when I was tempted to quit. Without him, I might never have started this research.

Lastly, I would like to thank my friends and family, for always believing in me.

1 Theories from physics

Effective resistance is a term used in electric circuit analysis to define the resistance between two points in an electric network. In graph theory this is also known as the resistance distance between two vertices of a weighted, simple (a graph with no double edges or loops), connected graph. Here the graph can be viewed as an electric network, where edge (i, j) with edge weight $w_{ij} \in \mathbb{R}_{>0}$ corresponds to a resistor with resistance $r_{ij} = \frac{1}{w_{ij}}$, with a voltage source connected between a pair of vertices.

The basic idea of the effective resistance is established by letting a voltage source be connected between vertices a and b. Let I_{ab} be the net electric current from a to b. The voltage source effectively sees the network between a and b as a single edge connecting a and b, through which the electric current flows. The effective resistance is basically the resistance of that single edge.

1.1 Electrical laws and the effective resistance

In the study of electrical circuits, one is interested in the behaviour of idealized electrical components in a circuit in terms of voltage (or potential difference) and current [13, p.11]. It turns out that these voltages and currents satisfy important laws in electrical circuit analysis, namely Kirchhoff's circuit laws and Ohm's law.

Let I_{ab} be the net current from a to b as a result of connecting a voltage source between a and b and let i_{xy} denote the current from x to y for $x, y \in V$, with $i_{xy} = -i_{yx}$ for all $x, y \in V$ and $i_{xy} = 0$ when $(x, y) \notin E$. We call the end points of our network, which is open for connecting to a voltage source or other external circuits, a terminal. In our case these are the vertices a, b and any vertex with degree one. A vertex that is not a terminal, and where two or more edges of the network come together is called a junction. The electrical laws state the following.

Kirchhoff's Current Law (KCL), also known as Kirchhoff's first law or Kirchhoff's junction rule, states that the algebraic sum of the currents into any junction is zero [17].

This could also be interpreted as, for any vertex in the network, the current into it equals the current out of it. So for $x \neq a, b$ we have $\sum_{y \in N(x)} i_{xy} = \sum_{y \in V} i_{xy} = 0$, where $N(x)$ denotes the set of neighbours of x. As for a and b, the current I_{ab} comes in a through an external source while leaving b back to the external source. So it holds that

$$\sum_{y \in V} i_{xy} = \begin{cases} I_{ab} & \text{for } x = a, \\ -I_{ab} & \text{for } x = b, \\ 0 & \text{otherwise.} \end{cases}$$

Kirchhoff's Voltage Law (KVL), also known as Kirchhoff's second law or Kirchhoff's loop rule, states that for any closed loop the sum of the voltages is zero [17].

Let U_{xy} denote the voltage or potential difference between vertex x and y (where $U_{xy} = -U_{yx}$). Then KVL states that for a cycle C in the network, we have

$$\sum_{(x,y) \in C} U_{xy} = 0.$$

Ohm's Law (OL) is generally stated as $U = IR$ where U is the potential difference between the endpoints of a resistor with resistance R through which a current I flows (from the end with higher potential to the end with lower potential).

So $U_{xy} = i_{xy} r_{xy}$ for all possible $x, y \in V$. This allows us to define a potential v_x on vertex x such that for any edge $(x, y) \in E$

$$v_x - v_y = i_{xy} r_{xy}.$$

Note that by changing the location of the voltage source (a and b) and/or the net current I_{ab}, the values of i_{xy} will change, and thus the potentials will change as well.

Definition 1.1. The effective resistance between vertices a and b (after connecting a voltage source between them) is defined as the potential difference between a and b per unit net current from a to b. In other words,

$$R_{ab} = \frac{v_a - v_b}{I_{ab}}.$$

Once we fix the current from a to b to be $I_{ab} = 1$, the effective resistance becomes $R_{ab} = v_a - v_b$. The values of the potentials v_x for $x \in V$ can be found by solving the system of equations with the potentials and currents as unknown variables defined by the three laws stated above.

Note, that this system of equations will yield one of the potentials as a free variable. This makes perfect sense, since a potential on its own means nothing. Only the potential differences matter. We can eliminate this free variable by setting $v_k = 0$ for a given vertex k.

Example 1.2. Consider the network given in Figure 1, where each edge (x, y) corresponds to a resistor with resistance $r_{xy} = 1$. We shall determine R_{12} using KCL and OL.

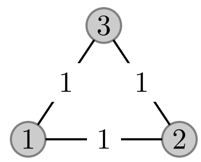


Figure 1: Configuration for Example 1.2. The numbers on the edges denote the edge resistances.

Let the net electric current from 1 to 2 be fixed as $I_{12} = 1$. Our unknown variables are the currents i_{12}, i_{13}, i_{23} and potentials v_1, v_2, v_3. From KCL we obtain the following three equations:

$$\begin{aligned} i_{12} + i_{13} &= 1 \\ i_{21} + i_{23} &= -1 \\ i_{31} + i_{32} &= 0. \end{aligned}$$

Applying OL to these equations yields the following equations:

$$\begin{aligned} 2v_1 - v_2 - v_3 &= 1 \\ -v_1 + 2v_2 - v_3 &= -1 \\ -v_1 - v_2 + 2v_3 &= 0. \end{aligned}$$

Solving this system of equations translates to solving $A\boldsymbol{v} = \boldsymbol{c}$ with

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \text{ and } \boldsymbol{c} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

This yields $\boldsymbol{v}^\top = \left(v_3 + \frac{1}{3}, v_3 - \frac{1}{3}, v_3\right)$. Thus $R_{12} = v_1 - v_2 = \left(v_3 + \frac{1}{3}\right) - \left(v_3 - \frac{1}{3}\right) = \frac{2}{3}$. ■

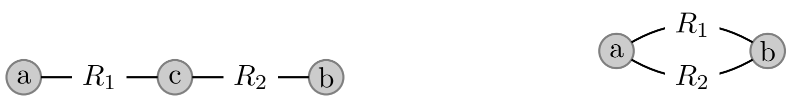
Remarks: The values we have found for the potentials are invalid for calculating effective resistances other than R_{12}. R_{13} is definitely not equal to $v_3 - v_1 = \frac{1}{3}$ (by symmetry of this network it holds that $R_{13} = R_{12}$). This is because we have different voltage source locations when determining R_{12} and R_{13}. So the step which made use of KCL yields a different vector $\boldsymbol{c}^\top = (1, 0, -1)$. This leads to different solution for $A\boldsymbol{v} = \boldsymbol{c}$.

1.2 Techniques for determining the effective resistance

One does not always have to solve the full system of equations obtained from KCL and OL as in Example 1.2. There are techniques to diminish the workload by reducing the size of the graph and thus simplifying the calculation of the effective resistances. However, depending on the choice of a and b, one will have to take different steps in determining R_{ab} using these techniques.

The first and probably the most well-known technique to aid in the computation of the effective resistance is the series and parallel laws, which allow us to replace resistors in series and in parallel by a single resistor without changing the nature of the network. Details can be seen in Chapter 1.2.1. Another technique is the Y-Δ transform, which allows us to replace a star-shaped configuration network (denoted as Y) with a triangle configuration network (denoted as Δ) and vice versa. Details can be found in Chapter 1.2.2. In Chapter 1.2.3, we discuss the performances of these techniques and the availability for other techniques.

1.2.1 Resistors in series and in parallel



(a) Resistors in series

(b) Resistors in parallel

Figure 2: Configurations for resistors in series and in parallel. R_1 and R_2 denote the resistances of the labelled edges.

For a circuit with resistances R_1 and R_2 in series or in parallel as in Figure 2, the effective resistance can be calculated by using the following rules.

Lemma 1.3 (Series law)**.** *For resistors in series as in Figure 2a, the resistance R_{ab} is given by*

$$R_{ab} = R_1 + R_2.$$

Proof. Let I_{ab} be the current from a to b. From OL we find $v_a - v_c = R_1 I_{ab}$ and $v_c - v_b = R_2 I_{ab}$. Adding these 2 equations yields $v_a - v_b = (R_1 + R_2)I_{ab}$. Thus $R_{ab} = R_1 + R_2$. □

Lemma 1.4 (Parallel law)**.** *For resistors in parallel as in Figure 2b, the resistance R_{ab} is given by*

$$\frac{1}{R_{ab}} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Proof. Let I_{ab} be the current from a to b and denote the current from a to b through the wire with resistance R_1 by i_1 and the current through the other wire by i_2. From KCL we have $i_1 + i_2 = I_{ab}$. From OL we have $v_a - v_b = i_1 R_1 = i_2 R_2$. Manipulating these 2 equations gives us $v_a - v_b = \frac{R_1 R_2}{R_1 + R_2} I_{ab}$. Thus, $R_{ab} = \frac{R_1 R_2}{R_1 + R_2}$, and so $\frac{1}{R_{ab}} = \frac{1}{R_1} + \frac{1}{R_2}$. □

Example 1.5. With these rules, determining R_{12} in the triangle from Example 1.2 can be done in two steps as seen in Figure 3.

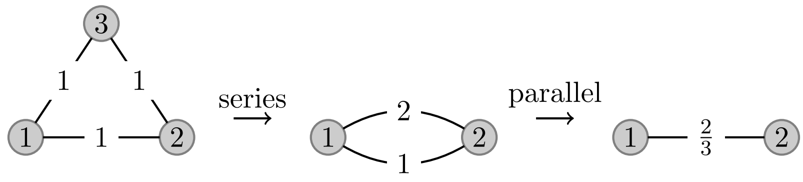


Figure 3: Determining R_{12} on a triangle using series law and parallel law. The numbers on the edges denote the edge resistances.

■

1.2.2 The Y-Δ transform

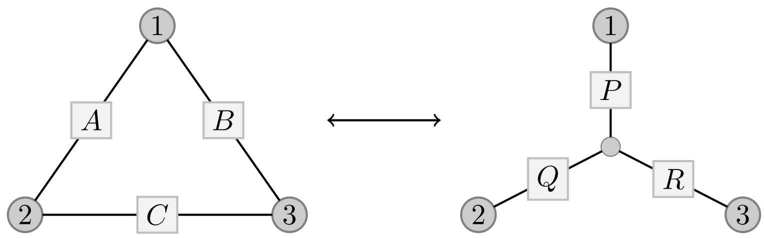


Figure 4: The interchangable Δ and Y configurations with resistances A, B, C, P, Q and R.

Sometimes it is not immediately clear, that we can use the series and parallel laws to determine the effective resistance. For instance: what are the resistors in series and in parallel in K_4, the complete graph with 4 vertices? In order to analyse the network further, we can interchange network parts that look like the configurations in Figure 4. We denote the triangular configuration as the Δ configuration and the star-shaped configuration as the Y configuration.

Lemma 1.6. *Let us be given one of the configurations (Y or Δ) depicted in Figure 4. Upon switching to the other configuration without altering the effective resistances between any pair of vertices in the network, the resistances in the new configuration as a function of the resistances on the former configuration translate as follows:*

$$\begin{aligned} A &= \frac{PQ+PR+QR}{R} & \qquad P &= \frac{AB}{A+B+C} \\ B &= \frac{PQ+PR+QR}{Q} & \qquad Q &= \frac{AC}{A+B+C} \\ C &= \frac{PQ+PR+QR}{P} & \qquad R &= \frac{BC}{A+B+C}. \end{aligned}$$

Proof. By using the series/parallel laws on the Δ configuration we obtain

$$R_{12} = \frac{A(B+C)}{A+B+C} \qquad R_{13} = \frac{B(A+C)}{A+B+C} \quad \text{and} \quad R_{23} = \frac{C(A+B)}{A+B+C}. \tag{1.1}$$

From the Y configuration we find

$$R_{12} = P+Q \qquad R_{13} = P+R \qquad \text{and} \qquad R_{23} = Q+R. \tag{1.2}$$

Here we see that resistance of the resistor branching away from of the main path is not a part of the equations. This is because KCL tells us that no current will flow through this resistor in the Y configuration, so it does not contribute to the effective resistance.

Adding all the resistances yields

$$\frac{R_{12}+R_{13}+R_{23}}{2} \quad = \quad \frac{AB+AC+BC}{A+B+C} \quad = \quad P+Q+R. \tag{1.3}$$

By isolating P, Q or R in Equation 1.3 (depending on which one we need) and substituting the resulting expressions by Equations 1.2 and 1.1, we find

$$P = \frac{AB}{A+B+C}, \qquad Q = \frac{AC}{A+B+C} \quad \text{and} \quad R = \frac{BC}{A+B+C}.$$

On the other hand, from the above equations we find

$$PC = QB = RA = \frac{ABC}{A+B+C}.$$

Manipulating this equation gives us

$$\begin{aligned} PQR &= \left(\frac{ABC}{A+B+C}\right)^3 \frac{1}{ABC}, \\ \frac{1}{P}+\frac{1}{Q}+\frac{1}{R} &= \left(\frac{A+B+C}{ABC}\right)(A+B+C). \end{aligned}$$

Thus

$$PQ+PR+QR = \left(\frac{1}{P}+\frac{1}{Q}+\frac{1}{R}\right)PQR = \frac{ABC}{A+B+C}.$$

So we also have

$$PC = QB = RA = PQ+PR+QR$$

from which the equations for A, B and C follow. □

Example 1.7. Determining the effective resistance between any two points in K_4 (for example R_{12}) can be done as in Figure 5.

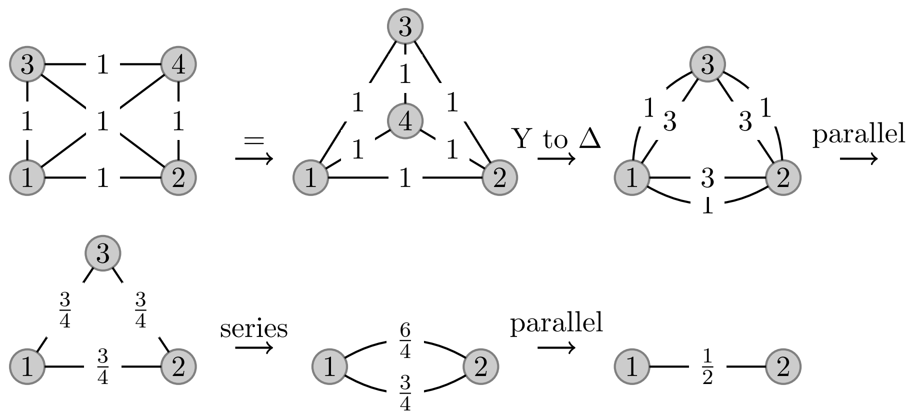


Figure 5: Determining R_{12} on K_4 using the Y-Δ transform, series law and parallel law. The numbers on the edges denote the edge resistances.

