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Analysis of the Whittle index in a single server model with service control and customer impatience

Master thesis

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Contents

1 Introduction

In high-dimensional queueing systems, optimal strategies may have a complicated structure. Due to high-dimensionality, it is numerically intractable to compute such an optimal strategy. Instead, it may be beneficial to use a good approximating strategy that is easily computed.

An example of such a controlled queueing system, is the K-competing queueing model, with service control. In the case of a linear cost structure, it is well-known that the optimal strategy has a simple structure: always serve the queue yielding the largest cost reduction per unit time (cf. [4]).

In case the cost per unit time is not a linear function of the state, or if customers may abandon the system before having finished service, the optimal service control strategy does not have this simple form in general.

In the case of linear cost and abandonment, there may still be a simple optimal strategy under extra conditions on the parameters (cf. [5]). However, if the cost is not linear, such a simple strategy may not exist. The paper [1] suggests to use the so-called Whittle index strategy as a relatively simply computed approximation. The Whittle index strategy assigns to every queue a function of the number of customers in that queue. The queue with the highest function value will be served, unless this index is negative.

In this thesis we will study the Whittle index and some of its properties as discussed in [1]. We will clarify and elaborate on some of the proofs from [1], and provide alternative proofs in other cases.

The structure of this thesis is as follows: first we will introduce the model we study. We will analyse the constraints on the cost functions and the applications to specific forms of these.

Since the Whittle index is based on a single customer type, we will give a recap on the single queue model and we will describe the concept of a threshold strategy. We analyse the stationary distribution of the Markov chain associated with a given threshold strategy and give a closed form formula for this distribution.

After some definitions that we need to give a meaningful definition of the Whittle index, we introduce this index and we see that it is possible to index all queues in the model. We will give another proof as the proof in [1]. We also discuss different possibilities to compute the Whittle index. This is not immediately clear. In [1] a simple formula was conjectured to hold. Unfortunately, we have not succeeded in proving its validity. In the end we will give some examples of the Whittle index strategy and we finish with some conclusions and recommendations for further research.

2 Model

In the section we describe the K-competing queues model from [1] in more detail. This is a queueing model with a single server. The server has capacity 1, and so at most one customer can be served at any moment in time. There are K customer types, that compete for service. Arrivals of type k customers occur according to a Poisson process with rate λ_k, $k \in \{1, \cdots, K\}$. Each type k customer may leave impatiently after an $\exp(\theta_k)$ distributed amount of time. When a type k customer is being served, he may leave impatiently after an $\exp(\theta'_k)$ distributed amount of time, or he finishes service after an $\exp(\mu'_k)$ distributed amount of time. We define $\mu_k = \theta'_k + \mu'_k$ as the departure rate for a customer of type k in service. We assume that $\mu'_k + \theta'_k \geq \theta_k$, which means that on average a customer in service leaves faster than a queued customer.
Let x_k be the number of type k customers in the system. We assume that there is a cost for having customers in the system. For each customer type we define two cost functions: $C_k(x_k, 0)$ is the cost per unit time when there are x_k customers in the system and class k is not being served; $C_k(x_k, 1)$ is the cost per unit time when there are x_k customers in the system and class k is being served. In [1] it is assumed that both functions are convex and non-decreasing, and satisfy

$$C_k(x_k, 0) - C_k(\max\{0, x_k - 1\}, 0) \leq C_k(x_k + 1, 1) - C_k(x_k, 1) \leq C_k(x_k + 1, 0) - C_k(x_k, 0). \quad (1)$$

Further, we assume that there is a lump cost d_k for an impatient departure of a queued type k customer and a lump cost d'_k for an impatient departure of a type k customer in service.
It is convenient to model the combined costs as a cost per unit time. Thus, we have to remodel the lump cost d_k, d'_k per type k customer as a cost per unit time. This can be done as follows. Remark that a type k customer in the queue abandons the queue after an $\exp(\theta_k)$ distributed amount of time. Thus on average he abandons the queue after $1/\theta_k$ time units. So, when there are x_k customers in queue k, the lump cost due to impatient departures from this queue can be modelled as an average cost of $x_k \theta_k d_k$ per unit time. When the server is serving queue k, an expression for the average cost per unit time for departing customers in service follows with the same reasoning. Combining, yields the following cost per unit time for type k customers,

$$\tilde{C}_k(x_k, 0) = C_k(x_k, 0) + d_k \theta_k x_k,$$

and

$$\tilde{C}_k(x_k, 1) = C_k(x_k, 1) + d_k \theta_k \max(0, x_k - 1) + d'_k \theta'_k \min(1, x_k).$$

We wish to determine a service assignment strategy for this model that minimises the average expected total cost per unit time. The form of the optimal strategy is expected to be very complex. The idea is to use a heuristic instead. For this heuristic, we need the so called Whittle index. This index will be determined per queue and depends only on the number of customers per type. This motivates to study the single server model with one customer type. We will refer to this model as the single queue model. We will do so in Section 4, but first we will elaborate on the cost restrictions in Equation (1).

3 Analyses of the restrictions on the cost

It is not immediately clear, that the condition in Equation (1) is very restrictive. We will analyse several cost functions, and we will see that the leading terms in $C(x,0)$ and $C(x,1)$ have to be equal. Remark, that since we look at the difference of two function values, any constants, if present, will disappear. It is therefore clear, that these can take any value, independently of the other function. So, we can add any constant we like, without destroying the condition in Equation (1). Therefore, in the proofs we ignore these constants.

3.1 The linear case

Lemma 1. *Let $C(x,0) = ax + b$, $C(x,1) = a'x + b$, with $a, a' > 0$. If Equation (1) holds, then $a = a'$.*

Proof We may assume, that $C(x,0) = ax$ and $C(x,1) = a'x$. Our functions have to satisfy Equation (1). Thus, we need $a, a' \in \mathbb{R}$ to satisfy

$$ax - a(x-1) \leq a'(x+1) - a'x \leq a(x+1) - ax.$$

Remark, that this implies that

$$a \leq a' \leq a,$$

and so it follows immediately that $a = a'$. ☐

3.2 The quadratic case

Lemma 2. *Let $C(x,0) = ax^2 + bx + c$ and $C(x,1) = a'x^2 + b'x + c'$ be non-decreasing functions for $x \geq 0$. If Equation (1) holds, then $a = a'$ and $b - 2a \leq b' \leq b$.*

Proof In this case, we may assume, that $C(x,0) = ax^2 + bx$ and $C(x,1) = a'x^2 + b'x$. These functions have to be convex and non-decreasing, so $a, a' > 0$. Equation (1) has to hold, so a, a', b, b' have to satisfy the following inequalities

$$ax2+bx-(a(x-1)^2+b(x-1)) \leq a'(x+1)^2+b'(x+1)-(a'x^2+b'x) \leq a(x+1)^2+b(x+1)-(ax^2+bx).$$

After rewriting, we get

$$b + 2ax - a \leq b' + 2a'x + a' \leq b + 2ax + a. \tag{2}$$

The first inequality of Equation (2) implies that

$$2ax - 2a'x \leq a' + a + b' - b, \;\; x = 0, 1, ...,$$

and the second that

$$2a'x - 2ax \leq a - a' + b - b', \;\; x = 0, 1, ...$$

Both inequalities have to be satisfied for all $x \in \mathbb{N}$. This therefore implies, that $a = a'$. When $a = a'$, Equation (2) becomes

$$b - a \leq b' + a \leq b + a.$$

This gives us the following condition for b'

$$b - 2a \leq b' \leq b.$$

☐

3.3 The exponential case

Lemma 3. *Let $C(x,0) = a\exp(bx)+c$ and $C(x,1) = a'\exp(b'x)+c'$ be non-decreasing functions. If Equation (1) holds, then $b = b'$ and $a\exp(-b) \leq a' \leq a$.*

Proof We may assume, that $C(x,0) = a\exp(bx)$ and $C(x,1) = a'\exp(b'x)$. These functions have to be convex and non-decreasing, so that $a, a', b, b' > 0$.
Our functions have to satisfy Equation (1) as well. The first inequality of this equation leads to

$$\begin{aligned}
&a(\exp(bx) - \exp(b(x-1)) \leq a'(\exp(b'(x+1)) - \exp(b'x)) \\
&\Rightarrow \frac{a(\exp(bx) - \exp(b(x-1))}{a'(\exp(b'(x+1)) - \exp(b'x))} \leq 1 \\
&\Rightarrow \frac{\exp(bx) - \exp(b(x-1)}{\exp(b'(x+1)) - \exp(b'x)} \leq \frac{a'}{a} \\
&\Rightarrow \frac{\exp(b'(x+1)) - \exp(b'x)}{\exp(bx) - \exp(b(x-1))} \geq \frac{a}{a'}, \quad x \in \{1,2,...\}.
\end{aligned}$$

The second inequality of Equation (1) leads to

$$\begin{aligned}
&a'(\exp(b'(x+1)) - \exp(b'x)) \leq a(\exp(b(x+1)) - \exp(bx)) \\
&\Rightarrow \frac{a'(\exp(b'(x+1)) - \exp(b'x))}{a(\exp(b(x+1)) - \exp(bx))} \leq 1 \\
&\Rightarrow \frac{\exp(b'(x+1)) - \exp(b'x)}{\exp(b(x+1)) - \exp(bx)} \leq \frac{a}{a'}, \quad x \in \{1,2,...\}.
\end{aligned}$$

So we get

$$\frac{\exp(b'(x+1)) - \exp(b'x)}{\exp(b(x+1)) - \exp(bx)} \leq \frac{a}{a'} \leq \frac{\exp(b'(x+1)) - \exp(b'x)}{\exp(bx) - \exp(b(x-1))}, \quad x \in \{1,2,...\}. \tag{3}$$

Note that $0 < \frac{a}{a'} < \infty$. With this fact, we can deduce, that $b = b'$. This is easy to show. Indeed, if $b < b'$ we see that

$$\lim_{n\to\infty} \frac{\exp(b'(x+1)) - \exp(b'x)}{\exp(b(x+1)) - \exp(bx)} = \infty,$$

which is impossible, because the first inequality will not hold in this case.
Similarly, if $b > b'$, then

$$\lim_{x\to\infty} \frac{\exp(b'(x+1)) - \exp(b'x)}{\exp(bx) - \exp(b(x-1))} = 0.$$

