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Isometric lattice isomorphisms associated with convolution actions of measures on L^p-spaces

Master thesis

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Abstract

Let G be a locally compact Hausdorff group and let ν be a fixed Haar measure on G. If μ is a bounded positive Radon measure on G and $f \in L^p(G, \nu)$, their convolution product $\mu * f$ exists. Thus every bounded Radon measure acts on $L^p(G, \nu)$ as a convolution operator.

It is easy to see that the action of a bounded Radon measure on $L^p(G, \nu)$ gives rise to an injective Banach algebra homomorphism from the Banach algebra of bounded Radon measures on G to the Banach algebra of regular operators from $L^p(G, \nu)$ to $L^p(G, \nu)$. We will investigate when this algebra homomorphism is also a (isometric) lattice isomorphism.

For each $a \in G$ we have a right translation operator ρ_a on $L^p(G, \nu)$. The regular commutant of the right translation operators is the set of regular operators T from $L^p(G, \nu)$ to $L^p(G, \nu)$ such that $\rho_a T = T \rho_a$ for all $a \in G$. We will show that the regular commutant of the right translation operators is also a Banach lattice. The bounded Radon measures on G also form a Banach lattice. It is then investigated when the injective Banach algebra homomorphism is actually a lattice isomorphism from the bounded Radon measures on G onto the regular commutant of the right translation operators. This is the main result of this thesis.

We start by showing that each positive regular operator in the regular commutant of the right translation operators is in fact a convolution operator related to a positive Radon measure. For $p = 1$, we improve the result by showing that the Radon measure must be bounded and we show that our algebra isomorphism is then an isometric lattice isomorphism. For $p \in (1, \infty)$ the same results follow under the extra assumption that G is also amenable.

Contents

1 Introduction and overview

Let G be a locally compact Hausdorff group and let ν be a fixed Haar measure on G. If μ is a bounded positive Radon measure on G and $f \in L^p(G,\nu)$, we show that their convolution product $\mu * f$ exists. Thus every bounded Radon measure acts on $L^p(G,\nu)$ as a convolution operator.

This thesis is the result of research on the question when the action of a bounded Radon measure on $L^p(G,\nu)$ leads to an isometric lattice isomorphism. It is easy to see that the action of a bounded Radon measure on $L^p(G,\nu)$ gives rise to an injective Banach algebra homomorphism from the Banach algebra of bounded Radon measures on G to the Banach algebra of regular operators from $L^p(G,\nu)$ to $L^p(G,\nu)$. We will investigate when this algebra homomorphism is also a (isometric) lattice isomorphism. The strategy of this investigation is to show that the injective Banach algebra homomorphism is also surjective, and after that, also bipositive.

For each $a \in G$ we have a right translation operator ρ_a on $L^p(G,\nu)$. The regular commutant of the right translation operators is the set of regular operators T from $L^p(G,\nu)$ to $L^p(G,\nu)$ such that $\rho_a T = T\rho_a$ for all $a \in G$. We will show that the regular commutant of the right translation operators is also a Banach lattice. To see this, we will introduce a well-chosen lattice automorphism of the space of regular operators from $L^p(G,\nu)$ to $L^p(G,\nu)$ and show that it is bipositive. The bounded Radon measures on G also form a Banach lattice. It is then investigated when the injective Banach algebra homomorphism is actually a lattice isomorphism from the bounded Radon measures on G onto the regular commutant of the right translation operators. This is the main result of this thesis.

We start by showing that each positive regular operator in the regular commutant of the right translation operators is in fact a convolution operator related to a positive Radon measure. We need several steps to come to this result. It is then clear that the injective Banach algebra homomorphism is also surjective and it is easy to see that this bijection is a bipositive map. For $p = 1$, we improve the result by showing that the Radon measure must be bounded and we show that our algebra isomorphism is then an isometric lattice isomorphism. For $p \in (1,\infty)$ the same results follow under the extra assumption that G is also amenable, which is shown by Gilbert [11].

We will now give an overview of the structure of this thesis, at the same time giving a summary of the necessary equipment and a complete formal statement of the result.