■

1.2.3 Limitations of the techniques

Both methods mentioned in the previous subsections generally get us very far in analysing electrical circuits. Often one can reduce the whole network into a single resistor (as seen in Examples 1.2 and 1.7), thus avoiding having to solve any system of equations obtained from KCL and OL. We find ourselves wondering, whether we could do such a reduction between any two vertices in any graph, using only these techniques.

Let us focus merely on how far the graph reduces by using the parallel law, series law and the Y-Δ transformation, while ignoring the values of the resistances on each edge, as we use these techniques. We can then roughly view them as the following four modification operations on a graph:

① (Parallel law) Make the graph simple by removing double edges and loops.

② (Series law) For a vertex i with degree $d_i = 2$, remove vertex i.

③ (Y to Δ) For a vertex i with degree $d_i = 3$, add three edges to connect the neighbours of i into a triangle. Then remove vertex i and the three edges connected to i.

④ (Δ to Y) For a triangle with vertices a, b and c, create a new vertex i connecting to these vertices and remove the edges of the triangle.

Let δ be the sum of the degrees of the vertices in our graph. ① reduces the number of edges in the graph, and thus reduces δ. ② removes a vertex and reduces δ by 2. ③ removes a vertex while keeping δ unchanged. While ④ reduces the degrees of edges a, b and c by 1, it adds a new edge with degree 3 and thus also leaves δ unchanged. As long as we can use ① and ②, δ will decrease. By successively using ③ and ④ to manipulate the graph in such way that we could use ① and ②, we can potentially end up with just one edge and two vertices.

However, there are graphs on which we can not perform any of the four operations above. Consider, for example, the complete bipartite graph $K_{4,4}$ (see Figure 6). This graph is already simple since it is unweighted with no double edges or loops, all vertices have degree 4, and there are no triangles. So it is impossible to reduce this graph using any of the four operations stated above.

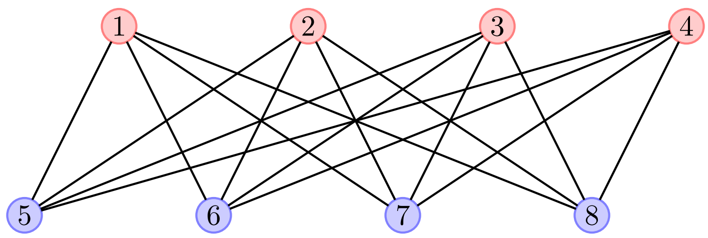


Figure 6: The complete bipartite graph $K_{4,4}$

Since the Y-Δ transform interchanges S_3 (a star graph with 3 edges) and C_3 (a cyclic graph with 3 edges/vertices) or K_3 (the complete graph with 3 vertices), we wonder whether it is possible define a new method allowing to interchange S_n and C_n or K_n. Further investigation reveals that it is generally not possible to interchange S_n and C_n.

Example 1.8. Let $V_n = \{1, 2, \ldots, n\}$ with $n \geqslant 4$ and let r_{ij} denote the resistance of edge (i, j). Let G be in a star configuration S_n for which all edges correspond to a resistor with resistance 1. So $G = (V_n \cup \{*\}, E')$ with $E' = \{(v, *) : v \in V_n\}$ and $r_{v*} = 1$ for $v \in V_n$. For this configuration we find that the resistance between any two vertices $i, j \in V_n$ is $R_{ij} = r_{i*} + r_{j*} = 2$.
Assume it is possible to replace G by a cyclic network $C_n = (V_n, E)$ with $E = \{(1, 2), (2, 3), \ldots, (n-1, n), (n, 1)\}$. Then for $(a, b) \in E$ we find by using the series and parallel laws that

$$R_{ab} = \frac{r_{ab} \cdot \sum_{(i,j)\in E\setminus\{(a,b)\}} r_{ij}}{\sum_{(i,j)\in E} r_{ij}}.$$

By pairwise comparing these resistances for any pair of edges in E and using the fact that $R_{ij} = 2$ for all $i, j \in V_n$, we find that for $(a, b), (k, l) \in E$

$$\begin{aligned} r_{ab} \sum_{(i,j)\in E\setminus\{(a,b)\}} r_{ij} &= r_{kl} \sum_{(i,j)\in E\setminus\{(k,l)\}} r_{ij} \\ r_{ab}r_{kl} + r_{ab} \sum_{(i,j)\in E\setminus\{(a,b),(k,l)\}} r_{ij} &= r_{ab}r_{kl} + r_{kl} \sum_{(i,j)\in E\setminus\{(a,b),(k,l)\}} r_{ij}. \end{aligned}$$

Thus, $r_{ab} = r_{kl}$ and so the values of all r_{ij} for $(i, j) \in E$ are the same, say $r_{ij} = r$.

Then, by using the series and parallel laws again, we see in this cyclic graph that $R_{12} = \frac{(n-1)r}{n}$ and $R_{13} = \frac{2(n-2)r}{n}$. It holds that $R_{12} = R_{13}$ if and only if $n = 3$. Sine $n \geqslant 4$, it follows that $R_{12} \neq R_{13}$, which contradicts $R_{ij} = 2$ for all $i, j \in V_n$.

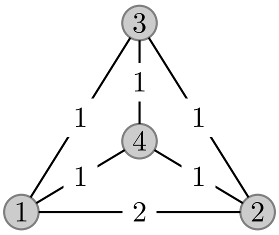
If G is in a cyclic configuration for which all edges correspond to a resistor with resistance 1, we find a similar contradiction. ■

What about interchanging S_n with K_n? Replacing S_n by K_n seems counterproductive, since the number of edges and the sum of the vertex degrees increases as we do so. As for replacing K_n by S_n, we show in Example 1.9 that for $n = 4$ it is not always possible to do so. For $n > 4$ we do not yet have a shortcut to determine effective resistances between vertices in K_n.

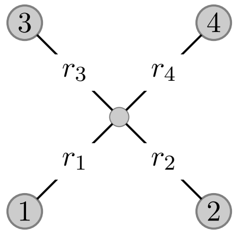
Example 1.9. Let $G = (V, E)$ be a network with configuration K_4 and with resistances $r_{12} = 2$, and $r_{ij} = 1$ for all other $i, j \in V$ (see Figure 7a). Then by using the series law, parallel law and the Y-Δ transform, we find, that $R_{12} = \frac{2}{3}$, $R_{13} = R_{14} = R_{23} = R_{24} = \frac{13}{24}$ and $R_{34} = \frac{1}{2}$.

Assume, that we can replace G by a network with star configuration S_4 without altering any pairwise effective resistances. Let the resistance of the edge connecting to vertex $i \in V$ be r_i (see Figure 7b). Then $R_{ij} = r_i + r_j$ for all possible $i, j \in V$. Combined with the fact that $R_{13} = R_{14} = R_{23} = R_{24}$, we find $r_1 = r_2$ and $r_3 = r_4$. From $r_1 + r_2 = 2r_1 = R_{12} = \frac{2}{3}$ follows that $r_1 = \frac{1}{3}$. Now $r_1 + r_4 = R_{14} = \frac{13}{24}$ yields $r_4 = \frac{5}{24}$. However, then $R_{34} = r_3 + r_4 = 2r_4 = \frac{5}{12}$, which contradicts $R_{34} = \frac{1}{2}$. ■

Since we were unsuccessful in defining more methods similar to the series law, parallel law and Y-Δ transformation, we wonder what the necessary and sufficient conditions are to be able to reduce a graph to one edge. However, we did not investigate this further.



(a) Configuration for the starting graph G in K_4 shape.



(b) Star configuration S_4 by which we would like to replace G.

Figure 7: Configurations for Example 1.9. Here the labels on the edges denote their resistances.

1.3 Energy dissipation

When a current I flows through a resistor with resistance R, losing voltage U in the process, the energy dissipated (or the power consumed by the resistor) is $P = IU = I^2R = \frac{U^2}{R}$. There is a relation between the total energy dissipation of a network and the effective resistance between two points in it. Although the results of the relation are mainly used to study random walks, we include this in the current chapter due to the physics involved. This subsection is an adaptation of chapters in [5] and [11] of corresponding topics.

Throughout this chapter, we work with the graph $G = (V, E)$ as a network with resistances $r = \{r_{xy} : (x, y) \in E\}$ and a net current flowing from a to b with $a, b \in V$, unless stated otherwise. From Chapter 1.1 we know that there is a current i_{xy} for each edge $(x, y) \in E$ (with $i_{xy} = -i_{yx}$ for all $x, y \in V$ and $i_{xy} = 0$ when $(x, y) \notin E$) and a potential v_x for each vertex $x \in V$ satisfying KCL and OL. For an edge $(x, y) \in E$ the energy dissipated along the edge is $i_{xy}(v_x - v_y)$. The total energy dissipation of the network is thus

$$E = \sum_{(x,y)\in E} i_{xy}(v_x - v_y) = \frac{1}{2}\sum_{x,y\in V} i_{xy}(v_x - v_y) = \frac{1}{2}\sum_{x,y\in V} i_{xy}^2 r_{xy} = \frac{1}{2}\sum_{x,y\in V} \frac{(v_x - v_y)^2}{r_{xy}}. \tag{1.4}$$

Furthermore, denoting the net current from a to b as I_{ab}, we have

$$\begin{aligned} E = \frac{1}{2}\sum_{x,y\in V} i_{xy}(v_x - v_y) &= \frac{1}{2}\sum_{x,y\in V} (i_{xy}v_x + i_{yx}v_y) \\ &= \frac{1}{2}\sum_{x\in V} v_x \sum_{y\in V} i_{xy} + \frac{1}{2}\sum_{y\in V} v_y \sum_{x\in V} i_{yx} \\ &= (v_a - v_b)I_{ab}, \end{aligned} \tag{1.5}$$

where we use KCL in the last equation. This equation makes sense, since by viewing the network as a single edge between a and b with resistance R_{ab} through which a current I_{ab} flows, we should find that the energy dissipation on this single edge is $(v_a - v_b)I_{ab}$. Another way to look at this is by viewing $(v_a - v_b)I_{ab}$ as the energy supplied by the voltage source. The principle of conservation of energy then states that the total energy dissipated along the network must be equal to the energy supplied. Furthermore, by using OL, we can alternatively write the total energy dissipation on a network with a net current I_{ab} from a to b as

$$E = (v_a - v_b)I_{ab} = I_{ab}^2 R_{ab} = \frac{(v_a - v_b)^2}{R_{ab}}. \tag{1.6}$$

For general flows, this principle of conservation of energy also holds. Let us first define a flow.

Definition 1.10. A flow j from a to b on G is a function $j : V \times V \to \mathbb{R}$ with $j(x, y) = j_{xy}$ and

(1) $j_{xy} = -j_{yx}$ for all $x, y \in V$,

(2) $j_x = \sum_{y \in V} j_{xy} = 0$ for all $x \in V \backslash \{a, b\}$,

(3) $j_{xy} = 0$ when $(x, y) \notin E$.

Note that the electrical current i is a flow. For our electric current we have $i_a = I_{ab} = -i_b$. For general flows j, it also holds that $j_a = -j_b$, since

$$j_a + j_b = \sum_{x \in V} j_x = \sum_{x \in V} \sum_{y \in V} j_{xy} = \frac{1}{2} \sum_{x,y \in V} (j_{xy} + j_{yx}) = 0.$$

If $j_a = -j_b = 1$ we call j a unit flow. We define $\mathcal{F}^1_{ab}$ as the set of unit flows from a to b on G.

Lemma 1.11 (Conservation of energy). *For any function $w : V \to \mathbb{R}$ and any flow j from a to b on G, it holds that*

$$(w_a - w_b) j_a = \frac{1}{2} \sum_{x,y \in V} (w_x - w_y) j_{xy}.$$

Proof. Similarly to the manipulations done leading to Equation 1.5, using properties (1) and (2) from Definition 1.10 and the fact that $j_a = -j_b$, yields the result that needs proving. □

We have seen in Equation 1.4, that we can write the total energy dissipation as a function of currents and potentials, or currents and resistances. So, given a flow j from a to b on G with resistances r, we can write the total energy dissipation associated with this flow as

$$E^r_f(j) = \frac{1}{2} \sum_{x,y \in V} j^2_{xy} r_{xy}.$$

Nature always seem to find a way to handle things most efficiently. Out of all the possible flows through G with the same total flow, the energy dissipated along the network associated to the flow that follows the electrical laws is least.

Theorem 1.12 (Thomson Principle). *Suppose, that $i \in \mathcal{F}^1_{ab}$ satisfies the electrical laws (KCL, KVL and OL). Then,*

$$E^r_f(i) = \min_{j \in \mathcal{F}^1_{ab}} E^r_f(j).$$

Proof. We copy the proof from [11]. Fix $j \in \mathcal{F}^1_{ab}$ and define $\delta_{xy} = j_{xy} - i_{xy}$ for all $x, y \in V$. Then δ is a flow with $\delta_a = 0$, since $j_a = i_a = 1$. We have

$$\begin{aligned}
E^r_f(j) &= \frac{1}{2} \sum_{x,y \in V} (i_{xy} + \delta_{xy})^2 r_{xy} \\
&= \frac{1}{2} \sum_{x,y \in V} i^2_{xy} r_{xy} + \frac{1}{2} \sum_{x,y \in V} \delta^2_{xy} r_{xy} + \sum_{x,y \in V} i_{xy} \delta_{xy} r_{xy} \\
&= E^r_f(i) + E^r_f(\delta) + \sum_{x,y \in V} (v_x - v_y) \delta_{xy}.
\end{aligned}$$

From Theorem 1.11 it follows that $\sum_{x,y \in V} (v_x - v_y) \delta_{xy} = 2(v_a - v_b)\delta_a = 0$. So it holds that $E^r_f(j) = E^r_f(i) + E^r_f(\delta)$. Since $E^r_f(\delta) \geqslant 0$ with equality if and only only if $\delta = 0$, the claim holds. □

We can also write the total energy dissipation as a function of potentials and resistances. So, for any potential function $u : V \to \mathbb{R}$ on G with resistances r, we can write the total energy dissipation associated with this potential as

$$E_p^r(u) = \frac{1}{2} \sum_{x,y \in V} \frac{(u_x - u_y)^2}{r_{xy}}.$$

There is a dual form for Thomson's principle. Instead of comparing the energy dissipated along a network for different flows, one can compare the energy dissipated for different distributions of the potentials among the vertices V. Given the same voltage across a and b, the distribution satisfying the electrical laws is again the most efficient one. For a network with a voltage source over a and b, a potential function $u : V \to R$ is called a unit potential if $u_a = 1$ and $u_b = 0$ (so the voltage across a and b is one). Define $\mathcal{P}_{ab}^1$ as the set of unit potentials.