This cannot be the case, because then the second inequality cannot hold. So, $b' = b$. Now it is easy to see, that we can simplify Equation (3) to

$$1 \leq \frac{a}{a'} \leq \frac{\exp(b) - 1}{1 - \exp(-b)},$$

and we can rewrite this as the following condition on a and a'

$$\frac{a(1 - \exp(-b))}{\exp(b) - 1} \leq a' \leq a.$$

We can simplify the first term, so we get

$$a\exp(-b) \leq a' \leq a.$$

□

4 Single queue model

As pointed out in Section 2, we will study the single queue model. We consider one customer type. From now on we ignore the subscript k. This model has state space $S = \{0, 1, 2, ...\}$, where the state denotes the number of customers in the system.
Customers arrive according a Poisson process with rate λ. The server can be turned on or off. When the server is turned on, he serves one customer at a time, if there is any. Customers may leave impatiently after an $\exp(\theta)$ distributed amount of time, when they are in the queue, and after an $\exp(\theta')$ distributed amount of time, when they are in service. Customers finish service after an $\exp(\mu')$ distributed amount of time. Remark, that we assume that $\theta \leq \theta' + \mu' = \mu$, just as in Section 2.
The objective is how to optimally turn the server on or off, so as to minimize the total expected cost per unit time. We model this as a Markov decision process with state space $S = \{0, 1, 2, ...\}$, with action space $A(x) = \{0, 1\}$, where 0 stands for the server being turned off and 1 for the server being turned on. The corresponding transition rates are

$$q_{xy}(a) = \begin{cases} \lambda, & y = x + 1 \\ x\theta \mathbb{1}_{\{a=0\}} + ((x-1)\theta + \mu)\mathbb{1}_{\{a=1\}}, & y = x - 1, x \geq 1 \\ -\lambda - x\theta \mathbb{1}_{\{a=0\}} - ((x-1)\theta + \mu)\mathbb{1}_{\{a=1\}}, & y = x, x \geq 1 \\ -\lambda & y = 0, x = 0. \end{cases} \tag{4}$$

A strategy prescribes in which states to turn the server on or off. Thus, a strategy is a function mapping each state to an action. One special strategy is the so called threshold strategy: when the number of customers in the system is higher than the threshold, the server is on and when the number of customers is lower than or equal to the threshold, the server is off. So a threshold strategy is fully characterized by a threshold $\phi \in S$, such that the server is turned off in state $x \leq \phi$ and on in state $x > \phi$. $\phi = -1$ stands for the strategy where the server is never switched off. Figure 1 shows us the Markov processes associated with threshold strategies ϕ and $\phi + 1$, with the transition rates given by Equation (4).

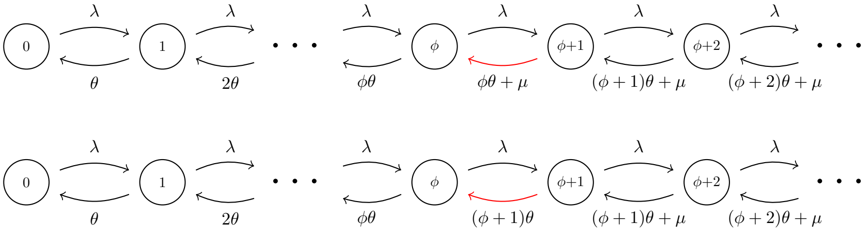


Figure 1: Markov processes that correspond to threshold strategies ϕ and $\phi + 1$ respectively.

Note that only one rate in this Markov process is different for these two strategies.
By virtue of [1, Proposition 1] there is always a threshold optimal strategy. Thus the average expected cost g^ϕ per unit time under threshold ϕ is given by

$$g^\phi = \sum_{x \leq \phi} \pi^\phi(x)\tilde{C}(x, 0) + \sum_{x > \phi} \pi^\phi(x)\tilde{C}(x, 1),$$

with π^ϕ the corresponding stationary distribution of the Markov process associated with threshold strategy ϕ. The Markov process associated with ϕ is a birth death process, so we can explicitly determine the stationary distribution.

5 Stationary distribution under a threshold strategy

We will determine the stationary distribution π^{ϕ} of the Markov process operating under strategy ϕ. To do so, we will rewrite the transition rates as in Equation (4). Let

$$\theta_x^{\phi} = \begin{cases} x\theta, & x \leq \phi \\ (x-1)\theta + \mu, & x > \phi \end{cases} \tag{5}$$

be the transition rate from state x to state $x-1$ in the case that the threshold is ϕ. Let $P_0^{\phi} = 1$ and $P_x^{\phi} = \prod_{j=1}^{x} \frac{\lambda}{\theta_j^{\phi}}$ for all ϕ. When we use Equation (5), we get

$$P_x^{\phi} = \begin{cases} \dfrac{\lambda^x}{x!\theta^x}, & x \leq \phi \\ \dfrac{\lambda^{\phi}}{\phi!\theta^{\phi}} \cdot \prod_{i=\phi+1}^{x} \dfrac{\lambda}{(i-1)\theta + \mu} & x > \phi. \end{cases} \tag{6}$$

Then,

$$\pi^{\phi}(0) = \left(\sum_{x=0}^{\infty} P_x^{\phi} \right)^{-1}, \tag{7}$$

and

$$\pi^{\phi}(x) = P_x^{\phi} \pi^{\phi}(0).$$

Using Equation (6), Equation (7) becomes

$$(\pi^{\phi}(0))^{-1} = 1 + \sum_{x=1}^{\phi} \frac{\lambda^x}{x!\theta^x} + \frac{\lambda^{\phi}}{\phi!\theta^{\phi}} \sum_{x=\phi+1}^{\infty} \prod_{j=\phi+1}^{x} \frac{\lambda}{(j-1)\theta + \mu}.$$

A closed form expression for $(\pi^{\phi}(0))^{-1}$ is provided by Lemma 4. This expression was obtained with Wolfram Alpha, but we prove the correctness below.

Lemma 4.

$$(\pi^{\phi}(0))^{-1} = \frac{\exp\left(\frac{\lambda}{\theta}\right) \left[\left(\frac{\lambda}{\theta}\right)^{1-\frac{\mu}{\theta}} \gamma\left(\frac{\mu}{\theta} + \phi, \frac{\lambda}{\theta}\right) + \Gamma\left(\phi + 1, \frac{\lambda}{\theta}\right) \right]}{\phi!}, \tag{8}$$

with $\Gamma(s, x)$ *as in Equation (19) and* $\gamma(s, x)$ *as in Equation (20).*

Proof We will verify the correcteness of Equation (8).

$$\begin{aligned}
&\frac{\exp\left(\frac{\lambda}{\theta}\right) \left[\left(\frac{\lambda}{\theta}\right)^{1-\frac{\mu}{\theta}} \gamma\left(\frac{\mu}{\theta} + \phi, \frac{\lambda}{\theta}\right) + \Gamma\left(\phi + 1, \frac{\lambda}{\theta}\right) \right]}{\phi!} \\
&= \frac{\exp\left(\frac{\lambda}{\theta}\right) \left[\left(\frac{\lambda}{\theta}\right)^{1-\frac{\mu}{\theta}} \left(\frac{\lambda}{\theta}\right)^{\phi+\frac{\mu}{\theta}} \exp\left(-\frac{\lambda}{\theta}\right) \sum_{k=0}^{\infty} \frac{\left(\frac{\lambda}{\theta}\right)^k}{\prod_{i=0}^{k}(\frac{\mu}{\theta}+\phi+i)} + \phi! \exp\left(-\frac{\lambda}{\theta}\right) \sum_{k=0}^{\phi} \frac{\left(\frac{\lambda}{\theta}\right)^k}{k!} \right]}{\phi!} \\
&= \frac{\exp\left(\frac{\lambda}{\theta}\right) \left(\frac{\lambda}{\theta}\right)^{\phi+1} \exp\left(-\frac{\lambda}{\theta}\right) \sum_{k=0}^{\infty} \frac{\left(\frac{\lambda}{\theta}\right)^k}{\prod_{i=0}^{k}(\frac{\mu}{\theta}+\phi+i)}}{\phi!} + \frac{\exp\left(\frac{\lambda}{\theta}\right) \phi! \exp\left(-\frac{\lambda}{\theta}\right) \sum_{k=0}^{\phi} \frac{\left(\frac{\lambda}{\theta}\right)^k}{k!}}{\phi!} \\
&= \frac{\left(\frac{\lambda}{\theta}\right)^{\phi}}{\phi!} \sum_{k=0}^{\infty} \frac{\left(\frac{\lambda}{\theta}\right)^{k+1}}{\prod_{i=0}^{k}(\frac{\mu}{\theta} + \phi + i)} + \sum_{k=0}^{\phi} \frac{\left(\frac{\lambda}{\theta}\right)^k}{k!}
\end{aligned}$$

$$
\begin{aligned}
&= 1 + \sum_{k=1}^{\phi} \frac{\lambda^k}{k!\theta^k} + \frac{\lambda^\phi}{\phi!\theta^\phi} \sum_{k=\phi+1}^{\infty} \left(\frac{1}{\theta}\right)^{k+1-\phi-1} \prod_{i=\phi+1}^{k} \frac{\lambda}{\frac{1}{\theta}(\mu + \theta(\phi + i - \phi - 1))} \\
&= 1 + \sum_{k=1}^{\phi} \frac{\lambda^k}{k!\theta^k} + \frac{\lambda^\phi}{\phi!\theta^\phi} \sum_{k=\phi+1}^{\infty} \prod_{i=\phi+1}^{k} \frac{\lambda}{((i-1)\theta + \mu)} \\
&= (\pi^\phi(0))^{-1}.
\end{aligned}
$$

□

Remark that the first equality follows by virtue of Equations (23) and (24). Thus, the closed expression from Wolfram Alpha is indeed correct. This provides closed form expressions for the other values of the stationary distribution.

