LEIDEN UNIVERSITY

MASTER PROJECT

MATHEMATICS

Closure Data of Étale schemes

Author:
Marius STEKELENBURG

Supervisors:
Dr. Owen BIESEL
Dr. David HOLMES

December 1, 2016

Stack of Closure Data

Marius Stekelenburg

December 1, 2016

Contents

1 Introduction **3**

2 Galois Closure data for free ring extensions **5**
2.1 The Galois closure of a field extension 5
2.2 Closure data and the Ferrand map 6
2.3 Inducing closure data 8

3 Étale morphisms **10**
3.1 Grothendieck topologies 10
3.2 Étale morphisms 12
3.3 Finite étale morphisms 15

4 Closure Data on schemes **17**
4.1 The Ferrand morphism on schemes of constant rank 17
4.2 Exponentiation 19
4.3 The Ferrand morphism on finite étale covers 21
4.4 The sheaf of Closure data 24
4.5 Extra structures 28

5 Connected schemes **31**
5.1 The fundamental group 31
5.2 Explicit computations 32

6 Algebraic Spaces **35**
6.1 Representability 35
6.2 Algebraic spaces 36
6.3 Gluing Properties 38

7 Stacks and Algebraic Stacks **40**
7.1 Fibred categories 40
7.2 Stacks 43
7.3 Algebraic Stacks 45

8 The stack of closure data **47**
8.1 The stack of closure data defined 47
8.2 The stack of vector bundles 48
8.3 The stack of locally free algebras 50
8.4 The stack of finite étale covers 52
8.5 The final step 53

1 Introduction

Overview

Let $k \to l$ be a separable field extension of degree n. Usually, the Galois closure N of l is defined to be a minimal extension of l that is Galois over k. However, there is another definition. Using the Theorem of the Primitive element, write $l = k(\alpha)$, and let f_α be the minimal polynomial of α. Let G be the Galois group of l. By regarding on how it acts on the roots of f_α, this can be viewed as a subgroup of S_n. For some appropriately defined k-algebra map $\sigma \colon (l^{\otimes n})^G \to k$, the following identity holds:

$$N = k \otimes_{(l^{\otimes n})^G} l^{\otimes n}$$

This turns out to be the case if $\sigma \colon (l^{\otimes n})^G \to k$ is *normative*. This gives rise to a *closure datum*: a pair (H, σ) with $H \subseteq S_n$ a subgroup, and $\sigma \colon (l^{\otimes n})^H \to k$ a normative k-algebra morphism.

This generalizes to schemes in the following way. Let $X \to S$ be a scheme that is finite and of rank n. Then a closure datum is a pair (H, σ) where $\sigma S \to (X^n)_H$ is normative. If $X \to S$ is finite étale, being of rank n means being *locally isomorphic* to the scheme nS. This leads to the following generalization: if $X \to S$ and $Y \to S$ are locally isomorphic finite étale, then one considers the *exponential* X^Y, and defines a closure datum to be a pair (H, σ), where $H \subseteq Y^{Y*}$ is a subgroup scheme, and σ is a normative morphism $(X^Y)_H \to S$.

Given two locally isomorphic finite étale $X, Y \to S$, one would like to compute the set $\mathrm{Data}(X, Y)$, the set of all closure data. This set can actually be made into a sheaf. It turns out that this sheaf is *representable*, i.e. it corresponds to a scheme over S!

There are many pairs of locally isomorphic finite étale schemes over S. Instead of computing each case individually, one could instead capture all cases together. This gives rise to the notion of the *stack of closure data*. When dealing with moduli problems, it is interesting to see whether this stack is *algebraic*, so that it can be treated as a scheme. The answer to this question turns out to be positive, and a full proof will be provided in chapter 8.

Acknowledgments

This thesis wouldn't exist without the help of Owen Biesel and David Holmes. I am especially thankful to Owen for sharing his knowledge about closure data, for his good knowledge about commutative algebra, and his constant enthusiasm. He also supervised my bachelor thesis. I am also very thankful for David Holmes for supervising me during the later session. He corrected my thesis, had good faith, and had a very good knowledge about stacks.
I am also thankful to my family and friends for their support.

Notations and conventions

We start with conventions regarding sets.

- If G is a group acting on a set X, then X^G denotes the invariant space, whereas X_G denotes the orbit space. These notions also extend to categories, as shown in Definition 4.1.1 and Definition 4.2.8.
- The notation $\mathbf{n}$ denotes the set $\{1, ..., n\}$.
- If X and Y are sets, then we denote the set of maps by $\mathrm{Map}(X, Y)$. We use the notation $\mathrm{Bij}(X, Y)$ for the set of bijections. The set $\mathrm{Bij}(\mathbf{n}, \mathbf{n})$ will be denoted by S_n.

The next part will be about rings. Let R be a ring.

- All rings in this thesis are commutative, and contain the element 1. This also goes for ring algebras.
- Let M be an R-module. The tensor product $M \otimes_S ... \otimes_S M$ of n copies of M is denoted by $M^{\otimes n}$.
- Suppose $f\colon M \to N$ is a morphism of R-modules. If S is an R-algebra, then the map $f_S\colon M \otimes_R S \to N \otimes_R S$ denotes the base extension of f. Sometimes, it is denoted by $f \otimes_R S$. If there is too much confusion about the base, it is denoted by $M^{\otimes_R n}$.

The last part will be about category theory. Let $\mathcal{C}$ ba a category.

- All categories in this thesis contain a terminal object 1, and are closed under fibred products.
- For any set S, and $X \in \mathcal{C}$, we denote the repeated product $\prod_{i \in S} X$ by X^S and the repeated sum $\coprod_{i \in S} X$ by $S.X$. If $S = \mathbf{n}$, then we simply use the notations X^n and nX.
- If $f\colon X \to Y$ is a morphism, and $S \to Y$ is another morphism, then $f_S\colon X \times_Y S \to S$ will denote the base extension. If f is of the form $Y_i \to Y$, then the base extension is also denoted by $f_i\colon S_i \to S$.

The last part will be about algebraic geometry. Lastly, the definitions regarding algebraic geometry are always copied from Stacks Project ([5]). I do, however, avoid using topologies that are not Zariski or étale. This makes Definition 6.2.1, for instance, slightly different, but still equivalent.
The definitions regarding category theory are mostly copied from [5] as well, but I also uses some results from [3].

2 Galois Closure data for free ring extensions

This chapter will be a heavily condensed introduction to closure data for ring extensions. For a long version, I'd like to refer to Biesel's thesis [1].

2.1 The Galois closure of a field extension

Let $k \to l$ be a separable field extension of degree n, and let $k \to \overline{k}$ be the algebraic closure. By the Theorem of the Primitive Element, we can write $l = k(\alpha)$. Let $f_\alpha \in k[x]$ be the minimum polynomial of α. Then f_α is of degree n, and therefore has n roots in $\overline{k}$; call them $\alpha_1, ..., \alpha_n$. Let $N = k[\alpha_1, ..., \alpha_n]$, the *Galois closure* of l. We obtain a k-algebra homomorphism $\Phi\colon l^{\otimes n} \to N$.
In order to define this Φ explicitly, we need some extra notation. The k-algebra $l^{\otimes n}$ is generated by elements in the form $1 \otimes ... \otimes 1 \otimes \alpha \otimes ... \otimes 1$. Such element where the i-th component is α will be denoted $\alpha^{(i)}$. Then Φ is the following:

$$\Phi\colon l^{\otimes n} \to N, \alpha^{(i)} \mapsto \alpha_i$$

Now let G be the group of all k-invariant automorphisms of N; the *Galois group* of N. All automorphisms are permutations on the set $\{\alpha_1, ..., \alpha_n\}$, and the automorphism is completely determined by this permutation. As a result, G can be viewed as a subgroup of S_n.
On $l^{\otimes n}$, we define an S_n-action given by:

$$\pi(x_1 \otimes ... \otimes x_n) = x_{\pi^{-1}(1)} \otimes ... \otimes x_{\pi^{-1}(n)}$$

Under this action, $l^{\otimes n}$ also becomes a G-set, and Φ is a morphism of G-sets. We know that an element of N is G-invariant if and only if it is part of k, which means that Φ gives rise to a morphism $(l^{\otimes n})^G \to k$. This gives rise to the following commutative diagram:

$$\begin{tikzcd}
(l^{\otimes n})^G \arrow[r, "\Phi"] \arrow[d] & k \arrow[d] \\
l^{\otimes n} \arrow[r, "\Phi"] & N
\end{tikzcd}$$

Lemma 2.1.1. *This diagram is a tensor product diagram.*

Proof. We have to prove that the induced map $\Psi\colon l^{\otimes n} \otimes_{(l^{\otimes n})^G} k \to N$ is bijective. It is clearly surjective, as $\Psi(\alpha^{(i)} \otimes 1) = \alpha_i$ for each i. So it remains to prove that it is injective.
Each element of $l^{\otimes n} \otimes_{(l^{\otimes n})^G} k$ is of the form $\beta \otimes 1$. So let $\beta \otimes 1 \in \ker \Psi$. For each $g \in G$, we have $\Psi(g\beta \otimes 1) = 0$ as well, as g also represents an

automorphism of N. Now let $m = |G|$. For each elementary symmetric polynomial $S_i \in \mathbb{Z}[x_1, ..., x_m]$ of nonzero degree, we have $\Psi(S_i(g_1\beta, ..., g_m\beta)\otimes 1) = 0$. But $S_i(g_1\beta, ..., g_m\beta) \in (l^{\otimes n})_G$. As a result, $S_i(g_1\beta, ..., g_m\beta) \otimes 1 = 0$. So we find, for each i:

$$\begin{aligned}[g_i\beta \otimes 1]^m &= [g_i\beta^m - S_1(g_1\beta, ..., g_m\beta)g_i\beta^{m-1} + ... + (-1)^m S_m(g_1\beta, ..., g_m\beta)] \otimes 1 \\ &= [(g_i\beta - g_1\beta)...(g_i\beta - g_m\beta)] \otimes 1 \\ &= 0\end{aligned}$$

This proves that $\ker \Psi$ consists of nilpotent elements only. So it remains to prove that $l^{\otimes n} \otimes_{(l^{\otimes n})^G} k$ is reduced. Using *Galois theory*, it follows that $l^{\otimes n} \otimes_{(l^{\otimes n})^G} k$ is a product of finite separable field extensions of k. Hence the ring is indeed reduced. So Ψ is injective, hence bijective. □

This motivates us to find "correct" morphisms $(l^{\otimes n})^G \to k$ to make sure $l^{\otimes n} \otimes_{(l^{\otimes n})^G} k$ becomes the Galois closure. It turns out that this is the case when the morphism is *normative*. In that case, the morphism $(l^{\otimes n})^G \to k$ becomes *normative*, and therefore a *G-closure datum*. This notion can be generalized to arbitrary subgroups of S_n. The result ultimately becomes the following:

Theorem 2.1.2. *Let $k \to l$ be a separable field extension of rank n, with Galois group G. The minimal closure data(H, σ) are equivalent to the closure datum (G, Φ), where $\Phi\colon (l^{\otimes n})^G \to k$ is the morphism defined earlier this section.*

Some notions of above statement have not yet been introduced. What is a closure datum? What is a minimal closure datum? And when are closure data equivalent? This will be explained in the next sections.

2.2 Closure data and the Ferrand map

This section provides a general definition for closure data of free ring extensions. In order to give a definition, we first need the definition of normative maps. But in order to define those, we need the Ferrand map. This will all be introduced in this section.
First of all, the tensor permutation action also carries over to rings. To be precise, it works as follows.

Definition 2.2.1. Let A be an R algebra. On $A^{\otimes n}$, we define an S_n-action by:

$$\pi(a_1 \otimes ... \otimes a_n) = a_{\pi_1^{-1}(1)} \otimes ... \otimes a_{\pi_n^{-1}(n)}$$

The Ferrand map will be defined by the following theorem.

Theorem 2.2.2. *Let A be an R-algebra that is isomorphic to R^n as an R-module. Let R-Alg be the category of R-algebras. There is a unique collection of morphisms $\{\Phi_{A_S/S}\colon (A_S^{\otimes n})^{S_n} \to S\}_{S \in R\text{-}\mathsf{Alg}}$ satisfying the following properties:*

- *$\Phi_{A_S/S}$ is a morphism of S-algebras.*
- *For all $a \in A_S, \Phi_{A_S/S}(a \otimes ... \otimes a) = \det(M_a)$, with $M_a \in Mat_{n\times n}(S)$ being the matrix representing the multiplication map $m \mapsto ma$.*
- *For each R-algebra morphism $S \to S'$, we have $\Phi_{A_{S'}/S'} = S' \otimes_S \Phi_{A_S/S}$.*

Proof. See [1], Lemma 2.3.3, p. 23. $\square$

This theorem motivates the following definition.

Definition 2.2.3. Let $A \to R$ be a free rank n algebra. The *Ferrand map* $(A^{\otimes n})^{S_n} \to R$ is the unique map of Theorem 2.2.2.

The following example gives an idea about how to compute the Ferrand map explicitly.

Example 2.2.4. Let $R = \mathbb{Z}$ and let $A = \mathbb{Z}[\sqrt{2}]$. We will compute the Ferrand map $\Phi_{A/\mathbb{Z}}$. The algebra A is a free $\mathbb{Z}$-module of rank 2, with basis $\{1, \sqrt{2}\}$. Therefore, $(A^{\otimes 2})^{S_2}$ is a free $\mathbb{Z}$-module with basis:

$$\{1 \otimes 1, 1 \otimes \sqrt{2} + \sqrt{2} \otimes 1, \sqrt{2} \otimes \sqrt{2}\}$$

The matrix associated with 1 is obviously the identity matrix I. As a result, $\Phi_{A/\mathbb{Z}}(1 \otimes 1) = \det I = 1$. The matrix associated with $\sqrt{2}$ is the following matrix:

$$M_{\sqrt{2}} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

As a result, we have $\Phi_{A/\mathbb{Z}}(\sqrt{2} \otimes \sqrt{2}) = \det M_{\sqrt{2}} = -2$.
It remains to compute $\Phi_{A/\mathbb{Z}}(1 \otimes \sqrt{2} + \sqrt{2} \otimes 1)$. In order to do so, consider the ring $A[x, y]$, as a $\mathbb{Z}[x, y]$-algebra. We have $\Phi_{A[x,y]/\mathbb{Z}[x,y]} = \Phi_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[x, y]$. The element $1 \otimes \sqrt{2} + \sqrt{2} \otimes 1$ is the xy-term of α, which is given by:

$$\alpha\colon = (x + \sqrt{2}y) \otimes_{\mathbb{Z}[x,y]} (x + \sqrt{2}y)$$

As a result, $\Phi_{A/\mathbb{Z}}(1 \otimes \sqrt{2} + \sqrt{2} \otimes 1)$ is the xy-term of $\Phi_{A[x,y]/\mathbb{Z}[x,y]}(\alpha)$.
So we will compute $\Phi_{A[x,y]/\mathbb{Z}[x,y]}(\alpha)$ first. The $\mathbb{Z}[x, y]$-algebra $A[x, y]$ has basis $\{1, \sqrt{2}\}$. The matrix $M_{x+\sqrt{2}y}$ with respect to that basis is given by:

$$M_\alpha = \begin{pmatrix} x & 2y \\ y & x \end{pmatrix}$$

As a result, $\Phi_{A[x,y]/\mathbb{Z}[x,y]}(\alpha) = x^2 - 2y^2$. Hence $\Phi_{A/\mathbb{Z}}(1 \otimes \sqrt{2} + \sqrt{2} \otimes 1) = 0$.

Definition 2.2.5. Let A be an R-algebra that is free of rank n. Let $B \subseteq A^{\otimes n}$ be a subalgebra that contains $(A^{\otimes n})^{S_n}$. An R-algebra morphism $f\colon B \to R$ is said to be *normative* if $f|_{(A^{\otimes n})^{S_n}} = \Phi_{A/R}$.

This gives us the definition of an H-closure datum, for subgroups $H \subseteq S_n$.

Definition 2.2.6. Let A be a free R-algebra of rank n, and let $H \subseteq S_n$ be a subgroup. A *closure datum* is a pair (H, σ) such that $H \subseteq S_n$ is a subgroup, and $\sigma\colon (A^{\otimes n})^H \to R$ is a normative R-algebra morphism

The set of closure data is denoted by $\mathrm{Data}_{A/R}$. In the next section, we will see that this set has some additional structure, namely of a *partially ordered* S_n*-set*.

2.3 Inducing closure data

There are several ways ways of inducing new closure data out of old ones. The first one is very straightforward.

Lemma 2.3.1. *Let A be a free rank n R-algebra and let $H, H' \subseteq S_n$ be subgroups such that $H \subseteq H'$. Let $\sigma\colon (A^{\otimes n})^H \to R$ be a closure datum. Then $(A^{\otimes n})^{H'} \subseteq (A^{\otimes n})^H$, and the restriction $\sigma|_{(A^{\otimes n})_{H'}}\colon (A^{\otimes n})_{H'} \to R$ becomes a closure datum as well.*

Because of this, $\mathrm{Data}_{A/R}$ becomes a *partially ordered set* by putting $(H, \sigma) \leqslant (H', \sigma')$ if $H \subseteq H'$ and $\sigma' = \sigma|_{(A^{\otimes n})^{H'}}$.
The second way of inducing new closure data out of old ones is the following.

Lemma 2.3.2. *Let A be a free rank n R-algebra and let $H \subseteq S_n$ be a subgroup. Let $\pi \in S_n$ be a permutation, and let $\sigma\colon (A^{\otimes n})^H \to R$ be a closure datum. Consider the map $\pi(\sigma)\colon (A^{\otimes n})^{\pi H \pi^{-1}} \to R$ given by:*

$$\pi(\sigma)(x) = \sigma(\pi x)$$

Then $(\pi H \pi^{-1}, \pi(\sigma))$ is also a closure datum.

This motivates the following extra structure on $\mathrm{Data}_{A/R}$.

Definition 2.3.3. On $\mathrm{Data}_{A/R}$, we define an S_n-action, given by $\pi(H, \sigma) = (\pi H \pi^{-1}, \pi(\sigma))$. Two closure data (H, σ) and (H', σ') are *equivalent* if they are in the same orbit under said action.

Clearly, if $(H, \sigma) \leqslant (H', \sigma')$, then $\pi(H, \sigma) \leqslant \pi(H', \sigma')$ for any $\pi \in S_n$. This makes $\mathrm{Data}_{A/R}$ a *partially ordered S_n-set.*
It is clear that $\mathrm{Data}_{A/R}$ has precisely one maximal closure datum that is also S_n-invariant. Namely the Ferrand map. Theorem 2.1.2 effectively says that if k is a field, and l is a separable extension with Galois group G, then $\mathrm{Data}_{l/k}$ has one orbit of minimal closure data, which contains a G-closure datum.
There is a third way of inducing closure data. This is a straight consequence of the fact that the Ferrand map is stable under base extension.

Lemma 2.3.4. *Let A be a free rank n R-algebra, and let $f\colon R \to S$ be any ring homomorphism. Let $(H, \sigma) \in Data_{A/R}$. Let $f^*\sigma\colon (A_S^{\otimes n})^H \to S$ be the base extension of σ. Then $(H, f^*\sigma)$ is a closure datum. As a result, we gain a map $f^*\colon Data_{A/R} \to Data_{A_S/S}$.*

This motivates a final extra structure on $\mathrm{Data}_{A/R}$.