Let G be any locally compact Hausdorff group, not necessarily abelian. In what follows, we will limit ourself to the field of the real numbers $\mathbb{R}$, sometimes the extended real numbers $\overline{\mathbb{R}}$.

In Section 2 we introduce our real-valued function spaces on G. The most important one is $C_c(G)$, the vector space consisting of the continuous functions on G of compact support. For $K \subseteq G$ compact, the vector space

$$C_c(K,G) = \{f \in C_c(G)\colon \operatorname{supp}(f) \subseteq K\}$$

is a Banach subspace of $C_c(G)$ with respect to the supremum norm. The topology of $C_c(G)$ is the inductive limit topology of these subspaces, which is discussed in A.2. We follow the work of Edwards [8] in a comprehensive functional analytic approach to Radon measures on G. We define $M_b(G)$ to be the Banach lattice algebra consisting of all bounded Radon measures on G. If μ is a positive Radon measure on G we define $L^p(G,\mu)$, for $p \in [1,\infty)$, to be the space of p-fold μ-integrable functions on G. We end with a couple of results about vector-valued integration.

In Section 3 we will summarize some results concerning left Haar measures on G. A Haar measure on G is unique, up to a constant. This constant defines the right Haar modulus Δ, a positive and continuous group homomorphism from G to $\mathbb{R}$. We use ν to denote our fixed Haar measure throughout this thesis and, among others, we abbreviate $L^p(G,\nu)$ to $L^p(G)$.

In Section 4 we introduce the translation operators on $\mathbb{R}^G$ given by

$$\lambda_a\colon \mathbb{R}^G \to \mathbb{R}^G, \lambda_a f(x) = f\big(a^{-1}x\big)$$

and

$$\rho_a\colon \mathbb{R}^G \to \mathbb{R}^G, \rho_a f(x) = f\big(xa^{-1}\big),$$

where $a \in G$. These operators are extended to the space of Radon measures on G.

In Section 5 we follow Edwards [8] once more to dive into the theory of the convolution products $\mu_1 * \mu_2$, where μ_1 and μ_2 are positive Radon measures. As a special case, with the assumption that they exist, we derive the following pointwise expressions of the convolution products

$$\mu * f(x) = \int f\big(y^{-1}x\big)\, d\mu(y) = \int \lambda_y f(x)\, d\mu(y)$$

and

$$f * g(x) = \int f(y) g\big(y^{-1}x\big)\, dy = \int f(y) \lambda_y g(x)\, dy,$$

where μ is a positive Radon measure and $f, g \in \mathbb{R}^G$ are at least ν-integrable. These expressions hold ν-almost everywhere on G, in special cases everywhere on G. After the definition of each of these convolution products, we prove some properties regarding existence and continuity. We will show that the convolution $f * g$ also exists as a vector-valued integral in $C_c(G)$. We will see that the translation of a convolution is again a convolution.

In Section 6 we are ready to prove the first result. If a positive operator $T\colon L^p(G) \to L^q(G)$, with $p \in [1,\infty)$, commutes with all right translation operators, we will show the existence of a positive Radon measure μ such that $Tf = \mu * f$ holds ν-a.e. on G for all $f \in L^p(G)$. The assumption that T commutes with all right translation operators appears to be crucial for proving the main result of this thesis.

In Section 7 we extend the results of Section 6, we will first show that the measure μ that was found must be bounded. This result is almost immediate for $p = 1$. In fact, this means that the operator T is the convolution operator $T_\mu^{p,q}$, the unique continuous extension to $L^p(G)$ of the continuous linear map

$$C_c(G) \to L^q(G), f \mapsto \mu * f.$$

The operator norm of $T_\mu^{1,q}$ equals $\|\mu\|$. Both results are true for $p \in (1,\infty)$, provided that G satisfies an additional assumption. We will introduce amenable groups and we cite the results from Gilbert [11].