6 Whittle's index

In [2] Whittle provides us with a heuristic strategy for the model described in Section 2. For every threshold ϕ we can compute the so-called Whittle index per state.
To come to this index, it is necessary that we have a reason to turn a server off. Therefore, [2] and [1] add a subsidy for passivity, denoted by W. In other words, it depends on W and the number of customers in the system, whether a server is on or off. States in which the server is off, and so customers are not served, are called passive. The set of all passive states is denoted by $D(W)$.
Paper [1] uses a value function approach to prove results for this model. The value function is a solution to the optimality equation. If the cost functions are non-negative, the minimal solution (up to a constant), is the solution we are looking for. The optimality equation is given by

$$\begin{aligned}
&(\mu' + \theta' + x\theta + \lambda)V^W(x) + g^W \\
&\quad = \lambda V^W(x+1) + \theta(x-1)V^W(\max\{0, x-1\}) + \min\{\tilde{C}(x,0) - W(x) + (\mu' + \theta')V^W(x) \\
&\qquad + \theta V^W(\max\{0, x-1\}), \tilde{C}(x,1) + (\mu' + \theta')V^W(\max\{0, x-1\}) + \theta V^W(x)\},
\end{aligned}$$

where g^W is the average cost incurred under an optimal strategy, with subsidy for passivity W, and the value function $V^W(x)$ is the minimal total expected reduced cost in state x. From [2], it follows that the Whittle index $\tilde{W}(x)$ is given by the value $W(x)$ such, that

$$\begin{aligned}
&\tilde{C}(x,0) - W(x) + (\mu' + \theta')V^W(x) + \theta V^W(\max\{0, x-1\}) \\
&\quad = \tilde{C}(x,1) + (\mu' + \theta')V^W(\max\{0, x-1\}) + \theta V^W(x),
\end{aligned}$$

that is, in state x the controller is indifferent to turning the server on or off. We can rewrite this, so that

$$\tilde{W}(x) = \tilde{C}(x,0) - \tilde{C}(x,1) + (\mu' + \theta' - \theta)(V^W(x) - V^W(\max\{0, x-1\})).$$

This is not very well-defined, because $\tilde{W}(x)$ is not a priori unique. To come to a well-defined index, we need the concept of indexability (cf. [1]).

Definition 1. *The single queue model is indexable if* $W' < W$ *implies that* $D(W') \subseteq D(W)$.

[1, Proposition 1] shows that the optimal solution of the given problem is a threshold strategy. Let $\phi(W)$ be the optimal threshold for subsidy of passivity W. Then we can rewrite Definition 1 as follows.

Definition 2. *The single queue model is indexable if* $W' < W$ *implies that* $\phi(W') \leq \phi(W)$.

When the single queue model is indexable, we define the Whittle index in state x as the smallest value W such, that it is optimal to be passive in state x.

Definition 3. *For an indexable single queue model, the Whittle index in state* x *is defined by* $W(x) := \inf\{W : x \leq \phi(W)\}$.

We now have enough to give a definition of the Whittle index strategy.

Definition 4. *In a* K*-competing queues model, the Whittle index strategy serves the non-empty class, with the highest non-negative Whittle index at the moment.*

6.1 Indexability of all classes

[1, Proposition 2] proves, that all classes are indexable. In this subsection we will provide another proof, using Lemma 5. We will see that this proof leads automatically to [1, Theorem 1].

Proposition 1. *All classes are indexable.*

The proof requires that $\phi \to \sum_{x=0}^{\phi} \pi^{\phi}(x)$ is increasing. The advantage of our proof is, that we can explicitly estimate the difference between $\sum_{x=0}^{\phi} \pi^{\phi}(x)$ and $\sum_{x=0}^{\phi+1} \pi^{\phi+1}(x)$. We will show this first.

6.1.1 Increasingness of $\sum_{x=0}^{\phi} \pi^{\phi}(x)$

Lemma 5. $\sum_{x=0}^{\phi} \pi^{\phi}(x) < \sum_{x=0}^{\phi+1} \pi^{\phi+1}(x)$.

Proof To prove this lemma, we will rewrite first the above inequality.

$$\begin{aligned}
&\sum_{x=0}^{\phi} \pi^{\phi}(x) < \sum_{x=0}^{\phi+1} \pi^{\phi+1}(x) \\
&\Leftrightarrow \frac{1}{\pi^{\phi+1}(0)} \sum_{x=0}^{\phi} \frac{\lambda^x}{x!\theta^x} < \frac{1}{\pi^{\phi}(0)} \sum_{x=0}^{\phi+1} \frac{\lambda^x}{x!\theta^x} \\
&\Leftrightarrow \frac{\exp\left(\frac{\lambda}{\theta}\right)\left[\left(\frac{\lambda}{\theta}\right)^{1-\frac{\mu}{\theta}} \gamma\left(\frac{\mu}{\theta}+\phi+1, \frac{\lambda}{\theta}\right) + \Gamma\left(\phi+2, \frac{\lambda}{\theta}\right)\right]}{(\phi+1)!} \sum_{x=0}^{\phi} \frac{\lambda^x}{x!\theta^x} \\
&\quad < \frac{\exp\left(\frac{\lambda}{\theta}\right)\left[\left(\frac{\lambda}{\theta}\right)^{1-\frac{\mu}{\theta}} \gamma\left(\frac{\mu}{\theta}+\phi, \frac{\lambda}{\theta}\right) + \Gamma\left(\phi+1, \frac{\lambda}{\theta}\right)\right]}{\phi!} \sum_{x=0}^{\phi+1} \frac{\lambda^x}{x!\theta^x} \\
&\Leftrightarrow \frac{\left[\left(\frac{\lambda}{\theta}\right)^{1-\frac{\mu}{\theta}} \gamma\left(\frac{\mu}{\theta}+\phi+1, \frac{\lambda}{\theta}\right) + \Gamma\left(\phi+2, \frac{\lambda}{\theta}\right)\right]}{\phi+1} \sum_{x=0}^{\phi} \frac{\lambda^x}{x!\theta^x} \\
&\quad < \left[\left(\frac{\lambda}{\theta}\right)^{1-\frac{\mu}{\theta}} \gamma\left(\frac{\mu}{\theta}+\phi, \frac{\lambda}{\theta}\right) + \Gamma\left(\phi+1, \frac{\lambda}{\theta}\right)\right] \sum_{x=0}^{\phi+1} \frac{\lambda^x}{x!\theta^x}. \qquad (9)
\end{aligned}$$

By virtue of Equations (21) and (22), it follows that

$$\begin{aligned}
&\left(\frac{\lambda}{\theta}\right)^{1-\frac{\mu}{\theta}} \gamma\left(\frac{\mu}{\theta}+\phi+1, \frac{\lambda}{\theta}\right) + \Gamma\left(\phi+2, \frac{\lambda}{\theta}\right) \\
&= \left(\frac{\lambda}{\theta}\right)^{1-\frac{\mu}{\theta}} \left[\left(\frac{\mu}{\theta}+\phi\right) \gamma\left(\frac{\mu}{\theta}+\phi, \frac{\lambda}{\theta}\right) - \left(\frac{\lambda}{\theta}\right)^{\frac{\mu}{\theta}+\phi} \exp\left(-\frac{\lambda}{\theta}\right)\right] + (\phi+1)\Gamma\left(\phi+1, \frac{\lambda}{\theta}\right) \\
&\quad + \left(\frac{\lambda}{\theta}\right)^{\phi+1} \exp\left(-\frac{\lambda}{\theta}\right) \\
&= \left(\frac{\lambda}{\theta}\right)^{1-\frac{\mu}{\theta}} \left(\frac{\mu}{\theta}+\phi\right) \gamma\left(\frac{\mu}{\theta}+\phi, \frac{\lambda}{\theta}\right) + (\phi+1)\Gamma\left(\phi+1, \frac{\lambda}{\theta}\right) - \left(\frac{\lambda}{\theta}\right)^{\phi+1} \exp\left(-\frac{\lambda}{\theta}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \left(\frac{\lambda}{\theta}\right)^{\phi+1} \exp\left(-\frac{\lambda}{\theta}\right)\\
&= \left(\frac{\lambda}{\theta}\right)^{1-\frac{\mu}{\theta}} \left(\frac{\mu}{\theta}+\phi\right)\gamma\left(\frac{\mu}{\theta}+\phi,\frac{\lambda}{\theta}\right) + (\phi+1)\Gamma\left(\phi+1,\frac{\lambda}{\theta}\right).
\end{aligned}
$$

Using this, we obtain that

$$
\begin{aligned}
\text{Inequality (9)} \Leftrightarrow &\frac{\left[\left(\frac{\lambda}{\theta}\right)^{1-\frac{\mu}{\theta}} \left(\frac{\mu}{\theta}+\phi\right)\gamma\left(\frac{\mu}{\theta}+\phi,\frac{\lambda}{\theta}\right) + (\phi+1)\Gamma\left(\phi+1,\frac{\lambda}{\theta}\right)\right]}{\phi+1}\sum_{x=0}^{\phi}\frac{\lambda^x}{x!\theta^x}\\
&< \left[\left(\frac{\lambda}{\theta}\right)^{1-\frac{\mu}{\theta}}\gamma\left(\frac{\mu}{\theta}+\phi,\frac{\lambda}{\theta}\right)+\Gamma\left(\phi+1,\frac{\lambda}{\theta}\right)\right]\sum_{x=0}^{\phi+1}\frac{\lambda^x}{x!\theta^x}\\
\Leftrightarrow &\frac{\left(\frac{\lambda}{\theta}\right)^{1-\frac{\mu}{\theta}} \left(\frac{\mu}{\theta}+\phi\right)\gamma\left(\frac{\mu}{\theta}+\phi,\frac{\lambda}{\theta}\right)}{\phi+1}\sum_{x=0}^{\phi}\frac{\lambda^x}{x!\theta^x}\\
&< \left(\frac{\lambda}{\theta}\right)^{1-\frac{\mu}{\theta}}\gamma\left(\frac{\mu}{\theta}+\phi,\frac{\lambda}{\theta}\right)\sum_{x=0}^{\phi+1}\frac{\lambda^x}{x!\theta^x}+\frac{\lambda^{\phi+1}}{(\phi+1)!\theta^{\phi+1}}\Gamma\left(\phi+1,\frac{\lambda}{\theta}\right). \qquad (10)
\end{aligned}
$$