Definition 2.3.5. On $\mathrm{Data}_{A/R}$, we define an $\mathrm{Aut}_R(A)$-action that is given by $f(H, \sigma) = (H, f^*\sigma)$.

Again, this action is compatible with the ordering on $\mathrm{Data}_{A/R}$. So we have seen that there exist two group actions on $\mathrm{Data}_{A/R}$, both of which are compatible with the ordering. Later in this thesis, we will see that there is a correlation between those when A is an *étale* algebra.

3 Étale morphisms

This chapter will be a review about étale morphisms of schemes, and about Grothendieck topologies.

3.1 Grothendieck topologies

When people learn about algebraic geometry for the first time, they learn how to view schemes as *topological spaces*. While this approach is convenient for starters, it falls flat under certain applications. Fortunately, there are better topologies available. However, they are not the kind of topologies math students learn during their first class of topology. Instead, they are *Grothendieck topologies*. This chapter will be an introduction to those.

Definition 3.1.1. Let $\mathcal{C}$ be a category. A *Grothendieck topology* is a collection of collections in the form $\{U_i \to U\}_i$ called *covering sieves* (or simply coverings), subject to the following properties.

1. Suppose $V \to U$ is an isomorphism. Then the singleton $\{V \to U\}$ is a covering sieve.
2. Suppose $V \to U$ is a morphism, and $\{U_i \to U\}_i$ is a covering sieve. Then collection of base changes $\{V_i \to V\}_i$ is a covering sieve.
3. Suppose $\{U_i \to U\}_i$ is a covering, and for each i, the collection $\{U_{ij} \to U_i\}_j$ is a covering as well. Then the collection of compositions $\{U_{ij} \to U\}$ is also a covering sieve.

A category equipped with a Grothendieck topology is a *site*. This is a generalization to topological spaces in the following way.

Example 3.1.2. Any topological space X arises into a site in a canonical way. To elaborate, let the underlying category be $Op(X)$, the category of opens of X, where the morphisms are inclusions. A family of inclusions $\{U_i \subseteq U\}$ is a covering if $\bigcup_{i\in I} U_i = U$.

Example 3.1.3. Let S be a scheme, and let Sch/S be the category of schemes over S. The *Zariski topology* is the topology where $\{U_i \to U\}$ is a covering if each $U_i \to U$ is an open immersion, and the disjoint union $\coprod_i U_i \to U$ is surjective.

Sheaves on topological spaces generalize in a natural way to sheaves on sites. The definition is as follows:

Definition 3.1.4. Let $\mathcal{C}$ be a site.

- A *presheaf (of sets)* is a functor of categories $\mathcal{F}\colon \mathcal{C}^{opp} \to \mathsf{Set}$.

- A *sheaf* if a presheaf $\mathcal{F}$ such that for each covering $\{U_i \to U\}_{i\in I}$, the following diagram is an equalizer diagram:
$$\mathcal{F}(U) \longrightarrow \prod_{i\in I}\mathcal{F}(U_i) \underset{\mathcal{F}(\pi_2)}{\overset{\mathcal{F}(\pi_1)}{\rightrightarrows}} \prod_{(i,j)\in I^2}\mathcal{F}(U_i \times_U U_j)$$

- A *morphism of (pre)sheaves* $\mathcal{F} \to \mathcal{G}$ is a natural transformation of the functors, with usual composition. This arises to the category of sheaves on $\mathcal{C}$, denoted by $\mathrm{Sh}(\mathcal{C})$.

There is one important property sites can have. To define this property, we will review the *Yoneda lemma* first.

Lemma 3.1.5. *Let $\mathcal{C}$ be a category, and let $\mathsf{Set}^{\mathcal{C}^{opp}}$ be the category of presheaves on $\mathcal{C}$. Consider the following functor, called the* Yoneda embedding*:*

$$h\colon \mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{opp}}, X \mapsto Hom(-, X)$$

This functor is full and faithful. That is, for each $X, Y \in \mathcal{C}$, the induced map $Hom(X,Y) \to Hom(h(X), h(Y))$ is a bijection.

This lemma tells us that $\mathcal{C}$ can be viewed as a *subcategory* of $\mathsf{Set}^{\mathcal{C}^{opp}}$. If a presheaf is (up to isomorphism) situated in $\mathcal{C}$ under this inclusion, we call $\mathcal{F}$ *representable.* It is very common to actually interchange the notions of representable presheaves with objects of $\mathcal{C}$.
This gives rise to the definition of a *subcanonical topology.*

Definition 3.1.6. Let $\mathcal{C}$ be a site. Its topology is *subcanonical* if for each $X \in \mathcal{C}$, the presheaf $h(X)$ is a sheaf.In In other words, the Yoneda functor is a functor $h : \mathcal{C} \to Sh(\mathcal{C})$.

I would like to end this section with the notion of a *slice site.* It is possible to give a slice category a very straightforward structure of a site. Here is the definition in full.

Definition 3.1.7. Let $\mathcal{C}$ be a site. Let $T \in \mathcal{C}$. The *slice site* $\mathcal{C}/T$ is defined as follows.

- The objects are morphisms $U \to T$.

- The morphisms $(U_1 \to T) \to (U_2 \to T)$ are morphisms $U_1 \to U_2$ such that the following diagram commutes.
$$\begin{array}{ll} U_1 & \\ \downarrow & \searrow \\ U_2 & \to T \end{array}$$

- A collection of morphisms $\{(U_i \to T) \to (U \to T)\}_i$ is a covering if $\{U_i \to U\}_i$ is a covering in $\mathcal{C}$.

A sheaf on $\mathcal{C}$ also gives rise to a sheaf on $\mathcal{C}/T$ and vice versa.

Definition 3.1.8. Let $\mathcal{C}$ be a site, and let $T \in \mathcal{C}$. Let $t : T \to 1$ be the unique arrow.

- Suppose $\mathcal{F}$ is a sheaf on $\mathcal{C}/T$. Then the *pushforward sheaf* $t_*(\mathcal{F})$ is the sheaf on $\mathcal{C}$ given by $X \mapsto \mathcal{F}(X \times T)$.
- Suppose $\mathcal{F}$ is a sheaf on $\mathcal{C}$. Then the *pullback sheaf* $t^*(\mathcal{F})$ is the sheaf on $\mathcal{C}/T$ given by $(X \to T) \mapsto \mathcal{F}(X)$. We will also denote this sheaf by $\mathcal{F}|_T$.

One of the definitions works nicely with the Yoneda embedding.

Lemma 3.1.9. *In the situation of previous definition, suppose $\mathcal{F} \cong h(X)$ for some $X \in \mathcal{C}$. Then the pullback sheaf $t^*\mathcal{F}$ is isomorphic to $h(T \times X)$. There is no similar statement for pushforward sheaves.*

The definition also allows us to define the *Hom sheaf.*

Definition 3.1.10. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on a site $\mathcal{C}$. Then the sheaf $\mathsf{Hom}(\mathcal{F}, \mathcal{G})$ is given by:

$$X \mapsto \mathrm{Hom}_{Sh(\mathcal{C}/X)}(\mathcal{F}|_X, \mathcal{G}|_X)$$

This shows that if $\mathcal{C}$ is a site, that the category $\mathsf{Sh}(\mathcal{C}^{opp})$ is closed under *exponentiation.* I will define this notion in full in 4.2.

3.2 Étale morphisms

Before we define étale morphisms, we will first define *smooth* morphisms of schemes. I will do it this way because étale morphisms are special cases of smooth morphisms, and because smooth morphisms are ultimately needed when defining *algebraic stacks.* It is easier to define the notion of smoothness after having defined *standard smoothness* first.

Definition 3.2.1. Let $f\colon R \to A$ be morphism of rings. We say f is *standard smooth of relative dimension* n if f turns A into an R-algebra of the form $R[x_1, ..., x_{m+n}]/(f_1, ..., f_m)$ such that the following matrix is invertible in $\mathrm{Mat}_{m\times m}(A)$:

$$\begin{pmatrix} \frac{\delta f_1}{\delta x_1} & \dots & \frac{\delta f_1}{\delta x_m} \\ \dots & & \dots \\ \frac{\delta f_m}{\delta x_1} & \dots & \frac{\delta f_m}{\delta x_m} \end{pmatrix}$$

If we don't want to specify n, then we say f is *standard smooth.*

A smooth morphism of schemes is, informally, a morphism that is locally standard smooth. The formal definition is the following.

Definition 3.2.2. Let $f\colon X \to S$ be a morphism of schemes. Let $x \in X$. We say f is *smooth* if there exists an affine covering $(U_i)_i$ of S, and for each i an affine covering $(V_{ij})_j$ of $f^{-1}(U_i)$, such that for all i, j, the induced morphism $V_{ij} \to U_i$ is standard smooth.

This notion is very similar to the notion of smoothness on manifolds. *Étale morphisms* are special versions of smooth morphisms. They are analogues of *local diffeomorphisms*.

Definition 3.2.3. Let $f\colon X \to S$ be a morphism of schemes. We say f is *étale* if there exists an affine covering $(U_i)_i$ of S, and for each i an affine covering $(V_{ij})_j$ of $f^{-1}(U_i)$, such that for all i, j, the induced morphism $V_{ij} \to U_i$ is standard smooth of relative dimension 0 (i.e. it is *standard étale*).

Example 3.2.4. Open immersions are étale.

Proof. Let $U \to S$ be an open immersion. For simplicity, view the immersion as an inclusion $U \subseteq S$. Let $(V_i)_i$ be an affine open cover of S. Then for each i, $V_i \cap U$ is an open subscheme of V_i, so it can be covered by distinguished opens $(D(f_{ij})_j$ of V_i. It remains to prove that $D(f_{ij}) \to V_i$ is standard étale. This is the same as proving that for any ring R, and any $f \in R$, the map $R \to R_f$ is Standard étale. The ring R_f can be written as $R[x]/(xf - 1)$. The derivitive of the polynomial $xf - 1$, which is f, is invertible in R_f by definition. So indeed, $R \to R_f$ is Standard étale. □

For fields, étaleness can be characterized as follows.

Example 3.2.5. Let $k \to l$ be any field extension. Then the induced morphism $\operatorname{Spec} l \to \operatorname{Spec} k$ is étale if and only if the extension is finite and separable.

Étale morphisms have the following properties.

Lemma 3.2.6. *Let X, Y, Z be schemes.*

1. *Let $X \to Y$ and $Y \to Z$ be morphisms of schemes. Suppose both of them are étale. Then the composition $X \to Z$ is also étale.*

2. *Let $X \to Z$ and $Y \to Z$ be morphisms of schemes. Suppose $X \to Z$ is étale. Then the base change $X_Y \to Y$ is also étale.*

3. *Let $f\colon X \to Y$ be a morphism of schemes, and let $(U_i)_i$ of Y be an open covering. Then f is étale if and only if for each i, the induced morphism $f^{-1}(U_i) \to U_i$ is étale.*

4. *Let $f\colon X \to Y$ be a morphism of schemes, and let $U \to Y$ be a surjective étale morphism. Then f is étale if and only if the base extension f_U is étale.*

Property (1) says that being étale is *stable under composition.* Property (2) says that being étale is *stable under base change.* Property (3) says that being étale is *Zariski local.* Property (4) says that being étale is *étale local.* These properties are also true for smooth morphisms, among many other properties of morphisms of schemes.
With these properties, it is possible to define the étale site!

Definition 3.2.7. Let S be any scheme. The *(big) étale site over S* is given by the slice category Sch/S. We say a set $\{U_i \to U\}$ is a covering if the disjoint union $\coprod_i U_i \to U$ is a surjective étale morphism.

During this thesis, this will always the topology we will be using. Other authors, however, often like to use other topologies, like the fppf topology, or the fpqc topology. To avoid this confusion, I still often write $(\mathsf{Sch}/S)_{ét}$.

Remark 3.2.8. The reason this is called the big étale site is because there is also a *small étale site.* The underlying category will be limited to the elements of Sch/S where the morphism $X \to S$ is in itself étale. The coverings are the same.

Fortunately, this topology is subcanonical. To be precise:

Theorem 3.2.9. *Let $X \in \mathsf{Sch}/S$ be an object. Then for each X, the presheaf $h(X)$ is a sheaf.*

Proof. See [5], Tag 023Q, and use the fact that sheaves on the fpqc topology are also sheaves on the étale topology. □

This has some nice consequences.

Corollary 3.2.10. *Let S be a scheme. The global section functor $\mathcal{O}_S\colon (\mathsf{Sch}/S)_{ét} \to \mathsf{Set}, X \mapsto \mathcal{O}(X)$ is a sheaf.*

Proof. This follows from the fact that the global section functor is representable by $S \times_{\mathrm{Spec}\ \mathbb{Z}} \mathrm{Spec}\ \mathbb{Z}[x]$. □

Corollary 3.2.11. *Let S be a scheme. Let $X, Y \in \mathsf{Sch}/S$ be schemes over S. The presheaf $\mathsf{Hom}(X,Y)$ given by $(\mathsf{Sch}/S) \to \mathsf{Set}, U \mapsto Hom_U(X \times_S U, Y \times_S U)$ is a sheaf on $(\mathsf{Sch}/S)_{ét}$.*

Proof. It is easy to see that this sheaf is isomorphic to the presheaf $\mathsf{Hom}(h(X), h(Y))$ as defined earlier. Since $h(X)$ and $h(Y)$ are sheaves, so is $\mathsf{Hom}(h(X), h(Y))$. □

The sheaf $\mathcal{O}_S$ of Corollary 3.2.10 is in fact a sheaf of rings. As a result, one has the notion of an $\mathcal{O}_S$-module. Among those, there are also *quasi-coherent* modules. The definition is as follows.

Definition 3.2.12. Let S be a scheme.

- Suppose $S = \text{Spec } R$, and let M be an R-module. Then $\widetilde{M}$ is the sheafification of the presheaf $U \mapsto \mathcal{O}(U) \otimes_R M$. An $\mathcal{O}_S$ module is said to be *quasi-coherent* if it is isomorphic to the sheaf $\widetilde{M}$ for some R-module M.

- In general, an $\mathcal{O}_S$-module $\mathcal{F}$ is said to be *quasi-coherent* if for each $U \to S$ with U affine, the restriction $\mathcal{F}|_U$ is quasi-coherent under previous definition.

Quasi-coherent modules $\mathcal{O}_S$ are usually defined in terms of sheaves on topological spaces. However, there is a one-to-one correspondence between the two versions. The advantage of this version is, however, that the inverse image sheaf is much easier to define. Indeed, for any morphism $f\colon X \to S$ of schemes and any quasi-coherent module $\mathcal{F}$ on $\mathcal{O}_S$, the inverse image module is now simply the restriction $\mathcal{F}|_X$.
All definitions and results of this chapter also work when (Sch/S) is equipped with the *Zariski* topology of Example 3.1.3. However, the next chapter reveals a nice characterization of *finite étale morphisms* in terms of sheaves on the étale topology, which cannot be accomplished with the other topology.

3.3 Finite étale morphisms

This section will be an introduction to finite étale morphisms of schemes. These are the algebra-geometric analogue of *covering spaces*. As such, topologists can use these as a reference during this introduction.
As the name suggests, a finite étale morphism of schemes is a morphism that is both finite and étale. So it makes sense to recall the definition of finite morphisms first.

Definition 3.3.1. Let $f\colon X \to S$ be an affine morphism of schemes.

- If $S = \text{Spec } R$ and $X = \text{Spec } A$, then f is étale if the corresponding ring map $R \to A$ turns A into a finitely generated R-module.

- In general, f is étale if for each affine open $U \subseteq X$, the induced morphism $f^{-1}(U) \to U$ is finite by above definition.

There are a few examples of finite morphisms.

Example 3.3.2. Closed immersions are finite morphisms.

Example 3.3.3. Suppose $k \to l$ is a finite field extension. Then the induced morphism $\text{Spec } l \to \text{Spec } k$ is finite.

We will now head to finite étale morphisms.

Definition 3.3.4. A morphism $X \to S$ is finite étale if it is finite and étale.

Finite étale morphisms enjoy the following properties of Lemma 3.2.6.

Lemma 3.3.5. *Let X, Y, Z be schemes.*

1. *Let $X \to Y$ and $Y \to Z$ be morphisms of schemes. Suppose both of them are finite étale. Then the composition $X \to Z$ is also finite étale.*

2. *Let $X \to Z$ and $Y \to Z$ be morphisms of schemes. Suppose $X \to Z$ is finite étale. Then the base change $X_Y \to Y$ is also finite étale.*

3. *Suppose $f: X \to Y$ is a morphism of schemes, and let $(U_i)_i$ of Y be an open (Zariski) covering. Then f is finite if and only if for each i, the induced morphism $f^{-1}(U_i) \to U_i$ is finite étale.*

4. *Let $f: X \to Y$ be a morphism of schemes, and let $U \to Y$ be a surjective étale morphism. Then f is finite étale if and only if the base extension f_U is finite étale.*

However, there is a much nicer characterization of finite étale morphisms. This relies on the following result.

Theorem 3.3.6. *Let $X \to S$ be finite étale. Then there exists a covering $(U_i)_i$ of X such that the $U_i \times_S X \cong n_i U_i$ for some integer $n_i \in \mathbb{N}$.*

Proof. See [5], Tag 04HN. □

Now view the morphism $X \to S$ as a sheaf on $(\mathsf{Sch}/S)_{\acute{e}t}$. Then Theorem 3.3.6 effectively tells that if $X \to S$ is étale, then $h(X)$ is a locally constant finite sheaf on $(\mathsf{Sch}/S)_{\acute{e}t}$.
It turns out that all localy constant sheaves on $(\mathsf{Sch}/S)_{\acute{e}t}$ arise this way.

Theorem 3.3.7. *Let S be a scheme, and let $\mathcal{F}$ be a sheaf on $(\mathsf{Sch}/S)_{\acute{e}t}$. The following are equivalent:*

- *$\mathcal{F}$ is locally constant finite.*
- *$\mathcal{F}$ is representable by a finite étale $X \to S$.*

Proof. See [5], Tag 03RV. □

This result is very fundamental for this thesis. It is in fact more convenient to actually *characterize* étale morphism $X \to S$ as locally constant sheaves on $(\mathsf{Sch}/S)_{\acute{e}t}$.

4 Closure Data on schemes

This section will put everything introduced in chapter 2 into a scheme-theoretic setting.

4.1 The Ferrand morphism on schemes of constant rank

The aim of this chapter is to generalize the notion of a closure datum to schemes. This starts by putting the Ferrand map of rings into a scheme-theoretic setting, defining the *Ferrand morphism*. But in order to do so, we first need to generalize the notion of a group action to arbitrary categories. This is done as follows.

Definition 4.1.1. Let $\mathcal{C}$ be a category, let $X \in \mathcal{C}$ be an object, and let G be a group (of sets).