In Section 8 we get to our main result, combining the results of Brainerd & Edwards [4] and Gilbert [11]. We will define $\mathcal{L}_r(L^p(G))$ to be the Banach lattice consisting of all regular operators $L^p(G) \to L^p(G)$. We will show that the map

$$\tau_p'\colon M_b(G) \to \mathcal{L}_r(L^p(G)), \mu \mapsto T_\mu^{p,p}$$

is an injective Banach algebra homomorphism for every $p \in [1,\infty)$. Next, we replace $\mathcal{L}_r(L^p(G))$ by the sublattice $\mathcal{L} = \{T \in \mathcal{L}_r(L^p(G))\colon \rho_a T = T\rho_a \text{ for all } a \in G\}$, the regular commutant of the right translation operators. We will see that τ_p' is surjective onto $\mathcal{L}$. Let us now formulate Theorem 8.4.

Theorem [2, Theorem 3.3]. *Let $p \in [1, \infty)$ and consider the map*

$$\tau_p \colon M_b(G) \to \mathcal{L}, \mu \mapsto T_\mu^{p,p}.$$

If $p = 1$ or if G is amenable, then τ_p is an isometric algebra and lattice isomorphism.

For the proof, we will use an equivalent description of the regular commutant of the right translation operators. In the proof we show that τ_p is a bipositive map.

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2 Preliminaries

In this thesis we will consider functions and, after that, measures on a locally compact topological group. In this section we will formulate the definitions and some facts that we will use in the following sections. One may review these facts in Conway [5], Larsen [14] or Rudin [19].

2.1 Groups and topology

A *topological group* is a group G that is equipped with a topology such that the group operations $\cdot\colon G\times G\to G, (x,y)\mapsto xy$ and ${}^{-1}\colon G\to G, x\mapsto x^{-1}$ are continuous maps. The identity element (with respect to multiplication) of G will be denoted by e. Next, we assume this topology on G to be *locally compact*, meaning that every point $x\in G$ has a compact neighborhood. Let $\mathcal{N}_x$ denote the set of all neighborhoods of $x\in G$. Furthermore, as is common in literature, we will also assume that the topology on G is Hausdorff.

The set $\mathcal{N}_x$ may be partially ordered using reversed set inclusion, thus $U\leqslant V$ if and only if $V\subseteq U$. Then $\mathcal{N}_x$ becomes a directed set and we may and will consider any directed subset $V\subseteq\mathcal{N}_x$ as a *net* $(V_\alpha)_\alpha$ indexed by some directed set $\mathbb{A}$.

2.2 Function spaces

Let $\mathbb{R}^G$ be the space of all real-valued functions on G. Let $C(G)\subseteq\mathbb{R}^G$ denote the set of all real-valued continuous functions on G. Endowed with pointwise addition and scalar multiplication, it becomes a real vector space. We define the subspace $C_b(G)$, inheriting the addition and multiplication, to be all continuous functions $f\in C(G)$ that are bounded, thus having finite *supremum norm*

$$\|f\|_\infty \coloneqq \sup_{x\in G}|f(x)|.$$

We will focus on two even smaller subspaces of $C_b(G)$ using the same norm and vector space operations. Firstly, let $C_0(G)\subseteq C(G)$ denote the subspace formed by the functions that *vanish at infinity*, that is, all continuous functions f on G such that the set $\{x\in G\colon |f(x)|\geqslant\varepsilon\}$ is compact for all $\varepsilon>0$. Secondly, let $C_c(G)\subseteq C(G)$ denote the subspace consisting of the continuous functions on G with *compact support*, that is, all continuous functions f on G such that

$$\operatorname{supp}(f)\coloneqq\overline{\{x\in G\colon f(x)\neq 0\}}$$

is compact.

The spaces $(C_0(G),\|\cdot\|_\infty)$ and $(C_b(G),\|\cdot\|_\infty)$ are both Banach spaces and $\overline{C_c(G)}=C_0(G)$ for any locally compact Hausdorff group. For any group G we have $C_c(G)\subseteq C_0(G)\subseteq C_b(G)\subseteq C(G)\subseteq\mathbb{R}^G$ as proper subspaces except when G is compact, then we have $C_c(G)=C(G)$.

