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Segal Objects in Homotopical Categories & K-theory of Proto-exact Categories

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Contents

Introduction

> To be real in the scientific sense means to be an element of the system.
>
> *Empiricism, semantics, and ontology*
> Carnap

A homotopical category $\mathcal{C}$ captures the idea of a category endowed with a notion of weak equivalences. Think for example of the category $\mathcal{T}op$ of topological spaces together with weak homotopy equivalences, or the category of chain complexes in some given abelian category together with quasi-isomorphisms. Now a d-Segal object X in such a homotopical category $\mathcal{C}$ is a simplicial object in $\mathcal{C}$ that satisfies some symmetry conditions, stating that up to weak equivalence the simplices of X in higher degrees can be expressed in terms of simplices in lower degrees (determined by the integer d).

For example, $\mathcal{S}et$ turns out to be a homotopical category, with the bijections as weak equivalences. Then a 1-Segal object in $\mathcal{S}et$ is the same thing as the nerve of a small category. This points towards the fact that 1-Segal objects in $\mathcal{T}op$ can be interesting objects. After all, one replaces some strict associativity conditions in the setting of nerves of honest categories with a weak version, 'coherent up to homotopy'. And indeed, the result turns out to be a model for so-called ∞-categories.

The purpose of the present study is to formulate and investigate the notion of 1- and 2-Segal objects in the setting of homotopical categories. A major resource for this has been [DK12], where 1- and 2-Segal objects have been defined in combinatorial model categories. The advantage of the more general setting of homotopical categories is mostly aesthetic: since Segal objects only refer to weak equivalences, it is nice to develop as much of the theory as possible while only using those weak equivalences.

As an application of the theory of Segal objects, some K-theory of proto-exact categories is developed. Dyckerhoff and Kapranov introduce these non-additive analogues of exact categories in [DK12]. They associate to such a category $\mathcal{P}$ a 2-Segal object in $\mathcal{C}at$, which we then use to define and probe the higher K-groups of $\mathcal{P}$.

Segal objects in homotopical categories

Let us now be a bit more precise. A *homotopical category* is a category $\mathcal{C}$ endowed with a subcategory $\mathcal{W}$ of *weak equivalences*, such that $\mathcal{W}$ satisfies 2-of-6 and contains

all isomorphisms. If one for example starts with a model category $\mathcal{M}$ and forgets the (co)fibrations, then the result is a homotopical category.

As in the setting of model categories, one can define the notion of a homotopy (co)limit in $\mathcal{C}$ as a best approximation to the ordinary (co)limit such that the result does preserve weak equivalences. More generally, one can define right- and left derived functors of a given functor between homotopical categories as homotopical approximations to that given functor.

Write $\Delta[\mathcal{I}_n]$ for the union of the edges $[0,1],\dots,[n-1,n]$ of $\Delta[n]$. A *triangulation* of $[n]$ is a subset $\mathcal{T} \subset 2^{[n]}$ that corresponds to a triangulation of a convex $n+1$-gon in the obvious way. For such a $\mathcal{T}$ one writes $\Delta[\mathcal{T}]$ for the union inside $\Delta[n]$ of all $\Delta[I]$ with $I \in \mathcal{T}$. Then define

$$\begin{aligned}
&\mathcal{S}_1 \coloneqq \{\Delta[\mathcal{I}_n] \hookrightarrow \Delta[n] \mid n \geq 2\}\,;\\
&\mathcal{S}_2 \coloneqq \{\Delta[\mathcal{T}] \hookrightarrow \Delta[n] \mid n \geq 3 \text{ and } \mathcal{T} \text{ is a triangulation of } [n]\}\,,
\end{aligned}$$

and call these maps *1-* and *2-Segal coverings* respectively.

Throughout, write $\mathcal{S}\mathrm{et}$ for the category of simplicial sets. Let $\mathcal{C}$ be a homotopical category, and consider the Yoneda embedding $\Delta[-] : \mathbf{\Delta} \to \mathcal{S}\mathrm{et}$. Then the right Kan extension $\Delta_* : \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}} \to \mathcal{C}^{\mathcal{S}\mathrm{et}^{\mathrm{op}}}$ of Δ^{op} is called the *Yoneda extension functor*, and the right derived functor $\mathbb{R}\Delta_*$ of Δ_* is the *homotopy Yoneda extension functor*. Call a simplicial object $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ in $\mathcal{C}$ a *d-Segal object* ($d = 1, 2$) if its homotopy Yoneda extension $\mathbb{R}\Delta_*(X) : \mathcal{S}\mathrm{et}^{\mathrm{op}} \to \mathcal{C}$ maps d-Segal coverings to weak equivalences.

For example, a simplicial set is a 2-Segal object in $\mathcal{S}\mathrm{et}$ if all of its simplices of dimension ≥ 2 are degenerate.

Now if the homotopical category $\mathcal{C}$ has *natural membranes* then the (homotopy) Yoneda extensions Δ_* and $\mathbb{R}\Delta_*$ come with, for every diagram of simplicial sets $(D_b)_{b\in\mathbb{B}}$ with colimit D and every simplicial object X in $\mathcal{C}$

(NM1) A natural isomorphism $\Delta_*(X)(D) \cong \lim_{b\in\mathbb{B}^{\mathrm{op}}} \Delta_*(X)(D_b)$;

(NM2) A natural weak equivalence $\mathbb{R}\Delta_*(X)(D) \simeq \operatorname{holim}_{b\in\mathbb{B}^{\mathrm{op}}} \mathbb{R}\Delta_*(X)(D_b)$, provided $(D_b)_{b\in\mathbb{B}}$ is acyclic (i.e. locally contractible in a certain sense).

A main theorem shown in the present work is that simplicial model categories have natural membranes. Having natural membranes is a desirable property, since one can then write a 1-Segal covering map $\mathbb{R}\Delta_*(X)(\Delta[n]) \to \mathbb{R}\Delta_*(X)(\Delta[\mathcal{I}_n])$ associated to a given simplicial object X in $\mathcal{C}$ in the form

$$X_n \to X_{\{0,1\}} \times^R_{X_{\{1\}}} X_{\{1,2\}} \times^R_{X_{\{2\}}} \cdots \times^R_{X_{\{n-1\}}} X_{\{n-1,n\}}\,,$$

with the term on the right a homotopy fiber product, i.e. a homotopy limit of the obvious diagram. One has a similar formula for 2-Segal maps, this time involving homotopy fiber products of X_2's over X_1's. Employing such formulae, it is shown a 1-Segal object in a homotopical category with natural membranes is automatically a 2-Segal object.

K-theory of proto-exact categories

Let $\mathcal{P}$ be a proto-exact category, which is a non-additive analogue of an exact category in the sense of Quillen. Then one can carry out an $\mathcal{S}_\bullet$ construction on $\mathcal{P}$ and get the *Waldhausen simplicial groupoid* $\mathcal{S}_\bullet\mathcal{P}$ in $\mathcal{C}\mathrm{at}$. Here, $\mathcal{C}\mathrm{at}$ is considered homotopical by taking the equivalences of categories as weak equivalences. It is shown $\mathcal{S}_\bullet\mathcal{P}$ is in fact 2-Segal.

One can also carry out Quillen's Q-construction on $\mathcal{P}$ and get the category $Q\mathcal{P}$. Now for $n \geq 0$, the *K-groups in the sense of Waldhausen* are defined as $\pi_{n+1}|\mathcal{S}_\bullet\mathcal{P}|$, with $|\mathcal{S}_\bullet\mathcal{P}|$ the geometric realization of the simplicial space $[n] \mapsto B\mathcal{S}_n\mathcal{P}$. Likewise, the *$K$-groups in the sense of Quillen* are defined as $\pi_{n+1}BQ\mathcal{P}$, with $BQ\mathcal{P}$ the classifying space of $Q\mathcal{P}$, pointed by 0.

Following the case of exact categories, it is shown that these two approaches in fact yield the same K-groups. It is further shown that the zeroth K-group of $\mathcal{P}$ is canonically isomorphic to the Grothendieck group of $\mathcal{P}$. A surprising fact here is that these latter groups need not be abelian. We close with an additivity theorem for proto-exact categories, and employ this theorem in the Eilenberg-Mazur swindle: the latter shows that proto-exact categories that have infinite coproducts tend to have trivial K-groups.

Notation

Let $\mathbf{\Delta}$ be the simplex category. For $[n] \in \mathbf{\Delta}$ let $\Delta[n]$ be the combinatorial standard n-simplex, and Δ^n the topological standard n-simplex. Write $\mathrm{s}\mathcal{S}\mathrm{et}$ for the simplicial model category of simplicial sets, and $\mathcal{T}\mathrm{op}$ for the simplicial model category of nice topological spaces.[1] Unless otherwise stated, categories written as $\mathbb{C}$ are assumed to be small.

For $f : [n] \to [m]$ in $\mathbf{\Delta}$ and any simplicial object $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ for a given category $\mathcal{C}$, write σf for the image of a $\sigma \in X_m$ under the map $f^* : X_m \to X_n$. When $\mathcal{C} = \mathcal{S}\mathrm{et}$, we seamlessly identify an n-simplex in X with the corresponding simplicial map $\Delta[n] \to X$, by Yoneda. As such, the notation σf is in fact composition of simplicial maps. On the other hand, we write the structure maps of X as d_i and s_j. Since a given d_i is a function $X_n \to X_{n-1}$ for some n, for a simplex $\sigma \in X_n$ its image in X_{n-1} under d_i is written just as $d_i\sigma$. We write $d^i : [n-1] \to [n]$ for the associated map such that $d^{i*} = d_i$. Then in the previous notation it holds $d_i\sigma = \sigma d^i$.

For any category $\mathbb{A}$, when convenient, identify $\mathbb{A}$ with its simplicial nerve $N(\mathbb{A})$. For $m \geq 0$ and $\sigma = a_0 \to \cdots \to a_m \in \mathbb{A}_m$, write σ_i for a_i. Note that for $f : [n] \to [m]$ in $\mathbf{\Delta}$ and $\sigma \in \mathbb{A}_m$, we have induced maps $\sigma_0 \to \sigma f_0$ and $\sigma f_n \to \sigma_m$.

The classifying space $B\mathbb{A}$ of $\mathbb{A}$ is the geometric realization $|N\mathbb{A}|$ of the nerve of $\mathbb{A}$.

For an object a_0 in $\mathbb{A}$ write $\mathbb{A}/a_0$ for the *over category* or *slice category*. Recall that $\mathbb{A}/a_0$ has as objects $a \to a_0$. A morphism from $(a \to a_0)$ to $(a' \to a_0)$ is given by an arrow $a \to a'$ that makes the obvious triangle commutative. Note that this construction is natural in a_0.

More generally, for a functor $F : \mathbb{A} \to \mathbb{B}$ and $b \in \mathbb{B}$, we have the *comma category* F/b. It has as objects pairs of the form (a, φ), where a is an object of $\mathbb{A}$ and φ an arrow

[1] See appendix B for a short discussion of what this means.

$Fa \to b$ in $\mathbb{B}$. A morphism $(a, \varphi) \to (a', \varphi')$ in F/b is an arrow $u : a \to a'$ such that $\varphi' \circ Fu = \varphi$. A stricter version is the fiber category $F^{-1}b$. It is the subcategory of $\mathbb{A}$ consisting of those arrows in $\mathbb{A}$ that are mapped to the identity on b. We also have the obvious dual notions $a_0/\mathbb{A}$ and b/F.

Let $\mathcal{C}, \mathbb{A}$ be given categories. Then an $\mathbb{A}$-shaped diagram in $\mathcal{C}$, i.e. a functor $\mathbb{A} \to \mathcal{C}$, may be written as $(X_a)_{a \in \mathbb{A}}$ or $X_\bullet$ if the diagram category $\mathbb{A}$ is clear from the context. We may even write just X if it is clear X is a diagram. For a colimit $\operatorname{colim}_{\mathbb{A}} X$ of X we write the inclusions as $\iota_a : X_a \to \operatorname{colim}_{\mathbb{A}} X$. Likewise the projections are written as $\pi_a : \lim_{\mathbb{A}} X \to X_a$.

A terminal object may be written as $*$, when it is clear from context what we mean. For example in $\mathcal{S}\mathrm{et}$ it is a one-element set, in $\mathrm{s}\mathcal{S}\mathrm{et}$ it is the constant simplicial set $[n] \mapsto *$, etc.

A *zero object* 0 in a category $\mathcal{C}$ is an object in $\mathcal{C}$ which is at the same time initial as well as terminal. A *pointed category* is a category $\mathcal{C}$ with a chosen zero object in $\mathcal{C}$. In such a category, the unique map $X \to Y$ that factorizes over 0 is called the *zero map*.

Kan extensions

The following is classical, see e.g. [Mac71, §X], where it is famously asserted that Kan extensions subsume 'all other fundamental concepts of category theory'. We shall use Kan extensions in our definition of homotopy (co)limits, and also directly in the definition of Segal objects.

Let $\alpha : \mathbb{A} \to \mathbb{B}$ be a functor, and $\mathcal{C}$ a category. Write $\alpha^* : \mathcal{C}^{\mathbb{B}} \to \mathcal{C}^{\mathbb{A}}$ for the pullback functor $Y \mapsto Y \circ \alpha$. A left adjoint of α^*, if it exists, is written as $\alpha_!$ and is called the *left Kan extension operating along* α.

First suppose $\alpha_!$ exists, and let $X \in \mathcal{C}^{\mathbb{A}}$ be given. Then $\alpha_! X$ is determined by the following universal property. Write Φ_X for the category of pairs (Y, τ) such that $Y \in \mathcal{C}^{\mathbb{B}}$ and with τ a natural transformation $X \Rightarrow Y\alpha$. A morphism $\varphi : (Y', \tau') \to (Y, \tau)$ is an arrow $\varphi : Y' \Rightarrow Y$ that makes the following diagram commutative

$$\begin{tikzcd} X \arrow[rr, Rightarrow, "\tau"] \arrow[dr, Rightarrow, "\tau'"'] & & Y\alpha \\ & Y'\alpha \arrow[ur, Rightarrow, "\varphi\alpha"'] & \end{tikzcd}$$

Let η be the unit $\mathrm{id}_{\mathcal{C}^{\mathbb{A}}} \Rightarrow \alpha^*\alpha_!$ of the adjunction $\alpha_! \dashv \alpha^*$. Now the universal property of the left Kan extension is the statement that $(\alpha_! X, \eta_X)$ is initial in Φ_X. This is witnessed by the fact that for any other object (Y, τ) in Φ_X the transpose $\overline{\tau} : \alpha_! X \Rightarrow Y$ gives a unique morphism $(\alpha_! X, \eta_X) \to (Y, \tau)$.

Conversely, suppose that for each $X \in \mathcal{C}^{\mathbb{A}}$ the category Φ_X has an initial object, suggestively written as $(\alpha_! X, \eta_X)$. Let $\psi : X \Rightarrow X'$ be a natural transformation. Then $(\alpha_! X', \eta_{X'}\psi)$ is an object in Φ_X. We hence get a unique arrow $\alpha_!(\psi) : \alpha_! X \to \alpha_! X'$ in Φ_X. These arrows make $X \mapsto \alpha_! X$ into a functor $\mathcal{C}^{\mathbb{A}} \to \mathcal{C}^{\mathbb{B}}$, which is in fact a left adjoint to α^*, with unit given by the η_X. For the counit, let Y in $\mathcal{C}^{\mathbb{B}}$ be given. Then $(Y, \mathrm{id}_{Y\alpha})$ is

an object of $\Phi_{Y\alpha}$, which gives us a unique arrow $\epsilon_Y : \alpha_!\alpha^*Y \to Y$ in $\Phi_{Y\alpha}$. We take these ϵ_Y as the counit $\alpha_!\alpha^* \Rightarrow \mathrm{id}_{\mathcal{C}^{\mathbb{B}}}$.

In any case, if $X \in \mathcal{C}^{\mathbb{A}}$ is given such that Φ_X has an initial object $(\alpha_!X, \eta_X)$, then we call $\alpha_!X$ the *left Kan extension of X along α*.

Dually, a right adjoint to α^* is written as α_*, and is called the *right Kan extension operating along α*. For $X \in \mathcal{C}^{\mathbb{A}}$, let Ψ_X be the category of pairs (Z, θ) with $Z \in \mathcal{C}^{\mathbb{B}}$ and $\theta : Z\alpha \Rightarrow X$. Then a right adjoint α_* to α^* with counit ϵ exists iff Ψ_X has terminal object (α_*X, ϵ_X) for all $X \in \mathcal{C}^{\mathbb{A}}$. Again, for such an object in Ψ_X the functor α_*X is called the *right Kan extension of X along α*.

Reading guide

There is no big theorem that unifies the present work. In stead, what lies before you is a journey from abstract homotopy theory to algebraic K-theory, going through some simplicial ideas in geometry. We of course build on enough existing material. In particular, the three main constituents that are recent are the homotopical categories from [DHKS04], and the 2-Segal spaces and proto-exact categories from [DK12]. The layer we add consists of two main parts: 2-Segal objects in homotopical categories and K-groups of proto-exact categories.

In the first part, the main result is the formulation of a niceness condition on a given homotopical category $\mathcal{C}$ that guarantees 1-Segal objects in $\mathcal{C}$ to be also 2-Segal (Thm. 2.4.1). We assure ourselves this condition is a reasonable one by showing in Thm. 2.3.6 that it is satisfied by simplicial model categories. The main work done in the second part consists of giving two equivalent descriptions of higher K-groups of proto-exact categories (Thm. 3.5.3).

In order to get this far we need however to lay some groundwork in Chap. 1, on homotopy (co)limits in homotopical categories. This chapter is perhaps the most technical one. In it, we cover more than is strictly necessary: the most important parts for understanding Chap. 2 are section 1.1 and Def. 1.2.4; for Chap. 3 one only really needs paragraph 1.5 and Prop. 2.5.1.

The text should be accessible to a reader with a working knowledge in category theory and with some familiarity with algebraic topology. To aid the reader and myself I have added appendices on model categories, nice topological spaces and some homological algebra.

A remark on size

Let us satisfy our inner logician by reflecting for a moment upon foundations. Lurie identifies three possible strategies for dealing with issues of size in [Lur06, §1.2.15]: working with universes; only working with sets and keeping track of size; ignoring the issue altogether. He 'officially' adopts the first strategy, as is also done in [DHKS04] and exposited for example in [Low13]. The attractiveness of this approach is that it is mostly

invisible in the background, but still allows for certain constructions where the difference between 'small' and 'large' needs to be played out.

In practice, I have adopted the third strategy of ignoring the issue altogether. For the most part this should be safe, as our constructions do not hinge on any notion of size. Of course, when discussing (co)limits we do assume the necessary smallness conditions on our indexing categories. But the most notable exception to the rule that size does not matter for us, is the fact that the homotopy category $\mathrm{Ho}\,\mathcal{C}$ of a given homotopical category $\mathcal{C}$ need not have small hom-sets. Be that as it may, we only use the universal property of the localization functor $\gamma : \mathcal{C} \to \mathrm{Ho}\,\mathcal{C}$. This property should be preserved regardless of the convention one adopts to deal with these issues.

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1. Some categorical homotopy theory

> And what can life be worth if the first rehearsal for life is life itself?
>
> *The Unbearable Lightness of Being*
> Kundera

The goal of this chapter is to give a framework in which we can formulate the notion of Segal objects as discussed by Dyckerhoff & Kapranov in [DK12]. In the work of these authors, Segal objects live in combinatorial model categories. We however choose the more general setting of homotopical categories.[1] The philosophical reason for this is that Segal objects only involve weak equivalences, so that it should be more natural to define these objects in a setting in which one only has weak equivalences. We also get the practical advantage of a somewhat more lean theory.

1.1 Homotopical categories

Recently in homotopy-land Dwyer, Hirshhorn, Kan & Smith have isolated a key part of model theory that revolves only around weak equivalences, as explained in [DHKS04, Part II]. It turns out that this is exactly the kind of framework we need. Riehl gives a presentation of these ideas in [Rie14, §2.1], which I found accessible also to a novice. It is for this reason I have mainly followed her in what comes below.

Definition 1.1.1. A *homotopical category* is a category $\mathcal{C}$, with a subcategory $\mathcal{W}$ of *weak equivalences* that contains all isomorphisms and satisfies the *2-of-6 property*: if hg, gf are in $\mathcal{W}$ then so are f, g, h, hgf, for all composable f, g, h in $\mathcal{C}$.

Let $(\mathcal{C}, \mathcal{W})$ be a homotopical category. Then there is a *homotopy category* $\operatorname{Ho}\mathcal{C}$ associated to $\mathcal{C}$, with the same objects as $\mathcal{C}$, but wherein all weak equivalences are formally inverted. For this, one takes the free category on the directed graph $\mathcal{C} + \mathcal{W}[-1]$ with $\mathcal{W}[-1]$ formal inverses to $\mathcal{W}$, and then quotients out the congruence relation coming from the composition laws.

[1] The interested reader may find a definition of combinatorial model categories in [Lur06, Def. A.2.6.1], although we won't be using this. It is told the notion of combinatorial model categories goes back to Smith, who introduced it at a conference in Barcelona in 1998, and whose publication on the matter is forthcoming.

It turns out the precise construction of $\operatorname{Ho}\mathcal{C}$ does not matter much for us. What is important is the *localization functor*, i.e. the canonical projection $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$. This γ is determined by the universal property that it induces an isomorphism between the category of all functors $\operatorname{Ho}\mathcal{C} \to \mathcal{D}$, and the category of those functors $\mathcal{C} \to \mathcal{D}$ that turn weak equivalences into isomorphisms.

Remark 1.1.2. Since 2-of-6 implies 2-of-3, a homotopical category $(\mathcal{D}, \mathcal{W})$ can be endowed with a model structure if there are classes of maps $\mathcal{C}, \mathcal{F}$ such that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are both weak factorization systems (Def. A.2).

Note that conversely, 2-of-3 does not imply 2-of-6. For a minimal example, consider the following category $\mathcal{D}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "gf"'] & B \arrow[d, "hg"] \arrow[dl, "g"'] \\ C \arrow[r, "h"'] & D \end{tikzcd}$$

Let $\mathcal{W}$ be the identities, together with the arrows gf and hg. Then for any diagram in $\mathcal{D}$ of the form

$$\begin{tikzcd} & Y \arrow[dr] & \\ X \arrow[ur] \arrow[rr] & & Z \end{tikzcd}$$

where two of the three arrows are in $\mathcal{W}$, we must have that at least one of them is the identity. Hence $\mathcal{W}$ satisfies 2-of-3 vacuously, but clearly does not satisfy 2-of-6.

Call a homotopical category $\mathcal{C}$ *saturated* when a morphism in $\mathcal{C}$ is a weak equivalence iff it is an isomorphism in $\operatorname{Ho}\mathcal{C}$. The two most important properties of homotopical categories that we shall be using are the following:

Lemma 1.1.3. *For $\mathcal{C}$ a homotopical category and $\mathbb{A}$ a category, the functor category $\mathcal{C}^{\mathbb{A}}$ is homotopical, with weak equivalences taken pointwise. This functor category is saturated whenever $\mathcal{C}$ is.*

Proof. See [DHKS04, §33.2, 36.4]. □

Lemma 1.1.4. *Any model category $\mathcal{M}$ is homotopical and saturated.*[2]

Proof. See [Rie14, Rem. 2.1.3, Rem. 2.1.8]. □

From the above lemma it follows that the category $\mathcal{T}\mathrm{op}$ of nice topological spaces is a homotopical category, with weak homotopy equivalences as weak equivalences. Likewise, the simplicial model category $\mathrm{s}\mathcal{S}\mathrm{et}$ of simplicial sets is a homotopical category. In what follows, we always take this homotopical structure on $\mathcal{T}\mathrm{op}$ and on $\mathrm{s}\mathcal{S}\mathrm{et}$.

[2]See the appendix A for a reminder on model categories.

Example 1.1.5. Let $\mathcal{C}$ be a category. Then there are two trivial ways we can make $\mathcal{C}$ into a homotopical category. In the first, we endow $\mathcal{C}$ with the *minimal* homotopical structure, i.e. we let the isomorphisms be $\mathcal{W}$. Indeed, if we are given $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ with gf, hg isomorphisms, then g is monic since hg is an isomorphism, and furthermore $g \circ f(gf)^{-1} = \mathrm{id}_C$. Therefore g is monic and split epic, hence an isomorphism. It follows that f, h, hgf are isomorphisms as well.

In the second way, we take all arrows of $\mathcal{C}$ as $\mathcal{W}$, which gives us the *maximal* homotopical structure. Note that for $\mathcal{C}$ with this maximal structure, it holds that $\mathrm{Ho}\,\mathcal{C}$ is the groupoid obtained from $\mathcal{C}$ by formally inverting all the arrows.

Example 1.1.6. Let $\mathcal{A}$ be an abelian category. Write $\mathrm{Ch}^\bullet \mathcal{A}$ for the associated category of cochain complexes $\cdots \to C^i \to C^{i+1} \to \dots$. Recall that a chain map $f : A^\bullet \to B^\bullet$ is called a chain homotopy equivalence if there is a chain map $g : B^\bullet \to A^\bullet$ with $fg \sim \mathrm{id}_B$ and $gf \sim \mathrm{id}_A$, where $\sim$ denotes chain homotopy. I claim $\mathrm{Ch}^\bullet \mathcal{A}$ with chain homotopy equivalences as weak equivalences is homotopical.

Indeed, let $A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{h} D^\bullet$ be chain maps such that gf and hg are chain homotopy equivalences. Take $u : C^\bullet \to A^\bullet$ and $v : D^\bullet \to B^\bullet$ such that u resp. v is a homotopy inverse of gf resp. hg. Then it holds

$$fu = \mathrm{id}_B\, fu \sim vhgfu \sim vh\, \mathrm{id}_C = vh\,,$$

from which it follows that $fu \circ g \sim vhg \sim \mathrm{id}_B$. Since by assumption $g \circ fu \sim \mathrm{id}_C$, we see that fu is a homotopy inverse of g. It follows that ug and gv are homotopy inverses of f and h respectively.

This example can be generalized to a setting where $\mathcal{C}$ is a category endowed with a congruence relation, i.e. an equivalence relation on each hom-set which is well-behaved with respect to composition. Now call a morphism f in $\mathcal{C}$ a weak equivalence if there is a g such that fg, gf are congruent to the identities, and carry out the above procedure to observe the result is a homotopical category.

Suppose we have a homotopical category $\mathcal{C}'$ with weak equivalences $\mathcal{W}'$. Then a functor $F : \mathcal{C} \to \mathcal{C}'$ from any category $\mathcal{C}$ induces a homotopical structure on $\mathcal{C}$, by declaring a $\mathcal{C}$-morphism g to be a weak equivalence iff $Fg \in \mathcal{W}'$.

Example 1.1.7. The category $\mathrm{Ch}^\bullet \mathcal{A}$ endowed with quasi-isomorphisms as weak equivalences is also a homotopical category.

1.1.a Derived functors

Throughout, fix homotopical categories $\mathcal{C}$ and $\mathcal{D}$ with localizations $\gamma : \mathcal{C} \to \mathrm{Ho}\,\mathcal{C}$ and $\delta : \mathcal{D} \to \mathrm{Ho}\,\mathcal{D}$ respectively.

Objects X, Y in $\mathcal{C}$ are called *weakly equivalent*, notation $X \simeq Y$, when there is a finite zig-zag of weak equivalences between X and Y in $\mathcal{C}$. If $\mathcal{C}$ is saturated, then $X \simeq Y$ holds in $\mathcal{C}$ iff $X \cong Y$ holds in $\mathrm{Ho}\,\mathcal{C}$.

We call a given functor $F : \mathcal{C} \to \mathcal{D}$ *homotopical* if it preserves weak equivalences. If this is the case, it descends uniquely to a functor $\bar{F} : \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$ that makes the obvious square commutative. This $\bar{F}$ is called the *descent* of F.

Example 1.1.8. Let F, F' be two homotopical functors $\mathcal{C} \to \mathcal{D}$. Then by the universal property of the localization γ, natural transformations $\delta F \Rightarrow \delta F'$ correspond bijectively to natural transformations $\bar{F} \Rightarrow \bar{F}'$.

Take for example $\mathcal{C} = *$. Then a natural transformation $\delta F \Rightarrow \delta F'$ is just a morphism $F(*) \to F'(*)$ in $\operatorname{Ho}\mathcal{D}$, which is indeed the same as a natural transformation $\bar{F} \Rightarrow \bar{F}'$. But observe, such a morphism $F(*) \to F'(*)$ in $\operatorname{Ho}\mathcal{D}$ cannot in general be lifted to a corresponding single morphism in $\mathcal{D}$.

Let $F : \mathcal{C} \to \mathcal{D}$ be any functor. Then a *left derived functor* of F is determined by the following data. It is a homotopical functor $\mathbb{L}F : \mathcal{C} \to \mathcal{D}$, together with a natural transformation $\lambda : \mathbb{L}F \Rightarrow F$, such that the descent $\mathbf{L}F$ of $\mathbb{L}F$ is a right Kan extension of δF along γ. Unpacking the definition, we see this means that $(\mathbf{L}F, \delta\lambda)$ is a terminal object in the category of pairs (G, α) with $G : \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$ and $\alpha : G\gamma \Rightarrow \delta F$ (with obvious morphisms). In a diagram a left derived functor of F looks as follows:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "\mathbb{L}F"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow\lambda"] \arrow[d, "\gamma"] & \mathcal{D} \arrow[d, "\delta"] \\
\operatorname{Ho}\mathcal{C} \arrow[r, "\mathbf{L}F"] & \operatorname{Ho}\mathcal{D}
\end{tikzcd}$$

Example 1.1.9. Let $\lambda : \mathbb{L}F \Rightarrow F$ be a left derived functor of F, and suppose we are given a natural weak equivalence $\sigma : \mathbb{L}'F \Rightarrow \mathbb{L}F$, i.e. a natural transformation that is pointwise a weak equivalence. Note that this implies $\mathbb{L}'F$ is also a homotopical functor. Let us show $(\mathbf{L}'F, \delta\lambda\sigma) \cong (\mathbf{L}F, \delta\lambda)$, with $\mathbf{L}'F$ the descent of $\mathbb{L}'F$. For this, observe that σ descends to a natural isomorphism $\bar{\sigma} : \mathbf{L}'F \Rightarrow \mathbf{L}F$. Hence it suffices to show $\bar{\sigma}$ is a morphism in the category of pairs (G, α) as before, i.e. we need to show that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{L}'F\gamma = \delta\mathbb{L}'F \arrow[rr, Rightarrow, "\delta\lambda\sigma"] \arrow[dr, Rightarrow, "\bar{\sigma}\gamma=\delta\sigma"] & & \delta F \\
& \mathbf{L}F\gamma = \delta\mathbb{L}F \arrow[ur, Rightarrow, "\delta\lambda"'] &
\end{tikzcd}$$

which indeed it does by construction. This implies that $\lambda\sigma : \mathbb{L}'F \Rightarrow F$ is also a left derived functor of F.

Similarly, if we have a natural weak equivalence $\tau : \mathbb{L}F \Rightarrow \mathbb{L}''F$, and a natural transformation $\lambda'' : \mathbb{L}''F \Rightarrow F$ such that $\lambda''\tau = \lambda$, then $\lambda'' : \mathbb{L}''F \Rightarrow F$ is also a left derived functor of F.

Note that by the universal property of $\operatorname{Ho}\mathcal{C}$, the following two things are the same: to give a pair (G, α) as before; or to give a functor $g : \mathcal{C} \to \operatorname{Ho}\mathcal{D}$ that sends weak equivalences to isomorphisms, together with a natural transformation $\beta : g \Rightarrow \delta F$. It

follows that a left derived functor of F determines a terminal object in the category of such pairs (g, β). Note that conversely, a terminal object (g, β) in the latter category cannot in general be lifted to a left derived functor of F, since we have no guarantee that a pointwise lift $g' : \mathcal{C} \to \mathcal{D}$ of g on objects can be made into a functor.

Example 1.1.10. Suppose $\mathcal{C}$ has the minimal homotopical structure. Then any functor $F : \mathcal{C} \to \mathcal{D}$ is homotopical, as it preserves isomorphisms. Furthermore, since in constructing $\operatorname{Ho}\mathcal{C}$ we are only inverting isomorphisms, the result is again $\mathcal{C}$ with δ the identity. It follows that $\mathrm{id} : F \Rightarrow F$ is a left derived functor of F.

Let $\lambda_i : \mathbb{L}F_i \Rightarrow F$ (for $i = 1, 2$) be left derived functors of F. Then as terminal objects in the category of pairs (G, α) as above, $(\mathbf{L}F_1, \delta\lambda_1)$ and $(\mathbf{L}F_2, \delta\lambda_2)$ are uniquely isomorphic. In particular, if $\mathcal{D}$ is saturated, the value $\mathbb{L}FC$ with C an object of $\mathcal{C}$ is unique up to weak equivalence in $\mathcal{D}$.

Example 1.1.11. Suppose $F : \mathcal{C} \to \mathcal{D}$ is itself already homotopical. Then $\mathrm{id} : F \Rightarrow F$ is a left derived functor. Now let $\lambda : \mathbb{L}F \Rightarrow F$ be any other left derived functor. Then this induces a natural transformation $\bar{\lambda} : \mathbf{L}F \Rightarrow \bar{F}$. By the above remark, it is an isomorphism. Hence, when $\mathcal{D}$ is saturated, the natural transformation $\lambda : \mathbb{L}F \Rightarrow F$ is itself a pointwise weak equivalence.

We have the following convenient and important method for computing left derived functors.

Lemma 1.1.12. *Let F be a functor $\mathcal{C} \to \mathcal{D}$. Suppose that we have a functor $Q : \mathcal{C} \to \mathcal{C}_Q$, with $\mathcal{C}_Q$ a full subcategory of $\mathcal{C}$ such that F is homotopical on $\mathcal{C}_Q$, and that we also have a natural weak equivalence $q : Q \Rightarrow \mathrm{id}_{\mathcal{C}}$. Then $Fq : FQ \Rightarrow F$ is a left derived functor of F.*

Proof. See [Rie14, Thm. 2.2.8]. □

In the above situation, $q : Q \Rightarrow \mathrm{id}_{\mathcal{C}}$ is called a *left deformation* for F.

Example 1.1.13. The following can also be found in [Rie14, §2.3]. Let $\mathcal{A}$ be an abelian category, and write $\mathrm{Ch}_+\mathcal{A}$ for the category of chain complexes $\cdots \leftarrow C_{i-1} \leftarrow C_i \leftarrow \ldots$, which are concentrated at positive degree. In this example, we consider this category to be homotopical by looking at the quasi-isomorphisms.

Suppose that we have an endofunctor Q on $\mathrm{Ch}_+\mathcal{A}$ that sends a chain complex $A_\bullet$ to a chain complex $P_\bullet$ of projectives, and that we also have a natural transformation $q : Q \Rightarrow \mathrm{id}_{\mathrm{Ch}_+\mathcal{A}}$ such that $QA_\bullet \to A_\bullet$ is a quasi-isomorphism for each $A_\bullet$ in $\mathrm{Ch}_+\mathcal{A}$.

Now let $F : \mathcal{A} \to \mathcal{A}'$ be an additive and right exact functor, where $\mathcal{A}'$ is some other abelian category. Recall that the classical notion of a left derived functor of F is calculated as follows. First fix, for each $A \in \mathcal{A}$, a projective resolution $P_\bullet$ of A. Then put $L_iF = \mathcal{H}_i(FP_\bullet)$, where $\mathcal{H}_i(FP_\bullet)$ is the homology group at degree i of the chain complex $FP_\bullet$ (see e.g. [Har77, §III.1]).

On the other hand, since F is additive, it induces a functor $F_\bullet : \mathrm{Ch}_+\mathcal{A} \to \mathrm{Ch}_+\mathcal{A}'$, which one can show preserves quasi-isomorphisms between complexes of projectives.

Hence the natural transformation $q : Q \Rightarrow \mathrm{id}_{\mathrm{Ch}_+ \mathcal{A}}$ is a left deformation for $F_\bullet$, and according to Lem. 1.1.12 we can calculate the left derived functor $\mathbb{L}F_\bullet$ of $F_\bullet$ as $F_\bullet Q$. So we see that the classical notion of a left derived functor is revived as the compositions

$$\mathcal{A} \xrightarrow{\deg_0} \mathrm{Ch}_+ \mathcal{A} \xrightarrow{\mathbb{L}F_\bullet} \mathrm{Ch}_+ \mathcal{A}' \xrightarrow{\mathcal{H}^i(-)} \mathcal{A}',$$

where $\deg_0$ sends A to the complex $\cdots \to 0 \to A$. This is because the quasi-isomorphism $q : Q \deg_0 A \to \deg_0 A$ establishes that the complex $Q \deg_0 A$ is a projective resolution of A.

Definition 1.1.14. Dually, a *right derived functor* of a given functor $F : \mathcal{C} \to \mathcal{D}$ is a homotopical functor $\mathbb{R}F : \mathcal{C} \to \mathcal{D}$, together with a natural transformation $\rho : F \to \mathbb{R}F$, that satisfy the following property: the pair $(\mathbf{R}F, \delta\rho)$, with $\mathbf{R}F$ the descent of $\mathbb{R}F$, is initial in the category Ψ of pairs (H, β), with $H : \mathrm{Ho}\,\mathcal{C} \to \mathrm{Ho}\,\mathcal{D}$ and $\beta : \delta F \Rightarrow H\gamma$.

Example 1.1.15. Here is another description of derived functors in the classical setting of homological algebra. Let again $\mathcal{A}$ and $\mathcal{A}'$ be abelian categories, and this time $F : \mathcal{A} \to \mathcal{A}'$ an additive left exact functor. Let $K^+\mathcal{A}$ be the quotient category of $\mathrm{Ch}^+ \mathcal{A}$ of cochain complexes $C^0 \to C^1 \to \dots$, with morphisms taken modulo chain homotopy. Write $D^+\mathcal{A}$ for the localization by quasi-isomorphisms of $K^+\mathcal{A}$. Then in fact $D^+\mathcal{A}$ is what we have called the homotopy category $\mathrm{Ho}(\mathrm{Ch}^+ \mathcal{A})$, where $\mathrm{Ch}^+ \mathcal{A}$ has quasi-isomorphisms as weak equivalences (see [GM03, Prop. III.4.2]). Write its localization as $\gamma : \mathrm{Ch}^+ \mathcal{A} \to D^+\mathcal{A}$. Likewise for $\mathcal{A}'$.

Now in [GM03, Def. III.6.6], a total right derived functor of F is defined as an exact functor $RF : D^+\mathcal{A} \to D^+\mathcal{A}'$, together with a natural transformation $\epsilon : \gamma' K^+(F) \Rightarrow RF\gamma$, where $K^+(F)$ is the functor $\mathrm{Ch}^+ \mathcal{A} \to \mathrm{Ch}^+ \mathcal{A}'$ induced by F in the obvious way. This pair (RF, ϵ) needs to be initial in the category category Ψ of pairs (H, β) from the above definition.

We see that for a right derived functor $\mathbb{R}F$ of F as defined in 1.1.14, the descent $\overline{\mathbb{R}F}$ gives a total right functor as defined in [GM03, Def. III.6.6]. The converse again need not hold, since for a given exact functor $RF : D^+\mathcal{A} \to D^+\mathcal{A}'$ there may fail to be a lift $F' : \mathrm{Ch}_+ \mathcal{A} \to \mathrm{Ch}_+ \mathcal{A}'$ that descends to RF.

Example 1.1.16. Suppose that a functor $F : \mathcal{C} \to \mathcal{D}$ is given, and that it has a right derived functor $\rho : F \Rightarrow \mathbb{R}F$. Suppose further that $G : \mathcal{C} \to \mathcal{D}$ is a given homotopical functor, and that $\sigma : F \Rightarrow G$ is a natural transformation.

Let Ψ be the category of pairs (H, β) as in the above definition. Then since $(\mathbf{R}F, \delta\rho)$ is initial in Ψ and by the universal property of the localization γ, we have a unique natural transformation $\varphi : \delta\mathbb{R}F \Rightarrow \delta G$ that fits in the following commutative diagram

$$\begin{tikzcd} \delta F \arrow[rr, Rightarrow, "\delta\sigma"] \arrow[dr, Rightarrow, "\delta\rho"'] & & \delta G \\ & \delta\mathbb{R}F \arrow[ur, Rightarrow, "\varphi"'] & \end{tikzcd}$$

We reiterate that φ need not lift to a natural transformation $\varphi' : \mathbb{R}F \Rightarrow G$ that makes the above diagram, but without the δ's, commutative.

Remark 1.1.17. We also have a statement dual to Lem. 1.1.12. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between homotopical categories. Then a *right deformation* for F is a functor $R : \mathcal{C} \to \mathcal{C}_R$, with $\mathcal{C}_R$ a full subcategory of $\mathcal{C}$ such that F is homotopical on $\mathcal{C}_R$, together with a natural weak equivalence $r : \mathrm{id}_{\mathcal{C}} \Rightarrow R$. If we have such a deformation, then $Fr : F \Rightarrow FR$ is a right derived functor of F. Note that a right deformation R is alway homotopical, as can easily be checked.

1.2 Homotopy (co)limits

In the following, let $\mathcal{C}$ still be our homotopical category with localization $\gamma : \mathcal{C} \to \mathrm{Ho}\,\mathcal{C}$, and let $\mathbb{A}$ be some indexing category. Suppose that $\mathcal{C}$ has all colimits of shape $\mathbb{A}$.

Definition 1.2.1. A left derived functor of $\mathrm{colim} : \mathcal{C}^{\mathbb{A}} \to \mathcal{C}$ is called a *homotopy colimit*, and is written as $\lambda : \mathrm{hocolim} \Rightarrow \mathrm{colim}$.

Remark 1.2.2. In the literature, one sometimes takes $\mathbf{L}\,\mathrm{colim}$ as the homotopy colimit functor, and calls $\mathbb{L}\,\mathrm{colim}$ a model for the homotopy colimit. Let us stress however that we take $\mathrm{hocolim}_{\mathbb{A}} = \mathbb{L}\,\mathrm{colim}_{\mathbb{A}}$, hence a given $\mathrm{hocolim}_{\mathbb{A}}$ is a functor $\mathcal{C}^{\mathbb{A}} \to \mathcal{C}$. In doing so, we follow [DHKS04, §47.1] and [Rie14, Thm. 5.1.1].

Our convention comes with the following notational subtlety. In writing $\mathrm{hocolim}_{\mathbb{A}}$ in a given formula, we can either mean that this formula holds for any given left derived functor of $\mathrm{colim}_{\mathbb{A}}$, or that it holds for a specific left derived functor of $\mathrm{colim}_{\mathbb{A}}$. We agree to handle this in the same way as one handles 'a colimit' versus 'the colimit'; that is, the difference should be clear from the context. This will not cause any trouble, as long as we remember that homotopy colimits can only be unique up to weak equivalence.