Theorem 1.13 (Dirichlet Principle)**.** *Let v be a unit potential on G that satisfies the electrical laws (KCL, KVL and OL). Then*

$$E_p^r(v) = \min_{u \in \mathcal{P}_{ab}^1} E_p^r(u).$$

Proof. Fix $u \in \mathcal{P}_{ab}^1$ and let $\delta_x = u_x - v_x$ for all $x \in V$. Then δ is a function $\delta : V \to \mathbb{R}$ with $\delta_a = \delta_b = 0$ since $u_a = v_a = 1$ and $u_b = v_b = 0$. Let i be the current through G associated with the potential v. Similarly to the proof of Theorem 1.12, we find

$$\begin{aligned} E_p^r(u) &= E_p^r(v) + E_p^r(\delta) + \sum_{x,y \in V} \frac{(v_x - v_y)(\delta_x - \delta_y)}{r_{xy}} \\ &= E_p^r(v) + E_p^r(\delta) + \sum_{x,y \in V} i_{xy}(\delta_x - \delta_y). \end{aligned}$$

Because i is also a flow, it follows from Theorem 1.11 that $\sum_{x,y \in V} i_{xy}(\delta_x - \delta_y) = (\delta_a - \delta_b) i_a = 0$. So it holds that $E_p^r(u) = E_p^r(v) + E_p^r(\delta)$. Since $E_p^r(\delta) \geqslant 0$ with equality if and only only if $\delta = 0$, the claim holds. $\square$

Using Equations 1.4 and 1.5 and the same unit flow i and unit potential v as in Theorem 1.12 and Theorem 1.13 respectively, we find that

$$E_f^r(i) = i_a^2 R_{ab} = R_{ab} \qquad \text{and} \qquad E_p^r(v) = \frac{(v_a - v_b)^2}{R_{ab}} = \frac{1}{R_{ab}}.$$

Combining this result with Theorems 1.12 and 1.13 yiels the following corollary.

Corollary 1.14. *For any $a, b \in V$, the effective resistance R_{ab} between vertices a and b satisfies*

$$\max_{u \in \mathcal{P}_{ab}^1} \frac{1}{E_p^r(u)} = R_{ab} = \min_{j \in \mathcal{F}_{ab}^1} E_f^r(j).$$

This result is useful for estimating the effective resistances on complex networks that one would rather not compute directly.

An interesting by-product of the Thomson Principle is the Rayleigh Monotonicity Law.

Theorem 1.15 (Rayleigh Monotonicity Law)**.** *If the resistances of a network are increased (or decreased), then the effective resistance between any two points in the network can only increase (or decrease).*

Proof. Let a network $G = (V, E)$ be given with resistances $r = \{r_{xy} : (x, y) \in E\}$. Let $r' = \{r'_{xy} : (x, y) \in E\}$ denote the increased resistances on G. So $r'_{xy} \geqslant r_{xy}$ for all $(x, y) \in E$. Let i and j be the unit electric current flow from a to b associated to the network with resistances r and r' respectively. Then for arbitrary $a, b \in V$ it holds that

$$R'_{ab} = E_f^{r'}(j) = \frac{1}{2} \sum_{x,y \in V} j_{xy}^2 r'_{xy} \geqslant \frac{1}{2} \sum_{x,y \in V} j_{xy}^2 r_{xy} = E_f^r(j).$$

From Thomson's Principle (Theorem 1.12) follows that

$$E_f^r(j) \geqslant E_f^r(i) = R_{ab}.$$

So $R'_{ab} \geqslant R_{ab}$. Because $a, b \in V$ were arbitrary, it follows that the effective resistance between any two points in the network can only increase. Similarly the effective resistance between any two points in the network can only decrease when the resistances are decreased. □

Remarks: In the previous proof, if $r'_{xy} > r_{xy}$ for some $(x, y) \in E$ but not all $(x, y) \in E$, it does not immediately follow that $R'_{ab} > R_{ab}$, for arbitrary $a, b \in V$. This is due to the fact that j_{xy} could be zero.

For example, consider the network in Figure 8. Then for $a, b \in \{1, 2, 3\}$ it holds that $j_{34} = 0$ for any unit flow j from a to b. Thus if $r'_{34} > r_{34}$, while $r'_{xy} = r_{xy}$ for all other combinations of x and y, it does not hold that $R'_{ab} > R_{ab}$ for $a, b \in \{1, 2, 3\}$. In fact they remain completely unchanged. However, R_{4a} does increase for any $a \in \{1, 2, 3\}$.

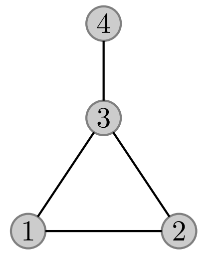


Figure 8: $j_{34} = 0$ for any flow j, when $a, b \neq 4$.

Theorem 1.16. *If the resistances of a network are increased (or decreased), then the total effective resistance $R_{tot} = \frac{1}{2} \sum_{i,j} R_{ij}$ of the network strictly increases (or decreases).*

Proof. We follow the notations from the proof of Theorem 1.15. Let $(x, y) \in E$ and assume $r'_{xy} > r_{xy}$ while $r'_{ij} = r_{ij}$ for all other possible $i, j \in V$. We can view the resistor with resistance r_{xy} as two resistors in parallel, one with resistance r'_{xy} and one with resistance $r = \frac{r_{xy} r'_{xy}}{r'_{xy} - r_{xy}}$. Hence, the old network is the new network in parallel to an edge with resistance r. Thus, it holds that

$$\frac{1}{R_{xy}} = \frac{1}{R'_{xy}} + \frac{1}{r} > \frac{1}{R'_{xy}}.$$

Thus, $R'_{xy} > R_{xy}$. Combining this with Theorem 1.15 leads to

$$R'_{tot} \quad = \quad \frac{1}{2} \sum_{i,j\,,\,ij \neq xy} R'_{ij} + R'_{xy} > \frac{1}{2} \sum_{i,j\,,\,ij \neq xy} R'_{ij} + R_{xy} \geqslant \frac{1}{2} \sum_{i,j\,,\,ij \neq xy} R_{ij} + R_{xy} = R_{tot}$$

Hence, $R'_{tot} > R_{tot}$ when increasing the resistance of one edge. If more edge resistances are increased, we can repeat the argument above. Thus, $R'_{tot} > R_{tot}$ when resistances of a network are increased. Similarly, $R'_{tot} < R_{tot}$ when resistances of a network are decreased.

□

2 The Laplacian

From this section onward, we only consider connected, simple graphs. One can define a graph by an $n \times n$ matrix that fully characterizes the graph. An example of such a matrix is the adjacency matrix A, defined by $A_{ij} = 1$ if $(i,j) \in E$ and $A_{ij} = 0$ otherwise. Another example is the (weighted) Laplacian L. For a weighted graph with edge weights $w_{ij} \in \mathbb{R}_{>0}$ for edge $(i,j) \in E$ and $w_{ij} = 0$ otherwise, the Laplacian L is defined by $L = S - W$ where S is the diagonal matrix of strengths defined by $S_{ii} = s_i = \sum_{k=1}^n w_{ik}$ and W is the matrix of weights defined by $W_{ij} = w_{ij}$.

Definition 2.1. The Laplacian L of a weighted graph $G = (V, E)$ with edge weights $w_{ij} \in \mathbb{R}_{>0}$ for edge $(i,j) \in E$ and $w_{ij} = 0$ when $(i,j) \notin E$, is an $n \times n$ matrix defined by

$$L_{ij} = \begin{cases} s_i = \sum_{k=1}^n w_{ik} & \text{if } i = j, \\ -w_{ij} & \text{otherwise.} \end{cases}$$

By defining the weights on the edges as the reciprocal of the resistances, the Laplacian can help us determine the effective resistance between two vertices in a graph. We have already encountered the Laplacian. The matrix A in Example 1.2 is the Laplacian of the given triangular graph. In general, given a graph with resistances $r_{xy} = \frac{1}{w_{xy}}$ for $(x,y) \in E$, we know from Kirchhoff's current law that

$$\sum_{y \in V} i_{xy} = \begin{cases} I_{ab} & \text{for } x = a, \\ -I_{ab} & \text{for } x = b, \\ 0 & \text{otherwise.} \end{cases}$$

Applying Ohm's law to this equation and using the fact that $r_{xy} = \frac{1}{w_{xy}}$ yields

$$\sum_{y \in V} (v_x - v_y) w_{xy} = v_x \sum_{y \in V} w_{xy} + \sum_{y \in V} v_y(-w_{xy}) = \begin{cases} I_{ab} & \text{for } x = a, \\ -I_{ab} & \text{for } x = b, \\ 0 & \text{otherwise.} \end{cases}$$

Since $w_{xx} = 0$, the previous equation can be written in matrix form as

$$L\boldsymbol{v} = I_{ab}(\boldsymbol{e}_a - \boldsymbol{e}_b). \tag{2.1}$$

Here, for vertex $i \in V$, the vector $\boldsymbol{e}_i$ is the standard basis vector of $\mathbb{R}^n$ with value one on the position associated to vertex i (and zeroes everywhere else). Solving Equation 2.1 to find v_a and v_b gives us the effective resistance $R_{ab} = \frac{(v_a - v_b)}{I_{ab}}$.

2.1 Properties of the Laplacian

Before continuing to solve Equation 2.1, let us first list a few properties of the Laplacian, we use in the upcoming subsections.

One of the first things we notice is that the row- and column-sums of L are equal to zero. Hence, we see that $\mathbf{1} = (1, 1, \ldots, 1)^\top$ is an eigenvector of L to eigenvalue zero. This means that L is not invertible.

Theorem 3.1 in [6] states that the multiplicity of the eigenvalue zero of the Laplacian for a graph $G = (V, E)$ corresponds to the number of connected components of G. Since we only consider connected graphs, we know that eigenvalue zero of L has multiplicity one.

Furthermore, since the graph $G = (V, E)$ is undirected, $w_{ij} = w_{ji}$ for all possible i and j. Thus, L is a symmetric matrix. From the Principle Axis Theorem, we know that $n \times n$ symmetric matrices have a set of eigenvectors forming an orthonormal basis of $\mathbb{R}^n$. Due to this fact, it holds that L is orthogonally diagonalizable.

From now on, let us refer to the eigenvalues of L as $\lambda_1, \lambda_2 \ldots, \lambda_n$ where $\lambda_1 = 0$ and $\lambda_i \neq 0$ for all other i. Denote the orthonormal basis of eigenvectors by $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ such that $L\boldsymbol{b}_i = \lambda_i \boldsymbol{b}_i$ for all i. Define U as the matrix having vector $\boldsymbol{b}_i$ as its $i-$th column and define D as the diagonal matrix with $D_{ii} = \lambda_i$. Then $U^\top = U^{-1}$ and we can write $L = UDU^\top$.

Because $\lambda_1 = 0$ has multiplicity one, we know that $\ker(L) = \text{span}\{\mathbf{1}\}$. For assume, that this were not the case, then there is a vector $\boldsymbol{v} \in \mathbb{R}^n$, $\boldsymbol{v} \notin \text{span}\{\mathbf{1}\}$ such that $L\boldsymbol{v} = 0$. By definition of a basis, we also have that $\boldsymbol{v} = \sum_{i=2}^n a_i \boldsymbol{b}_i$ with a_i not all equal to zero. So $L\boldsymbol{v} = \sum_{i=2}^n a_i L\boldsymbol{b}_i = \sum_{i=2}^n a_i \lambda_i \boldsymbol{b}_i = 0$. But that would mean that there are vectors among $\boldsymbol{b}_2, \ldots, \boldsymbol{b}_n$ that are linearly dependent. This contradicts the fact that they are linearly independent as basis vectors. So $\dim\ker(L) = 1$. Thus, the rank of L is $n - 1$.

2.2 The Laplacian pseudo-inverse

Let us look back at Equation 2.1,

$$L\boldsymbol{v} = I_{ab}(\boldsymbol{e}_a - \boldsymbol{e}_b).$$

If L were invertible, we could isolate $\boldsymbol{v}$ and thus easily determine R_{ab}. Sadly, L is not invertible. However, L, as a linear transformation, is invertible if we restrict it to the space perpendicular to $\ker(L)$, which is $\text{span}\{\boldsymbol{b}_2, \boldsymbol{b}_3, \ldots, \boldsymbol{b}_n\}$. Since $L\boldsymbol{b}_i = \lambda_i \boldsymbol{b}_i$ with $\lambda_i \neq 0$ for $i \neq 0$, the inverse of L on $\text{span}\{\boldsymbol{b}_2, \boldsymbol{b}_3, \ldots, \boldsymbol{b}_n\}$ would map $\boldsymbol{b}_i$ to $\lambda_i^{-1}\boldsymbol{b}_i$. We define a pseudo-inverse of L that does exactly this on $\text{span}\{\boldsymbol{b}_2, \boldsymbol{b}_3, \ldots, \boldsymbol{b}_n\}$, while being the zero map on $\ker(L) = \text{span}\{\mathbf{1}\}$.

Definition 2.2. Let $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n\}$ be an orthonormal basis of eigenvectors of the Laplacian L. A Laplacian pseudo-inverse L^+ is an $n \times n$ matrix defined by

$$L^+\boldsymbol{b}_1 = 0 \qquad \text{and} \qquad L^+\boldsymbol{b}_i = \frac{1}{\lambda_i}\boldsymbol{b}_i \text{ for } i \neq 0.$$

This definition could also be intepreted as "$L^+\mathbf{1} = 0$ and for every $\boldsymbol{w} \perp \mathbf{1}$, $L^+\boldsymbol{w} = \boldsymbol{v}$ with $\boldsymbol{v} \perp \mathbf{1}$ and $L\boldsymbol{v} = \boldsymbol{w}$.". This is the definition of the pseudo-inverse in [6]. In matrix form, L^+ could be written as $L^+ = UD^+U^\top$, where U has $\boldsymbol{b}_i$ as its $i-$th column as before and D^+ is the diagonal matrix with $D^+_{11} = 0$ and $D^+_{ii} = \lambda_i^{-1}$ for $i \neq 0$.