For each of the previous spaces of continuous functions there is a natural topology. For $C(G)$ this is the topology of locally uniform convergence and for $C_0(G)$ this is the topology of uniform convergence (by restricting the supremum norm to $C_0(G)$). For $C_c(G)$ this is the topology that appears when considering $C_c(G)$ as the internal inductive limit (see Appendix A.2 for more details) of the subspaces

$$C_c(K,G)=\{f\in C_c(G)\colon\ \operatorname{supp}(f)\subseteq K\}$$

where $K \subseteq G$ is any of the compact subsets of G, considering each of the $C_c(K, G)$ as a Banach space using the supremum norm. It is clear that the topologies of $C(G), C_0(G)$ and $C_c(G)$ coincide whenever G is compact.

For any $f \in \mathbb{R}^G$, we define the *reflection of* f, denoted by $\check{f} \in \mathbb{R}^G$, to be the map $\check{f}(x) := f(x^{-1})$. Because $\check{f}$ is in fact the composition of f with the continuous inverse map $^{-1}$, we have $\check{f} \in C(G)$ when $f \in C(G)$ and also $\check{f} \in C_c(G)$ if $f \in C_c(G)$.

2.3 A functional analytic approach to Radon measures

Traditionally, a *(positive) Borel measure* on G is a $[0, \infty]$-valued function on the Borel σ-algebra $\mathcal{B}(G)$ that maps $\varnothing$ to 0 and is also σ-additive. In this way, for example on $G = \mathbb{R}$, we will not be able to have a Borel measure with density $\sin(x)$. The problem is that it not a positive density. There are functions on $\mathbb{R}$ that would give a finite integral when integrated against the Lebesgue measure, but the integral of $\sin(x)$ over arbitrary measurable subsets need not exist. The solution is to consider functions h on $\mathbb{R}$ such that the integral of $h(x)\sin(x)$ exist, for example continuous functions of compact support. In this part, we follow a functional analytic approach to set up the theory of Radon measures on any group G that uses this solution. In that way we will be able to relate a functional to the density function $\sin(x)$ on $\mathbb{R}$.

We will consider the real-valued linear forms on $C_c(G)$, however, there are too many of them. We need some form of continuity to distinguish the special ones from the others. In [8, Definition 4.3.1] we find the following pair of definitions of a Radon measure.

2.1. Definition. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $C_c(G)$. The sequence (f_n) *converges to* 0 *in the sense of* $C_c(G)$ if there exists a compact set $K \subseteq G$ such that $\operatorname{supp}(f_n) \subseteq K$ for all n and f_n converges uniformly to 0 on G. We will denote this by $f_n \twoheadrightarrow 0$.

2.2. Definition. A *(real-valued) Radon measure on* G is a linear form $\varphi \colon C_c(G) \to \mathbb{R}$ that is continuous in the sense that $\lim_n \langle f_n, \varphi\rangle = 0$ for all $(f_n)_{n\in\mathbb{N}} \subseteq C_c(G)$ with $f_n \twoheadrightarrow 0$.

We introduced the notation $\langle f, \varphi\rangle = \varphi(f)$, a commonly used notation in linear functional analysis. We will also use $\langle \varphi, \mathrm{ev}_f\rangle$ when appropriate.

On the other hand, there is one obvious choice, let the Radon measures be the linear functionals on $C_c(G)$ that are continuous in the sense of the inductive limit topology. Let φ be such a linear functional. We will show that this is equivalent to the previous definitions. To show that, we have the following general theorem on continuity on inductive limit spaces.

2.3. Theorem [8, Theorem 6.3.2]. *Let X be a topological vector space and let $(Y_i)_{i\in I}$ be a family of topological vector spaces. Consider linear maps $\psi_i \colon Y_i \to X$ such that*

$$\bigcup_{i\in I} \psi_i(Y_i) = X.$$

Let (X, τ) be the inductive limit of the spaces Y_i relative to the maps ψ_i. Let $\chi \colon X \to Z$ be a linear map where Z is any locally convex topological vector space. Then χ is continuous if and only if $\chi \circ \psi_i \colon Y_i \to Z$ is continuous for all $i \in I$.