- A *G-action* on X is a group homomorphism $\lambda\colon G \to \mathrm{Aut}(X)$.
- If a G-action on X is given, then the *orbit space* X_G is the co-equalizer of the morphisms:

$$\{\lambda_g\colon X \to X; g \in G\}$$

Recall that the Ferrand map is a morphism $(A^{\otimes n})^{S_n} \to R$, with A being an R-algebra that, as a module, is free of rank n. This corresponds to a morphism Spec $R \to$ Spec $((A^{\otimes n})^{S_n})$. We know that Spec $(A^{\otimes n}) =$ (Spec $A)^n$. But do we also have Spec $(A^{\otimes n})^{S_n} =$ Spec $(A^{\otimes n})_{S_n}$? The answer is yes, as the following lemma shows.

Lemma 4.1.2. *Let $X =$ Spec A be an affine scheme, and let G be a finite group acting on it. Then $X_G =$ Spec A^G.*

Proof. Recall that schemes are ringed spaces, and that morphisms of schemes are morphisms of ringed spaces. So it suffices to prove that Spec $A^G = X_G$ in the category of ringed spaces.
In this category, we have that X_G is the following. The underlying topological space of X_G is the orbit space with quotient topology. The sheaf structure on X_G is given by $X_G(U) = X(q^{-1}(U))^G$ (as the G-action is also defined on $q^{-1}(U)$ as an open subscheme of X).
The inclusion map $A^G \to A$ is integral. This is the case since each element $a \in A$ is a root of the monic polynomial $\prod_{g\in G}(X - ga) \in A^G(X)$. As a result, the induced morphism Spec $A \to$ Spec A^G is closed and surjective; this follows from Going Up and Lying Over properties for integral extensions, see [2], p. 99. Henceforth, the induced map $X_G \to$ Spec A^G (which exists as Spec A^G is clearly G-invariant) also satisfies the two properties.
We will now prove it is also injective. Suppose $\mathfrak{p} \cap A^G = \mathfrak{p}' \cap A^G$. For any

$x \in \mathfrak{p}$, we gave $\prod_{g \in G} gx \in \mathfrak{p}$. This element also appears in $\mathfrak{p} \cap A^G$ and henceforth in $\mathfrak{p}' \cap A^G$. But $\mathfrak{p}'$ is prime, so for some g, $gx \in \mathfrak{p}'$. So $x \in \bigcup_{g \in G} g\mathfrak{p}'$. This proves that $\mathfrak{p} \subseteq \bigcup_{g \in G} g\mathfrak{p}'$. By Prime Avoidance (see [2], p. 79), this means that $\mathfrak{p} \subseteq g\mathfrak{p}'$ and hence $\mathfrak{p} = g\mathfrak{p}'$. So $\mathfrak{p}$ and $\mathfrak{p}'$ lie in the same orbit, proving injectivity.
So the map is a closed continuous bijection. So it must be a homeomorphism. As ringed spaces, they also coincide: for any $f \in A^G$, we have:

$$X_G(D(f)) = [A_f]^G = [A^G]_f = (\text{Spec } A^G)(D(f))$$

This proves the desired result. □

Corollary 4.1.3. *Let $X \to S$ be an affine morphism of schemes, and let G be a finite group acting on X (that is compatible with S). Then X_G exists, and $X_G \to S$ is also an affine morphism of schemes.*

This proves that the Ferrand map corresponds to a morphism Spec $R \to ((\text{Spec } A)^n)_{S_n}$. We can now generalize it as follows.

Theorem 4.1.4. *Let S be a scheme, $n \in \mathbb{N}$, and let $f\colon X \to S$ be an affine morphism such that there exists an Zariski cover $(U_i)_i$ of affine schemes such that $\mathcal{O}(f^{-1}(U_i))$ is a free $\mathcal{O}(U_i)$-module of rank n. Then there exists a (unique) morphism $S \to (X^n)_{S_n}$ such that the restriction to each U_i corresponds to the Ferrand map $U_i \to (f^{-1}(U_i)^n)_{S_n}$.*

Proof. Let $f_i\colon U_i \to (f^{-1}(U_i)^n)_{S_n}$ be the Ferrand map. This by composition becomes a map $U_i \to (X^n)_{S_n}$. It suffices to prove that for each i, j, that $f_i|_{U_{ij}} = f_j|_{U_{ij}}$. Then we can glue to a morphism $S \to (X^n)_{S_n}$.
This already follows from Lemma 2.3.4. To elaborate, let i, j be indices, and let $(V_k)_k$ be a covering. Thanks to the base-change property of the Ferrand map, it follows that $f_i|_{V_k}$ and $f_j|_{V_k}$ are both the Ferrand map $V_k \to ((V_k)^n)_{S_n}$. So f_i and f_j agree on V_k. This holds for all k, so f_i and f_j agree on U_{ij}. So $f_i = f_j$ This holds for all i, j, so the f_i glue to an f. □

This motivates the following definition.

Definition 4.1.5. Let $X \to S$ as in Theorem 4.1.4. Then the *Ferrand morphism* $\Phi_{X/S}\colon S \to (X^n)_{S_n}$ is the unique morphism of Theorem 4.1.4.

Thanks to Corollary 4.1.3, $(X^n)_G$ exists for any $G \subseteq S_n$. As a result, one could define a closure datum to be a pair (H, σ) where $H \subseteq S_n$ is a subgroup, and $S \to (X^n)_H$ a morphism that factors through the Ferrand morphism. Then everything said in section 2.3 will carry over.
However, this is not going to be the direction of the thesis. Instead, we want to generalize the Ferrand map to a special class of finite covers, namely *finite étale covers*. But in order to do so, we first need some additional machinery.

4.2 Exponentiation

This section will be an introduction to the concept of *exponentiation*, which is a generalization of "taking the n-th power of something." On Set, the set Map (Y, X) is often denoted by X^Y. This is also known as the Y-th power of X. When $Y = \{1, ..., n\}$, this just becomes X^n. This will be formalized as follows.

Definition 4.2.1. Let $\mathcal{C}$ be a category that is closed under taking products.

- Let $X, Y \in \mathcal{C}$ be two objects. A Y-th power of X is an object, X^Y, together with an evaluation map $\mathrm{ev}\colon X^Y \times Y \to X$, satisfying the following universal property: for each $f\colon Z \times Y \to X$, there exists a *unique* morphism $\tilde{f}\colon Z \to X^Y$ such that $f = \mathrm{ev} \circ (\tilde{f}, \mathrm{id}_Y)$.
- An object $Y \in \mathcal{C}$ is *exponential* if for each $X \in \mathcal{C}$, the object X^Y exists.

Example 4.2.2. Let $X, Y \in \mathsf{Set}$ be two sets. Then we have $X^Y = \mathrm{Map}(Y, X)$, with the evaluation map being the usual one. More generally, let $\mathcal{F}, \mathcal{G} \in \mathsf{Sh}(\mathcal{C})$ be two sheaves, with $\mathcal{C}$ being a site. Then $\mathcal{F}^{\mathcal{G}}$ is given by $\mathsf{Hom}(\mathcal{F}, \mathcal{G})$ from Definition 3.1.10.

Example 4.2.3. Let $\mathcal{C}$ be a *distributive* category closed under products. That is, for each $X, Y, Z \in \mathcal{C}$, we have $X \times (Y \sqcup Z) \cong (X \times Y) \sqcup (X \times Z)$. Let T be its terminal object, and let $nT = \coprod_{i \in \{1,...,n\}} T$ (assuming it exists). Then $X^{nT} = X^n$.

In this regard, exponentiation really is a generalization to taking the n-th power in the ordinary sense.
One very convenient property is that the Yoneda lemma preserves exponentials.

Theorem 4.2.4. *Let $\mathcal{C}$ be a category. The Yoneda embedding $h\colon \mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{opp}}$ preserves exponentiation; i.e. for $X, Y \in \mathcal{C}$, we have $h(X^Y) = (h(X))^{h(Y)}$ (assuming the objects in question exist).*

Proof. Let $X, Y \in \mathcal{C}$. By the universal property of exponentiation, there is an isomorphism of presheaves on $\mathcal{C}$:

$$\mathrm{Hom}_{\mathcal{C}}(- \times Y, X) \cong \mathrm{Hom}_{\mathcal{C}}(-, X^Y)$$

For each $U \in \mathcal{C}$, there is a canonical identification $\mathrm{Hom}_{\mathcal{C}}(U \times Y, X) \cong \mathrm{Hom}_{\mathcal{C}/U}(U \times Y, U \times X)$. Under this identification, precomposing a morphism $U \times Y \to X$ with a morphism in the form (f, id_Y) a morphism $U \times Y \to U \times X$ over U by f. So we gain an isomorphism of sheaves:

$$\mathrm{Hom}_{\mathcal{C}/-}(- \times Y, - \times X) \cong \mathrm{Hom}_{\mathcal{C}}(- \times Y, X)$$

For any U, the sheaf $h(U \times Y)$ on $sh(\mathsf{Sch}/U)$ is isomorphic to the sheaf $h(Y)|_U$ by Lemma 3.1.9. So by the Yoneda lemma, we have an identification $\mathrm{Hom}_{\mathcal{C}/U}(U \times Y, U \times X) \cong \mathrm{Hom}_{sh(\mathcal{C}/U)}(h(Y)|_U, h(X)|_U)$. This gives rise to an isomorphism:

$$(h(X))^{h(Y)} = \mathsf{Hom}(h(Y), h(X)) \cong \mathrm{Hom}_{\mathcal{C}/-}(- \times Y, - \times X)$$

So, indeed, $(h(X))^{h(Y)} \cong h(X^Y)$. This isomorphism is also compatible with the evaluation maps, so indeed, h preserves exponentiation. □

After having generalized the notions of $\mathrm{Map}(X, Y)$ in Set and $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ in $\mathsf{Set}^{\mathcal{C}^{opp}}$, is it also possible to formalize the notions of $\mathrm{Iso}(X, Y)$ and $\mathsf{Iso}(\mathcal{F}, \mathcal{G})$ in $\mathsf{Set}^{\mathcal{C}^{opp}}$? The answer is yes, as the following lemma shows.

Lemma 4.2.5. *Let $\mathcal{C}$ be a category. Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves. The presheaf $\mathsf{Iso}(\mathcal{F}, \mathcal{G}) \subseteq \mathcal{F}^{\mathcal{G}}$, given by $U \mapsto Iso(\mathcal{F}|_U, \mathcal{G}|_U)$, can be constructed by taking limits and exponentials.*

Proof. When three presheaves $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are given, we have the *composition map* $\circ\colon \mathsf{Hom}(\mathcal{G}, \mathcal{H}) \times \mathsf{Hom}(\mathcal{F}, \mathcal{G}) \to \mathsf{Hom}(\mathcal{F}, \mathcal{H})$, as well as a *unit map* $e\colon 1 \to \mathsf{Hom}(\mathcal{F}, \mathcal{F})$. Using this, we can define the sheaf $\mathsf{Sec}(\mathcal{F}, \mathcal{G})$, the sheaf *sections*, i.e. of left invertible morphisms. We not only gain an inclusion $\iota\colon \mathsf{Sec}(\mathcal{F}, \mathcal{G}) \to \mathcal{G}^{\mathcal{F}}$, but also a reversion map $\mu\colon \mathsf{Sec}(\mathcal{F}, \mathcal{G}) \to \mathcal{F}^{\mathcal{G}}$. In a similar vein, we can define the presheaf $\mathsf{Ret}(\mathcal{F}, \mathcal{G})$ of *retractions*, with similar maps. The following diagrams commute, and are Cartesian:

$$\begin{tikzcd}
\mathsf{Sec}(\mathcal{F},\mathcal{G}) \arrow[r] \arrow[d, "{(\mu,\iota)}"'] & 1 \arrow[d, "e"] \\
\mathsf{Hom}(\mathcal{G},\mathcal{F}) \times \mathsf{Hom}(\mathcal{F},\mathcal{G}) \arrow[r, "\circ"] & \mathsf{Hom}(\mathcal{G},\mathcal{G})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Ret}(\mathcal{F},\mathcal{G}) \arrow[r] \arrow[d, "{(\iota,\mu)}"'] & 1 \arrow[d, "e"] \\
\mathsf{Hom}(\mathcal{F},\mathcal{G}) \times \mathsf{Hom}(\mathcal{G},\mathcal{F}) \arrow[r, "\circ"] & \mathsf{Hom}(\mathcal{F},\mathcal{F})
\end{tikzcd}$$

Isomorphisms are morphisms that are both left and right invertible. As a result, the following diagram is Cartesian as well:

$$\begin{tikzcd}
\mathsf{Iso}(\mathcal{F},\mathcal{G}) \arrow[r] \arrow[d] & \mathsf{Ret}(\mathcal{F},\mathcal{G}) \arrow[d] \\
\mathsf{Sec}(\mathcal{F},\mathcal{G}) \arrow[r] & \mathsf{Hom}(\mathcal{F},\mathcal{G})
\end{tikzcd}$$

□

Given the fact that the Yoneda lemma preserves limits and exponentials, we can now define the notion that generalizes the notion of the set of bijections in Set.

Definition 4.2.6. Let $\mathcal{C}$ be a category, and let $X, Y \in \mathcal{C}$ be objects such that X^Y and Y^X both exist. Then we define the *Isom*-object X^{Y*} by taking above construction.

Remark 4.2.7. This is not an official notation. The reason behind this choice of notation is because it is also used for denoting the unit group of a ring.

Using the Yoneda lemma, one sees that for any $X \in \mathcal{C}$, X^{X*} is a group object. Moreover, there is both an X^{X*}- and Y^{Y*}-action on X^{Y*}, for any $X, Y \in \mathcal{C}$. But first, we need a general definition of group actions. This is the following.

Definition 4.2.8. Let $\mathcal{C}$ be a category, $X \in \mathcal{C}$ be an object, and G be a group object.

- A *G-action* on X is a morphism $\lambda\colon G \times X \to X$ such that for all $T \in \mathcal{C}$, the induced map $\mathrm{Hom}(T, G) \times \mathrm{Hom}(T, X) \to \mathrm{Hom}(T, X)$ is a group action in the usual sense.
- If a G-action on X is given, the *orbit space* X_G is the co-equalizer of the diagram:
  $$X \times G \underset{\pi_1}{\overset{\lambda}{\rightrightarrows}} X$$

By applying the Yoneda embedding, one sees easily that there exist two group actions on X^Y. One of them is an Y^{Y*}-action, that arises of the pre-composition action. The other is an X^{X*}-action, that arises from the post-composition action.

Remark 4.2.9. This is a generalization of Definition 4.1.1. Indeed, if $\mathcal{C}$ is a *distributive* category with terminal object 1, then each G-action on X (with G being a set) gives rise to a $G.1$-action as in Definition 4.2.8, with the orbit space being the same under either definition.

4.3 The Ferrand morphism on finite étale covers

Now let S be a scheme and let $X, Y \to S$ be *both* finite étale (and being locally of the same rank). We wish to find a way to define the Ferrand morphism as a morphism $S \to (X^Y)_{Y^{Y*}}$. The scheme Y no longer needs to be of the form nS for some $n \in \mathbb{N}$. The only condition we need is that X and Y are locally isomorphic.

Definition 4.3.1. Let $X, Y \in \mathsf{Sch}/S$ be schemes over S. We say X and Y are *locally isomorphic* if there exists an epimorphism $U \to S$ such that $X \times_S U \cong Y \times_S U$ as schemes over U.

When $X, Y \to S$ are Étale, this means that there exists an Étale covering $(U_i)_i$ of S such that $X \times_S U_i \cong Y \times_S U_i \cong n_i U_i$ for some $n_i \in \mathbb{N}$.
But we haven't even proven that the desired exponential exist! This needs to be done first. Theorem 3.3.7 tells that finite étale morphisms are in correspondence to locally constant finite sheaves on $(\mathsf{Sch}/S)_{\acute{e}t}$. By Theorem 4.2.4,

the yoneda embedding preserves exponentials. So in order to prove the existence of an exponential object $X^Y \to S$, it suffices to prove the following.

Theorem 4.3.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be locally constant finite sheaves on $(\mathsf{Sch}/S)_{\acute{e}t}$. Then $\mathsf{Hom}(\mathcal{F},\mathcal{G})$ is also locally constant finite.*

Proof. First assume $\mathcal{F}$ and $\mathcal{G}$ are constant. Then they are isomorphic to sheafifications of sets $\mathbf{m}$ resp. $\mathbf{n}$ for some $m, n \in \mathbb{N}$. We will prove that $\mathsf{Hom}(\mathcal{F},\mathcal{G})$ is the sheafification of $\mathrm{Map}(\mathbf{m},\mathbf{n}) = \mathbf{n}^{\mathbf{m}}$.
Consider the morphism of presheaves $\phi\colon \mathbf{n}^{\mathbf{m}} \to \mathsf{Hom}(\mathcal{F},\mathcal{G})$ that sends each $f \in \mathbf{n}^{\mathbf{m}}(U)$ to the induced map of sheaves $f_s\colon \mathbf{m}_s|_U \to \mathbf{n}_s|_U$. Unless $U = \emptyset$, in which case all sheaves are the same, this morphism is certainly injective. To prove ϕ_s becomes an iso, it suffices to prove that ϕ is *locally surjective*. In order to do so, let $U \to S$, and let $f \in \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$. For each $i \in \{i, ..., m\}$, let $(V_{ij})_{j\in J(i)}$ be a covering of U such that $f(i)|_{V_{ij}}$ is an element of $\mathbf{n}$. Now for tuples $(j_1, ..., j_m)$ with $j_i \in J(I)$, write

$$V_{(j_1,...,j_m)} = V_{1j_1} \times_U \times_U ... \times_U V_{mj_m}$$

Then $(V_{(j_1,...,j_m)})_{(j_1,...,j_m)}$ is a covering of U. On top of that, $f|_{V_{(j_1,...,j_m)}}$ is constant. This proves that f is locally in the image of Φ. Hence Φ_s is surjective.
So $\mathsf{Hom}(\mathcal{F},\mathcal{G})$ is the sheafification of $\mathrm{Map}(\mathbf{m},\mathbf{n})$. In particular, $\mathsf{Hom}(\mathcal{F},\mathcal{G})$ is constant if $\mathcal{F}$ and $\mathcal{G}$ are constant.
Now assume that $\mathcal{F}$ and $\mathcal{G}$ are locally constant finite. Then there exists a covering (U_i) such that $\mathcal{F}|_{U_i}$ and $\mathcal{G}|_{U_i}$ are constant finite. Then $\mathsf{Hom}(\mathcal{F},\mathcal{G})|_{U_i}$ is also constant finite. This holds for all i, so $\mathsf{Hom}(\mathcal{F},\mathcal{G})$ is locally constant finite. $\square$

This proves the existence of an exponentiation $X^Y \to S$ if $X, Y \to S$ are finite étale. The scheme Y^{Y*} also exists. On top of that, $(X^Y)_{Y^{Y*}}$ also exists, as co-equalizers of locally constant finite sheaves are also locally constant finite (this can be proven in a same fashion). Also, by Remark 4.2.9, if $Y = nS$, this corresponds to $(X^n)_{S_n}$.
It is now time to define the Ferrand morphism. Using the local-global principle, it suffices to examine the trivial case. Namely when $X = Y = nS$.