Example 1.2.3. A well-known but instructive example is when we let $\mathbb{A}$ be the category $\cdot \leftarrow \cdot \rightarrow \cdot$, and put $\mathcal{C} = \mathcal{T}\mathrm{op}$, i.e. we are going to take homotopy pushouts of spaces. Let $\mathbb{L}\,\mathrm{colim} : \mathcal{T}\mathrm{op}^{\mathbb{A}} \to \mathcal{T}\mathrm{op}$ be the functor that sends a diagram $X \xleftarrow{f} A \xrightarrow{g} Y$ to the space obtained from $X \amalg (A \times I) \amalg Y$ by identifying $(a, 0) \sim f(a)$ and $(a, 1) \sim g(a)$. It is a classical result that this gives a homotopical functor $\mathbb{L}\,\mathrm{colim}$, together with a natural transformation $\lambda : \mathbb{L}\,\mathrm{colim} \Rightarrow \mathrm{colim}$ (see e.g. [Dug08, Exm. 2.2]). Let us see this construction also agrees with our notion of homotopy colimits, i.e. that $\lambda : \mathbb{L}\,\mathrm{colim} \Rightarrow \mathrm{colim}$ is a left derived functor of colim.

Suppose we are given $G : \mathrm{Ho}(\mathcal{T}\mathrm{op}^{\mathbb{A}}) \to \mathrm{Ho}\,\mathcal{T}\mathrm{op}$ and $\alpha : G\gamma \Rightarrow \delta\,\mathrm{colim}$, where γ, δ are the localization functors of $\mathcal{T}\mathrm{op}^{\mathbb{A}}$ and of $\mathcal{T}\mathrm{op}$ respectively. Then we need a unique functor $\sigma : G \Rightarrow \mathbf{L}\,\mathrm{colim}$ which makes the following diagram commute

$$\begin{tikzcd}
G\gamma \arrow[rr, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\sigma\gamma"] & & \delta\,\mathrm{colim} \\
& \mathbf{L}\,\mathrm{colim}\,\gamma = \delta\mathbb{L}\,\mathrm{colim} \arrow[ur, Rightarrow, "\delta\lambda"'] &
\end{tikzcd}$$

For a diagram $D_\bullet$ of the form $X \xleftarrow{f} A \xrightarrow{g} Y$, write $QD_\bullet$ for the resulting diagram $M_f \leftarrow A \rightarrow M_g$, where M_f, M_g are the mapping cylinders of f and of g respectively.

Note that this gives an endofunctor Q on $\mathcal{T}\mathrm{op}^{\mathbb{A}}$ such that $\mathbb{L}\operatorname{colim} D_\bullet = \operatorname{colim} QD_\bullet$, and also a natural weak equivalence $Q \Rightarrow \mathrm{id}$. Hence we can define σ on $D_\bullet$ as the composition

$$G(D_\bullet) \cong G(QD_\bullet) \overset{\alpha}{\Rightarrow} \mathbf{L}\operatorname{colim}(QD_\bullet) \cong \mathbf{L}\operatorname{colim}(D_\bullet)\,.$$

It is clear that this σ satisfies the requirements.

Note that a homotopy colimit does not always exist and, if it exists, is not always unique. It is however unique up to weak equivalence in $\mathcal{C}$ when $\mathcal{C}$ is saturated. Furthermore, Exm. 1.1.9 shows that if we can replace $\operatorname{hocolim} X_\bullet$ by a weakly equivalent object $HX_\bullet$ functorially in $X_\bullet$, then we are justified in taking $HX_\bullet$ as model for the homotopy colimit of $X_\bullet$.

Dually, assume now that $\mathcal{C}$ has all limits of shape $\mathbb{A}$.

Definition 1.2.4. A right derived functor of $\lim : \mathcal{C}^{\mathbb{A}} \to \mathcal{C}$ is called a *homotopy limit*, and is written as $\rho : \lim \Rightarrow \operatorname{holim}$.

Example 1.2.5. Let $\mathbb{A}$ be a category with initial object a. Then $\lim_{\mathbb{A}}$ is just the evaluation functor $\mathrm{ev}_a : \mathcal{C}^{\mathbb{A}} \to \mathcal{C}$. Clearly, ev_a is its own right derived functor, so that $X \mapsto X_a$ is a homotopy limit $\operatorname{holim}_{\mathbb{A}}$. If furthermore $\mathcal{C}$ is saturated, then for every diagram $X \in \mathcal{C}^{\mathbb{A}}$ it holds $\operatorname{holim}_{\mathbb{A}} X \simeq X_a$.

Example 1.2.6. For a diagram $X_1 \to Y_1 \leftarrow X_2 \to Y_2 \leftarrow \ldots Y_n \leftarrow X_{n+1}$ write its homotopy limit as $X_1 \times^R_{Y_1} X_2 \times^R_{Y_2} \cdots \times^R_{Y_n} X_{n+1}$, and call it the *homotopy fiber product*. When $\mathcal{C}$ is saturated, then identity arrows in homotopy fiber products cancel, e.g. when $X_1 \to Y_1$ is the identity, then $X_1 \times^R_{Y_1} X_2 \times^R_{Y_2} \cdots \times^R_{Y_n} X_n$ is up to weak equivalence just $X_2 \times^R_{Y_2} \cdots \times^R_{Y_n} X_n$, assuming these homotopy limits exist.

This follows from the following observation. Write $\mathbb{A}$ for the indexing category $a_1 \to b_1 \leftarrow a_2 \to b_2 \leftarrow \ldots b_n \leftarrow a_{n+1}$, and $\mathbb{A}'$ for the subcategory of $\mathbb{A}$ with a_1 removed. Let $\rho : \lim_{\mathbb{A}} \Rightarrow \operatorname{holim}_{\mathbb{A}}$ be a right derived functor. Write ι for the functor $\mathcal{C}^{\mathbb{A}'} \to \mathcal{C}^{\mathbb{A}}$ that extends a diagram $\mathbb{A}' \to \mathcal{C}$ to one of the form $\mathbb{A} \to \mathcal{C}$ in the obvious way, i.e. by adding an identity. Likewise, let $\pi : \mathcal{C}^{\mathbb{A}} \to \mathcal{C}^{\mathbb{A}'}$ the functor that restricts diagram $\mathbb{A} \to \mathcal{C}$ to $\mathbb{A}'$. Then we have an adjunction $\pi \dashv \iota$; write its counit and unit as $\epsilon : \pi\iota \Rightarrow \mathrm{id}$ and $\eta : \mathrm{id} \Rightarrow \iota\pi$ respectively.

Now the claim is that $\rho\iota : \lim_{\mathbb{A}} \iota \Rightarrow \operatorname{holim}_{\mathbb{A}} \iota$ is a right derived functor. Write Ψ' for the category of pairs (G, β) with G a functor $\mathcal{C}^{\mathbb{A}'} \to \operatorname{Ho}\mathcal{C}$ that sends weak equivalences to isomorphisms and with β a natural transformation $\gamma \lim_{\mathbb{A}} \iota \Rightarrow G$. Our claim comes down to showing $(\gamma \operatorname{holim}_{\mathbb{A}} \iota, \gamma\rho\iota)$ is initial in Ψ'.

For the latter claim, by the universal property of $(\gamma \operatorname{holim}_{\mathbb{A}}, \gamma\rho)$ we have a unique natural transformation $\sigma : \gamma \operatorname{holim}_{\mathbb{A}} \Rightarrow G\pi$ such that $\beta\pi \circ \gamma \lim_{\mathbb{A}} \eta = \sigma \circ \gamma\rho$. Then the induced natural transformation $\sigma\iota : \gamma \operatorname{holim}_{\mathbb{A}} \iota \Rightarrow G\pi\iota = G$ is such that $\sigma\iota \circ \gamma\rho\iota = \beta$, and is furthermore unique with this property. For the latter facts, one uses that $\pi\iota$ is the identity, that $\iota\epsilon \circ \eta\iota$ is the identity $\iota \Rightarrow \iota$ by adjointness, and that ϵ is just the identity transformation.

Now for a diagram $D \coloneqq Y_1 \leftarrow X_2 \to Y_2 \leftarrow \ldots Y_n \leftarrow X_{n+1}$ it holds

$$X_2 \times^R_{Y_2} \cdots \times^R_{Y_n} X_n \simeq \operatorname{holim}_{\mathbb{A}'} D = \mathbb{R}\lim_{\mathbb{A}} \iota D \simeq \operatorname{holim}_{\mathbb{A}} \iota D = X_1 \times^R_{Y_1} \cdots \times^R_{Y_n} X_n\,,$$

where in the last fiber product the arrow $X_1 \to Y_1$ is the identity. Note in the first weak equivalence we used the previous example, and in the second one we used that in $\mathcal{C}$ homotopy limits are unique up to weak equivalence, as $\mathcal{C}$ is saturated.

1.2.a In simplicial model categories

There is a large class of homotopical categories wherein homotopy (co)limits always exist, and wherein we can even give explicit formulae. Indeed, from hereon let $\mathcal{M}$ be a simplicial model category. Then we can calculate the homotopy colimit of a diagram $X_\bullet : \mathbb{A} \to \mathcal{M}$ as the geometric realization of the simplicial replacement of $X_\bullet$, using the bar construction. To understand what this means, we need a series of definitions.

Definition 1.2.7 (Coend)**.** Let $X : \mathbb{A} \to \mathcal{M}$ and $K : \mathbb{A}^{\mathrm{op}} \to \mathrm{sSet}$ be given diagrams. Then the *coend* $X \otimes_{\mathbb{A}} K$ is the coequalizer in $\mathcal{M}$ of the maps

$$\coprod_{\sigma \in \mathbb{A}_1} X_{\sigma_0} \otimes K_{\sigma_1} \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \coprod_{a \in \mathbb{A}_0} X_a \otimes K_a \,, \tag{1.1}$$

where on the summand $X_{\sigma_0} \otimes K_{\sigma_1}$ indexed by $\sigma \in \mathbb{A}_1$:

- The map φ is defined as $\mathrm{id}_{X_{\sigma_0}} \otimes (\sigma_0 \to \sigma_1)^*$ followed by the inclusion ι_{σ_0};
- The map ψ is defined as $(\sigma_0 \to \sigma_1)_* \otimes \mathrm{id}_{K_{\sigma_1}}$ followed by the inclusion ι_{σ_1}.

Example 1.2.8. Let $\mathbb{A}$ be the diagram $a \xrightarrow{f} b$ and take $\mathcal{M} := \mathcal{T}\mathrm{op}$. Let $X_\bullet$ be the $\mathbb{A}$-diagram $X_a \xrightarrow{f_*} X_b$ of the inclusion $S^1 \to D^2$ of the unit circle into the unit disk as its boundary. Let $K_\bullet$ be the diagram $K_b \xrightarrow{f^*} K_a$ of the inclusion $\Delta[1] \to \Delta[2]$ induced by $\{0,1\} \subset \{0,1,2\}$. As for a space X and a simplicial set K it holds $X \otimes K$ is, by definition of the tensor product in $\mathcal{T}\mathrm{op}$, the space $X \times |K|$, we see that $X_\bullet \otimes_{\mathbb{A}} K_\bullet$ is the coequalizer of the diagram

$$(S^1 \times \Delta^2) \amalg (S^1 \times \Delta^1) \amalg (D^2 \times \Delta^1) \underset{\psi}{\overset{\varphi}{\rightrightarrows}} (S^1 \times \Delta^2) \amalg (D^2 \times \Delta^1)$$

with φ, ψ as above.

Now φ, ψ are both the identity on the solid torus $S^1 \times \Delta^2$ and on the solid cylinder $D^2 \times \Delta^1$. Furthermore, φ maps the tube $S^1 \times \Delta^1$ to a strip on the boundary of $S^1 \times \Delta^2$, which is on the inside of— and goes around the hole of $S^1 \times \Delta^2$. Likewise, ψ is the obvious inclusion of the tube into the cylinder. Since we are taking the coequalizer of φ, ψ, the image of $S^1 \times \Delta^1$ under these maps in $(S^1 \times \Delta^2) \amalg (D^2 \times \Delta^1)$ is identified. Hence we are sewing in the cylinder in the hole of our torus, which results in a solid sphere.

When we write $X_\bullet \otimes_{\mathbb{A}} K_\bullet$ for diagrams as in the above definition, then unless otherwise stated it is implicitly understood that we are taking the coend in the category $\mathcal{M}$ wherein $X_\bullet$ is a diagram.

Example 1.2.9 ([Rie14, Exm. 4.1.3])**.** The functor $- \otimes_{\mathbb{A}} *$ is isomorphic to $\operatorname{colim}_{\mathbb{A}} : \mathcal{M}^{\mathbb{A}} \to \mathcal{M}$, with $*$ the functor that sends each a to the point $*$ in $\mathcal{S}\mathrm{et}$. This easily follows from the formula of colimits as coequalizers of coproducts.

Lemma 1.2.10 (Adjointness)**.** *For diagrams $X : \mathbb{A} \to \mathcal{M}$ and $K : \mathbb{A}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, and $\mathcal{M}$-object Z, we have a natural isomorphism*

$$\mathcal{M}(X_\bullet \otimes_{\mathbb{A}} K_\bullet, Z) \cong \mathcal{S}\mathrm{et}^{\mathbb{A}^{\mathrm{op}}}(K_\bullet, \operatorname{Map}(X_\bullet, Z)) .$$

Proof. See [Hir14, Prop. 7.11]. □

If we let $K_\bullet$ be the $\mathbb{A}^{\mathrm{op}}$-diagram $a \mapsto *$ in $\mathcal{S}\mathrm{et}$, then the above lemma gives us

$$\mathcal{M}(\operatorname{colim}_{\mathbb{A}} X_\bullet, Z) \cong \mathcal{M}(X_\bullet \otimes_{\mathbb{A}} *, Z) \cong \mathcal{S}\mathrm{et}^{\mathbb{A}^{\mathrm{op}}}(*, \operatorname{Map}(X_\bullet, Z)) ,$$

with $\operatorname{Map}(X_\bullet, Z)$ the $\mathbb{A}^{\mathrm{op}}$-diagram of simplicial sets that sends $a \in \mathbb{A}$ to the mapping space $\operatorname{Map}(X_a, Z)$ in $\mathcal{S}\mathrm{et}$. Now for any simplicial set S, we know that giving a simplicial map $* \to S$ is the same thing as pointing to an element in S_0. Hence, as $\operatorname{Map}_0(X_a, Z)$ is just the set $\mathcal{M}(X_a, Z)$, an element on the right-hand side of the above equation is equivalent to a family of maps $X_a \to Z$, natural in $a \in \mathbb{A}$. Thus we retrieve the ordinary adjunction between $\operatorname{colim}_{\mathbb{A}}$ and the diagonal functor.

Definition 1.2.11 (Geometric Realization)**.** Let $H : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{M}$ be a simplicial object in $\mathcal{M}$. Then the *geometric realization* $|H|$ of H is the coend $H \otimes_{\mathbf{\Delta}^{\mathrm{op}}} \Delta$.

Example 1.2.12. Let D be a simplicial set. Consider D as diagram $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{T}\mathrm{op}$ by taking the discrete topology on each D_n. Then the coend $D \otimes_{\mathbf{\Delta}^{\mathrm{op}}} \Delta$ in $\mathcal{T}\mathrm{op}$ is exactly the classical geometric realization of D.

Example 1.2.13. Recall that a bisimplicial set is a functor $X : \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$. Let X be such a set. Note for $n \geq 0$ that $Y_n \coloneqq X_{n,\bullet}$ is a simplicial set, which gives us a simplicial object $Y_\bullet$ in $\mathcal{S}\mathrm{et}$. Also, define the *diagonal* $d(X)$ of X as the simplicial set $[n] \mapsto X_{nn}$. Then $d(X)$ is the geometric realization of $Y_\bullet$. To see this, let (σ, θ) be an r-simplex in $\coprod_{k \geq 0} Y_k \otimes \Delta[k]$. Then we have an induced function $\theta^* : X_{kr} \to X_{rr}$, and we put $\mu(\sigma, \theta) \coloneqq \theta^*(\sigma)$. It is straightforward to show this μ makes $d(X)$ into a coequalizer of diagram (1.1). A similar, but more involved argument is given in more detail in section 1.4.

With the notion of geometric realization in a general simplicial model category under our belt, we can define the bar construction.

Definition 1.2.14 (Bar Construction)**.** Let $X : \mathbb{A} \to \mathcal{M}$ and $K : \mathbb{A}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be given diagrams. Then for $n \geq 0$ define

$$B_n(K, \mathbb{A}, X) \coloneqq \coprod_{\sigma \in \mathbb{A}_n} X_{\sigma_0} \otimes K_{\sigma_n} .$$

This gives a simplicial object $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{M}$ as follows. Let $[n] \xrightarrow{f} [m]$ in $\mathbf{\Delta}$ be given. Then f^* is defined on the summand indexed by $\sigma \in \mathbb{A}_m$ by the following diagram

$$\begin{tikzcd} B_m(K,\mathbb{A},X) \arrow[r, "f^*"] & B_n(K,\mathbb{A},X) \\ X_{\sigma_0}\otimes K_{\sigma_m} \arrow[u, "\iota_\sigma"] \arrow[r, "(\sigma_0\to\sigma f_0)_*\otimes(\sigma f_n\to\sigma_m)^*"] & X_{\sigma f_0}\otimes K_{\sigma f_n} \arrow[u, "\iota_{\sigma f}"] \end{tikzcd}$$

We call $B_\bullet(K,\mathbb{A},X)$ the *two-sided bar construction.* The *bar construction* $B(K,\mathbb{A},X)$ is the geometric realization $B_\bullet(K,\mathbb{A},X)\otimes_{\mathbf{\Delta}^{\mathrm{op}}}\Delta$ of $B_\bullet(K,\mathbb{A},X)$.

Example 1.2.15 ([Rie14, Exm. 8.3.8])**.** For $K\in \mathrm{sSet}^{\mathbb{A}^{\mathrm{op}}}$ and $X\in\mathcal{M}^{\mathbb{A}}$ we have that the colimit of $B_\bullet(K,\mathbb{A},X)$ is isomorphic to $X\otimes_{\mathbb{A}}K$, natural in both K and X. This follows from the fact that the inclusion F of the category $[1]\rightrightarrows[0]$ into $\mathbf{\Delta}^{\mathrm{op}}$ is final, since for every $n\geq 0$ the category $[n]/F$ is nonempty and connected. Hence we can compute the colimit of $B_\bullet(K,\mathbb{A},X)$ on only $[1]\rightrightarrows[0]$, which is readily seen to result in $X\otimes_{\mathbb{A}}K$.

For $a\in\mathbb{A}$, let $\mathbf{y}_a$ be the functor $\mathbb{A}(-,a):\mathbb{A}^{\mathrm{op}}\to\mathrm{sSet}$, and $\mathbf{y}^a$ the functor $\mathbb{A}(a,-):\mathbb{A}\to\mathrm{sSet}$, both considered as constant simplicial sets. Note these assignments are natural in a.

Example 1.2.16 (Yoneda, found in [Rie14, Exm. 4.1.4])**.** The functor $-\otimes_{\mathbb{A}}\mathbf{y}_a$ is isomorphic to the evaluation functor $\mathcal{M}^{\mathbb{A}}\to\mathcal{M}$ at a. Similarly, $\mathbf{y}^a\otimes_{\mathbb{A}}-$ is isomorphic to the evaluation $\mathrm{sSet}^{\mathbb{A}^{\mathrm{op}}}\to\mathrm{sSet}$ at a. To see the first claim, let $X\in\mathcal{M}^{\mathbb{A}}$ be given. Then for each object Z in $\mathcal{M}$ we have

$$\mathcal{M}(X\otimes_{\mathbb{A}}\mathbf{y}_a,Z)\simeq\mathrm{sSet}^{\mathbb{A}^{\mathrm{op}}}(\mathbf{y}_a,\mathrm{Map}(X_\bullet,Z))\,.$$

Now since $\mathbf{y}_a$ is constant, an element on the right-hand side is equivalent to a family of maps $\mathbb{A}(a',a)\to\mathcal{M}(X_{a'},Z)$, natural in a', which in turn is completely determined by a single morphism $X_a\to Z$. It follows that $X\otimes_{\mathbb{A}}\mathbf{y}_a\cong X_a$.

For the second claim, use that $\mathbf{y}^a$ is isomorphic to the functor $\mathbf{y}'_a=\mathbb{A}^{\mathrm{op}}(-,a)$, and that $-\otimes_{\mathbb{A}^{\mathrm{op}}}\mathbf{y}'_a$ is the same as $\mathbf{y}^a\otimes_{\mathbb{A}}-$. It follows that this is just a special case of the previous claim.

For $X\in\mathcal{M}^{\mathbb{A}}$ we have a functor

$$B(\mathbb{A},\mathbb{A},X_\bullet):\mathbb{A}\to\mathcal{M}:a\mapsto B(\mathbf{y}_a,\mathbb{A},X)\,,$$

which is in fact natural in X. Hence this gives a functor $B(\mathbb{A},\mathbb{A},-):\mathcal{M}^{\mathbb{A}}\to\mathcal{M}^{\mathbb{A}}$. We can further construct a natural transformation

$$\epsilon:B(\mathbb{A},\mathbb{A},-)\Rightarrow\mathrm{id}_{\mathcal{M}^{\mathbb{A}}}\tag{1.2}$$

as follows. First note that the unique map $\Delta\to *$ induces a map

$$B(\mathbf{y}_a,\mathbb{A},X)=B_\bullet(\mathbf{y}_a,\mathbb{A},X)\otimes_{\mathbf{\Delta}^{\mathrm{op}}}\Delta\to B_\bullet(\mathbf{y}_a,\mathbb{A},X)\otimes_{\mathbf{\Delta}^{\mathrm{op}}}*\,.$$

Now observe that the right-hand side is $\mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}}B_\bullet(\mathbf{y}_a,\mathbb{A},X)$. This colimit is isomorphic to $X\otimes_{\mathbb{A}}\mathbf{y}_a$, which in turn is isomorphic to X_a (see Exms. 1.2.9, 1.2.15, 1.2.16). This functor will reappear later on.

Definition 1.2.17 (Simplicial Replacement)**.** Let $X : \mathbb{A} \to \mathcal{M}$ be a diagram in $\mathcal{M}$. Then $\amalg_\bullet X \coloneqq B_\bullet(*, \mathbb{A}, X)$ is called the *simplicial replacement* of X.

Recall that $\mathcal{M}$ comes with a cofibrant replacement functor Q, which naturally associates a weak equivalence $QX \to X$ with QX cofibrant to any X in $\mathcal{M}$ (see Prop. A.8). We assume Q is the identity on cofibrant objects. Note that Q induces an endofunctor on $\mathcal{M}^{\mathbb{A}}$ by composition. We write this functor also as Q, but one should not get confused and think this is a cofibrant replacement functor on $\mathcal{M}^{\mathbb{A}}$.

Theorem 1.2.18 (Homotopy Colimit)**.** *For $X : \mathbb{A} \to \mathcal{M}$ we can calculate the homotopy colimit of X as*

$$\operatorname{hocolim} X = B(*, \mathbb{A}, QX) = |\amalg_\bullet QX| \,.$$

If we unpack the definition of $|\amalg_\bullet -|$ and use that $- \otimes K$ commutes with colimits for $K \in \mathrm{s}\mathcal{S}\mathrm{et}$, then we find that the homotopy colimit of $X : \mathbb{A} \to \mathcal{M}$ is the coequalizer of the diagram

$$\coprod_{\substack{[n]\leftarrow[m] \\ \sigma\in\mathbb{A}_n}} Y_{\sigma_0} \otimes \Delta[m] \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \coprod_{\substack{k\geq 0 \\ \tau\in\mathbb{A}_k}} Y_{\tau_0} \otimes \Delta[k] \tag{1.3}$$

where $Y = Q \circ X$, and for given $f : [m] \to [n]$ in $\mathbf{\Delta}$ and $\sigma \in \mathbb{A}_n$ the map φ on $Y_{\sigma_0} \otimes \Delta[m]$ is given by the composition

$$Y_{\sigma_0} \otimes \Delta[m] \xrightarrow{\mathrm{id}_{Y_{\sigma_0}} \otimes f_*} Y_{\sigma_0} \otimes \Delta[n] \xrightarrow{\iota_\sigma} \coprod_{\substack{k\geq 0 \\ \tau\in\mathbb{A}_k}} Y_{\tau_0} \otimes \Delta[k] \,,$$

while ψ on the same summand is given by the composition

$$Y_{\sigma_0} \otimes \Delta[m] \xrightarrow{(\sigma_0 \to \sigma f_0)_* \otimes \mathrm{id}_{\Delta[m]}} Y_{\sigma f_0} \otimes \Delta[m] \xrightarrow{\iota_{\sigma f}} \coprod_{\substack{k\geq 0 \\ \tau\in\mathbb{A}_k}} Y_{\tau_0} \otimes \Delta[k] \,.$$

In this construction, I follow [Rie14, Thm. 5.1.1]. The difference with [Hir14] and [Dug08] is that we first apply the cofibrant replacement functor pointwise before executing the bar construction. This has the convenient effect that results taken from [Hir14] hold without assuming the appropriate diagrams are pointwise cofibrant. It also has the advantage that $\operatorname{hocolim}_{\mathbb{A}}$ indeed becomes the left derived functor of colim, for which it needs to be homotopical.

Let us give a sketch of the argument given in [Rie14, Thm. 5.1.1] for Thm. 1.2.18. Write δ for the localization $\mathcal{M} \to \operatorname{Ho}\mathcal{M}$ and ϵ_Q for the natural transformation $B(\mathbb{A}, \mathbb{A}, Q-) \Rightarrow Q$ induced by ϵ from (1.2). Then the first step is to show that $q\epsilon_Q : B(\mathbb{A}, \mathbb{A}, Q-) \Rightarrow \mathrm{id}$ is a left deformation for colim, which implies that

$$\operatorname{colim} q\epsilon_Q : \operatorname{colim} B(\mathbb{A}, \mathbb{A}, Q-) \Rightarrow \operatorname{colim}$$

is a homotopy colimit by Lem. 1.1.12. Then by the commutativity of coends,[3] one can show that there is an isomorphism

$$B(\mathbb{A}, \mathbb{A}, -) \otimes_{\mathbb{A}} * \cong B(*, \mathbb{A}, -)\,.$$

Therefore, since the left-hand side is isomorphic to $\operatorname{colim}_{\mathbb{A}} B(\mathbb{A}, \mathbb{A}, -)$, the homotopy colimit of $X \in \mathcal{M}^{\mathbb{A}}$ can be computed as $B(*, \mathbb{A}, QX) = |\amalg_{\bullet} QX|$.

Remark 1.2.19. Let $X : \mathbb{A} \to \mathcal{M}$ be given. Then the homotopy colimit $|\amalg_{\bullet} QX|$ is naturally isomorphic to $QX \otimes_{\mathbb{A}} N(-/\mathbb{A})^{\mathrm{op}}$ (see [Hir14, Def. 8.1, Thm. 9.5]). Ignoring Q for the moment, one can use Lem. 1.2.10 and the adjointness from Def. A.4 to show that this latter colimit is left adjoint to the functor $\mathcal{M} \to \mathcal{M}^{\mathbb{A}}$ which sends $Z \in \mathcal{M}$ to the diagram $a \mapsto Z^{N(-/\mathbb{A})^{\mathrm{op}}}$. If we think of $\operatorname{colim}_{\mathbb{A}}$ as the left adjoint of the diagonal functor, this shows that, in a sense, $\operatorname{hocolim}_{\mathbb{A}}$ is the best homotopical approximation of $\operatorname{colim}_{\mathbb{A}}$.

1.2.b The cobar construction

Still suppose $\mathcal{M}$ is a simplicial model category. We can compute homotopy limits in $\mathcal{M}$ by means of the cobar construction as the totalization of a cosimplicial replacement. To see what this means, we again need a series of definitions.

Definition 1.2.20 (End)**.** Let $X : \mathbb{A} \to \mathcal{M}$ and $K : \mathbb{A} \to \mathrm{s}\mathcal{S}\mathrm{et}$ be given diagrams. Then the *end* $\hom^{\mathbb{A}}(K, X)$ is the equalizer in $\mathcal{M}$ of φ, ψ which are defined by the diagram

$$\prod_{a \in \mathbb{A}_0} X_a^{K_a} \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \prod_{\sigma \in \mathbb{A}_1} X_{\sigma_1}^{K_{\sigma_0}}\,,$$

where on the factor $X_{\sigma_1}^{K_{\sigma_0}}$ indexed by $\sigma \in \mathbb{A}_1$:

- The map φ is the projection π_{σ_0} followed by $(\sigma_0 \to \sigma_1)_*^{\mathrm{id}_{K_{\sigma_0}}}$;
- The map ψ is the projection π_{σ_1} followed by $\mathrm{id}_{X_{\sigma_1}}^{(\sigma_0 \to \sigma_1)_*}$.

Definition 1.2.21 (Totalization)**.** Let $H : \mathbf{\Delta} \to \mathcal{M}$ be a cosimplicial object in $\mathcal{M}$. Then its *total object* $\operatorname{Tot} H$ is the end $\hom^{\mathbf{\Delta}}(\Delta, H)$.

Definition 1.2.22 (Cobar Construction)**.** Let $X : \mathbb{A} \to \mathcal{M}$ and $K : \mathbb{A} \to \mathrm{s}\mathcal{S}\mathrm{et}$ be given diagrams. Then for $n \geq 0$ define

$$C^n(K, \mathbb{A}, X) \coloneqq \prod_{\sigma \in \mathbb{A}_n} X_{\sigma_n}^{K_{\sigma_0}}\,.$$

This gives a cosimplicial object $\mathbf{\Delta} \to \mathcal{M}$ as follows. Let $[n] \xrightarrow{f} [m]$ in $\mathbf{\Delta}$ be given. Then f_* is defined on the factor indexed by $\sigma \in \mathbb{A}_m$ by the following diagram

[3] This is well-know and goes back to at least [Yon60]. See e.g. [Lor15] for a fun overview of facts of this sort.

$$\begin{tikzcd} C^n(K,\mathbb{A},X) \arrow[r, "f_*"] \arrow[d, "\pi_{\sigma f}"] & C^m(K,\mathbb{A},X) \arrow[d, "\pi_\sigma"] \\ X_{\sigma f_n}^{K_{\sigma f_0}} \arrow[r, "(\sigma f_n \to \sigma_m)_*^{(\sigma_0 \to \sigma f_0)_*}"] & X_{\sigma_m}^{K_{\sigma_0}} \end{tikzcd}$$

We call $C^\bullet(K,\mathbb{A},X)$ the *two-sided cobar construction.* The *cobar construction* $C(K,\mathbb{A},X)$ is the totalization of $C^\bullet(K,\mathbb{A},X)$.

Definition 1.2.23 (Cosimplicial Replacement)**.** Let $X : \mathbb{A} \to \mathcal{M}$ be a diagram in $\mathcal{M}$. Then $\Pi^\bullet X \coloneqq C^\bullet(*,\mathbb{A},X)$ is called the *cosimplicial replacement* of X.

Recall that $\mathcal{M}$ comes with a fibrant replacement functor R, which naturally associates a weak equivalence $X \to RX$ with RX fibrant to any object X in $\mathcal{M}$. We assume R is the identity on fibrant objects. We again have an induced endofunctor R on $\mathcal{M}^{\mathbb{A}}$ by composition.

Theorem 1.2.24 (Homotopy Limit)**.** *For $X : \mathbb{A} \to \mathcal{M}$ we can calculate the homotopy limit of X as*

$$\operatorname{holim} X = C(*,\mathbb{A},RX) = \operatorname{Tot}(\Pi^\bullet RX)\,.$$

As in the case of homotopy colimits, the proof involves a right deformation for lim, which we report here for later reference. So let X be a given $\mathbb{A}$-diagram in $\mathcal{M}$. Then we have a functor

$$C(\mathbb{A},\mathbb{A},X) : \mathbb{A} \to \mathcal{M} : a \mapsto C(\mathbf{y}^a,\mathbb{A},X)$$

that depends functorially on X. Hence this gives an endofunctor $C(\mathbb{A},\mathbb{A},-)$ on $\mathcal{M}^{\mathbb{A}}$. We can again construct a natural transformation $\eta : \mathrm{id}_{\mathcal{M}^{\mathbb{A}}} \Rightarrow C(\mathbb{A},\mathbb{A},-)$ as follows. First we use the map $\Delta \to *$ to get maps

$$\hom^{\mathbf{\Delta}}(*, C^\bullet(\mathbf{y}^a,\mathbb{A},X)) \to \hom^{\mathbf{\Delta}}(\Delta, C^\bullet(\mathbf{y}^a,\mathbb{A},X)) = C(\mathbf{y}^a,\mathbb{A},X)\,,$$

which are natural in a. Then one observes

$$\hom^{\mathbf{\Delta}}(*, C^\bullet(\mathbf{y}^a,\mathbb{A},X)) \cong \lim_{\mathbf{\Delta}} C^\bullet(\mathbf{y}^a,\mathbb{A},X) \cong \hom^{\mathbb{A}}(\mathbf{y}^a,X) \cong X_a$$

holds, using $\hom^{\mathbb{B}}(*,-) \cong \lim_{\mathbb{B}}$ for any $\mathbb{B}$ for the first isomorphism, the dual version of Exm. 1.2.15 for the second one and the natural isomorphism between $\mathcal{M}(Z, \hom^{\mathbb{A}}(\mathbf{y}^a, X_\bullet))$ and $\mathrm{sSet}^{\mathbb{A}}(\mathbf{y}^a, \operatorname{Map}(Z, X_\bullet))$ for all objects Z in $\mathcal{M}$ from [Hir14, Prop. 7.11] for the third. Hence this gives the η that we wanted.

Then one shows, with the natural transformation $r : \mathrm{id}_{\mathcal{M}^{\mathbb{A}}} \Rightarrow R$ induced by our fibrant replacement on $\mathcal{M}$, that the composition

$$\mathrm{id}_{\mathcal{M}^{\mathbb{A}}} \overset{r}{\Rightarrow} R \overset{\eta_R}{\Longrightarrow} C(\mathbb{A},\mathbb{A},R-) \tag{1.4}$$

is a right deformation for $\lim_{\mathbb{A}}$, and finally that $C(*,\mathbb{A},RX)$ is the limit of the $\mathbb{A}$-diagram $C(\mathbb{A},\mathbb{A},RX)$ (again see [Rie14, Thm. 5.1.1] for details).

By similar arguments as before, we can explicitly calculate the homotopy limit of $X \in \mathcal{M}^{\mathbb{A}}$ as the equalizer of the diagram

$$\prod_{\substack{k\geq 0\\ \sigma\in\mathbb{A}_k}} Y_{\sigma_k}^{\Delta[k]} \overset{\varphi}{\underset{\psi}{\rightrightarrows}} \prod_{\substack{[n]\to[m]\\ \tau\in\mathbb{A}_m}} Y_{\tau_m}^{\Delta[n]}$$

where $Y = RX$ and for given $f : [n] \to [m]$ in $\mathbf{\Delta}$ and $\tau \in \mathbb{A}_m$ the map φ on the factor $Y_{\tau_m}^{\Delta[n]}$ is given by the composition

$$\prod_{\substack{k\geq 0\\ \sigma\in\mathbb{A}_k}} Y_{\sigma_k}^{\Delta[k]} \xrightarrow{\pi_{\tau f}} Y_{\tau f_n}^{\Delta[n]} \xrightarrow{(\tau f_n\to\tau_m)_*^{\mathrm{id}_{\Delta[n]}}} Y_{\tau_m}^{\Delta[n]} ,$$

while ψ is given on the same factor by the composition

$$\prod_{\substack{k\geq 0\\ \sigma\in\mathbb{A}_k}} Y_{\sigma_k}^{\Delta[k]} \xrightarrow{\pi_{\tau}} Y_{\tau_m}^{\Delta[m]} \xrightarrow{\mathrm{id}_{Y_{\tau_m}}^{f_*}} Y_{\tau_m}^{\Delta[n]} .$$

Remark 1.2.25. As in the case of homotopy colimits, one has an adjunction

$$\mathcal{M}(Z, \hom^{\mathbb{A}}(K_\bullet, X_\bullet)) \simeq \mathrm{s}\mathcal{S}\mathrm{et}^{\mathbb{A}}(K_\bullet, \mathrm{Map}(Z, X_\bullet)) ,$$

where Z is an object in $\mathcal{M}$ and X, K are $\mathbb{A}$-shaped diagram in $\mathcal{M}$ and in $\mathrm{s}\mathcal{S}\mathrm{et}$ respectively. Also, one can show that $\mathrm{Tot}(\Pi^\bullet RX)$ is isomorphic to $\hom^{\mathbb{A}}(N(\mathbb{A}/-), RX)$ (see e.g. [Hir14, Prop. 7.11, Def. 12.2, Thm. 12.5]). Hence, ignoring R for the moment, one sees that the above constructed $\mathrm{holim}_{\mathbb{A}}$ is right adjoint to the functor that sends an object Z in $\mathcal{M}$ to the diagram $Z \otimes N(\mathbb{A}/-)$. This again gives us a homotopical analogy to the adjunction of $\lim_{\mathbb{A}}$ with the diagonal functor.

1.3 The case of topological spaces

Let us now turn to the concrete setting of our simplicial model category $\mathcal{T}\mathrm{op}$ of nice topological spaces. In the following, let $\mathbb{A}$ be a given category. Observe, since all objects in $\mathcal{T}\mathrm{op}$ are fibrant, the fibrant replacement functor on $\mathcal{T}\mathrm{op}$ is the identity, and hence the functors $\mathrm{holim}_{\mathbb{A}}$ and $\mathrm{Tot}\,\Pi^\bullet(-)$ from $\mathcal{T}\mathrm{op}^{\mathbb{A}}$ to $\mathcal{T}\mathrm{op}$ are the same.

On the other hand, one can show that also $|\amalg_\bullet -| : \mathcal{T}\mathrm{op}^{\mathbb{A}} \to \mathcal{T}\mathrm{op}$ is homotopical (see e.g. [Dug08, Rem. 4.9] and [Rie14, §14.4]). Using the cofibrant replacement functor Q, it follows that we have a natural weak equivalence

$$\mathrm{hocolim}_{\mathbb{A}} = B(*, \mathbb{A}, Q-) \Rightarrow B(*, \mathbb{A}, -) = |\amalg_\bullet -| .$$

The upshot is that we can and will use the formula $B(*, \mathbb{A}, -)$ in stead of $B(*, \mathbb{A}, Q-)$ for computation of the homotopy colimit in $\mathcal{T}\mathrm{op}$, which is justified by Exm. 1.1.9.

Let $X : \mathbb{A} \to \mathcal{T}\mathrm{op}$ be given. Write $\mathbb{A}'_n$ for all nondegenerate simplices of $\mathbb{A}_n$. Then $\mathrm{holim}\, X$ is isomorphic to the subspace of those points

$$(x_\sigma : \Delta^n \to X_{\sigma_n})_{n\geq 0, \sigma\in\mathbb{A}'_n} \in \prod_{n\geq 0, \sigma\in\mathbb{A}'_n} X_{\sigma_n}^{\Delta[n]}$$

such that for all $h : [m] \to [n]$ and $\sigma \in \mathbb{A}'_n$ with $\sigma h \in \mathbb{A}'_m$, the following diagram commutes:

$$\begin{tikzcd} \Delta^m \arrow[r, "h_*"] \arrow[d, "x_{\sigma h}"] & \Delta^n \arrow[d, "x_h"] \\ X_{\sigma h_m} \arrow[r, "(\sigma h_m \to h_n)_*"] & X_{\sigma_n} \end{tikzcd}$$

To show this, one observes that for a given $\sigma \in \mathbb{A}_m$ there is a unique $n \geq 0$, a $\tau \in \mathbb{A}_n$, and a degeneracy $g : [m] \to [n]$, such that $\tau g = \sigma$. This is a direct application of the Eilenberg-Zilber lemma (see e.g. [Hir14, Prop. 3.12]). Now observe that $\tau g_m \to \tau_n$ must be the identity in $\mathbb{A}$, hence a possible coefficient $x_\sigma : \Delta[m] \to X_{\sigma_m}$ of a point in $\operatorname{holim} X$ is completely determined by the coefficient $x_\tau : \Delta[n] \to X_{\tau_n}$.

Example 1.3.1. To appreciate the point of the remark above, let us compute the homotopy limit of a diagram $X \xrightarrow{f} Y \xleftarrow{g} Z$. Let $\mathbb{A}$ be the underlying category $a_0 \to a_1 \leftarrow a_2$. Note $\mathbb{A}'_1$ are the simplices $a_0 \to a_1, a_2 \to a_1$, and there are no higher nondegenerate simplices. Hence the homotopy limit of our diagram consists of those points (x, y, z, η, μ) in $X \times Y \times Z \times Y^{\Delta^1} \times Y^{\Delta^1}$ such that η is a path from $f(x)$ to y, and μ a path from $g(z)$ to y.

We can even do a little bit better. Let P be the space of those points (x, z, γ) in $X \times Z \times Y^{\Delta^1}$ such that γ is a path from $f(x)$ to $g(z)$. Then consider the map

$$\varphi : P \to \operatorname{holim}(X \xrightarrow{f} Y \xleftarrow{g} Z) : (x, z, \gamma) \mapsto (x, \gamma(1/2), z, \gamma^+, \gamma^-)\,,$$

where γ^+ is half of the path γ, connecting $f(x)$ to $\gamma(1/2)$, while γ^- is the other half, in the opposite direction, connecting $g(z)$ to $\gamma(1/2)$. Now note that this map is an isomorphism: it is clearly surjective, and an inverse is given by concatenation of γ^+ with γ^- for a given point $(x, \gamma(1/2), z, \gamma^+, \gamma^-)$ in $\operatorname{holim}(X \to Y \leftarrow Z)$. Since φ is natural, we identify the latter space with P.

Similarly we have that

$$\operatorname{hocolim} X \cong \coprod_{n \geq 0, \sigma \in \mathbb{A}_n} X_{\sigma_0} \times \Delta^n / \sim\,,$$

where $\sim$ is the equivalence relation generated by the rule

$$X_{\sigma_0} \times \Delta^n \ni (x, f_* t) \sim ((\sigma_0 \to \sigma f_0)_* x, t) \in X_{\sigma f_0} \times \Delta^m\,,$$

for all $f : [m] \to [n]$ in $\mathbf{\Delta}$, $\sigma \in \mathbb{A}_n$ and all $(x, t) \in X_{\sigma_0} \times \Delta^m$. And by a similar argument as for the homotopy limit, the quotient space above is the image of the natural projection out of $\coprod_{n \geq 0, \sigma \in \mathbb{A}'_n} X_{\sigma_0} \times \Delta^n$.