The Laplacian pseudo-inverse exists by definition and it is uniquely defined, since it is a special case of the Moore-Penrose pseudo-inverse (see Appendix A). This pseudo-inverse gives us a systematic, unique way to determine the effective resistance R_{ab} as we shall see in the following theorem.

Theorem 2.3. *For a graph $G = (V, E)$ with edge weights w_{ij} for edge $(i, j) \in E$, define the resistances on each edge $(i, j) \in E$ as $r_{ij} = w_{ij}^{-1}$. Then the effective resistance between vertex a and b is*

$$R_{ab} = (\boldsymbol{e}_a - \boldsymbol{e}_b)^\top L^+(\boldsymbol{e}_a - \boldsymbol{e}_b) = L^+_{aa} - 2L^+_{ab} + L^+_{bb}.$$

Proof. Since $(\boldsymbol{e}_a - \boldsymbol{e}_b) \perp \mathbf{1}$, we know that solutions $\boldsymbol{v}$ of Equation 2.1 are of the form

$$\boldsymbol{v} = I_{ab} L^+ (\boldsymbol{e}_a - \boldsymbol{e}_b) + c\mathbf{1},$$

with $c \in \mathbb{R}$ a constant. Combining this with the fact that $R_{ab} = \frac{v_a - v_b}{I_{ab}} = (\boldsymbol{e}_a - \boldsymbol{e}_b)^\top \frac{\boldsymbol{v}}{I_{ab}}$ (Definition 1.1) yields the first equation

$$R_{ab} = (\boldsymbol{e}_a - \boldsymbol{e}_b)^\top L^+ (\boldsymbol{e}_a - \boldsymbol{e}_b).$$

The latter equation follows from this and the fact that $L^+ = UD^+U^\top$ is symmetric. $\square$

In much literature on effective resistance or resistance distance, one will find the formula given in Theorem 2.3. It gives a uniquely defined way to determine the effective resistance between any pair of vertices while only having to determine L^+. This is very appealing compared to the computations using only the physics laws. It also gives rise to the following neat expression for the total effective resistance $R_{tot} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} R_{ij}$ (found in [6], originally from [9]).

Theorem 2.4. *For a graph with Laplacian L having eigenvalues $\lambda_2, \lambda_3, \ldots, \lambda_n \neq 0$, the total effective resistance is*

$$R_{tot} = n \sum_{i=2}^{n} \frac{1}{\lambda_i}.$$

Proof. We follow the proof in [6]. By definition of the total effective resistance, we have

$$R_{tot} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} R_{ij} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} R_{ij}.$$

From Theorem 2.3 it follows that

$$\begin{aligned} R_{tot} &= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} L^+_{ii} - 2L^+_{ij} + L^+_{jj} \\ &= n \sum_{i=1}^{n} L^+_{ii} - \mathbf{1}^\top L^+ \mathbf{1} \\ &= n \operatorname{Tr}(L^+). \end{aligned}$$

Because $L^+ = UD^+U^\top$ and because similar matrices have the same trace, it follows that

$$R_{tot} = n \operatorname{Tr}(L^+) = n \operatorname{Tr}(D^+) = n \sum_{i=2}^{n} \frac{1}{\lambda_i}.$$

$\square$

2.3 An alternative for the Laplacian pseudo-inverse

In the previous section, we have seen the Laplacian pseudo-inverse L^+ as a means to solve Equation 2.1. However, to determine L^+ we have to find all eigenvalues and eigenvectors of L, which is computationally inefficient. Previously, we only resulted to L^+ due to the need of having some sort of inverse matrix for L to solve Equation 2.1. We wonder if the traditional method of finding a matrix inverse by performing elementary row operations on $(L|I)$ would yield similar but more practical results.

Performing elementary row operations on the augmented matrix $(L|I)$ will yield a new pair of augmented matrix $(A|B)$ with $BL = A$. The matrix of interest is B, since this would have been the inverse if L were invertible and A were I. Depending on the order and number of row operations we perform, we will end up with different matrices. One way to get a unique result is to go all the way until $(L|I)$ reaches its reduced row echelon form, which is unique. However, by doing so we lose sight of the meaning of our matrix B. To prevent this, we find our alternative inverse of L by performing elementary row operations on $(L|I)$ in the following order:

1. Add the top $(n-1)$ rows to the bottommost row.
2. Continue performing row operations on the top $(n-1)$ rows of the matrix until the top $(n-1)$ rows have reached its reduced row echelon form.

We denote the resulting matrix by $(T|H)$. This final result is unique since the bottommost row will always consist of n zeroes and n ones regardless or the order we do step 1 in, and the top $n-1$ rows are unique, because the reduced row echelon form is so. We illustrate our method with an example.

Example 2.5. Consider the triangle graph with all edge weights equal to the one that we have used before (Figure 1). We start off with

$$(L|I) = \left(\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right).$$

After adding the top two rows to the bottommost row we have

$$\left(\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right).$$

Lastly, reducing the previous top two rows into until they are in reduced row echelon form gives us

$$(T|H) = \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -1 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right).$$

■

Let us focus on the appearances of T and H for general L.

As stated before and seen in Example 2.5, the first step would replace the bottommost row of $(L|I)$ by zeroes on the left-hand side and ones on the right-hand side. This is due to the fact that L has column sum 0 and I has column sum 1. Furthermore, due to the fact that the top $(n-1)$ rows of L are linearly independent, since L has rank $(n-1)$, and L has row sum 0, T satisfies $T_{ii} = 1$, $T_{in} = -1$ for $i = 1, \dots, n-1$, and $T_{ij} = 0$ for all other i and j.

As for H, let us first define $A(i)$ to be the $(n-1)\times(n-1)$ submatrix of the matrix A obtained by removing row i and column i from A. We also let $A(i)$ denote the submatrix of A obtained by removing the row and column associated to vertex i from A. Because $T(n)$ is the identity matrix, and $HL = T$ by construction of H and T, it follows that $L(n)$ is invertible and $H(n) = L(n)^{-1}$.

By denoting the $(n-1)\times(n-1)$ identity matrix by I_{n-1}, the matrices T and H obtained from the row operations above thus have the form

$$T = \left(\begin{array}{ccc|c} & & & -1 \\ & I_{n-1} & & \vdots \\ & & & -1 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right), \qquad H = \left(\begin{array}{ccc|c} & & & 0 \\ & L(n)^{-1} & & \vdots \\ & & & 0 \\ \hline 1 & \cdots & 1 & 1 \end{array}\right). \tag{2.2}$$

Our target matrix H can be used to determine the effective resistance much like L^{+}.

Theorem 2.6. *Let L be the Laplacian characterising the graph $G = (V, E)$ and H be as in Equation 2.2. Then the effective resistance between any pair of vertices $a, b \in V$ is equal to*

$$R_{ab} = (\boldsymbol{e}_a - \boldsymbol{e}_b)^{\top} H(\boldsymbol{e}_a - \boldsymbol{e}_b).$$

Proof. Multiplying both sides of Equation 2.1 by H yields

$$HL\boldsymbol{v} = T\boldsymbol{v} = \boldsymbol{v} - v_n\mathbf{1} = I_{ab}H(\boldsymbol{e}_a - \boldsymbol{e}_b).$$

Thus we have

$$\boldsymbol{v} = I_{ab}H(\boldsymbol{e}_a - \boldsymbol{e}_b) + v_n\mathbf{1}.$$

As in Theorem 2.3, using the fact that $R_{ab} = (\boldsymbol{e}_a - \boldsymbol{e}_b)^{\top}\frac{\boldsymbol{v}}{I_{ab}}$ gives us

$$R_{ab} = (\boldsymbol{e}_a - \boldsymbol{e}_b)^{\top} H(\boldsymbol{e}_a - \boldsymbol{e}_b).$$

□

Let the subscripts of $L(k)^{-1}_{ij}$ denote the row and column of associated to vertex i and j respectively. The previous theorem is equivalent to the following description of the effective resistance.

Theorem 2.7. *Let L be the Laplacian characterising the graph $G = (V, E)$ and let $k \in V$. The effective resistance between any pair of vertices $a, b \in V$ is*

$$R_{ab} = \begin{cases} L(k)^{-1}_{aa} & \text{for } a \neq k, b = k \\ L(k)^{-1}_{bb} & \text{for } a = k, b \neq k \\ L(k)^{-1}_{aa} + L(k)^{-1}_{bb} - 2L(k)^{-1}_{ab} & \text{for } a, b \neq k. \end{cases}$$

Proof. Without loss of generality, we may assume the vertices of the graph are labelled in such way that vertex k corresponds to the $n-$th vertex used in L. Let H be the matrix as in Equation 2.2. From Theorem 2.6 we have

$$\begin{aligned} R_{ab} &= (\boldsymbol{e}_a - \boldsymbol{e}_b)^{\top} H(\boldsymbol{e}_a - \boldsymbol{e}_b) \\ &= H_{aa} - H_{ab} - H_{ba} + H_{bb}. \end{aligned}$$

The matrix H satisfies $H_{ik} = 0$ for $i \neq k$ and $H_{kj} = 1$ for all possible j. Furthermore, $H(k) = L(k)^{-1}$, which is a symmetric matrix because $L(k)$ is symmetric. Thus $H_{ij} = H_{ji} = L(k)^{-1}_{ij}$ for $i, j \neq k$. By using this information about H and considering all possible pairs of a and b we find the formula for R_{ab} as in the statement. □

Furthermore, Theorem 2.7 gives another explicit formula for the total effective resistance.

Corollary 2.8. *Let L be the Laplacian characterising the graph $G = (V, E)$ and let $k \in V$. The total effective resistance is equal to*

$$R_{tot} = n \sum_{i \in V i \neq k} L(k)^{-1}_{ii} - \mathbf{1}_{n-1} L(k)^{-1} \mathbf{1}_{n-1}.$$

Compared to the formula given in Theorem 2.4, this might not be the most elegant nor compact formula to describe R_{tot}. However, it does provide an easier method to determine the value of R_{tot}, since one only has to compute the inverse of a single matrix, which can be done with elementary row operations.

2.4 Generalized inverse of the Laplacian

In the previous two subsections we presented two ways to partially solve Equation 2.1 in order to compute the value of R_{ab}. In both cases we make use of a specific uniquely defined matrix M that mimics an inverse of L and find that $R_{ab} = (\boldsymbol{e}_a - \boldsymbol{e}_b)^\top M(\boldsymbol{e}_a - \boldsymbol{e}_b)$. It turns out that this formula for R_{ab} holds for a more general set of matrices, namely for all generalized inverses of L ([2] Lemma 3).

For a matrix A, the matrix A^g is called a generalized inverse of A if $AA^gA = A$. We know that such matrix A^g exists since the Moore-Penrose pseudo-inverse is also a generalized inverse. However, the generalized inverse does not have to be unique.

Theorem 2.9. *Let L be the Laplacian characterising the graph $G = (V, E)$ and let X be a generalized inverse of L. For any $a, b \in V$ the effective resistance between a and b is*

$$R_{ab} = (\boldsymbol{e}_a - \boldsymbol{e}_b)^\top X(\boldsymbol{e}_a - \boldsymbol{e}_b).$$

Proof. Let $K = XL - I$, then $XL = I + K$, thus $LXL = L + LK$. Since X is a generalized inverse of L, it holds that $LXL = L$. Hence $LK = 0$. This is only possible if the columns of K are vectors in $\ker(L) = \operatorname{span}\{\mathbf{1}\}$. So column j of K is in the form $c_j\mathbf{1}$ with $c_j \in \mathbb{R}$.

We know that $L\boldsymbol{v} = I_{ab}(\boldsymbol{e}_a - \boldsymbol{e}_b)$ (Equation 2.1) for a certain current I_{ab} from a to b. This yields $XL\boldsymbol{v} = (I + K)\boldsymbol{v} = I_{ab}X(\boldsymbol{e}_a - \boldsymbol{e}_b)$. Thus

$$\boldsymbol{v} = I_{ab}X(\boldsymbol{e}_a - \boldsymbol{e}_b) - K\boldsymbol{v} = I_{ab}X(\boldsymbol{e}_a - \boldsymbol{e}_b) - c\mathbf{1},$$

with $c = \sum_{j=1}^n c_j v_j$. Again, by using $R_{ab} = (\boldsymbol{e}_a - \boldsymbol{e}_b)^\top \frac{\boldsymbol{v}}{I_{ab}}$ we find

$$R_{ab} = (\boldsymbol{e}_a - \boldsymbol{e}_b)^\top X(\boldsymbol{e}_a - \boldsymbol{e}_b).$$

□

Instead of L^+ or H, at times an other generalized inverse X could be more useful for determining the effective resistance as we can see in [2] Theorem 5.

2.5 The determinant of the Laplacian submatrices

Instead of the Laplacian generalized inverses or the inverse of the Laplacian submatrix, we can also use the determinants of submatrices of the Laplacian to calculate the effective resistance. We extend the definition of our submatrix with one row and column removed. For $i, j \in V$, define $L(i, j)$ to be the $(n-2) \times (n-2)$ submatrix of the Laplacian L obtained by removing the rows and columns associated to vertices i and j from L. Then the effective resistance between vertices $a, b \in V$ can be written as in the following theorem.

Theorem 2.10. *Let $G(V, E)$ be a simple weighted connected graph with $n \geqslant 3$ vertices and with edge weights $w_{ij} \in \mathbb{R}_{>0}$ for edge $(i, j) \in E$, where $w_{ij} = 0$ if $(i, j) \notin E$. Define the resistance of edge $e = (i, j) \in E$ to be $R_e = \frac{1}{w_{ij}}$. Then, the effective resistance between any pair of vertices $a, b \in V$ is*

$$R_{ab} = \frac{\det L(a, b)}{\det L(a)}.$$

The standard proof (found in [3]) makes use of Corollary 1.14, direct computations of $\det L(a)$ and Cramer's rule to find the inverse of $L(a, b)$. It can be also be found in Dutch in [8]. We provide an alternative proof.