Lemma 4.3.3. *Let R be a ring, and let $A = R^n$ be the free R-algebra of rank n. Then the Ferrand map is $\psi|_{(A^{\otimes n})^{S_n}}$, where $\psi\colon A^{\otimes n} \to R$ is given by:*

$$(r_{11}, r_{12}, ..., r_{1n}) \otimes ... \otimes (r_{n1}, r_{n2}, ..., r_{nn}) \mapsto r_{11}r_{22}...r_{nn}$$

Proof. The R-module $(A^{\otimes n})^{S_n}$ is a free, with a basis consisting of elements in the form

$$x_{i_1,...,i_n} := \sum_{\sigma \in S_n/\mathrm{inv}} e_{i_{\sigma(1)}} \otimes ... \otimes e_{i_{\sigma(n)}}$$

Now consider the ring $A[X_1, ..., X_n] = A \otimes_R R[X_1, ..., X_n]$. The element $x_{i_1,...,i_n}$ is the $X_{i_1}X_{i_2}...X_{i_n}$-term of the polynomial $x \in ((A[X_1, ..., X_n])^{\otimes_{R[X_1,...,X_n]} n})^{S_n}$, which is given by:

$$x := (e_1X_1 + ... + e_nX_n) \otimes ... \otimes (e_1X_1 + ... + e_nX_n) \in$$

So it suffices to compute $\phi_{A[X_1,...,X_n]/R[X_1,...,X_n]}(x)$. We know $A[X_1, ..., X_n]$ is a free $R[X_1, ..., X_n]$-algebra with basis $\{e_1, ..., e_n\}$. We have:

$$e_i(e_1X_1 + ... + e_nX_n) = e_iX_i = 0e_1 + ...0e_{i-1} + X_ie_i + 0e_{i+1} + ... + 0e_n$$

As a result, we find that:

$$\phi_{A[X_1,...,X_n]/R[X_1,...,X_n]}(x) = \begin{vmatrix} X_1 & 0 & ... & 0 \\ 0 & X_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & X_n \end{vmatrix} = X_1...X_n$$

Since $\Phi_{A[X_1,...,X_n]/R[X_1,...,X_n]} = \Phi_{A/R} \otimes R[X_1, ..., X_n]$, we find that:

$$\phi_{A/R}(x_{i_1,...,i_n}) = \begin{cases} 1 & \text{if } \{i_1, ..., i_n\} = \{1, ..., n\} \\ 0 & \text{else} \end{cases}$$

Now consider the given morphism $\psi \colon A^{\otimes n} \to R$. Then one can check easily that $f(x_{i_1,...,i_n}) = \Phi_{A/R}(x_{i_1,...,i_n})$ for all possible $x_{i_1,...,i_n}$ of the basis. As a result, $\psi|_{(A^{\otimes n})^{S_n}} = \Phi_{A/R}$. $\square$

Corollary 4.3.4. *Let S be a scheme. Let $X, Y \to S$ be finite étale. Assume $Y = nS$, and X is as in Theorem 4.1.4. We have $(X^{Y*})_{Y^{Y*}} = S$, and the Ferrand morphism is the inclusion $(X^{Y*})_{Y^{Y*}} \to (X^Y)_{Y^{Y*}}$.*

Proof. It suffices to look étale locally, given the fact that the Ferrand morphism is preserved by base change. So we may assume S is affine, and that $X = nS$ as well. The ψ of previous lemma is the morphism $(i_1, ..., i_n) \colon S \to (nS)^n$, which corresponds to $\mathrm{id}_{nS} \colon nS \to nS$.
Under this hypothesis, X and Y both correspond to the sheaf $\mathcal{F} = \mathbf{n}$ on $(\mathsf{Sch}/S)_{ét}$. Then the Ferrand morphism arises from $1 \to \mathrm{Hom}(\mathcal{F}, \mathcal{F}), 1_U \mapsto \mathrm{id}_{\mathcal{F}|_U}$. So it corresponds to

$$1 \to \mathrm{Hom}(\mathcal{F}, \mathcal{F})_{\mathcal{F}^{\mathcal{F}*}}, 1_U \mapsto \overline{\mathrm{id}_{\mathcal{F}|_U}}$$

It is immediate from the definitions that $(\mathcal{F}^{\mathcal{G}*})_{\mathcal{G}^{\mathcal{G}*}} = 1$ and that the Ferrand morphism now is the inclusion $(\mathcal{F}^{\mathcal{G}*})_{\mathcal{G}^{\mathcal{G}*}} \to (\mathcal{F}^{\mathcal{G}})_{\mathcal{G}^{\mathcal{G}*}}$. The result follows going back to schemes. $\square$

This motivates the following definition for the Ferrand morphism:

Definition 4.3.5. Let S be any algebraic stack,and let $X, Y \to S$ be locally isomorphic finite étale schemes. Then the *Ferrand morphism* is the inclusion $S = (X^{Y*})_{Y^{Y*}} \to (X^Y)_{Y^{Y*}}$.

This definition of the Ferrand morphism is much more natural!

Remark 4.3.6. The equality $S = (X^{Y*})_{Y^{Y*}}$ follows from X and Y being locally isomorphic. In fact, since the quotient arrow $X^{Y*} \to (X^{Y*})_{Y^{Y*}}$ is an epimorphism, and we have an isomorphism $X \times X^{Y*} \cong Y \times X^{Y*}$, this implication is actually an equivalence.

4.4 The sheaf of Closure data

After having defined the Ferrand morphism, the definition of closure data as in Definition 2.2.6 has an analogue as well. We start with the analogue of normative maps.

Definition 4.4.1. Let X, Y be locally finite free schemes over S, of which Y is finite étale. Let $H \in \mathsf{Sch}/S$ be a finite étale group object, and let $H \to Y^{Y*}$ be a monomorphism of group objects. A morphism of S-schemes $S \to (X^Y)_H$ is said to be *normative* if post composition with the morphism $(X^Y)_H \to (X^Y)_{Y^{Y*}}$ yields the Ferrand morphism.

However, there is an alternative definition, as the following lemma shows.

Lemma 4.4.2. *Let $X, Y \to S$ be locally isomorphic finite étale schemes, and let $H \subseteq Y^{Y*}$ be a clopen subgroup. Let $\sigma\colon S \to (X^Y)_H$ be any morphism of S-schemes. The following are equivalent.*

1. *The morphism σ is normative.*
2. *The morphism factors (uniquely) through the inclusion $(X^{Y*})_H \to (X^Y)_H$.*

Proof. Obvious when viewing the schemes as locally constant finite sheaves. □

One could define a closure data to be a pair (H, σ), where H is a finite étale group scheme equipped with a monomorphism of group objects $H \to Y^{Y*}$, and $\sigma\colon S \to (X^Y)_H$ is a normative morphism. Comparing to Definition 2.2.6, there is one major difference. Under that setting, closure data are pairs (H, σ) where H is a *subset* of S_n. Here we simply ask $H \to Y^{Y*}$ to be merely a *monomorphism*. In order to address this issue, one could use the category-theoretical notion of a *subobject*. In our setting, however, this turns out to be unnecessary, thanks to the following result:

Lemma 4.4.3. *Let $f\colon X \to Y$ and $g\colon Y \to S$ be morphisms of schemes. Assume that g is finite étale and f is a monomorphism. Then the following are equivalent:*

1. *f is a closed and open immersion (i.e. a clopen immersion).*

2. *The composition $g \circ f$ is finite étale.*

Proof. Suppose (1) holds. Closed immersions are finite, whereas open immersions are étale. As a result, f is finite étale. This property is stable under composition, so $g \circ f$ is also finite étale. This proves (2).
Now suppose (2) holds. Being a clopen immersion is an étale local property. So we may assume that $X = mS$ and $Y = nS$ for some $m, n \in S$. Using Theorem 4.3.2, we have that $\mathsf{Hom}(X, Y)$ is the sheafification of $\mathrm{Map}(\mathbf{m}, \mathbf{n})$. So by going even more local, we may assume even f corresponds to a morphism $\mathbf{m} \to \mathbf{n}$. This means that f is the morphism $(i, x) \mapsto (f(i), x)$. Since f is a monomorphism, it follows that $m \leqslant n$, and that f corresponds to the first inclusion of the decomposition $nS \cong mS \sqcup (n - m)S$. So f is a clopen immersion. This proves (1). $\square$

This means that closure data can be defined as follows.

Definition 4.4.4. Let $X \to S$ and $Y \to S$ be locally isomorphic finite étale S-schemes. A *closure datum* is a pair (H, σ). Here is $H \subseteq Y^{Y*}$ a clopen subgroup, and $\sigma \colon S \to (X^{Y*})_G$ is a morphism of schemes over S.

We call the set of closure data $\mathrm{Data}(X, Y)$. However, it would be nice if this could actually be a *presheaf* on $(\mathsf{Sch}/S)_{ét}$. Thankfully, this is the case. In the same vein as Lemma 2.3.4, the following is true:

Lemma 4.4.5. *Suppose $S \to (X^Y)_G$ is a closure datum. Suppose $T \to S$ is a morphism of schemes. Then the base extension $T \to (X^Y \times_S T)_{G_T} = ({X_T}^{Y_T})_{G_T}$ is also a closure datum. As a result, we obtain a map*

$$\mathit{Data}(X, Y) \to \mathit{Data}(X_T, Y_T)$$

This gives rise to a presheaf!

Definition 4.4.6. Let $X, Y \to S$ be locally isomorphic finite étale S-schemes. The *presheaf of closure data* $\mathsf{Data}(X, Y)$ is the presheaf on $(\mathsf{Sch}/S)_{ét}$ given by $T \mapsto \mathrm{Data}(X_T, Y_T)$.

It seems intuitively true that this is indeed a sheaf. However, it has yet to be proven! While this could be done straight from the definitions, it is more useful to construct it out of sheaves we already know.
So we will spend the rest of this section proving the following.

Theorem 4.4.7. *Let $X, Y \to S$ be finite étale S-schemes. Then $\mathsf{Data}(X, Y)$ is a sheaf on $(\mathsf{Sch}/S)_{et}$ that is locally constant of finite rank.*

In order to prove the statement, we first need an alternative definition of closure data.

Lemma 4.4.8. *Let $X \to S$ be a finite étale scheme. We define the* power sheaf PX *on* $(\mathsf{Sch}/S)_{ét}$ *to be the following:*

$$PX(U) = \{\textit{clopen subschemes } F \subseteq X_U\}$$

This sheaf is isomorphic to the sheaf $Hom(X, 2S)$. As a result, it is a locally constant sheaf of finite rank.

Proof. The proof is a direct copy of the case with sets. Let $F \in PX(U)$ be a clopen subscheme. Let $F \to 2U$ be the morphism $F \to U$, composed with the first inclusion $U \to 2U$. Let $F^c \to 2U$ be the morphism $F^c \to U$ composed with the second inclusion $U \to 2U$. Since $X_U \cong F \sqcup F^c$, this yields an indicator morphism $i_F \colon X_U \to 2U$. This yield a map:

$$PX(U) \to \mathrm{Hom}(X, 2S)(U); F \mapsto i_F$$

Now write $2U = U_1 \sqcup U_2$. Then above map has an inverse given by $t \mapsto t^{-1}(U_1)$. As a result, above map is bijective. It is also compatible with restrictions, yielding an isomorphism $PX \cong \mathrm{Hom}(X, 2S)$. □

Now we can prove that the sheaf of subgroups is also locally constant.

Lemma 4.4.9. *Let $G \to S$ be finite étale group scheme. The presheaf $P_{grp}G$ on $(\mathsf{Sch}/S)_{ét}$ is given by:*

$$P_{grp}G(U) = \{\textit{clopen subgroups } H \subseteq G_U\}$$

This presheaf can be constructed out of already-constructed sheaves by taking finite limits, and is therefore a locally constant finite sheaf.

Proof. First consider the presheaf $P_{mult}G$, given by:

$$P_{mult}G(U) = \{\text{mult. closed clopen subschemes } H \subseteq G_U\}$$

Let $\mu \colon G \times_S G \to G$ be the multiplication morphism. A subscheme $H \subseteq G_U$ is multiplicatively closed if μ_U maps $H \times_U H$ into H. This means that $H \times_U H \subseteq \mu_U^{-1}(H)$, which is equivalent to $\mu_U^{-1}(H) \cap (H \times_U H) = H \times_U H$. As a result, $P_{mult}G$ is the equalizer of the morphisms $p, q \colon PG \to P(G \times_S G)$ given by:

$$\begin{aligned} p_U &\colon PG(U) \to P(G \times_S G)(U), H \mapsto \mu_U^{-1}(H) \cap (H \times_U H) \\ q_U &\colon PG(U) \to P(G \times_S G)(U), H \mapsto H \times_U H \end{aligned}$$

This proves that $P_{mult}G(U)$ is a locally constant finite sheaf. In an identical fashion, the presheaves $P_{inv}G$ and $P_{unit}G$ are also locally constant finite sheaves. These are given by:

$$\begin{aligned} P_{inv}G(U) &= \{\text{clopen subschemes } H \subseteq G_U \text{ that are closed under inversion}\} \\ P_{unit}G(U) &= \{\text{clopen subschemes } H \subseteq G_U \text{ that contain the unit elements}\} \end{aligned}$$

Since $P_{grp}G = P_{mult}G \times_{PG} P_{inv}G \times_{PG} P_{unit}G$, it follows that $P_{grp}G$ is also locally constant finite. □

This has the following consequence:

Corollary 4.4.10. *Let $X, Y \to S$ be finite étale. Then the following sheaf is also locally constant finite:*

$$P_{grp}(Y^{Y*}) \times X^{Y*}$$

Note that this sheaf is given by:

$$U \mapsto \{(H, \sigma) \colon H \subseteq (Y^{Y*})_U \text{ clopen subgroup}; \sigma \in \mathrm{Hom}(U, X^{Y*})\}$$

This already looks very similar to the presheaf $\mathsf{Data}(X, Y)$, which is given by:

$$U \mapsto \{(H, \sigma) \colon H \subseteq (Y^{Y*})_U \text{ clopen subgroup}; \sigma \in \mathrm{Hom}(U, (X^{Y*})_H)\}$$

However, they are not quite the same. However, it suggests that $\mathsf{Data}(X, Y)$ is a *quotient* of $P_{Grp}(Y^{Y*}) \times X^{Y*}$. Recall the following.

Lemma 4.4.11. *Let $H \subseteq Y^{Y*}$ be a clopen subgroup. Then the sheaf $Hom(-, (X^{Y*})_H)$ ($= (X^{Y*})_H$) is the sheaf $Hom(-, X^{Y*})/\mathcal{R}$, where $\mathcal{R}$ is the sheaf of equivalence relations:*

$$\mathcal{R} \colon U \mapsto \{(\sigma, \tau) \colon \sigma, \tau \in Hom(U, X^{Y*}) \colon \sigma^{-1} \circ \tau \in Hom(U, H)\}$$

In a similar fashion, we have:

Lemma 4.4.12. *Let $\mathcal{F}$ be the sheaf of Corollary 4.4.10. Then $\mathsf{Data}(X, Y)$ is the quotient $\mathcal{F}/\mathcal{R}$, with $\mathcal{R}$ being the sheaf of equivalence relations given by:*

$$\mathcal{R} \colon U \mapsto \{(H, \sigma, \tau) \colon H \subseteq (Y^{Y*})_U, \sigma, \tau \in Hom(U, X^{Y*}) \colon \sigma^{-1} \circ \tau \in Hom(U, H)\}$$

Remark 4.4.13. The "$\circ$" and "$(-)^{-1}$" operations used in Lemma 4.4.11 and Lemma 4.4.12 do not represent the usual composition and inversion operators of arrows! Instead, they represent the socalled "componentwise composition" and "componentwise inversion". To elaborate, let $\mu \colon Y^{X*} \times X^{Y*} \to Y^{Y*}$ be the (abstract) composition morphism. Then the componentwise composition map $\circ$ is given by:

$$\circ \colon \mathrm{Hom}(U, Y^{X*}) \times \mathrm{Hom}(U, X^{Y*}), (f, g) \mapsto \mu \circ (f, g)$$

The componentwise inversion map $(-)^{-1} \colon \mathrm{Hom}(U, X^{Y*}) \to \mathrm{Hom}(U, Y^{X*})$ is defined similarly.

This ends the proof of Theorem 4.4.7. Note that that the sheaf of relations of previous theorem is also locally constant finite, as it can also be constructed out of already-constructed locally constant sheaves by taking limits and exponentials. I leave the details to the reader.

4.5 Extra structures

In this section, I will prove that the sheaves $\mathsf{Data}(X,Y)$ and $\mathsf{Data}(Y,X)$ are isomorphic. On top of that, I will take a look at the extra structures from 2.3, and give them an analogue on $\mathsf{Data}(X,Y)$.
One could use the usual sheaf routine in order to get the desired results. This is doable, and not very hard, but very dull. On top of that, it is unnecessary.
Theorem 4.4.7 tells us that $\mathsf{Data}(X,Y)$ is locally constant of finite rank. Moreover, it can be constructed out of the schemes X and Y using finite limits and exponentials. Now, using the exact same construction in a simpler category, we get the following:

Definition 4.5.1. Let X and Y be finite sets of the same size. The set $\mathsf{Data}(X,Y)$ is given by:

$$\mathsf{Data}(X,Y) = \{(G,\overline{\sigma}) \colon G \subseteq Y^{Y*} \text{ subgroup}, \overline{\sigma} \in \operatorname{Iso}(X,Y)_G\}$$

We will now prove that there exists a bijection between $\mathsf{Data}(X,Y)$ and $\mathsf{Data}(Y,X)$. This may sound silly, since there already exists a bijection between X and Y. However, the bijection we will construct is *canonical*; it does not depend on the choice of the bijection between X and Y. This is convenient when going back to locally constant finite sheaves.

Theorem 4.5.2. *Let X and Y be sets of the same size. There exists a canonical isomorphism $\mathsf{Data}(X,Y) \cong \mathsf{Data}(Y,X)$.*

Proof. Consider the map given by:

$$\Phi \colon \mathsf{Data}(X,Y) \to \mathsf{Data}(Y,X), (G,\overline{\sigma}) \mapsto (\sigma G \sigma^{-1}, \overline{\sigma^{-1}})$$

We have to prove that it is well defined. So suppose $(G,\overline{\sigma_1}) = (G,\overline{\sigma_2})$. Then there exists a $g \in G$ such that $\sigma_1 g = \sigma_2$. So we already find:

$$\sigma_2 G \sigma_2^{-1} = (\sigma_1 g)G(\sigma_1 g)^{-1} = \sigma_1(gGg^{-1})\sigma_1^{-1} = \sigma_1 G \sigma_1^{-1}$$

So the first components of $\Phi(G,\overline{\sigma_1})$ and $\Phi(G,\overline{\sigma_2})$ match. We also have:

$$\sigma_2^{-1} = g^{-1}\sigma_1^{-1} = \sigma_1^{-1}.(\sigma_1 g^{-1} \sigma_1^{-1})$$

So there exists a $g' \in \sigma_1 G \sigma_1^{-1}$ such that $\sigma_2^{-1} = \sigma_1^{-1} g'$. So $\overline{\sigma_1^{-1}} = \overline{\sigma_2^{-1}}$. So $\Phi(G,\overline{\sigma_1}) = \Phi(G,\overline{\sigma_2})$.
This proves that Φ is well-defined. It also has an identically-defined inverse $\mathsf{Data}(Y,X) \to \mathsf{Data}(X,Y)$, so Φ is a bijection. □

Now let's go back to group actions. The actions of Definition 2.3.3 and Definition 2.3.5 have easy analogues in this setting.