In case $Z = \mathbb{R}$ we find that $\chi \circ \psi_i \colon Y_i \to \mathbb{R}$ is continuous if and only if $\chi \colon X \to \mathbb{R}$ is continuous. Now consider our setting. Let $(K_i)_{i\in I}$ be all compact subsets of G. Then, for the internal inductive limit

$C_c(G)$ of the $C_c(K_i, G)$ relative to the inclusion maps $C_c(K_i, G) \to C_c(G)$, any map $\varphi\colon C_c(G) \to \mathbb{R}$ is continuous if and only if the map $C_c(K_i, G) \to \mathbb{R}$ is continuous for all $i \in I$.

We conclude that any map $\varphi\colon C_c(G) \to \mathbb{R}$ is continuous if and only if the map $C_c(K, G) \to \mathbb{R}$ is continuous (with respect to the topology on $C_c(K, G)$ induced by the supremum norm restricted to K) for all compact $K \subseteq G$. That leads to the following proposition, summing up the results so far.

2.4. Proposition. *A linear functional $\varphi\colon C_c(G) \to \mathbb{R}$ is continuous in the sense of the inductive limit topology if and only if for every compact $K \subseteq G$ there exists a constant M_K such that*

$$|\langle f, \varphi\rangle| \leqslant M_K \|f\|_\infty \tag{1}$$

for all $f \in C_c(G)$ with $\operatorname{supp}(f) \subseteq K$.

The conditions on f mean precisely that f must be an element of $C_c(K, G)$ for which (1) must hold for any compact $K \subseteq G$. As stated before, the topology of $C_c(K, G)$ is the topology of uniform convergence. We continue with the following result.

2.5. Proposition. *The statements of Definition 2.2 and* (1) *are equivalent.*

Proof. First, suppose that φ satisfies (1) and let $(f_n)_{n\in\mathbb{N}} \subseteq C_c(G)$ be such that $f_n \twoheadrightarrow 0$. Let $K \subseteq G$ be a compact set such that $f_n \twoheadrightarrow 0$. Let $\varepsilon > 0$. Then, by assumption, there exists some constant M_K such that

$$|\langle f_n, \varphi\rangle| \leqslant M_K \|f_n\|_\infty.$$

for all $n \in \mathbb{N}$. Since the sequence (f_n) converges uniformly to 0, there exists some index N such that

$$|f_n| \leqslant \frac{\varepsilon}{M_K}$$

for each $n \geqslant N$. Thus, for all $n \geqslant N$ we have $|\langle f_n, \varphi\rangle| \leqslant \varepsilon$. Thus $\lim_n \langle f_n, \varphi\rangle = 0$.

Second, suppose that φ satisfies Definition 2.2. Let $K \subseteq G$ be compact and let $f \in C_c(K, G)$. Since $C_c(K, G)$ is a Banach space with the supremum norm, suppose that $(f_n)_{n\in\mathbb{N}} \subseteq C_c(K, G)$ converges uniformly to f. Then also $f_n - f \twoheadrightarrow 0$ in $C_c(G)$. Now it holds, by assumption, that $\lim_n \langle f_n - f, \varphi\rangle = 0$. Hence $\lim_n \langle f_n, \varphi\rangle = \langle f, \varphi\rangle$. This means that φ is continuous in the sense of a linear functional on a normed space, hence there exists a constant M_K such that

$$|\langle f, \varphi\rangle| \leqslant M_K \|f\|_\infty.$$

□

Thus, the type of continuity introduced in Definition 2.2 is equivalent to the one in Theorem 2.3. From now on, we use (1) as definition of a Radon measure. A Radon measure is called *positive* when the functional φ itself is positive.

Via the Riesz Representation Theorem, we know that any positive Radon measure (a positive functional φ on $C_c(G)$) corresponds uniquely to a regular Borel measure as we know it from Measure Theory. That is, for any positive Radon measure φ there exists a unique positive regular Borel measure μ such that

$$\langle f, \varphi\rangle = \int f\, d\mu = \int f(x)\, d\mu(x)$$

for $f \in C_c(G)$. We will use μ to denote both φ and μ as in the previous sense.