Next we use Equation (23) to obtain, that

$$
\begin{aligned}
&\left(\frac{\lambda}{\theta}\right)^{1-\frac{\mu}{\theta}} \left(\frac{\mu}{\theta}+\phi\right)\gamma\left(\frac{\mu}{\theta}+\phi,\frac{\lambda}{\theta}\right)\\
&= \left(\frac{\mu}{\theta}+\phi\right)\left(\frac{\lambda}{\theta}\right)^{\phi+1}\exp\left(-\frac{\lambda}{\theta}\right)\sum_{x=0}^{\infty}\frac{\lambda^x}{\theta^x\prod_{i=0}^{x}(\frac{\mu}{\theta}+\phi+i)}\\
&= \left(\frac{\lambda}{\theta}\right)^{\phi+1}\exp\left(-\frac{\lambda}{\theta}\right)\left(\frac{\mu}{\theta}+\phi\right)\left(\frac{1}{\frac{\mu}{\theta}+\phi}+\frac{\lambda}{\theta(\frac{\mu}{\theta}+\phi)(\frac{\mu}{\theta}+\phi+1)}\right.\\
&\quad\left.+\frac{\lambda^2}{\theta^2(\frac{\mu}{\theta}+\phi)(\frac{\mu}{\theta}+\phi+1)(\frac{\mu}{\theta}+\phi+2)}+...\right)\\
&= \left(\frac{\lambda}{\theta}\right)^{\phi+1}\exp\left(-\frac{\lambda}{\theta}\right)\left(1+\frac{\lambda}{\theta(\frac{\mu}{\theta}+\phi+1)}+\frac{\lambda^2}{\theta^2(\frac{\mu}{\theta}+\phi+1)(\frac{\mu}{\theta}+\phi+2)}+...\right)\\
&< \left(\frac{\lambda}{\theta}\right)^{\phi+1}\exp\left(-\frac{\lambda}{\theta}\right)\left(1+\frac{\lambda}{\theta(\frac{\mu}{\theta}+\phi)}+\frac{\lambda^2}{\theta^2(\frac{\mu}{\theta}+\phi)(\frac{\mu}{\theta}+\phi+1)}+...\right)\\
&= \left(\frac{\lambda}{\theta}\right)^{\phi+1}\exp\left(-\frac{\lambda}{\theta}\right)+\left(\frac{\lambda}{\theta}\right)^{\phi+1}\exp\left(-\frac{\lambda}{\theta}\right)\left(\frac{\lambda}{\theta(\frac{\mu}{\theta}+\phi)}+\frac{\lambda^2}{\theta^2(\frac{\mu}{\theta}+\phi)(\frac{\mu}{\theta}+\phi+1)}+...\right)\\
&= \left(\frac{\lambda}{\theta}\right)^{\phi+1}\exp\left(-\frac{\lambda}{\theta}\right)+\left(\frac{\lambda}{\theta}\right)^{\phi+2}\exp\left(-\frac{\lambda}{\theta}\right)\left(\frac{1}{(\frac{\mu}{\theta}+\phi)}+\frac{\lambda}{\theta(\frac{\mu}{\theta}+\phi)(\frac{\mu}{\theta}+\phi+1)}+...\right)\\
&= \left(\frac{\lambda}{\theta}\right)^{\phi+1}\exp\left(-\frac{\lambda}{\theta}\right)+\left(\frac{\lambda}{\theta}\right)^{\phi+2}\exp\left(-\frac{\lambda}{\theta}\right)\sum_{x=0}^{\infty}\frac{\lambda^x}{\theta^x\prod_{i=0}^{x}(\frac{\mu}{\theta}+\phi+i)}\\
&= \left(\frac{\lambda}{\theta}\right)^{\phi+1}\exp\left(-\frac{\lambda}{\theta}\right)+\left(\frac{\lambda}{\theta}\right)^{2-\frac{\mu}{\theta}}\gamma\left(\frac{\mu}{\theta}+\phi,\frac{\lambda}{\theta}\right),
\end{aligned}
$$

where for the last equality, we applied again Equation (23). Then,

$$
\begin{aligned}
&\frac{\left(\frac{\lambda}{\theta}\right)^{1-\frac{\mu}{\theta}}\left(\frac{\mu}{\theta}+\phi\right)\gamma\left(\frac{\mu}{\theta}+\phi,\frac{\lambda}{\theta}\right)}{\phi+1}\sum_{x=0}^{\phi}\frac{\lambda^x}{x!\phi^x}\\
&<\frac{\left(\frac{\lambda}{\theta}\right)^{2-\frac{\mu}{\theta}}\gamma\left(\frac{\mu}{\theta}+\phi,\frac{\lambda}{\theta}\right)}{\phi+1}\sum_{x=0}^{\phi}\frac{\lambda^x}{x!\phi^x}+\frac{\left(\frac{\lambda}{\theta}\right)^{\phi+1}\exp\left(-\frac{\lambda}{\theta}\right)}{\phi+1}\sum_{x=0}^{\phi}\frac{\lambda^x}{x!\phi^x}\\
&=\left(\frac{\lambda}{\theta}\right)^{1-\frac{\mu}{\theta}}\gamma\left(\frac{\mu}{\theta}+\phi,\frac{\lambda}{\theta}\right)\sum_{x=0}^{\phi}\frac{\lambda^{x+1}}{x!(\phi+1)\phi^{x+1}}+\frac{\left(\frac{\lambda}{\theta}\right)^{\phi+1}\exp\left(-\frac{\lambda}{\theta}\right)}{\phi+1}\sum_{x=0}^{\phi}\frac{\lambda^x}{x!\phi^x}\\
&=\left(\frac{\lambda}{\theta}\right)^{1-\frac{\mu}{\theta}}\gamma\left(\frac{\mu}{\theta}+\phi,\frac{\lambda}{\theta}\right)\sum_{x=1}^{\phi+1}\frac{\lambda^{x}}{(x-1)!(\phi+1)\phi^{x}}+\frac{\left(\frac{\lambda}{\theta}\right)^{\phi+1}\exp\left(-\frac{\lambda}{\theta}\right)}{\phi+1}\sum_{x=0}^{\phi}\frac{\lambda^x}{x!\phi^x}\\
&<\left(\frac{\lambda}{\theta}\right)^{1-\frac{\mu}{\theta}}\gamma\left(\frac{\mu}{\theta}+\phi,\frac{\lambda}{\theta}\right)\sum_{x=1}^{\phi+1}\frac{\lambda^{x}}{x!\phi^{x}}+\frac{\left(\frac{\lambda}{\theta}\right)^{\phi+1}\exp\left(-\frac{\lambda}{\theta}\right)}{\phi+1}\sum_{x=0}^{\phi}\frac{\lambda^x}{x!\phi^x}\\
&<\left(\frac{\lambda}{\theta}\right)^{1-\frac{\mu}{\theta}}\gamma\left(\frac{\mu}{\theta}+\phi,\frac{\lambda}{\theta}\right)\sum_{x=0}^{\phi+1}\frac{\lambda^{x}}{x!\phi^{x}}+\frac{\left(\frac{\lambda}{\theta}\right)^{\phi+1}\exp\left(-\frac{\lambda}{\theta}\right)}{\phi+1}\sum_{x=0}^{\phi}\frac{\lambda^x}{x!\phi^x}.
\end{aligned}
$$

To show that Equation (10) holds, it suffices to show, that

$$
\frac{\left(\frac{\lambda}{\theta}\right)^{\phi+1}\exp\left(-\frac{\lambda}{\theta}\right)}{\phi+1}\sum_{x=0}^{\phi}\frac{\lambda^x}{x!\theta^x}\leq\Gamma\left(\phi+1,\frac{\lambda}{\theta}\right)\frac{\lambda^{\phi+1}}{(\phi+1)!\theta^{(\phi+1)}}. \tag{11}
$$

We will rewrite the left-hand side of Equation (**??**) by using Equation (24). We get,

$$
\frac{\left(\frac{\lambda}{\theta}\right)^{\phi+1}\exp\left(-\frac{\lambda}{\theta}\right)}{\phi+1}\sum_{x=0}^{\phi}\frac{\lambda^x}{x!\theta^x}=\frac{\left(\frac{\lambda}{\theta}\right)^{\phi+1}\phi!\exp\left(-\frac{\lambda}{\theta}\right)}{(\phi+1)!}\sum_{x=0}^{\phi}\frac{\lambda^x}{x!\theta^x}=\frac{\lambda^{\phi+1}}{(\phi+1)!\theta^{\phi+1}}\Gamma\left(\phi+1,\frac{\lambda}{\theta}\right),
$$

which is equal to the right-hand side. So we see, that Equation (11) holds, thus that completes the proof of the lemma. □

6.1.2 Proof of Proposition 1

As mentioned before, there is a threshold optimal strategy for each subsidy of passivity W. Recall, that the average expected cost for fixed W and ϕ, is equal to

$$
g^{\phi}(W):=\sum_{x=0}^{\phi}\tilde{C}(x,0)\pi^{\phi}(x)+\sum_{x=\phi+1}^{\infty}\tilde{C}(x,1)\pi^{\phi}(x)-W\sum_{x=0}^{\phi}\pi^{\phi}(x). \tag{12}
$$

We are interested in the minimum average expected cost for given W. By threshold optimality, we may minimise over the threshold average cost values, so that the threshold ϕ that minimises $g^{\phi}(W)$ is the optimal threshold. Thus,

$$g(W) := \min_{\phi} g^{\phi}(W) \tag{13}$$

yields the desired minimum expected average cost. We will give an algorithm to find the Whittle index and we will see that this implies indexability of all classes.