Definition 4.5.3. Let X and Y be sets of the same cardinality.

- The Y^{Y*}-action on $\mathsf{Data}(X,Y)$ is given by
$$\pi(H,\overline{\sigma}) = (\pi H \pi^{-1}, \overline{\sigma \circ \pi^{-1}})$$
- The X^{X*}-action on $\mathsf{Data}(X,Y)$ is given by:
$$\pi(H,\overline{\sigma}) = (\overline{\pi \circ \sigma}, H)$$

This means that $\mathsf{Data}(X,Y)$ and $\mathsf{Data}(Y,X)$ are both X^{X*}- and Y^{Y*}-sets. The question is, are those sets isomorphic w.r.t. those actions. The answer is yes!

Theorem 4.5.4. *The bijection Φ constructed in Theorem* 4.5.2 *is an isomorphism of X^{X*}- and Y^{Y*}-sets.*

Proof. I will only prove that Φ it is an isomorphism of X^{X*}-sets. Since Φ^{-1} is defined identically, it then follows that Φ^{-1} is an isomorphism of Y^{Y*}-sets, and hence that both Φ and Φ^{-1} are isomorphisms of X^{X*}- and Y^{Y*}-sets.
To prove the claim, let $\pi \in X^{X*}$. Call the bijection Φ. We have to show that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{Data}(X,Y) \arrow[r, "\Phi"] \arrow[d, "\pi(-)"'] & \mathsf{Data}(Y,X) \arrow[d, "\pi(-)"'] \\
\mathsf{Data}(X,Y) \arrow[r, "\Phi"] & \mathsf{Data}(Y,X)
\end{tikzcd}$$

This is the case, as for any closure datum, we have:

$$\begin{tikzcd}
(H,\overline{\sigma}) \arrow[r, mapsto, "\Phi"] \arrow[d, mapsto, "\pi(-)"'] & (\sigma H \sigma^{-1}, \overline{\sigma^{-1}}) \arrow[d, mapsto, "\pi(-)"'] \\
(H,\overline{\pi\sigma}) \arrow[r, mapsto, "\Phi"] & (\pi\sigma H \sigma^{-1}\pi^{-1}, \overline{\sigma^{-1}\pi^{-1}})
\end{tikzcd}$$

This proves our claim. □

Lastly, we have seen a structure of a partial ordering. This has the following analog.

Definition 4.5.5. Let $(H_1,\overline{\sigma_1}), (H_2,\overline{\sigma_2}) \in \mathsf{Data}(X,Y)$ be closure data. We say $(H_1,\overline{\sigma_1}) \leqslant (H_2,\overline{\sigma_2})$ if $H_1 \subseteq H_2$ and $\overline{\sigma_1} = \overline{\sigma_2}$ in $\mathrm{Iso}(X,Y)_{H_2}$.

This turns $\mathsf{Data}(X,Y)$ and $\mathsf{Data}(Y,X)$ into partially ordered sets. It is evident that Φ is also order preserving.
Now let X, Y be locally isomorphic finite étale schemes over S. All constructions done in Theorem 4.5.2, Definition 4.5.3 and Definition 4.5.5 are canonical, so they also work for locally constant sheaves! To be precise, we get the following results.

Theorem 4.5.6. *Let $X, Y \to S$ be locally isomorphic finite étale schemes over S.*

- *The actions of Definition* 4.5.3 *give rise to X^{X*}- and Y^{Y*}-actions on* $\mathsf{Data}(X, Y)$.
- *The ordering of Definition* 4.5.5 *turns* $\mathsf{Data}(X, Y)$ *into a sheaf of posets.*
- *The bijection of Theorem* 4.5.2 *gives rise to an isomorphism* $\mathsf{Data}(X, Y) \cong \mathsf{Data}(Y, X)$ *preserving the three aforementioned structures.*

Remark 4.5.7. The ordering on $\mathsf{Data}(X, Y)$ is actually very easy to define explicitly. On $\mathsf{Data}(X, Y)(U)$, we have $(G, \sigma) \leqslant (G', \sigma')$ if $G \subseteq G'$, and $\sigma' \colon U \to (X^{Y*})_{G'}$ is obtained out of $\sigma \colon U \to (X^{Y*})_G$ by post composition with the projection $(X^{Y*})_G \to (X^{Y*})_{G'}$.

5 Connected schemes

Theorem 4.4.7 tells that $\mathsf{Data}(X, Y)$ is representable by a finite étale cover. In this section, I will provide ways to compute the sheaf of closure data explicitly. By doing so, one can automatically deduce what the minimal closure data actually are, and how they are in relation to each other. During this chapter, I limit myself to the case where S is *connected*. In that case, there is one useful invariant: the *fundamental group*.

5.1 The fundamental group

Theorem 5.1.1. *Let S be a connected scheme, and pick a geometric point $\Omega \to S$. Let $\mathsf{F.ET}_S$ be the category of finite étale coverings of S. Consider the functor:*

$$\begin{aligned} \Psi \colon \mathsf{F.ET}_S &\to \mathsf{F.ET}_\Omega \\ S &\mapsto S \times_Y \Omega \end{aligned}$$

This can be viewed as a functor to Set, and gives a 1-to-1 correspondence between finite étale coverings $X \to S$ and finite sets on which $Aut(\Psi)$ acts continuously (as $Aut(\Psi)$ is a profinite group).

Proof. See [4], chapter 3. □

Thanks to this theorem, we can work in a much easier category, namely the category $\hat{G} - \mathsf{Set}$, where $\hat{G}$ is a profinite group. This category consists of $\hat{G}$-sets as objects, and he morphisms are functions of sets that are compatible with the $\hat{G}$-actions. Limits and colimits work as expected.

Lemma 5.1.2. *Let $\hat{G}$ be a profinite group, and let $X, Y \in \hat{G} - \mathsf{Set}$.*

- *The product $X \times Y$ is the product of X and Y as sets, on which $\hat{G}$ acts component wise.*
- *If $f_1, f_2 \colon X \to Y$ are morphisms of $\hat{G}$-sets, the equalizer is given by $\{x \in X \colon f_1(x) = f_2(x)\}$, on which $\hat{G}$ acts the same as on X.*
- *The coproduct $X \sqcup Y$ is the disjoint union of X and Y, with $\hat{G}$ having the same respective action on each component.*
- *If $f_1, f_2 \colon X \to Y$ are morphisms, the co-equalizer is the same as on sets, with $\hat{G}$ acting on each class as if it is acting on each representative.*

Exponentiation works as follows.

Lemma 5.1.3. *Let $\hat{G}$ be a profinite group, and let $X, Y \in \hat{G} - \mathsf{Set}$. For each $g \in \hat{G}$, let $\lambda_{g,X} \colon X \to X$ and $\lambda_{g,Y} \colon Y \to Y$ be the associated bijections. Then X^Y is the set $Map(X, Y)$, with the action $g(f) = \lambda_{g,Y} \circ f \circ \lambda_{g,X}^{-1}$.*

Backtracking all constructions, the set $\mathsf{Data}(X,Y)$ is the following.

$$\mathsf{Data}(X,Y) = \{(H,\overline{\sigma}), H \subseteq Y^{Y*}, \overline{\sigma} \in X_H^{Y*}\}$$

With the $\hat{G}$-action given by:

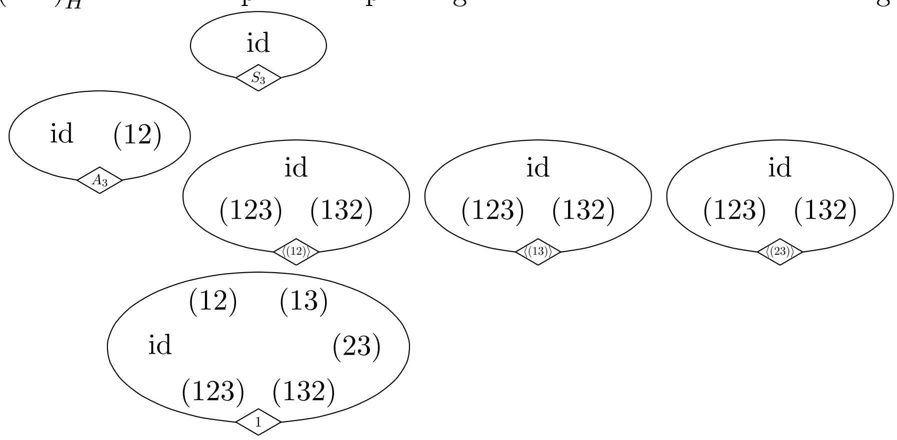
$$\pi(H,\overline{\sigma}) = (\lambda_{Y,\pi} H \lambda_{Y,\pi}^{-1}, \overline{\lambda_{X,\pi}\sigma\lambda_{Y,\pi}^{-1}})$$

5.2 Explicit computations

We will now focus on the case X and Y have cardinality 3, and make explicit computations. The first step is taking a look at the underlying set of $\mathsf{Data}(X,Y)$. This is independent of the $\hat{G}$-action. The set is given by:

$$\begin{aligned}\mathsf{Data}(X,Y) = \{&(S_3,\overline{\mathrm{id}}),\\ &(A_3,\overline{\mathrm{id}}),(A_3,\overline{(12)}),\\ &(\langle(12)\rangle,\overline{\mathrm{id}}),(\langle(12)\rangle,\overline{(123)}),(\langle(12)\rangle,\overline{(132)}),\\ &(\langle(13)\rangle,\overline{\mathrm{id}}),(\langle(13)\rangle,\overline{(123)}),(\langle(13)\rangle,\overline{(132)}),\\ &(\langle(23)\rangle,\overline{\mathrm{id}}),(\langle(23)\rangle,\overline{(123)}),(\langle(23)\rangle,\overline{(132)}),\\ &(1,\mathrm{id}),(1,(12)),(1,(13)),(1,(23)),(1,(123)),(1,(132))\}\end{aligned}$$

This set consists of 15 elements! In order to compute the $\hat{G}$-actions on it, we need to display the elements in some sort of diagram. This is done as follows. First draw an ellipse for each subgroup $H \subseteq X^{Y*}$. Then put each element of $(X^Y)^*_H$ into the ellipse corresponding to H. The result is the following:



How $\mathsf{Data}(X,Y)$ looks as a $\hat{G}$-set depends on X and Y. However, there is a very easy algorithm of computing the set. This is done as follows.

Lemma 5.2.1. *The set of closure data $\mathsf{Data}(X,Y)$ can be computed as follows.*

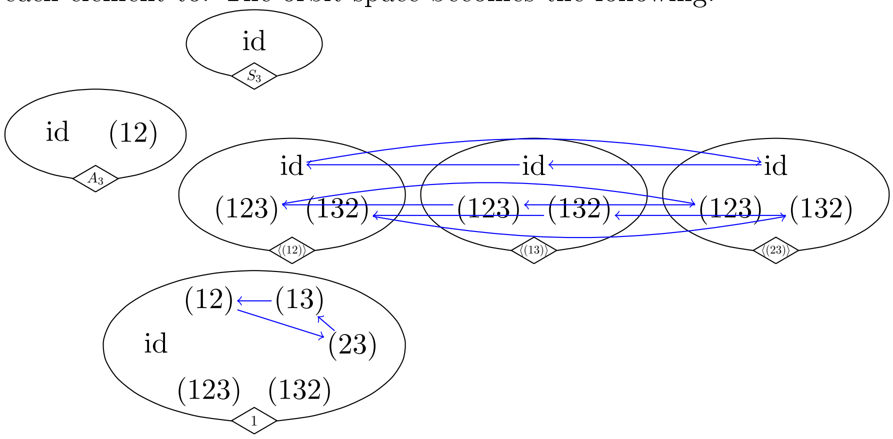
1. *Verify how $\hat{G}$ acts on the subgroups of Y^{Y*}.*

2. *Verify how $\hat{G}$ acts on the set of 1-closure data.*

3. *Derive the $\hat{G}$-action on other closure data from step 1-2.*

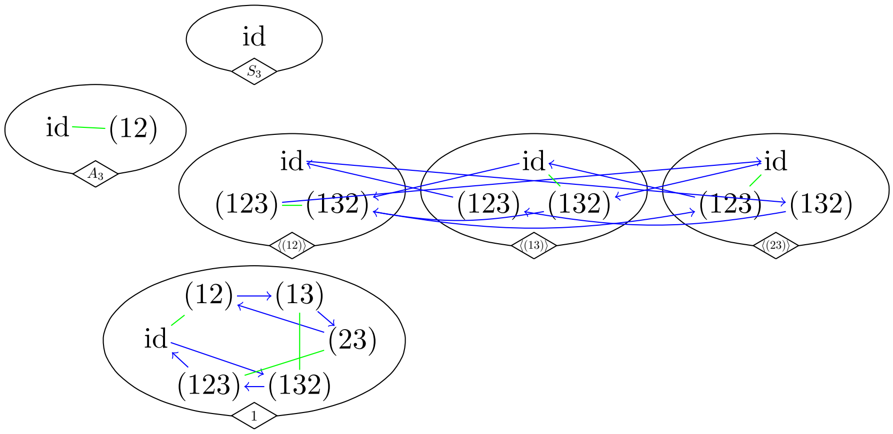
Since this chapter is not the main focus of this thesis, I will not work out every step in detail. Instead, I will just state the results.

Example 5.2.2. Consider the $X = Y = \text{Spec}\,\mathbb{F}_8$ over the scheme $S = \text{Spec}\,\mathbb{F}_2$. The fundamental group is now given by $\hat{G} = \hat{\mathbb{Z}}$. The sets X and Y are the 3-element sets on which $\hat{G}$ acts cyclically through the projection $\hat{G} \to C_3$. In particular, we can view X and Y as C_3-sets. The group C_3 is generated by (123). So the arrows denote where that permutation maps each element to. The orbit space becomes the following.



It follows that $\mathsf{Data}(X, Y)$ has 4 orbits of 3 elements, and 6 orbits of 1 element. As a result, we find $\mathsf{Data}(X, Y) = \text{Spec}\,(\mathbb{F}_8^4 \times \mathbb{F}_2^6)$. In particular, there are 6 closure data $1 \to \mathsf{Data}(X, Y)$.

Example 5.2.3. Now let $X = \text{Spec}\,(\mathbb{F}_4 \times \mathbb{F}_2)$ and let $Y = \text{Spec}\,(\mathbb{F}_8)$, over $S = \text{Spec}\,\mathbb{F}_2$. Again, the fundamental group $\hat{G}$ is $\hat{\mathbb{Z}}$. This time, however, X corresponds to the 3-element set on which $\hat{G}$ acts by permuting 1 and 2 through the projection $\hat{G} \to C_2$. In particular, we can view X and Y both as $C_2 \times C_3$-sets. We display the action of C_2 in green and the action of C_3 in blue. The result becomes:



It follows that there are two orbits of 6 elements, one orbit of 3 element, one orbit of 2 elements, and 1 orbit of 1 element. As a result, we find that $\mathsf{Data}(X, Y) = \text{Spec}\ (\mathbb{F}_{64}^2 \times \mathbb{F}_8 \times \mathbb{F}_4 \times \mathbb{F}_2)$. In particular, we find that the Ferrand map is the *only* closure datum!

It turns out that $\mathsf{Data}(X, Y)$ can become very weird. We will end this chapter with the easiest class of examples.

Example 5.2.4. For the easiest example, let S be any connected scheme, and assume $X = Y = 3S$. Then X and Y correspond to the $\hat{G}$-set of 3 elements on which $\hat{G}$ acts trivially. Then $\hat{G}$ also acts trivially on $\mathsf{Data}(X, Y)$. As a result, $\mathsf{Data}(X, Y)$ corresponds to the scheme $15S$. In particular, there are 15 closure data $1 \to \mathsf{Data}(X, Y)$.

6 Algebraic Spaces

In this chapter, I will give a short introduction to algebraic spaces, the first generalization of schemes. This notion is not only required to introduce algebraic stacks, but will also make them a lot easier to understand.

6.1 Representability

In order to define algebraic spaces, we need the notion of representability. Recall that a sheaf $\mathcal{F}$ is representable if it is isomorphic to the sheaf $\mathrm{Hom}(-, X)$ for some scheme $X \in \mathsf{Sch}/S$. It is usual to call such sheaves schemes themselves, and denote the sheaf $\mathrm{Hom}(-, X)$ by X.
However, morphisms of sheaves can be representable as well. This is as follows.

Definition 6.1.1. Let S be a scheme. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on $(\mathsf{Sch}/S)_{ét}$. The morphism f is *representable* if for each morphism $X \to \mathcal{G}$ with X a scheme, the base extension $X \times_{\mathcal{G}} \mathcal{F}$ is also a scheme.

Example 6.1.2. Let $f\colon X \to Y$ be a morphism of schemes. Then f is representable. More generally, suppose $f\colon \mathcal{F} \to X$ is a morphism of sheaves of which X is a scheme. Then f is representable if and only if $\mathcal{F}$ is a scheme.

It is possible to generalize properties of morphisms of schemes to properties of representable morphisms.

Definition 6.1.3. Let P be a property of morphisms of $(Sch/S)_{et}$. Assume that:

- P is stable under base change. That is, if $X \to Y$ has property P, then for all $U \to Y$ with $U \in \mathsf{Sch}/S$, the base change $X \times_Y U \to U$ also has property P.

- P is local on the base. That is, a morphism of schemes $X \to Y$ has property P if and only if there exists an étale covering $(U_i \to Y)_{i \in I}$ such that $X \times_Y U_i \to U_i$ has property P for all i.

Then we say that a representable morphism $\mathcal{F} \to \mathcal{G}$ has property P if for each $U \to \mathcal{G}$, the base change $\mathcal{F} \times_{\mathcal{G}} U \to U$ has property P.

Example 6.1.4. Let $\mathcal{F} \to \mathcal{G}$ be a representable morphism of sheaves.

- The morphism is étale if for each $U \to \mathcal{G}$ with U a scheme, the base extension $\mathcal{F} \times_{\mathcal{G}} U \to U$ is étale.

- The morphism is a (closed/open) immersion if for each $U \to \mathcal{G}$ with U a scheme, the base extension $\mathcal{F} \times_{\mathcal{G}} U \to U$ is a (closed/open) immersion.

- The morphism is surjective if for each $U \to \mathcal{G}$ with U a scheme, the base extension $\mathcal{F} \times_{\mathcal{G}} U \to U$ is surjective.

I will end this section with some straightforward, easy-to-prove properties.