Note that, when considering complex-valued continuous functions with compact support, the corresponding complex-valued linear functional φ decomposes uniquely into the form $\varphi_r + i\varphi_i$, where both φ_r, φ_i are real-valued linear functionals. Once more, this confirms our choice to consider the real-valued measures only, as the behavior of the original complex-valued measure depends completely on the combined behavior of the corresponding real-valued measures. The following theorem shows that we are able to decompose such a (real) Radon measure into its positive parts.

2.6. Theorem [8, Theorem 4.3.2]. *Let φ be a Radon measure on G. There exist unique positive Radon measures φ^+ and φ^- on G such that*

1. *$\varphi = \varphi^+ - \varphi^-$, and*
2. *if φ_1, φ_2 are positive Radon measures on G with $\varphi = \varphi_1 - \varphi_2$, then $\varphi_1 - \varphi^+$ and $\varphi_2 - \varphi^-$ are positive Radon measures on G.*

We call this the *minimal decomposition of* φ, and we will call φ^+ and φ^- the *positive* and *negative parts of* φ. We call $|\varphi| = \varphi^+ + \varphi^-$ the *modulus of* φ and it is clear that $|\varphi|$ is again a Radon measure on G.

2.4 Spaces of Radon measures

We will define some spaces consisting of Radon measures and their topologies. First of all, let $M(G)$ denote the space of all Radon measures on G; by our construction it is the topological dual of $C_c(G)$. We endow $M(G)$ with the *vague topology of measures* denoted by

$$\sigma(M(G), C_c(G)) = \sigma(C_c(G)^*, C_c(G)),$$

which is the weak*-topology such that the map $\mu \mapsto \langle f, \mu \rangle$ is continuous for all $f \in C_c(G)$. A sequence $(\mu_n)_{n\in\mathbb{N}} \subseteq M(G)$ *converges vaguely* to μ if $\langle f, \mu_n \rangle \to \langle f, \mu \rangle$ for all $f \in C_c(G)$. In this setting for a set $V \subseteq M(G)$ to be bounded means that V is *vaguely bounded*, that is,

$$\sup_{\mu \in V} |\langle f, \mu \rangle| < \infty$$

for all $f \in C_c(G)$.

Let $\mu \in M(G)$. Using the minimal decomposition of μ, we have unique positive functionals μ^+ and μ^- such that $\mu = \mu^+ - \mu^-$. Via the Riesz Representation Theorem, μ^+ and μ^- are regular Borel measures on G. Let $M_b(G) \subseteq M(G)$ be the subspace of Radon measures such that both μ^+ and μ^- are bounded regular Borel measures, that is,

$$\|\mu\| := |\mu|(G) = \mu^+(G) + \mu^-(G) < \infty.$$

Using the Riesz Representation Theorem, we know that $(M_b(G), \|\cdot\|) = (C_0(G), \|\cdot\|_\infty)^*$. Hence it is a Banach space.

2.5 Spaces of integrable functions

Now that we know what a Radon measure is, we will move forward to explore its usage. We will end up with some special class of real-valued functions on G, not necessarily of compact support or even continuous. We will follow the work of Edwards in [8, §4.5,§4.6,§4.11]. We skip the proofs, but will summarize results and definitions needed. Alternatively, one may follow the work of Nachbin in [15, §1.8].

To that end, for this section, let μ be a fixed positive Radon measure (functional) on G. In this section we will, amongst others, define what $\mu(A)$ means for any $A \subseteq G$, getting close to the fact that this linear functional μ from the previous part is indeed a measure in the measure theoretic sense. Let $\Phi(G)$ denote the set of positive lower semicontinuous functions on G. Any lower semicontinuous function may actually have values in $\overline{\mathbb{R}}$, the extended real numbers. We assume that such a function never attains the value $-\infty$. Furthermore, all continuous real-valued functions on G are lower semicontinuous. For any $\varphi \in \Phi(G)$ we define the supremum[1]