1.4 Simplicial stuff

Let us turn now to the simplicial model category s$\mathcal{S}$et of simplicial sets. In this case, we have a particularly nice description of homotopy colimits. Indeed, for a given diagram

$X : \mathbb{A} \to \mathrm{s}\mathcal{S}\mathrm{et}$, define the simplicial set $H_\bullet X$ as

$$H_n X := \coprod_{\sigma \in \mathbb{A}_n} X_{\sigma_0, n}\,,$$

for $n \geq 0$, with obvious structure maps. I claim this is a coequalizer of the diagram in (1.3), where it holds that $Y = QX = X$ since all objects are cofibrant in $\mathrm{s}\mathcal{S}\mathrm{et}$. To this end, consider the following diagram

$$\coprod_{\substack{[n]\leftarrow[m] \\ \sigma\in\mathbb{A}_n}} X_{\sigma_0} \otimes \Delta[m] \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \coprod_{\substack{k\geq 0 \\ \tau\in\mathbb{A}_k}} X_{\tau_0} \otimes \Delta[k] \xrightarrow{\mu} H_\bullet X\,, \tag{1.5}$$

where φ, ψ are defined as in (1.3), and μ is the map defined as follows. It sends an r-simplex $(x, \theta : [r] \to [k])$ from the summand $X_{\tau_0} \times \Delta[k]$ indexed by $\tau \in \mathbb{A}_k$ to $(\tau_0 \to \tau\theta_0)_*(x)$ in $X_{\tau\theta_0, r}$, which is a summand of $H_r X$ indexed by $\tau\theta$.

It is straightforward to check that μ is a simplicial map: for $f : [r] \to [s]$ in $\mathbf{\Delta}$ and an s-simplex $(x, \theta : [s] \to [k])$ in $X_{\tau_0} \times \Delta[k]$ it holds that $\mu_r f^*(x, \theta)$ is equal to

$$\begin{aligned}
\mu_r(f^*(x), \theta f) &= (\tau_0 \to \tau\theta f_0)_*(f^*(x)) \\
&= (\tau\theta_0 \to \tau\theta f_0)_*(\tau_0 \to \tau\theta_0)_*(f^*(x)) \\
&= (\tau\theta_0 \to \tau\theta f_0)_*(f^*((\tau_0 \to \tau\theta_0)_*(x))) = f^*\mu_s(x, \theta)\,.
\end{aligned}$$

To see that $\mu\varphi$ is equal to $\mu\psi$, let $(x, \theta : [r] \to [m])$ be an r-simplex of the summand $X_{\sigma_0} \otimes \Delta[m]$, indexed by $g : [m] \to [n]$ and by $\sigma \in \mathbb{A}_n$. Then we indeed have that

$$\begin{aligned}
\mu\varphi(x, \theta) = \mu(x, g\theta) &= (\sigma_0 \to \sigma g\theta_0)_*(x) \\
&= (\sigma g_0 \to \sigma g\theta_0)_*((\sigma_0 \to \sigma g_0)_*(x)) = \mu\psi(x, \theta)\,.
\end{aligned}$$

Proposition 1.4.1. *The above map μ is a coequalizer of the diagram (1.3). Consequently, the simplicial set $H_\bullet(X)$ is a homotopy colimit of X.*

Proof. Let α be a simplicial map $\coprod_{k\geq 0, \tau\in\mathbb{A}_k} X_{\tau_0} \otimes \Delta[k] \to Z$ such that $\alpha\varphi = \alpha\psi$. Then we construct a map $\beta : H_\bullet X \to Z$ as follows. Let $x \in X_{\sigma_0, r}$ for some $\sigma \in \mathbb{A}_r$ be given. Then it holds that $\mu(x, \mathrm{id}_{[r]}) = x$, so we can put $\beta(x) := \alpha(x, \mathrm{id}_{[r]})$, and immediately see that if $\beta\mu$ is indeed α, then β is unique with the property.

Let us show that $\beta\mu = \alpha$. So let $(x, \theta : [r] \to [k]) \in X_{\tau_0} \times \Delta[k]$ be given. Then observe that $(x, \mathrm{id}_{[r]})$ is an element of the summand $X_{\tau_0} \otimes \Delta[r]$, indexed by $\tau \in \mathbb{A}_k$ and $\theta : [r] \to [k]$, from the leftmost coproduct in (1.5). It therefore holds that

$$\begin{aligned}
\beta\mu(x, \theta) &= \alpha((\tau_0 \to \tau\theta_0)_*(x), \mathrm{id}_{[r]}); \\
&= \alpha\psi(x, \mathrm{id}_{[r]}) = \alpha\varphi(x, \mathrm{id}_{[r]}) = \alpha(x, \theta)\,,
\end{aligned}$$

by the fact that $\alpha\varphi = \alpha\psi$. It is again straightforward to check that β is a simplicial map, which establishes the claim. $\square$

For a given diagram $X : \mathbb{A} \to \mathrm{sSet}$, write $BE_{\bullet,\bullet}X$ for the bisimplicial set that sends $([m],[n])$ to $\coprod_{\sigma\in\mathbb{A}_m} X_{\sigma_0,n}$. Observe that $H_\bullet$ is the diagonal of this $BE_{\bullet,\bullet}X$. In the light of this and of the previous proposition, the following lemma establishes some nice homotopical behavior of taking the diagonal.

Lemma 1.4.2. *If $f : Y \to Z$ is a map of bisimplicial sets such that for all $n \geq 0$ the maps $f_n : Y_{\bullet,n} \to Z_{\bullet,n}$ are weak equivalences, then the induced map $d(Y) \to d(Z)$ is a weak equivalence as well.*

Proof. See [GJ09, Prop. IV.1.7]. □

Example 1.4.3. Let X be a simplicial set. Then the homotopy colimit of the functor $\pi : \Delta/X \to \mathrm{sSet}$ that sends $\Delta[p] \to X$ to $\Delta[p]$ is weakly equivalent to X.

To see this, consider the bisimplicial set $BE\pi$ as above. Also, let Y be the bisimplicial set that sends $([p],[q])$ to X_q. We construct a map $f : BE\pi \to Y$ as follows. A typical (m,n)-simplex of $BE\pi$ is a datum $(\Delta[\sigma_0] \to \cdots \to \Delta[\sigma_m] \to X, \Delta[n] \to \Delta[\sigma_0])$. By composition this gives an n simplex $\Delta[n] \to X$ of X, i.e. an element of Y_{mn}.

Now let us show that the induced maps $f_n : BE_{\bullet,n}\pi \to Y_{\bullet,n}$ are weak equivalences. Note that $Y_{\bullet,n}$ is just the constant simplicial set X_n. As such, it is the nerve of the discrete category $\mathbb{X}$ that has X_n as objects. On the other hand, $BE_{\bullet,n}\pi$ is the nerve of the category $\mathbb{B}$ that has maps of the form $\Delta[n] \to \Delta[s] \to X$ as objects; a morphisms from $\Delta[n] \to \Delta[s] \to X$ to $\Delta[n] \to \Delta[t] \to X$ in $\mathbb{B}$ is a map $\Delta[s] \to \Delta[t]$ which makes the obvious diagram commutative. What is more, the map f_n is induced from the functor $f'_n : \mathbb{B} \to \mathbb{X}$ that sends $\Delta[n] \to \Delta[s] \to X$ to the composition $\Delta[n] \to X$.

Now we are in a position to apply Quillen's theorem A ([Qui73]), which states that a functor $g : \mathbb{C} \to \mathbb{C}'$ such that each g/Y is contractible for every Y of $\mathbb{C}'$ induces a homotopy equivalence on the classifying spaces, hence certainly a weak equivalence on the nerves. So let $\sigma : \Delta[n] \to X$ be an object of the discrete category $\mathbb{X}$. Then f'_n/σ is the category with objects those $\Delta[n] \to \Delta[s] \to X$ which compose into σ. But note that $\Delta[n] \xrightarrow{\mathrm{id}} \Delta[n] \xrightarrow{\sigma} X$ is an initial object in this latter category, which implies it is indeed contractible.

From the above it follows that the maps f_n are weak equivalences. By the previous proposition, this implies that the map on the diagonals $d(BE\pi) \to d(Y)$ induced by f is a weak equivalence. But we recognize $d(BE\pi)$ as the homotopy colimit of π, and $d(Y)$ as just X itself, which is what we wanted.

Example 1.4.4. Let again X be a simplicial set. We are going to show that also $N(\Delta/X) \simeq X$ holds. To this end, let $\pi : \Delta/X \to \mathrm{sSet}$ and $BE\pi$ be as before. Let also Z be the bisimplicial set that sends $([m],[n])$ to $(\Delta/X)_m$.

Consider the obvious map $f : BE\pi \to Z$ of bisimplicial set. Note that for fixed m, the induced map

$$f_m : BE_{m,\bullet}\pi \to Z_{m,\bullet}$$

is the map $\coprod_{\sigma\in(\Delta/X)_m} \Delta[\sigma_0] \to \coprod_{(\Delta/X)_m} \Delta[0]$ that is given by projection on each summand. But this map is a disjoint union of weak equivalences, hence itself a weak

equivalence. It follows from Lem. 1.4.2 and from the previous example that f induces a weak equivalence $X \simeq d(BE\pi) \to d(Z) = N(\Delta/X)$.

One can also give an explicit natural weak equivalence $N(\Delta/X) \to X$, by using barycentric subdivision, as done in e.g. [Lat77]. The construction is as follows. Let a k-simplex $\sigma := \Delta[\sigma_0] \to \cdots \to \Delta[\sigma_k] \to X$ of $N(\Delta/X)$ be given. This induces maps $\varphi_i : [\sigma_i] \to [\sigma_k]$. Then send σ to the k-simplex $\Delta[k] \to \Delta[\sigma_k] \to X$ of X, where the first arrow is induced by sending $i \in [k]$ to $\varphi_i(\sigma_i)$.

1.4.a Acyclic diagrams

Let $D_\bullet = (D_b)_{b\in\mathbb{B}}$ be a diagram of simplicial sets with colimit $D \in s\mathcal{S}\mathrm{et}$. For $n \geq 0$ and $\sigma \in D_n$, let B_σ be the category that has

- As objects pairs (b, τ), with $b \in \mathbb{B}$ and $\tau \in D_b(n)$, such that $\tau \mapsto \sigma$ under the canonical map $D_b \to D$;
- As morphisms $(b, \tau) \to (b', \tau')$ arrows $b \to b'$ in $\mathbb{B}$, such that $\tau \mapsto \tau'$ under the induced map $D_b \to D_{b'}$.

In diagrams, a morphism $(b, \tau) \to (b', \tau')$ in B_σ is given by a $\gamma : b \to b'$ which makes the following diagram commute

$$\begin{tikzcd}
 & D & \\
 & \Delta[n] \arrow[u, dashed, "\sigma"] & \\
D_b \arrow[ur, "\iota_b"] \arrow[rr, "\gamma_*"] & & D_{b'} \arrow[ul, "\iota_{b'}"'] \\
 & \Delta[n] \arrow[ul, dashed, "\tau"] \arrow[ur, dashed, "\tau'"'] &
\end{tikzcd}$$

For $n \geq 0$, consider $D_\bullet(n) = (D_b(n))_{b\in\mathbb{B}}$ as a diagram $\mathbb{B} \to \mathcal{T}\mathrm{op}$ of discrete spaces.

Definition 1.4.5. We call $D_\bullet$ *acyclic* if for all $n \geq 0$ the natural map in $\mathcal{T}\mathrm{op}$

$$\mathrm{hocolim}_{b\in\mathbb{B}} D_b(n) \to \mathrm{colim}_{b\in\mathbb{B}} D_b(n)$$

is a weak equivalence.

Proposition 1.4.6. *The diagram $D_\bullet$ is acyclic iff for all $n \geq 0$ and $\sigma \in D_n$ the category B_σ is weakly contractible.*

Proof. Observe that D_n equals $\mathrm{colim}_{b\in\mathbb{B}} D_b(n)$ (as discrete spaces), since colimits of diagrams are calculated pointwise. Write π for the map $\mathrm{hocolim}_{b\in\mathbb{B}} D_b(n) \to D_n$, and let $\sigma \in D_n$ be given. Let us first show that $\pi^{-1}\sigma$ is the classifying space of B_σ.

Note that an m-simplex $(b_0, \tau_0) \to \cdots \to (b_m, \tau_m)$ in NB_σ is completely determined by the data $\tau_0 \in D_{b_0}(n)$ and $\epsilon = b_0 \to \cdots \to b_m \in \mathbb{B}_m$. It follows that $N_m B_\sigma$ is the subset of those points $\tau \in B_m(*, \mathbb{B}, D_\bullet(n))$ such that, if τ is in the summand $D_{\epsilon_0}(n)$ indexed by

$\epsilon \in \mathbb{B}_m$, then it is sent to σ under the map $D_{\epsilon_0} \to D$. It follows that $|NB_\sigma|$ is a subspace of the geometric realization of the simplicial object $B_\bullet(*, \mathbb{B}, D_\bullet(n))$ in $\mathcal{T}op$. More precisely, it is the subspace of those points in $\operatorname{hocolim}_{b\in\mathbb{B}} D_b(n)$ that can be written as $[\epsilon, \tau, t]$, with $\epsilon \in \mathbb{B}_m, \tau \in D_{\epsilon_0}(n)$ and $t \in \Delta^m$, such that (ϵ_0, τ) is an object of B_σ. It is straightforward to show that this is indeed the fiber of π above σ.

Now since each $D_b(n)$ is discrete, so is its colimit D_n. It follows that π is a discrete sum of maps $\pi^{-1}\sigma \to \{\sigma\}$ for $\sigma \in D_n$. Hence π is a weak equivalence iff all of these maps $\pi^{-1}\sigma \to \{\sigma\}$ are iff each $\pi^{-1}\sigma = |NB_\sigma|$ is weakly contractible, which was to be shown. □

1.5 Projective 2-limits of categories

Let $\mathcal{C}at$ be the category of small categories. It is straightforward to check that $\mathcal{C}at$ is a homotopical category, which has equivalences of categories as its weak equivalences. The goal of this section is to give an explicit description of homotopy limits in $\mathcal{C}at$. This can be done by means of a simplicial model structure on $\mathcal{C}at$, as in [Rez00]. We give however a direct approach, using only the homotopical structure on $\mathcal{C}at$, which is a cathartic exercise in homotopy limits as derived functors. These homotopy limits will be used in our study of the $\mathcal{S}_\bullet$-construction on proto-exact categories in Chap. 2.

Notation 1.5.1. When convenient we write a family $(x_a)_{a\in\mathbb{A}}$ simply as (x_a).

Definition 1.5.2. Let $\mathcal{C} = (\mathcal{C}_a)_{a\in\mathbb{A}}$ be a diagram in $\mathcal{C}at$. Then define its *projective 2-limit*, notation $2\lim \mathcal{C}$, as the following category. An object in $2\lim \mathcal{C}$ is a datum $(y_a, y_u)_{a\in\mathbb{A}_0, u\in\mathbb{A}_1}$, where y_a is an object of $\mathcal{C}_a$ and $y_u : u_*(y_a) \to y_b$ is an isomorphism in $\mathcal{C}_b$ for $u : a \to b$ in $\mathbb{A}$, subject to the compatibility condition that for $a \xrightarrow{u} b \xrightarrow{v} c$ we have $y_{vu} = y_v \circ v_*(y_u)$. A morphism $(y_a, y_u) \to (z_a, z_u)$ between two such objects is a family of morphisms $(f_a : y_a \to z_a)_{a\in\mathbb{A}_0}$ that commute with the y_u and the z_u's.

Example 1.5.3. Let $\mathcal{D} \xrightarrow{f} \mathcal{E} \xleftarrow{g} \mathcal{C}$ be a diagram of categories. Then it is not hard to show that the projective 2-limit is naturally equivalent to the category which has as objects triples (c, d, σ), where c and d are objects of $\mathcal{C}$ and $\mathcal{D}$ respectively and σ is an isomorphism $f(d) \cong g(c)$ in $\mathcal{E}$. Morphisms between such triples are the obvious ones. This category is called the *2-fiber product*.

Observe that an object of the ordinary limit of $(\mathcal{C}_a)_{a\in\mathbb{A}}$ is a datum $(y_a)_{a\in\mathbb{A}}$, where each y_a is an object of $\mathcal{C}_a$, such that $u_*(y_a) = y_b$ holds for all $u : a \to b$ in $\mathbb{A}$. A morphism $(y_a) \to (z_a)$ between such objects is a family of morphisms $(f_a : y_a \to z_a)_{a\in\mathbb{A}}$ such that $u_*(f_a) = f_b$ holds for all $u : a \to b$ in $\mathbb{A}$. This gives us an obvious functor $\lim \mathcal{C} \to 2\lim \mathcal{C}$, which extends to a natural transformation $\rho : \lim \to 2\lim$. I claim this ρ is a right derived functor of $\lim : \mathcal{C}at^{\mathbb{A}} \to \mathcal{C}at$.

Let us first show that the functor $2\lim$ is homotopical. To this end, let a natural transformation $(\varphi_a : \mathcal{C}_a \to \mathcal{C}'_a)_{a\in\mathbb{A}}$ of diagrams be given that is pointwise an equivalence of categories. We define a functor φ from $2\lim \mathcal{C}$ to $2\lim \mathcal{C}'$ as follows. Let (y_a, y_u) be a

given object of $2\lim \mathcal{C}$. Then simply put

$$\varphi(y_a, y_u) := (\varphi_a(y_a), \varphi_b(y_{u:a\to b}))\,.$$

To see this is well-defined, observe that for $u : a \to b$ it holds $\varphi_b(y_u)$ is a morphism

$$u_*(\varphi_a(y_a)) = \varphi_b(u_*(y_a)) \to \varphi_b(y_b)\,,$$

by naturality of (φ_a). Furthermore, for $a \xrightarrow{u} b \xrightarrow{v} c$ it holds that

$$\varphi_c(y_{vu}) = \varphi_c(y_v \circ v_*(y_u)) = \varphi_c(y_v) \circ v_*(\varphi_b(y_u))\,.$$

Next let $f = (f_a : y_a \to z_a)$ be a morphism $(y_a, y_u) \to (z_a, z_u)$ in $2\lim \mathcal{C}$. Define the morphism $\varphi(f) : \varphi(y_a, y_u) \to \varphi(z_a, z_u)$ in $2\lim \mathcal{C}'$ as $(\varphi_a(f_a))_{a\in\mathbb{A}}$, which is a family of morphisms $\varphi_a(y_a) \to \varphi_a(z_a)$. Again by the naturality of (φ_a), it is straightforward to show that this yields a morphism in $2\lim \mathcal{C}'$.

Lemma 1.5.4. *The functor $\varphi : 2\lim \mathcal{C} \to 2\lim \mathcal{C}'$ as constructed in the above is an equivalence of categories.*

Proof. It is clear that φ is injective on hom-sets. Now let (y_a, y_u) and (z_a, z_u) be objects of $2\lim \mathcal{C}$, and (g_a) a morphism $\varphi(y_a, y_u) \to \varphi(z_a, z_u)$ in $2\lim \mathcal{C}'$. Using the fact that each φ_a is an equivalence, take unique $f_a : y_a \to z_a$ such that $\varphi_a(f_a) = g_a$ holds for all $a \in \mathbb{A}$. To see these (f_a) form a morphism $(y_a, y_u) \to (z_a, z_u)$ in $2\lim \mathcal{C}$, we need to check that $f_b \circ y_u = z_u \circ u_*(f_a)$ holds for all $u : a \to b$. It suffices to check this equality after applying φ_b, and the latter indeed holds by naturality of (φ_a) and the fact that (g_a) is a $2\lim \mathcal{C}'$-morphism.

To see that φ is essentially surjective, let (x_a, x_u) be a given object of $2\lim \mathcal{C}'$. By using that all the φ_a are essentially surjective, take objects $y_a \in \mathcal{C}_a$ and isomorphisms $\sigma_a : \varphi_a(y_a) \to x_a$, one for each $a \in \mathbb{A}$. Now to give a map $y_u : u_*(y_a) \to y_b$ for some $u : a \to b$, observe that by fully faithfulness it suffices to give a map $\varphi_b(u_*(y_a)) \to \varphi_b(y_b)$. For this latter map we take the composition

$$\varphi_b(u_*(y_a)) = u_*(\varphi_a(y_a)) \xrightarrow{u_*(\sigma_a)} u_*(x_a) \xrightarrow{x_u} x_b \xrightarrow{\sigma_b^{-1}} \varphi_b(y_b)\,.$$

By a simple diagram chase, using naturality of (φ_a) and the equality $x_{vu} = x_v \circ v_*(x_u)$, one shows that $\varphi_c(y_{vu}) = \varphi_c(y_v \circ v_*(y_u))$ holds. By fully faithfulness of φ_c, this suffices to show that (y_a, y_u) is an object of $2\lim \mathcal{C}$.

Now by construction, $(\sigma_a)_{a\in\mathbb{A}}$ is a morphism $\varphi(y_a, y_u) \to (x_a, x_u)$, which is furthermore clearly an isomorphism. This implies that φ is an equivalence of categories, which was to be shown. $\square$

Let $\mathcal{C}$ be a diagram in $\mathcal{C}\mathrm{at}$. Then its 2-limit is the same as the ordinary limit of an associated diagram $(R\mathcal{C}_a)_{a\in\mathbb{A}}$. To see this, for $a \in \mathbb{A}$ define the category $R\mathcal{C}_a$ as follows. As objects take families $(y_a, y_u)_{u:a\to b}$, where $y_a \in \mathcal{C}_a$ is fixed and y_u is an isomorphism $u_*(y_a) \to y_b$ in $\mathcal{C}_b$ for a certain y_b, one for each $u : a \to b$. We demand that such an y_u is

the identity whenever u is. A morphism from $(y_a, y_u)_{u:a\to b}$ to $(z_a, z_u)_{u:a\to b}$ in $R\mathcal{C}_a$ is just a morphism $f_a : y_a \to z_a$ in $\mathcal{C}_a$.

To make this into a diagram, let $w : a \to a'$ be given. Then define the functor $R\mathcal{C}(w) : R\mathcal{C}_a \to R\mathcal{C}_{a'}$ as follows. Let $(y_a, y_u)_{u:a\to b}$ be a given object of $R\mathcal{C}_a$. Let $y_{a'} \in \mathcal{C}_{a'}$ be the target of the isomorphism $y_w : w_*(y_a) \to y_{a'}$. For a morphism $u' : a' \to b$, define the morphism $y_{u'} : u'_*(y_{a'}) \to y_b$ as the composition

$$u'_*(y_{a'}) \xrightarrow{u'_*(y_w^{-1})} u'_* w_*(y_a) \xrightarrow{y_{u'w}} y_b\,.$$

Then the family $(y_{a'}, y_{u'})_{u':a'\to b}$ is the object of $R\mathcal{C}_{a'}$ whereto we send $(y_a, y_u)_{u:a\to b}$ under $R\mathcal{C}(w)$. The action of $R\mathcal{C}(w)$ on morphisms is straightforward. It is a fun exercise in abstract nonsense to show that this indeed makes $(R\mathcal{C}_a)_{a\in\mathbb{A}}$ into a diagram.

Lemma 1.5.5. *With $(\mathcal{C}_a)_{a\in\mathbb{A}}$ and $(R\mathcal{C}_a)_{a\in\mathbb{A}}$ as above, it holds that $\lim R\mathcal{C}$ is the 2-limit of $(\mathcal{C}_a)_{a\in\mathbb{A}}$.*

Proof. Observe that $\lim R\mathcal{C}$ has as objects families of families $((y_a, y_u)_{u:a\to b})_{a\in\mathbb{A}}$, i.e. for each $a \in \mathbb{A}$ we take one y_a, and then for each $u : a \to b$ we take an y_u. This is the same as just a family $(y_a, y_u)_{a\in\mathbb{A},u:a\to b}$. Since we are taking the limit, such a family must be subject to the condition that for all $w : a \to a'$ it holds $R\mathcal{C}(w)(y_a, y_u)_{u:a\to b}$ is equal to $(y_{a'}, y_{u'})_{u':a'\to b}$, which precisely means that for all $a \xrightarrow{w} a' \xrightarrow{u'} b$ it holds that $y_{u'} \circ u'_*(y_w) = y_{u'w}$. This implies that the induced family $(y_a, y_u)_{a\in\mathbb{A},u\in\mathbb{A}_1}$ is indeed an object of $2\lim\mathcal{C}$.

Now a morphism from $((y_a, y_u)_{u:a\to b})_{a\in\mathbb{A}}$ to $((x_a, x_u)_{u:a\to b})_{a\in\mathbb{A}}$ in $\lim R\mathcal{C}$ is given by a family of morphisms $f_a : y_a \to x_a$ such that $R\mathcal{C}(w)(f_a) = f_{a'}$ holds for all $w : a \to a'$. This exactly means that it is a family of morphisms $f_a : y_a \to x_a$ that commute with the y_u and the x_u, which is what we wanted. □

The rule that assigns $(R\mathcal{C}_a)_{a\in\mathbb{A}}$ to a given diagram $(\mathcal{C}_a)_{a\in\mathbb{A}}$ gives us a functor R on $\mathcal{C}\mathrm{at}^{\mathbb{A}}$ in the obvious way.

Proposition 1.5.6. *The projective 2-limit is a homotopy limit in $\mathcal{C}\mathrm{at}$.*

Proof. Let $\mathcal{C}_R^{\mathbb{A}}$ be the essential image of the functor R, i.e. the smallest subcategory of $\mathcal{C}^{\mathbb{A}}$ that is closed under isomorphisms and contains the image of R. Note that for a give diagram $\mathcal{C}$, we have a functor $\mathcal{C}_a \to R\mathcal{C}_a$ that sends y_a to $(y_a, \mathrm{id}_{u_*(y_a)})_{u:a\to b}$. These functors are in fact all equivalences of categories, and collect into a natural transformation $r_{\mathcal{C}} : \mathcal{C} \Rightarrow R\mathcal{C}$. It follows that $\mathcal{C}_R^{\mathbb{A}}$ is full.

By the previous lemma and by Rem. 1.1.17, we see that it suffices to show R is a right deformation for lim. The $r_{\mathcal{C}}$ give a natural weak equivalence $\mathrm{id} \Rightarrow R$. And the previous two lemmata clearly imply that $\lim_{\mathbb{A}}$ is homotopical on $\mathcal{C}_R^{\mathbb{A}}$, so we are done. □

Because homotopy fiber products are naturally equivalent to 2-fiber products in $\mathcal{C}\mathrm{at}$ (Exm. 1.5.3), from hereon we shall identify the two.

2. Segal objects

> The arrow points only in the application that a living being makes of it.
>
> *Philosophical Investigations*
> Wittgenstein

In the following, fix a homotopical category $\mathcal{C}$ with localization $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$.

2.1 Homotopy Kan extensions

Let α be a functor $\mathbb{A} \to \mathbb{B}$, and write $\alpha^* : \mathcal{C}^{\mathbb{B}} \to \mathcal{C}^{\mathbb{A}}$ for pulling back along α. For every $b \in \mathbb{B}$ we write π_b for the projection $b/\alpha \to \mathbb{A}$, when α is understood.

Recall that a right adjoint to α^*, presuming it exists, is written as α_* and is called the right Kan extension functor. It is well-known that if $\mathcal{C}$ is complete, then α_* is computed as

$$\alpha_* : \mathcal{C}^{\mathbb{A}} \to \mathcal{C}^{\mathbb{B}} : X_\bullet \mapsto \left(b \mapsto \lim(b/\alpha \xrightarrow{\pi_b} \mathbb{A} \xrightarrow{X_\bullet} \mathcal{C})\right) . \tag{2.1}$$

In the following, we use the notions of right derived functors and of right deformations as discussed in Def. 1.1.14 and in Rem. 1.1.17 respectively.

Definition 2.1.1. A right derived functor of a right Kan extension functor is called a *right homotopy Kan extension functor.*

Later on we shall see that right homotopy Kan extensions play a pivotal role in the definition of Segal objects. To have some control on these extensions, we will need $\mathcal{C}$ to be well-behaved in a certain sense. Formulating this in a precise way is the last technicality we will have to deal with before we can get to the really fascinating stuff in the next section.

Definition 2.1.2 ([DHKS04, 42.3])**.** Let $\mathcal{C}' \xrightarrow{F'} \mathcal{C}'' \xrightarrow{F''} \mathcal{C}$ be a pair of functors between homotopical categories. Then (F', F'') is called *locally right deformable* if there are right deformations $r' : \mathrm{id}_{\mathcal{C}'} \Rightarrow R'$ and $r'' : \mathrm{id}_{\mathcal{C}''} \Rightarrow R''$ for F' and for F'' respectively, such that $r' : \mathrm{id}_{\mathcal{C}'} \Rightarrow R'$ is also a right deformation for $F'' \circ F'$.

Example 2.1.3. Let $\mathcal{A}' \xrightarrow{F'} \mathcal{A}'' \xrightarrow{F''} \mathcal{A}$ be additive, left exact functors between abelian categories. Write $\mathcal{C}$ for the homotopical category $\mathrm{Ch}^+\mathcal{A}$ from Exm. 1.1.15, which has quasi-isomorphisms as weak equivalences. Similarly for $\mathcal{A}', \mathcal{A}''$. Then we have induced functors on complexes, also written as $\mathcal{C}' \xrightarrow{F'} \mathcal{C}'' \xrightarrow{F''} \mathcal{C}$. Call a complex *acyclic* when it is quasi-isomorphic to the zero complex.

Suppose there are right deformations $r' : \mathrm{id}_{\mathcal{C}'} \Rightarrow R'$ and $r'' : \mathrm{id}_{\mathcal{C}''} \Rightarrow R''$ as in the above definition. Let A be a given object of $\mathcal{A}'$, and write $I^\bullet$ for $R'(\deg_0 A)$. Then A is a subobject of I^0, as witnessed by the quasi-isomorphism $r' : \deg_0 A \to I^\bullet$. Furthermore, it can be shown that I^0 is F'-acyclic, i.e. that $R^iF(I^0) = 0$ for all $i \neq 0$, by the fact that F' is homotopical on the image of R'.

From [GM03, III.6.3, III.6.16] it follows that r' induces a class $\mathcal{A}_{R'}$ of objects *adopted to* F', i.e. a class closed under finite sums, such that any object of $\mathcal{A}'$ is a subobject of an object of $\mathcal{A}_{R'}$, and any acyclic complex with terms in $\mathcal{A}_{R'}$ is mapped under F' to an acyclic complex in $\mathcal{C}''$. With the same argument, we get such a class $\mathcal{A}_{R''} \subset \mathcal{A}''$.

Now assume $F'(\mathcal{A}_{R'}) \subset \mathcal{A}_{R''}$ holds. Because $\mathcal{A}_{R'}$ and $\mathcal{A}_{R''}$ are adopted to F' and to F'' respectively, a known result in homological algebra (e.g. [GM03, III.7.1]) says that in this case for the total derived functors (Exm. 1.1.15) of F', F'' and of $F''F'$, it holds that $R(F''F')$ is isomorphic to $RF'' \circ RF'$.

On the other hand, write $\mathcal{C}_{R'}$ and $\mathcal{C}_{R''}$ for the images of R' and R'' respectively, and suppose $F'(\mathcal{C}_{R'}) \subset \mathcal{C}_{R''}$ holds. Then this means that F', F'' are in fact *right deformable* [DHKS04, 42.3]. A general result on homotopical categories then says that any composition $\mathbb{R}F'' \circ \mathbb{R}F'$ of right derived functors of F' and F'' respectively, gives us a right derived functor of $F''F'$ [DHKS04, 42.4].

Definition 2.1.4. Call $\mathcal{C}$ *sufficiently nice* when it is complete and saturated, when for every functor $\alpha : \mathbb{A} \to \mathbb{B}$ the pair $(\alpha_*, \lim_{\mathbb{B}})$ is locally right deformable, and when furthermore it holds:

- We have a right deformation $r_\alpha : \mathrm{id}_{\mathcal{C}^{\mathbb{A}}} \Rightarrow R_\alpha$ for α_*,
- And for all $b \in \mathbb{B}$ a right deformation $r_b : \mathrm{id}_{\mathcal{C}^{b/\alpha}} \Rightarrow R_b$ for $\lim_{b/\alpha}$,
- Such that it holds $\pi_b^* R_\alpha \cong R_b \pi_b^*$, and also in the following diagram

$$\begin{tikzcd}
\mathcal{C}^{\mathbb{A}} \arrow[r, bend left=50, "\mathrm{id}"] \arrow[r, bend right=50, "R_\alpha"'] \arrow[r, phantom, "r_\alpha \Downarrow"] \arrow[d, "\pi_b^*"'] & \mathcal{C}^{\mathbb{A}} \arrow[d, "\pi_b^*"] \\
\mathcal{C}^{b/\alpha} \arrow[r, bend left=50, "\mathrm{id}"] \arrow[r, bend right=50, "R_b"'] \arrow[r, phantom, "r_b \Downarrow"] & \mathcal{C}^{b/\alpha}
\end{tikzcd} \tag{2.2}$$

  the natural transformations $\pi_b^* r_\alpha : \pi_b^* \Rightarrow \pi_b^* R_\alpha$ and $r_b \pi_b^* : \pi_b^* \Rightarrow R_b \pi_b^*$ commute with the isomorphism $\pi_b^* R_\alpha \cong R_b \pi_b^*$.

Example 2.1.5. When $\mathcal{C}$ is complete and has the minimal homotopical structure, then it is sufficiently nice.

From hereon, suppose $\mathcal{C}$ is sufficiently nice, and fix a functor $\alpha : \mathbb{A} \to \mathbb{B}$. Let $b \to b'$ be a morphism in $\mathbb{B}$. Then, by functoriality of α_*, we have a natural transformation

$$\lim_{b/\alpha} R_b \pi_b^* \cong \lim_{b/\alpha} \pi_b^* R_\alpha \Rightarrow \lim_{b'/\alpha} \pi_{b'}^* R_\alpha \cong \lim_{b'/\alpha} R_{b'} \pi_{b'}^* . \tag{2.3}$$

Recall that $\lim_{b/\alpha} r_b : \lim_{b/\alpha} \Rightarrow \lim_{b/\alpha} R_b$ is a homotopy limit. With this homotopy limit, the natural transformations (2.3) hence induce a functor

$$H : \mathcal{C}^{\mathbb{A}} \to \mathcal{C}^{\mathbb{B}} : X_\bullet \mapsto \Big(b \mapsto \operatorname{holim}(b/\alpha \xrightarrow{\pi_b} \mathbb{A} \xrightarrow{X_\bullet} \mathcal{C}) \Big) . \tag{2.4}$$

Lemma 2.1.6. *In the situation above, H is a right derived functor of α_*.*

Proof. Let $b \in \mathbb{B}$ be given and write ev_b for the functor $\mathcal{C}^{\mathbb{B}} \to \mathcal{C}$ that evaluates diagrams in b. Observe that we have

$$\mathrm{ev}_b\, \alpha_* R_\alpha = \lim_{b/\alpha} \pi_b^* R_\alpha \cong \lim_{b/\alpha} R_b \pi_b^* = \operatorname{holim}_{b/\alpha} \pi_b^* = \mathrm{ev}_b\, H .$$

It follows that $\alpha_* R_\alpha \cong H$. Because R_α is a right deformation for α_* by assumption, we see that H is indeed a right derived functor of α_*. $\square$

Write $\rho : \alpha_* \Rightarrow H$ for the natural transformation induced by $\alpha_* r_\alpha : \alpha_* \Rightarrow \alpha_* R_\alpha$, which exhibits H as right derived functor of α_*. Take any homotopy limit $\pi : \lim_{\mathbb{B}} \Rightarrow \operatorname{holim}_{\mathbb{B}}$, and consider the following diagram

$$\begin{tikzcd} \mathcal{C}^{\mathbb{A}} \arrow[r, bend left, "\alpha_*"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\rho"] & \mathcal{C}^{\mathbb{B}} \arrow[r, bend left, "\lim_{\mathbb{B}}"] \arrow[r, bend right, "\operatorname{holim}_{\mathbb{B}}"'] \arrow[r, phantom, "\Downarrow\pi"] & \mathcal{C} \end{tikzcd}$$

Let D be the diagonal functor $\mathcal{C} \to \mathcal{C}^{\mathbb{B}}$. Since we now have adjunctions $\alpha^* \dashv \alpha_*$ and $D \dashv \lim_{\mathbb{B}}$, and since adjunctions are preserved by composition and are unique up to isomorphism, it follows that $\lim_{\mathbb{B}} \alpha_* \cong \lim_{\mathbb{A}}$.

Write σ for the natural transformation $\pi\rho : \lim_{\mathbb{B}} \alpha_* \Rightarrow \operatorname{holim}_{\mathbb{B}} H$. Then by using that $(\alpha_*, \lim_{\mathbb{B}})$ is locally right deformable, we have the following powerful result, shown in [DHKS04, 47.4.ii].

Lemma 2.1.7. *Still in the above situation, σ is a right derived functor of* $\lim_{\mathbb{B}} \alpha_*$.

Note that by [DHKS04, 42.4], it does not matter which homotopy limit $\operatorname{holim}_{\mathbb{B}}$ we take in the above lemma. We will not prove this lemma here, but it will play an important part in Lem. 2.3.7. The latter will imply that Segal objects in for example simplicial model categories are well-behaved, in a sense that will become clear. In the proof of the above lemma, one only uses that $\mathcal{C}$, and hence $\mathcal{C}^{\mathbb{A}}$ and $\mathcal{C}^{\mathbb{B}}$ are saturated, and that $(\lim_{\mathbb{B}}, \alpha_*)$ is locally right deformable.

On the face of it, being sufficiently nice is a rather strong property. However, in the light of [DK12, Def. 5.2.2], where 1- and 2-Segal spaces are defined using the functor

H as right derived functor of the right Kan extension of the Yoneda embedding in the context of combinatorial model categories, it is clear that being sufficiently nice is a desirable property when one wants to define 1- and 2-Segal objects without invoking a combinatorial model structure. Luckily, we have the following remarkable result.

Proposition 2.1.8. *Simplicial model categories are sufficiently nice.*

Proof. Let $\mathcal{M}$ be a simplicial model category, $\alpha : \mathbb{A} \to \mathbb{B}$ a functor. Because $\mathcal{M}$ is complete and saturated, it suffices to show that we have a right deformation $R_\alpha : \mathcal{M}^{\mathbb{A}} \to \mathcal{M}^{\mathbb{A}}$ for α_*, and right deformations $R_b : \mathcal{M}^{b/\alpha} \to \mathcal{M}^{b/\alpha}$ for $\lim_{b/\alpha}$ for all $b \in \mathbb{B}$, such that they satisfy the appropriate commutativity property as in (2.2), and furthermore such that $R_\alpha : \mathcal{M}^{\mathbb{A}} \to \mathcal{M}^{\mathbb{A}}$ is also a right deformation for $\lim_{\mathbb{A}} \cong \lim_{\mathbb{B}} \alpha_*$.

Let $r : \mathrm{id}_{\mathcal{M}} \Rightarrow R$ be a fibrant replacement functor on $\mathcal{M}$, and put $R_b := C(b/\alpha, b/\alpha, R-)$. Recall from (1.4) that for $b \in \mathbb{B}$ we have a right deformation for $\lim_{b/\alpha}$

$$\mathrm{id}_{\mathcal{M}^{b/\alpha}} \overset{r}{\Rightarrow} R \overset{\eta_R}{\Longrightarrow} R_b \,.$$

It is straightforward to check that $\pi_b^* \circ C(\mathbb{A}, \mathbb{A}, R-) \cong C(b/\alpha, b/\alpha, R-) \circ \pi_b^*$ holds for all b. Indeed, let $\theta : b \to \alpha(a)$ be an element of b/α. Then for $n \geq 0$ and $X \in \mathcal{M}^{\mathbb{A}}$ it holds

$$C^n(\mathbf{y}^a, \mathbb{A}, RX) = \prod_{\sigma \in \mathbb{A}_n} RX_{\sigma_n}^{\mathbf{y}^a(\sigma_0)} = \prod_{\substack{\sigma \in \mathbb{A}_n \\ a \to \sigma_0}} RX_{\sigma_n} \,,$$

by the fact that $\mathbf{y}^a(\sigma_0)$ is $\mathbb{A}(a, \sigma_0)$ as discrete simplicial set. Now for the product on the right, to give an index $\sigma \in \mathbb{A}_n, a \to \sigma_0$ is the same thing as to give a datum $\sigma \in \mathbb{A}_n, g : a \to \sigma_0, f : b \to \alpha(\sigma_0)$ such that $\alpha g \circ \theta : b \to \alpha(a) \to \alpha(\sigma_0)$ is $f : b \to \alpha(\sigma_0)$. But to give a datum of this latter form is the same as to give a simplex $\tau \in (b/\alpha)^n$ and a morphism $\theta \to \tau_0$ in b/α. We therefore have that

$$\prod_{\substack{\sigma \in \mathbb{A}_n \\ a \to \sigma_0}} RX_{\sigma_n} \cong \prod_{\tau \in (b/\alpha)^n} (R\pi_b^* X)_{\tau_n}^{\mathbf{y}^\theta(\tau_0)} = C^n(\mathbf{y}^\theta, b/\alpha, R\pi_b^* X) \,.$$

From this it follows that

$$\begin{aligned} \pi_b^* C(\mathbb{A}, \mathbb{A}, RX)(\theta) &= C(\mathbf{y}^a, \mathbb{A}, RX) = \hom^{\mathbf{\Delta}}(\Delta, C^\bullet(\mathbf{y}^a, \mathbb{A}, RX)) \\ &\cong \hom^{\mathbf{\Delta}}(\Delta, C^\bullet(\mathbf{y}^\theta, b/\alpha, R\pi_b^* X)) = C(b/\alpha, b/\alpha, R\pi_b^* X)(\theta) \,. \end{aligned}$$

Hence a candidate for the right deformation for α_* that satisfies the required commutativity property is, with η'_R again defined as in (1.4),

$$\mathrm{id}_{\mathcal{M}^{\mathbb{A}}} \overset{r}{\Rightarrow} R \overset{\eta'_R}{\Longrightarrow} R_\alpha := C(\mathbb{A}, \mathbb{A}, R-) \,,$$

provided that it holds $\eta_R r \pi_b^* \cong \pi_b^* \eta'_R r$, and that we can find a full subcategory of $\mathcal{M}^{\mathbb{A}}$ that contains the image of $C(\mathbb{A}, \mathbb{A}, R-)$ and on which α_* is homotopical.

For the first claim, note that the maps

$$\eta'_{RX}(a) : RX_a \cong \hom^{\mathbf{\Delta}}(*, C^\bullet(\mathbf{y}^a, \mathbb{A}, RX)) \to \hom^{\mathbf{\Delta}}(\Delta, C^\bullet(\mathbf{y}^a, \mathbb{A}, RX))$$
$$\eta_{R\pi_b^* X}(\theta) : RX_a \cong \hom^{\mathbf{\Delta}}(*, C^\bullet(\mathbf{y}^\theta, b/\alpha, R\pi_b^* X)) \to \hom^{\mathbf{\Delta}}(\Delta, C^\bullet(\mathbf{y}^\theta, b/\alpha, R\pi_b^* X))$$

are both induced by $\Delta \to *$, hence commute by naturality of $\hom^{\mathbf{\Delta}}$ with the above given isomorphism $\pi_b^* C(\mathbb{A}, \mathbb{A}, R-) \cong C(b/\alpha, b/\alpha, R-)\pi_b^*$.