Algorithm 1 Algorithm to determine the Whittle index

```
Step 1
Determine $g^0(W)$ as a function of $W \in \mathbb{R}$
OptThreshold($W$) := 0 $\forall W$
OptCurve($W$) := $g^0(W)$
NonDif:= $\emptyset$
$n := 1$
Step 2
for $n = 1$ to $\infty$ do
    Determine $g^n(W)$ as a function of $W \in \mathbb{R}$
    for all $W$ do
        if OptCurve($W$) $>$ $g^n(W)$ then
            OptThreshold($W$) = $n$
            OptCurve($W$) = $g^n(W)$
        end if
    end for
    Determine $W' = \min\{W : \text{OptCurve}(W) = g^n(W)\}$
    for $W$ $\in$NonDif do
        if $W > W'$ then
            NonDif=NonDif$\setminus\{W\}$
        end if
    end for
    NonDif=NonDif$\cup\{W'\}$
end for
Step 3
for all $W$ do
    $n(W)$ :=OptThreshold($W$)
    $g(W)$ :=OptCurve($W$)
end for
```

In the i^{th} iteration of step 2, the following can happen. Since $W \to g^n(W)$ has the (strictly) minimum gradient of the functions $W \to g^k(W)$, $k \leq n$, it intersects OptCurve(W) at some point W_n. So,

$$\text{OptCurve}(W) := \begin{cases} \text{OptCurve}(W), & W \leq W_n \\ g^n(W), & W \geq W_n. \end{cases}$$

Thus, iteratively, it follows that OptCurve(W) is a piecewise linear, non-increasing function, consisting of successive linear pieces of the functions $g^{n_1}(W), ..., g^{n_k}(W) = g^n(W)$, where $n_1 < ... < n_k = n$, since the gradients are strictly decreasing in n.

The set NonDif consist of the points where OptCurve(W) is not differentiable. These are exactly the points $W_{n_1} < ... < W_{n_k}$ where $g^{n_1}(W_{n_1}) = g^{n_2}(W_{n_1}), ..., g^{n_{k-1}}(W_{n_k}) = g^{n_k}(W_{n_k}) = g^n(W_{n_k})$

respectively. This clearly holds for the final set NonDif.
So it follows that we also have a concave function and we now can conclude that we satisfy the requirement in Definition 2. So indeed, Algorithm 1 gives us, that each single queue is indexable. This concludes the proof of Proposition 1. □

6.1.3 Computation of the Whittle index

Algorithm 1 not only proves Proposition 1, but it also constructs the Whittle index. Remark that the points where OptCurve(W) is not differentiable, are exactly the points where $W(x) := \inf\{W : x \leq \text{OptThreshold}(W)\}$. This is the Whittle index in state x by Definition 3. The set NonDif consist of the intersection points of two linear functions. We see that [1, Theorem 1] calculates this points in another way. As a shorthand notation we define

$$\eta(\phi) = \sum_{x=0}^{\phi} \pi^{\phi}(x),$$

and

$$\delta(\phi) = \sum_{x=0}^{\phi} \pi^{\phi}(x)\tilde{C}(x,0) + \sum_{x=\phi+1}^{\infty} \pi^{\phi}(x)\tilde{C}(x,1),$$

with $\delta(-1) = \sum_{x=0}^{\infty} \pi^{-1}(x)\tilde{C}(x,1)$. The algorithm of [1] is as follows.

Algorithm 2 Algorithm of [1] to compute the Whittle index

Step 1
Compute

$$W_0 = \inf_{\phi \in \mathbb{N} \cup \{0\}} \frac{\mathbb{E}(\delta(\phi) - \delta(-1))}{\eta(\phi)},$$

and call ϕ_0 the largest minimizer. Then, define $W(\phi) := W_0$ for all $\phi < \phi_0$. If $\phi_0 = \infty$ define $W(\phi) := W_0$ for all ϕ, otherwise go to step 2, for $i = 1$.

Step 2 $i = 1, 2, ...$

$$W_i = \inf_{\phi \in \mathbb{N} \setminus \{0,1,2,\ldots,\phi_{i-1}\}} \frac{\delta(\phi) - \delta(\phi_{i-1})}{\eta(\phi) - \eta(\phi_{i-1})},$$

and call ϕ_i the largest minimizer. Define $W(\phi) := W_i$ for all $\phi_{i-1} < \phi < \phi_i$. If $\phi_i = \infty$ define $W(\phi) := W_i$ for all ϕ, otherwise repeat step 2 for $i = i + 1$.

Theorem 1. *Algorithm 2 determines the Whittle index for each state x in a single queue model.*

The validity of Theorem 1 follows immediately from the construction in Algorithm 1. The set $\{W_0, W_1, ...\}$ is precisely the set NonDif from Algorithm 1, but now computed in increasing order. In our opinion, Algorithm 1 is a more elegant way to prove this theorem, than the present proof in [1]. In particular, [1] did not prove that the optimal curve $g(W)$ is indeed concave and non-increasing.

7 Other possibilities to compute the Whittle index

Remark that $\delta(\phi)$ is the expected average cost of the system under threshold strategy ϕ. The way to compute the Whittle index is presented in Algorithm 2. Corollary 1 of [1] implies that the Whittle index in state ϕ is given by

$$I(\phi) = \frac{\delta(\phi) - \delta(\phi - 1)}{\eta(\phi) - \eta(\phi - 1)}, \tag{14}$$

provided $I(\phi)$ is non-decreasing in ϕ. Unfortunately, [1] does not prove that Equation (14) is non-decreasing in ϕ. We have tried to prove that Equation (14) is non-decreasing as a function of ϕ, by using our expression for the stationary distribution. We have not been able to do, in spite of intensive effort. The numerical results below show that indeed $\phi \to I(\phi)$ is a non-decreasing function.

7.1 Abandonment rate equals service rate

In the next subsections, we will look at some special cases from [1] with $I(\phi)$ non-decreasing in ϕ. It is easy to see, that for every strategy ϕ the Markov process has the transition rates presented in Figure 2.

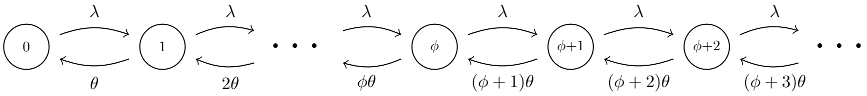


Figure 2: Markov process for every strategy when $\theta = \mu$.

From this fact, it follows automatically that $\pi^{\phi}(x) = \pi^{\phi-1}(x)$ for every x. So the denominator of Equation (14) is equal to

$$\sum_{x=0}^{\phi} \pi^{\phi}(x) - \sum_{x=0}^{\phi-1} \pi^{\phi-1}(x) = \sum_{x=0}^{\phi} \pi^{\phi}(x) - \sum_{x=0}^{\phi-1} \pi^{\phi}(x) = \pi^{\phi}(\phi).$$

The numerator is equal to

$$\begin{aligned}
&\delta(\phi) - \delta(\phi - 1) \\
&= \sum_{x=0}^{\phi} \pi^{\phi}(x)\tilde{C}(x,0) + \sum_{x=\phi+1}^{\infty} \pi^{\phi}(x)\tilde{C}(x,1) - \sum_{x=0}^{\phi-1} \pi^{\phi-1}(x)\tilde{C}(x,0) - \sum_{x=\phi}^{\infty} \pi^{\phi-1}(x)\tilde{C}(x,1) \\
&= \sum_{x=0}^{\phi} \pi^{\phi}(x)\tilde{C}(x,0) + \sum_{x=\phi+1}^{\infty} \pi^{\phi}(x)\tilde{C}(x,1) - \sum_{x=0}^{\phi-1} \pi^{\phi}(x)\tilde{C}(x,0) - \sum_{x=\phi}^{\infty} \pi^{\phi}(x)\tilde{C}(x,1) \\
&= \pi^{\phi}(\phi)\tilde{C}(\phi,0) - \pi^{\phi}(\phi)\tilde{C}(\phi,1).
\end{aligned}$$

We see now, that in this case Equation (14) is equal to

$$\frac{\pi^{\phi}(\phi)\tilde{C}(\phi,0) - \pi^{\phi}(\phi)\tilde{C}(\phi,1)}{\pi^{\phi}(\phi)} = \tilde{C}(\phi,0) - \tilde{C}(\phi,1). \tag{15}$$

We can rewrite the second inequality of Equation (1) to

$$\begin{aligned}
&C(\phi,0)-C(\phi,1)\leq C(\phi+1,0)-C(\phi+1,1)\\
&\Leftrightarrow C(\phi,0)-C(\phi,1)+d\theta-d'\theta'\leq C(\phi+1,0)-C(\phi+1,1)+d\theta-d'\theta'\\
&\Leftrightarrow C(\phi,0)-C(\phi,1)+d\theta(\phi-(\phi-1))-d'\theta'\leq C(\phi+1,0)-C(\phi+1,1)+d\theta(\phi+1-\phi)-d'\theta'\\
&\Leftrightarrow \tilde{C}(\phi,0)-\tilde{C}(\phi,1)\leq \tilde{C}(\phi+1,0)-\tilde{C}(\phi+1,1).
\end{aligned}$$

Thus the expression in Equation (14) is non-decreasing in ϕ and the Whittle index is thus given by Equation (14).