Proof of Theorem 2.10. Theorem 2.7 with $k = a$ implies that $R_{ab} = L(a)^{-1}_{bb}$. Apart from using Gauss-Jordan elimination on $(L|I)$ or $(L(a)|I)$, we can also use Cramer's rule to determine $L(a)^{-1}_{bb}$. Recall that for an invertible matrix A, its inverse can be written explicitly by Cramer's rule as $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$ ($\operatorname{adj}(A)$ is the transpose of the cofactor matrix of A). In particularly, for diagonal entries of A^{-1}, we have $A^{-1}_{ii} = \frac{\det(A(i))}{\det(A)}$. Thus,

$$R_{ab} = L(a)^{-1}_{bb} = \frac{\det L(a, b)}{\det L(a)}.$$

□

Example 2.11. Consider the graph given in Figure . Then by labelling vertices a, b and c as 1, 2 and 3 respectively, we have

$$L = \begin{pmatrix} 4 & -1 & -3 \\ -1 & 3 & -2 \\ -3 & -2 & 5 \end{pmatrix}.$$

It holds that

$$L(a) = \begin{pmatrix} 3 & -2 \\ -2 & 5 \end{pmatrix} \quad \text{and} \quad L(a, c) = \begin{pmatrix} 3 \end{pmatrix}.$$

Thus, by Theorem 2.10

$$R_{ac} = \frac{\det L(a, c)}{\det L(a)} = \frac{3}{11}.$$

■

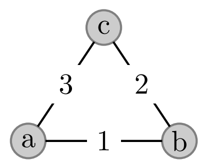


Figure 9: Configuration for Example 2.11. The numbers on the edges denote the edge weights.

3 Spanning trees

In Theorem 2.10 we saw that $R_{ab} = \frac{\det(L(a,b))}{\det(L(a))}$. One might recall from graph theory that, for an unweighted graph $G = (V, E)$, the number of spanning trees in G is equal to $\det(L(v))$ for any $v \in V$ (The Matrix-Tree Theorem). So there is certainly a connection between the effective resistance between vertices in a graph and the number of spanning trees present in the graph. In fact, it turns out that $\det(L(a, b))$ is the number of spanning trees of G containing the edge (a, b), regardless of the existence of (a, b) in E. We could say that

$$R_{ab} = \frac{\#\text{ spanning trees through edge } (a, b) \text{ in the graph } G \text{ with added edge } (a, b)}{\#\text{ spanning trees of } G}.$$

This is a special case of a similar result for weighted graphs. We follow the notations of [14]. Let $G = (V, E)$ we a weighted graph with edge weights w_{ij} for $(i, j) \in E$ or w_e for $e \in E$. We let $\mathcal{T}(G)$ denote the set of spanning trees of G. For $S \subseteq E$ define $w^S = \prod_{e \in S} w_e$.

Define $T(G) = \sum_{T \in \mathcal{T}(G)} w^T$, also called the spanning tree enumerator. Note that for unweighted graphs, all edges have edgeweight equal to 1. Thus $w^T = 1$ for all $T \subseteq E$. This means that for unweighted graphs, $T(G)$ denotes the number of spanning trees of G. According to Theorem 5 in [14], there is a similar result to the Matrix-Tree Theorem for weighted graphs.

Theorem 3.1 (Weighted Matrix-Tree Theorem)**.** *For any graph $G = (V, E)$ with Laplacian L and any vertex $v \in V$, it holds that*

$$T(G) = \det(L(v)).$$

For $a, b \in V$, we define G/ab as the graph obtained by merging vertex a and b into a single vertex and removing all edges that were between them. For $e \in E$, we define $G - e$ as the graph obtained by removing edge e from G and $\mathcal{T}_e(G)$ as the set of spanning trees of G through edge e. For $e \notin E$, define $G + e$ as the graph obtained by adding edge e with edge weight $w_e = 1$ to G.

Theorem 3.2. *For a weighted graph $G = (V, E)$ with Laplacian L and any pair of vertices $a, b \in V$ it holds that*

$$T(G/ab) = \det L(a, b).$$

Proof. We consider the following two possibilities: edge $e := (a, b) \in E$ and $e \notin E$.

If $e \in E$, then we can divide $\mathcal{T}(G)$, the set of spanning trees of G, into two disjunct sets, namely the subset of trees consisting edge e ($\mathcal{T}_e(G)$) and the subset of trees that does not ($\mathcal{T}(G - e)$). Hence it holds that

$$\sum_{T \in \mathcal{T}(G)} w^T = \sum_{T \in \mathcal{T}_e(G)} w^T + \sum_{T \in \mathcal{T}(G-e)} w^T. \tag{3.1}$$

Furthermore, note that for $T \in \mathcal{T}(G/ab)$ it holds that $T \cup \{e\} \in \mathcal{T}_e(G)$ and for $T \in \mathcal{T}_e(G)$ it holds that $T \backslash \{e\} \in \mathcal{T}(G/ab)$. So the sets $\mathcal{T}(G/ab)$ and $\mathcal{T}_e(G)$ are equinumerous and differ only in the existence of the edge e. This gives us

$$\sum_{T \in \mathcal{T}_e(G)} w^T = \sum_{T \in \mathcal{T}(G/ab)} w^T \cdot w_e. \tag{3.2}$$

Recall that all the sums involved are in fact spanning trees enumerators. Combining Equations 3.1 and 3.2 yields

$$T(G/ab) = \frac{1}{w_e} \left(T(G) - T(G-e) \right). \tag{3.3}$$

Let L be the Laplacian of G and L' be the Laplacian of $G - e$. Note that is almost identical to L and differs only in the four entries related to vertex a and b, namely $L'_{ii} = L_{ii} - w_i$ for $i \in \{a, b\}$ and $L'_{ab} = L'_{ba} = 0$. So $L'(a)$ only differs from $L(a)$ at entry L'_{bb}. Without loss of generality, we can assume that vertex a and b are labelled as vertex 1 and 2 respectively. Applying Theorem 3.1 at vertex a to Equation 3.3 gives us

$$\begin{aligned} T(G/ab) &= \frac{1}{w_e} \big(\det(L(a)) - \det(L'(a)) \big) \\ &= \frac{1}{w_e} \left(\det \left(\begin{array}{c|c} L_{bb} & \boldsymbol{l}^\top \\ \hline \boldsymbol{l} & L(a,b) \end{array} \right) - \det \left(\begin{array}{c|c} L_{bb} - w_e & \boldsymbol{l}^\top \\ \hline \boldsymbol{l} & L(a,b) \end{array} \right) \right), \end{aligned}$$

with $\boldsymbol{l}$ being the appropriate $n-2$ dimensional vector. By expanding both determinants in the first row we find

$$T(G/ab) = \frac{1}{w_e} \cdot w_e \det(L(a,b)) = \det(L(a,b)).$$

So $T(G/ab) = \det(L(a,b))$ for $e \in E$.

If $e \notin E$, consider the graph $G' = G + e$. We can then follow the proof of the case $e \in E$ for the graph G' up until Equation 3.3. This gives us $T(G'/ab) = \frac{1}{w_e} (T(G') - T(G' - e))$. Note that $G'/ab = (G + e)/ab = G/ab$ since the edge e gets removed after merging nodes a and b together. Furthermore $G' = G + e$ and $G' - e = G$. So we have

$$T(G/ab) = \frac{1}{w_e} \left(T(G+e) - T(G) \right),$$

from which we can continue with the proof after Equation 3.3 in a similar manner and find that $T(G/ab) = \det(L(a,b))$ for $e \notin E$ as well. □

The previous two theorems leads us to the next theorem.

Theorem 3.3. *For a graph $G = (V, E)$ and $a, b \in V$, the effective resistance between vertex a and b is*

$$R_{ab} = \frac{T(G/ab)}{T(G)}.$$

Proof. This is a direct result of Theorem 2.10 combined with Theorems 3.1 and 3.2. In [14], this theorem corresponds to Theorem 8, for which they have two other proofs. The first proof is essentially the same as ours while the second proof introduces a linear transformation that magically yields the result. □

3.1 Some applications to unweighted graphs

As stated earlier, an unweighted graph is just a special case of a weighted graph in which all edges have edge weights equal to 1. So the value of $T(G) = \sum_{T\in\mathcal{T}(G)} w^T$ is just the number of spanning trees in G. This means that Theorem 3.3 states that

$$R_{ab} = \frac{\#\text{ spanning trees of } G/ab}{\#\text{ spanning trees of } G}.$$

In the proof of Theorem 3.2, we have also seen that the spanning trees of G/ab and G or $G + (a, b)$ differ only in the existence of the edge (a, b). Thus the number of spanning trees in G/ab is the same as the number of spanning trees through edge (a, b) in the graph G with added edge (a, b) . This leads us back to the formula we started with

$$R_{ab} = \frac{\#\text{ spanning trees through edge } (a,b) \text{ in the graph } G \text{ with added edge } (a,b)}{\#\text{ spanning trees of } G}. \qquad (3.4)$$

In Chapter 1.2.3, we have used $K_{4,4}$ as an example of a graph for which the series/parallel laws and the Y-Δ transform is nonfunctional. This made us very curious of the values of the effective resistances between vertices in $K_{n,n}$ or $K_{m,n}$ in general. Thankfully, using Theorem 2.7 and Matlab (See Appendix C), it is easy to determine them. Upon trying for different values of n, it seems that the effective resistance between vertices a, b in $K_{n,n}$ follows the formula

$$R_{ab} = \begin{cases} \frac{2n-1}{n^2} & \text{for } a,b \text{ in different parts} \\ \frac{2}{n} & \text{for } a,b \text{ in the same part.} \end{cases}$$

Here the parts denote the bipartition of the vertex set of $K_{n,n}$. By using Equation 3.4 we can prove the correctness of this formula (see Example 3.5) as well as determine the effective resistance between vertices in other interesting graphs. As an example we look at K_n, the complete graph with n vertices, and $K_{m,n}$, the complete bipartite graph with vertex sets of size m and n.

Example 3.4. Let a, b be two vertices in K_n. Using our Theorem 3.1 and Lemma 1 from [12] we find that the number of spanning trees of K_n is n^{n-2} (also known as Cayley's Formula). According to Corollary 7′ in [12], the number of spanning trees of K_n through (a, b) is $2n^{n-3}$. Thus we find that

$$R_{ab} = \frac{2n^{n-3}}{n^{n-2}} = \frac{2}{n}.$$

■

The number of spanning trees found in Example 3.4 has been derived by using Theorems 3.1 and 3.2, from which Theorem 3.3 follows. We would also like to present an example where we determine the effective resistance between two points by actually counting the number of spanning trees. We shall do so for $K_{m,n}$ in Appendix B.

Example 3.5. The effective resistance between vertices a, b in $K_{m,n}$ can be found by using Equation 3.4 and the theorems from Appendix B. It is equal to

$$R_{ab} = \begin{cases} \frac{m+n-1}{mn} & \text{for } a,b \text{ in different parts,} \\ \frac{2}{m} & \text{for } a,b \text{ in the same part with size } m, \\ \frac{2}{n} & \text{for } a,b \text{ in the same part with size } n. \end{cases}$$

■

4 Random walks

In the previous sections we have seen different approaches to determine the effective resistance. In this section we shall present known relationships between the effective resistance and random walks.

The fact that there is a connection between electrical networks and random walks is nothing new. In fact, many people studying random walks tend to use electrical networks as a tool to give a better understanding to somewhat complicated concepts in random walks.

Let an undirected graph $G = (V, E)$ with edge weights w_{ij} for $i, j \in V$ be given. Note that if $(i, j) \notin E$ then $w_{ij} = 0$. And let $s_i = \sum_j w_{ij}$. As in all previous chapters, we can view this graph as an electrical network with each edge $(i, j) \in E$ being a resistor with resistance $r_{ij} = \frac{1}{w_{ij}}$. We shall define a random walk on this graph by defining the transition matrix P to satisfy $P_{ij} = \frac{w_{ij}}{s_i}$.

4.1 Commute time

The commute time between vertices a and b is the expected time for a random walk starting in a, to reach b and return to a (or the other way around). Let $\tau(a, b)$ be the expected first hittng time in b starting in a. Then, the commute time between a and b is $\tau(a, b) + \tau(b, a)$.

According to Chandra et al. [4], the commute time between vertices a and b in an unweighted graph G, is equal to $2mR_{ab}$, where m is the number of edges in G. For weighted graphs, this relation translates as follows

Theorem 4.1. *Let a graph $G = (V, E)$ with edge weights w_{ij} for edge $(i, j) \in E$ be given. Define an electric network on this graph by setting each edge $(i, j) \in E$ to be a resistor with resistance $r_{ij} = \frac{1}{w_{ij}}$. Define a random walk on the graph by setting the transition probabilities to be $p_{ij} = \frac{w_{ij}}{s_i}$. Then for any pair of $a, b \in V$ the effective resistance satisfies*

$$R_{ab} = \frac{\tau(a, b) + \tau(b, a)}{\sum_k s_k}.$$

This theorem and its proof is given in [6] Chapter 4.4. We would like to present a different proof based on Theorem 2.7. Before we go on to the proof, a few notations and lemmas need to be discussed.

Recall that the Laplacian as in Definition 2.1 was also defined as $L = S - W$ where S is the diagonal matrix of strengths defined by $S_{ii} = s_i = \sum_j w_{ij}$ and W is the weight matrix defined by $W_{ij} = w_{ij}$. It holds that $S(I - P) = L$. And so

$$S(b)(I - P(b)) = L(b). \tag{4.1}$$

Here for any matrix A and vertex $b \in V$, $A(b)$ is defined as the sub-matrix of A with the row and columns regarding vertex b removed. We have seen in Section 2.3 that $L(b)$ is invertible. $S(b)$ is invertible too, since it is a diagonal matrix without any zeroes on the diagonal. So $(S(b))^{-1}_{ii} = \frac{1}{S(b)_{ii}}$. This means $(I - P(b))$ must be invertible also.