For the latter claim, let $\mathcal{N}$ be the full subcategory of $\mathcal{M}^{\mathbb{A}}$ generated by the essential image of R_α. Then clearly R_α lands in $\mathcal{N}$. Note further more that α_* is homotopical on $\mathcal{N}$ iff $\mathrm{ev}_b\, \alpha_* = \lim_{b/\alpha} \pi_b^*$ is so for each $b \in \mathbb{B}$. And the latter holds by the fact that each object X in $\mathcal{N}$ is isomorphic to an object of the form $R_\alpha(X')$, and since $\lim_{b/\alpha} \pi_b^* R_\alpha \cong \lim_{b/\alpha} R_b \pi_b^*$ is homotopical.

To finish our proof, observe that $R_\alpha : \mathcal{M}^{\mathbb{A}} \to \mathcal{M}^{\mathbb{A}}$ is indeed also a right deformation for $\lim_{\mathbb{A}} \cong \lim_{\mathbb{B}} \alpha_*$, so that $(\alpha_*, \lim_{\mathbb{B}})$ is locally right deformable. □

Remark 2.1.9. Call $\mathcal{C}$ *sufficiently conice* when $\mathcal{C}^{\mathrm{op}}$ is sufficiently nice. If $\mathcal{C}$ is sufficiently conice, then the above theory can be dualized in an obvious way. Also, recall that the notion of model categories is self-dual in the sense that for a given model category $\mathcal{M}$ the opposite $\mathcal{M}^{\mathrm{op}}$ is again a model category, after interchanging the fibrations with the cofibrations. It thus holds any model category is also sufficiently conice. These things are nice to know, but will not be used in what follows.

A right Kan extension α_* is called *pointwise* when it can be computed by the formula given in (2.1).

Definition 2.1.10. If a right Kan extension α_* is pointwise, then we call a right homotopy Kan extension $\rho : \alpha_* \Rightarrow \mathbb{R}\alpha_*$ *pointwise* when for each $b \in \mathbb{B}$ there is a right derived functor $\kappa_b : \lim_{b/\alpha} \Rightarrow \mathrm{holim}_{b/\alpha}$ such that $\mathbb{R}\alpha_*$ can be computed with these homotopy limit by means of the formula (2.4), and when furthermore for each $b \in \mathbb{B}$ the maps $\lim_{b/\alpha} \pi_b^* \Rightarrow \mathrm{holim}_{b/\alpha} \pi_b^*$ induced by ρ and by κ_b are in fact isomorphic.

Remark 2.1.11. It is straightforward to check that a sufficiently nice homotopical category has pointwise (homotopy) Kan extensions. Now the point of the above rather heavy definition is the following. Suppose $\mathcal{C}$ is saturated and has pointwise (homotopy) Kan extensions. Further suppose $\alpha : \mathbb{A} \to \mathbb{B}$ is such that for each $b \in \mathbb{B}$ it holds that b/α has an initial object $\theta_b : b \to \alpha(a_b)$.

Clearly, $\lim_{b/\alpha}$ is now homotopical. Since $\mathcal{C}$ is saturated, it follows that the κ_b in the above definition are all weak equivalences (Exm. 1.1.11). But now for the pointwise right homotopy Kan extension $\rho : \alpha_* \Rightarrow \mathbb{R}\alpha_*$ it holds that the induced maps $\rho_X(b) : \alpha_*(X)(b) \to \mathbb{R}\alpha_*(X)(b)$ are weak equivalences as well. And the upshot is: these weak equivalences are all natural in $b \in \mathbb{B}$.

Observe that α induces a natural transformation $\lim_{\mathbb{B}} \Rightarrow \lim_{\mathbb{A}} \alpha^*$. Since $\lim_{\mathbb{A}}$ and $\lim_{\mathbb{B}}$ are themselves right Kan extension, the fact that $\mathcal{C}$ is sufficiently nice gives us right deformation $R_{\mathbb{A}}$ and $R_{\mathbb{B}}$ for these limits respectively. These deformations induce the

following diagram

$$\begin{tikzcd} \lim_{\mathbb{B}} \arrow[r] \arrow[d] & \lim_{\mathbb{A}} \alpha^* \arrow[d] \\ \lim_{\mathbb{B}} R_{\mathbb{B}} \arrow[d] & \lim_{\mathbb{A}} R_{\mathbb{A}}\alpha^* \arrow[d, "\simeq"] \\ \lim_{\mathbb{A}} \alpha^* R_{\mathbb{B}} \arrow[r] & \lim_{\mathbb{A}} R_{\mathbb{A}}\alpha^* R_{\mathbb{B}} \end{tikzcd} \tag{2.5}$$

which is commutative by whiskering. The downwards arrow is a weak equivalence, as indicated, since $\mathrm{id}_{\mathcal{C}^{\mathbb{B}}} \Rightarrow R_{\mathbb{B}}$ is a pointwise weak equivalence and since $\lim_{\mathbb{A}} R_{\mathbb{A}}\alpha^*$ is homotopical.

Definition 2.1.12. Say that homotopy limits in $\mathcal{C}$ are *preserved under homotopy initial functors* when for all homotopy initial functors $\alpha : \mathbb{A} \to \mathbb{B}$ (Def. A.12) the functor $\lim_{\mathbb{A}} \alpha^* R_{\mathbb{B}} \to \lim_{\mathbb{A}} R_{\mathbb{A}}\alpha^* R_{\mathbb{B}}$ in the above diagram is a weak equivalence.

The point of this definition is the following: if the conditions are satisfied, then $\lim_{\mathbb{B}} \Rightarrow \lim_{\mathbb{A}} \alpha^*$ is an isomorphism, so we have weak equivalences

$$\mathrm{holim}_{\mathbb{B}} \to \mathrm{holim}_{\mathbb{A}} \alpha^* R_{\mathbb{B}} \leftarrow \mathrm{holim}_{\mathbb{A}} \alpha^* .$$

The fact that these are natural will prove important later on.

In the case of the Yoneda embedding $\Upsilon : \mathbb{A} \to \mathcal{S}\mathrm{et}^{\mathbb{A}^{\mathrm{op}}}$, we call the right Kan extension functor Υ_* of Υ^{op} the *Yoneda extension functor*, and its right derived functor $\mathbb{R}\Upsilon_*$ the *homotopy Yoneda extension functor*. Note that in this case we have the formula, for $X \in \mathcal{C}^{\mathbb{A}^{\mathrm{op}}}$ and $D \in \mathcal{S}\mathrm{et}^{\mathbb{A}^{\mathrm{op}}}$

$$\Upsilon_*(X)(D) = \lim((\Upsilon/D)^{\mathrm{op}} \to \mathbb{A}^{\mathrm{op}} \xrightarrow{X} \mathcal{C}) ,$$

provided the limit on the right exists. If $\mathcal{C}$ is sufficiently nice, then we get a description of the homotopy Yoneda extension $\mathbb{R}\Upsilon_*$ by replacing the above limit with the appropriate homotopy limit.

2.2 First encounter

Let $n \geq 0$. For $\mathcal{I}$ a subset of the powerset $2^{[n]}$ of $[n]$, let $\Delta[\mathcal{I}]$ be the union

$$\Delta[\mathcal{I}] := \bigcup_{I \in \mathcal{I}} \Delta[I] \subset \Delta[n] ,$$

where $\Delta[I]$ is the simplicial subset of $\Delta[n]$ that has as k-simplices those maps $[k] \to [n]$ which have their image in I.

Example 2.2.1. Let $\mathcal{I} \subset 2^{[n]}$ be given. Suppose $\mathcal{I}$ is closed under taking intersections. Then $\Delta[\mathcal{I}]$ can also be computed as $\mathrm{colim}_{I \in \mathcal{I}} \Delta[I]$. Note however that without the assumption on $\mathcal{I}$, this formula need not hold. We can for example take $\mathcal{I} \subset 2^{[2]}$ to be $\{\{0,1\},\{0,2\}\}$. Then $\mathrm{colim}_{I \in \mathcal{I}} \Delta[I]$ is isomorphic to $\Delta[1] \amalg \Delta[1]$, which is not a simplicial subset of $\Delta[2]$.

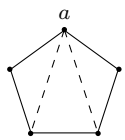
Example 2.2.2. Write $\mathcal{I}_n$ for the *biangulation* of $[n]$, i.e. the subset of $2^{[n]}$ consisting of the sets $\{0,1\},\{1,2\},\dots\{n-1,n\}$. Then $\Delta[\mathcal{I}_n]$ is the union in $\Delta[n]$ of all the edges $[i,i+1]$ for $0 \leq i < n$.

For each $n \geq 2$ write P_n for any strictly convex $n+1$-gon in $\mathbb{R}^2$ with a numbering $0,1,\dots,n$ of its vertices in the anti-clockwise direction.

Definition 2.2.3. A *triangulation* of $[n]$ is a subset $\mathcal{T} \subset 2^{[n]}$ of 3-element sets such that the corresponding subsets of vertices of any P_n induce a triangulation of P_n.

Observe that, combinatorially, it does not matter which $n+1$-gon we take: for any fixed P_n it holds that a subset of $2^{[n]}$ is a triangulation iff it induces a triangulation of this fixed P_n.

Example 2.2.4. The set $[2]$ has only one triangulation: the trivial one. The set $[3]$ has two triangulation: the obvious ones. The set $[4]$ has five triangulations. These are all of the form



where a takes a value in $\{0,1,2,3,4\}$. In general, the number of triangulations of $[n]$ is equal to the $(n-1)$-th Catalan number. There are closed formulae available for these numbers, which apparently go back to 18th century China (see [Lar99]).

We are finally ready to define the notion of 1- and 2-Segal objects. Write

$$\begin{aligned}\mathcal{S}_1 &\coloneqq \{\Delta[\mathcal{I}_n] \hookrightarrow \Delta[n] \mid n \geq 2\}\,;\\ \mathcal{S}_2 &\coloneqq \{\Delta[\mathcal{T}] \hookrightarrow \Delta[n] \mid n \geq 3 \text{ and } \mathcal{T} \text{ is a triangulation of } n\}\,,\end{aligned}$$

and call these maps the *1-* and *2-Segal coverings* respectively.

Definition 2.2.5. Let $\mathcal{C}$ be a homotopical category such that the (homotopy) Yoneda extension functors Δ_* and $\mathbb{R}\Delta_*$ exist. Then for $d = 1,2$, a simplicial object $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ is called a *d-Segal object* in $\mathcal{C}$ if $\mathbb{R}\Delta_*(X) : \mathrm{s}\mathcal{S}\mathrm{et}^{\mathrm{op}} \to \mathcal{C}$ maps d-Segal coverings to weak equivalences. The image of the d-Segal coverings in $\mathcal{C}$ under X are called the *d-Segal maps*.

Example 2.2.6. Consider the homotopical category $\mathrm{s}\mathcal{S}\mathrm{et}^{\mathrm{op}}$, and let X be the simplicial object $\Delta[-]^{\mathrm{op}} : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{s}\mathcal{S}\mathrm{et}^{\mathrm{op}}$. Then since $\mathrm{s}\mathcal{S}\mathrm{et}$ is a simplicial model category, it is sufficiently conice. It follows that Δ_* and $\mathbb{R}\Delta_*$ are pointwise, so for $D \in \mathrm{s}\mathcal{S}\mathrm{et}$ we can compute $\Delta_*(X)(D)$ as the following limit in $\mathrm{s}\mathcal{S}\mathrm{et}^{\mathrm{op}}$

$$\Delta_*(X)(D) = \lim_{(\Delta[p]\to D)\in(\Delta/D)^{\mathrm{op}}} X_p = \lim_{(\Delta[p]\to D)\in(\Delta/D)^{\mathrm{op}}} \Delta[p]\,,$$

which is the colimit of the (Δ/D)-diagram in s$\mathcal{S}$et that sends $\Delta[p] \to D$ to $\Delta[p]$. It follows that in this case $\Delta_*(X)(D) \cong D$. Likewise, since we have

$$\mathbb{R}\Delta_*(X)(D) = \operatorname{holim}_{(\Delta[p]\to D)\in(\Delta/D)^{\mathrm{op}}} \Delta[p]\,,$$

by Exm. 1.4.3 it follows that in this case $\mathbb{R}\Delta_*(X)(D) \simeq D$.

When Δ_* on a given homotopical category $\mathcal{C}$ is pointwise, then for $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ and $D \in$ s$\mathcal{S}$et we can compute $\Delta_*(X)(D)$ as a limit in $\mathcal{C}$ over the category $(\Delta/D)^{\mathrm{op}}$. It is therefore convenient to have some tools for computing such limits. Specifically, it would be nice if we could only consider nondegenerate simplices of D. It turns out this is too ambitious in general, but in the cases in which we are interested this can be done.[1]

Definition 2.2.7. Let D be a simplicial set. A p-simplex in D is called *nonsingular* iff the corresponding map $\Delta[p] \to D$ is monic, i.e. injective in each degree. Now D is called *regular* or *nonsingular* if all of its nondegenerate simplices are nonsingular.

Note that a nonsingular simplex is always nondegenerate. For a simplicial set D write Δ/D_{nd} for the category of nondegenerate simplices in D. Observe that a morphism in Δ/D_{nd} is necessarily monic.

Remark 2.2.8. A simplicial set D is *finite* when Δ/D_{nd} has a finite number of objects, or equivalently when $|D|$ is compact. One can associate to such a finite D a regular simplicial set ID, together with a map $D \to ID$ that induces a weak equivalence on the geometric realizations. This construction is functorial on finite simplicial sets and is called the *improvement functor*: see [WJR13, Thm. 2.5.2] for details. We only mention this to indicate there are strategies available to deal with irregular simplicial sets in a case where regular simplicial sets are needed.

Example 2.2.9. For $n \geq 0$ the simplex $\Delta[n]$ is regular. More generally, any simplicial subset $D' \subset D$ of a regular simplicial set D is itself regular. Indeed, let $\sigma : \Delta[p] \to D'$ be nondegenerate. Then suppose $\Delta[p] \to D' \to D$ factorizes as $\Delta[p] \xrightarrow{f} \Delta[q] \xrightarrow{\tau} D$ with f epic and τ nondegenerate, hence monic. Take some mono $g : \Delta[q] \to \Delta[p]$ such that $fg = \mathrm{id}_{\Delta[q]}$. Then observe, with $\iota : D' \to D$ the inclusion we have that

$$\iota\sigma gf = \tau fgf = \tau f = \iota\sigma\,,$$

hence that $\sigma gf = \sigma$ holds since ι is monic. Because σ is nondegenerate and gf is a morphism $\sigma \to \sigma$ in Δ/D'_{nd}, it follows that gf, hence f and therefore σ, is monic.

Lemma 2.2.10. *Let D be a regular simplicial set. Then the inclusion of Δ/D_{nd} into Δ/D is final.*

Proof. Write F for the inclusion and let $\sigma : \Delta[p] \to D$ be given. Factorize σ uniquely as $\Delta[p] \twoheadrightarrow \Delta[p'] \hookrightarrow D$. It is straightforward to show, by the regularity of D, that this factorization determines an element of σ/F that is mapped to any other given element of σ/F. □

[1] See the errata [Hov15, 9] why this does not work in general.

Example 2.2.11. Consider the case where $\mathcal{C}$ is $\mathcal{S}\mathrm{et}$ with minimal homotopical structure. Call 1-Segal objects in $\mathcal{S}\mathrm{et}$ *1-Segal sets*. Let us show that 1-Segal sets are exactly the nerves of small categories. For this first note that $\mathbb{R}\Delta_*$ is just Δ_*, since we are working in a minimal homotopical structure.

Let $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a 1-Segal set and $n \geq 0$. Observe that $\Delta/\Delta[n]$ has a terminal object, given by $\mathrm{id} : \Delta[n] \to \Delta[n]$, and hence that $\Delta_*(X)(\Delta[n])$ is just X_n. I claim that $\Delta_*(X)(\Delta[\mathcal{I}_n])$ is the limit of the diagram

$$X_1 \xrightarrow{d_0} X_0 \xleftarrow{d_1} X_1 \to \ldots \to X_0 \xleftarrow{d_1} X_1\,,$$

i.e. that it is the n-fold fiber product $X_1 \times_{X_0} \cdots \times_{X_0} X_1$.

Note that $\Delta[\mathcal{I}_n]$ is the simplicial subset of $\Delta[n]$ consisting of its n edges $[0,1], \ldots, [n-1,n]$. Hence it is regular, and has as category of nondegenerate simplices

$$\begin{tikzcd}
\Delta\{0\} \arrow[dr] & & \Delta\{1\} \arrow[dl] \arrow[dr] & & \Delta\{2\} \arrow[dl] \arrow[dr] & & \ldots \\
& \Delta\{0,1\} \arrow[dr] & & \Delta\{1,2\} \arrow[dl] & & \ldots & \\
& & \Delta[\mathcal{I}_n] & & & &
\end{tikzcd}$$

where the maps $\Delta\{i\} \to \Delta[\mathcal{I}_n]$ and $\Delta\{i, i+1\} \to \Delta[\mathcal{I}_n]$ are the objects of $\Delta/\Delta[\mathcal{I}_n]_{\mathrm{nd}}$. By Exm. 2.2.9 and Lem. 2.2.10 it follows that $\Delta_*(X)(\Delta[\mathcal{I}_n])$ is the limit in $\mathcal{S}\mathrm{et}$ of the functor on the opposite of the above category, that sends $\Delta\{i\} \to \Delta[\mathcal{I}_n]$ to X_0 and $\Delta\{i, i+1\} \to \Delta[\mathcal{I}_n]$ to X_1. This is indeed the fiber product that we wanted.

It is not hard to show that the 1-Segal map $\Delta_*(X)(\Delta[n]) \to \Delta_*(X)(\Delta[\mathcal{I}_n])$ sends an n-simplex $\Delta[n] \to X$ to the element in $X_1 \times_{X_0} \cdots \times_{X_0} X_1$ which on the i-th factor is given by the 1-simplex $\Delta\{i-1, i\} \to \Delta[n] \to X$. By assumption on X, these maps are all bijections.

Now let $\mathcal{C}(X)$ be the category which has X_0 as objects, and with $\mathcal{C}(X)(x, y)$ for $x, y \in X_0$ the set of $f \in X_1$ such that $d_1 f = x$ and $d_0 f = y$. To give the composition we use the bijection $X_2 \cong X_1 \times_{X_0} X_1$ associated to X that maps a $\sigma \in X_2$ to $(d_2\sigma, d_0\sigma)$. Then for $x \xrightarrow{f} y \xrightarrow{g} z$ in $\mathcal{C}(X)$, take a unique $\sigma \in X_2$ such that $(d_2\sigma, d_0\sigma) = (f, g)$, and put $g \circ f \coloneqq d_1\sigma$. It is easily seen that $s_0 x \in \mathcal{C}(X)(x, x)$ is a unit for this composition law.

To show associativity of this composition law, let $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$ in $\mathcal{C}(X)$ be given. The isomorphism $X_3 \cong X_1 \times_{X_0} X_1 \times_{X_0} X_1$ gives us a unique $\tau \in X_3$ that has the following edges and vertices

$$\begin{tikzcd}
& w & \\
& z \arrow[u, dotted, "h"] & \\
x \arrow[ur, dotted] \arrow[rr, "f"] & & y \arrow[ul, dotted, "g"'] \arrow[uul]
\end{tikzcd}$$

From the picture it is readily seen that $h \circ (g \circ f) = d_1 d_1 \tau$, as witnessed by $d_1\tau$, and that likewise $(h \circ g) \circ f = d_1 d_2 \tau$, as witnessed by $d_2\tau$. Thus the proposed composition is associative.

It is easy to see that, conversely, the nerve of a category is a 1-Segal set. Observe that any 1-Segal set $X_\bullet$ is isomorphic to the simplicial set $X'_\bullet$ which has the n-fold fiber product of X_1 over X_0 in degree n. It readily follows that taking the nerve of a category is, up to isomorphism, inverse to the above construction $X_\bullet \mapsto \mathcal{C}(X_\bullet)$. What is more, with this strategy we even see that simplicial maps $X \to Y$ between 1-Segal sets correspond bijectively to functors $\mathcal{C}(X) \to \mathcal{C}(Y)$ between the associated categories in an obvious way. We hence have an equivalence from the category of 1-Segal sets to $\mathcal{C}$at.

Remark 2.2.12. By the above example, one can imagine what a 1-Segal space (i.e. a 1-Segal object in $\mathcal{T}$op) might be, namely a simplicial space that behaves like the nerve of a category, but where composition of morphisms is only defined 'up to homotopy'. And indeed these 1-Segal spaces are a road to higher category theory; see for example the overview given in [Ber06] or some in-depth look in [JT06]. We however won't be going in that direction.

Call a 2-Segal object that is not a 1-Segal object *strict*. As we shall see below in Thm. 2.4.1, under some reasonable conditions on the homotopical category $\mathcal{C}$, it holds that a 1-Segal object in $\mathcal{C}$ is also 2-Segal. So it is nice to know whether the notion of strict 2-Segal objects is not void. To this question we can give an affirmative answer. To this end, let an *oriented graph* be a 1-skeletal simplicial set, i.e. one such that all of its simplices of dimension ≥ 2 are degenerate.

Proposition 2.2.13. *An oriented graph $X_\bullet$ is a 2-Segal set.*

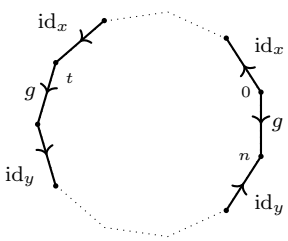
Proof. We need to show that for every triangulation $\mathcal{T}$ of P_n, with $n \geq 3$, the 2-Segal map $f_{\mathcal{T}} : X_n \to X_{\mathcal{T}}$ is a bijection, where $X_{\mathcal{T}}$ is defined as $\Delta_*(X)(\Delta[\mathcal{T}])$. We induct on n.

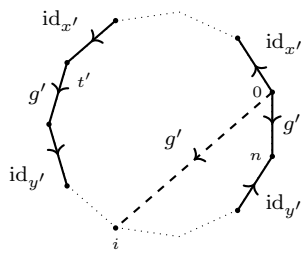
Let $0 < i < n$ and $n \geq 3$ be given. Consider the map

$$f_i : X_n \to X_{\{0,1,\ldots,i\}} \times_{X_{\{0,i\}}} X_{\{0,i,i+1,\ldots,n\}} \,.$$

I claim that f_i is a bijection.

Let $\sigma \in X_n$ be given. Write $g : x \to y$ for the 1-simplex on the edge of σ between the vertices 0 and n. If this g is nondegenerate, then there is a unique vertex t such that the edge $[t, t+1]$ is also given by $g : x \to y$. By the assumption on X, the simplex σ is completely determined by g and t, or only by any vertex x of σ when g is degenerate. Hence σ looks like the picture on the left





The picture on the right is a possible object σ' in $X_{\{0,1,\ldots,i\}} \times_{X_{\{0,i\}}} X_{\{0,i,i+1,\ldots,n\}}$, with g', t' defined similarly as g, t. In this picture, we have assumed $i > t'$. Of course, $i \leq t'$

gives a similar picture, but with id_x on the dashed line. As we can see, σ' is also completely determined by g', t' when it contains a nondegenerate edge, or by any on of its vertices x' otherwise. Since f_i maps the above simplex σ on the left to the simplex σ' on the right iff $g = g'$ and $t = t'$ holds, we see that f_i is indeed a bijection.

With a similar argument, one can also show for $0 < j < n$ that the map

$$f_j : X_n \to X_{\{0,1,\dots,j,n\}} \times_{X_{\{j,n\}}} X_{\{j,j+1,\dots,n\}} \,.$$

is a bijection.

By the same strategy as employed in Exm. 2.2.11, we note that $X_{\mathcal{T}}$ can be calculated as the limit of $([p] \to \Delta[\mathcal{T}]) \mapsto X_p$ over the category of nondegenerate simplices of $\Delta[\mathcal{T}]$. As we shall see later on, we can even look at only the nondegenerate simplices of dimension ≥ 1. In the base case $n = 3$ there are two triangulations of P_3 for which we need to check that the corresponding 2-Segal map $f_{\mathcal{T}}$ is a bijection. It follows that these two maps are precisely given by the above bijections f_i and f_j, with $i = 2$ and $j = 1$.

For the induction step, let $\mathcal{T}$ be a triangulation of P_n for $n > 3$. Consider the unique $0 < i < n$ such that $\{0, i, n\}$ is a triangle of $\mathcal{T}$. We want to chop up P_n in two pieces along $\{0, i\}$ and apply our induction hypothesis. This indeed works when the edge $\{0, i\}$ is not an outer edge on the boundary of P_n, i.e. when $i \neq 1$. Assume that we are in this case. Then cutting P_n along $\{0, i\}$ gives us two polygons P_i, P_{n-i+1}, of which $\mathcal{T}$ induces triangulations $\mathcal{T}_i, \mathcal{T}_{n-i+1}$. Since $i, n - i + 1 < n$, the induction hypothesis applies.

It is not hard to show that for $\sigma : \Delta[n] \to X$ it holds that $f_{\mathcal{T}}(\sigma) \in X_{\mathcal{T}}$ is given by the composition $\Delta\{u, v, w\} \to \Delta[n] \to X$ on the factor $X_{\{u,v,w\}}$, for any triangle $\{u, v, w\}$ of $\mathcal{T}$. It follows that we can factorize $f_{\mathcal{T}}$ as

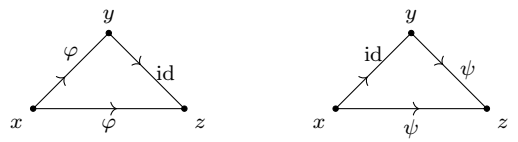
$$X_n \xrightarrow{f_i} X_{\{0,1,\dots,i\}} \times_{X_{\{0,i\}}} X_{\{0,i,i+1,\dots,n\}} \xrightarrow{(f_{\mathcal{T}_i}, f_{\mathcal{T}_{n-i+1}})} X_{\mathcal{T}_i} \times_{X_{\{0,i\}}} X_{\mathcal{T}_{n-i+1}} \cong X_{\mathcal{T}} \,,$$

and hence that $f_{\mathcal{T}}$ is a bijection.

In the case $i = 1$ we can cut along $\{1, n\}$ and proceed similarly. □

The proposition above and the example below can be found in [DK12, Exm. 3.1.1]. We have given some more detail in our proofs, to illustrate a few key ideas in what is still to follow.

Example 2.2.14. Consider an oriented graph $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$. Then X is 1-Segal iff it has no nontrivial composable arrows. To see this, let $x \xrightarrow{\varphi} y \xrightarrow{\psi} z$ be given. Then the only possible 2-simplices in X mapped to (φ, ψ) under $f_2 : X_2 \to X_1 \times_{X_0} X_1$ are as follows:



From this it follows that if X is 1-Segal then it has no pairs of nontrivial composable arrows. For the implication the other way around, we note that if X satisfies this latter property

then elements from $X_1 \times_{X_0} \cdots \times_{X_0} X_1$ are of the form $\eta = (\mathrm{id}_x, \ldots, \mathrm{id}_x, f, \mathrm{id}_y, \ldots, \mathrm{id}_y)$ with $f : x \to y$ a 1-simplex. Using X is 1-skeletal, it is clear such an element η determines a unique $\sigma \in X_n$ that is mapped to η under the 1-Segal map f_n.

We thus have a whole range of examples of strict 2-Segal objects, namely oriented graphs with nontrivial composable arrows, such as e.g. $\mathcal{I}_n$ for $n > 1$.

Remark 2.2.15. A *double category* $\mathcal{D}$ has objects and two kinds of morphisms: *horizontal* and *vertical* ones. Part of the structure of $\mathcal{D}$ is also a set S of *squares* in $\mathcal{D}$ that have horizontal arrows in the horizontal directions and vertical arrows in the vertical one. The horizontal arrows compose, as do the vertical arrows, as do the squares in two directions. These data of course need to satisfy some compatibility conditions that we won't go in to.

Now such a double category $\mathcal{D}$ is called *stable* if every square in S is completely determined by its bottom and its right arrow, and likewise by its top and left arrow. Furthermore, $\mathcal{D}$ is called *augmented* when there is a set A of objects in $\mathcal{D}$ such that for every object d in $\mathcal{D}$ there is a unique horizontal arrow $a \to d$ and a unique vertical one $d \to a'$ such that $a, a' \in A$.

A 2-Segal object X in a given homotopical category $\mathcal{C}$ is called *unital* if for all $n \geq 0$ and $0 \leq i \leq n-1$ the map $u : X_{n-1} \to X_n \times^R_{X_{\{i,i+1\}}} X_{\{i\}}$ is a weak equivalence, where u is the degeneracy map s_i on the factor X_n. In [BOORS16] it is shown that the full subcategory of s$\mathcal{S}$et consisting of unital 2-Segal sets is equivalent to the category of augmented, stable double categories.

The equivalence acts on a given unital 2-Segal set X as follows. One constructs a double category $\mathcal{D}$ which has X_1 as objects, X_2 as both the horizontal and the vertical arrows, and X_3 as squares. Here, an $\xi \in X_3$ is the square

$$\begin{array}{ccc} f & \xrightarrow{\sigma} & g \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ h & \xrightarrow{\tau} & k \end{array}$$

where f, g, h, k are the edges $[0,2], [0,3], [1,2], [1,3]$. The arrow σ corresponds to the face in ξ that has f, g as edges, i.e. the face $[0,2,3]$. And likewise for the other arrows.

2.3 Natural membranes

Let $\mathcal{C}$ be a homotopical category such that the (homotopy) Yoneda extension functors Δ_* and $\mathbb{R}\Delta_*$ both exist. We have the following convenient terminology.

Definition 2.3.1. For a simplicial object X in $\mathcal{C}$ and a simplicial set D, write (D, X) for $\Delta_*(X)(D)$ and $(D, X)_R$ for $\mathbb{R}\Delta_*(X)(D)$, and call it the *D-membrane object in X* resp. the *derived D-membrane object in X* (or simply *(derived) D-membranes*).

Clearly, these constructions define functors $(-,-)$ and $(-,-)_R$ from s$\mathcal{S}$et$^{\mathrm{op}} \times \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$ to $\mathcal{C}$, where $(-,-)_R$ is homotopical in the second coordinate. Now fix a simplicial object X in $\mathcal{C}$. In the case where $D = \Delta[\mathcal{P}]$ for $\mathcal{P}$ either $\mathcal{I}_n$ or $\mathcal{T}$ we put:

$$X_{\mathcal{P}} \coloneqq (\Delta[\mathcal{P}], X) = \Delta_*(X)(\Delta[\mathcal{P}])\,; \quad \& \quad \mathbb{R}X_{\mathcal{P}} \coloneqq (\Delta[\mathcal{P}], X)_R = \mathbb{R}\Delta_*(X)(\Delta[\mathcal{P}])\,.$$

Example 2.3.2. One should think of (derived) D-membranes as (derived) mapping spaces. To justify this intuition, first consider the case $\mathcal{C} = \mathcal{S}\mathrm{et}$ with minimal homotopical structure. Since $\mathcal{S}\mathrm{et}$ is complete, the right Kan extension Δ_* is pointwise. For simplicial sets D, X we thus have

$$(D, X) = \lim_{\Delta[p]\to D} X_p \,,$$

where the indexing category of the above limit is $(\Delta/D)^{\mathrm{op}}$. Now an element of this limit is given by the following data: for each $\sigma : \Delta[p] \to D$ an $f_p(\sigma) : \Delta[p] \to X$ such that for all $h : [p] \to [q]$ and all $\tau : \Delta[q] \to D$ it holds $f_p(\tau h) = f_q(\tau)h$. Clearly this is the same as a simplicial map $D \to X$, so that $(D, X) \cong \mathrm{s}\mathcal{S}\mathrm{et}(D, X)$.

Likewise, consider the case where $\mathcal{C} = \mathcal{T}\mathrm{op}$, and let simplicial spaces X, X' be given. Consider the cosimplicial object

$$\mathbf{\Delta} \to \mathcal{T}\mathrm{op} : n \mapsto \prod_{\sigma\in\mathbf{\Delta}_n} X_{\sigma_0}^{X'_{\sigma_n}} \,,$$

with obvious structure maps. Now define the derived mapping space $R\,\mathrm{Map}(X', X)$ as the totalization of this object.

In this case we have that $(D, X)_R \cong R\,\mathrm{Map}(\langle D\rangle, X)$, with $\langle D\rangle$ the discrete simplicial space associated to $D \in \mathrm{s}\mathcal{S}\mathrm{et}$. This follows from the following computation:

$$(D, X)_R = \mathrm{Tot} \prod_{\sigma\in(\Delta/D)_n^{\mathrm{op}}} X_{\sigma_n} \cong \mathrm{Tot} \prod_{\substack{\tau\in\mathbf{\Delta}_n \\ \Delta[\tau_n]\to D}} X_{\tau_0} \cong \mathrm{Tot} \prod_{\tau\in\mathbf{\Delta}_n} X_{\tau_0}^{\langle D\rangle_{\tau_n}} = R\,\mathrm{Map}(\langle D\rangle, X) \,,$$

where we have used that for $\tau \in \mathbf{\Delta}_n$ it holds that $\prod_{\Delta[\tau_n]\to D} X_{\tau_0} \cong X_{\tau_0}^{\langle D\rangle_{\tau_n}}$, since $\langle D\rangle_{\tau_n}$ is discrete.

To get some more interesting theory of Segal objects off the ground, it turns out that we need some assumptions on the functors $(-,-)$ and $(-,-)_R$. In need of a name, we introduce the following

Definition 2.3.3. A homotopical category $\mathcal{C}$ is said to have *natural membranes* if it is saturated and when the (homotopy) Yoneda extensions Δ_* and $\mathbb{R}\Delta_*$ both exist, are pointwise, and furthermore come with, for every diagram of simplicial sets $D_\bullet = (D_b)_{b\in\mathbb{B}}$ with colimit D and every simplicial object X in $\mathcal{C}$:

(NM1) An isomorphism $(D, X) \cong \lim_{b\in\mathbb{B}^{\mathrm{op}}}(D_b, X)$;

(NM2) A weak equivalence $(D, X)_R \simeq \mathrm{holim}_{b\in\mathbb{B}^{\mathrm{op}}}(D_b, X)_R$, provided $D_\bullet$ is acyclic;

Both natural in X, the first natural in all $D_\bullet$ and the second in acyclic $D_\bullet$.

Example 2.3.4. Consider again the case where $\mathcal{C} = \mathcal{S}\mathrm{et}$ with minimal homotopical structure. Then for simplicial sets D, X it holds that $\mathbb{R}\Delta_*$ is just Δ_*, and also that $(D, X) \cong \mathrm{s}\mathcal{S}\mathrm{et}(D, X)$ by Exm. 2.3.2. Because $\mathrm{s}\mathcal{S}\mathrm{et}(-, X)$ sends colimits to limits, it is clear $\mathcal{S}\mathrm{et}$ thus has natural membranes.

Example 2.3.5. Let $\mathcal{C}$ be a homotopical category with natural membranes, and X a simplicial object in $\mathcal{C}$. Note that since $(\Delta/\Delta[n])^{\mathrm{op}}$ has the initial object $\mathrm{id} : \Delta[n] \to \Delta[n]$, we see that

$$(\Delta[n], X) \cong X_n \simeq (\Delta[n], X)_R \,.$$

The weak equivalence is induced by the natural transformation $\Delta_* \Rightarrow \mathbb{R}\Delta_*$ which establishes $\mathbb{R}\Delta_*$ as a pointwise right derived functor of Δ_* (Rem. 2.1.11). As such, this weak equivalence is natural in n. What is more, a map $\mathbb{R}X_n \to Y$ for a given object Y is a weak equivalence iff the composition $X_n \to \mathbb{R}X_n \to Y$ is.

Now let D be a simplicial set. Then since $D = \mathrm{colim}_{\Delta[p]\to D}\, \Delta[p]$ and by the assumption on $\mathcal{C}$ we get our pointwise formulae back:

$$(D, X) \cong \lim_{\Delta[p]\to D}(\Delta[p], X) \cong \lim_{\Delta[p]\to D} X_p \,;$$
$$(D, X)_R \simeq \mathrm{holim}_{\Delta[p]\to D}(\Delta[p], X)_R \simeq \mathrm{holim}_{\Delta[p]\to D} X_p \,,$$

where in the second weak equivalence we have used that $\mathrm{holim}_{(\Delta/D)^{\mathrm{op}}}$ preserves weak equivalences.

To see the definition of natural membranes makes sense, let us show the following

Theorem 2.3.6. *Simplicial model categories have natural membranes.*

We prove this in two steps. The second and largest step will be

Lemma 2.3.7. *Let $\mathcal{C}$ be a sufficiently nice homotopical category such that homotopy limits in $\mathcal{C}$ are preserved under homotopy initial functors (Def. 2.1.12). Then $\mathcal{C}$ has natural membranes.*

But let us first show this indeed suffices.

Proof of Thm. 2.3.6. Let $\mathcal{M}$ be a simplicial model category. Then it is sufficiently nice by Prop. 2.1.8. Furthermore, it is a known result that homotopy limits in $\mathcal{M}$ are preserved up to weak equivalence under homotopy initial functors (e.g. [Rie14, Thm. 8.5.6], [Hir14, Thm. 13.7]). Hence the above lemma applies, which shows that $\mathcal{M}$ has natural membranes. □

We finish with a proof of Lem. 2.3.7. So let $\mathcal{C}$ be a homotopical category that satisfies the conditions of our lemma. Since $\mathcal{C}$ is sufficiently nice, it has pointwise (homotopy) Kan extensions by Rem. 2.1.11. It remains to be shown we have isomorphisms and weak equivalences as in (NM1) and (NM2).

So suppose we are given a simplicial object X in $\mathcal{C}$ and a diagram of simplicial sets $(D_b)_{b\in\mathbb{B}}$ that has colimit D. The idea of the proof is the following. Observe that we can express (D, X) as

$$(D, X) = \lim_{\Delta[p]\to D} X_p \,.$$

Now since D is the colimit of $D_\bullet$, an element $\Delta[p] \to D$ in the indexing category $(\Delta/D)^{\mathrm{op}}$ of the above limit factorizes as $\Delta[p] \to D_b \to D$ for a certain $b \in \mathbb{B}$. Hence this limits should

be computable over some category $\mathbb{A}$ that has elements of the form $\Delta[p] \to D_b \to D$. The goal is to find such an $\mathbb{A}$, and then to show we can compute the limit over $\mathbb{A}$ 'piecewise':

$$\lim_{(\Delta[p]\to D_b\to D)\in\mathbb{A}} X_p \cong \lim_{b\in\mathbb{B}^{\mathrm{op}}} \lim_{\Delta[p]\to D_b} X_p\,.$$

We will make this precise, and show that it also work in the derived case. In the latter, we must take some care with naturality.

Proof (NM1). We have a diagram of functors

$$\Delta/D \xrightarrow{f} \mathrm{sSet}/D \xleftarrow{g} \mathbb{B}\,,$$

where f is the inclusion and g sends $b \in \mathbb{B}$ to the map $\iota_b : D_b \to D$ associated to the colimit D. This gives rise to the comma category f/g. In this case, an object in f/g is a pair (α, b), where b is an object in $\mathbb{B}$ and α is a map $\Delta[p] \to D_b$. A morphism from (α, b) to (α', b') is given by a morphism $p \to p'$ in $\mathbf{\Delta}$ and a morphism $b \to b'$ in $\mathbb{B}$, which make the following induced diagram commutative

$$\begin{tikzcd} \Delta[p] \arrow[r] \arrow[d, "\alpha"'] & \Delta[p'] \arrow[d, "\alpha'"] \\ D_b \arrow[r] & D_{b'} \end{tikzcd}$$

Now consider the following functors

$$\begin{aligned} F &: \mathbb{B}^{\mathrm{op}} \to \mathcal{C} : b \mapsto (D_b, X)\,; \\ G &: (f/g)^{\mathrm{op}} \to \mathcal{C} : (\alpha, b) \mapsto X_p\,; \\ H &: (\Delta/D)^{\mathrm{op}} \to \mathcal{C} : (\Delta[p] \to D) \mapsto X_p\,. \end{aligned}$$

Observe that (D, X) is the limit of H, and clearly $\lim_{b\in\mathbb{B}^{\mathrm{op}}}(D_b, X)$ is the limit of F. Hence it suffices to show that $\lim_{\mathbb{B}^{\mathrm{op}}} F \cong \lim_{(f/g)^{\mathrm{op}}} G \cong \lim_{(\Delta/D)^{\mathrm{op}}} H$ holds.

Consider the unique functor $\pi : \mathbb{B}^{\mathrm{op}} \to *$ and also the obvious functor

$$\kappa : (f/g)^{\mathrm{op}} \to \mathbb{B}^{\mathrm{op}} : (\alpha, b) \mapsto b\,.$$

Then by definition of right Kan extensions, we have that $\pi_* = \lim_{\mathbb{B}^{\mathrm{op}}}$. It follows that

$$\lim_{\mathbb{B}^{\mathrm{op}}} \circ \kappa_* \cong \lim_{(f/g)^{\mathrm{op}}}\,.$$

Hence it holds that $\lim_{(f/g)^{\mathrm{op}}} G \cong \lim_{\mathbb{B}^{\mathrm{op}}} \kappa_* G$. We are going to show that $\kappa_* G \cong F$.

Let $b \in \mathbb{B}^{\mathrm{op}}$ be given. Note that an object in b/κ is given by a morphism $b_1 \to b$ in $\mathbb{B}$ together with a simplex $\alpha_1 : \Delta[r_1] \to D_{b_1}$. Now consider the functor

$$\epsilon_b : b/\kappa \to (\Delta/D_b)^{\mathrm{op}}$$

that sends such a datum $(b_1 \to b, \alpha_1)$ in b/κ to the composition $\Delta[r_1] \to D_{b_1} \to D_b$ in Δ/D_b. I claim that ϵ_b is initial.