7.1.1 Example of the case that the abandonment rate equals the service rate

In this section we will provide an example of the case $\mu=\mu'+\theta'=\theta$. We choose the following parameters:

$$\lambda=1,\ \mu=\theta=\frac{1}{4},\ \theta'=\frac{1}{16},\ \mu'=\frac{3}{16}.$$

Let $C(x,0)=x^2+x$ and $C(x,1)=x^2$. It's easy to check that these cost functions satisfy Equation (1).
Take $d=5$ and $d'=10$ to be the lump costs, so the combined cost becomes

$$\tilde{C}(x,0)=x^2+x+5\cdot\frac{1}{4}x=x^2+\frac{9}{4}x,$$

and

$$\tilde{C}(x,1)=x^2+5\cdot\frac{1}{4}\max(0,x-1)+10\cdot\frac{1}{16}\min(1,x)=x^2+\frac{5}{4}\max(0,x-1)+\frac{5}{8}\min(1,x).$$

Remark, that we can view this Markov process as an $M/M/\infty$ queue. We can use the analysis of [3] to calculate the stationary distribution. This distribution is independent of strategy ϕ as we saw before, so we may omit de index ϕ in the stationary distribution. We get

$$\pi(0)=\left(\sum_{x=0}^{\infty}\frac{\left(\frac{\lambda}{\mu}\right)^x}{x!}\right)^{-1}=\left(\sum_{x=0}^{\infty}\frac{\left(\frac{1}{\frac{1}{4}}\right)^x}{x!}\right)^{-1}=(\exp(4))^{-1}=\exp(-4).$$

With [3], it follows that

$$\pi(m)=\frac{\left(\frac{\lambda}{\mu}\right)^m}{m!}\pi(0)=\frac{4^m}{m!}\pi(0).$$

This allows to calculate $\eta(\phi)$ and $\delta(\phi)$ for every ϕ. The results or $0\leq\phi\leq 10$ are given in Figure 3. Figure 4 is a plot of $I(\phi)$, and we see that it is indeed linear in ϕ.

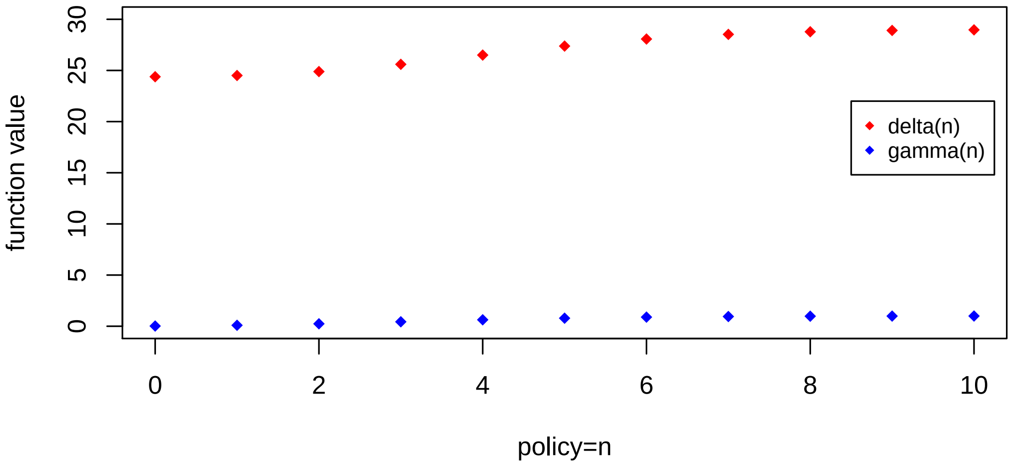


Figure 3: Values of $\eta(n)$ and $\delta(n)$ for $0 \leq n \leq 10$.

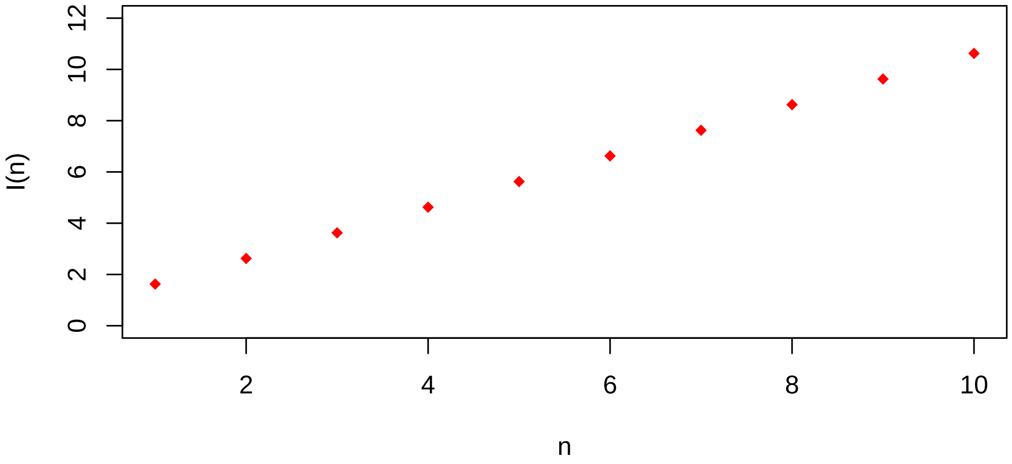


Figure 4: Value of $I(n)$.

7.2 Linear cost functions

In this subsection we assume that the cost functions are linear. So, let $c, c', d, d' \in \mathbb{R}$, and put

$$\tilde{C}(x, 0) = cx + d\theta x = \tilde{c}x,$$

and

$$\tilde{C}(x, 1) = c(x-1) + c' + \theta d(x-1) + d'\theta' = \tilde{c}(x-1) + \tilde{c}'.$$

We will prove that the Whittle index in this case is a constant. The idea of this proof is based on the proof in Appendix 3 of [1].

The crucial argument in this proof is, that the fraction of impatient departures f plus the fraction of departures g due to service completion is equal to one. This is because of the fact that a customer leaves the system once and either leaves impatiently or served. Let f be the fraction of the customers that leaves the queue impatiently and let g be the fraction of the customers that leaves served, so

$$f + g = 1.$$

The first part of the analysis is independent of the cost functions. Let $\mathbb{E}[N]$ be the average number of customers in the system. Remark, that the total departure rate is equal to the arrival rate λ. When we condition on the state, it follows that

$$\begin{aligned}\lambda f &= \sum_{x=0}^{\phi} \theta x \pi^{\phi}(x) + \sum_{x=\phi+1}^{\infty} (\theta' + (x-1)\theta)\pi^{\phi}(x) \\ &= \sum_{x=0}^{\infty} \theta x \pi^{\phi}(x) + \sum_{x=\phi+1}^{\infty} \theta' \pi^{\phi}(x) - \sum_{x=\phi+1}^{\infty} \theta \pi^{\phi}(x) \\ &= \theta \mathbb{E}[N] + \sum_{x=\phi+1}^{\infty} \theta' \pi^{\phi}(x) - \sum_{x=\phi+1}^{\infty} \theta \pi^{\phi}(x).\end{aligned}$$

Further, we have that

$$\lambda g = \mu' \sum_{x=\phi+1}^{\infty} \pi^{\phi}(x).$$

Summing these two equations, we get

$$\begin{aligned}&\lambda f + \lambda g = \theta \mathbb{E}[N] + \sum_{x=\phi+1}^{\infty} \theta' \pi^{x}(x) - \sum_{x=\phi+1}^{\infty} \theta \pi^{\phi}(x) + \sum_{x=\phi+1}^{\infty} \mu' \pi^{\phi}(x) \\ &\Rightarrow \lambda = \theta \mathbb{E}[N] + (\theta' - \theta + \mu') \sum_{x=\phi+1}^{\infty} \pi^{\phi}(x) \\ &\Rightarrow \mathbb{E}[N] = \frac{\lambda}{\theta} - \frac{\theta' - \theta + \mu'}{\theta} \sum_{x=\phi+1}^{\infty} \pi^{\phi}(x). \qquad (16)\end{aligned}$$

From now on we will consider the expected average cost. We have

$$\begin{aligned}\delta(\phi) &= \sum_{x=0}^{\phi} \tilde{c} x \pi^{\phi}(x) + \sum_{x=\phi+1}^{\infty} (\tilde{c}(x-1) + \tilde{c}')\pi^{\phi}(x) \\ &= \sum_{x=0}^{\infty} \tilde{c} x \pi^{\phi}(x) + \sum_{x=\phi+1}^{\infty} (\tilde{c}' - \tilde{c})\pi^{\phi}(x) \\ &= \tilde{c}\mathbb{E}[N] + (\tilde{c}' - \tilde{c}) \sum_{x=\phi+1}^{\infty} \pi^{\phi}(x) \\ &= \tilde{c}\left(\frac{\lambda}{\theta} - \frac{\theta' - \theta + \mu'}{\theta} \sum_{x=\phi+1}^{\infty} \pi^{\phi}(x)\right) + (\tilde{c}' - \tilde{c}) \sum_{x=\phi+1}^{\infty} \pi^{\phi}(x).\end{aligned}$$

The last step follows from Equation (16).
This yields

$$
\begin{aligned}
I(\phi) &= \frac{\tilde{c}\left(\frac{\lambda}{\theta} - \frac{\theta' - \theta + \mu'}{\theta}\sum_{x=\phi+1}^{\infty}\pi^{\phi}(x)\right) + (\tilde{c}' - \tilde{c})\sum_{x=\phi+1}^{\infty}\pi^{\phi}(x)}{\sum_{x=0}^{\phi}\pi^{\phi}(x) - \sum_{x=0}^{\phi-1}\pi^{\phi-1}(x)} \\
&\quad - \frac{\tilde{c}\left(\frac{\lambda}{\theta} - \frac{\theta' - \theta + \mu'}{\theta}\sum_{x=\phi}^{\infty}\pi^{\phi-1}(x)\right) + (\tilde{c}' - \tilde{c})\sum_{x=\phi}^{\infty}\pi^{\phi-1}(x)}{\sum_{x=0}^{\phi}\pi^{\phi}(x) - \sum_{x=0}^{\phi-1}\pi^{\phi-1}(x)} \\
&= \frac{\left(-\tilde{c}\frac{\theta' - \theta + \mu'}{\theta} + \tilde{c}' - \tilde{c}\right)\left(\sum_{x=\phi+1}^{\infty}\pi^{\phi}(x) - \sum_{x=\phi}^{\infty}\pi^{\phi-1}(x)\right)}{\sum_{x=0}^{\phi}\pi^{\phi}(x) - \sum_{x=0}^{\phi-1}\pi^{\phi-1}(x)} \\
&= \frac{\left(-\tilde{c}\frac{\theta' - \theta + \mu'}{\theta} + \tilde{c}' - \tilde{c}\right)\left(1 - \sum_{x=0}^{\phi}\pi^{\phi}(x) - (1 - \sum_{x=0}^{\phi-1}\pi^{\phi-1}(x))\right)}{\sum_{x=0}^{\phi}\pi^{\phi}(x) - \sum_{x=0}^{\phi-1}\pi^{\phi-1}(x)} \\
&= \frac{\left(-\tilde{c}\frac{\theta' - \theta + \mu'}{\theta} + \tilde{c}' - \tilde{c}\right)\left(- \sum_{x=0}^{\phi}\pi^{\phi}(x) + \sum_{x=0}^{\phi-1}\pi^{\phi-1}(x)\right)}{\sum_{x=0}^{\phi}\pi^{\phi}(x) - \sum_{x=0}^{\phi-1}\pi^{\phi-1}(x)} \\
&= -\left(-\tilde{c}\frac{\theta' - \theta + \mu'}{\theta} + \tilde{c}' - \tilde{c}\right) \\
&= \tilde{c}\frac{\theta' + \mu'}{\theta} - \tilde{c}'. \qquad (17)
\end{aligned}
$$

We see that $I(\phi)$ is a constant function and so it is non-decreasing in ϕ. Hence, the Whittle index in state ϕ is given by $I(\phi)$, and thus equal to a constant. In Section 8.1, we discuss a 2-competing queues model, with linear cost for both queues.