Lemma 6.1.5. *Suppose $f\colon \mathcal{F} \to \mathcal{G}$ and $g\colon \mathcal{G} \to \mathcal{H}$ are representable morphisms of schemes. Then the composition $g \circ f$ is also representable. Moreover, if f and g have property P as in Definition* 6.1.3*, so does the composition.*

Proof. Let $X \to \mathcal{H}$ be any morphism, with X a scheme. A formal argument shows that $X \times_{\mathcal{H}} \mathcal{F} = (X \times_{\mathcal{H}} \mathcal{G}) \times_{\mathcal{G}} \mathcal{F}$. Since f and g are both representable, it follows that $X \times_{\mathcal{H}} \mathcal{F}$ is a scheme. This proves that $g \circ f$ is representable. Now suppose f and g have property P. For any scheme X, the projection $X \times_{\mathcal{H}} \mathcal{F} \to X$ is the composition of $(X \times_{\mathcal{H}} \mathcal{G}) \times_{\mathcal{G}} \mathcal{F} \to X \times_{\mathcal{H}} \mathcal{G}$ and $X \times_{\mathcal{H}} \mathcal{G} \to X$. Both of them have property P. Since P is stable under composition, the projection $X \times_{\mathcal{H}} \mathcal{F} \to X$ also has property P. This proves that $g \circ f$ has property P. $\square$

Lemma 6.1.6. *Suppose $f\colon \mathcal{F} \to \mathcal{H}$ and $g\colon \mathcal{G} \to \mathcal{H}$ are morphisms of which f is representable. Then the base extension $f_{\mathcal{G}}\colon \mathcal{F} \times_{\mathcal{H}} \mathcal{G} \to \mathcal{G}$ is representable as well. Furthermore, if $\mathcal{F} \to \mathcal{H}$ has property P as in Definition* 6.1.3*, so does the base extension.*

Proof. Let any $X \to \mathcal{G}$ be any morphism, with X a scheme. Then we have $(\mathcal{F} \times_{\mathcal{H}} \mathcal{G}) \times_{\mathcal{G}} X = \mathcal{F} \times_{\mathcal{H}} X$. Since f is representable, we find $X \times_{\mathcal{H}} \mathcal{F}$ is a scheme. This proves that $f_{\mathcal{G}}$ is representable.
Now suppose f has property P. For any $X \to \mathcal{G}$, the base extension of $f_{\mathcal{G}}$ is just the projection $\mathcal{F} \times_{\mathcal{H}} X \to X$. Since f has property P, so has this projection. This proves that $f_{\mathcal{G}}$ has property P. $\square$

6.2 Algebraic spaces

In this section, I introduce algebraic spaces, and prove some easy properties.

Definition 6.2.1. Let $\mathcal{F}$ be a sheaf on $(\mathsf{Sch}/S)_{\acute{e}t}$. We say $\mathcal{F}$ is an *algebraic space over S* if the following conditions are met.

- The diagonal $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is representable. Equivalently, for each two morphisms $X \to \mathcal{F}$ and $Y \to \mathcal{F}$ with X and Y schemes, the fibred product $X \times_{\mathcal{F}} Y$ is also a scheme.
- There exists an étale surjective $U \to \mathcal{F}$, with U being a scheme.

Remark 6.2.2. Many authors require the diagonal to be quasi-compact. This is unnecessary, as that hypothesis can also be just added when necessary. Besides, that approach also prevents algebraic spaces to be a replacement to all schemes.

Much like schemes, algebraic spaces are closed under taking finite limits. Recall that all finite limits can be constructed out of fibred products and terminal objects. The terminal object is naturally the scheme S itself. So it suffices to prove that the fibred product of algebraic spaces is also an algebraic space.

Lemma 6.2.3. *Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be algebraic spaces. Let $\mathcal{F} \to \mathcal{H}$ and $\mathcal{G} \to \mathcal{H}$ be (not necessarily representable) morphisms of sheaves. Then the fibred product $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ is also an algebraic space.*

Proof. We have to prove that the diagonal is representable, and that there exists an étale cover. For the first property, let $X \to \mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ and $Y \to \mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ be morphisms with X and Y schemes. A formal argument shows that:

$$X \times_{\mathcal{F} \times_{\mathcal{H}} \mathcal{G}} Y = (X \times_{\mathcal{F}} Y) \times_{X \times_{\mathcal{H}} Y} (X \times_{\mathcal{G}} Y)$$

The right hand side is a fibred product of schemes, since $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are algebraic spaces. This proves that the diagonal is representable.
For the existence of an étale cover, let $U_1 \to \mathcal{F}$ and $U_2 \to \mathcal{G}$ be surjective étale morphisms. Consider the following composition:

$$U_1 \times_{\mathcal{H}} U_2 \to U_1 \times_{\mathcal{H}} \mathcal{G} \to \mathcal{F} \times_{\mathcal{H}} \mathcal{G}$$

Being surjective étale is stable under base extension and composition, so this composition yields an étale surjection. Also, $U_1 \times_{\mathcal{H}} U_2$ is a scheme, since $\mathcal{H} \to \mathcal{H} \times \mathcal{H}$ is representable. This proves the existence of an étale cover. $\square$

The following property is very convenient for proving that a sheaf is an algebraic space. The algebraic stack analogue will play an important role in this thesis.

Lemma 6.2.4. *Let $\mathcal{F} \to \mathcal{G}$ be a representable morphism of sheaves. Suppose $\mathcal{G}$ is an algebraic space. Then so is $\mathcal{F}$.*

Proof. We start by proving that the diagonal $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is representable. The diagonal $\mathcal{F} \to \mathcal{F} \times_{\mathcal{G}} \mathcal{F}$ is representable; it is a base extension of the morphism $\mathcal{F} \to \mathcal{G}$. The inclusion $\mathcal{F} \times_{\mathcal{G}} \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is also representable; it is a base extension of the diagonal $\mathcal{G} \to \mathcal{G} \times \mathcal{G}$. As a result, the composition $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$, which is the diagonal, is representable as well.
To prove the existence of an étale cover, let $U \to \mathcal{G}$ be an étale cover of $\mathcal{G}$. Then the base extension $U_{\mathcal{F}} \to \mathcal{F}$ is also étale surjective. Since $\mathcal{F} \to \mathcal{G}$ is representable, the sheaf $U_{\mathcal{F}}$ is also a scheme. $\square$

Lastly, schemes

6.3 Gluing Properties

Algebraic spaces can be obtained out of schemes through some sort of gluing procedure. This section shows how. After knowing how the gluing works, it is very easy to construct some algebraic spaces.
In order to do so, one needs the notion of an *equivalence relation* on sheaves. This is the following.

Definition 6.3.1. Let $\mathcal{R}, \mathcal{F}$ be a sheaves on $(\mathsf{Sch}/S)_{ét}$, and let $\mathcal{R} \to \mathcal{F} \times \mathcal{F}$ be a monomorphism. We say $\mathcal{R}$ is an *equivalence relation* if for each $T \to S$, the injection $\mathcal{R}(T) \to \mathcal{F}(T) \times \mathcal{F}(T)$ denotes an equivalence relation on $\mathcal{F}(T)$. The *quotient* $\mathcal{F}/\mathcal{R}$ is the co-equalizer of the morphisms $\mathcal{R} \to \mathcal{F}$ (in $Sh(\mathsf{Sch}/S)_{ét}$).

Definition 6.3.2. An equivalence relation $\mathcal{R} \to \mathcal{F} \times \mathcal{F}$ is said to be an *étale equivalence relation* if both morphisms $\mathcal{R} \to \mathcal{F}$ are étale.

It is easy to see that each algebraic space gives rise to an étale equivalence relation on schemes.

Lemma 6.3.3. *Let $\mathcal{F}$ be an algebraic space. Let $U \to \mathcal{F}$ be an étale surjective morphism with U a scheme. Putting $R = U \times_{\mathcal{F}} U$, the inclusion $R \to U \times U$ is an étale equivalence relation on U. Furthermore, we have $\mathcal{F} = U/R$*

This shows that each algebraic space is the quotient space of an étale equivalence relation. However, it turns out that *each* étale equivalence relation on schemes gives rise to an algebraic space.

Theorem 6.3.4. *Let $\mathcal{F}$ and $\mathcal{R}$ be algebraic spaces, and suppose $\mathcal{R} \to \mathcal{F} \times \mathcal{F}$ is an étale equivalence relation. Then the quotient $\mathcal{F}/\mathcal{R}$ is an algebraic space.*

Proof. See [5], Tag 02WW. □

A special case of this kind of gluing is sheaf gluing. I will provide a full proof of that.

Theorem 6.3.5. *Let $\mathcal{F}$ be a scheme on (Sch/S). Assume there exists an étale surjection $U \to S$ such that the restricted sheaf $\mathcal{F}|_U$ on $(\mathsf{Sch}/U)_{ét}$ is a scheme. Then $\mathcal{F}$ is an algebraic space.*

Proof. Let X be the scheme over U corresponding to $\mathcal{F}|_U$. Using Lemma 3.1.9, $\mathcal{F}$ is obtained out of X through a gluing isomorphism $\alpha\colon X \times_S U \to U \times_S X$ of schemes over $U \times_S U$. Consider the following co-equalizer diagram of schemes:

$$X \times_S U \times_S U \underset{\alpha^{-1} \circ (\pi_2, \pi_3) \circ \pi_{13}^*(\alpha)}{\overset{\alpha^{-1} \circ (\pi_1, \pi_3) \circ \pi_{13}^*(\alpha)}{\rightrightarrows}} X \times_S U \xrightarrow{\pi_2 \circ \alpha} X$$

The arrows have been chosen so that by taking the projections $\pi_3\colon X \times_S U \times_S U \to U$ and $\pi_2\colon X \times_S U \to U$, this diagram becomes a co-equalizer

diagram of schemes over U.
The scheme X over S is obtained out of $X \times_S U$ over U through the gluing isomorphism:

$$(\pi_3, \pi_1, \pi_2) \colon X \times_S U \times_S U \to U \times_S X \times_S U$$

Likewise, the scheme $X \times_S U$ is obtained out of $X \times_S U \times_S U$ through the gluing isomorphism

$$(\pi_4, \pi_1, \pi_2, \pi_3) \colon X \times_S U \times_S U \times_S U \to U \times_S X \times_S U \times_S U$$

So we have to verify that the arrows on the equalizer diagram are compatible with the gluing isomorphisms. We start with the arrow $\pi_2 \circ \alpha$. We have to verify that the following diagram is commutative:

$$\begin{tikzcd}
X \times_S U \times_S U \arrow[r, "(\pi_2, \pi_3) \circ (\alpha, \mathrm{id}_U)"] \arrow[d, "(\pi_3, \pi_1, \pi_2)"'] & X \times_S U \arrow[d, "\alpha"] \\
U \times_S X \times_S U \arrow[r, "(\pi_1, \pi_3) \circ (\mathrm{id}_U, \alpha)"] & U \times_S X
\end{tikzcd}$$

The composition of the arrows on the right hand side is $(\pi_2, \pi_3) \circ \pi_{23}^*(\alpha) \circ \pi_{12}^*(\alpha)$. The composition of the arrows on the left hand side is $(\pi_2, \pi_3) \circ \pi_{13}^*(\alpha)$. The gluing conditions state that $\pi_{23}^*(\alpha) \circ \pi_{12}^*(\alpha) = \pi_{13}^*(\alpha)$. So those compositions indeed match. So this diagram is indeed commutative.
This proves that $\pi_2 \circ \alpha$ is indeed compatible with the gluing isomorphisms. Checking that the other two arrows of the diagram are compatible is very straightforward and will be left to the reader.
Gluing yields a new co-equalizer diagram:

$$X \times_S U \rightrightarrows X \longrightarrow \mathcal{F}$$

The fact the left two arrows make $X \times_S U \to X \times_S X$ an étale equivalence relation follows already from the fact that the base extension with U satisfies that property. As a result, $\mathcal{F}$ is a quotient of an étale equivalence relation. As a result, $\mathcal{F}$ is an algebraic space. $\square$

This leads to an important example of algebraic spaces, which we need in chapter 8.

Example 6.3.6. Let M be an $\mathcal{O}_S$-module that is (Étale) locally free of finite rank (i.e. a *vector bundle*). Then M is an algebraic space.

7 Stacks and Algebraic Stacks

In this chapter, I will introduce the notion of an algebraic stacks, as a generalization of schemes. Unlike algebraic spaces, those are in general not sheaves. Instead, they are, as the name suggests, *stacks*, which are *fibred categories*. The first three sections will be an introduction to those.

7.1 Fibred categories

Definition 7.1.1. Let $F\colon \mathcal{F} \to \mathcal{C}$ be a functor of categories. We say that the functor is a *category groupoids* when the following conditions are met:

- For each arrow $u\colon X \to Y$ in $\mathcal{C}$ and each $\tilde{Y} \in \mathcal{F}$ such that $F(\tilde{Y}) = Y$, there exists a lifting $\tilde{u}\colon \tilde{X} \to \tilde{Y}$ such that $F(\tilde{u}) = u$.
- Suppose we have given three arrows $u\colon X \to Y, v\colon Y \to Z$ and $w\colon X \to Z$ in $\mathcal{C}$ such that $w = v \circ u$. Suppose in addition that we have been given liftings $\tilde{v}\colon \tilde{Y} \to \tilde{Z}$ and $\tilde{w}\colon \tilde{X} \to \tilde{Z}$ of v resp. w. Then there exists a *unique* lifting $\tilde{u}\colon X \to Y$ of u such that $\tilde{w} = \tilde{v} \circ \tilde{u}$.

Categories fibred in groupoids are, informally speaking, presheaves $\mathcal{F}$ such that each section $\mathcal{F}(U)$ a groupoid, as opposed to a set. The following definitions show how.

Definition 7.1.2. Let $F\colon \mathcal{F} \to \mathcal{C}$ be a fibred category in groupoids. Let $U \in \mathcal{C}$. The *fibre of* U, called $\mathcal{F}(U)$, is the subcategory of $\mathcal{F}$, whose objects are $X \in \mathcal{F}$ where $F(X) = U$, and whose morphisms are $f \in \mathcal{F}$ where $F(f) = \mathrm{id}_U$.

Remark 7.1.3. The category $\mathcal{F}(U)$ is always a groupoid. Hence the name fibred in groupoids.
To prove this, let $U \in \mathcal{C}$, and let $f\colon X \to Y$ be an arrow in $\mathcal{F}(U)$. Using the framework of the second condition in Definition 7.1.1, let $u = v = w = \mathrm{id}_U$, with liftings $\tilde{w} = \mathrm{id}_Y$ and $\tilde{v} = f$. Then $\tilde{u}$ will be the right inverse of f.
So each morphism in $\mathcal{F}(U)$ has a right inverse. An elementary proof then shows that those right inverses are also left inverses. So $\mathcal{F}(U)$ is a groupoid.

Definition 7.1.4. Let $f\colon V \to U$ be a morphism in $\mathcal{C}$. Choose for each $X \in \mathcal{F}(U)$ an arrow g_X with codomain X such that $Fg_X = f$ (this is called *cleavage*). The *restriction functor* $-|_V\colon \mathcal{F}(U) \to \mathcal{F}(V)$ maps each object X to the domain of g_X, and each morphism $u\colon X \to Y \in \mathcal{F}(U)$ to the unique morphism $u' \in \mathcal{F}(V)$ such that the following diagram commutes:

$$\begin{tikzcd} X|_V \arrow[r, "u'"] \arrow[d, "g_X"'] & Y|_V \arrow[d, "g_Y"] \\ X \arrow[r, "u"] & Y \end{tikzcd}$$

Remark 7.1.5. This notation is unusual. It can be rather confusing as well, as it suggests that it only depends on V. A more common notation therefore is f^*. However, I prefer this notation as it avoids the need of brackets, it avoids the need of naming f, and is more natural in the case f resembles some sort of inclusion.

Remark 7.1.6. With these definitions in mind, one obtains a *pseudo functor* $\mathcal{C} \to \mathsf{Cat}$, given by $U \mapsto \mathcal{F}(U), f \mapsto f^*$. It is not a functor, as it usually is not compatible with composition.
That pseudofunctor determines the fibration completely, however. In fact, one can prove that pseudofunctors correspond to fibrations. However, the precise definition of such functors is rather horrible, and therefore not the definition I prefer.

This is how presheaves generalize to fibred categories.

Example 7.1.7. Let $\mathcal{F}\colon \mathcal{C}^{opp} \to \mathsf{Set}$ be a functor (i.e. a presheaf). Then the *object category* $\mathsf{Ob}(\mathcal{F})$ is as follows:

- The objects are pairs (U, x) such that $x \in \mathcal{F}(U)$.
- The morphisms $(V, y) \to (U, x)$ are morphisms $f\colon U \to V$ such that $\mathcal{F}f(x) = y$ (i.e. $x|_V = y$).

This comes with an obvious morphism $\mathsf{Ob}(\mathcal{F}) \to \mathcal{C}$, making $\mathcal{C}$ a category fibred in groupoids.

But there are more examples.

Example 7.1.8. Let $\mathcal{C}$ be a site. The fibred category Sh consists of the following:

- The objects are pairs $(U, \mathcal{F})$, where $U \in \mathcal{C}$ is an objects, and $\mathcal{F} \in Sh(\mathcal{C}/U)$ a sheaf.
- The morphisms $(U', \mathcal{F}') \to (U, \mathcal{F})$ are pairs (f, β) such that $f\colon U' \to U$ is an arrow, and $\beta\colon \mathcal{F}' \cong f^*(\mathcal{F})$ is an isomorphism.

The projection $Sh \to \mathsf{Sch}/S$ makes it a fibred category in groupoids.

Example 7.1.9. Let S be a scheme. Let P be a property of schemes that is stable under base extension. We define Mor_P category consists of the following:

- The objects are pairs (U, X), where $U \in \mathsf{Sch}/S$ is a scheme, and $X \to U$ a morphism of schemes with property P.
- The morphisms $(U', X') \to (U, X)$ are pairs (f, β) such that $f\colon U' \to U$ is an arrow, and $\beta\colon X' \cong X \times_U U'$ is an isomorphism of schemes over U'.

The projection $\mathsf{Mor}_P \to \mathsf{Sch}/S$ makes it a fibred category in groupoids.

When a new object is introduced, there is also the notion of a morphism. This is the following.

Definition 7.1.10. Let $F\colon \mathcal{F} \to \mathcal{C}$ and $G\colon \mathcal{G} \to \mathcal{C}$ be fibred in groupoids. A functor $\psi\colon \mathcal{F} \to \mathcal{G}$ is a *morphism of fibred categories* if $F = G \circ \psi$.

Remark 7.1.11. In general, if $\mathcal{F} \to \mathcal{C}$ and $\mathcal{G} \to \mathcal{C}$ are just functors, then ψ would be a functor of $\mathcal{C}$-categories if there exists a natural isomorphism $F \cong G \circ \psi$. In this setting, however, it is unnecessary as $\mathcal{F}$ is then equivalent (as a functor of categories) to a functor $\tilde{F}$ such that $\tilde{F} = G \circ \psi$.

This gives rise to the *category fibred in groupoids over* $\mathcal{C}$, $\mathsf{Fib}(\mathcal{C})$. Much like the category of categories Cat, this is a *(2,1)-category*. Indeed, we have the notion of equivalence of functors. This is the following.

Definition 7.1.12. Let $F\colon \mathcal{F} \to \mathcal{C}$ and $G\colon \mathcal{G} \to \mathcal{C}$ be categories fibred in groupoids. Let $\phi, \psi\colon \mathcal{F} \to \mathcal{G}$ be morphisms of fibred categories. We say $\mathcal{F}$ and $\mathcal{G}$ are *equivalent* (denoted by $\mathcal{F} \sim \mathcal{G}$) if there exists a natural isomorphism of functors $\eta\colon \mathcal{F} \cong \mathcal{G}$ such that for each $U \in \mathcal{C}$ and each $X \in \mathcal{F}(U)$, we have that $\eta_X \in \mathcal{G}(U)$.