To see this, let $\sigma : \Delta[p] \to D_b$ in $(\Delta/D_b)^{\mathrm{op}}$ be given. Then an element y_1 in the category ϵ_b/σ is determined by a datum of the form

$$\begin{tikzcd} \Delta[r_1] \arrow[r, "\alpha_1"] & D_{b_1} \arrow[r, "(b_1 \to b)_*"] & D_b \\ \Delta[p] \arrow[u, "\theta_1"] \arrow[ur, "\sigma"] & & \end{tikzcd} \tag{2.6}$$

A morphism $y_1 \to y_2$ between such data is given by morphisms $b_2 \to b_1$ and $\Delta[r_2] \to \Delta[r_1]$ in $\mathbb{B}$ and s$\mathcal{S}$et respectively, such that they make the following diagrams commutative:

$$\begin{tikzcd} b_2 \arrow[rr] \arrow[dr] & & b_1 \arrow[dl] \\ & b & \end{tikzcd} \qquad \begin{tikzcd} \Delta[r_2] \arrow[rr] \arrow[d] & & \Delta[r_1] \arrow[d] \\ D_{b_2} \arrow[rr] \arrow[dr] & & D_{b_1} \arrow[dl] \\ & D_b & \\ & \Delta[p] \arrow[u] \arrow[uuu, bend left] \arrow[uuu, bend right] & \end{tikzcd}$$

Now let x be the element in ϵ_b/σ induced by the arrows $\Delta[p] \xrightarrow{\theta} \Delta[p'] \xrightarrow{\sigma'} D_b \xrightarrow{\mathrm{id}} D_b$ with θ epic and σ' nondegenerate, and such that $\sigma'\theta = \sigma$. Then let an object y_1 in ϵ_b/σ as above be given, and factorize $\theta_1 : \Delta[p] \to \Delta[r_1]$ as $\Delta[p] \xrightarrow{\theta_2} \Delta[r_2] \xrightarrow{\tau} \Delta[r_1]$ with θ_2 epic and τ nondegenerate. Then by factorizing $\Delta[r_2] \to D_b$ as an epi followed by a nondegenerate, and using the uniqueness of such factorizations, we get a morphism $\Delta[r_2] \to \Delta[p']$ that fits in the following commutative diagram in s$\mathcal{S}$et

$$\begin{tikzcd} & \Delta[r_1] \arrow[r] & D_{b_1} \arrow[r] & D_b \\ \Delta[r_2] \arrow[ur, hook, "\tau"] \arrow[urrr, dotted, bend right] & & & \\ & \Delta[p] \arrow[uu, "\theta_1"] \arrow[uur, "\sigma"] \arrow[ul, two heads, "\theta_2"] \arrow[rr, two heads, "\theta"] & & \Delta[p'] \arrow[uu, "\sigma'"] \end{tikzcd}$$

Now θ_2 induces an object y_2 of ϵ_b/σ, and furthermore the dotted arrow induces an ϵ_b/σ-morphism from x to this object y_2. Likewise, τ induces an ϵ_b/σ-morphism $y_1 \to y_2$. Hence we see that ϵ_b/σ is non-empty and connected, so that ϵ_b is initial as claimed.

By construction of G and the fact that ϵ_b is initial, it now holds that

$$\lim(b/\kappa \to (f/g)^{\mathrm{op}} \xrightarrow{G} \mathcal{C}) = \lim(b/\kappa \xrightarrow{\epsilon_b} (\Delta/D_b)^{\mathrm{op}} \to \mathcal{C}) \cong \lim((\Delta/D_b)^{\mathrm{op}} \to \mathcal{C}), \tag{2.7}$$

where the arrow $(\Delta/D_b)^{\mathrm{op}} \to \mathcal{C}$ sends $\Delta[p] \to D_b$ to X_p. Observe that the left-hand side of the above equation is $\kappa_*(G)(b)$, while the right-hand side is $F(b)$. Now note ϵ_b is natural in b, from which naturality of the above isomorphism follows, so that indeed

$\kappa_*(G) \cong F$. This isomorphism is natural in both $X_\bullet$ and $D_\bullet$, as can easily be checked by the fact that changing an indexing category of a limit under an initial functor is natural.

To show that $\lim_{(f/g)^{\mathrm{op}}} G \cong \lim_{(\Delta/D)^{\mathrm{op}}} H$, consider the obvious functor λ from $(f/g)^{\mathrm{op}}$ to $(\Delta/D)^{\mathrm{op}}$. I claim this functor is initial. So let $\sigma : \Delta[p] \to D$ be given in Δ/D. We are going to show that λ/σ is non-empty and connected. To this end, observe that since colimits of presheaves are calculated pointwise, we have some factorization of $\Delta[p] \to D$ as $\Delta[p] \to D_b \to D$. Then $\Delta[p] \xrightarrow{\mathrm{id}} \Delta[p] \to D_b \to D$ gives an element x of λ/σ. It is straightforward to show, considering D_p as the disjoint union $\coprod_{b\in\mathbb{B}}(D_b)_p$ modulo the equivalence relation coming from $D_\bullet$, that there is a zig-zag of maps from x to every other element of λ/σ.

Because λ is initial, the natural map $\lim_{(\Delta/D)^{\mathrm{op}}} H \to \lim_{(f/g)^{\mathrm{op}}} \lambda^* H$ is an isomorphism. Since $\lambda^* H$ is clearly G, this is what we wanted.

It is again straightforward to show that the construction of G, H is natural in $X_\bullet$ with respect to taking the limits, and that the construction of λ is natural in $D_\bullet$. Since we are natural in every step of the way, we thus have a natural isomorphism $(D, X) \cong \lim_{b\in\mathbb{B}^{\mathrm{op}}}(D_b, X)$, which was to be shown. □

Proof (NM2). Now suppose $D_\bullet$ is acyclic. Take all homotopy limits to be given by the right deformations that witness $\mathcal{C}$ as sufficiently nice, except $\mathrm{holim}_{(f/g)^{\mathrm{op}}}$ which we will specifically define later on for naturality. We retain the notation of the above proof, but this time also consider the functor

$$F' : \mathbb{B}^{\mathrm{op}} \to \mathcal{C} : b \mapsto (D_b, X)_R\,.$$

Observe that we then have $\mathrm{holim}_{b\in\mathbb{B}^{\mathrm{op}}} F' = \mathrm{holim}_{b\in\mathbb{B}^{\mathrm{op}}}(D_b, X)_R$ by construction. We also have, by Lem. 2.1.6, that $(D, X)_R$ is $\mathrm{holim}_{(\Delta/D)^{\mathrm{op}}} H$. It therefore suffices to show that $\mathrm{holim}_{\mathbb{B}^{\mathrm{op}}} F' \simeq \mathrm{holim}_{(f/g)^{\mathrm{op}}} G \simeq \mathrm{holim}_{(\Delta/D)^{\mathrm{op}}} H$ holds, with $\mathrm{holim}_{(f/g)^{\mathrm{op}}}$ the one to be defined.

Let $\mathbb{R}\kappa_*$ be a pointwise right derived functor of κ_*. The strategy for the first weak equivalence is to show that $\mathbb{R}\kappa_*(G) \simeq F'$. For this we first show that the functor $\epsilon_b : b/\kappa \to (\Delta/D_b)^{\mathrm{op}}$ for $b \in \mathbb{B}$ is also homotopy initial.

As in the previous proof, fix a $\sigma : \Delta[p] \to D_b$ in $(\Delta/D_b)^{\mathrm{op}}$, and let x be the element in ϵ_b/σ induced by the factorization of σ into an epi followed by a nondegenerate simplex. For an element y_1 in ϵ_b/σ as in (2.6), write again y_2 for the element in ϵ_b/σ induced by the factorization of θ_1 as an epi followed by a mono. Recall that we have morphisms $x \to y_2 \leftarrow y_1$. I claim this construction is in fact functorial, which will imply ϵ_b/σ is contractible.

Let a morphism $y_1 \to y_1'$ be given in ϵ_b/σ. Then by factorizing θ_1 resp. θ_1' as an epi followed by a mono, we get the following diagram

$$\begin{tikzcd}
 & \Delta[r_1'] \arrow[rr] \arrow[d] & & \Delta[r_1] \arrow[d] & \\
\Delta[r_2'] \arrow[ur, bend left] & D_{b_1'} \arrow[rr] \arrow[dr] & & D_{b_1} \arrow[dl] & \Delta[r_2] \arrow[ul, bend right] \\
 & & D_b & & \\
 & & \Delta[p] \arrow[u] \arrow[uull, bend left, "\theta_2'"] \arrow[uul, bend left, "\theta_1'"] \arrow[uur, bend left, "\theta_1"] \arrow[uuur, bend left, "\theta_2"] & &
\end{tikzcd}$$

Then, by factorizing $\Delta[r_2'] \to \Delta[r_1]$ as as epi followed by a mono, we get a unique arrow $\Delta[r_2'] \to \Delta[r_2]$ that fits in the above diagram. Since y_2, y_2' are induced by θ_2, θ_2' respectively, we hence have an arrow $y_2 \to y_2'$ in ϵ_b/σ. By uniqueness, it is clear this construction is functorial.

Let φ be the functor $y_1 \mapsto y_2$. From the uniqueness of the map $x \to y_2$, it follows that we have a natural transformation $c \Rightarrow \varphi$, where c is the constant functor on x. Also, from the above construction it follows that we have a functor $\mathrm{id}_{\epsilon_b/\sigma} \Rightarrow \varphi$. Since natural transformations induce homotopies on classifying spaces by [Qui73, §1], the identity functor on ϵ_b/σ is homotopic to the constant functor c, and therefore ϵ_b/σ is contractible.

Now take right deformations $R_{D_b}, R_{b/\kappa}$ and R_{Δ_*} for $\lim_{(\Delta/D_b)^{\mathrm{op}}}, \lim_{b/\kappa}$ and for Δ_* respectively: the ones that exist by the fact that $\mathcal{C}$ is sufficiently nice. With π_{D_b} the projection $(\Delta/D_b)^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}}$, the diagram from (2.5) becomes

$$\begin{tikzcd}
\lim_{(\Delta/D_b)^{\mathrm{op}}} \pi_{D_b}^* \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \lim_{b/\kappa} \epsilon_b^* \pi_{D_b}^* \arrow[d, Rightarrow] \\
\lim_{(\Delta/D_b)^{\mathrm{op}}} R_{D_b} \pi_{D_b}^* \arrow[d, Rightarrow] & \lim_{b/\kappa} R_{b/\kappa} \epsilon_b^* \pi_{D_b}^* \arrow[d, Rightarrow, "\simeq"] \\
\lim_{b/\kappa} \epsilon_b^* R_{D_b} \pi_{D_b}^* \arrow[r, Rightarrow] & \lim_{b/\kappa} R_{b/\kappa} \epsilon_b^* R_{D_b} \pi_{D_b}^*
\end{tikzcd} \tag{2.8}$$

Write μ for the obvious functor $(f/g)^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}}$. Recall that it holds $R_{D_b} \pi_{D_b}^* \cong \pi_{D_b}^* R_{\Delta_*}$ since $\mathcal{C}$ is sufficiently nice. Using this and Lem. 2.1.6, the above diagram becomes

$$\begin{tikzcd}
(D_b, -) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathrm{ev}_b\, \kappa_* \mu^* \arrow[d, Rightarrow] \\
(D_b, -)_R \arrow[d, Rightarrow] & \mathrm{ev}_b\, \mathbb{R}\kappa_* \mu^* \arrow[d, Rightarrow, "\simeq"] \\
\mathrm{ev}_b\, \kappa_* \mu^* R_{\Delta_*} \arrow[r, Rightarrow] & \mathrm{ev}_b\, \mathbb{R}\kappa_* \mu^* R_{\Delta_*}
\end{tikzcd}$$

But now it is clear that all of the above natural transformations are natural in b. By the assumption that homotopy limits in $\mathcal{C}$ are preserved by homotopy initial functors we therefore get weak equivalences

$$F'(b) = (D_b, X)_R \to \mathrm{ev}_b\, \mathbb{R}\kappa_* \mu^* R_{\Delta_*}(X) \leftarrow \mathrm{ev}_b\, \mathbb{R}\kappa_* \mu^*(X) = \mathbb{R}\kappa_*(G)(b) \tag{2.9}$$

which are natural in b. As such, this implies $\mathbb{R}\kappa_*(G) \simeq F'$.

With Lem. 2.1.7, it holds that $\operatorname{holim}_{\mathbb{B}^{\mathrm{op}}} \mathbb{R}\kappa_*$ is a right derived functor of the functor $\lim_{\mathbb{B}^{\mathrm{op}}} \kappa_* \cong \lim_{(f/g)^{\mathrm{op}}}$. From hereon, we take this one as $\operatorname{holim}_{(f/g)^{\mathrm{op}}}$, which gives us

$$\operatorname{holim}_{\mathbb{B}^{\mathrm{op}}} F' \simeq \operatorname{holim}_{\mathbb{B}^{\mathrm{op}}} \mathbb{R}\kappa_* G = \mathbb{R}(\lim_{\mathbb{B}^{\mathrm{op}}} \kappa_*)G = \operatorname{holim}_{(f/g)^{\mathrm{op}}} G\,.$$

The point of taking this $\operatorname{holim}_{(f/g)^{\mathrm{op}}}$, is that the above weak equivalence is given by specific maps $\cdot \to \cdot \leftarrow \cdot$ induced by (2.9). And these maps are clearly natural in X. They are also natural in $D_\bullet$. This follows from the fact that $\mathbb{R}\kappa_*$ is pointwise, hence can also be computed as $\kappa_* R_\kappa$, with R_κ a right deformation for κ_*. Then one proceeds by a similar construction as in (2.8) to show naturality in $D_\bullet$.

Consider again the functor $\lambda : (f/g)^{\mathrm{op}} \to (\Delta/D)^{\mathrm{op}}$. Let us show that λ is homotopy initial. For convenience, we this time redefine λ as $f/g \to \Delta/D$, and observe it suffices to show that this new λ is homotopy final, i.e. that for all $\sigma : \Delta[p] \to D$ it holds that σ/λ is weakly contractible. So let such an σ be given. Recall that $\lambda^{-1}(\sigma)$ is the category which has as objects those elements of f/g which are sent to σ by λ. A morphism $u \to v$ between such objects is an f/g-map from u to v which is sent to the identity on σ by λ. Write ψ for the obvious map $\lambda^{-1}(\sigma) \to \sigma/\lambda$, and φ for the obvious map $\sigma/\lambda \to \lambda^{-1}(\sigma)$. I claim that ψ is left adjoint to φ.

Let x be an object $\Delta[p] \xrightarrow{\eta} D_{b_1} \to D$ of $\lambda^{-1}(\sigma)$, and y an object $\Delta[p] \xrightarrow{h} \Delta[q] \xrightarrow{\tau} D_{b_2} \to D$ of σ/λ. Now a σ/λ-map α from $\psi(x)$ to y is given by the dotted arrows in the following diagram, with the lower one induced by a $\mathbb{B}$-map.

$$\begin{tikzcd}
 & \Delta[p] \arrow[dl, equal] \arrow[d, dashed, "\sigma"] \arrow[dr, "h"] & \\
\Delta[p] \arrow[r, dotted] \arrow[d, "\eta"'] & D & \Delta[q] \arrow[d, "\tau"] \\
D_{b_1} \arrow[ur] \arrow[rr, dotted] & & D_{b_2} \arrow[ul]
\end{tikzcd}$$

From the diagram we see that the top dotted arrow can only by $h : \Delta[p] \to \Delta[q]$. It follows that giving such an α is equivalent to giving a morphism $b_1 \to b_2$ such that the induced simplicial map fits in the diagram

$$\begin{tikzcd}
\Delta[p] \arrow[r, equal] \arrow[d, "\eta"] & \Delta[p] \arrow[d, "\tau h"] \\
D_{b_1} \arrow[r, dotted] & D_{b_2}
\end{tikzcd}$$

which is exactly an $\lambda^{-1}(\sigma)$-map from x to $\varphi(y)$.

Since $\psi \dashv \varphi$, it holds that σ/λ is homotopy equivalent to $\lambda^{-1}(\sigma)$ by [Qui73]. Now it is not difficult to see that the category B_σ from Par. 1.4.a is equivalent to $\lambda^{-1}(\sigma)$. Since $D_\bullet$ is acyclic, by Prop. 1.4.6 it holds that B_σ and hence $\lambda^{-1}(\sigma)$ and σ/λ are weakly contractible.

From the above it follows that the functor $(f/g)^{\mathrm{op}} \to (\Delta/D)^{\mathrm{op}}$ is homotopy initial. By our assumption on $\mathcal{C}$ this implies that $\operatorname{holim}_{(f/g)^{\mathrm{op}}} G \simeq \operatorname{holim}_{(\Delta/D)^{\mathrm{op}}} H$. Since

naturality is similar as before, we hence have a natural weak equivalence $(D, X)_R \simeq \operatorname{holim}_{b \in \mathbb{B}^{\mathrm{op}}} (D_b, X)_R$, which was to be shown. $\square$

The strategy of the above proof is inspired by [DK12, Prop. 5.1.10], where in our terminology it is shown combinatorial simplicial model categories have natural membranes.

2.4 A closer look

Fix a homotopical category $\mathcal{C}$ with natural membranes. Our goal is to prove the following

Theorem 2.4.1. *A 1-Segal object in $\mathcal{C}$ is 2-Segal.*

To this end we need two lemmata. In this strategy, I mainly follow [DK12, §2, 5], safe the necessary adjustments to accommodate for our more general framework. We use the notion of regular simplicial sets and the category of nondegenerate simplices (Def. 2.2.7).

Lemma 2.4.2. *For a regular simplicial set D the membrane object (D, X) is isomorphic to $\lim_{\Delta[p] \to D \in (\Delta/D_{\mathrm{nd}})^{\mathrm{op}}} X_p$, while $(D, X)_R$ is weakly equivalent to $\operatorname{holim}_{\Delta[p] \to D \in (\Delta/D_{\mathrm{nd}})^{\mathrm{op}}} X_p$.*

Proof. Observe, since D is regular, Lem. 2.2.10 gives us

$$D \cong \operatorname{colim}_{\Delta[p] \to D \in \Delta/D} \Delta[p] \cong \operatorname{colim}_{\Delta[p] \to D \in \Delta/D_{\mathrm{nd}}} \Delta[p] \,.$$

By the assumption on $\mathcal{C}$ and Exm. 2.3.5 we hence have

$$(D, X) \cong \lim_{\Delta[p] \to D \in (\Delta/D_{\mathrm{nd}})^{\mathrm{op}}} (\Delta[p], X) \cong \lim_{\Delta[p] \to D \in (\Delta/D_{\mathrm{nd}})^{\mathrm{op}}} X_p \,.$$

Employing the same strategy, for the second claim it suffices to show that the diagram $E : \Delta/D_{\mathrm{nd}} \to \mathrm{s}\mathcal{S}\mathrm{et}$ that sends $\Delta[p] \to D$ to $\Delta[p]$ is acyclic. To see this, let a p-simplex σ in $\operatorname{colim}_{\Delta/D_{\mathrm{nd}}} E = D$ be given. Factorize σ uniquely as $\Delta[p] \to \Delta[q] \xrightarrow{\sigma'} D$, with the first arrow epic and σ' nondegenerate. Then by using that D is regular, we see that this factorization determines an initial object in the category B_σ from before, hence that E is an acyclic diagram by Prop. 1.4.6, which was to be shown. $\square$

For simplicial sets $D, D' \subset \Delta[n]$, recall that we have simplicial subsets $D \cup D'$ and $D \cap D'$ of $\Delta[n]$, with sets of p-simplices $D_p \cup D'_p$ and $D_p \cap D'_p$ respectively. Observe that this induces maps of simplicial sets

$$D \leftarrow D \cap D' \to D' \,,$$

and hence a homotopy fiber product $(D, X)_R \times^R_{(D \cap D', X)_R} (D', X)_R$.

Lemma 2.4.3. *With D, D' as above it holds that $(D \cup D', X)_R$ is weakly equivalent to $(D, X)_R \times^R_{(D \cap D', X)_R} (D', X)_R$.*

Proof. Since $\mathcal{C}$ has natural membranes, it suffices to show that the diagram

$$D \leftarrow D \cap D' \rightarrow D'$$

is acyclic and has colimit $D \cup D'$. The latter is clear. For the former, note that a simplex $\sigma : \Delta[n] \rightarrow D \cup D'$ either factorizes only through D or D', or through both. In the first case, B_σ consists of one point, in the latter it is the category $\cdot \leftarrow \cdot \rightarrow \cdot$: both are clearly contractible. We are done again by Prop. 1.4.6. $\square$

Let $\mathcal{I}, \mathcal{I}'$ be subsets of $2^{[n]}$ for a fixed $n \geq 0$. Write $\mathcal{I} \sqcap \mathcal{I}'$ for the subset of $2^{[n]}$ formed by taking pairwise intersections of elements from $\mathcal{I}$ with elements from $\mathcal{I}'$. Observe that $\Delta[\mathcal{I} \cup \mathcal{I}']$ is the union, while $\Delta[\mathcal{I} \sqcap \mathcal{I}']$ is the intersection of $\Delta[\mathcal{I}]$ with $\Delta[\mathcal{I}']$. By the above lemma we hence have

$$\mathbb{R}X_{\mathcal{I} \cup \mathcal{I}'} \simeq \mathbb{R}X_{\mathcal{I}} \times^R_{\mathbb{R}X_{\mathcal{I} \sqcap \mathcal{I}'}} \mathbb{R}X_{\mathcal{I}'} \,. \tag{2.10}$$

Now let $\mathcal{T}$ be a triangulation of $[n]$ for $n \geq 3$. By induction on the number of triangles in $\mathcal{T}$ and by applying (2.10) in each step, we see that $\mathbb{R}X_{\mathcal{T}}$ is actually weakly equivalent to the homotopy limit of the diagram over all nondegenerate simplices $\Delta[p] \rightarrow \Delta[\mathcal{T}]$ with $p \geq 1$: a result promised back in the proof of Prop. 2.2.13 and also used in the proof below of the fact that 1-Segal objects are 2-Segal.

Proof of Thm. 2.4.1. Let $X_\bullet$ be a 1-Segal object, $\mathcal{T}$ a triangulation of $[n]$ for $n \geq 3$. Then the inclusions $\Delta[\mathcal{I}_n] \subset \Delta[\mathcal{T}] \subset \Delta[n]$ induce a commutative triangle

$$\begin{tikzcd} X_n \arrow[r, "f_{\mathcal{T}}"] \arrow[dr, "f_n"'] & \mathbb{R}X_{\mathcal{T}} \arrow[d, "h"] \\ & \mathbb{R}X_{\mathcal{I}_n} \end{tikzcd}$$

with $f_n, f_{\mathcal{T}}$ the Segal maps. Since X is 1-Segal, f_n is a weak equivalence. By 2-of-3, it hence suffices to show h is one as well.

We are going to induct on n. For the base case $n = 3$, suppose without loss of generality that $\mathcal{T}$ is given by the triangles $\{0,1,2\}, \{0,2,3\}$. Then by the fact that X is 1-Segal, that $\mathbb{R}X_p$ is naturally weak equivalent to X_p, and that $\mathbb{R}X_{\mathcal{T}}$ can be calculated on nondegenerate simplices of dimension ≥ 1, it holds

$$\begin{aligned} \mathbb{R}X_{\mathcal{T}} &\simeq X_{\{0,1,2\}} \times^R_{X_{\{0,2\}}} X_{\{0,2,3\}} \,; \\ &\simeq (X_{\{0,1\}} \times^R_{X_{\{1\}}} X_{\{1,2\}}) \times^R_{X_{\{0,2\}}} (X_{\{0,2\}} \times^R_{X_{\{2\}}} X_{\{2,3\}}) \,; \\ &\simeq X_{\{0,1\}} \times^R_{X_{\{1\}}} X_{\{1,2\}} \times^R_{X_{\{2\}}} X_{\{2,3\}} = X_{\mathcal{I}_3} \,, \end{aligned}$$

since identity arrows cancel in homotopy fiber products. Because these weak equivalences are induced by the same inclusions in $\mathbf{\Delta}$ as h is, we see that h is a weak equivalence.

For the induction step, suppose $n > 3$. Then take a triangle of $\mathcal{T}$ with edge not on the boundary of P_n. Without loss of generality, we may assume this triangle is $\{0, i, n\}$ with

$\{0, i\}$ not on the boundary, i.e. such that $i \neq 1$. Then $\mathcal{T}$ induces triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of $\{0, 1, \ldots, i\}$ and $\{i, i+1, \ldots, n\}$ respectively. Now after twice applying the formula (2.10), we see that the morphism h is weakly equivalent to

$$\mathbb{R}X_{\mathcal{T}_1} \times^R_{X_{\{0,i\}}} X_{\{0,i,n\}} \times^R_{X_{\{i,n\}}} \mathbb{R}X_{\mathcal{T}_2} \to \mathbb{R}X_{\mathcal{I}_n} .$$

By naturality and since X is 1-Segal, this in turn is weakly equivalent to the map

$$\mathbb{R}X_{\mathcal{T}_1} \times^R_{X_{\{i\}}} \mathbb{R}X_{\mathcal{T}_2} \to \mathbb{R}X_{\mathcal{I}_i} \times^R_{X_{\{i\}}} \mathbb{R}X_{\mathcal{I}_{n-i}} ,$$

which is indeed a weak equivalence by the induction hypothesis. □

Example 2.4.4. With this theorem, we see that the converse of Prop. 2.2.13 does not hold in general, i.e. that a 2-Segal set might not be an oriented graph. Take for example the nerve X of a category $\mathbb{C}$. Then X is 1-Segal, and therefore also 2-Segal. By Exm. 2.2.14 it holds that if X is an oriented graph, then $\mathbb{C}$ has no nontrivial composable arrows: a condition that clearly does not hold in general.

2.5 Polygonal subdivision and the pullback condition

Let still $\mathcal{C}$ be a homotopical category with natural membranes. The goal of this part is to give a more tractable way of recognizing 2-Segal objects in $\mathcal{C}$.

Let $n \geq 0$. Analogous to the notion of a triangulation of $[n]$, we define a *polygonal subdivision* of $[n]$ as a subset $\mathcal{P} \subset 2^{[n]}$ such that the corresponding subsets of vertices of P_n induce a polygonal subdivision. If we order all polygonal subdivision of $[n]$ by refinement, we see that the triangulations of $[n]$ are exactly the maximal polygonal subdivisions.

Recall that a square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is called *homotopy cartesian* if the morphism $A \to C \times_D B \to C \times^R_D B$ is a weak equivalence. Now fix a simplicial object $X_\bullet$ in $\mathcal{C}$. Note that a polygonal subdivision $\mathcal{P}$ of $[n]$ induces an inclusion $\Delta[\mathcal{P}] \subset \Delta[n]$, hence a morphism

$$f_{\mathcal{P}} : X_n \to \mathbb{R}X_{\mathcal{P}} = (\Delta[\mathcal{P}], X)_R .$$

Proposition 2.5.1. *The following conditions are equivalent:*

1. *$X_\bullet$ is a 2-Segal object in $\mathcal{C}$;*
2. *For every polygonal subdivision $\mathcal{P}$ of $[n]$ the map $f_{\mathcal{P}}$ is a weak equivalence;*
3. *For $n \geq 3$ and $0 \leq i < j \leq n$ the following square is homotopy cartesian:*

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & X_{\{i,\dots,j\}} \arrow[d] \\ X_{\{0,\dots,i,j\dots,n\}} \arrow[r] & X_{\{i,j\}} \end{tikzcd}$$

4. The same condition as (3), but with only $i = 0$ or $j = n$.

We mimic the proof of [DK12, Prop. 2.3.2], where the above statement in the case $\mathcal{C} = \mathcal{T}\mathrm{op}$ is shown.

Proof. The implications (1) $\Leftarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are clear. For (1) $\Rightarrow$ (2) we induct on n. Observe that for $n \leq 3$ there is nothing to prove. For the induction step, let $\mathcal{P}$ be a polygonal subdivision of $[n]$. By adding in edges, we complete this to a triangulation $\mathcal{T}$, which gives us the inclusions $\Delta[\mathcal{T}] \subset \Delta[\mathcal{P}] \subset \Delta[n]$, hence a diagram

$$\begin{tikzcd} X_n \arrow[r, "f_{\mathcal{P}}"] \arrow[dr, "f_{\mathcal{T}}"'] & \mathbb{R}X_{\mathcal{P}} \arrow[d, "h"] \\ & \mathbb{R}X_{\mathcal{T}} \end{tikzcd}$$

By assumption, $f_{\mathcal{T}}$ is a weak equivalence. By 2-of-3, it suffices to show h is as well.

Suppose $\mathcal{P}, \mathcal{T}$ have no common edge in the interior of P_n. Then in fact $\mathcal{P}$ is the trivial polygonal subdivision, i.e. $\Delta[\mathcal{P}]$ is just $\Delta[n]$, in which case surely $f_{\mathcal{P}}$ is a weak equivalence. So assume we have such a common edge, and take one, say $\{s, t\}$. Divide P_n along this edge $\{s, t\}$ into two polygons P_n', P_n'', endowed with polygonal subdivisions $\mathcal{P}', \mathcal{P}''$ and triangulations $\mathcal{T}', \mathcal{T}''$ induced by $\mathcal{P}$ and $\mathcal{T}$. Then h is weakly equivalent to the map

$$\mathbb{R}X_{\mathcal{P}'} \times^R_{X_{\{s,t\}}} \mathbb{R}X_{\mathcal{P}''} \to \mathbb{R}X_{\mathcal{T}'} \times^R_{X_{\{s,t\}}} \mathbb{R}X_{\mathcal{T}''} ,$$

which is a weak equivalence by the induction hypothesis. This implies h is one as well.

For the implication (4) $\Rightarrow$ (1), we employ a similar induction as the previous one, this time using that a triangulation of $[n]$ either has an edge of the form $\{0, j\}$ or of $\{i, n\}$ in the interior of P_n. $\square$

Example 2.5.2. Let Y be a compact Riemann surface. Consider the simplicial space X, defined as $X_n \coloneqq Y^{\Delta^n}$, with obvious structure maps. I claim X is 2-Segal.

Let I be the unit interval, and let $n \geq 3$ and $0 \leq i \leq n$ be given. Write P for the space $\Delta^{\{0,1,\dots,i\}} \amalg_{\{0\}\times\Delta^{\{0,i\}}} (I \times \Delta^{\{0,i\}}) \amalg_{\{1\}\times\Delta^{\{0,i\}}} \Delta^{\{0,i,\dots,n\}}$. Take a projection

$$\pi : \Delta^n \to P$$

as follows. First take the projection $\Delta^n \to \Delta^{\{0,1,\dots,i\}} \amalg_{\Delta^{\{0,i\}}} \Delta^{\{0,i,\dots,n\}}$, and then collapse a small cylinder around the axis $[0, i]$ in the latter space.

Now consider the Segal map

$$f : X_n \to X_{\{0,1,\dots,i\}} \times^R_{X_{\{0,i\}}} X_{\{0,i,\dots,n\}} .$$

By the previous proposition and by symmetry, it suffices to show that the above map is a homotopy equivalence.

Observe that a point in the space on the right is given by a map

$$\varphi : P \to Y\,.$$

Let g be the map $X_{\{0,1,\ldots,i\}} \times^R_{X_{\{0,i\}}} X_{\{0,i,\ldots,n\}} \to X_n$ that sends such a point φ to the composition $\varphi\pi$. Now I claim g is a homotopy inverse of f.

Let $\psi \in X_n$ be given. Then $gf(\psi)$ is the composition

$$\Delta^n \to \Delta^{\{0,1,\ldots,i\}} \amalg_{\Delta^{\{0,i\}}} \Delta^{\{0,i,\ldots,n\}} \to \Delta^n \xrightarrow{\psi} Y\,,$$

with the first arrow the projection and the second one the inclusion. Because Y is a surface, the above map is homotopical to ψ itself. It is clear that, with some care, these homotopies $gf(\psi) \simeq \psi$ combine into a single homotopy $gf \simeq \mathrm{id}_{X_n}$.

Likewise, let $\varphi \in X_{\{0,1,\ldots,i\}} \times^R_{X_{\{0,i\}}} X_{\{0,i,\ldots,n\}}$ be given. Then $fg(\varphi)$ is the map $P \to Y$ that is constant on $I \times \Delta^{\{0,i\}}$ and is the restriction of $\varphi\pi$ on $\Delta^{\{0,1,\ldots,i\}}$ and on $\Delta^{\{0,i,\ldots,n\}}$. These latter two restrictions are homotopic to the restrictions of φ itself on $\Delta^{\{0,1,\ldots,i\}}$ and on $\Delta^{\{0,i,\ldots,n\}}$. Therefore $fg(\varphi) \simeq \varphi$ holds, and again these homotopies can be combined to give a homotopy between fg and the identity on $X_{\{0,1,\ldots,i\}} \times^R_{X_{\{0,i\}}} X_{\{0,i,\ldots,n\}}$.

3. Non-additive 2-Segal K-theory

> When God calculates and cogitates,
> the world is made.
>
> Margin of *Dialogus*
> Leibniz

Let $\mathcal{M}$ be a pointed model category such that all objects in $\mathcal{M}$ are cofibrant, and let $\mathcal{E}$ be an exact category (see Def. C.2). Write $\mathrm{co}\mathcal{M}$ for the class of cofibrations in $\mathcal{M}$, and $\mathfrak{M}$ for the class of admissible monos in $\mathcal{E}$. Note that $(\mathcal{M}, \mathrm{co}\mathcal{M})$ and $(\mathcal{E}, \mathfrak{M})$ have some similarities. For one, $\mathrm{co}\mathcal{M}$ and $\mathfrak{M}$ both contain all isomorphisms. Furthermore, both $\mathcal{M}$ and $\mathcal{E}$ are pointed, say with point 0. And in both cases we have that the unique map $0 \to X$ for any X in $\mathcal{M}$ resp. in $\mathcal{E}$ is in the class $\mathrm{co}\mathcal{M}$ resp. in $\mathfrak{M}$. Also, both $\mathrm{co}\mathcal{M}$ and $\mathfrak{M}$ admit cobase change along all maps. Indeed, let $A \to B$ be a given cofibration in $\mathcal{M}$, and $A \to C$ an arbitrary map. Then $C \to B \cup_A C$ has the left lifting property with respect to any trivial fibration, since $A \to B$ has this property and by the universal property of the pushout. Hence $C \to B \cup_A C$ is again a cofibration by [GJ09, Lem. II.1.1].

We have just shown $(\mathcal{M}, \mathrm{co}\mathcal{M})$ and $(\mathcal{E}, \mathfrak{M})$ are both cases of *categories with cofibrations*. We can further add to $\mathcal{M}$ the weak equivalences $\mathrm{w}\mathcal{M}$ and to $\mathcal{E}$ the isomorphisms $\mathrm{i}\mathcal{E}$ to get instances of *Waldhausen categories*, as formulated in [Wal85]. Essentially, this is a category $\mathcal{C}$ endowed with cofibrations $\mathrm{co}\mathcal{C}$ and weak equivalences $\mathrm{w}\mathcal{C}$, which satisfy some broad requirements that the above two examples have in common, but see Def. C.3 for details.

Now Waldhausen associates to $\mathcal{C}$ a simplicial object $\mathrm{w}\mathcal{S}_\bullet\mathcal{C}$ in $\mathcal{C}\mathrm{at}$ in order to define the K-groups of $\mathcal{C}$. It would be nice if $\mathrm{w}\mathcal{S}_\bullet\mathcal{C}$ were always 2-Segal, but alas this is not so. It is however the case that if $\mathcal{C}$ comes from an exact category by forgetting some structure as done in the above, then $\mathrm{w}\mathcal{S}_\bullet\mathcal{C}$ is 2-Segal. A question then arises: is there an analogue to Waldhausen categories that generalizes the notion of exact categories, but in such a way that it allows for an $\mathcal{S}_\bullet$-construction which is 2-Segal? A positive answer is provided by [DK12] in the form of proto-exact categories. And as we shall see below, the theory of Segal objects is a nice stepping stone towards some K-theory on such proto-exact categories.

3.1 Classical $\mathrm{w}\mathcal{S}_\bullet$-construction

Let $\mathcal{C}$ be a Waldhausen category. Cofibrations are written as $A \rightarrowtail B$. For such a cofibration we write $B \twoheadrightarrow B/A$ for its pushout along $A \to 0$:

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[d, two heads] \\ 0 \arrow[r] & B/A \end{tikzcd}$$

and call $A \rightarrowtail B \twoheadrightarrow B/A$ a *cofibration sequence*, and $B \twoheadrightarrow B/A$ a *quotient map*.

For $n \geq 0$ write T_n for the category of functors $[1] \to [n]$. Note that we can identify T_n with the poset of pairs (i,j) such that $0 \leq i \leq j \leq n$, with the ordering given by $(i,j) \leq (k,l)$ if $i \leq k$ and $j \leq l$. Hence, a functor $F : T_n \to \mathcal{C}$, written as $(i,j) \mapsto F_{ij}$, gives a nice 'staircase'

$$\begin{tikzcd} F_{00} \arrow[r] & F_{01} \arrow[r] \arrow[d] & \cdot \arrow[r, dashed] \arrow[d, dashed] & \cdot \arrow[r] \arrow[d, dashed] & F_{0,n-1} \arrow[r] \arrow[d] & F_{0n} \arrow[d] \\ & F_{11} \arrow[r] & \cdot \arrow[r, dashed] \arrow[d, dashed] & \cdot \arrow[r] \arrow[d, dashed] & F_{1,n-1} \arrow[r] \arrow[d] & F_{1n} \arrow[d] \\ & & \cdot \arrow[r, dashed] & \cdot \arrow[r, dashed] \arrow[d, dashed] & \cdot \arrow[r, dashed] \arrow[d, dashed] & \cdot \arrow[d, dashed] \\ & & & \cdot \arrow[r, dashed] & \cdot \arrow[r, dashed] \arrow[d] & \cdot \arrow[d] \\ & & & & F_{n-1,n-1} \arrow[r] & F_{n-1,n} \arrow[d] \\ & & & & & F_{nn} \end{tikzcd} \tag{3.1}$$

Let $\mathrm{w}\mathcal{S}_n\mathcal{C}$ be the subcategory of $\mathcal{C}^{T_n}$ on all such staircases $F : T_n \to \mathcal{C}$ such that for all $i \leq j \leq k$ the composition $F_{ij} \to F_{ik} \to F_{jk}$ is a cofibration sequences, and with all F_{ii} equal to 0. As morphisms we take the pointwise weak equivalences. A consequence of this definition is that, for an $F \in \mathrm{w}\mathcal{S}_n\mathcal{C}$, each square is a pushout in the above staircase diagram.

Observe that, by post-composition, the T_n form a cosimplicial object $T : \mathbf{\Delta} \to \mathcal{C}\mathrm{at}$. For $[n] \to [m]$ this gives us an obvious functor $\mathrm{w}\mathcal{S}_m\mathcal{C} \to \mathcal{C}^{T_n}$, and in fact this functor lands in $\mathrm{w}\mathcal{S}_n\mathcal{C}$ since pushout squares compose into pushout squares and cofibrations are preserved under composition. We hence have a simplicial object $\mathrm{w}\mathcal{S}_\bullet\mathcal{C}$ in $\mathcal{C}\mathrm{at}$.

For $0 < i < n$ and $F \in \mathrm{w}\mathcal{S}_{n+1}\mathcal{C}$, note that d_iF is obtained from F by composing at the vertical line $(F_{si})_s$ and at the horizontal line of $(F_{it})_t$, while d_0F resp. d_nF is given by removing the top row resp. the right column. Likewise, for $0 \leq j \leq n-1$ and $G \in \mathrm{w}\mathcal{S}_{n-1}\mathcal{C}$, it holds that s_jG is obtained from G by introducing identities between

the $(G_{sj})_s$ and the $(G_{s,j+1})_s$ columns and between the $(G_{jt})_t$ and the $(G_{j+1,t})_t$ rows. In general, for $H \in \mathrm{w}\mathcal{S}_m\mathcal{C}$ and $f : [n] \to [m]$ it holds that $f^*(H)_{ij} = H_{f(i),f(j)}$.

Proposition 3.1.1. *If* $\mathrm{w}\mathcal{S}_\bullet\mathcal{C}$ *is 2-Segal, then the class of quotient maps in* $\mathcal{C}$ *is closed under composition.*

Proof. Suppose that $\mathcal{S}_\bullet := \mathrm{w}\mathcal{S}_\bullet\mathcal{C}$ is 2-Segal. Let σ and τ be two cofibration sequences $A \rightarrowtail B \twoheadrightarrow B/A$ and $C \rightarrowtail B/A \twoheadrightarrow (B/A)/C$ respectively. By adding zeros, σ and τ can be completed to 'staircases', i.e. we get an element (σ, τ) in $\mathcal{S}_{\{0,1,3\}} \times^R_{\mathcal{S}_{\{1,3\}}} \mathcal{S}_{\{1,2,3\}}$. Since $\mathcal{S}_\bullet$ is 2-Segal, we have an element F in $\mathcal{S}_3$ together with an isomorphism $\varphi : (d_2F, d_0F) \to (\sigma, \tau)$ in $\mathcal{S}_{\{0,1,3\}} \times^R_{\mathcal{S}_{\{1,3\}}} \mathcal{S}_{\{1,2,3\}}$.

Observe that φ gives us the following two squares

$$\begin{tikzcd} F_{03} \arrow[r] \arrow[d] & B \arrow[d, two heads] \\ F_{13} \arrow[r, "f"] & B/A \end{tikzcd} \qquad \begin{tikzcd} F_{13} \arrow[r, "f'"] \arrow[d] & B/A \arrow[d, two heads] \\ F_{23} \arrow[r] & (B/A)/C \end{tikzcd}$$

with the horizontal maps part of φ, hence isomorphisms, and the vertical maps part of F resp. of σ, τ. Furthermore, the isomorphisms $d_2F \to \sigma$ and $d_0F \to \tau$ given by φ must agree on $\mathcal{S}_{\{1,3\}}$, from which it follows that f and f' in the above diagram are actually the same. The given quotient maps therefore fit into the commutative diagram

$$\begin{tikzcd} F_{02} \arrow[r, tail] \arrow[dd] & F_{03} \arrow[r, equal, "\sim"] \arrow[d] & B \arrow[d, two heads] \\ & F_{13} \arrow[r, equal, "\sim"] \arrow[d] & B/A \arrow[d, two heads] \\ 0 \arrow[r] & F_{23} \arrow[r, equal, "\cong"] & (B/A)/C \end{tikzcd}$$

As the square on the left is a pushout square, the large square is one as well. Furthermore, since $\mathrm{co}\mathcal{C}$ is closed under composition and contains all isomorphisms, $F_{02} \to B$ is a cofibration. It follows that $B \to (B/A)/C$ is a quotient map, which was to be shown. $\square$

Example 3.1.2. Consider the category $\mathcal{S}\mathrm{et}_*$ of pointed sets, and let $\mathrm{co}\mathcal{S}\mathrm{et}_*$ be the inclusions. Then it is straightforward to show that this makes $\mathcal{S}\mathrm{et}_*$ into a category with cofibrations, for which the class of quotient maps is closed under composition.