7.2.1 Example of a queue with linear cost functions

In this section we will discuss an example with linear cost functions. We choose the following parameters:

$$\lambda = 1, \theta = \frac{1}{8}, \theta' = \frac{1}{25}, \mu' = \frac{1}{2},$$

we further put $C(x, 0) = 5x + 5$ and $C(x, 1) = 5x$ and lump cost $d = 5$ and $d' = 10$. This yields

$$\tilde{C}(x, 0) = 5x + 5 + 5 \cdot \frac{1}{8}x = \frac{45}{8}x + 5,$$

and

$$\tilde{C}(x, 1) = 5x + 5 \cdot \frac{1}{8}\max(0, x - 1) + 10 \cdot \frac{1}{25}\min(1, x) = \frac{45}{8}x - \frac{9}{40}\min(1, x).$$

In this case, it is very hard to calculate $\eta(\phi)$ and $\delta(\phi)$ explicitly. Therefore, we have write a program in R, to compute the desired results. This program can be found in the appendix, in Section 10.2. It follows that the summation in Equation (7) converges very fast, so we will take into account states x with $P_x^{\phi} > 10^{-16}$. Figure 5 shows resulting values of $\delta(\phi)$ and $\eta(\phi)$. Figure 6 confirms that $I(\phi)$ is independent of the value ϕ.

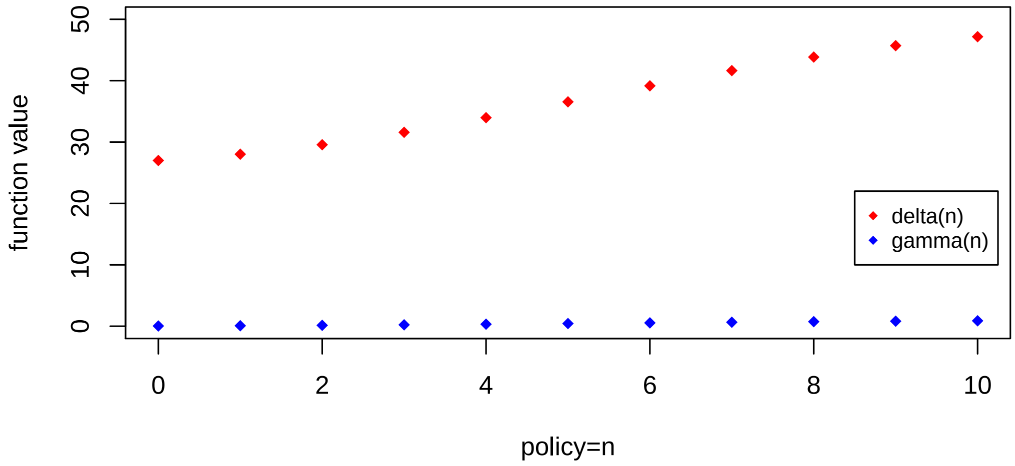


Figure 5: Values of $\eta(n)$ and $\delta(n)$ for $0 \leq n \leq 10$.

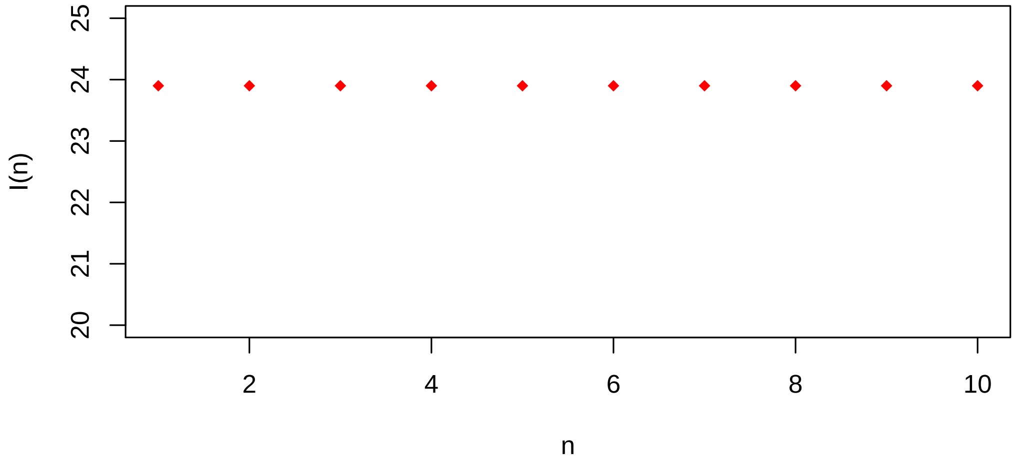


Figure 6: Value of $I(n)$.

8 Two-competing queues

In this section we will discuss some examples illustrating the Whittle index strategy, as defined in Definition 4.

8.1 Two queues with linear cost functions

In this example both queues have linear cost functions. In Section 7.2 we have computed the Whittle index for this case. For each queue, it is a constant independent of the state of the queue.
Queue 1 has parameters

$$\lambda_1 = 1, \theta_1 = \frac{1}{4}, \theta_1' = \frac{1}{20}, \mu_1' = \frac{1}{3},$$

and the cost-functions are

$$\tilde{C_1}(x,0) = 5x \text{ and } \tilde{C_1}(x,1) = 5x + 3.$$

Queue 2 has parameters

$$\lambda_2 = 1, \theta_2 = \frac{3}{4}, \theta_2' = \frac{1}{5}, \mu_2' = \frac{4}{5},$$

with cost functions

$$\tilde{C_2}(x,0) = 0.5x \text{ and } \tilde{C_2}(x,1) = 0.5x + 2.$$

We can use Equation (17) to calculate the Whittle index in both queues. For queue 1 we get

$$I_1 = 5 \cdot \frac{\frac{1}{20} + \frac{1}{3}}{\frac{1}{4}} - 3 = \frac{14}{3},$$

and for queue 2

$$I_2 = 0.5 \cdot \frac{\frac{1}{5} + \frac{4}{5}}{\frac{3}{4}} - 2 = -\frac{4}{3}.$$

Hence, the Whittle index strategy prescribes to always serve queue 1 when type 1 customers are present.
Remark that this case is always easy to analyse. In the case of the K-competing queues model, with linear cost associated with each customer type, the Whittle index strategy will always prescribe to serve the non-empty queue with the highest index, independent of the number of customers in that queue. Queues with

$$\tilde{c}\frac{\theta' + \mu'}{\theta} < \tilde{c}',$$

are never served.

8.2 One queue with linear cost functions and one queue with $\theta = \mu$

In this example, we assume that for type 1 customers a linear cost is incurred, and the service rate of type 2 customers is equal to the impatience rate. The parameters chosen for type 1 customers are equal to the parameters for type 1 customers in Section 8.1. For type 2 customers we use the same parameters as for customers in Section 7.1.1.

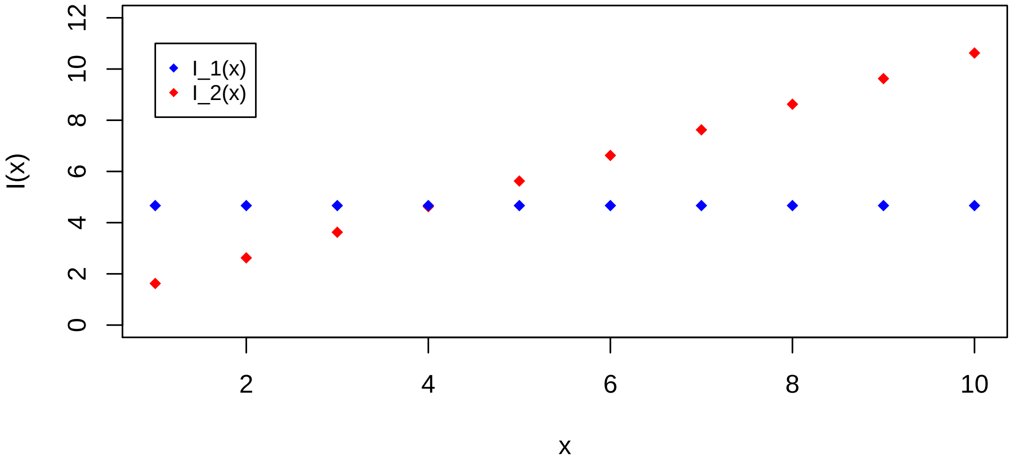


Figure 7: The Whittle index for queue 1 and queue 2.