Lemma 4.2. *Let $G = (V, E)$ be a connected weighted graph with edge weights $w_{ij} \neq 0$ for edge $(i, j) \in E$. Define a random walk on the graph by setting the transition probabilities to be $p_{ij} = \frac{w_{ij}}{s_i}$ and let P be the transition matrix. Then for any $b \in V$ the sub-matrix $P(b)$ of P, obtained by removing the column and row from P related to vertex b, satisfies*

$$(I - P(b))^{-1} = \sum_{k=0}^{\infty} P^k(b).$$

Proof. We can prove this in the same fashion as proving the equation $\sum_{k=0}^{\infty} x^k = (1-x)^{-1}$ for $x \in \mathbb{R}$, $|x| < 1$. Firstly, for any matrix A, we have $(I - A)\sum_{k=0}^{n} A^k = (I - A^{n+1})$. Secondly, because $(I - P(b))$ is invertible, as explained above, $M := (I - P(b))^{-1}$ exist. So we can view the following equations

$$\begin{aligned} (I - P(b)) \cdot \sum_{k=0}^{n} P^k(b) &= I - P^{n+1}(b) \\ M(I - P(b)) \cdot \sum_{k=0}^{n} P^k(b) &= M\left(I - P^{n+1}(b)\right) \\ \sum_{k=0}^{n} P^k(b) &= M\left(I - P^{n+1}(b)\right). \end{aligned}$$

By viewing $P(b)$ as the probability matrix of transitioning from any transient state to another in an absorbing Markov chain with absorbing state b, we know for $n \to \infty$ that $P^{n+1}(b) \to 0$. Hence, $\lim_{k\to\infty} P^k(b) = 0$. Thus, by taking the limit for n to ∞ on the last equation, it follows that

$$\sum_{k=0}^{\infty} P^k(b) = M(I - 0) = M = (I - P(b))^{-1}.$$

□

We know that $P^k(b)_{ij}$ is the probability to reach j in k steps, starting in i, before reaching b. So the infinite sum $\sum_{k=0}^{\infty} P^k(b)_{ij}$ can interpreted as the expected number of hits on j starting in i before reaching b. Let us denote this expectation by $\tau_{[b]}(i, j)$. Thus we have

$$\tau_{[b]}(a, k) = \sum_{n=0}^{\infty} (P^n(b))_{ak}. \tag{4.2}$$

Furthermore, $\sum_{k\neq b} \tau_{[n]}(a, k)$ is the sum of the expected hits on all vertices except for b during a random walk starting in a before hitting b. Since in each step of a random walk, we hit a vertex, this sum is simply the expected first hit time in b starting in a. In other words,

$$\tau(a, b) = \sum_{k\neq b} \tau_{[b]}(a, k). \tag{4.3}$$

From Equation 4.1 and Lemma 4.2 we have

$$\begin{aligned} (L(b))^{-1} &= (I - P(b))^{-1}(S(b))^{-1} \\ &= \sum_{n=0}^{\infty} P^n(b)(S(b))^{-1}. \end{aligned}$$

Using Equation 4.2 we thus find that for $a, b, k \in V$ with $a \neq b$ or $k \neq b$,

$$(L(b))^{-1}_{ak} = \frac{\tau_{[b]}(a,k)}{s_k}. \tag{4.4}$$

With the use of the previous equations we can finally give our proof of Theorem 4.1.

Proof of Theorem 4.1. Let $a, b \in V$ and $a \neq b$. Then according to Equations 4.3 and 4.4

$$\begin{aligned} \tau(a,b) &= \sum_{k \neq b} \tau_{[b]}(a,k) \\ &= \sum_{k \neq b} L(b)^{-1}_{ak} s_k. \end{aligned}$$

According to Theorem 2.7 it holds that for $k \neq a$ and $k \neq b$

$$R_{ak} = L(b)^{-1}_{aa} + L(b)^{-1}_{kk} - 2L(b)^{-1}_{ak}.$$

Thus

$$\begin{aligned} L(b)^{-1}_{ak} &= \frac{1}{2}\left(L(b)^{-1}_{aa} + L(b)^{-1}_{kk} - R_{ak}\right) \\ &= \frac{1}{2}\left(R_{ab} + R_{kb} - R_{ak}\right). \end{aligned}$$

Here we use Theorem 2.7 again for the latter equation. Combining this result with Equation 4.5, we find

$$\begin{aligned} \tau(a,b) &= \sum_{k \neq b} L(b)^{-1}_{ak} s_k \\ &= \sum_{k \neq a, k \neq b} L(b)^{-1}_{ak} s_k + L(b)^{-1}_{aa} s_a \\ &= \sum_{k \neq a, k \neq b} \frac{1}{2}\left(R_{ab} + R_{kb} - R_{ak}\right) s_k + R_{ab} \cdot s_a. \end{aligned}$$

The same equation with a and b interchanged can be found for $\tau(b,a)$. Adding $\tau(a,b)$ and $\tau(b,a)$ together and using the fact that $R_{ij} = R_{ji}$ for all possible $i, j \in V$, we find

$$\begin{aligned} \tau(a,b) + \tau(b,a) &= \sum_{k \neq a, k \neq b} \frac{1}{2}\left(R_{ab} + R_{kb} - R_{ak}\right) s_k + R_{ab} \cdot s_a \\ &\quad + \sum_{k \neq a, k \neq b} + \frac{1}{2}\left(R_{ba} + R_{ka} - R_{bk}\right) s_k + R_{ba} \cdot s_b \\ &= \sum_{k \neq a, k \neq b} \left(R_{ab} \cdot s_k\right) + R_{ab}(s_a + s_b) \\ &= R_{ab} \sum_k s_k. \end{aligned}$$

Thus

$$R_{ab} = \frac{\tau(a,b) + \tau(b,a)}{\sum_k s_k}.$$

□

4.2 Escape probability

For $a, b \in V$, we define $p_{esc}(a \to b)$ as the probability that, starting in a the random walk reaches b before returning to a. We can also call this the escape probability at a. Let us view the graph of the random walk starting in a as though it only has two vertices a and b with probability p to jump back to a and probability $(1-p)$ to jump to b. Then $p_{esc}(a \to b) = (1-p)$.

In this situation, we find $\tau_{[b]}(a,a) = \sum_{n=0}^{\infty}(n+1)p^n(1-p)$. We can argue this as follows: the probability to reach b after k steps is the probability to return to vertex a in a total of $k-1$ times before finally taking a step to vertex b. In other words, this probability is $p^{k-1}(1-p)$. During each step back to a, we hit vertex a once, and including the start in a we hit vertex a in a total of $(k-1)+1 = k$ times. Thus the expected hits on a before reaching b in k steps is $kp^{k-1}(1-p)$. Summing over all possible number of steps before reaching b, we find $\tau_{[b]}(a,a) = \sum_{n=0}^{\infty}(n+1)p^n(1-p)$.

Since $\sum_{n=0}^{\infty}(n+1)p^n = \frac{1}{(1-p)^2}$ we find $\tau_{[b]}(a,a) = \frac{1}{(1-p)^2}(1-p) = \frac{1}{(1-p)}$ and thus

$$\tau_{[b]}(a,a) = \frac{1}{p_{esc}(a \to b)}. \tag{4.5}$$

The following Theorem has been discussed and proven in [5] and [11]. We give an alternative proof.

Theorem 4.3. *On a graph $G = (V, E)$ with edge weights w_{ij} for edge $(i,j) \in E$. Define an electric network on this graph by setting each edge $(i,j) \in E$ to be a resistor with resistance $r_{ij} = \frac{1}{w_{ij}}$. Define a random walk on the graph by setting the transition probabilities to be $p_{ij} = \frac{w_{ij}}{s_i}$. Then for any pair of $a, b \in V$ the effective resistance satisfies*

$$R_{ab} = \frac{1}{s_a \cdot p_{esc}(a \to b)}.$$

Proof. Using Theorem 2.7, Equation 4.4 and 4.5 we find

$$R_{ab} = L(b)^{-1}_{aa} = \frac{\tau_{[b]}(a,a)}{s_a} = \frac{1}{s_a \cdot p_{esc}(a \to b)}.$$

□

5 The difference in effective resistance due to edge weight change

One of the properties of the effective resistance that makes it a viable option for determining the robustness of a network, is the fact that resistances do not increase when adding an edge or increasing an edge weight. This fact is actually one case of the Rayleigh Monotonicity Law (Theorem 1.15), where the resistance of an edge is the reciprocal of the edge weight. However, this theorem does not give us insight on how much the resistances decrease. So we would like to present another proof of the Rayleigh Monotonicity Law. To this end, we need the following Lemma.

Lemma 5.1 (Sherman-Morrison formula)**.** *For any invertible $n\times n$ matrix A and column vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, with $1 + \boldsymbol{v}^\top A^{-1}\boldsymbol{u} \neq 0$, it holds that*

$$\left(A + \boldsymbol{u}\boldsymbol{v}^\top\right)^{-1} = A^{-1} - \frac{A^{-1}\boldsymbol{u}\boldsymbol{v}^\top A^{-1}}{1 + \boldsymbol{v}^\top A^{-1}\boldsymbol{u}}.$$

Proof. This lemma can be proven by straightforward computation. □

Theorem 5.2 (Reformulation of the Rayleigh Monotonicity Law, Theorem 1.15)**.** *The effective resistances in a network do not increase (decrease) when an edge is added (removed) or the weight of an edge is increased (decreased).*

Proof. Let a connected weighted graph $G = (V, E)$ with edge weights $w_{ij} \neq 0$ for edge $(i, j) \in E$ and $w_{ij} = 0$ for $(i, j) \notin E$ be given. Let L be the Laplacian associated with this graph. For $a, b \in V$ arbitrary, we can add the edge (a, b) with weight α or raise the weight if $(a, b) \in E$, by raising the edge weight w_{ab} by a certain positive constant α. By doing so we gain a new graph $G' = (V, E \cup \{(a, b)\})$ with edge weights $w'_{ab} = w_{ab} + \alpha$ and $w'_{ij} = w_{ij}$ for all other $i, j \in V$. The Laplacian L' associated to G' satisfies

$$L'_{ij} = \begin{cases} L_{ij} + \alpha & \text{for } i = j \text{ with } i, j \in \{a, b\} \\ L_{ij} - \alpha & \text{for } i \neq j \text{ with } i, j \in \{a, b\} \\ L_{ij} & \text{for all other } i, j \in V. \end{cases}$$

So $L'(b)$ satisfies $L'(b)_{aa} = L(b)_{aa} + \alpha$ and $L'(b)_{ij} = L(b)_{ij}$ for all other $i, j \in V$.

Let the column vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^{n-1}$ be defined as $\boldsymbol{u} = \alpha\boldsymbol{e}_a$ and $\boldsymbol{v} = \boldsymbol{e}_a$. Then $L(b)+\boldsymbol{u}\boldsymbol{v}^\top = L'(b)$. Since $L(b)$ is invertible, we can use Lemma 5.1, which gives us

$$\begin{aligned} L'(b)^{-1} = (L(b) + \boldsymbol{u}\boldsymbol{v}^\top)^{-1} &= L(b)^{-1} - \frac{L(b)^{-1}\boldsymbol{u}\boldsymbol{v}^\top L(b)^{-1}}{1 + \boldsymbol{v}^\top L(b)^{-1}\boldsymbol{u}} \\ &= L(b)^{-1} - \frac{B}{1 + \alpha L(b)^{-1}_{aa}} \end{aligned}$$

with B an $(n-1) \times (n-1)$ matrix satisfying $B_{ij} = \alpha L(b)^{-1}_{ai} L(b)^{-1}_{ja}$. By defining $\beta = \frac{\alpha}{1+\alpha L(b)^{-1}_{aa}}$ we get

$$L'(b)^{-1}_{ij} = L(b)^{-1}_{ij} - \beta L(b)^{-1}_{ai} L(b)^{-1}_{ja}. \tag{5.1}$$

Note that $\alpha > 0$ by definition of α and $L(b)^{-1}_{aa} = \frac{\tau_{[b]}(a,a)}{s_a}$ (Equation 4.4). Since $\tau_{[b]}(a, a) \geqslant 1$ and $s_a > 0$, we find that $L(b)^{-1}_{aa} > 0$. Another way to see this, is that resistances are positive and $L(b)^{-1}_{aa}$ denotes the effective resistance between a and b (Theorem 2.7). Hence, $\beta > 0$.

Combining Equation 5.1 with the fact that $L(b)^{-1}$ is a symmetric matrix, since $L(b)$ is symmetric, we find for any $k \in V$ and for any $i, j \in V$ with $i \neq j$ that

$$\begin{aligned} L'(b)^{-1}_{kk} &= L(b)^{-1}_{kk} - \beta \left(L(b)^{-1}_{ak}\right)^2, && (5.2) \\ L'(b)^{-1}_{ii} + L'(b)^{-1}_{jj} - 2L'(b)^{-1}_{ij} &= L(b)^{-1}_{ii} + L(b)^{-1}_{jj} - 2L(b)^{-1}_{ij} - \beta \left(L(b)^{-1}_{ai} - L(b)^{-1}_{aj}\right)^2. && (5.3) \end{aligned}$$

Let us denote R_{ij} and R'_{ij} as the effective resistance between $i, j \in V$ on G and G' respectively. Recall from Theorem 2.7 that

$$R_{ij} = \begin{cases} L(b)^{-1}_{ii} & \text{for } i \neq b, j = b \\ L(b)^{-1}_{jj} & \text{for } i = b, j \neq b \\ L(b)^{-1}_{ii} + L(b)^{-1}_{jj} - 2L(b)^{-1}_{ij} & \text{for } i, j \neq b. \end{cases}$$

Needless to say, this also holds when replacing R and L by R' and L' respectively. So with the use of Theorem 2.7 and Equations 5.2 an 5.3, we find

$$R'_{ij} = \begin{cases} R_{ij} - \beta \left(L(b)^{-1}_{ai}\right)^2 & \text{for } i \neq b, j = b \\ R_{ij} - \beta \left(L(b)^{-1}_{aj}\right)^2 & \text{for } i = b, j \neq b \\ R_{ij} - \beta \left(L(b)^{-1}_{ai} - L(b)^{-1}_{aj}\right)^2 & \text{for } i, j \neq b. \end{cases} \tag{5.4}$$

Because $\beta > 0$ and all $x^2 \geqslant 0$ for any $x \in \mathbb{R}$, we see that $R'_{ij} \leqslant R_{ij}$. Thus the effective resistances do not increase when an edge is added or the weight of an edge is increased.

On the other hand, if we remove or reduce the edge weight of an edge $(a, b) \in E$ instead, the whole proof can be copied with the only difference being the limitation $-w_{ab} \leqslant \alpha < 0$ and thus the value of $\beta = \frac{\alpha}{1+\alpha L(b)^{-1}_{aa}}$ in Equation 5.4. The limitation on α is due to the fact that we can not reduce the weight of an edge more than its current weight.