One can vary the above example to a number of categories of 'pointed objects', such as the category of pointed simplicial sets with pointed simplicial inclusions as cofibrations. Nonetheless, not every category with cofibrations has a class of quotient maps that is closed under composition, as illustrated in the following

Example 3.1.3. We will construct a Waldhausen category $\mathcal{C}$ in stages as follows. I claim we have a sequence of full pointed subcategories

$$\mathcal{C}_0 \subset \mathcal{C}_1 \subset \mathcal{C}_2 \subset \dots$$

with for each $\mathcal{C}_i$ a class of maps $\mathrm{co}\mathcal{C}_i$, closed under composition and satisfying $\mathrm{co}\mathcal{C}_0 \subset \mathrm{co}\mathcal{C}_1 \subset \dots$, such that:

1. All maps in $\mathcal{C}_i$ of the form $0 \to W$ and all identities are in $\mathrm{co}\mathcal{C}_i$;
2. A square in $\mathcal{C}_i$ is a pushout in $\mathcal{C}_i$ iff it is so in $\mathcal{C}_{i+1}$;
3. Each $\mathrm{co}\mathcal{C}_i$ is closed in $\mathcal{C}_{i+1}$ under pushouts along arbitrary maps in $\mathcal{C}_i$;
4. Each $\mathcal{C}_i$ is skeletal: i.e. has no nontrivial isomorphisms.

For the first step, let $\mathcal{C}_0$ be the following category:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] & \\ 0 \arrow[r] & B/A \arrow[d] & C \arrow[l] \arrow[d] \\ & (B/A)/C & 0 \arrow[l] \end{tikzcd} \tag{3.2}$$

together with morphisms $0 \to X$ and $X \to 0$ for $X \in \mathcal{C}$ in such a way that $\mathcal{C}_0$ becomes pointed by 0. Let $\mathrm{co}\mathcal{C}_0$ be the identity arrows, the arrows $0 \to X$ and the two maps $A \to B$ and $C \to B/A$. Then clearly $\mathrm{co}\mathcal{C}_0$ is closed under composition, and contains the maps of the form $0 \to Z$ and the identities. It is also clear that $\mathcal{C}_0$ is skeletal.

For the induction step, suppose that $\mathcal{C}_0, \dots, \mathcal{C}_n$ and $\mathrm{co}\mathcal{C}_0, \dots, \mathrm{co}\mathcal{C}_n$ are constructed. Then let $\mathcal{C}_{n+1}$ be the closure of $\mathcal{C}_n$ under pushouts of maps in $\mathrm{co}\mathcal{C}_n$ along arbitrary maps in $\mathcal{C}_n$. Hence, for each $X \to Y$ in $\mathrm{co}\mathcal{C}_n$ and $X \to Z$ in $\mathcal{C}_n$, if the pushout $Z \cup_X Y$ does not exist in $\mathcal{C}_n$, then we add a unique square

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & Z \cup_X Y \end{tikzcd} \tag{3.3}$$

together with the necessary arrows $Z \cup_X Y \to W$ to make such squares into pushouts in $\mathcal{C}_{n+1}$. Then we close this construction under composition. We let $\mathrm{co}\acute{}\,\mathcal{C}_{n+1}$ be $\mathrm{co}\mathcal{C}_n$, together with all maps of the form $Z \to Z \cup_X Y$ or $0 \to Z \cup_X Y$ or $Z \cup_X Y = Z \cup_X Y$ for X, Y, Z as above. Then we take the closure of $\mathrm{co}\acute{}\,\mathcal{C}_{n+1}$ under composition as $\mathrm{co}\mathcal{C}_{n+1}$.

A consequence of this construction is that for X, Y, Z as above and V in $\mathcal{C}_n$, the only arrows $V \to Z \cup_X Y$ in $\mathcal{C}_{n+1}$ are the ones that factorize over Z or Y via an arrow which is already in $\mathcal{C}_n$. This immediately implies that $\mathcal{C}_n$ is a full subcategory of $\mathcal{C}_{n+1}$. Let us verify this construction satisfies the requirements.

For the first point, let W in $\mathcal{C}_{n+1}$ be given. Then either $W \in \mathcal{C}_n$ or W is of the form $Z \cup_X Y$ as above. In both cases it holds that $0 \to W$ and $W = W$ are in $\mathrm{co}\mathcal{C}_{n+1}$: by induction or by construction respectively.

For the second point, let a square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Q \arrow[d] \\ R \arrow[r] & S \end{tikzcd}$$

in $\mathcal{C}_n$ be given. First suppose it is a pushout in $\mathcal{C}_n$. To show that it remains so in $\mathcal{C}_{n+1}$, since $\mathcal{C}_n$ is full in $\mathcal{C}_{n+1}$ we only need to check the universal property with respect to the newly added objects of the form $Z \cup_X Y$. So suppose we have arrows $Q \to Z \cup_X Y \leftarrow R$, that make the obvious square commute. Since they make this square commute, they either both factorize over Z or both over Y by arrows in $\mathcal{C}_n$: suppose over Z. In a picture we get

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Q \arrow[d] \arrow[ddr, bend left] \arrow[ddrr, bend left] & & \\ R \arrow[r] \arrow[drr, bend right] \arrow[drrr, bend right] & S \arrow[dr, dotted] \arrow[drr, dotted] & & \\ & & Z \cup_X Y & Z \arrow[l] \end{tikzcd}$$

The dotted arrow $S \to Z$ exists uniquely since the square is assumed to be a pushout in $\mathcal{C}_n$. By composition, the dotted arrow $S \to Z \cup_X Y$ exists uniquely as well. Hence our square remains a pushout in $\mathcal{C}_{n+1}$. The converse also holds by fullness of $\mathcal{C}_n$ in $\mathcal{C}_{n+1}$.

The third point clearly holds by construction.

For the fourth point, suppose a nontrivial isomorphism $W' \cong W$ in $\mathcal{C}_{n+1}$ is given. By induction, we may assume it is of the form $Z \cup_X Y \to W$, with X, Y, Z as above. Then this map must be induced by a diagram of the form

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d] & Y \arrow[d] \arrow[ddr, bend left] & \\ Z \arrow[r] \arrow[drr, bend right] & Z \cup_X Y \arrow[dr] & \\ & & W \end{tikzcd} \tag{3.4}$$

Since the map $Z \cup_X Y \to W$ is an isomorphism, the outer square is a pushout. Then if W is in $\mathcal{C}_n$, the pushout $Z \cup_X Y$ would not have been added. Hence W is in $\mathcal{C}_{n+1}$, and the arrow $Z \cup_X Y \to W$ is the identity because the pushout squares that we add are unique.

This finishes our construction by induction. Now let $\mathcal{C}$ be the category $\cup_{i \geq 0} \mathcal{C}_i$, which is well-defined since our sequence $\mathcal{C}_0 \subset \mathcal{C}_1 \subset \dots$ consists of full subcategories. Likewise, let $\mathrm{co}\mathcal{C}$ be $\cup_{i \geq 0} \mathrm{co}\mathcal{C}_i$. Then since any diagram of the form $Z \leftarrow X \to Y$ in $\mathcal{C}$ with $X \to Y$ in $\mathrm{co}\mathcal{C}$ is constructed in some stage n, the pushout of this diagram exists in the next stage, and hence in every subsequent stage, and hence in $\mathcal{C}$. Also, since all of the $\mathcal{C}_i$ are skeletal, so is $\mathcal{C}$. Since each $\mathrm{co}\mathcal{C}_i$ contains all identities and all maps of the form $0 \to W$, it follows that $\mathrm{co}\mathcal{C}$ contains all identities, hence all isomorphisms, and also all maps of the form $0 \to W$. It follows that $\mathcal{C}$ is indeed a Waldhausen category, with $\mathrm{co}\mathcal{C}$ as cofibrations.

However, the arrow $B \to (B/A)/C$ with A, B, C the fixed objects in $\mathcal{C}_0$ cannot be a quotient map in $\mathcal{C}$. Indeed, suppose that we have a pushout square in $\mathcal{C}$ of the form

$$\begin{tikzcd} K \arrow[r, tail] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & (B/A)/C \end{tikzcd}$$

Observe that we already have pushouts

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[d] & B \arrow[d, equal] \\ 0 \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, equal] \arrow[d] & B \arrow[d] \\ 0 \arrow[r, equal] & 0 \end{tikzcd}$$

Because $\mathcal{C}$ is skeletal, it follows that $K \neq 0$ and that $K \to B$ is not id_B.

Suppose $K \rightarrowtail B$ is the zero map. Then consider the following diagram

$$\begin{tikzcd} K \arrow[r, tail] \arrow[d] & B \arrow[d] \arrow[ddr, bend left, "\mathrm{id}"] & \\ 0 \arrow[r] \arrow[drr, bend right] & (B/A)/C \arrow[dr, dotted] & \\ & & B \end{tikzcd}$$

where the dotted arrow exists because we have assumed our square is a pushout. But the only arrow $(B/A)/C \to B$ is the zero map, from which it follows that id_B is the zero map, which is absurd.

With similar reasoning, one shows that $K \neq A$. It follows that K cannot be in $\mathcal{C}_0$.

But K can neither be in $\mathcal{C}_n$ for $n > 0$. For else it must be a composition of the form $K \rightarrowtail K' \rightarrowtail B$ with $K' \rightarrowtail B$ as the lower arrow $K' \rightarrowtail K' \cup_L M$ in some pushout square, and not equal to id_B. But observe, if this pushout was added at stage n, then since B is in $\mathcal{C}_0$ it would already be a pushout in $\mathcal{C}_n$, which is impossible unless $n = 0$. By construction of $\mathrm{co}\mathcal{C}_0$, it then follows $K' = 0$. Hence the map $K \rightarrowtail B$ is a zero map, which we saw is impossible.

Now we finish by the observation that the squares in (3.2) are both pushouts in $\mathcal{C}_0$, hence pushouts in $\mathcal{C}$. The arrows $B \to B/A$ and $B/A \to (B/A)/C$ are therefore both quotient maps. But as we have seen, their composition is not.

3.2 Proto-exact categories

Consider a given commutative square in some category

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "q"'] & B \arrow[d, "p"] \\ C \arrow[r, "j"] & D \end{tikzcd} \tag{3.5}$$

We call (3.5) *cartesian* if it is a pullback, *cocartesian* when it is a pushout and *bicartesian* when it is both.

Definition 3.2.1. A *proto-exact category* is a category $\mathcal{P}$ together with two classes of morphisms $\mathfrak{M}$ and $\mathfrak{E}$, called the *admissible monos* and the *admissible epis* respectively. We ask that $\mathfrak{M}$ and $\mathfrak{E}$ contain all isomorphisms, are closed under composition, and satisfy the following properties:

(Pointed) $\mathcal{P}$ is pointed in such a way that for the point $0 \in \mathcal{P}$ it holds that all $0 \to A$ are admissibly monic and all $A \to 0$ admissibly epic;

(Bicartesian) Every square (3.5) in $\mathcal{P}$ with $i, j \in \mathfrak{M}$ and $p, q \in \mathfrak{E}$ is cartesian iff it is cocartesian;

(Pullback) Every $C \xrightarrow{j} D \xleftarrow{p} B$ with $j \in \mathfrak{M}$ and $p \in \mathfrak{E}$ can be completed to a bicartesian square (3.5) such that $i \in \mathfrak{M}$ and $q \in \mathfrak{E}$;

(Pushout) Every $C \xleftarrow{q} A \xrightarrow{i} B$ with $i \in \mathfrak{M}$ and $q \in \mathfrak{E}$ can be completed to a bicartesian square (3.5) such that $j \in \mathfrak{M}$ and $p \in \mathfrak{E}$.

In a proto-exact category, admissible monos are written as $A \rightarrowtail B$, while admissible epis are written as $A \twoheadrightarrow B$. A bicartesian square (3.5) with $i, j \in \mathfrak{M}$ and $p, q \in \mathfrak{E}$ is called an *admissible square*. The notation $A \rightarrowtail B \twoheadrightarrow B/A$ again means these arrows fit in an admissible square (3.5), with $D = B/A$ and with $C = 0$.

Example 3.2.2. Let $\mathcal{P}$ be a proto-exact category which is skeletal, i.e. such that each isomorphism class in $\mathcal{P}$ consists of a single object. Then $\mathcal{P}$ is an augmented, stable double category (Rem. 2.2.15), with $\mathfrak{M}$ and $\mathfrak{E}$ as horizontal and vertical arrows respectively, with admissible squares as squares, and with $\{0\}$ as augmentation.

What is more, the equivalence from the category of augmented, stable double categories to the category of unital 2-Segal sets as given in [BOORS16] is exhibited by a 'generalized $S_\bullet$ construction'. For such a double category $\mathcal{D}$ with squares T and augmentation A one lets $S_\bullet\mathcal{D}$ be the simplicial set that has in degree n staircases of the form (3.1) that have all of their squares in T, and with objects in A on the anti-diagonal.

Clearly, the notion of proto-exact categories is self-dual. It is also convenient to know that the admissible epis are completely determined by the admissible monos, and the other way around. Indeed, any admissible epi $B \twoheadrightarrow C$ fits as the right arrow in an admissible square that has $0 \rightarrowtail C$ at the bottom. Therefore, the class of those admissible epis that are of the form $B \twoheadrightarrow B/A$ for some admissible mono $A \rightarrowtail B$, is in fact all of $\mathfrak{E}$.

Admissible monos in a proto-exact category $\mathcal{P}$ are always monic (in the ordinary sense), which can be seen by taking the pushout of an admissible mono $X \rightarrowtail Y$ along the admissible epi $X \twoheadrightarrow 0$ and by then exploiting $X = 0 \times_{Y/X} Y$ in the obvious way. Dually, admissible epis are always epic.

Example 3.2.3 ([DK12, Exm. 2.4.4])**.** Let $\mathcal{S}\mathrm{et}_*$ be the category of pointed sets. Recall one sometimes says that a sequence $(X,x) \xrightarrow{f} (Y,y) \xrightarrow{g} (Z,z)$ of pointed sets is exact if $f(X) = g^{-1}(z)$. To endow $\mathcal{S}\mathrm{et}_*$ with the structure of a proto-exact category, we would want such exact sequences to fit in admissible squares with lower-left term equal to 0. If indeed the latter holds, then $X = g^{-1}(z) = f(X)$ would imply f must be injective. And it turns out $\mathcal{S}\mathrm{et}_*$ with the inclusions as $\mathfrak{M}$ is proto-exact. It has the maps that admit unique sections as admissible epis.

Recall the notion of an exact category as in [Qui73], i.e. an additive category $\mathcal{E}$ with a class $\mathfrak{E}$ consisting of sequences of $\mathcal{E}$-arrows

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0\,, \tag{3.6}$$

which are called the *short exact sequences*. One asks that this structure satisfies some axioms capturing the notion of short exact sequences (see the appendix C for details). In a short exact sequence as above, i is again called an *admissible mono* and p an *admissible epi*.

An example of an exact category $\mathcal{E}$ is when $\mathcal{E}$ is a full subcategory of an abelian category $\mathcal{A}$ which is closed under extensions in $\mathcal{A}$ in the sense that for any short exact sequence of the form $0 \to A' \to A \to A/A' \to 0$ in $\mathcal{A}$ with $A', A/A' \in \mathcal{E}$ it holds $A \in \mathcal{E}$.

A powerful result on exact categories says that, in fact, all exact categories arise in this way. More precisely, let $\mathcal{E}$ be an exact category. Then a functor $F : \mathcal{E}^{\mathrm{op}} \to \mathcal{A}\mathrm{b}$ is left exact when it carries exact sequences of the form (3.6) to exact sequences $0 \to FM'' \to FM \to FM'$ in $\mathcal{A}\mathrm{b}$. Now let $\mathcal{F}$ be the category of left exact functors $\mathcal{E}^{\mathrm{op}} \to \mathcal{A}\mathrm{b}$. Then $\mathcal{F}$ is an abelian category in such a way that the Yoneda embedding $\mathcal{E} \to \mathcal{F}$ embeds $\mathcal{E}$ as a full subcategory of $\mathcal{F}$. Furthermore, this image of $\mathcal{E}$ in $\mathcal{F}$ is closed under extensions, and a sequence in $\mathcal{E}$ is short exact iff the corresponding sequence in $\mathcal{F}$ is. This $\mathcal{F}$ is called the *abelian envelope* of $\mathcal{E}$.[1]

Example 3.2.4 ([DK12, Exm. 2.4.3])**.** Any exact category $\mathcal{E}$ is proto-exact in the obvious way. Indeed, by assumption the classes $\mathfrak{M}$ and $\mathfrak{E}$ of admissible monos and admissible epis in $\mathcal{E}$ respectively are closed under composition and contain all isomorphisms. Since $\mathcal{E}$ is additive, the empty biproduct is a zero object. To see e.g. $0 \to A$ is admissibly monic, note that

$$0 \to 0 \to 0 \oplus A \to A \to 0$$

is a short exact sequence in $\mathcal{E}$ which is isomorphic to $0 \to 0 \to A \to A \to 0$.

Now suppose we have a square (3.5) in $\mathcal{E}$ with $i, j \in \mathfrak{M}$ and with $p, q \in \mathfrak{E}$. Then employing the embedding of $\mathcal{E}$ into its abelian envelope, once can show (3.5) is cartesian iff it is cocartesian (see e.g. [Pre07, Prop. 4.10]). It also follows that admissible monos are monos.

[1] Details can be found in [Pre07, Thm. 4.5]. An early reference is Quillen himself in [Qui73, §2], where he attributes the idea to [Gab62].

Next suppose there is given a pair of morphisms $C \xrightarrow{j} D \xleftarrow{p} B$ in $\mathcal{E}$ with $j \in \mathfrak{M}$ and with $p \in \mathfrak{E}$. We need to complete this into a bicartesian square (3.5) such that $i \in \mathfrak{M}$ and $q \in \mathfrak{E}$. By the previous remark, it suffices to complete this into a cartesian square with $i \in \mathfrak{M}$ and $q \in \mathfrak{E}$. For this consider the following diagram

$$\begin{tikzcd} & A \arrow[r, "i"] \arrow[d, "q"] & B \arrow[d, "p"] & & \\ 0 \arrow[r] & C \arrow[r, "j"] & D \arrow[r, "k"] & E \arrow[r] & 0 \end{tikzcd}$$

where the square is a pullback with $q \in \mathfrak{E}$, induced by the base change axiom, and $k \in \mathfrak{E}$ exhibits j as an admissible mono. It suffices to show that $i \in \mathfrak{M}$. For this one uses that kp is again admissibly epic, and that the admissible mono $B' \to B$ witnessing this fact factorizes through i by the fact that j is a kernel of k, and using the property of A as a pullback. Next one shows that the induced morphism $B' \to A$ is in fact an isomorphism, by showing that B' is also a pullback of the left square, this time using that $B' \to B$ is a kernel of kp and that j is a mono. Since $B' \to B$ is admissibly monic and $\mathfrak{M}$ is closed under isomorphisms, i is admissibly monic.

The pushout axiom is dual to the previous one.

3.3 The Waldhausen simplicial groupoid

Call 2-Segal objects in $\mathcal{C}\mathrm{at}$ *2-Segal categories*. Let $\mathcal{P}$ be a proto-exact category. We will emulate the classical $w\mathcal{S}_\bullet$-construction as given in §3.1, and show that this results in a 2-Segal category. In doing so, we follow [DK12, §2.4].

Recall that T_n for $n \geq 0$ is the poset of pairs (i, j) with $0 \leq i \leq j \leq n$. A functor $F : T_n \to \mathcal{P}$ is again written as $(i, j) \mapsto F_{ij}$, and gives a staircase as in (3.1). Let $\mathcal{W}_n\mathcal{P}$ be the full subcategory of $\mathcal{P}^{T_n}$ on all such staircases $F : T_n \to \mathcal{P}$ for which the F_{ii}'s are all 0, with the horizontal maps all admissibly monic and the vertical maps all admissibly epic, and such that each square in the resulting diagram is bicartesian. Let further $\mathcal{S}_n\mathcal{P}$ be the subcategory of $\mathcal{W}_n\mathcal{P}$ with the same objects, but with only the isomorphisms as morphisms.

The $\mathcal{S}_n\mathcal{P}$ again collect into a simplicial object $\mathcal{S}_\bullet\mathcal{P}$ in $\mathcal{C}\mathrm{at}$, with the structure maps given in the same way as in the $w\mathcal{S}_\bullet$-construction on Waldhausen categories. We call this the *Waldhausen (simplicial) groupoid* of $\mathcal{P}$.

Example 3.3.1. Recall the equivalence from augmented, stable double categories to unital 2-Segal sets, given by a generalized $S_\bullet$-construction, that we mentioned in Rem. 2.2.15 and Exm. 3.2.2. Observe that if we start out with a skeletal, proto-exact category $\mathcal{P}$, then this generalized $S_\bullet$-construction on $\mathcal{P}$ (considered as such a double category) is a decategorified version of our $\mathcal{S}_\bullet$-construction, in that $S_n\mathcal{P}$ is the underlying set of $\mathcal{S}_n\mathcal{P}$.

Proposition 3.3.2. *The Waldhausen groupoid $\mathcal{S}_\bullet\mathcal{P}$ of a proto-exact category $\mathcal{P}$ is a 2-Segal category.*

We want to use Prop. 2.5.1 in the proof of the proposition above. To this end, we need to check that $\mathcal{C}\mathrm{at}$ has natural membranes. We in fact know this to be true, since $\mathcal{C}\mathrm{at}$ can be endowed with the structure of a simplicial model category, which in turn induces the homotopical structure on $\mathcal{C}\mathrm{at}$.[2]

Let us first show a convenient lemma. Observe, since $\mathfrak{E}$ and $\mathfrak{M}$ are closed under composition, they induce subcategories of $\mathcal{P}$ with the same objects as $\mathcal{P}$, also written as $\mathfrak{E}$ and $\mathfrak{M}$ respectively. Now for $n \geq 0$ write $\mathfrak{M}_n$ for the groupoid of functors $[n] \to \mathfrak{M}$, with isomorphisms between them. Likewise, write $\mathfrak{E}_n$ for the groupoid of functors $[n] \to \mathfrak{E}$. Note that these constructions induce simplicial objects $\mathfrak{M}_\bullet$ and $\mathfrak{E}_\bullet$ in $\mathcal{C}\mathrm{at}$ in the obvious way.

Now define (not necessarily simplicial) maps $\mu : \mathcal{S}_\bullet\mathcal{P} \to \mathfrak{M}_\bullet$ and $\epsilon : \mathcal{S}_\bullet\mathcal{P} \to \mathfrak{E}_\bullet$ as follows. For $n \geq 0$ and $F := (F_{ij})_{0\leq i\leq j\leq n}$ in $\mathcal{S}_n\mathcal{P}$ put

$$\mu_n(F) := F_{00} \rightarrowtail F_{01} \rightarrowtail \ldots \rightarrowtail F_{0n}\,;$$

$$\epsilon_n(F) := F_{0n} \twoheadrightarrow F_{1n} \twoheadrightarrow \ldots \twoheadrightarrow F_{nn}\,.$$

It is straightforward to show that μ is natural with respect to all $[n] \to [m]$ that preserve 0, while ϵ is natural with respect to all $[n] \to [m]$ that send n to m.

Lemma 3.3.3. *The above maps μ, ϵ are fully faithful for all n, with essential images in degree n the sequences of the form $0 \rightarrowtail M_1 \rightarrowtail \ldots \rightarrowtail M_n$ in $\mathfrak{M}_n$ resp. those of the form $E_0 \twoheadrightarrow E_1 \twoheadrightarrow \ldots \twoheadrightarrow E_{n-1} \twoheadrightarrow 0$ in $\mathfrak{E}_n$.*

Proof. Let $\mathfrak{M}'_n$ be the full subcategory of those sequences in $\mathfrak{M}_n$ that start with 0. Likewise, let $\mathfrak{E}'_n$ be the sequences in $\mathfrak{E}_n$ that end with 0. We need to show μ_n and ϵ_n give equivalences of categories onto $\mathfrak{M}'_n$ and onto $\mathfrak{E}'_n$ respectively. Let us start with μ_n.

To show μ_n is fully faithful, let $F, G \in \mathcal{S}_n\mathcal{P}$ be given and consider diagrams of the form

$$\begin{tikzcd}
 & G_{ij} \arrow[rr, tail] \arrow[dd, two heads] & & G_{i,j+1} \arrow[dd, two heads] \\
F_{ij} \arrow[ur] \arrow[rr, tail] \arrow[dd, two heads] & & F_{i,j+1} \arrow[ur] \arrow[dd, two heads] & \\
 & G_{i+1,j} \arrow[rr, tail] & & G_{i+1,j+1} \\
F_{i+1,j} \arrow[ur] \arrow[rr, tail] & & F_{i+1,j+1} \arrow[ur, dotted] &
\end{tikzcd} \tag{3.7}$$

where the horizontal and vertical maps are induced from F and G, the solid diagonal maps are isomorphisms and the solid diagram commutes. Observe that we have $F_{tt} = 0 = G_{tt}$, and also $\varphi_{0t} = \psi_{0t}$ for all $0 \leq t \leq n$ and all $\varphi, \psi \in \mathcal{S}_n\mathcal{P}(F, G)$ such that $\mu_n(\varphi) = \mu_n(\psi)$. Hence, by induction from left to right and then from top to bottom in the staircase composed of the above cubes, to show μ_n induces a bijection on hom-sets $\mathcal{S}_n\mathcal{P}(F, G) \to \mathfrak{M}_n(\mu_n F, \mu_n G)$, it suffices to show there is a unique isomorphism $F_{i+1,j+1} \to G_{i+1,j+1}$

[2]The simplicial model structure on $\mathcal{C}\mathrm{at}$ is well-known. See e.g. [Rez00].

that fits in the commutative diagram above as indicated by the dotted arrow. To give such an arrow one uses the fact that the front and the back square in the diagram are both pushouts and the fact that pushouts are unique up to unique isomorphism.

To show μ_n is essentially surjective onto $\mathfrak{M}'_n$, let $0 \rightarrowtail F_{01} \rightarrowtail F_{02} \rightarrowtail \ldots \rightarrowtail F_{0n}$ in $\mathfrak{M}'_n$ be given. Put $F_{ii} := 0$ for all $0 \leq i \leq n$. Then by a similar induction as the previous one, it suffices to show we can complete the following diagram into a bicartesian square

$$\begin{tikzcd} F_{ij} \arrow[r, tail] \arrow[d, two heads] & F_{i,j+1} \arrow[d, dotted] \\ F_{i+1,j} \arrow[r, dotted] & F_{i+1,j+1} \end{tikzcd}$$

where the top arrow is admissibly monic, the left arrow is admissibly epic, and we want the result to be such that also the bottom arrow is in $\mathfrak{M}$ and the right arrow in $\mathfrak{E}$. And indeed this can be done by the pushout axiom.

The claim on ϵ now follows from duality. $\square$

We are ready for the

Proof of Prop. 3.3.2. Write $\mathcal{S}_\bullet$ for the simplicial groupoid $\mathcal{S}_\bullet\mathcal{P}$. Throughout, we use Exm. 1.5.3 for computing homotopy fiber products in $\mathcal{C}\mathrm{at}$. By Prop. 2.5.1, it suffices to show that for all $0 < j \leq n$ and all $0 \leq i < n$ the following maps are equivalences

$$\begin{aligned} &\mathcal{S}_n \to \mathcal{S}_{\{0,j,j+1,\ldots,n\}} \times^R_{\mathcal{S}_{\{0,j\}}} \mathcal{S}_{\{0,1,\ldots,j\}}\,; \\ &\mathcal{S}_n \to \mathcal{S}_{\{0,1,\ldots,i,n\}} \times^R_{\mathcal{S}_{\{i,n\}}} \mathcal{S}_{\{i,i+1,\ldots,n\}}\,. \end{aligned} \tag{3.8}$$

Using μ is natural with respect to the inclusions of $\{0,j\}$ into $\{0,j,\ldots,n\}$ and into $\{0,\ldots,j\}$, we note that the first arrow fits into a commutative diagram

$$\begin{tikzcd} \mathcal{S}_n \arrow[r] \arrow[d, "\mu_n"'] & \mathcal{S}_{\{0,j,\ldots,n\}} \times^R_{\mathcal{S}_{\{0,j\}}} \mathcal{S}_{\{0,\ldots,j\}} \arrow[d, "\mu_{\{0,j,\ldots,n\}} \times \mu_{\{0,\ldots,j\}}"] \\ \mathfrak{M}'_n \arrow[r, "\varphi"] & \mathfrak{M}'_{\{0,j,\ldots,n\}} \times^R_{\mathfrak{M}'_{\{0,j\}}} \mathfrak{M}'_{\{0,\ldots,j\}} \end{tikzcd}$$

Since the vertical arrows are equivalences, it suffices to show that the bottom arrow is.

For essential surjectivity of φ, note that an object in $\mathfrak{M}'_{\{0,j,\ldots,n\}} \times^R_{\mathfrak{M}'_{\{0,j\}}} \mathfrak{M}'_{\{0,\ldots,j\}}$ is given by two sequences $0 \rightarrowtail M_j \rightarrowtail \ldots \rightarrowtail M_n$ and $0 \rightarrowtail M'_1 \rightarrowtail \ldots \rightarrowtail M'_j$ of admissible monos, together with an isomorphism $(0 \to M_j) \cong (0 \to M'_j)$ in $\mathfrak{M}_{\{0,j\}}$, i.e. just an isomorphism $M_j \cong M'_j$. Such an object is isomorphic in $\mathfrak{M}'_{\{0,j,\ldots,n\}} \times^R_{\mathfrak{M}'_{\{0,j\}}} \mathfrak{M}'_{\{0,\ldots,j\}}$ to the datum

$$\Big(0 \rightarrowtail M'_j \rightarrowtail M_{j+1} \rightarrowtail \ldots \rightarrowtail M_n, \quad M'_j \xrightarrow{\mathrm{id}} M'_j, \quad 0 \rightarrowtail M'_1 \rightarrowtail \ldots \rightarrowtail M'_j\Big)$$

where $M'_j \rightarrowtail M_{j+1}$ is the composition $M'_j \cong M_j \rightarrowtail M_{j+1}$. But the latter element is the image under φ of $0 \rightarrowtail M'_1 \rightarrowtail \ldots \rightarrowtail M'_j \rightarrowtail M_{j+1} \rightarrowtail \ldots \rightarrowtail M_n$ from $\mathfrak{M}_n$, which was to be shown.

For full faithfulness, let $M := 0 \rightarrowtail M_1 \rightarrowtail \ldots \rightarrowtail M_n$ and $N := 0 \rightarrowtail N_1 \rightarrowtail \ldots \rightarrowtail N_n$ be given in $\mathfrak{M}'_n$. A morphism from $\varphi(M)$ to $\varphi(N)$ is given by commuting morphisms $f_s : M_s \to N_s$ for $j \leq s \leq n$ and commuting morphisms $g_t : M_t \to N_t$ for $1 \leq t \leq j$ such that they induce the same morphism $(0 \rightarrowtail M_j) \to (0 \rightarrowtail N_j)$ in $\mathfrak{M}'_{\{0,j\}}$, i.e. such that $f_j = g_j$. Hence these maps combine into a unique set of commuting arrows $h_u : M_u \to N_u$ for $1 \leq u \leq n$, i.e. a unique morphism $h : M \to N$ such that $\varphi(h) = (f_s, g_t)_{st}$.

One uses a similar strategy to show that the second arrow in (3.8) is also an equivalence. This time one employs ϵ to reduce to the case of showing that

$$\mathfrak{E}'_n \to \mathfrak{E}'_{\{0,\ldots,i,n\}} \times^R_{\mathfrak{E}'_{\{i,n\}}} \mathfrak{E}'_{\{i,\ldots,n\}}$$

is an equivalence. The latter can be seen by remarking that an object on the right is a pair of sequences $E_0 \twoheadrightarrow E_1 \twoheadrightarrow \ldots \twoheadrightarrow E_i \twoheadrightarrow 0$ plus $E'_i \twoheadrightarrow E'_{i+1} \twoheadrightarrow \ldots \twoheadrightarrow E'_{n-1} \twoheadrightarrow 0$ together with an isomorphism $E'_i \cong E_i$, and by then continuing in the same fashion as before. $\square$

3.4 Interlude on comparison

Recall that a functor $\mathcal{E}' \to \mathcal{E}$ between exact categories is called *exact* if it is additive and carries exact sequences in $\mathcal{E}'$ to such sequences in $\mathcal{E}$. Write $\mathcal{E}\mathrm{x}$ for the resulting category of exact categories. Likewise, a functor $\mathcal{P}' \to \mathcal{P}$ between proto-exact categories is called *exact* if it preserves $0, \mathfrak{M}', \mathfrak{E}'$, and all pushouts of admissible monos and pullbacks of admissible epis along arbitrary maps.

For an exact category $\mathcal{E}$ let the proto-exact category $\mathcal{P}(\mathcal{E})$ be given by forgetting the additive structure (Exm. 3.2.4). Now we know that a morphism between exact categories preserves pushouts of admissible monos and pullbacks of admissible epis along all maps (see e.g. [Bue08, Prop. 5.2]). It is hence clear that the rule $\mathcal{E} \mapsto \mathcal{P}(\mathcal{E})$ gives us a functor $\mathcal{P}(-) : \mathcal{E}\mathrm{x} \to \mathcal{P}\mathrm{ro}$.

A functor $\mathcal{C}' \to \mathcal{C}$ between Waldhausen categories is called *exact* if it preserves the zero object and all cofibrations, pushouts of cofibrations and weak equivalences. This gives us the category $\mathcal{W}\mathrm{ald}$. Write $\mathrm{i}\mathcal{W}\mathrm{ald}$ for the full subcategory of $\mathcal{W}\mathrm{ald}$ consisting of those Waldhausen categories which have isomorphisms as weak equivalences, i.e. just the categories with cofibrations. Also, let $\mathrm{c}\mathcal{P}\mathrm{ro}$ be the full subcategory of those proto-exact categories for which admissible monos admit cobase change along all morphisms. Note by the cobase change axiom for exact categories, the functor $\mathcal{P}(-)$ lands in $\mathrm{c}\mathcal{P}\mathrm{ro}$.

For an exact category $\mathcal{E}$ we let $\mathcal{C}(\mathcal{E})$ be the Waldhausen category which has the admissible monos of $\mathcal{E}$ as cofibrations and the isomorphisms as weak equivalences. Note that an exact functor $\mathcal{E}' \to \mathcal{E}$ gives a morphism of Waldhausen categories $\mathcal{C}(\mathcal{E}') \to \mathcal{C}(\mathcal{E})$, because it preserves admissible monos and their pushouts. We therefore have a functor $\mathcal{C}(-) : \mathcal{E}\mathrm{x} \to \mathrm{i}\mathcal{W}\mathrm{ald}$.

If $\mathcal{P}$ is a proto-exact category for which the admissible monos admit cobase change along all morphisms, then after calling the admissible monos 'cofibrations', this results

in a Waldhausen category $\mathcal{C}(\mathcal{P})$. Since morphisms of proto-exact categories preserve admissible monos and their pushouts by assumption, it is clear we thus have a functor $\mathcal{C}(-) : \mathrm{c}\mathcal{P}\mathrm{ro} \to \mathrm{i}\mathcal{W}\mathrm{ald}$. The notation should not cause any confusion, since for an exact category $\mathcal{E}$ we have $\mathcal{C}(\mathcal{P}(\mathcal{E})) = \mathcal{C}(\mathcal{E})$.

To reiterate, we have the following functors

$$\begin{tikzcd} \mathcal{E}\mathrm{x} \arrow[rr, "\mathcal{C}(-)"] \arrow[dr, "\mathcal{P}(-)"'] & & \mathrm{i}\mathcal{W}\mathrm{ald} \\ & \mathrm{c}\mathcal{P}\mathrm{ro} \arrow[ur, "\mathcal{C}(-)"'] & \end{tikzcd}$$

Note that for any proto-exact category $\mathcal{P}$ in $\mathrm{c}\mathcal{P}\mathrm{ro}$ it holds that $\mathcal{S}_\bullet\mathcal{P}$ and $\mathrm{w}\mathcal{S}_\bullet\mathcal{C}(\mathcal{P})$ are naturally isomorphic. To see this, one uses the facts that in $\mathcal{P}$ a square where the horizontal arrows are admissibly monic and the vertical ones admissibly epic is a pushout iff it is bicartesian and that in $\mathcal{C}(\mathcal{P})$ the quotient maps are precisely the admissible epis.

None of the above functors are essentially surjective. Indeed, $\mathcal{S}\mathrm{et}_*$ from Exm. 3.2.3 is a proto-exact category which has almost no finite biproducts, hence is not in the essential image of $\mathcal{P}(-)$. Furthermore, we have seen $\mathcal{S}_\bullet\mathcal{P}$ is always 2-Segal for $\mathcal{P} \in \mathcal{P}\mathrm{ro}$, while $\mathrm{w}\mathcal{S}_\bullet\mathcal{C}$ for $\mathcal{C} \in \mathrm{i}\mathcal{W}\mathrm{ald}$ might not be (Prop. 3.3.2, 3.1.1 and Exm. 3.1.3).

The functor $\mathcal{P}(-)$ is however fully faithful. To prove this, it suffices to show a given functor $f : \mathcal{E}' \to \mathcal{E}$ between exact categories is a morphism in $\mathcal{E}\mathrm{x}$ if it is a morphism in $\mathcal{P}\mathrm{ro}$ when considered as functor $\mathcal{P}(\mathcal{E}') \to \mathcal{P}(\mathcal{E})$. So suppose the latter holds. In $\mathcal{E}'$ the finite biproducts are just the finite coproducts, hence f is additive since it preserves finite coproducts. Also, for a sequence $A' \to A \to A''$ in $\mathcal{E}'$ it holds that the square

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d] \\ 0 \arrow[r] & A'' \end{tikzcd}$$

is bicartesian iff the given sequence is exact by [Bue08, Prop. 2.12]. Since f preserves admissible squares in $\mathcal{P}(\mathcal{E}')$, it hence preserves exact sequences in $\mathcal{E}'$.

By the same token, the functor $\mathcal{C}(-)$ from $\mathcal{E}\mathrm{x}$ to $\mathrm{i}\mathcal{W}\mathrm{ald}$ is fully faithful. The passage from $\mathrm{c}\mathcal{P}\mathrm{ro}$ to $\mathrm{i}\mathcal{W}\mathrm{ald}$ however is not full. To see this, take for example the proto-exact category $\mathcal{S}\mathrm{et}_*$ from Exm. 3.2.3, and the functor $\mathcal{S}\mathrm{et}_* \to \mathcal{S}\mathrm{et}_*$ that sends a pointed set X to the pushout $X \cup_0 X$. Then although this is a morphism of Waldhausen categories, it does not preserve all products, hence not all pullbacks of cofibrations along arbitrary maps.

3.5 Higher K-groups

We are going to define higher K-groups for proto-exact categories. We have seen proto-exact categories are related to exact categories on the one hand and to Waldhausen categories on the other. Now the $\mathrm{w}\mathcal{S}_\bullet$-construction on Waldhausen categories leads to a definition of higher K-groups in that world, while Quillen's Q-construction (recalled below) gives such a definition in the world of exact categories. So we have two competing

options to mimic in the present case of proto-exact categories. We shall see these options actually result in the same thing.

Fix in the following a proto-exact category $\mathcal{P}$. Let $Q\mathcal{P}$ be the category with the same objects as $\mathcal{P}$, but with $Q\mathcal{P}(C, C')$ the equivalence classes of spans of the form $C \twoheadleftarrow D \rightarrowtail C'$, where two such spans $C \twoheadleftarrow D_1 \rightarrowtail C'$ and $C \twoheadleftarrow D_2 \rightarrowtail C'$ are to be considered equivalent when they differ by an isomorphism $D_1 \cong D_2$ that makes the obvious diagram commutative. Now for given spans $C \twoheadleftarrow D \rightarrowtail C'$ and $C' \twoheadleftarrow D' \rightarrowtail C''$, we define their composition from C to C'' in $Q\mathcal{P}$ as

$$C \twoheadleftarrow D \leftarrow D \times_{C'} D' \longrightarrow D' \rightarrowtail C''$$

Observe, by the pullback axiom, it holds that $D \leftarrow D \times_{C'} D'$ is admissibly epic while $D \times_{C'} D' \to D'$ is admissibly monic. Since the admissible epis resp. monos are closed under composition, the above span is of the right form, i.e. gives a morphism in $Q\mathcal{P}(C, C'')$. It is straightforward to show that this indeed results in a category $Q\mathcal{P}$. Arrows from C to C' in this category are written as $C \rightsquigarrow C'$.

The above construction is called the *Q-construction on* $\mathcal{P}$. The nerve of $Q\mathcal{P}$ is written as $Q_\bullet\mathcal{P}$. We have an associated simplicial groupoid $\mathcal{Q}_\bullet\mathcal{P}$, with $\mathcal{Q}_n\mathcal{P}$ the category that has $Q_n\mathcal{P}$ as objects and as morphisms from $C_0 \rightsquigarrow C_1 \rightsquigarrow \cdots \rightsquigarrow C_n$ to $C'_0 \rightsquigarrow C'_1 \rightsquigarrow \cdots \rightsquigarrow C'_n$ isomorphisms $(C_i \rightsquigarrow C'_i)_{0 \le i \le n}$ which make the obvious diagram in $Q\mathcal{P}$ commutative.

Notation 3.5.1. For a simplicial groupoid $G_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}$ write $B_\bullet G$ for the simplicial space $[n] \mapsto B(G_n)$.

Definition 3.5.2. Let $\mathcal{P}$ still be our proto-exact category. For $n \ge 0$ the *K-groups in the sense of Waldhausen* are defined as

$$K_n^W(\mathcal{P}) := \pi_{n+1}|B_\bullet \mathcal{S}\mathcal{P}|\,,$$

where $|B_\bullet \mathcal{S}\mathcal{P}|$ is pointed by $0 \in B_0\mathcal{S}\mathcal{P}$. Likewise, the *$K$-groups in the sense of Quillen* are defined as

$$K_n^Q(\mathcal{P}) := \pi_{n+1} BQ\mathcal{P}\,,$$

with $BQ\mathcal{P}$ the classifying space of $Q\mathcal{P}$, pointed by 0.

The following result was shown by Waldhausen in [Wal85] in the case of exact categories. We follow the same strategy, also found in [Wei13, §IV.8].

Theorem 3.5.3. *For $\mathcal{P}$ as above it holds $K_n^W(\mathcal{P}) \cong K_n^Q(\mathcal{P})$ for $n \ge 0$.*

For the proof of this, we need some more terminology and preliminary results. First, let δ be the *doubling map*, i.e. the endofunctor on $\mathbf{\Delta}$ that sends $[n]$ to $[2n+1]$ and which on morphisms is given by

$$[0, 1, \ldots, n] \overset{\delta}{\mapsto} [n', \ldots, 1', 0', 0, 1, \ldots, n]\,,$$

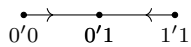
i.e. δ sends d^i to $d^{n+1+i}d^{n-i}$ and s^j to $s^{n+1+j}s^{n-j}$. For a simplicial object X in any category, the *edgewise subdivision* $\operatorname{Sub} X$ of X is defined as $X \circ \delta$.