This gives rise to equivalence of fibred categories. It is identical to the notion of equivalence of categories, or homotopy equivalence of topological spaces.

Definition 7.1.13. Two fibred categories $\mathcal{F} \to \mathcal{C}$ and $\mathcal{G} \to \mathcal{C}$ are *equivalent* if there exist a morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{F}$ such that $\phi \circ \psi \sim \mathrm{id}_{\mathcal{G}}$ and $\psi \circ \phi \sim \mathrm{id}_{\mathcal{F}}$.

There are also limits of fibred categories. These do not satisfy the universal properties of limits on ordinary categories. Instead, they are *2-limits*. While I could define them through universal properties, it is easier to just give the explicit constructions.

Definition 7.1.14. Let $F\colon \mathcal{F} \to \mathcal{C}$, $G\colon \mathcal{G} \to \mathcal{C}$ and $H\colon \mathcal{H} \to \mathcal{C}$ be fibrations in groupoids. Furthermore, let $\phi\colon \mathcal{F} \to \mathcal{H}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$ be morphisms of fibred categories. Then the fibred product $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ is the following category:

- The objects are quartets (U, X, Y, α) such that $U \in \mathcal{C}$, $X \in \mathcal{F}(U)$, $Y \in \mathcal{G}(U)$, and $\alpha\colon \phi(X) \cong \phi(Y)$ is an isomorphism in $\mathcal{H}(U)$.
- The morphisms $(U', X', Y', \alpha') \to (U, X, Y, \alpha)$ are triples (f, g, h) such that $f \in \mathrm{Hom}_{\mathcal{C}}(U', U)$, $g \in \mathrm{Hom}_{\mathcal{F}}(X', X)$ and $h \in \mathrm{Hom}_{\mathcal{G}}(Y', Y)$ both

lying over f. On top of that, the following diagram must commute:

$$\begin{array}{ccc} X' & \xrightarrow{\alpha'} & Y' \\ \Big\downarrow g & & \Big\downarrow h \\ X & \xrightarrow{\alpha} & Y \end{array}$$

The first coordinate projection $\mathcal{F}\times_{\mathcal{H}}\mathcal{G} \to \mathcal{C}$ makes it a fibration in groupoids. The projections $\pi_1 : \mathcal{F} \times_{\mathcal{H}} \mathcal{G} \to \mathcal{F}$ and $\pi_2 : \mathcal{F} \times_{\mathcal{H}} \mathcal{G} \to \mathcal{G}$ are morphisms of fibred categories.

The terminal object is the identity $\mathcal{C} \to \mathcal{C}$. The product $\mathcal{F} \times \mathcal{G}$ is the fibred product $\mathcal{F} \times_{\mathcal{C}} \mathcal{G}$. Other 2-limits, such as equalizers, are constructed similarly. 2-colimits, such as sums and co-equalizers, also admit a similar construction.
On presheaves, we have also defined limits and colimits. Do these definitions conflict with the respective definitions on their object categories? For limits, this is not the case, as the following lemma shows.

Lemma 7.1.15. *Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be presheaves. Let $\mathcal{F} \to \mathcal{H}$ and $\mathcal{G} \to \mathcal{H}$ be morphisms of sheaves. Then the fibred categories $\mathsf{Ob}(\mathcal{F} \times_{\mathcal{G}} \mathcal{H})$ and $\mathsf{Ob}(\mathcal{F}) \times_{\mathsf{Ob}(\mathcal{H})} \mathsf{Ob}(\mathcal{G})$ are equivalent.*

For sums of presheaves, this is not the case either.

Lemma 7.1.16. *Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves. Then $\mathsf{Ob}(\mathcal{F} \sqcup \mathcal{G})$ and $\mathsf{Ob}(\mathcal{F}) \sqcup \mathsf{Ob}(\mathcal{G})$ are equivalent.*

For co-equalizers, this does *not* hold in general.
I end this section with the *2-Yoneda lemma*. Recall that the Yoneda embedding $\mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{opp}}$ that is full and faithful, the *Yoneda embedding*. But this now becomes an embedding $\mathcal{C} \to \mathsf{Fib}(\mathcal{C})$. This functor sends each object $U \in \mathcal{C}$ to the slice category $\mathcal{C}/U$. The Yoneda lemma now tells us that there is a bijection $\mathrm{Hom}_{\mathsf{Fib}(\mathcal{C})}(\mathcal{C}/U, \mathcal{C}/V) \to Hom_{\mathcal{C}}(U, V)$. However, this generalizes further with the *2-Yoneda lemma*.

Lemma 7.1.17. *Let $\mathcal{F} \to \mathcal{C}$ be a fibred category. Let $U \in \mathcal{C}$ be an object. Then the following functor is an equivalence of categories:*

$$Hom_{\mathsf{Fib}(\mathcal{C})}(\mathcal{C}/U, \mathcal{F}) \to \mathcal{F}(U), \Phi \mapsto \Phi(\mathrm{id}_U)$$

7.2 Stacks

Just like presheaves generalize to fibred categories, sheaves generalize to *stacks*. In order to define this, we need the notion of a *descent*, which formalizes the notion of gluing.

Definition 7.2.1. Let $\mathcal{F} \to \mathcal{C}$ be a fibred category, and let $C = \{U_i \to U\}_{i \in I}$ be a family of morphisms. The category of *descent data*, denoted by $\mathsf{Desc}(C)$, is the following:

- The objects are families $\{X_i, \phi_{ij}\}_{i \in I, (i,j) \in I^2}$. Here we have $X_i \in \mathcal{F}(U_i)$ being objects and ϕ_{ij} being an isomorphism $X_i|_{U_{ij}} \to X_j|_{U_{ij}}$ such that for all i, j, k, we have $\phi_{ik}|_{U_{ijk}} = \phi_{jk}|_{U_{ijk}} \circ \phi_{ij}|_{U_{ijk}}$ and $\phi_{ij} = \phi_{ji}^{-1}$.
- The morphisms $\{X_i, \phi_{ij}\} \to \{Y_i, \psi_{ij}\}$ are families $\{f_i\}_i$, with $f_i \colon X_i \to Y_i \in \mathcal{F}(U_i)$ being morphisms such that the following diagram commutes:

$$\begin{tikzcd}
(X_i)|_{U_{ij}} \arrow[r, "f_i|_{U_{ij}}"] \arrow[d, "\phi_{ij}"'] & (Y_i)|_{U_{ij}} \arrow[d, "\psi_{ij}"] \\
(X_j)|_{U_{ij}} \arrow[r, "f_j|_{U_{ij}}"] & (Y_j)|_{U_{ij}}
\end{tikzcd}$$

There is one special kind of descent data: the *effective* descent data.

Example 7.2.2. In above situation, let $X \in \mathcal{F}(U)$. Let $X_i = X|_{U_i}$. Let $\Phi_{ij} \colon X_i|_{U_{ij}} \to X_j|_{U_{ij}}$ be the unique isomorphism such that:

$$\begin{tikzcd}
X_i|_{U_{ij}} \arrow[dr] \arrow[dd, dotted] & & \\
& X_i \arrow[dr] & \\
X_j|_{U_{ij}} \arrow[r] & X_j \arrow[r] & X
\end{tikzcd}$$

Then $\{X_i, \Phi_{ij}\}$ is a descent datum.

With this example, we gain a functor $\mathcal{F}(U) \to \mathsf{Desc}(C)$. Now we have the machinery to define stacks!

Definition 7.2.3. Let $\mathcal{F} \to \mathcal{C}$ be a fibred category (resp. fibred in groupoids), with $\mathcal{C}$ being a site as well.

- The fibred category is a *prestack* (resp. *prestack in groupoids*) if for each covering $C = \{U_i \to U\}_i$, the functor $\mathcal{F}(U) \to \mathsf{Desc}(S)$ is full and faithful.
- The fibred category is a *stack* (resp. *stack in groupoids*) if for each covering $C = \{U_i \to U\}_i$, the functor $\mathcal{F}(U) \to \mathsf{Desc}(S)$ is an equivalence.

This really formalizes gluing, as the following examples show.

Example 7.2.4. Let $\mathcal{C}$ be a site, and consider $sh(\mathcal{C})$ of Example 7.1.8. This is a stack. To prove this, let $\{U_i \to U\}$ be a covering. Let $\{(U_i, \mathcal{F}_i), \phi_{ij}\}$ be a descent datum. Then the ϕ_{ij} are sheaf isomorphisms $\mathcal{F}_i|_{U_{ij}} \cong \mathcal{F}_j|_{U_{ij}}$ such that for all i, j, k, we have:

$$\phi_{ij} = \phi_{ji} \text{ and } \phi_{ik}|_{U_{ijk}} = \phi_{jk}|_{U_{ijk}} \circ \phi_{ij}|_{U_{ijk}}$$

As a result, these sheaves glue to a sheaf $\mathcal{F}$ on U. So the descent datum is effective; it corresponds to the pair $(U, \mathcal{F})$.

Example 7.2.5. Let S be a scheme, and consider $\mathsf{Mor}_P \to \mathsf{Sch}/S$ of Example 7.1.9. First equip Sch/S with the Zariski topology. Then, if P is Zariski local on the base, the fibred category $\mathsf{Mor}_P \to \mathsf{Sch}/S$ is a stack.
If we put the étale topology on Sch/S, the fibred category $\mathsf{Mor}_P \to \mathsf{Sch}/S$ is *not* a stack in general.

Much like sheaves, limits of stacks are stacks.

Lemma 7.2.6. *Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be stacks, and let $\mathcal{F} \to \mathcal{H}$ and $\mathcal{G} \to \mathcal{H}$ be morphisms of fibred categories. Then the fibred product $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ is also a stack.*

For colimits, however, this is different. In the same vein as sheaves, we need the notion of *stackification*. The proof goes beyond the scope of the thesis.

Theorem 7.2.7. *Let $\mathcal{C}$ be a site. Let $\mathcal{F} \to \mathcal{C}$ be any fibration. Then there exists a stack $\mathcal{F}_s$, along with a morphism of fibred categories $i\colon \mathcal{F} \to \mathcal{F}_s$, having the following universal property: for each morphism of fibrations $f\colon \mathcal{F} \to \mathcal{G}$ with $\mathcal{G}$ a stack, there exists a* unique *morphism $\tilde{f}\colon \mathcal{F}_s \to \mathcal{G}$ such that $f = \tilde{f} \circ i$.*

Lastly, the notion of a *restricted prestack* also caries over to stacks.

Definition 7.2.8. Let $\mathcal{F} \to \mathcal{C}$ be a stack. Then the *restricted stack* $\mathcal{F}|_U$ on $\mathcal{C}/U$ is given by the composition $\mathcal{F} \to \mathcal{C} \to \mathcal{C}/U$, where the morphism $\mathcal{C} \to \mathcal{C}/U$ is given by $X \mapsto X \times U$.

7.3 Algebraic Stacks

After introducing some machinery in the previous part, we can now go back to the algebraic geometric setting, and introduce the notion of *algebraic stacks.* All properties listed here are direct copies of the ones listed in chapter 6.
The notion for *representability* generalizes immediately. Recall that from the 2-Yoneda lemma, a scheme X corresponds to the slice stack Sch/X.

Definition 7.3.1. Let S be a scheme. A morphism $\mathcal{F} \to \mathcal{G}$ of stacks over (Sch/S) is *representable by algebraic spaces* if for each scheme $X \in \mathsf{Sch}/S$, the fibred product $X \times_{\mathcal{F}} \mathcal{G}$ is an algebraic space over X.

Remark 7.3.2. A more logical definition would be asking for each algebraic space X, that the fibred product $X \times_{\mathcal{F}} \mathcal{G}$ is representable. It is easy to see that these definitions are equivalent. However, above definition makes proofs of representability slightly shorter.

The way of assigning properties to representable morphisms is completely identical to Definition 6.1.3. As a result, I will not state the definition here. This gives us the machinery to define algebraic stacks. Aping Definition 6.2.1 would give the definition of a *Deligne-Mumford stack*. This is insufficient for the thesis! Instead, we need to content ourselves with the existence of a smooth cover. This gives rise to an *Artin stack*, which is as follows.

Definition 7.3.3. Let $\mathcal{F} \to (\mathsf{Sch}/S)_{\acute{e}t}$ be a stack in groupoids. We say $\mathcal{F}$ is an *Artin stack* when the following conditions are met.

- The diagonal $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is representable by algebraic spaces. Equivalently, for each two schemes U, V over S, the fibre product $U \times_{\mathcal{F}} V$ is an algebraic space over $U \times_S V$.
- There exists a surjective *smooth* morphism $U \to \mathcal{F}$, with U being an algebraic space.

I will now state the most important results. Their proofs are identical to their counterparts of chapter 6: Lemma 6.2.3 and Lemma 6.2.4.

Lemma 7.3.4. *Suppose $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$ are algebraic stacks, and $\mathcal{F} \to \mathcal{H}$ and $\mathcal{G} \to \mathcal{H}$ are morphisms. Then the fibred product $\mathcal{F} \times_{\mathcal{G}} \mathcal{H}$ is also algebraic.*

Lemma 7.3.5. *Let $\mathcal{F} \to \mathcal{G}$ be a representable morphism of stacks in groupoids on $(\mathsf{Sch}/S)_{\acute{e}t}$. Assume $\mathcal{G}$ is algebraic. Then $\mathcal{F}$ is also algebraic.*

8 The stack of closure data

8.1 The stack of closure data defined

In this chapter, we define the stack of closure data, and prove that it is algebraic. The definition is as follows.

Definition 8.1.1. Let S be a scheme. The category Data_S is as follows.

- The objects are tuples (U, X, Y, G, σ) such that U is a scheme over S, X, Y are finite étale schemes over U, $G \subseteq Y^{Y*}$ is a clopen subgroup, and $\sigma\colon U \to (Y^{Y*})_G$ is a morphism making (G, σ) a closure datum.
- The morphisms $(U', X', Y', G', \sigma') \to (U, X, Y, G, \sigma)$ are tuples $(f, \alpha, \beta, \gamma)$ such that $f\colon U' \to U$ is a morphism of schemes, $\alpha\colon X_{U'} \to X'$, $\beta\colon Y_{U'} \to Y'$ are isomorphisms, and $\gamma\colon G_{U'} \to G'$ is an isomorphism of group schemes. On top of that, the following diagrams must commute:

$$\begin{tikzcd} G_{U'} \arrow[r, "\subseteq"] \arrow[d, "\gamma"'] & {Y_U}^{Y_U*} \arrow[d, "\beta \circ - \circ \beta^{-1}"] \\ G' \arrow[r, "\subseteq"] & {Y'}^{Y'*} \end{tikzcd} \qquad \begin{tikzcd} U' \arrow[r, "\sigma_{U'}"] \arrow[dr, "\sigma'"] & ({Y_U}^{Y_U*})_{G_U} \arrow[d, "\alpha \circ - \circ \beta^{-1}"] \\ & ({Y'}^{Y'*})_{G'} \end{tikzcd}$$

This comes with a projection $\mathsf{Data}_S \to \mathsf{Sch}/S$, making it a stack over $(\mathsf{Sch}/S)_{ét}$.

Remark 8.1.2. The definitions for $\beta \circ - \circ \beta^{-1}$ and $\alpha \circ - \circ \beta^{-1}$ are very straightforward when using the Yoneda embedding.

Proving that the stack is algebraic will be done as follows. In section 8.2, we introduce the stack of vector bundles Bund, and prove that that is algebraic. In 8.3, we define the stack of locally free algebras, $\mathsf{BundAlg}$, and prove that that is algebraic. In section 8.4, we introduce the stack of finite étale covers FinEt, and prove that that is algebraic. Using that, and the result of the end of chapter 4, it follows trivially that Data_S is algebraic. During all proofs, the following lemma plays a major role.

Lemma 8.1.3. *Let $\mathcal{C}$ be a site. Let $\mathcal{G} \to \mathcal{C}$ be a category fibred in groupoids, and let $\mathcal{F}$ be a presheaf on $\mathcal{G}$. Let $\mathsf{Ob}(\mathcal{F}) \to \mathcal{G}$ be the object category of $\mathcal{F}$, and let $H\colon \mathcal{H} \to \mathcal{G}$ be a morphisms of stacks over $\mathcal{C}$. Then the base extension $\mathsf{Ob}(\mathcal{F}) \times_{\mathcal{G}} \mathcal{H} \to \mathcal{H}$ is equivalent to the object category of the presheaf $X \mapsto \mathcal{F}(H(X))$ on $\mathcal{H}$.*

Proof. The category $\mathsf{Ob}(\mathcal{F}) \times_{\mathcal{G}} \mathcal{H}$ consists of the following.

- The objects are tuples $(U, (X, x), Y, \sigma)$ where $U \in \mathcal{C}$, $X \in \mathcal{G}(U)$, $x \in \mathcal{F}(X)$, $Y \in \mathcal{H}(U)$, and $\sigma\colon H(Y) \cong X$ is an isomorphism in $H(U)$.

- The morphisms $(U', (X', x'), Y', \sigma') \to (U, (X, x), Y, \sigma)$ are triples (f, g, h) such that $f\colon U' \to U$ is a morphism, $g\colon X' \to X$ and $h\colon Y' \to Y$ are morphisms lying over f. We must have the condition that $x' = g^*(x)$, and that f, g are compatible with σ and σ'.

For each tuple $(U, (X, x), Y, \sigma)$, we have a canonical isomorphism:

$$(\mathrm{id}_U, \sigma, \mathrm{id}_Y)\colon (U, (X, x), Y, \sigma) \cong (U, (H(Y), \sigma^*(x)), Y, \mathrm{id}_{H(Y)})$$

As a result, we may limit ourselves to the tuples in latter form. Henceforth, we find that the category is equivalent to the following category:

- The objects are tuples (Y, x) where $U \in \mathcal{C}$, $Y \in \mathcal{H}(U)$, and $x \in \mathcal{F}(H(Y))$. The components U and X are left out as they are completely determined by Y.
- The morphisms $(Y', x') \to (Y, x)$ are morphisms $h\colon Y' \to Y$ such that $(Hh)^*(x) = x'$. I left out the components f and g as they are completely determined by h.

But this is the object category of the desired presheaf! □

8.2 The stack of vector bundles

In section 8.3, we will prove that the stack of finite étale covers is representable. Recall that an affine S-scheme can be regarded as an $\mathcal{O}_S$-algebra, and therefore an $\mathcal{O}_S$-module. Thanks to Theorem 3.3.6, this module will be locally free, i.e. a *vector bundle*. We have seen in Example 6.3.6 that such modules are in fact algebraic spaces.
This gives rise to the following stack.

Definition 8.2.1. The category Bund_S is defined as follows:

- The objects are pairs (U, M), where U is a scheme, and M is a vector bundle over U.
- The morphisms $(U', M') \to (U, M)$ are pairs (f, α) where $f\colon U' \to U$ is a morphism of schemes, and $\alpha\colon f^*(M) \to M'$ is an isomorphism of $\mathcal{O}_{U'}$-modules.