We shall prove that $\beta < 0$. $\beta < 0$ if and only if $\alpha L(b)^{-1}_{aa} > -1$. We know from Theorem 2.7 that $L(b)^{-1}_{aa} = R_{ab}$. Define $\hat{G}$ as the graph G without the edge (a, b) and define R as the effective resistance between vertex a and b in $\hat{G}$. We can then view the network between a and b in G as two resistors in parallel between a and b, one with resistance R and the other with resistance $r_{ab} = \frac{1}{w_{ab}}$. So, by using the parallel law, $R_{ab} = \frac{r_{ab}R}{r_{ab}+R}$. This gives us

$$\alpha L(b)^{-1}_{aa} \geqslant -w_{ab}R_{ab} = \frac{-R}{r_{ab} + R} > \frac{-R}{R} = -1.$$

Hence $\beta < 0$ and thus, from Equation 5.4, it follows that the effective resistances do not decrease when an edge is removed or the weight of an edge is decreased. □

In Equation 5.4 of the proof we see a direct formula showing the resistance reduction when increasing an edge weight. In $L(b)^{-1}$, there are no elements associated to vertex b such as $L(b)^{-1}_{ab}$. However, we can define them to satisfy $L(b)^{-1}_{kb} = L(b)^{-1}_{bk} = 0$ for all $k \in V$. Equation 5.4 can then be written as

$$R'_{ij} = R_{ij} - \beta \left(L(b)^{-1}_{ai} - L(b)^{-1}_{aj}\right)^2 \text{ for all } i, j \in V.$$

Thus the change in effective resistance between vertices $i, j \in V$ when raising edgeweight w_{ab} by α is

$$\Delta R_{ij}(a,b,\alpha) = \frac{\alpha}{1+\alpha L(b)^{-1}_{aa}} \left(L(b)^{-1}_{ai} - L(b)^{-1}_{aj} \right)^2 .$$

Thus the change in total effective resistance of G when raising edgeweight w_{ab} by α is

$$\Delta R_{tot}(a,b,\alpha) = \frac{\alpha}{1+\alpha L(b)^{-1}_{aa}} \sum_{i,j\in V} \frac{1}{2} \left(L(b)^{-1}_{ai} - L(b)^{-1}_{aj} \right)^2 . \tag{5.5}$$

The value of α could be viewed as a budget. We assume for now that we can add any edge to our network regardless of the budget α. In reality, however, the budget dictates the possible moves. Equation 5.5 gives us a direct comparison between different options for extending a given network. Given a value of α and a point b in the graph, we could directly compare the result of extending the graph from b towards any pair of points without having to determine more than $L(b)^{-1}$.

Example 5.3. Consider the network in Figure 10. The inverse of the Laplacian submatrix $L(b)^{-1}$ associated to this graph is

$$L(b)^{-1} = \frac{1}{41} \begin{pmatrix} 17 & 9 & 6 \\ 9 & 12 & 8 \\ 6 & 8 & 19 \end{pmatrix}.$$

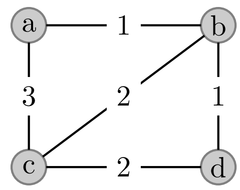


Figure 10: Configuration for Example 5.3. The numbers on the edges denote the edge weights.

The 1st, 2nd and 3rd columns and rows are associated to vertex a, b and c respectively. The values of ΔR_{tot} when demanding one endpoint to be b and setting $\alpha = 1$ are

$$\begin{aligned} \Delta R_{tot}(a,b,1) &= \frac{600}{2378} \approx 0.252 \\ \Delta R_{tot}(c,b,1) &= \frac{315}{2173} \approx 0.145 \\ \Delta R_{tot}(d,b,1) &= \frac{755}{2460} \approx 0.307 \end{aligned}$$

Hence, the best edge to add, for $\alpha = 1$ and a fixed begin or end point b is (d, b), since this yields the biggest change in R_{tot}. ■

In general, for any given $\alpha > 0$, we are interested in the pairs $(x, y) \in V \times V$ yielding the highest value for $\Delta R_{tot}(x, y, \alpha)$. We can order the pairs in $V \times V$ by their ΔR_{tot} value. For each $\alpha \in \mathbb{R}$, we define an order $\geqslant_\alpha$ on $V \times V$ by $(x_1, y_1) \geqslant_\alpha (x_2, y_2)$ when $\Delta R_{tot}(x_1, y_1, \alpha) \geqslant \Delta R_{tot}(x_2, y_2, \alpha)$. When a pair (x, y) is bigger than another one by this ordering, it means the network becomes more robust when adding edge (x, y) (with edge weight α than when adding the other. In Example 5.3, we see that $(d, b) \geqslant_1 (a, b) \geqslant_1 (c, b)$.

If the value of α changes, we wonder whether the ordering of the elements in $V \times V$ changes as well. In particular, we wonder if the biggest pair, remains the biggest pair still. Upon testing for a few random graphs using Matlab (see Appendix C), we find that the best edge to add seems to remain the same for many values of α. This led us to formulate and attempt to prove the following hypothesis.

Hypothesis 5.4. *Let $(x_1, y_1), (x_2, y_2) \in V \times V$ and $\alpha^* \in \mathbb{R}_{>0}$ be given. If $(x_1, y_1) \geqslant_{\alpha^*} (x_2, y_2)$, then $(x_1, y_1) \geqslant_\alpha (x_2, y_2)$ for all $\alpha \in \mathbb{R}_{>0}$.*

If the hypothesis were true, we can disregard α when determining the optimal edge to add to our network and only focus on the limitations from the budget in a real life setting. However, it is not true.

Consider the network in Figure 11. For this network, it holds that

$$(c,b) \geqslant_\alpha (a,b) \quad \text{for} \quad \alpha \in \left(0, \frac{102}{31}\right)$$
$$(a,b) \geqslant_\alpha (c,b) \quad \text{for} \quad \alpha \in \left(\frac{102}{31}, \infty\right).$$

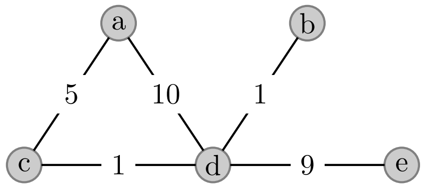


Figure 11: Counterexample configuration for Hypothesys 5.4. The numbers on the edges denote the edge weights.

In this network it holds that $\Delta R_{tot}(a,b,\alpha) = \Delta R_{tot}(c,b,\alpha)$ for $\alpha = 0$ and $\alpha = \frac{102}{31}$. In general, any two different functions of α, $\Delta R_{tot}(x_1,y_1,\alpha)$ and $\Delta R_{tot}(x_2,y_2,\alpha)$, intersect each other at most twice in α. For α beyond the highest intersection, the ordering with respect to $\geqslant_\alpha$ between (x_1,y_1) and (x_2,y_2) remains the same.

Since there are finitely many elements in $V \times V$, there exists an $\hat{\alpha} \in \mathbb{R}$, for which it does hold that if $(x_1,y_1) \geqslant_{\hat{\alpha}} (x_2,y_2)$, then $(x_1,y_1) \geqslant_\alpha (x_2,y_2)$ for all $\alpha \geqslant \hat{\alpha}$ for all possible $(x_1,y_1),(x_2,y_2) \in V \times V$.

However, it is still an open question what characteristics of a network make the network satisfy Hypothesis 5.4. Furthermore, the assumption that one can add any edge regardless of the budget α is not realistic. So we can still study how to choose the best edge to add when considering the limited possibilities in reality due to a lack of budget α.

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A Moore-Penrose pseudo-inverse

The Moore-Penrose pseudo-inverse is defined as follows.

Definition A.1. (An adaptation of [15]) Let A be an $m \times n$ matrix over a field $\mathbb{K}$, which is either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. Any matrix A^+ satisfying the following four properties is called a Moore-Penrose pseudo-inverse of A:

1. $AA^+A = A$
2. $A^+AA^+ = A^+$
3. $(AA^+)^* = AA^+$
4. $(A^+A)^* = A^+A$.

For any matrix A one can construct a Moore-Penrose pseudo-inverse by using the singular value decomposition. So the Moore-Penrose pseudo-inverse exists for all matrices A. Even though the singular value decomposition is not unique, the Moore-Penrose pseudo-inverse is. We will show these above properties in the next two proofs. For the rest of this section we denote the Moore-Penrose pseudo-inverse as pseudo-inverse.

Lemma A.2. *Any matrix A as in Definition A.1 has a Moore-Penrose pseudo-inverse.*

Proof. We prove this in two stages, first for general $m \times n$ diagonal matrices Σ over $\mathbb{K}$ with zeroes off the diagonal and then for arbitrary $m \times n$ matrices A over $\mathbb{K}$. The proof follows the uncompleted proof presented in [16].

For general diagonal matrices Σ:
Let Σ be an $m \times n$ matrix with values in $\mathbb{K}$ on the diagonal and zeroes off the diagonal. Define Σ^+ as the $n \times m$ matrix with

$$\Sigma^+_{ij} = \begin{cases} \frac{1}{\Sigma_{ij}} & \text{for } i = j \text{ and } \Sigma_{ij} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\Sigma\Sigma^+$ is a diagonal $m \times m$ matrix with

$$(\Sigma\Sigma^+)_{ij} = \begin{cases} 1 & \text{for } i = j \text{ and } \Sigma_{ij} \neq 0 \text{ (and thus } \Sigma^+_{ij} \neq 0), \\ 0 & \text{otherwise.} \end{cases} \tag{A.1}$$

Similarly $\Sigma^+\Sigma$ is a diagonal $n \times n$ matrix with $(\Sigma^+\Sigma)_{ii} = 1$ if and only if $\Sigma_{ii} \neq 0$ (and zeroes everywhere else).

We now show that Σ^+ is a pseudo-inverse of Σ:
For any pair (i, j) satisfying the dimensions of the upcoming matrices, we have

$$(\Sigma\Sigma^+\Sigma)_{ij} = (\Sigma\Sigma^+)_{ij}\Sigma_{ij} = \Sigma_{ij}.$$

The first equality follows from the fact that both $\Sigma\Sigma^+$ and Σ are diagonal matrices, and the product of diagonal matrices corresponds to multiplying diagonal elements together while leaving the rest of the matrix be zero. The latter equality is due to Equation A.1. Analogously, we obtain $(\Sigma^+\Sigma\Sigma^+)_{ij} = \Sigma^+_{ij}(\Sigma\Sigma^+)_{ij} = \Sigma^+_{ij}$. Hence $\Sigma\Sigma^+\Sigma = \Sigma$ and $\Sigma^+\Sigma\Sigma^+ = \Sigma^+$. Furthermore, it holds that $(\Sigma\Sigma^+)^* = \Sigma\Sigma^+$ and $(\Sigma^+\Sigma)^* = \Sigma^+\Sigma$ since they are both symmetric and real-valued matrices.

For arbitrary matrices A:
Let A be an arbitrary matrix as in Definition A.1. The singular value decomposition theorem states that there exists a factorization of the form $A = U\Sigma V^*$, where:

U is an $m \times m$ unitary matrix over $\mathbb{K}$;
Σ is an $m \times n$ matrix over $\mathbb{K}$ with non-negative real numbers on the diagonal and zeroes off the diagonal;
V is an $n \times n$ unitary matrix over $\mathbb{K}$.

The four equations below show that $A^+ := V\Sigma^+U^*$ is a pseudo-inverse of A:

$$\begin{array}{rrcl}
1. & AA^+A & = & U\Sigma V^*V\Sigma^+U^*U\Sigma V^* = U\Sigma\Sigma^+\Sigma V^* = U\Sigma V^* = A; \\
2. & A^+AA^+ & = & V\Sigma^+U^*U\Sigma V^*V\Sigma^+U^* = V\Sigma^+\Sigma\Sigma^+U^* = V\Sigma^+U^* = A^+; \\
3. & (AA^+)^* & = & (U\Sigma V^*V\Sigma^+U^*)^* = (U\Sigma\Sigma^+U^*)^* = U(\Sigma\Sigma^+)^*U^*; \\
 & & = & U(\Sigma\Sigma^+)U^* = U\Sigma V^*V\Sigma^+U^* = AA^+; \\
4. & (A^+A)^* & = & (V\Sigma^+U^*U\Sigma V^*)^* = (V\Sigma^+\Sigma V^*)^* = V(\Sigma^+\Sigma)^*V^*; \\
 & & = & V(\Sigma^+\Sigma)V^* = V\Sigma^+U^*U\Sigma V^* = A^+A.
\end{array}$$
□

Lemma A.3. *The Moore-Penrose pseudo-inverse is unique.*

Proof. This same proof can be found in [10], [16] and probably many more sources.
Let B and C both be Moore-Penrose pseudo-inverses of A. By using the four criteria of the Moore-Penrose pseudo-inverse, we find that

$$\begin{aligned}
AB &= (AB)^* = B^*A^* = B^*(ACA)^* = B^*A^*C^*A^* \\
&= (AB)^*(AC)^* = (AB)(AC) = (ABA)C = AC.
\end{aligned}$$

And similarly we also find that $BA = CA$. Therefore we have

$$B = BAB = BAC = CAC = C.$$

Thus the Moore-Penrose pseudo-inverse is unique. □

In Section 2 we have defined the weighted Laplacian pseudo-inverse L^+ and stated that it is a special case of the Moore-Penrose pseudo-inverse and thus unique. Here we present the proof of our statement.

Lemma A.4. *The weighted Laplacian pseudo-inverse L^+ as defined in Definition 2.2 is a special case of the Moore-Penrose pseudo-inverse.*

Proof. Since L is a real valued matrix for which we can write $L = UDU^\top$, where U is a unitary matrix with columns of a set of orthonormal basis of eigenvectors $b_1, b_2, \ldots, b_n$ of L and D is a diagonal matrix with $D_{ii} = \lambda_i$, the corresponding eigenvalue for eigenvector b_i. We see that $L = UDU^\top$ is a singular value decomposition of L. Thus by the same proof as in Lemma A.2 we see that $L^+ = UD^+U^\top$ is a Moore-Penrose pseudo-inverse of L. □

B Counting spanning trees in $K_{m,n}$

The sole purpose of this appendix is to count spanning trees in $K_{m,n}$ needed for Example 3.5. The number of spanning trees of $K_{m,n}$ can be determined by the using the Matrix-Tree Theorem and is known to be $T(K_{m,n}) = m^{n-1}n^{m-1}$. However, we have decided to follow the proof and technique for counting the number of trees in $K_{m,n}$ presented in [1], which we later adapt for determining the number of spanning trees in $K_{m,n}$ through the edge (a, b) for a certain pair of vertices $a, b \in V$,

Let $G = (V, E)$ be a graph with $n \geqslant 1$ vertices. Then we know from graph theory that any spanning tree T of G has exactly $n-1$ edges and no cycles. Furthermore, for any vertex $w \in V$, there is exactly one path from w to v in T. For assume there are different paths $P_1 \subset T$ and $P_2 \subset T$ from w to v, then $P_1 \cup P_2 \subset T$ is a cycle. This contradicts the fact that there are no cycles in trees.

We denote $\mathcal{T}(G)$ as the set of all spanning trees of G and for $v \in V$ we define $\mathcal{D}(G, v)$ as the set of all directed spanning trees of G with v as sink. For a finite set S, we write $|S|$ as the number of elements in S.