In general, $\operatorname{Sub}\Delta[n]$ has as k-simplices the points $[t'_k, \dots, t'_0, t_0, \dots, t_k] \in \mathbf{\Delta}(2k+1, n)$, i.e. so that $0 \le t'_k \le \dots \le t_k \le n$. Observe that for $0 \le j \le k-1$ and a $k-1$-simplex $[u'_{k-1}, \dots, u_{k-1}]$ in $\operatorname{Sub}\Delta[n]$ it holds that

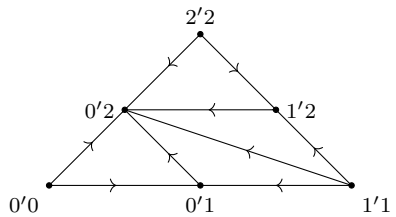
$$s_j([u'_{k-1}, \dots, u_{k-1}]) = [u'_{k-1}, \dots, u'_j, u'_j, \dots, u_j, u_j, \dots, u_{k-1}] \,.$$

It follows that a k-simplex $\sigma = [t'_k, \dots, t_k]$ of $\operatorname{Sub}\Delta[n]$ is nondegenerate iff the following holds: $t_j = t_{j+1}$ iff $t'_j \ne t'_{j+1}$ for all $0 \le j < k$. Furthermore σ has vertices $[t'_0, t_0], \dots, [t'_k, t_k]$, which are written as $t'_0 t_0, \dots, t'_k t_k$.

The subdivision $\operatorname{Sub}\Delta[1]$ of $\Delta[1]$ for example looks like



while the one of $\Delta[2]$ is



The 2-simplices of $\operatorname{Sub}\Delta[2]$ in the above picture are as follows. Starting in the lower-left triangle and in the anti-clockwise direction we have $[0', 0', 0', 0, 1, 2]$ and $[0', 0', 1', 1, 1, 2]$ at the bottom, then $[0', 1', 1', 1, 2, 2]$ in the right and $[0', 1', 2', 2, 2, 2]$ at the top.

In general, $\operatorname{Sub}\Delta[n]$ only has nondegenerate simplices of dimension $\le n$. To see this, one needs to check that for any given sequence $0 \le t'_{n+1} \le \dots \le t_{n+1} \le n$ it holds there is some $0 \le j \le n$ such that $t_j = t_{j+1}$ and also $t'_j = t'_{j+1}$. If $t_n \le n-1$ this indeed holds by induction; else we must have $t_n = t_{n+1} = n$. Then either $0 = t'_{n+1} = t'_n$, in which case we are done, or $1 \le t'_n$, in which case the sequence $0 \le t'_n - 1 \le \dots \le t_n - 1 \le n-1$ has such a double term by induction.

Furthermore, $\operatorname{Sub}\Delta[n]$ has 2^n nondegenerate n-simplices. This can be shown by observing a nondegenerate simplex $[t'_n, \dots, t_n]$ is actually completely determined by the sequence $0 \le t'_n \le \dots \le t'_0 \le n$. And furthermore, this sequence must be so that $t'_n = 0$ and with $t'_{i+1} - t'_i \le 1$. Hence giving such a sequence is the same as making n binary choices, of which there are 2^n possibilities.

Finally, it holds that the geometric realizations of $\operatorname{Sub}\Delta[n]$ and $\Delta[n]$ are isomorphic. This follows from the following

Lemma 3.5.4. *Let X be a simplicial space $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{T}\mathrm{op}$. Then X and $\operatorname{Sub} X$ are homeomorphic in their geometric realizations.*

Proof. This is a classical result. See for example [Seg73, Prop. A.1], where it is 'more or less' attributed to Quillen. The idea is to consider the maps $\operatorname{Sub}_n X \times \Delta^n \to X_{2n+1} \times \Delta^{2n+1}$ given by

$$(x, t_0, \dots, t_n) \mapsto (x, t_n/2, \dots, t_0/2, t_0/2, \dots, t_n/2) \ ,$$

where a geometric simplex Δ^k is considered as the set of points $(s_0, \dots, s_k)$ in $\mathbb{R}^{k+1}$ such that $\sum s_i = 1$. Then one shows that these maps respect the equivalence relations on $\bigsqcup_{n\geq 0} \operatorname{Sub}_n X \times \Delta^n$ and on $\bigsqcup_{n\geq 0} X_n \times \Delta^n$ that are induced by taking the geometric realizations of $\operatorname{Sub} X$ and of X respectively. Since these maps are all homeomorphisms, they therefore combine into a single homeomorphism $|\operatorname{Sub} X| \to |X|$. □

Lemma 3.5.5. *A morphism $C \rightsquigarrow C'$ in $Q\mathcal{P}$ is an isomorphism iff it can be written in the form $C \cong D \cong C'$ in $\mathcal{P}$.*

Proof. Let an isomorphism $C \rightsquigarrow C'$ with inverse $C' \rightsquigarrow C$ in $Q\mathcal{P}$ be given, say represented by the spans

$$C \overset{p}{\twoheadleftarrow} D \overset{i}{\rightarrowtail} C' \overset{q}{\twoheadleftarrow} D' \overset{j}{\rightarrowtail} C$$

By composing these spans and by using that they are inverse to each other, we get the following commutative diagrams

$$\begin{tikzcd}
C & D \arrow[l, two heads, "p"'] & D \times_{C'} D' \arrow[l, two heads, "\beta"'] \arrow[r, tail, "\alpha"] \arrow[d, dashed, "\sigma"] & D' \arrow[r, tail, "j"] & C \\
& & C \arrow[ull, "\mathrm{id}"] \arrow[urr, "\mathrm{id}"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
C' & D' \arrow[l, two heads, "q"'] & D' \times_{C} D \arrow[l, two heads, "\psi"'] \arrow[r, tail, "\varphi"] & D \arrow[r, tail, "i"] & C' \\
& & C' \arrow[ull, "\mathrm{id}"] \arrow[urr, "\mathrm{id}"'] \arrow[u, dashed, "\sigma'"] & &
\end{tikzcd}$$

with σ, σ' isomorphisms and $\alpha, \beta, \varphi, \psi$ the projections. It is easy to see that this induces an isomorphism $\tau : D \times_{C'} D' \to D' \times_C D$ such that $\varphi\tau = \beta$ and $\psi\tau = \alpha$, with inverse τ' such that $\alpha\tau' = \psi$ and $\beta\tau' = \varphi$. For this one uses the universal property of pullbacks and the fact that admissible monos are monic. It follows that β exhibits an isomorphism between the given span that represents $C \rightsquigarrow C'$, and the span

$$C \overset{\sigma}{\leftarrow} D \times_{C'} D' \overset{i\varphi\tau}{\longrightarrow} C'.$$

And indeed, this span is of the desired form, since σ is an isomorphism by assumption, and $i\varphi\tau$ has inverse $\tau'\sigma'$. To show β is an isomorphism one can use e.g. that φ is monic and split epic. □

Suppose now that we have a commutative square in $Q\mathcal{P}$

$$\begin{tikzcd}
A \arrow[r, squiggly] \arrow[d, squiggly, "\wr"] & C \arrow[d, squiggly, "\wr"] \\
A' \arrow[r, squiggly] & C'
\end{tikzcd}$$

where the vertical arrows are isomorphisms. Taking representatives of the horizontal arrows and using the previous lemma, this gives us the solid diagram

$$\begin{tikzcd} A & B \arrow[l, two heads] \arrow[r, tail] \arrow[d, dotted] & C \arrow[d] \\ A' \arrow[u] & B' \arrow[l, two heads] \arrow[r, tail] & C' \end{tikzcd}$$

with again the vertical arrows isomorphisms, this time in $\mathcal{P}$ itself. Since the compositions $A \rightsquigarrow C \rightsquigarrow C'$ and $A \rightsquigarrow A' \rightsquigarrow C$ are identical, we have an isomorphism $B \to B'$ that fits in the above diagram as the dotted arrow. It further holds that this arrow is unique with this property, as the admissible mono $B' \rightarrowtail C'$ in $\mathcal{P}$ is monic.

Lemma 3.5.6. *Let $\mathcal{C}_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}$ be a simplicial category. Then $|B_\bullet \mathcal{C}|$ is isomorphic to the geometric realization of the simplicial set $[n] \mapsto N_n \mathcal{C}_n$.*

Proof. Consider the bisimplicial space $T : \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{T}\mathrm{op}$ that sends (n, m) to $N_m \mathcal{C}_n$, considered as discrete space. A well-know result mentioned in [Qui73, p. 94] states that

$$|[n] \mapsto |[m] \mapsto T_{nm}|| \cong |[n] \mapsto T_{nn}| \,.$$

Now since the geometric realization of a discrete space is just the geometric realization of the underlying set, the expression on the left is $|B_\bullet \mathcal{C}|$, while the one on the right is indeed the geometric realization of the simplicial set $[n] \mapsto N_n \mathcal{C}_n$. □

The isomorphism between $|B_\bullet \mathcal{C}|$ and $|[n] \mapsto N_n \mathcal{C}_n|$ from the above lemma is natural in $\mathcal{C}$, because the mentioned well-known result is functorial in T. Using this, the following is not too difficult.

Corollary 3.5.7. *Let $\mathcal{C}_\bullet, \mathcal{C}'_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}$ be simplicial categories and $\varphi : \mathcal{C} \to \mathcal{C}'$ a simplicial map between them, which is an equivalence of categories in each degree. Then the induced map $|B_\bullet \mathcal{C}| \to |B_\bullet \mathcal{C}'|$ is a homotopy equivalence.*

Proof. As in the precious proof, let T be the bisimplicial space $(n, m) \mapsto N_m \mathcal{C}_n$, and T' likewise for $\mathcal{C}'_\bullet$. Consider φ as map of bisimplicial spaces $T \to T'$. Because an equivalence of categories is a homotopy equivalence on classifying spaces, for each $n \geq 0$ the induced map $\varphi_n : T_{n,\bullet} \to T'_{n,\bullet}$ is a homotopy equivalence. Hence, by Lem. 1.4.2 and the previous lemma, it follows that the map

$$|B_\bullet \mathcal{C}| \cong |d(T)| \to |d(T')| \cong |B_\bullet \mathcal{C}'|$$

is a homotopy equivalence, which was to be shown. □

Recall that we aim to show: the approach via the $\mathcal{S}_\bullet$-construction and via the Q-construction give the same K-groups on proto-exact categories (Def. 3.5.2). We are now in a position to prove this.

Proof of Thm. 3.5.3. Let us first construct a map $\sigma : \mathrm{Sub}_\bullet \mathcal{SP} \to \mathcal{Q}_\bullet \mathcal{P}$ of simplicial groupoids. The crucial step here is writing an element F in $\mathrm{Sub}_\bullet \mathcal{SP}$ in the form (F_{ij}) with i, j both running from n' to $0'$ and then from 0 to n, in that order. Hence, the top row of F, written as a staircase as in (3.1), is the sequence

$$F_{n'n'} \to F_{n',n'-1} \to \cdots \to F_{n'0'} \to F_{n'0} \to F_{n'1} \to \cdots \to F_{n'n} \,.$$

The squares on the anti-diagonal of F look like

$$\begin{tikzcd} F_{i'+1,i} \arrow[r, tail] \arrow[d, two heads] & F_{i'+1,i+1} \arrow[d, two heads] \\ F_{i'i} \arrow[r, tail] & F_{i',i+1} \end{tikzcd}$$

Such a square induces a span $F_{i'i} \rightsquigarrow F_{i'+1,i+1}$. We map F to the sequence of these spans

$$F_{0'0} \rightsquigarrow F_{1'1} \rightsquigarrow \cdots \rightsquigarrow F_{n'n}\,.$$

It is straightforward to show that this indeed gives a simplicial map $\sigma : \mathrm{Sub}_\bullet \mathcal{SP} \to \mathcal{Q}_\bullet \mathcal{P}$, using the structure maps on $\mathrm{Sub}_\bullet \mathcal{SP}$ act on $[0, 1, \ldots, n]$ and on $[0', 1', \ldots, n']$ separately.

Next observe that σ gives an equivalence of categories in each degree. Indeed, let $n \geq 0$ and a sequence C of spans

$$C := (F_{0'0} \rightsquigarrow F_{1'1} \rightsquigarrow \cdots \rightsquigarrow F_{n'n})$$

in $\mathcal{Q}_n \mathcal{P}$ be given. Then one constructs an element F in $\mathrm{Sub}_\bullet \mathcal{SP}$ such that $\sigma F \cong C$ as follows. First one takes representatives of the above spans

$$F_{0'0} \twoheadleftarrow F_{1'0} \rightarrowtail F_{1'1} \twoheadleftarrow F_{2'1} \rightarrowtail \ldots \twoheadleftarrow F_{n',n-1} \rightarrowtail F_{n'n}$$

Then one puts $F_{ii} = 0 = F_{j'j'}$ for $0 \leq i \leq n$ and $0' \leq j' \leq n'$. Finally, by applying the pullback resp. the pushout lemma multiple times, one fills the triangle of the staircase as in (3.1) above resp. below the anti-diagonal $F_{0'0}, F_{1'1}, \ldots, F_{n'n}$.

It is easy to see that the above construction indeed gives an element F in $\mathrm{Sub}_\bullet \mathcal{SP}$ which is mapped to C under σ. Hence σ_n is essentially surjective. Furthermore σ_n is fully faithful. To see this, let F, G in $\mathrm{Sub}_n \mathcal{SP}$ be given, and suppose we have a morphism $\varphi : \sigma_n F \to \sigma_n G$ in $\mathcal{Q}_n \mathcal{P}$. Then consider the following cubes

$$\begin{tikzcd} & G_{i'+1,i} \arrow[rr, tail] \arrow[dd, two heads] & & G_{i'+1,i+1} \arrow[dd, two heads] \\ F_{i'+1,i} \arrow[ur, dotted] \arrow[rr, tail] \arrow[dd, two heads] & & F_{i'+1,i+1} \arrow[ur] \arrow[dd, two heads] & \\ & G_{i'i} \arrow[rr, tail] & & G_{i',i+1} \\ F_{i'i} \arrow[ur] \arrow[rr, tail] & & F_{i',i+1} \arrow[ur, dotted] & \end{tikzcd}$$

where the front and back squares are part of F and G respectively, and the solid diagonal arrows are isomorphisms induces by φ, using Lem. 3.5.5. Following the remark below this lemma, we find a unique isomorphism that fits in the diagram as the top-left dotted arrow. Then we also find a unique isomorphism at the bottom-right dotted arrow, by the same argument as given in (3.7). But now we can continue upwards and downwards anti-diagonally in the obvious way, using $F_{tt} = 0 = G_{tt}$, to get a morphism $\varphi' : F \to G$ that is sent to φ under σ_n.

The morphism φ' is furthermore unique with this property. To prove this, it clearly suffices to show the maps $F_{i'i} \to G_{i'i}$ induced by φ are unique, since φ' is uniquely

determined by these maps. In other words, given an isomorphism $C \rightsquigarrow C'$ the induced isomorphism $C \to C'$ in $\mathcal{P}$ must be independent of the chosen representative. This is however clear, since two such isomorphisms $f, f' : C \to C'$ differ by an automorphism τ of C, i.e. $f = f'\tau$, for which it holds that τ commutes with the identity on C, i.e. $\tau\,\mathrm{id}_C = \mathrm{id}_C$. But this just means $f = f'$.

The above arguments show that $\sigma : \mathrm{Sub}_\bullet\, \mathcal{SP} \to \mathcal{Q}_\bullet\mathcal{P}$ is an equivalence of categories in each degree. Hence it induces a homotopy equivalence $|B_\bullet \mathrm{Sub}\, \mathcal{SP}| \to |B_\bullet \mathcal{QP}|$ by Cor. 3.5.7. We also have that $B_\bullet \mathrm{Sub}\, \mathcal{SP}$ is just $\mathrm{Sub}_\bullet\, B\mathcal{SP}$, which is homeomorphic to $B_\bullet \mathcal{SP}$ in its geometric realization by Lem. 3.5.4. Summarizing, we have shown the following relations

$$|B_\bullet \mathcal{SP}| \cong |\mathrm{Sub}_\bullet\, B\mathcal{SP}| = |B_\bullet \mathrm{Sub}\, \mathcal{SP}| \simeq |B_\bullet \mathcal{QP}|\,.$$

It hence remains to be shown $|B_\bullet \mathcal{QP}|$ and $BQ\mathcal{P}$ are weakly equivalent.

To see $|B_\bullet \mathcal{QP}| \simeq BQ\mathcal{P}$, first observe that by Lem. 3.5.6 it holds $|B_\bullet \mathcal{QP}|$ is isomorphic to the geometric realization of the diagonal of the bisimplicial set $N_\bullet \mathcal{Q}_\bullet \mathcal{P}$, i.e.

$$|B_\bullet \mathcal{QP}| \cong |[n] \mapsto N_n \mathcal{Q}_n \mathcal{P}| = |d((n,m) \mapsto N_m \mathcal{Q}_n \mathcal{P})|\,.$$

I claim the space on the right-hand side is homotopy equivalent to $|d((n,m) \mapsto Q_n\mathcal{P})|$. Write Y for the bisimplicial set $(n,m) \mapsto N_m \mathcal{Q}_n \mathcal{P}$ and Z for $(n,m) \mapsto Q_n \mathcal{P}$. Then an element in Y_{nm} is a commutative diagram in $Q\mathcal{P}$ of the form

$$\begin{array}{ccccccc}
C_0^{(0)} & \rightsquigarrow & C_1^{(0)} & \rightsquigarrow & \dots & \rightsquigarrow & C_n^{(0)} \\
\downarrow & & \downarrow & & & & \downarrow \\
\vdots & & \vdots & & & & \vdots \\
\downarrow & & \downarrow & & & & \downarrow \\
C_0^{(m)} & \rightsquigarrow & C_1^{(m)} & \rightsquigarrow & \dots & \rightsquigarrow & C_n^{(m)}
\end{array}$$

where the vertical arrows are all isomorphisms in $\mathcal{P}$. We have a map of bisimplicial sets $\varphi : Z \to Y$, mapping $C_0 \rightsquigarrow \cdots \rightsquigarrow C_n$ to a diagram of the above form, with $C_0 \rightsquigarrow \cdots \rightsquigarrow C_n$ on each row and identities in each column.

Fix $m \geq 0$, and consider the map of simplicial sets $\varphi_m : Z_{\bullet,m} \to Y_{\bullet,m}$. Note that $Z_{\bullet,m}$ is the nerve of the category $Q\mathcal{P}$. Also, $Y_{\bullet,m}$ is the nerve of the category $\mathcal{C}$ which has as objects strings of composable isomorphisms $C^{(0)} \to \cdots \to C^{(m)}$ and with obvious morphisms. Furthermore, φ_m is induced by the functor φ'_m that sends C in $Q\mathcal{P}$ to $C = \cdots = C$ in $\mathcal{C}$.

Now let $C^* = (C^{(0)} \xrightarrow{f^{(1)}} \dots \xrightarrow{f^{(m)}} C^{(m)})$ be a given object in $\mathcal{C}$. Then in the category φ'_m / C^* we have an object $(C^{(0)}, \theta)$, where θ is the morphism $(\theta_k : C^{(0)} \to C^{(k)})_{0 \leq k \leq m}$ from $C^{(0)} = \cdots = C^{(0)}$ to C^* given by $\theta_k := f^{(k)} f^{(k-1)} \cdots f^{(1)}$. It is straightforward to show that this object is in fact terminal in φ'_m / C^*. Hence φ_m is a homotopy equivalence by Quillen's theorem A. Now since this holds for all $m \geq 0$, Lem. 1.4.2 implies that φ induces a weak equivalence $d(Z) \to d(Y)$, which was to be shown. $\square$

Since the K-groups in the sense of Waldhausen and in the sense of Quillen agree, from here on we will simply write $K_n(\mathcal{P})$ in stead of $K_n^W(\mathcal{P})$ resp. $K_n^Q(\mathcal{P})$.

3.5.a Elementary properties of K-groups

Let us verify that some of the elementary properties of K-groups of exact categories as mentioned by Quillen in [Qui73] also hold in the present setting of proto-exact categories.

Definition 3.5.8. A functor between proto-exact categories is called *proto-exact* when it preserves all admissible squares.

Recall that an exact functor preserves pushouts of admissible monos and pullbacks of admissible epis along all maps. Hence an exact functor is proto-exact. The converse need not hold: $X \mapsto X \amalg X$ on $\mathcal{S}\mathrm{et}_*$ is proto-exact but fails to be exact. The latter we have seen, and the former follows from the fact that each admissible square in $\mathcal{S}\mathrm{et}_*$ is of the form

$$\begin{tikzcd} A \amalg D' \arrow[r, tail] \arrow[d, two heads] & D \amalg D' \arrow[d, two heads] \\ A \arrow[r, tail] & D \end{tikzcd}$$

where $D \amalg D' \twoheadrightarrow D$ is the identity on D and sends D' to the point in D, and likewise for $A \amalg D' \twoheadrightarrow A$.

Let $F : \mathcal{P}' \to \mathcal{P}$ be such a proto-exact functor. Using that it preserves admissible monos and epis, one sees that F induces a function from $Q\mathcal{P}'(C, C')$ to $Q\mathcal{P}(FC, FC')$ for all $C, C' \in \mathcal{P}'$. Since F preserves admissible squares, we see these functions on hom-sets combine into a functor $Q\mathcal{P}' \to Q\mathcal{P}$, and hence give a homomorphism of K-groups $F_* : K_n(\mathcal{P}') \to K_n(\mathcal{P})$. It follows that $K_n(-)$ is a functor from the category of proto-exact categories with proto-exact functors between them to the category of groups. This functor maps isomorphic proto-exact functors to identical homomorphism by [Qui73, Prop. 2].

Now let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two proto-exact categories. Their product $\mathcal{P}_1 \times \mathcal{P}_2$ is again proto-exact when we endow it with $\mathfrak{E}_1 \times \mathfrak{E}_2$ as admissible epis and with $\mathfrak{M}_1 \times \mathfrak{M}_2$ as admissible monos. Then $Q(\mathcal{P}_1 \times \mathcal{P}_2)$ is just $Q\mathcal{P}_1 \times Q\mathcal{P}_2$, and we know the functor $B(-)$ preserves products as mentioned in [Qui73, §2, (8)]. We therefore get that $K_n(\mathcal{P}_1 \times \mathcal{P}_2)$ is isomorphic to $K_n(\mathcal{P}_1) \times K_n(\mathcal{P}_2)$ in the obvious way.

3.6 The Grothendieck group of a proto-exact category

Throughout, let $\mathcal{P}$ be a proto-exact category. Write $G(\mathcal{P})$ for the group generated on the set of symbols $[M]$, one for each M in $\mathcal{P}$, and subject to the relation $[X] = [A] \cdot [B]$ for all bicartesian squares

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, two heads] & X \arrow[d, two heads] \\ 0 \arrow[r, tail] & B \end{tikzcd}$$

Notation 3.6.1. If we know $G(\mathcal{P})$ to be abelian, then we write it additively.

We call $G(\mathcal{P})$ the *Grothendieck group* of $\mathcal{P}$. It is straightforward to show that in $G(\mathcal{P})$ it holds $[A] \cdot [C]^{-1} = [B] \cdot [B \cup_A C]^{-1}$ for all admissible squares as in (3.5), by using the pasting lemma for pullbacks and by the bicartesian axiom on $\mathcal{P}$. If $G(\mathcal{P})$ is abelian, then it follows that $[B \cup_A C] = [B] + [C] - [A]$.

Now let $\mathcal{E}$ be an exact category. Recall $\mathcal{E}$ can be considered as a proto-exact category in the obvious way. Then the relation $[B][C] = [B \amalg C] = [C \amalg B] = [C][B]$ for all $B, C \in \mathcal{E}$ forces the result $G(\mathcal{E})$ to be abelian, so we write it additively. Further note, the condition $[M] = [M'] + [M'']$ for M, M', M'' in $\mathcal{E}$ is exactly asking that we have a short exact sequence of the form $M' \to M \to M''$. Hence the Grothendieck group $G(\mathcal{E})$ as defined in the above on $\mathcal{E}$ considered as proto-exact category is the same as the one mentioned in [Qui73, Thm. 1] on $\mathcal{E}$ considered as exact category.

In [Qui73, Thm. 1] Quillen proved the Grothendieck group of an exact category $\mathcal{E}$ is isomorphic to $K_0(\mathcal{E})$. Hence it is natural to ask whether this also holds true in the setting of proto-exact categories. It turns out this is indeed the case, as shall be shown below following the strategy given in [Qui73, Thm. 1].

Theorem 3.6.2. *The Grothendieck group $G(\mathcal{P})$ of the proto-exact category $\mathcal{P}$ is isomorphic to the K-group $K_0(\mathcal{P}) = \pi_1 BQ\mathcal{P}$ of $\mathcal{P}$.*

For the proof of the above theorem we will need to know a bit more about the category $Q\mathcal{P}$. Note that for $i : A \rightarrowtail B$ we have a morphism $i_! : A \rightsquigarrow B$, and for $p : B \twoheadrightarrow C$ a morphism $p^! : C \rightsquigarrow B$. Arrows in $Q\mathcal{P}$ of the form $i_!$ resp. $p^!$ are called *injections* resp. *surjections*.

Now let $u : C \rightsquigarrow C'$ be a given morphism in $Q\mathcal{P}$, say given by the span $C \overset{p}{\twoheadleftarrow} D \overset{i}{\rightarrowtail} C'$. Then u factorizes as $i_! p^!$. This factorization of u as a surjection followed by an injection is unique up to unique isomorphism at the middle term D. For any bicartesian square

$$\begin{tikzcd} D \arrow[r, tail, "i"] \arrow[d, two heads, "p"'] & C' \arrow[d, two heads, "q"] \\ C \arrow[r, tail, "j"] & X \end{tikzcd} \tag{3.9}$$

we further have that u also factorizes as $q^! j_!$, which follows from $D = C \times_X C'$. The latter factorization is again unique up to unique isomorphism, this time at X.

Observe that the rule $i \mapsto i_!$ gives rise to a covariant functor $\mathfrak{M} \to Q\mathcal{P}$. Likewise, $p \mapsto p^!$ gives a contravariant functor $\mathfrak{E} \to Q\mathcal{P}$. Further note that the above factorization satisfies the property that for bicartesian squares as in (3.9) it holds $q^! j_! = i_! p^!$. And in fact, $Q\mathcal{P}$ is universal with respect to these properties in the following sense.

Let Φ be the following category: a Φ-object is a triple $(\mathcal{C}, f, g)$ of a category $\mathcal{C}$ together with a covariant functor $f : \mathfrak{M} \to \mathcal{C}$ and a contravariant one $g : \mathfrak{E} \to \mathcal{C}$, such that for bicartesian squares as in (3.9) it holds $f(i)g(p) = g(q)f(j)$; a Φ-morphism from $(\mathcal{C}, f, g)$ to $(\mathcal{C}', f', g')$ is given by a functor $\mathcal{C} \to \mathcal{C}'$ that makes the obvious diagram commutative.

Now the *universal property of $Q\mathcal{P}$* is the statement that $Q\mathcal{P}$ is initial in the category Φ. To see this, let $(\mathcal{C}, f, g)$ in Φ be given. We need to find a unique functor $h : Q\mathcal{P} \to \mathcal{C}$ that fits in the following diagram

$$\begin{tikzcd} \mathfrak{M} \arrow[r, "(-)_!"] \arrow[dr, "f"'] & Q\mathcal{P} \arrow[d, dotted, "h"] & \mathfrak{E} \arrow[l, "(-)^!"'] \arrow[dl, "g"] \\ & \mathcal{C} & \end{tikzcd}$$

To construct h, let a span $u : C \rightsquigarrow C'$ be given. Factorize u as $i_! p^!$ as in (3.9), and put $h(u) \coloneqq f(i)g(p)$. Then one sees $h(u)$ does not depend on the factorization of u, by using that the factorization of u into $i_! p^!$ is unique up to an isomorphism σ at the middle term D, for which it holds $f(\sigma) = g(\sigma^{-1})$. To show that h respects the composition, one uses the fact that $f(i)g(p) = g(q)f(j)$ holds for bicartesian squares as in (3.9). Clearly h fits in the above diagram and is unique with this property.

The second ingredient in the proof of Thm. 3.6.2 is the following beautiful observation. Let $\mathcal{C}$ be a category such that $B\mathcal{C}$ is connected. Then employing general results on covering spaces, it can be shown that the category of $\pi_1(B\mathcal{C}, X)$-sets is equivalent to the category of morphism-inverting functors $F : \mathcal{C} \to \mathcal{S}\mathrm{et}$, for any X in $\mathcal{C}$. The equivalence is given by identifying $\pi_1(B\mathcal{C}, X)$ with the automorphism group of X inside the localized category $\mathcal{C}[\mathcal{C}^{-1}]$, and subsequently letting this group act of $F(X)$ in the obvious way, for a given morphism-inverting functor $F : \mathcal{C} \to \mathcal{S}\mathrm{et}$. See [Qui73, Prop. 1] for details, where the key ingredient is attributed to [GZ67].

Proof of Thm. 3.6.2. Write $\mathcal{F}$ for the category of morphism-inverting functors $Q\mathcal{P} \to \mathcal{S}\mathrm{et}$. By the previous observation, it suffices to show $\mathcal{F}$ is equivalent to the category of $G(\mathcal{P})$-sets, with $G(\mathcal{P})$ still the Grothendieck group of $\mathcal{P}$. For an object A in $\mathcal{P}$ write $i^A : 0 \rightarrowtail A$ and $p_A : A \twoheadrightarrow 0$.

Let $\mathcal{F}'$ be the full subcategory of $\mathcal{F}$ consisting of those $F' : Q\mathcal{P} \to \mathcal{S}\mathrm{et}$ for which $F'(C) = F'(0)$ and $F'(i_!^C) = \mathrm{id}_{F'(0)}$ holds for all C in $\mathcal{P}$. Then $\mathcal{F}'$ is equivalent to $\mathcal{F}$ since any F in $\mathcal{F}$ is isomorphic to an object in $\mathcal{F}'$. Indeed, for F in $\mathcal{F}$ one defines F' by sending $u : C \rightsquigarrow C'$ to $F(i_!^{C'})^{-1}F(u)F(i_!^C)$. Then F' is an element of $\mathcal{F}'$, and $(F(i_!^C))_{C \in \mathcal{P}}$ exhibits an isomorphism $F' \cong F$.

We have reduced our task to showing that $\mathcal{F}'$ is equivalent to the category of $G(\mathcal{P})$-sets. So let S be a $G(\mathcal{P})$-set. For $p : X \twoheadrightarrow B$ write $[\ker p]$ for the element $[0 \times_B X]$ in $G(\mathcal{P})$ given by pulling back p along i^B. Note $[\ker p]$ is independent of the choice of this pullback as in general $A \cong A'$ implies $[A] = [A']$. Then define a functor $f : \mathfrak{M} \to \mathcal{S}\mathrm{et}$ by sending all morphisms i to id_S, and a functor $g : \mathfrak{E} \to \mathcal{S}\mathrm{et}$ which is constantly S on objects and which on morphisms is given by sending p to the function $x \mapsto \ker[p] \cdot x$ on S. To check g is functorial, one uses that $[\ker q] \cdot [\ker p] = [\ker pq]$ for composable $p, q \in \mathfrak{E}$. And note the order of application here! As g needs to be contravariant, this comes out just right.

Let i, j and p, q be given as in the square (3.9). I claim $f(i)g(p) = g(q)f(j)$ holds. It suffices to show $g(p) = g(q)$. By the pasting lemma for pullbacks it holds $\ker p \cong \ker q$. Hence certainly the actions of $[\ker p]$ and of $[\ker q]$ are the same on S, so that $g(p) = g(q)$ indeed holds.

By the universal property of $Q\mathcal{P}$, the above construction gives us a functor $F_S : Q\mathcal{P} \to \mathcal{S}\mathrm{et}$ with $F_S(i_!) = \mathrm{id}_S$ for all $i \in \mathfrak{M}$ and with $F_S(p^!)$ for $p \in \mathfrak{E}$ the function on S given by multiplication with $[\ker p]$. To see that F_S is morphism-inverting, looking at the proof of

the universal property of $Q\mathcal{P}$ it suffices to show F_S is morphism-inverting on all injections and surjections. Both cases are clear by construction.

Observe a $G(\mathcal{P})$-equivariant map $S \to S'$ gives a natural transformation $F_S \Rightarrow F_{S'}$ in an obvious way. Hence we have a functor from $G(\mathcal{P})$-sets to $\mathcal{F}'$.

To see that this functor $S \mapsto F_S$ is an equivalence, let F in $\mathcal{F}'$ be given. Then we let $G(\mathcal{P})$ act on $F(0)$ by means of the function

$$G(\mathcal{P}) \to \mathrm{Aut}(F(0)) : [C] \mapsto F(p_C^!)\,.$$

The above map is a group-homomorphism. To see this, consider a given diagram of the following form, with the left-hand square a pullback

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, two heads, "p_A"'] & X \arrow[r, two heads, "p_X"] \arrow[d, two heads, "q"] & 0 \\ 0 \arrow[r, tail, "i^B"] & B \arrow[ur, two heads, "p_B"'] & \end{tikzcd}$$

Then we need to show $F(p_X^!) = F(p_A^!)F(p_B^!)$. For this first note

$$F(i_!) = F((i \circ i^A)_!) = F(i_!^X) = \mathrm{id}_{F(0)}\ .$$

We therefore have

$$F(q^!) = F(q^! i_!^B) = F(i_! p_A^!) = F(p_A^!)\,, \tag{3.10}$$

which implies $F(p_X^!) = F(q^! p_B^!) = F(p_A^!)F(p_B^!)$, which was to be shown.

For the resulting $G(\mathcal{P})$-group $F(0)$ it holds $F_{F(0)} = F$. To see this, it suffices to show for $q : X \twoheadrightarrow B$ that $F(q^!)$ is given by multiplication by $[\ker q]$ on $F(0)$. And indeed, by construction of the $G(\mathcal{P})$-action on $F(0)$, we have that for $x \in F(0)$ it holds

$$[\ker q](x) = F(p_{\ker q}^!)(x) = F(q^!)(x)\,,$$

using observation (3.10).

From the above reasoning it follows that the functor $S \mapsto F_S$ is essentially surjective. For full faithfulness, first note that clearly this functor is injective on hom-sets. For surjectivity, let a natural transformation $\epsilon : F_S \to F_{S'}$ for $G(\mathcal{P})$-sets S, S' be given. Then observe that, for all A in $Q\mathcal{P}$, it holds

$$\epsilon_A = \epsilon_A F_S(i_!^A) = F_{S'}(i_!^A)\epsilon_0 = \epsilon_0\,,$$

so in fact $(\epsilon_A)_{A\in\mathcal{P}}$ is just $(\epsilon_0)_{A\in\mathcal{P}}$. Since for all X in $\mathcal{P}$ it holds $[X] = [\ker p_X]$, it is clear that ϵ_0 is a $G(\mathcal{P})$-equivariant map $S \to S'$, which induces $\epsilon : F_S \Rightarrow F_{S'}$ under the functor $S \mapsto F_S$. Hence this functor is also fully faithful, which was to be shown. □

Example 3.6.3. It is very natural to ask whether the K_0-group of a proto-exact category is always abelian, as in the case of exact categories. The surprising answer to this question turns out to be negative.

To see this, we construct a proto-exact category as follows. First take objects $A, 0, B$, and for $n \in \mathbb{Z}$ add the following arrows

$$\begin{tikzcd} x^n \circlearrowright A \arrow[r, bend right, "p"'] & 0 \arrow[l, bend right, "i"'] \arrow[r, bend left, "j"] & B \circlearrowleft y^n \arrow[l, bend left, "q"] \end{tikzcd}$$

with the rules that $x^n \circ x^m = x^{n+m}$, that $x^n \circ i = i$, and that $p \circ x^n = p$ for $n, m \in \mathbb{Z}$, and likewise with respect to y. Then between any two objects of $\{A, B\}$, add a zero map z. Declare any composition of the above arrows that factorizes over 0 to be equal to this zero map z. It is clear that this gives us a category, which we shall write as $\mathcal{P}$. It is also clear this $\mathcal{P}$ is pointed by 0.

Endow $\mathcal{P}$ with the following structure: call all maps except p, q, z admissibly monic. Likewise, call all maps except i, j, z admissibly epic. In symbols:

$$\begin{aligned} \mathfrak{M} &\coloneqq \{\mathrm{id}_0, i, j, x^n, y^n \mid n \in \mathbb{Z}\}\,; \\ \mathfrak{E} &\coloneqq \{\mathrm{id}_0, p, q, x^n, y^n \mid n \in \mathbb{Z}\}\,. \end{aligned}$$

It is straightforward to show that this structure makes $\mathcal{P}$ into a proto-exact category. It has admissible squares of the form, with $n, m, k \in \mathbb{Z}$

$$\begin{tikzcd} A \arrow[r, tail, "x^m"] \arrow[d, two heads, "x^n"'] & A \arrow[d, two heads, "x^n"] \\ A \arrow[r, tail, "x^m"] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, tail, "x^k"] \arrow[d, two heads, "p"'] & A \arrow[d, two heads, "p"] \\ 0 \arrow[r, equal] & 0 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, tail, "i"] \arrow[d, equal] & A \arrow[d, two heads, "x^k"] \\ 0 \arrow[r, tail, "i"] & A \end{tikzcd} \tag{3.11}$$

and likewise for B, j, q, y, together with the identity squares.

Now let us consider the Q-construction on $\mathcal{P}$. It is again straightforward to show the category $Q\mathcal{P}$ is the following

$$\begin{tikzcd} x_!^n \circlearrowright A \arrow[r, bend right, "p^!"'] & 0 \arrow[l, bend right, "i_!"'] \arrow[r, bend left, "j_!"] & B \circlearrowleft y_!^n \arrow[l, bend left, "q^!"] \end{tikzcd}$$

with the rule that $x_!^n p^! = p^!$ and $x_!^n i_! = i_!$ for all $n \in \mathbb{Z}$, and likewise for y, q, j.

As we have seen, the zeroth K-group of $\mathcal{P}$ can be computed as $\pi_1 Q\mathcal{P}$, which in turn is isomorphic to the automorphism group $\mathrm{Aut}(0)$ of 0 inside the localized version $Q\mathcal{P}[Q\mathcal{P}^{-1}]$ of $Q\mathcal{P}$. I claim this latter group $\mathrm{Aut}(0)$ is a free group on two letters.

To localize $Q\mathcal{P}$, we need to add four new morphisms, namely the dashed ones in the following diagram (drawn with non-squiggly arrows for clarity):

$$\begin{tikzcd} x_!^n \circlearrowright A \arrow[r, bend right, "p^!"'] \arrow[r, bend right=60, dashed, "p^{!-1}"'] & 0 \arrow[l, bend right, "i_!"'] \arrow[l, bend right=60, dashed, "i_!^{-1}"'] \arrow[r, bend left, "j_!"] \arrow[r, bend left=60, dashed, "q^{!-1}"] & B \circlearrowleft y_!^n \arrow[l, bend left, "q^!"] \arrow[l, bend left=60, dashed, "j_!^{-1}"] \end{tikzcd}$$

Because $x_!^n p^! = p^!$, by composing with ${p^!}^{-1}$ to the right, we see that $x_n^!$ becomes the identity in $Q\mathcal{P}[Q\mathcal{P}^{-1}]$. Likewise for y.

Now put $u := j_!^{-1} q^!$, and $v := i_!^{-1} p^!$. Then ${q^!}^{-1} j_! = u^{-1}$ and ${p^!}^{-1} i_! = v^{-1}$, hence u, v generate all of $\mathrm{Aut}(0)$. Because we have added the above dashed arrows freely, there are no relations between u and v. Hence $\mathrm{Aut}(0)$ is the free group generated on u, v, which is clearly non-abelian.

3.7 Calculations

Example 3.7.1 (Eilenberg-Mazur swindle)**.** Let us show that $K_0(\mathcal{S}\mathrm{et}_*)$ is trivial. Let X be a given pointed set. Then in $\mathcal{S}\mathrm{et}_*$ it holds that $X \amalg \coprod_{n\geq 0} X$ is just $\coprod_{n\geq 0} X$. Furthermore, any pair of pointed sets A, B fit in an admissible square of the form

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, two heads] & A \amalg B \arrow[d, two heads] \\ 0 \arrow[r, tail] & B \end{tikzcd}$$

In $K_0(\mathcal{S}\mathrm{et}_*)$ we therefore get

$$\left[\coprod_{n\geq 0} X\right] = \left[X \amalg \coprod_{n\geq 0} X\right] = [X] \cdot \left[\coprod_{n\geq 0} X\right],$$

which clearly implies $[X] = 1$.

The key point in this example is that the functor $I : \mathcal{S}\mathrm{et}_* \to \mathcal{S}\mathrm{et}_*$ that sends X to $\coprod_{n\geq 0} X$ satisfies $X \amalg IX \cong IX$. It is natural to try and generalize this idea in two directions and ask: does this work for all K-groups on any proto-exact category $\mathcal{P}$ with such a functor I? It turns out that under some reasonable assumptions on $\mathcal{P}$ it does.

In the following, we will need an additivity theorem. For this, we let ourselves again be guided by the case of exact categories in [Qui73] and Waldhausen categories in [Wal85]. See also [Wei13, IV.1.9] for the idea of applying the additivity theorem to exact categories or Waldhausen categories that have such a functor I.

3.7.a Additivity theorem

Throughout, fix a proto-exact category $\mathcal{P}$. Call a sequence of morphisms of the form $A' \rightarrowtail A \twoheadrightarrow A''$ *exact* if it fits in a bicartesian square

$$\begin{tikzcd} A' \arrow[r, tail] \arrow[d, two heads] & A \arrow[d, two heads] \\ 0 \arrow[r, tail] & A'' \end{tikzcd}$$

Let $\mathcal{P}'$ be another proto-exact category. Recall that a functor $\mathcal{P}' \to \mathcal{P}$ is called proto-exact when it preserves all admissible squares. Let F', F, F'' be such proto-exact functors $\mathcal{P}' \to \mathcal{P}$. Then a sequence of natural transformations $F' \to F \to F''$ is called *exact* if for all A in $\mathcal{P}'$ the sequence $F'(A) \rightarrowtail F(A) \twoheadrightarrow F''(A)$ is exact in $\mathcal{P}$.

Definition 3.7.2. We say $\mathcal{P}$ has *admissible inclusions* if for all admissible monos $X \rightarrowtail A$ and $X \rightarrowtail B$ the pushout $A \cup_X B$ exists and furthermore the induced map $B \to A \cup_X B$ is admissibly monic. Likewise, $\mathcal{P}$ is said to have *admissible projections* if for all admissible epis $C \twoheadrightarrow Y$ and $D \twoheadrightarrow Y$ the pullback $C \times_Y D$ exists and the map $C \times_Y D \to C$ is admissibly epic.

It is clear that any proto-exact category that results from an exact category by forgetting its additive structure has admissible inclusions and projections. It is also straightforward to show that $\mathcal{S}\mathrm{et}_*$ has admissible inclusions. However, $\mathcal{S}\mathrm{et}_*$ does not have admissible projections. This is because for pointed sets X, Y, the projection $X \times_0 Y \to Y$ in general does not have a unique section. The following example shows that not all proto-exact categories have admissible inclusions.