$$\mu'(\varphi) = \sup\{\langle k, \mu\rangle \colon k \in C_c(G), k \leqslant \varphi\},$$

which is possibly ∞ and always positive since both μ and φ are. This supremum is indeed an extension, since $\mu'(f) = \langle f, \mu\rangle$ for all $f \in C_c(G)$. It is easy to see that $\mu'(c\varphi) = c\mu'(\varphi)$ for all $c \in \mathbb{R}_{>0}$ and $\varphi \in \Phi(G)$ and that $\mu'(\varphi_1) \leqslant \mu(\varphi_2)$ for $\varphi_1, \varphi_2 \in \Phi(G)$ with $\varphi_1 \leqslant \varphi_2$. We will actually extend μ' a bit more, namely to any function f on G taking values in $[0, \infty] \subseteq \overline{\mathbb{R}}$ by the infimum[2]

$$\mu^*(f) = \inf\{\mu'(\varphi) \colon \varphi \in \Phi(G), f \leqslant \varphi\}.$$

This infimum is also an extension, since $\mu^*(\varphi) = \mu'(\varphi)$ for all $\varphi \in \Phi(G)$. Let f_1, f_2 be defined on G taking values in $[0, \infty] \subseteq \overline{\mathbb{R}}$ and let any constant $c \geqslant 0$ be given. Then it follows that $0 \leqslant \mu^*(f_1) \leqslant \infty$ and $\mu^*(f_1 + f_2) \leqslant \mu^*(f_1) + \mu^*(f_2)$ and $\mu^*(cf_1) = c\mu^*(f_1)$. If $f_1 \leqslant f_2$ it also holds that $\mu^*(f_1) \leqslant \mu^*(f_2)$. That is, μ^* is a sublinear function on $[0, \infty]^G$. Furthermore, it is immediate from the above construction that

$$\mu^*(f) = \mu'(f) = \langle f, \mu\rangle$$

holds for $f \in C_c(G) \subseteq \Phi(G)$.

The next step is to use μ^*, which is now defined for any positive function on G, to construct a seminorm. For any map $f \in \mathbb{R}^G$ we define for $1 \leqslant p < \infty$ the value

$$N_p(f) = \mu^*(|f|^p)^{\frac{1}{p}}.$$

Let $\mathcal{F}^p = \{f \in \mathbb{R}^G \colon N_p(f) < \infty\}$. The aforementioned properties imply that $\mathcal{F}^p$ is a vector space with seminorm N_p (for $1 < p < \infty$ this proof depends on Minkowski's inequality). We will show that

$$|\langle f, \mu\rangle| \leqslant N_1(f) < \infty.$$

holds for any $f \in C_c(G)$. Let $f \in C_c(G)$ and let $f = f^+ - f^-$ where $f^+ \geqslant 0$ is the positive part of f and $f^- \geqslant 0$ is the negative part of f. It is obvious that f^+, f^- and $|f|$ are elements of $C_c(G)$. Then, by linearity and positivity of μ

$$|\langle f, \mu\rangle| \leqslant |\langle f^+, \mu\rangle| + |\langle f^-, \mu\rangle| = \langle |f|, \mu\rangle = \mu^*(|f|) = N_1(f).$$

We define $\mathcal{L}^p(G, \mu)$ to be the closure of $C_c(G)$ relative to the seminorm N_p on $\mathcal{F}^p$. The functions in $\mathcal{L}^p(G, \mu)$ are said to be *p-fold μ-integrable*, or just *μ-integrable* for $p = 1$. For any set $A \subseteq G$ we may abbreviate $\mu^*(\chi_A)$ to $\mu^*(A)$. By [8, §4.7.1], any relatively compact open set or any compact set is integrable. Moreover, by [8, Lemma 4.6.2] the values of μ' and μ coincide on positive lower semicontinuous functions on G, hence surely on open sets and on relatively compact open sets. Furthermore, if f is integrable, then $|f|$ is also integrable.

[1] Note that μ' is not necessarily a functional, so we will not use the notations $\langle \cdot, \cdot \rangle$ or $\int \cdot \, d\cdot$.