This comes with a natural functor $\mathsf{Bund}_S \to \mathsf{Sch}/S$, making it a stack.

The aim of this section is to prove the following.

Theorem 8.2.2. *The stack $\mathsf{Bund}_S \to (\mathsf{Sch}/S)$ is algebraic.*

In order to prove this, we will completely rely on the definitions of Definition 7.3.3. As a result, the proof will be done in two steps. But first, we need the following result.

Lemma 8.2.3. *Let S be a scheme, and let M_1 and M_2 be $\mathcal{O}_S$-modules. Then the sheaf $\mathsf{Iso}(M_1, M_2)\colon U \mapsto Iso_{\mathcal{O}_U}(M_1|_U, M_2|_U)$ is an algebraic space.*

Proof. The sheaves $\mathsf{Hom}(M_1, M_2)$ and $\mathsf{Hom}(M_2, M_1)$ are algebraic spaces, since they are also locally free modules. In the same fashion as Lemma 4.2.5, the sheaf $\mathsf{Iso}(M_1, M_2)$ can be constructed by taking finite limits. So by Lemma 6.2.3, $\mathsf{Iso}(M_1, M_2)$ is an algebraic space. □

We proceed to step 1.

Lemma 8.2.4. *The diagonal $\mathsf{Bund}_S \to \mathsf{Bund}_S \times \mathsf{Bund}_S$ is representable by algebraic spaces.*

Proof. Let X, Y be two schemes over S, and let $F_1\colon X \to \mathsf{Bund}_S$ and $F_2\colon Y \to \mathsf{Bund}_S$ be morphisms of stacks. We have to prove that $X \times_{\mathsf{Bund}_S} Y$ is representable.
Let $M_1 = F_1(\mathrm{id}_X)$ and $M_2 = F_2(\mathrm{id}_Y)$. Thanks to the 2-Yoneda lemma, we may assume that F_1 is the morphism $U \mapsto M_1|_U$ and F_2 is the morphism $U \mapsto M_2|_U$. The fibred product is as follows:

- The objects are tuples (U, α). Here is U a scheme over S, equipped with a morphisms $U \to X$ and $U \to Y$. The morphism $\alpha\colon (M_1|_U) \to (M_2|_U)$ is an isomorphism of $\mathcal{O}_U$-modules.
- The morphisms $(U', \alpha') \to (U, \alpha)$ are pairs (f_1, f_2) of morphisms $f_i\colon U_i' \to U_i$ that are compatible with the α.

In the tuples $(U_1, U_2, (\mathrm{id}, \alpha))$, the U_i are the same schemes over the base scheme. So we can view them as one scheme U over both X and Y; henceforth as a scheme over $X \times Y$. Then the α will be just an isomorphism $\pi_1^*(M_1) \cong \pi_2^*(M_2)$. So it follows that $\mathsf{Sch}/X \times_{\mathsf{CohFree}} \mathsf{Sch}/Y$ is the category consisting of:

- The objects are pairs (U, α). Here is U a scheme over $X \times Y$, and $\alpha\colon \pi_1^*(M_1)|_U \cong \pi_2^*(M_2)|_U$ is an isomorphism of $\mathcal{O}_U$-modules.
- The morphisms $(U', \alpha') \to (U, \alpha)$ are morphisms $f\colon U' \to U$ that are compatible with the α.

But this is the object category of the sheaf $\mathsf{Iso}(\pi_1^*(M_1), \pi_2^*(M_2))$ on $(\mathsf{Sch}/X \times_S Y)_{ét}$. So by Lemma 8.2.3, this is an algebraic space over $X \times_S Y$. This proves the claim. □

The next step is to show the existence of a smooth cover $X \to \mathsf{Bund}$. This will rely on the following lemma.

Lemma 8.2.5. *The sheaf GL_n on (Sch/S) mapping each U to the set of invertible $n \times n$-matrices is representable by a smooth and surjective scheme.*

Proof. See [5], Tag 022W. $\square$

With this, we can prove the existence of a smooth cover.

Lemma 8.2.6. *There exists a smooth cover $U \to \mathsf{Bund}$.*

Proof. For each $n \in \mathbb{N}$, let $U_n = \operatorname{Spec} \mathbb{Z}$, and let $\Phi_n \colon \mathsf{Sch}/U_n \to \mathsf{Bund}$ be given by $V \mapsto \mathcal{O}_V^n$. Let $U = \coprod_{n \in \mathbb{N}} U_n$. Our claim is that $\coprod_n \Phi_n \colon U_n \to \mathsf{Bund}$ is smooth and surjective.
Let $F \colon V \to \mathsf{Bund}$ be any scheme. Let $M = F(\mathrm{id}_V)$. By Lemma 8.2.4, it follows that for all $n \in \mathbb{N}$, that $V \times_{\mathsf{Bund}} U_n = \mathsf{Iso}(M, \mathcal{O}_V^n)$.
First assume that M is (globally) free, i.e. of the form $\mathcal{O}_V^m$. Then $\mathsf{Iso}(M, \mathcal{O}_V^n) = \mathsf{Iso}(\mathcal{O}_V^m, \mathcal{O}_V^n)$. Suppose first that $m \neq n$. Then for all $W \to V$, the modules $\mathcal{O}_W^m$ and $\mathcal{O}_W^n$ are *not* isomorphic unless $W = \emptyset$. As a result, it follows that $\mathsf{Iso}(M, \mathcal{O}_V^n)$ is representable by $\emptyset$ in that case. So $U_n \times_{\mathsf{Bund}} V = \emptyset$.
Now suppose $m = n$. Then it follows that $U_n \times_{\mathsf{Bund}} V = \mathsf{Iso}(\mathcal{O}_V^n, \mathcal{O}_V^n) = \mathrm{GL}_n(V)$. So it follows that:

$$(\coprod_{n \in \mathbb{N}} U_n) \times_{\mathsf{Bund}} V = \coprod_n (U_n \times_{\mathsf{Bund}} V) = \coprod_{n \neq m} \emptyset \sqcup \mathrm{GL}_n/(V) = \mathrm{GL}_n(V)$$

By Lemma 8.2.5, the scheme $\mathrm{GL}_n(V)$ is a smooth and surjective scheme over V. So $U \times_{\mathsf{Bund}} V \to V$ is smooth and surjective when M is free. Since being smooth and surjective is an étale local property, this also holds when M is locally free. $\square$

Combining Lemma 8.2.4 and Lemma 8.2.6 gives us a proof of Theorem 8.2.2.

Remark 8.2.7. This is not an étale cover! One can prove that there is no étale cover at all, which means that this is *not* a Deligne-Mumford stack.

8.3 The stack of locally free algebras

In this section, we will define the stack of locally free algebras and prove the algebraicity thereof.

Definition 8.3.1. Let S be a scheme. The category $\mathsf{BundAlg}_S$ consists of the following.

- The objects are pairs (U, A), where U is a scheme over S, and A is an $\mathcal{O}_U$-algebra that as an $\mathcal{O}_U$-module is a vector bundle over U.
- The morphisms $(U', A') \to (U, A)$ are pairs (f, α) where $f \colon U' \to U$ is a morphism of schemes, and $\alpha \colon f^*(A) \to A'$ is an isomorphism of $\mathcal{O}_{U'}$-algebras.

This comes with a natural functor $\mathsf{BundAlg} \to \mathsf{Sch}/S$, making it a stack.

Theorem 8.3.2. *The stack $\mathsf{BundAlg}_S \to (\mathsf{Sch}/S)_{ét}$ is algebraic.*

In order to prove the claim, we need to prove the following auxiliary result.

Lemma 8.3.3. *Let S be a scheme, and let M be a locally free $\mathcal{O}_S$-module. Let $\mathsf{Alg}(M)$ be the sheaf sending each $U \in \mathsf{Sch}/S$ to the set of algebra structures on $M|_U$. This is an algebraic space.*

Proof. Recall that an algebra structure on an $\mathcal{O}_T$-module N is given by a multiplication morphism $\mu\colon N \otimes_{\mathcal{O}_T} N \to N$ and a unit morphism $\eta\colon \mathcal{O}_T \to N$, satisfying the unit law, the associativity law, and the commutativity law. These laws are equivalent to saying that the following diagrams commute:

$$\begin{tikzcd}
 & N \arrow[dl, "\eta\otimes\mathrm{id}"'] \arrow[d, equal] \arrow[dr, "\mathrm{id}\otimes\eta"] & \\
N^{\otimes 2} \arrow[r, "\mu"] & N & N^{\otimes 2} \arrow[l, "\mu"']
\end{tikzcd}
\qquad
\begin{tikzcd}
N^{\otimes 3} \arrow[r, "\mu\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu"'] & N^{\otimes 2} \arrow[d, "\mu"] \\
N^{\otimes 2} \arrow[r, "\mu"] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
N^{\otimes 2} \arrow[d, "\sigma"'] \arrow[dr, "\mu"] & \\
N^{\otimes 2} \arrow[r, "\mu"] & N
\end{tikzcd}$$

Let $\mathcal{F}_u$ be the subsheaf of $\mathcal{F} := \mathrm{Hom}(M^{\otimes 2}, M) \times \mathrm{Hom}(\mathcal{O}_S, M)$ where each section consists of pairs (μ, η) satisfying the left diagram. Then $\mathcal{F}_u$ is obtained by equalizing the following morphisms:

$$\begin{aligned}
\mathrm{Hom}(M^{\otimes 2}, M) \times \mathrm{Hom}(\mathcal{O}_S, M) \to \mathrm{Hom}(M, M)(\mu, \eta) &\qquad \mapsto \mathrm{id} \\
(\mu, \eta) &\mapsto \mu \circ (\eta \otimes \mathrm{id}) \\
(\mu, \eta) &\mapsto \mu \circ (\mathrm{id} \otimes \eta)
\end{aligned}$$

The sheaves $\mathcal{F}_a, \mathcal{F}_c \subseteq \mathrm{Hom}(M^{\otimes 2}, M) \times \mathrm{Hom}(\mathcal{O}_S, M)$ consisting of tuples (μ, η) satisfying the second and third diagram respectively are constructed in a similar fashion. Ultimately, we have $\mathsf{Alg}(M) = \mathcal{F}_a \times_{\mathcal{F}} \times \mathcal{F}_u \times_{\mathcal{F}} \mathcal{F}_m$. So $\mathsf{Alg}(M)$ is constructed out of sheaves through limits of sheaves in the form $\mathrm{Hom}(M_1, M_2)$.
We know that those sheaves are vector bundles, hence they are algebraic spaces. Therefore, by Lemma 6.2.3, $\mathsf{Alg}(M)$ is also an algebraic space. $\square$

Lemma 8.3.4. *The stack $\mathsf{BundAlg}_S \to \mathsf{Sch}$ is algebraic.*

Proof. Consider the forgetful functor of stacks

$$\mathsf{BundAlg} \to \mathsf{Bund}, (U, A) \mapsto (U, A \text{ as a module})$$

By Theorem 8.2.2, the stack Bund is algebraic. So by Lemma 7.3.5, it suffices to prove that $\mathsf{HomBund} \to \mathsf{Bund}^2$ is representable by algebraic spaces.
First note that $\mathsf{HomBund} \to \mathsf{Bund}$ is actually equivalent to the object category of a presheaf on Bund, namely the presheaf $(U, M) \mapsto \mathsf{Alg}(M)$.
So let $F\colon \mathcal{C}/T \to \mathsf{Bund}^2$ be any functor of stacks over $\mathcal{C}$. Let $F(\mathrm{id}_U) = M$.

By the 2-Yoneda lemma, F is equivalent to the functor $U \mapsto (M|_U)$. So using Lemma 8.1.3, the functor $\mathsf{HomCoh} \times_{\mathsf{Coh}} \mathsf{Sch}/T \to \mathsf{Sch}/T$ is equivalent to the object category of the presheaf $U \mapsto \mathrm{Alg}(M|_U)$. But this is the sheaf $\mathsf{Alg}(M)$ on Sch/T. We have already seen that this is an algebraic space over $\mathcal{C}/T$. This proves that the functor is representable by algebraic spaces. $\square$

8.4 The stack of finite étale covers

In this section, we will prove that the following stack is algebraic.

Definition 8.4.1. Let S be a scheme. The stack of finite étale coverings $\mathsf{FinEt} \to (\mathsf{Sch}/S)_{ét}$ is the following.

- The objects are tuples (X, U) such that X is a scheme, and $U \to X$ is a finite étale scheme.
- The Morphisms $(X', U') \to (X, U)$ are pairs (f, α), where $f : X' \to X$ is a morphism of schemes, and where $\beta : U' \to U_{X'}$ is an isomorphism of schemes over U'.

Theorem 8.4.2. *The stack* $\mathsf{FinEt} \to (\mathsf{Sch}/S)_{ét}$ *is algebraic.*

In order to prove this claim, we need the following lemma.

Lemma 8.4.3. *Let $f : X \to Y$ be a finite morphism of schemes. Then there exists an open subscheme $U \subseteq Y$ such that for all $g : T \to Y$, the following are equivalent:*

1. *The base extension $f_T : X_T \to T$ is étale.*
2. *The morphism g maps into U.*

Proof. Consider the sheaf of differentials $\Omega_{X/Y}$ on $(\mathsf{Sch}/X)_{ét}$. The morphism f is finite, so it is also locally of finite presentation. So by [5], Tag 02GF, the following equivalence holds:

1. The morphism f is étale.
2. $\Omega_{X/Y} = 0$.

Now consider the support $F = Supp(\Omega_{X/Y}) \subseteq X$ of the module. As with quasi-coherent modules, this is closed. Let $U = f(F)^c \subseteq Y$. This is open, since finite morphisms are proper, and proper morphisms are closed. (see [5], tag 01W0).
Let $g : T \to Y$ be any morphism. We know that $g_X^* \Omega_{X/Y} = \Omega_{X_T/T}$ (see [5], tag 01V0). As f is finite, $\Omega_{X/Y}$ is of finite type (see [5], Tag 01V2). As a result, using Tag 07TZ, it follows that:

$$Supp(\Omega_{X_T/T}) = Supp(g_X^* \Omega_{X/Y}) = g_X^{-1}(Supp(\Omega_{X/Y}))$$

Suppose g maps T into U. Then $T \to Y$ can be viewed as a morphism $T \to U$. By definition of U, $f^{-1}U \cap \mathit{Supp}(\Omega_{X/Y}) = \emptyset$, so $\Omega_{f^{-1}U/U} = 0$. Hence f is étale on $f^{-1}U$. By base change, it follows that f_T is étale as well. Now suppose g does not map T into U. Then we can choose a $t \in T$ such that $g(t) \in f(\mathit{Supp}(\Omega_{X/Y}))$. Hence there exists a $x \in \mathit{Supp}_{X/Y}$ such that $f(x) = g(t)$. Now let $u \in X_T$ such that $g_X(u) = x$ and $f_T(u) = t$. Then $u \in \mathit{Supp}(\Omega_{X_T/T})$ by above. So $\mathit{Supp}(\Omega_{X_T/T}) \neq \emptyset$, so f_T is not étale.
Combining our result, we find that f_T is étale if and only if $g(T) \subseteq U$. □

Now we can prove that the stack $\mathsf{FinEt} \to \mathsf{Sch}$ is algebraic. Recall that each affine morphism $Y \to X$ of schemes corresponds to a quasi-coherent $\mathcal{O}_X$-algebra, namely the algebra $U \mapsto \mathcal{O}(U \times_X Y)$. ByTheorem 3.3.6, it follows that if $Y \to X$ is finite étale, that it is a locally free $\mathcal{O}_Y$-algebra. This gives rise to an inclusion $\mathsf{FinEt} \to \mathsf{BundAlg}$.

Lemma 8.4.4. *The inclusion $\mathsf{FinEt} \to \mathsf{BundAlg}$ is representable by an open subscheme.*

Proof. The inclusion $\mathsf{FinEt} \to \mathsf{BundAlg}$ actually makes FinEt a presheaf over $\mathsf{BundAlg}$. Indeed, it is the object category of presheaf where:

$$\mathsf{FinEt}(X, A) = \begin{cases} 1 & \text{if } A \text{ is a finite étale scheme over X} \\ \emptyset & \text{else} \end{cases}$$

So now let $F\colon \mathsf{Sch}/T \to \mathsf{BundAlg}$ be a morphism of stacks. Let $F(\mathrm{id}_T) = (T, A)$. By using the 2-Yoneda lemma, we may assume that F is equivalent to the functor $U \mapsto (U, A|_U)$. Letting $Y \to T$ be the scheme corresponding to A, this functor is the functor $X \mapsto (X \times_T A)$. Using Lemma 8.1.3, the fibre product is equivalent to the presheaf given by:

$$X \mapsto \begin{cases} 1 & \text{if } Y \times_T X \text{ is finite étale} \\ 0 & \text{else} \end{cases}$$

Now let U be from Lemma 8.4.3. Then it follows that above presheaf is the presheaf sending $X \to T$ to 1 if the image of X lies in U, and $\emptyset$ otherwise. But this is exactly the sheaf corresponding to U! So the fibre product $\mathsf{FinEt} \times_U \mathsf{Sch}/T$ is representable by an open subscheme. □

This ends the proof of Theorem 8.4.2.

8.5 The final step

It is now time to prove the fact that Data is algebraic. This has been essentially already done at the end of chapter 4.

Theorem 8.5.1. *The stack $\mathsf{Data}_S \to \mathsf{Sch}/S$ is algebraic, and is finite étale over FinEt^2.*

Proof. Consider the morphism $\mathsf{Data}_S \to \mathsf{FinEt}_S^2$ given by $(U, X, Y, G, \sigma) \mapsto (U, X, Y)$. This morphism is equivalent to the presheaf on FinEt^2, given by $(U, X, Y) \mapsto \mathrm{Data}(X, Y)$.
Now let $\mathsf{Sch}/T \to \mathsf{FinEt}_S^2$ be a test functor. Let (X, Y) be the image of id_T. Then by the 2-Yoneda lemma, this functor is equivalent to the functor $U \mapsto (X_U, Y_U)$. Using Lemma 8.1.3, it follows that $\mathsf{Data}_S \times_{\mathsf{FinEt}_S^2} \mathsf{Sch}/T$ is equivalent to the object category of the sheaf $U \mapsto \mathrm{Data}(X_U, Y_U)$. But this is the sheaf $\mathsf{Data}(X, Y)$ on Sch/T! We have seen at the end of chapter 4 that this is representable by an étale scheme. So the morphism $\mathsf{Data}_S \to \mathsf{FinEt}^2$ is representable and in fact finite étale. This proves that Data_S is an algebraic stack! $\square$

References

[1] Owen Biesel, *Galois Closures of Rings*, PHD thesis, Princeton University, 2013

[2] Gregor Kemper, *A Course in Commutative Algebra*, Springer, 2011

[3] Saunders Mac Lane & Ieke Moerdijk, *Sheaves in Geometry and Logic*, Springer, 1992

[4] Hendrik Lenstra, *Galois Theory for Schemes*, Course notes, 2008, electronic 3rd edition, available at:
`http://websites.math.leidenuniv.nl/algebra/GSchemes.pdf`

[5] Unknown author, *Stacks Project*, online, last accessed on 28-09-2016,
`http://stacks.math.columbia.edu/`