10	2	2	2	2	2	2	2	2	2	2	2
9	2	2	2	2	2	2	2	2	2	2	2
8	2	2	2	2	2	2	2	2	2	2	2
7	2	2	2	2	2	2	2	2	2	2	2
6	2	2	2	2	2	2	2	2	2	2	2
5	2	2	2	2	2	2	2	2	2	2	2
4	2	1	1	1	1	1	1	1	1	1	1
3	2	1	1	1	1	1	1	1	1	1	1
2	2	1	1	1	1	1	1	1	1	1	1
1	2	1	1	1	1	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1	1	1	1
	0	1	2	3	4	5	6	7	8	9	10

Number of customers in queue 2

Number of customers in queue 1

Figure 8: The Whittle index strategy per state.

Figure 7 shows that the Whittle index of queue 2 is smaller than the Whittle index of queue 1, when there are at most 4 customers in queue 2. The table in Figure 8 shows the details of the Whittle index strategy for this case.

8.3 Two queues with $\theta = \mu$

Next we consider the case of two queues with equal service rate and abandonment rate. For type 1 customers, we use again the parameters of Section 7.1.1. We add another queue with the following parameters.

$$\lambda = 1, \theta = \frac{1}{5}, \theta' = \frac{1}{20}, \mu' = \frac{3}{20},$$

and

$$\tilde{C}(x, 0) = x^3 + 3x \text{ and } \tilde{C}(x, 1) = x^3 + x + 1.$$

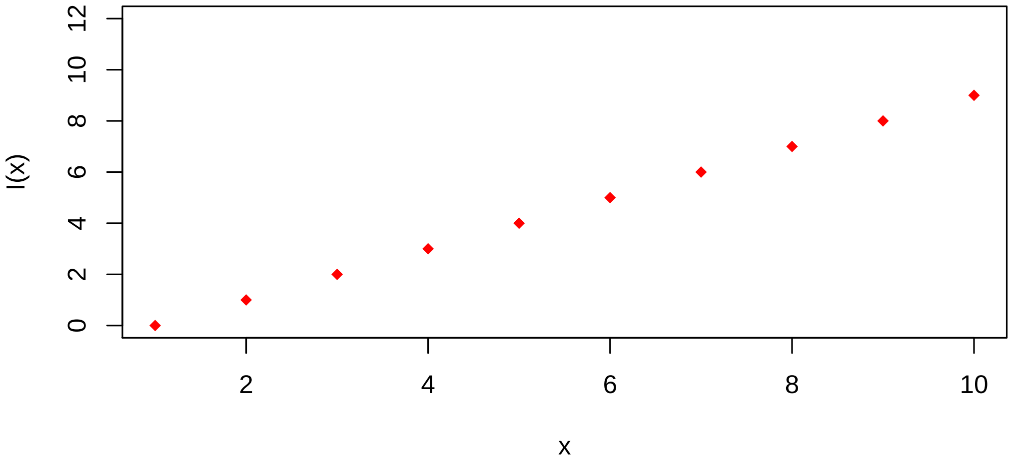
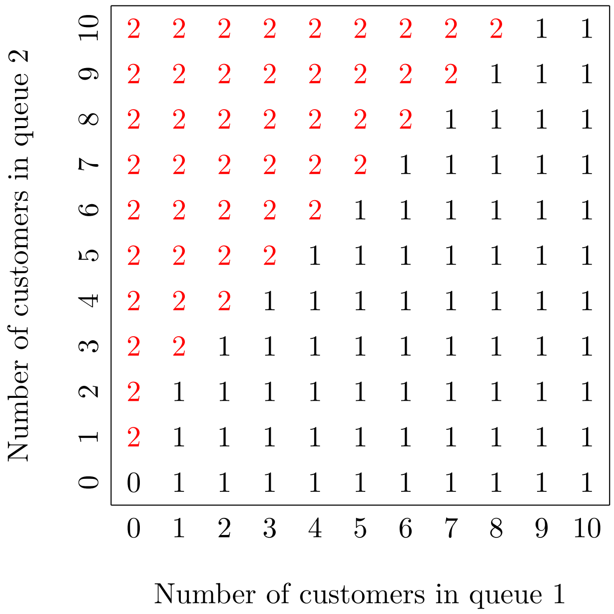


Figure 9: The Whittle index for queue 2.



10	2	2	2	2	2	2	2	2	2	1	1
9	2	2	2	2	2	2	2	2	1	1	1
8	2	2	2	2	2	2	2	1	1	1	1
7	2	2	2	2	2	2	1	1	1	1	1
6	2	2	2	2	2	1	1	1	1	1	1
5	2	2	2	2	1	1	1	1	1	1	1
4	2	2	2	1	1	1	1	1	1	1	1
3	2	2	1	1	1	1	1	1	1	1	1
2	2	1	1	1	1	1	1	1	1	1	1
1	2	1	1	1	1	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1	1	1	1
	0	1	2	3	4	5	6	7	8	9	10

Number of customers in queue 2

Number of customers in queue 1

Figure 10: The Whittle index strategy per state.

Figure 9 shows the Whittle index for queue 2. Figure 10 provides the table with the server assignments per state, when the Whittle index strategy is used. It is interesting to note, that the Whittle index strategy is characterised by a linear switching curve.

9 Conclusions and possibilities of further research

This thesis is mostly based on [1]. This article is not always very clear or easy to understand. So, in this thesis we have tried to clarify some of the proofs in [1].

We have further derived a closed form formula for the stationary distribution of the single queue model operating under a threshold strategy. Using this expression for the stationary distribution, we have provided another proof than the one in [1] to show that

$$\sum_{x=0}^{\phi} \pi^{\phi}(x) < \sum_{x=0}^{\phi+1} \pi^{\phi+1}(x).$$

This proof can be used to bound the difference between the two expressions.

It is not yet clear how, but this stationary distribution can be the key to prove that $I(\phi)$ is increasing in ϕ. This part is very interesting for further research. Unfortunately we have not succeeded in doing so.

Another thing we tried without success, is to show using value iteration, that the threshold increases, when W increases.

For further research, it could be interesting to relax very restrictive conditions on the cost functions.

A last thing that is not completely clear for us, is how well the Whittle index strategy performs compared the optimal strategy. In [1], it was proved that the Whittle index strategy is asymptotically optimal as the number of servers grows, and in the light and heavy traffic limits. However, nothing is known for fixed rates and a fixed number of servers.

10 Appendix

10.1 The Gamma function and some of its properties

The Gamma function is defined as

$$\Gamma(s) = \int_0^\infty t^{s-1} \exp(-t) dt. \tag{18}$$

This function is an extension of the factorial function. Specially we have

$$\Gamma(s) = (s-1)! \quad \text{if } s \in \mathbb{N}_{>0}.$$

Based on the integral in Equation (18), we define the incomplete Gamma function. The upper incomplete Gamma function is defined as

$$\Gamma(s, x) = \int_x^\infty t^{s-1} \exp(-t) dt. \tag{19}$$

Similarly the definition of the lower incomplete Gamma function is

$$\gamma(s, x) = \int_0^x t^{s-1} \exp(-t) dt. \tag{20}$$

Remark, that it automatically follows that

$$\Gamma(s) = \gamma(s, x) + \Gamma(s, x).$$

Using integration by parts, we get the following recurrence relations

$$\Gamma(s) = (s-1)\Gamma(s-1),$$

$$\Gamma(s, x) = (s-1)\Gamma(s-1, x) + x^{s-1} \exp(-x), \tag{21}$$

and

$$\gamma(s, x) = (s-1)\gamma(s-1, x) - x^{s-1} \exp(-x). \tag{22}$$

For the Gamma function, as well as for both incomplete Gamma functions a lot of other expressions exist. One of these is the power series expansion for $\gamma(s, x)$, namely

$$\gamma(s, x) = x^s \exp(-x) \sum_{k=0}^{\infty} \frac{x^k}{\prod_{i=0}^{k} (s+i)}. \tag{23}$$

If $s > 0$ is an positive integer, then we can write

$$\Gamma(s, x) = (s-1)! \exp(-x) \sum_{k=0}^{s-1} \frac{x^k}{k!}. \tag{24}$$

10.2 Code to determine $I(\phi)$

This is the code to determine $I(\phi)$ relative precise for specific cases. It is based on the fact that $\pi^\phi(x)$ tends to zero, in the limit $x \to \infty$.

```
lambda = #fill in lambda
theta = #fill in theta
theta.acc = #fill in theta'
mu.acc = #fill in mu'
P0 <- c(0,0,0,0,0,0,0,0,0,0,0)

for (n in 1:11){
  x = 1
  y = 0
  z=lambda^(n-1)/(theta^(n-1)*factorial(n-1))
  while (x > 0.0000000000000001){
    if (y < n){
      x = lambda^(y)/(theta^(y)*factorial(y))
      P0[n] = P0[n] + x
      y = y+1
    }
    else{
      z = z*lambda/((y-1)*theta+theta.acc+mu.acc)
      x = z
      P0[n] = P0[n] + z
      y = y+1
    }
  }
  P0[n] = 1/P0[n]

}#computes pi(0) for phi=n

eta <- c(0,0,0,0,0,0,0,0,0,0,0)

for (n in 1:11){
  for (m in 1:n){
    eta[n] = eta[n]+lambda^(m-1)/(theta^(m-1)*factorial(m-1))*P0[n]
  }
} #computes eta

C0 <- function(a){
 #fill in C(0,a)
}

C1 <- function(a){
  #fill in C(1,a)
}

delta <- c(0,0,0,0,0,0,0,0,0,0,0)

for (n in  1:11){
  for (m in 1:n){
    delta[n] <- delta[n]+P0[n]*lambda^(m-1)/(theta^(m-1)*factorial(m-1))*C0(m-1)
```

```
  }
  t <- lambda^(n-1)/(theta^(n-1)*factorial(n-1))*P0[n]
  s <- n
  while(t > 0.0000000000000001){
    t <- t*lambda/((s-1)*theta+theta.acc+mu.acc)
    delta[n] <- delta[n]+t*C1(s)
    s <- s+1
  } #computes delta

}

I <- c(0,0,0,0,0,0,0,0,0,0)
for(n in 1:10){
  I[n] <- (delta[n+1]-delta[n])/(eta[n+1]-eta[n])
} #computes I
```

11 References

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