Lemma B.1. *For any graph $G = (V, E)$ with at least one vertex $v \in V$,*

$$|\mathcal{T}(G)| = |\mathcal{D}(G, v)|.$$

Proof. Let $T \in \mathcal{T}(G)$. Define $\vec{T}$ as the graph in $\mathcal{D}(G, v)$ constructed out of T by assigning to each edge of T a direction as follows. For each $w \in V$ find the unique path from w to v and direct the edges according to the path. Note that all edges are assigned a direction. Furthermore, it is not possible for any edge $(a, b) \in E$ to be assigned both directions through this method. For if it were the case, then there would be two distinct paths from a to v in T (one going through b, and one not going through b) and again we would find a cycle in T contradicting the fact that T is a tree. Thus $\vec{T}$ is well-defined and there is exactly one way to assign each edge a direction towards the sink v.

Consider the function $f : \mathcal{T}(G) \to \mathcal{D}(G, v)$ with $f(T) = \vec{T}$. Different trees $T_1, T_2 \in \mathcal{T}(G)$ have at least one different edge, thus $\vec{T_1}$ and $\vec{T_2}$ will have different edges as well. Thus f is injective. Furthermore, any directed spanning tree $T \in \mathcal{D}(G, v)$ corresponds to a spanning tree $T' \in \mathcal{T}(G)$, by neglecting the edge direction. Because there is only one way to assign directions on T' in such way that v becomes a sink, it holds that $f(T') = T$. So f is also surjective and thus bijective. This leads to $|\mathcal{T}(G)| = |\mathcal{D}(G, v)|$. $\square$

The previous lemma and its proof shows that a tree in $\mathcal{T}(G)$ corresponds to exactly one tree in $\mathcal{D}(G)$ with the same (directed) edges. So instead of counting spanning trees of G, we can count directed spanning trees in G with sink v. This is useful, since we can construct trees in $\mathcal{D}(G, v)$ more systematically due to the structure present. For $x \in V$, let $d^+(x)$ denote the out-degree of x. Because v is a sink, it holds that $d^+(v) = 0$ and $d^+(x) = 1$ for all other vertices $x \in V \backslash \{v\}$. Since a spanning tree of G has $n-1$ edges, constructing one in $\mathcal{D}(G, v)$ is reduced to selecting one outgoing edge for each vertex in $V \backslash \{v\}$. We use this fact to construct and count spanning trees in $K_{m,n}$. For the rest of this section we denote $K = K_{m,n} = (A \cup B, E)$, where A and B are disjoint sets with sizes m and n respectively, giving the bipartition on the vertex set of K.

Theorem B.2. *The number of spanning trees of $K_{m,n}$ is*

$$m^{n-1}n^{m-1}.$$

Proof. For $m = 1$ or $n = 1$, the statement is true since there is only 1 possible spanning tree. We continue the proof for $m, n \geqslant 2$. Let $v \in B$ arbitrary and let $B' = B \backslash \{v\}$. We count the number of trees in $\mathcal{D}(K, v)$ by constructing them in two steps

1 Select the outgoing edges from B' to A.

2 Select the outgoing edges from A to B.

Step 1: Each edge in B' can be connected to any of the m edges in A. Thus in total there are m^{n-1} possible ways to select the $n-1$ outgoing edges from B' to A.

Step 2:. Let G be a subgraph of $K_{m,n}$ with $n-1$ outgoing edges from B'. The graph $G \backslash \{v\}$ (when viewed as an undirected graph) has m different components , each with their unique vertex $a \in A$ with $d^+(a) = 0$. We shall extend G into a tree in $\mathcal{D}(K, v)$ by selecting the m outgoing edges from A to B. This is the same as selecting t outgoing edges from A to B', with $0 \leqslant t \leqslant m-1$, and connecting the remaining vertices $a \in A$ with $d^+(a) = 0$ to v. The limitation on t is due to the fact that there needs to be at least one edge going from A to v. There is exactly one way to choose $t = 0$ outgoing edges, so we shall first focus on $t \geqslant 1$.

We select the t edges from A to B' as follows. First pick a vertex $b \in B'$. Then select an $a \in A$ with $d^+(a) = 0$ from another component in G than the one b is in. The edge we add to G is edge (a, b). There are $(n-1)$ choices for b. For the first edge there are m components in total and thus $(m-1)$ choices for a. This means that we have $(n-1)(m-1)$ choices for the first edge. By adding such edges to G, the number of components of G decreases by one for each edge added. This means that there are $(n-1)(m-2)$ choices for the second edge and ultimately $(n-1)(m-t)$ choices for the $t-$th edge. Since the order in which we choose the t edges is irrelevant, it follows that we have

$$\prod_{i=1}^{t} \frac{(n-1)(m-i)}{t!} = \frac{(m-1)!}{t!(m-1-t)!} \cdot (n-1)^t = \binom{m-1}{t}(n-1)^t$$

possible ways of selecting the t edges. This means there are

$$1 + \sum_{t=1}^{m-1} \binom{m-1}{t}(n-1)^t = \sum_{t=0}^{m-1} \binom{m-1}{t}(n-1)^t = n^{m-1}$$

ways to extend G to a directed tree in $\mathcal{D}(K, v)$.

Since there were m^{n-1} possible subgraphs of $K_{m,n}$ with $n-1$ outgoing edges from B' to A, there are $m^{n-1}n^{m-1}$ trees in $\mathcal{D}(K, v)$ in total. Hence, by Lemma B.1, there are $m^{n-1}n^{m-1}$ spanning trees in $K_{m,n}$. $\square$

Similarly, counting spanning trees going by a specific edge e can also be done through counting directed spanning trees passing e in both directions.

Theorem B.3. *The number op spanning trees of $K_{m,n}$ through a fixed edge in $K_{m,n}$ is*

$$\frac{m+n-1}{mn} \cdot m^{n-1}n^{m-1}.$$

Proof. Again the formula is trivial for $m, n = 1$, so let $m, n \geqslant 1$. Let us denote the fixed edge as $(a, b) \in E$ with $a \in A$ and $b \in B$ and let $v \in B$ with $v \neq b$. We count the number of trees through (a, b) in $\mathcal{D}(K, v)$ by following the same two steps as in the proof of Theorem B.2. There are 2 possible orientations for edge (a, b) and we shall consider them both separately.

Orientation 1: Edge (a, b) is directed from b to a.
Step 1: Since the outgoing edge of vertex b is already fixed (edge (a, b)), we only have to select the other $n - 2$ edges from $B' \backslash \{b\}$ to A. This can be done in m^{n-2} ways.
Step 2: This step is identical to the proof of Theorem B.2, yielding n^{m-1} ways to add the m edges from A to B (after having chosen $n - 1$ edges from B' to A).
Thus in total there are $m^{n-2}n^{m-1} = \frac{1}{m} \cdot m^{n-1}m^{n-1}$ possible ways to construct a tree in $\mathcal{D}(K, v)$ through (a, b) with this orientation of (a, b).

Orientation 2: Edge (a, b) is directed from a to b.
Step 1: In this case, none of the vertices in B' have an outgoing edge yet. So we have to select all $n - 1$ edges from B' to A. Vertex b can not have an outgoing edge towards a, since that will create a cycle. So from vertex b there are $(m - 1)$ possible outgoing edges. The other $n - 2$ edges in B' can have outgoing edges towards any vertex in A. Thus in total there are $(m - 1)m^{n-2}$ ways to choose the $n - 1$ outgoing edges from B' to A.
Step 2: This step is almost identical to the proof of Theorem B.2 with the only difference being the value of m. Since vertex a is connected to b which is connected to another vertex in A, G (the graph with $n - 1$ edges from B' to A) only has $m - 1$ different components instead of m. The rest of the counting argument is analogous to the proof of the proof of Theorem B.2. By substituting m by $m - 1$, we find n^{m-2} ways to add the edges from A to B (after having chosen $n - 1$ edges from B' to A).
Thus in total there are $(m - 1)m^{n-2}n^{m-2} = \left(\frac{1}{n} - \frac{1}{mn}\right) m^{n-1}n^{m-1}$ possible ways to construct a tree in $\mathcal{D}(K, v)$ through (a, b) in this orientation of (a, b).

This means that in total there are $\left(\frac{1}{m} + \frac{1}{n} - \frac{1}{mn}\right) m^{n-1}n^{m-1} = \frac{m+n-1}{mn} \cdot m^{n-1}n^{m-1}$ trees in $\mathcal{D}(K, v)$ going through (a, b). □

Theorem B.4. *Let e be an edge that is not present in $K_{m,n}$. The number of spanning trees in $K_{m,n}$ with edge e, through the edge e is*

$$\frac{2}{k} \cdot m^{n-1}n^{m-1},$$

with $k = m$ or $k = n$ if e connects vertices in parts of size m or n respectively.

Proof. We assume e connects vertices in the part with size m. So $e = (a_1, a_2)$ with $a_1, a_2 \in A$ and $m \geqslant 2$. The proof for the other case is identical by swapping m and n. Define K_e as K with edge e and let $v \in B$. We shall count the number of trees through (a_1, a_2) in $\mathcal{D}(K_e, v)$. Again we follow the two steps as in the proof of Theorem B.2 with the addition of edge (a_1, a_2) already being selected.

Step 1: The fact that a_1 and a_2 are connected has no influence on the option for outgoing edges from B' to A. Thus the $n - 1$ outgoing edges from B' can be chosen in m^{n-1} ways.
Step 2: Let G be the subgraph of K_e with $n - 1$ outgoing edges from B' and the edge (a_1, a_2). There are 2 possible orientations of the edge (a_1, a_2). Fix one for G. Furthermore, because a_1 and a_2 are connected, G only has $m - 1$ different components. Apart of this, the counting argument is analogous to one in the proof of Theorem B.2 as well. This yields that we have n^{m-2} ways to extend G with a fixed orientation of (a_1, a_2) into a tree in $\mathcal{D}(K_e, v)$. Thus, there are $2n^{m-2}$ ways to add the edges from A to B (after having chosen $n - 1$ edges from B' to A).

This means that in total there are there are $2m^{n-1}n^{m-2} = \frac{2}{m} \cdot m^{n-1}n^{m-1}$ trees in $\mathcal{D}(K_e, v)$ going through edge e . □

C Matlab codes

Here is a compilation of the Matlab codes we used to easily calculate different aspects of the effective resistance. We use "`format rat`" at the beginning of all our codes so that Matlab returns values in fractions.

C.1 The Laplacian of K_n and $K_{n,m}$ and the effective resistance

In Section 3.1, when deducing a general formula for the effective resistance between vertices in K_n and $K_{n,m}$, we used the following codes in matlab.

Code for creating the Laplacian for the unweighted graph K_n.

```
n=4;  %Manually change the value of n here.
L=[]; %Delete any existing matrix L.
for i=1:n
    for j=1:n
        if i==j
            L(i,j)=n-1;
        else
            L(i,j)=-1;
        end;
    end;
end;
```

Code for creating the Laplacian for the unweighted graph $K_{n,m}$.

```
n=4; m=4; %Manually change the value of n and m here.
L=[];     %Delete any existing matrix L.
for i=1:n+m
    for j=1:n+m
        if i<=m && j<=m
            if i==j
               L(i,j)=n;
            else
               L(i,j)=0;
            end;
        elseif i>m && j>m
            if i==j
               L(i,j)=m;
            else
               L(i,j)=0;
            end;
        else
            L(i,j)=-1;
        end;
    end;
end;
```

Code for determining the effective resistances between all pairs in V using $L(b)^{-1}$ for $b \in V$ as in Theorem 2.7.

```
L=                                              %insert Laplacian
b=                                              %insert b in V
Z=size(L);n=Z(1);                               %determine the size of L
B=L;  B(:,b)=[];  B(b,:)=[]; B=inv(B);          %calculate L(b)^{-1}
for i=1:n-1
   for j = i+1:n
       if i~=b && j==b
           if i<b
           R = B(i,i);
           else
           R = B(i-1,i-1);
           end
       elseif i==b && j~=b
           if j<b
           R = B(j,j);
           else
           R = B(j-1,j-1);
           end
       else
           if i<b && j<b
           R = B(i,i)+ B(j,j)-2*B(i,j);
           elseif i<b && j>b
           R = B(i,i)+ B(j-1,j-1)-2*B(i,j-1);
           elseif i>b && j<b
           R = B(i-1,i-1)+ B(j,j)-2*B(i-1,j);
           else
           R = B(i-1,i-1)+ B(j-1,j-1)-2*B(i-1,j-1);
           end
       end
       fprintf('R_{%d%d} = ',i,j)
       disp(R)
   end
end
```

C.2 Random Laplacian and the optimal edge to add with edge weight $\alpha > 0$.

In order to determine the plausibility of Statement 5.4 in Section 5, we let Matlab generate random Laplacian matrixes for connected graphs and let it find the optimal edge to add for different values of α.

Here is the code we use to generate a random weighted Laplacian associated to a connected graph. This code was inspired by the code to generate random graphs found in [6].

```
n=5;                %set the number of vertices of our graph
p=0.5;              %the probability of an edge to be chosen
Connected = 0;      %start with no graph, thus a non-connected graph
while Connected == 0
    L=zeros(n);
    for i=1:n
        for j=i+1:n
           if rand<p                  %randomly decide if the edge gets chosen
             L(i,j)=-1*randi(10);     %give the edge a random integer weight in [1,10]
             L(j,i)=L(i,j);           %make the Laplacian symmetric
             L(i,i) = L(i,i) -L(i,j); %taking care of the diagonals
             L(j,j) = L(j,j) -L(j,i);
           end
        end
    end
    Connected = 1;                    %assume the graph is connected
    if rank(L)<n-1
        Connected = 0;                %a connected graph has Laplacian rank n-1
    end
end
```

Code for determining the optimal edge to add using the Laplacian given above.

```
besta=0; bestb=0; bestdR=0;
for alpha = 1:20
    for a=1:n-1
        for b=a+1:n
            B=L; B(:,b)=[];  B(b,:)=[]; B=inv(B); %determine L(b)^{-1}
            v=B(a,:); v=[v(1:b-1),0,v(b:n-1)];    %L(b)^{-1}_{ak} = L(b)^{-1}_{ka} = 0
            A = B(a,a);
            beta = alpha/(1+alpha*A);
            sum =0;
            for i=1:n
                for j=i:n
                    sum=sum+(v(i)-v(j))^2;
                end
            end
            dR = beta*sum;
            if dR>=bestdR
                bestdR = dR; besta=a; bestb=b;
            end
        end
    end
    fprintf('For alpha is %d, the optimal edge to add is edge (%d,%d)
            \n',alpha,besta,bestb)
end
```