Example 3.7.3. Let $\mathcal{C}$ be any pointed category with pushouts and pullbacks. Then we have a minimal proto-exact structure on $\mathcal{C}$ by taking the isomorphisms plus the maps of the form $0 \to A$ as $\mathfrak{M}$, and the isomorphisms plus the maps of the form $B \to 0$ as $\mathfrak{E}$. It is easily seen $\mathcal{C}$ with this proto-exact structure can only have admissible inclusions in some very special cases. But note that $Q\mathcal{C}$ with this minimal structure always has 0 as an initial object, so that $K_0\mathcal{C}$ is trivial.

Let A, B be objects of $\mathcal{P}$. Consider the maps $\mathrm{id} : B \to B$ and $A \to 0 \to B$. These induce a map $A \cup_0 B \to B$ by uniqueness of $0 \to B$. Note $A \cup_0 B$ is just $A \amalg B$. Now I claim that if $\mathcal{P}$ has admissible inclusions, then

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A \amalg B \arrow[d] \\ 0 \arrow[r] & B \end{tikzcd}$$

is an admissible square. Clearly, the top arrow is admissibly monic since it is a pushout of an admissible mono along an admissible mono and $\mathcal{P}$ has admissible inclusions. Hence it suffices to show that the square is a pushout, which is straightforward.

Because for A, B as above it follows $[A] \cdot [B] = [A \amalg B] = [B \amalg A] = [B] \cdot [A]$, we have shown the following

Corollary 3.7.4. *If $\mathcal{P}$ has admissible inclusions, then $K_0(\mathcal{P})$ is abelian.*

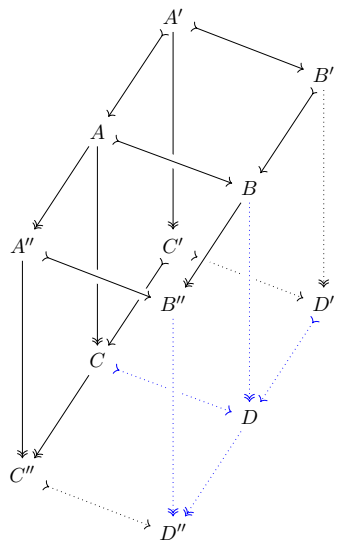
Let $\mathcal{X}$ be the category of exact sequences in $\mathcal{P}$ and with obvious morphisms between them. Call it the *extension category* of $\mathcal{P}$.

Lemma 3.7.5. *If $\mathcal{P}$ has admissible inclusions and projections, then $\mathcal{X}$ is a proto-exact category with the admissible monos resp. epis taken pointwise.*

Proof. Clearly, $\mathcal{X}$ is pointed. Let us begin with the pushout axiom. So consider the following diagram in $\mathcal{X}$

$$\begin{tikzcd} \alpha \arrow[r, tail] \arrow[d, two heads] & \beta \arrow[d, dotted, two heads] \\ \gamma \arrow[r, dotted, tail] & \delta \end{tikzcd}$$

where the solid arrows are given, and we need to complete the diagram into a pushout square, with admissble monos and epis as indicated. The above diagram induces the following diagram in $\mathcal{P}$



(3.12)

Here, the solid part is given by $\gamma \twoheadleftarrow \alpha \rightarrowtail \beta$, and the dotted part must be constructed.

To fill in the above diagram, let first D be the pushout $C \cup_A B$. Then using that $\mathcal{P}$ has admissible projections, let D'' be the pushout $D \cup_B B''$. Now let D' be the pullback $0 \times_{D''} D$. This gives the blue part of the above diagram, including the fact that the arrows are admissibly monic or epic as indicated.

Since the admissible epi $A' \twoheadrightarrow C'$ is epic, the composition

$$C' \rightarrowtail C \rightarrowtail D \twoheadrightarrow D''$$

is the zero map. By $D' = 0 \times_{D''} D$, this gives us a unique arrow $C' \to D'$ such that $C' \to D' \to D$ is the given map $C' \to C \to D$. Likewise, since the composition $B' \to D''$ is zero, we have a unique map $B' \to D'$ such that $B' \to D' \to D$ is the given map $B' \to B \to D$. Since by the same reasoning, we have a unique map $A' \to D'$ such that $A' \to D' \to D$ is the given map $A' \to D$, these maps $C' \to D' \leftarrow B'$ make the diagram commutative.

Likewise, it now holds that the composition $C' \to D''$ is zero. Hence, since $C'' = 0 \cup_{C'} C$, we have a unique arrow $C'' \to D''$ such that $C \to C'' \to D''$ is the given map $C \to D''$. Because we also have a unique arrow $A'' \to D''$ with $A \to A'' \to D''$ the given map $A \to D''$, this arrow $C' \to D''$ makes the above diagram commutative.

We have constructed the following diagram in $\mathcal{X}$

$$\begin{tikzcd} \alpha \arrow[r, tail] \arrow[d, two heads] & \beta \arrow[d] \\ \gamma \arrow[r] & \delta \end{tikzcd}$$

with δ the sequence $D' \rightarrowtail D \twoheadrightarrow D''$. Now one shows this is a pushout in $\mathcal{X}$ in a straightforward way, using admissible monos resp. epis are monic resp. epic in the ordinary sense. Observe this implies, that all the squares in (3.12) that are most parallel to the page are pushouts in $\mathcal{P}$, which implies $\gamma \to \delta$ is admissibly monic and $\beta \to \delta$ is admissibly epic.

By a dual argument, this time employing the assumption $\mathcal{P}$ has admissible inclusions, one shows the pullback axiom holds for $\mathcal{X}$. From these considerations it follows that a square in $\mathcal{X}$ is a pushout resp. a pullback iff it is so pointwise. The latter observation clearly implies the bicartesian axiom holds in $\mathcal{X}$ as well, which remained to be shown. □

Now let ℓ be the functor $\mathcal{P} \times \mathcal{P} \to \mathcal{X}$ that sends (A, B) to the 'split exact sequence' $A \rightarrowtail A \amalg B \twoheadrightarrow B$. Also, write σ for the functor $\mathcal{P} \times \mathcal{P} \to \mathcal{P}$ that sends (A, B) to $A \amalg B$.

Definition 3.7.6. Call $\mathcal{P}$ *fine* if it has admissible inclusions and projections and if furthermore the functor σ is proto-exact.

For example, any exact category considered as proto-exact category is fine.

From here on, we assume $\mathcal{P}$ is fine. Note σ_* can then be considered as a homomorphisms $K_n(\mathcal{P}) \times K_n(\mathcal{P}) \to K_n(\mathcal{P})$ by paragraph (3.5.a). As such, it is equal to $(\alpha, \beta) \mapsto \alpha + \beta$. This is because $A \mapsto \sigma(A, 0)$ and $B \mapsto \sigma(0, B)$ are isomorphic to the identity on $\mathcal{P}$, which implies

$$\sigma_*(\alpha, \beta) = \sigma_*(\alpha, 0) + \sigma_*(0, \beta) = \alpha + \beta \tag{3.13}$$

holds for all $\alpha, \beta \in K_n(\mathcal{P})$. Also note that σ being proto-exact implies that ℓ is proto-exact as well.

Theorem 3.7.7 (Additivity Theorem)**.** *For an exact sequence $F' \to F \to F''$ of proto-exact functors $\mathcal{P}' \to \mathcal{P}$ it holds $F_* = F'_* + F''_*$ as homomorphisms $K_n(\mathcal{P}') \to K_n(\mathcal{P})$.*

For the proof of this theorem we will need to know a bit more about exact sequences in $\mathcal{P}$. Observe we have functors s, t, q from $\mathcal{X}$ to $\mathcal{P}$, sending an exact sequence $A' \rightarrowtail A \twoheadrightarrow A''$ to the source A', the target A, and the quotient A'' respectively. These functors s, t, q are all proto-exact, because the admissible monos resp. epis in $\mathcal{X}$ are taken pointwise, and because the admissible squares in $\mathcal{X}$ are calculated pointwise.

The functors s, t, q furthermore enjoy the following *universal property.* Let $F' \to F \to F''$ be an exact sequence of proto-exact functors $\mathcal{P}' \to \mathcal{P}$. Then there is a unique proto-exact functor $G : \mathcal{P}' \to \mathcal{X}$ such that F', F, F'' factor as G followed by s, t, q respectively. This G is given by sending X in $\mathcal{P}'$ to the exact sequence $F'(X) \rightarrowtail F(X) \twoheadrightarrow F''(X)$.

Proposition 3.7.8. *The functor $(s, q) : Q\mathcal{X} \to Q\mathcal{P} \times Q\mathcal{P}$ is a homotopy equivalence.*

Proof. The proof given in [Qui73, §3, Thm. 2] for the case of exact categories also works in our setting. For brevity's sake we only sketch the argument. Let A, B be objects in $Q\mathcal{P}$, and consider the category $\mathcal{C} := (s, q)/(A, B)$. By Thm. A, it suffices to show that $\mathcal{C}$ is contractible.

Note that an object in $\mathcal{C}$ is a triple (σ, u, v), where σ is an exact sequence in $\mathcal{P}$ and u, v are morphisms $u : s\sigma \rightsquigarrow A$ and $v : q\sigma \rightsquigarrow B$ respectively. Now let $\mathcal{C}'$ resp. $\mathcal{C}''$ be the

full subcategory of such triples for which u is surjective, resp. for which u is surjective and v is injective.

The idea is to show first that the inclusion $\mathcal{C}' \to \mathcal{C}$ has a left adjoint. It is here one uses the assumption that $\mathcal{P}$ has admissible inclusions. In the second step, showing that $\mathcal{C}'' \to \mathcal{C}'$ also has a left adjoint, one uses the assumption that $\mathcal{P}$ has admissible projections.

Finally one shows that $\mathcal{C}''$ has initial object $(0, j_A^!, i_!^B)$. Indeed, let an object $(\sigma, j^!, i_!)$ in $\mathcal{C}''$ be given. Then it is straightforward to show that the diagram

$$\begin{tikzcd}
0 \arrow[r, equal] & 0 \arrow[r, equal] & 0 \\
s\sigma \arrow[u, two heads] \arrow[r, equal] \arrow[d, equal] & s\sigma \arrow[u, two heads] \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[u, equal] \arrow[d, tail] \\
s\sigma \arrow[r, tail] & t\sigma \arrow[r, two heads] & q\sigma
\end{tikzcd}$$

induces a unique arrow $(0, j_A^!, i_!^B) \to (\sigma, j^!, i_!)$.

Because $\mathcal{C}''$ has an initial object, it is contractible. Also, because adjoint functors are homotopy equivalences, $\mathcal{C}$ is weakly equivalent to $\mathcal{C}''$. It follows that $\mathcal{C}$ is contractible, hence that (s, q) is a homotopy equivalence. □

Proof of Thm. 3.7.7. By the universal property of the functors $s, t, q : \mathcal{X} \to \mathcal{P}$, it suffices to show $t_* = s_* + q_*$. Let ℓ, σ be the proto-exact functors from before. Then clearly $\sigma(s, q)\ell = t\ell$ holds, which by (3.13) gives us

$$t_*\ell_* = \sigma_*(s, q)_*\ell_* = (s_* + q_*)\ell_* \,.$$

But we also have $(s, q)\ell = \mathrm{id}$, and therefore $(s, q)_*\ell_* = \mathrm{id}$. Since $(s, q)_*$ is an isomorphism by the previous proposition, we can therefore cancel ℓ_* in the above equation to get the desired result. □

The following result is too much fun not to write down. It follows from the additivity theorem by induction.

Corollary 3.7.9. *Let $0 \to F_0 \to \cdots \to F_n \to 0$ be an exact sequence of proto-exact functors $\mathcal{P}' \to \mathcal{P}$. Then $\sum_{i=0}^n (-1)^i (F_i)_* = 0$ as morphisms $K_n(\mathcal{P}') \to K_n(\mathcal{P})$.*

3.7.b Flasque proto-exact categories

We are still under the assumption that $\mathcal{P}$ is a fine proto-exact category.

Definition 3.7.10. Call $\mathcal{P}$ *flasque* if there is a proto-exact functor $\infty : \mathcal{P} \to \mathcal{P}$ such that for all objects X in $\mathcal{P}$ it holds $\infty X \cong X \amalg \infty X$, natural in X.

Theorem 3.7.11. *A proto-exact category which is fine and flasque has trivial K-groups.*

Proof. Let Δ be the map $X \mapsto (X, X)$ from $\mathcal{P}$ to $\mathcal{P} \times \mathcal{P}$. By flasqueness of $\mathcal{P}$ we have

$$\sigma(\mathrm{id}, \infty)\Delta \cong \infty \,.$$

Clearly, the sequence $\mathrm{id} \to \sigma(\mathrm{id}, \infty)\Delta \to \infty$ is exact. By fineness of $\mathcal{P}$, the additivity theorem then tells us that ∞_* is $\mathrm{id}_* + \infty_*$, which implies $\mathrm{id}_* = 0$ and hence $K_n(\mathcal{P}) = 0$. □

3.7.c Quadratic spaces

A *quadratic space* is a finite-dimensional real vector space V endowed with a positive definite quadratic form ρ_V. A linear map $f : V \to W$ between such quadratic spaces is a *morphism of quadratic spaces* when $\rho_V \geq \rho_W f$ holds. We give a proto-exact structure on the resulting category $\mathcal{U}$ of quadratic spaces, following [DK12, Exm. 2.4.6], and then we compute the zeroth K-group $K_0(\mathcal{U})$ of $\mathcal{U}$.

Let $f : V \to W$ be a morphism of quadratic spaces. Then we call f

- An admissible mono when f is injective and when $\rho_V = \rho_W f$ holds;
- An admissible epi when f is surjective and when $\min_{f(v)=w} \rho_V(v) = \rho_W(w)$ holds.

Let $U \rightarrowtail W$ be an admissible mono. Then by employing the canonical form of quadratic forms, we may assume it is an inclusion of vector spaces in such a way that W is $U \oplus U'$ for a certain vector space U', for which it holds that $\rho_W(u, u') = \rho_W(u, 0) + \rho_W(0, u')$.

Now let also $g : U \twoheadrightarrow V$ be an admissible epi. Then $V \cup_U W$ is the vector space $V \oplus U'$ endowed with the quadratic form that is defined as

$$\rho_{V \cup_U W}(v, u') \coloneqq \rho_V(v) + \rho_W(0, u') .$$

Clearly the map $v \mapsto (v, 0)$ is an admissible mono $V \rightarrowtail V \oplus U'$. And it is straightforward to check $(u, u') \mapsto (g(u), u')$ is an admissible epi $W \twoheadrightarrow V \oplus U'$ by using that $\rho_W(u, u') = \rho_U(u) + \rho_W(0, u')$ holds.

Likewise, let a diagram $X \overset{f}{\rightarrowtail} Z \overset{g}{\twoheadleftarrow} Y$ in $\mathcal{U}$ be given. Then $X \times_Z Y$ is the space of those points $(x, y) \in X \times Y$ for which $f(x) = g(y)$ holds. It has quadratic form

$$\rho_{X \times_Z Y}(x, y) \coloneqq \rho_Y(y) .$$

Observe that $X \times_Z Y \to Y$ is admissibly monic and that $X \times_Z Y \to X$ is admissibly epic.

Let U, U', V, W be as above. Then it is not hard to see that $U = V \times_{V \oplus U'} (U \oplus U')$ and that $\rho_U = \rho_{V \times_{V \oplus U'} (U \oplus U')}$. Likewise, for X, Y, Z as above, let $Y' \subset Y$ be the unique subspace such that $Y = (X \times_Z Y) \oplus Y'$. Then it holds $Z = X \oplus Y'$ and also $\rho_Z = \rho_{X \cup_{X \times_Z Y} Y}$.

From the above remarks it follows that the proposed structure makes $\mathcal{U}$ into a proto-exact category.

Proposition 3.7.12. *The proto-exact category $\mathcal{U}$ is fine.*

Proof. This is straightforward by the observation that the admissible monos in $\mathcal{U}$ are of the form $A \rightarrowtail A \oplus A'$ with $\rho_{A \oplus A'} = \rho_A + \rho_{A'}$, and that the admissible epis in $\mathcal{U}$ are of the form $B \oplus B' \twoheadrightarrow B$ with $\rho_{B \oplus B'} = \rho_B + \rho_{B'}$. □

Let M be the commutative monoid that has the isomorphism classes $|U|$ of objects U of $\mathfrak{U}$ as underlying set and with addition $+$ on M defined by

$$|U| + |V| := |U \oplus V|\,,$$

for $U, V \in M$. The zero element of M is just $|0|$.

Recall the *group completion* of M is defined as the abelian group $M^{-1}M$ together with a homomorphism of monoids $\iota : M \to M^{-1}M$, with the universal property that for any other abelian group A and monoid-homomorphism $f : M \to A$ there is a unique group-homomorphism $\bar{f} : M^{-1}M \to A$ such that $\bar{f}\iota = f$.[3]

Proposition 3.7.13. *The group completion $M^{-1}M$ of M is isomorphic to $K_0(\mathfrak{U})$.*

Proof. From the description of $\mathfrak{M}, \mathfrak{E}$ in the proof of Prop. 3.7.12 it follows that all exact sequences in $\mathfrak{U}$ are split, i.e. are of the form $U \rightarrowtail U \oplus V \twoheadrightarrow V$. Since $\mathfrak{U}$ is fine, $K_0(\mathfrak{U})$ is abelian. It therefore holds by Thm. 3.6.2 that $K_0(\mathfrak{U})$ is the abelian group generated on the symbols $[U]$, one for each U in $\mathfrak{U}$, and subject to the relation $[U \oplus V] = [U] \oplus [V]$.

Let ι be the map $M \to K_0(\mathfrak{U})$ defined as $|U| \mapsto [U]$. Clearly ι is a homomorphism. It is straightforward to show that ι satisfies the universal property of a group completion by the above construction of $K_0(\mathfrak{U})$. Since $M^{-1}M$ is uniquely determined up to isomorphism by its universal property, the claim readily follows. $\square$

By the canonical form of quadratic spaces, we see that in fact $M \cong \mathbb{N}$. By the above proposition, this gives us

Corollary 3.7.14. *The zeroth K-group of $\mathfrak{U}$ is isomorphic to $\mathbb{Z}$.*

[3] A group completion of a commutative monoid always exists and is unique up to isomorphism. See e.g. [Wei13, §II.1] for some background.

Perspectives

Two roads diverged in a wood, and I—
I took the one less traveled by,
And that has made all the difference.

The Road Not Taken
Frost

Of course, as the notation suggests, d-Segal objects for $d = 1, 2$ are the beginning of a theory of such objects for $d \geq 0$. In [DK12], the authors promise these higher Segal objects are forthcoming. I suspect these objects are also going to be definable in homotopical categories.

Recall that a sufficiently nice homotopical category $\mathcal{C}$ has natural membranes, provided homotopy limits in $\mathcal{C}$ are preserved by homotopy initial functors (Def. 2.1.4, 2.3.3, Prop. 2.3.7). The two most important results shown for a homotopical category $\mathcal{C}$ that has natural membranes is the fact that 1-Segal objects in $\mathcal{C}$ are 2-Segal and the pullback condition (Thm. 2.4.1, Prop. 2.5.1). There is probably some redundancy in this route, and it would be interesting to see if one can do away with some of the assumptions. For example, in showing 1-Segal objects in $\mathcal{C}$ are 2-Segal, it seems one only needs the condition of natural membranes on simplicial sets of the form $D = \Delta[\mathcal{I}]$.

We haven't really explored the connection with higher categories. As mentioned in Rem. 2.2.12, 1-Segal spaces are one of the possible roads to ∞-categories. But ∞-categories also play a different role in this story: it turns out the theory of Segal objects can also itself be formulated in the setting of ∞-categories. This is the road taken in [GKT14] from the start and also eventually in [DK12, §7]. Now for us this is of interest for the following reason: starting with a homotopical category $\mathcal{C}$, there is also available a simplicial localization of $\mathcal{C}$ that results in an ∞-category $L\mathcal{C}$. The question then is: can we apply the theory of Segal objects in ∞-categories to $L\mathcal{C}$, in order to say something about Segal objects in $\mathcal{C}$? This would for example be a possible strategy in minimizing the assumptions on $\mathcal{C}$ needed to show Thm. 2.4.1 and Prop. 2.5.1.

Let $\mathcal{C}$ be a Waldhausen category. Then part of the work done in [Wal85] revolves around delooping $\mathcal{C}$: showing that the $\mathrm{w}\mathcal{S}_\bullet$-construction on $\mathcal{C}$ can be iterated to give n-simplicial categories $\mathrm{w}\mathcal{S}^n_\bullet\mathcal{C}$, such that their classifying spaces form an Ω-spectrum except at $n = 0$ (see also [Boy06] and [Car05] for an overview). The latter means that there are weak equivalences $|\mathrm{w}\mathcal{S}^n_\bullet\mathcal{C}| \to \Omega|\mathrm{w}\mathcal{S}^{n+1}_\bullet\mathcal{C}|$. Here, ΩX is the loop space of a given pointed

space X, i.e. the space of pointed maps $\Delta^1 \to X$. Since one always has $\pi_n \Omega X \cong \pi_{n+1} X$, one sees such an Ω-spectrum is a good thing to have.

In our present work, it would be interesting to see how far Waldhausen's approach goes through in the setting of proto-exact categories. In particular, since Waldhausen uses an additivity theorem in construction of his deloopings, one can ask to what extent the fineness condition in Thm. 3.7.7 is really necessary.

Appendices

I know the pieces fit
'cause I watched them tumble down.

Schism
Tool

A Homotopy theory

In this and the next appendices, we collect some basic definitions and results from abstract homotopy theory and homological algebra, mainly to fix notation and provide a convenient reminder if so necessary. In this appendix and the next two, I mainly follow the first two chapters of [GJ09], the appendices of [Lur06], Hirschhorn's preprint [Hir14] and some [Hov99]. I claim no self-contained exposition, so proofs, examples and explanations of the why are mostly left out.

Definition A.1. A *weak factorization system* on a category $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms such that

- For every $f : X \to Y$ in $\mathcal{C}$ there are $g \in \mathcal{L}, h \in \mathcal{R}$ such that $f = hg$;
- The class $\mathcal{L}$ are precisely those morphisms having the left lifting property with respect to $\mathcal{R}$, and the other way around.

Definition A.2. A *model structure* on a given category $\mathcal{M}$ is a triple $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ of classes of morphisms, called *weak equivalences*, *cofibrations* and *fibrations* respectively, such that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are both weak factorization systems and such that $\mathcal{W}$ contains all isomorphisms and satisfies 2-of-3. Now $\mathcal{M}$ is called a *model category* when it is (co)complete and is endowed with a model structure.

A weak equivalence which is also a (co)fibration is called a *trivial (co)fibration*. An object X for which $X \to 1$ is a fibration is called *fibrant*. An object Y for which $0 \to Y$ is a cofibration is called *cofibrant*.

Example A.3. The category $s\mathcal{S}\mathrm{et}$ of simplicial sets has the structure of a model category as follows.

- The cofibrations are inclusion maps;
- The weak equivalences are those maps $X \to Y$ for which the corresponding maps $|X| \to |Y|$ on the geometric realizations are weak homotopy equivalences;
- The fibrations are the Kan fibrations, i.e. those maps with the right lifting property with respect to all horn inclusions $\Lambda^k[n] \to \Delta[n]$, where $\Lambda^k[n]$ is obtained from $\Delta[n]$ by removing the k-th face.

This is called the *classical model structure* and is the only one that we use on s$\mathcal{S}$et. A fibrant object in s$\mathcal{S}$et is also called a *Kan complex*. Interestingly, for such Kan complexes X, Y the requirement of a map $X \to Y$ being a weak equivalence can be defined entirely combinatorially by means of simplicial homotopy groups. Note that all objects in s$\mathcal{S}$et are cofibrant, since $0 \to Y$ is always monic.

Definition A.4. A model category $\mathcal{M}$ is called a *simplicial model category* when it is an s$\mathcal{S}$et-enriched model category. This means that it is endowed with a Quillen adjunction of two variables, i.e. with functors

$$\mathcal{M} \times \mathrm{s}\mathcal{S}\mathrm{et} \xrightarrow{(-)\otimes(-)} \mathcal{M}; \qquad \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \xrightarrow{\mathrm{Map}_{\mathcal{M}}(-,-)} \mathrm{s}\mathcal{S}\mathrm{et}; \qquad \mathcal{M} \times \mathrm{s}\mathcal{S}\mathrm{et}^{\mathrm{op}} \xrightarrow{(-)^{(-)}} \mathcal{M}$$

such that $\mathrm{Map}_{\mathcal{M}}(M', M)$ is the set $\mathcal{M}(M', M)$ in degree 0, together with natural simplicial isomorphisms

$$\mathrm{Map}_{\mathcal{M}}(M', M^S) \cong \mathrm{Map}_{\mathrm{s}\mathcal{S}\mathrm{et}}(S, \mathrm{Map}_{\mathcal{M}}(M', M)) \cong \mathrm{Map}_{\mathcal{M}}(M' \otimes S, M)\,,$$

for all $M, M' \in \mathcal{M}$ and $S \in \mathrm{s}\mathcal{S}\mathrm{et}$, and with $\mathrm{Map}_{\mathrm{s}\mathcal{S}\mathrm{et}}(-,-)$ as in (A.6). This structure must also satisfy the condition that for all cofibrations $f : M \to M'$ in $\mathcal{M}$ and cofibrations $g : S \to S'$ in s$\mathcal{S}$et, the induced map from $(M \otimes S') \amalg_{M \otimes S} (M' \otimes S)$ to $M' \otimes S'$ is a cofibration in $\mathcal{M}$ which is trivial if either f or g is.

Note the advertised requirement on a simplicial model category $\mathcal{M}$ with respect to cofibrations $f : M \to M'$ and cofibrations $g : S \to S'$ is equivalent to asking $\mathrm{Map}_{\mathcal{M}}(B, X) \to \mathrm{Map}_{\mathcal{M}}(A, X) \times_{\mathrm{Map}_{\mathcal{M}}(A,Y)} \mathrm{Map}_{\mathcal{M}}(B, Y)$ is a fibration in s$\mathcal{S}$et for all cofibrations $j : A \to B$ and fibrations $q : X \to Y$ in $\mathcal{M}$, which again must be trivial if either j or q is. In any one of its incarnations, this requirement is called the *homotopy lifting extension axiom*. Although it seems awfully technical and does not seem to play any part in the main body of our work, it is in fact crucial in guaranteeing the model category structure and the simplicial enrichment on $\mathcal{M}$ interact nicely. For example, one needs this good interaction in order for the simplicial enrichment to be employable in constructing homotopy (co)limits. A straightforward result from the homotopy lifting extension axiom is the following

Lemma A.5. *Let A be cofibrant and X fibrant. Then*

- $\mathrm{Hom}(A, -)$ *preserves (trivial) fibrations while* $\mathrm{Hom}(-, X)$ *sends (trivial) cofibrations to (trivial) fibrations;*

- *$A \otimes (-)$ preserves (trivial) cofibrations;*
- *$X^{(-)}$ preserves (trivial) fibration.*

Example A.6. The model category on $\mathrm{s}\mathcal{S}\mathrm{et}$ can be enriched over $\mathrm{s}\mathcal{S}\mathrm{et}$, making it into a simplicial model category as follows. For simplicial set X, Y we define $X \otimes Y := X \times Y$. Also we let $\mathrm{Map}_{\mathrm{s}\mathcal{S}\mathrm{et}}(X, Y)$ and Y^X both be the simplicial set which is $\mathrm{s}\mathcal{S}\mathrm{et}(X \times \Delta[n], Y)$ in degree n, with obvious structure maps.

Definition A.7. Let $\mathcal{M}$ be a model category. Then a *cofibrant replacement functor* on $\mathcal{M}$ is an endofunctor Q on $\mathcal{M}$ together with a natural transformation $q : Q \Rightarrow \mathrm{id}_{\mathcal{M}}$ such that QM is cofibrant for each M in $\mathcal{M}$, and such that q is pointwise a trivial fibration. Likewise, a *fibrant replacement functor* on $\mathcal{M}$ is an endofunctor R on $\mathcal{M}$ with RN fibrant for all N in $\mathcal{M}$, plus an $r : \mathrm{id}_{\mathcal{M}} \Rightarrow R$ which is a trivial cofibration pointwise.

Proposition A.8. *Let $\mathcal{M}$ be a model category. Then there is a (co)fibrant replacement functor on $\mathcal{M}$.*

For example, when M is an object of $\mathcal{M}$, then $0 \to M$ can be factorized as $0 \to M' \to M$ with $0 \to M'$ a cofibration and $M' \to M$ a trivial fibration. Hence in constructing Q one can begin by taking M' for QM.

Definition A.9. Let $\mathcal{M}, \mathcal{N}$ be model categories and $F : \mathcal{M} \leftrightarrows \mathcal{N} : G$ an adjunction. Then this is called an *Quillen adjunction* when F preserves (trivial) cofibrations.

The adjunction $F \dashv G$ in the above definition is a Quillen adjunction iff G preserves (trivial) fibrations. If we have such a Quillen adjunction, then the total derived functor theorem tells us this situation induces an adjunction $\mathbf{L}F : \mathrm{Ho}\,\mathcal{M} \leftrightarrows \mathrm{Ho}\,\mathcal{N} : \mathbf{R}G$ between the homotopy categories. It is a result that for these $\bar{F}, \bar{G}$ it holds that one is an equivalence iff the other one is. If they indeed are equivalences, then the adjunction $F \dashv G$ is called an *Quillen equivalence*. The following lemma illustrates the importance of Quillen equivalences:

Lemma A.10. *A Quillen adjunction $F \dashv G$ as above is a Quillen equivalence iff the following holds: a morphism $M \to GN$ is a weak equivalence in $\mathcal{M}$ iff its transpose $FM \to N$ is a weak equivalence in $\mathcal{N}$.*

Example A.11. We have an Quillen equivalence $|-| : \mathrm{s}\mathcal{S}\mathrm{et} \leftrightarrows \mathcal{T}\mathrm{op} : \mathrm{Sing}$ between (nice) topological spaces (explained in the next appendix) and simplicial sets. Here, $\mathrm{Sing}\, X$ is the singular complex of a space X, which has the set $\mathcal{T}\mathrm{op}(\Delta^n, X)$ in degree n and obvious structure maps.

Now let $\mathbb{A}, \mathbb{B}$ be ordinary categories and $\alpha : \mathbb{A} \to \mathbb{B}$ a functor between them. Then α is called *final* if for each b in $\mathbb{B}$ the comma category b/α is non-empty and connected. Dually, α is called *initial* if for all $b \in \mathbb{B}$ the category α/b is non-empty and connected.

Let also $\mathcal{C}$ be a category. If α is final then for all diagrams $X_\bullet : \mathbb{B} \to \mathcal{C}$ it holds that the natural map $\mathrm{colim}_{\mathbb{A}}(X_\bullet \circ \alpha) \to \mathrm{colim}_{\mathbb{B}}(X_\bullet)$ is an isomorphism. Dually, if α is initial then for all $X_\bullet : \mathbb{B} \to \mathcal{C}$ the natural map $\lim_{\mathbb{B}}(X_\bullet) \to \lim_{\mathbb{A}}(X_\bullet \circ \alpha)$ is an isomorphism.

We also have a homotopical analogue to the statement above.

Definition A.12. Call α *homotopy final* resp. *homotopy initial* if for all $b \in \mathbb{B}$ the category b/α resp. α/b is contractible, i.e. if the nerve of these categories is weakly equivalent to a point.

Proposition A.13. *Let $\mathcal{M}$ be simplicial model category, $\alpha : \mathbb{A} \to \mathbb{B}$ a functor and $X_\bullet$ a diagram in $\mathcal{M}$ of shape $\mathbb{B}$. If α is homotopy final resp. homotopy initial then the natural map* $\operatorname{hocolim}_{\mathbb{A}}(X_\bullet \circ \alpha) \to \operatorname{hocolim}_{\mathbb{B}}(X_\bullet)$ *resp. the natural map* $\operatorname{holim}_{\mathbb{B}}(X_\bullet) \to \operatorname{holim}_{\mathbb{A}}(X_\bullet \circ \alpha)$ *is a weak equivalence.*

Let $\mathcal{M}$ be a model category and $\mathbb{A}$ an indexing category. Then the *projective model structure* on $\mathcal{M}^{\mathbb{A}}$ has pointwise weak equivalences resp. pointwise fibrations as weak equivalences resp. as fibrations. This model structure on $\mathcal{M}^{\mathbb{A}}$ need not exist: it does exist when $\mathcal{M}$ is cofibrantly generated (see e.g. [GJ09, Def. II.6.6] for what this means).

Now suppose $\mathcal{M}^{\mathbb{A}}$ has the projective model structure. Write σ for the functor $\mathcal{M} \to \mathcal{M}^{\mathbb{A}}$ that sends M to the constant diagram $a \mapsto M$. Then we have an adjunction $\operatorname{colim} \dashv \sigma$. But note, by construction of the model structure on $\mathcal{M}^{\mathbb{A}}$ it holds that σ preserves (trivial) fibrations. One can show this implies colim preserves (trivial) cofibrations and weak equivalences between cofibrant objects, as done in [GJ09, Lem. II.7.9] for example. Now Quillen's total derived functor theorem implies that we have an induced adjunction $\mathbf{L}\operatorname{colim} : \operatorname{Ho}(\mathcal{M}^{\mathbb{A}}) \leftrightarrows \operatorname{Ho}\mathcal{M} : \mathbf{R}\sigma$.

In the situation above, the functor $\mathbf{L}\operatorname{colim}$ is sometimes called the homotopy colimit. It can be computed by first taking a cofibrant replacement functor Q on $\mathcal{M}^{\mathbb{A}}$ and by then putting $\mathbb{L}\operatorname{colim} := \operatorname{colim} Q$. Now $\mathbf{L}\operatorname{colim}$ is isomorphic to the descent of $\mathbb{L}\operatorname{colim}$.

We can dualize the story above in the following way. Let $\mathbb{B}$ be another indexing category. Then the *injective model structure* on $\mathcal{M}^{\mathbb{B}}$ has pointwise weak equivalences and pointwise cofibrations. If $\mathcal{M}^{\mathbb{B}}$ has such a structure, then one can compute homotopy limits of shape $\mathbb{B}$ by employing a fibrant replacement on $\mathcal{M}^{\mathbb{B}}$. But note the assumption that $\mathcal{M}$ is cofibrantly generated is not sufficient to guarantee the injective model structure on $\mathcal{M}^{\mathbb{B}}$ exists. Assuming $\mathcal{M}$ is combinatorial however does suffice for this to work.

From the previous remark we see why one may prefer combinatorial model categories in a setting where one wishes to talk about homotopy limits. We also see why the theory of deformations on homotopical categories as discussed in paragraph 1.1.a is so nice: if we have the appropriate deformations then we can compute homotopy (co)limits. And in fact this is the whole crucial step which allowed us to formulate the theory of Segal objects in the more general setting of sufficiently nice homotopical categories.

B Spaces

Let $\mathcal{T}\mathrm{op}'$ be the category of all topological spaces. We endow it with the classical model structure, wherein

- Weak equivalences are weak homotopy equivalences;
- Fibrations are Serre fibrations, i.e. those maps with the right lifting property with respect to all maps of the form $(\mathrm{id}, 0) : I^n \to I^{n+1}$;

- Cofibrations are retracts of relative cell complexes.

See [Hir15] for a complete description and proof of this model structure.

It is an unfortunate fact that $\mathcal{T}op'$ with its classical model structure is not a simplicial model structure, when we take products as tensors and the standard compact-open topology on mapping spaces. This is because in general $(-) \times X$ is not a left adjoint to $(-)^X$. Luckily, there are remedies available. Informally, the category of nice topological spaces $\mathcal{T}op$ should have the following properties

- The classical model structure can be enriched to a simplicial model structure in a natural way;
- The underlying sets of limits and colimits in $\mathcal{T}op$ should be the same as those of the corresponding limits and colimits in $\mathcal{T}op'$;
- The result should be such that we have an Quillen adjunction between $\mathcal{T}op$ and $s\mathcal{S}et$.

These properties allow us to think of such a category $\mathcal{T}op$ as if it were just $\mathcal{T}op'$, with the added feature that it is a simplicial model category, so long as we take care in proceeding categorically. Below we give the standard resolution to this problem for definedness, although the precise construction is less important than the above informal description. In doing so, I mainly follow [May99, Ch. 5], [GJ09, Exm. II.3.14] and [Str09].

Definition B.1. Let X be a topological space. Then X is:

1. *Weak Hausdorff* if every continuous map $f : K \to X$ with K a compact Hausdorff space is closed;
2. A *k-space* or *compactly closed* if $U \subset X$ is closed iff $t^{-1}U$ is closed for any continuous map $t : C \to X$ with C a compact Hausdorff space;
3. *Compactly generated* if it is a weak Hausdorff k-space.

Let $\mathcal{T}op$ be the category of compactly generated topological spaces. We endow it with the same model category structure as was given on $\mathcal{T}op'$. Note all objects in $\mathcal{T}op$ are fibrant. We call objects of $\mathcal{T}op$ simply *(topological) spaces*.

The category $\mathcal{T}op$ comes with a functor $k : \mathcal{T}op' \to \mathcal{T}op$, called the *$k$-ification*. It associates to a space X the space kX which has the same underlying set as X, but with a topology given by declaring $F \subset kX$ to be closed iff $F \subset X$ is compactly closed in the original topology of X. It is immediate that k is just the identity on $\mathcal{T}op$ itself. Now (co)limits of diagrams in $\mathcal{T}op$ can be calculated by first computing the corresponding (co)limit in $\mathcal{T}op'$, and by then applying the functor k.

We enrich $\mathcal{T}op$ over $s\mathcal{S}et$ as follows. Let spaces X, Y be given. Then $\mathrm{Map}_{\mathcal{T}op}(X, Y)$ is the simplicial set which has $\mathcal{T}op(X \times \Delta^n, Y)$ at degree n and with obvious structure maps. Further, the space X^D for D a simplicial set is obtained by first endowing the set of continuous maps $|D| \to X$ with the compact-open topology, and by then applying the k-functor. The tensor product $X \otimes D$ is just the product $X \times |D|$ in $\mathcal{T}op$.

C Homological algebra

We collect some standard definitions. Write $\mathcal{A}\mathrm{b}$ for the category of abelian groups.

Definition C.1. Let $\mathcal{A}$ be a category. Then $\mathcal{A}$ is called:

- *Additive* if it is enriched over $\mathcal{A}\mathrm{b}$ and has all finite biproducts;
- *Abelian* if it is additive, if every map has a kernel and a cokernel and if furthermore every mono is a kernel and every epi is a cokernel.

We also have a generalization of abelian categories due to [Qui73].

Definition C.2. An *exact category* is an additive category $\mathcal{P}$ with a class $\mathfrak{E}$ of sequences of $\mathcal{P}$-arrows

$$0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0 \tag{3.14}$$

called the *short exact sequences*. In such a sequence, i is called admissibly monic and j admissibly epic. This structure must satisfy the following axioms:

(Closed) Any sequence in $\mathcal{P}$ isomorphic to one in $\mathfrak{E}$ is already in $\mathfrak{E}$. For any M', M'' in $\mathcal{E}$ the canonical sequence

$$0 \to M' \xrightarrow{(\mathrm{id},0)} M' \oplus M'' \xrightarrow{\pi_{M''}} M'' \to 0$$

is exact. For any sequence as in (3.14), i is a kernel of j while j is a cokernel for i.

(Composition) The classes of admissible monos and admissible epis are closed under composition.

(Base change) Any diagram $N \to M'' \leftarrow M$ with $M'' \leftarrow M$ admissibly epic has a pullback, such that $M \times_{M''} N \to N$ is admissibly epic.

(Cobase change) Dually, any diagram $N \leftarrow M' \to M$ with $M' \to M$ admissibly monic has a pushout, such that $N \to N \oplus_{M'} M$ is admissibly monic.

(Stability) Let $M \to M''$ be a map that has a kernel. Then if $N \to M \to M''$ is admissibly epic for any $N \to M$, then already $M \to M''$ is admissibly epic. Dually, if $M' \to M$ has a cokernel then $M' \to M \to N$ being admissible monic implies $M' \to M$ is.

The last axiom, also called the 'obscure axiom', turned out to be in fact redundant, see [Bue08, Rem. 2.17].

We furthermore have a non-additive generalization of exact categories (and of pointed model categories with only cofibrant objects), due to [Wal85].

Definition C.3. Let $\mathcal{C}$ be a pointed category, say with point 0. Then a subcategory $\mathrm{co}\mathcal{C}$, whose arrows we call *cofibrations*, makes $\mathcal{C}$ into a *category with cofibrations* if

- The isomorphisms in $\mathcal{C}$ are in $\mathrm{co}\mathcal{C}$;
- For every A the arrow $0 \to A$ is in $\mathrm{co}\mathcal{C}$;
- Cofibrations admit cobase change, i.e. $\mathrm{co}\mathcal{C}$ is closed under pushouts along arbitrary maps.

If we have such a subcategory $\mathrm{co}\mathcal{C}$, then a subcategory $\mathrm{w}\mathcal{C}$ in $\mathcal{C}$, whose arrows we call *weak equivalences*, is a *category of weak equivalences in* $(\mathcal{C}, \mathrm{co}\mathcal{C})$ if in addition we have:

- All isomorphisms in $\mathcal{C}$ are in $\mathrm{w}\mathcal{C}$;
- If in the below diagram the maps i, j are cofibrations and the vertical arrows are weak equivalences, then the induced morphism $B \amalg_A C \to B' \amalg_{A'} C'$ is a cofibration.

$$\begin{tikzcd} B \arrow[d] & A \arrow[l, "i"'] \arrow[r] \arrow[d] & C \arrow[d] \\ B' & A' \arrow[l, "j"] \arrow[r] & C' \end{tikzcd}$$

A category $\mathcal{C}$ with cofibrations $\mathrm{co}\mathcal{C}$ and weak equivalences $\mathrm{w}\mathcal{C}$ is called a *Waldhausen category*. We often suppress $\mathrm{co}\mathcal{C}$ and $\mathrm{w}\mathcal{C}$.

Observe the definition of a category with cofibrations is very broad. Indeed, Waldhausen calls it a 'perhaps even embarrassing' example that any pointed category $\mathcal{C}$ with finite colimits becomes a category with cofibrations by letting $\mathrm{co}\mathcal{C}$ be all of $\mathcal{C}$. Also note, for any given category with cofibrations $\mathcal{D}$, we always have a maximal and minimal option for endowing $\mathcal{D}$ with weak equivalences, namely by letting $\mathrm{w}\mathcal{D}$ be all of $\mathcal{D}$, or only the isomorphisms in $\mathcal{D}$ respectively.

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List of Symbols