[2] This μ^* is in fact an outer measure on G. However, we won't use (and won't copy) the suggestive notation suggested by Edwards [8, Remark following Theorem 4.8.1].

Any map $f \in \mathcal{F}^1$ is called *μ-negligible* if $N_1(f) = 0$. Any set $A \subseteq G$ for which χ_A is μ-negligible is called a *μ-negligible set.* A subset $A \subseteq G$ is called *locally negligible for μ* if $A \cap K$ is μ-negligible for each compact set $K \subseteq G$. Any property P on G is then true *(locally) μ-almost everywhere* if the subset of G where P does not hold is (locally) negligible for μ. We will abbreviate this to *μ-a.e.* (*μ-l.a.e.*). For more details, see [8, §4.7.3].

The seminorm N_p on $\mathcal{F}^p$ is certainly not a norm, there are in general multiple functions $f \in \mathcal{L}^p(G)$ that satisfy $N_p(f) = 0$ but $f \neq 0$. For more details, see [8, §4.11.10]. Let $\mathcal{N}$ be the set

$$\{f \in \mathcal{L}^p(G) \colon N_p(f) = 0\}$$

and we consider the quotient space $L^p(G, \mu) := \mathcal{L}^p(G, \mu)/\mathcal{N}$. Thus $L^p(G, \mu)$ is a collection of classes of functions on G that are μ-a.e. equal. We will not distinguish between a function in $\mathcal{L}^p(G, \mu)$ and its class modulo μ-negligible functions in $L^p(G, \mu)$ as is common and broadly accepted in the literature. We will also use the term integrable for a function f that is defined almost everywhere on G (possibly infinite at some points of the domain), if there exists a function g, defined and finite everywhere, and integrable as previously defined, such that $f = g$ almost everywhere. Furthermore, their integrals have the common value $\langle f, \mu\rangle = \langle g, \mu\rangle < \infty$, see [8, §4.6.11].

In addition to the above, we will call $f \in \mathbb{R}^G$ *locally μ-integrable* if $f\chi_K$ is μ-integrable for each compact $K \subseteq G$. Let $L^1_{loc}(G, \mu)$ denote the set of functions in $\mathbb{R}^G$ that are locally μ-integrable. For $1 \leqslant p < \infty$ it holds that $L^p(G, \mu) \subseteq L^1_{loc}(G, \mu)$. Let $K \subseteq G$ be compact, then $\chi_K \in L^q(G)$ for all $1 \leqslant q < \infty$ by construction. Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. It follows from Hölders Inequality [8, §4.11.2] that $N_1(f\chi_K) \leqslant N_p(f)N_q(\chi_K) < \infty$, as required.

For any function f defined almost everywhere on G, we define the *essential supremum of f (relative to ν)* by

$$\begin{aligned}\operatorname{ess\,sup} f &= \inf\{M \in \mathbb{R}_{>0} \colon |f(x)| \leqslant M \text{ for } \nu\text{-almost every } x \in G\}\\ &= \inf\{M \in \mathbb{R}_{>0} \colon \nu(\{x \in G \colon |f(x)| > M\}) = 0\}.\end{aligned}$$

For every $f \in C(G)$ it is true that $\sup|f| = \operatorname{ess\,sup} f$.

2.6 Product of a measure by a function

Let μ be a positive Radon measure on G and let $g \in L^1_{loc}(G, \mu)$ be given. It is easy to see that g is locally μ-integrable if and only if the pointwise product gf is μ-integrable for each *positive* $f \in C_c(G)$. Then gf is μ-integrable for each $f \in C_c(G)$ and, in that case, we construct a new Radon measure μ_g by defining

$$\langle f, \mu_g\rangle = \langle fg, \mu\rangle$$

with $f \in C_c(G)$. We will write $\mu_g = g \cdot \mu$, sometimes called the *measure with density g with respect to μ*. The measure μ_g is again positive if and only if g is μ-l.a.e. positive. That is, if and only if $g\chi_K$ is μ-a.e. positive for all compact $K \subseteq G$.

It is clear that we can write $g = g^+ - g^-$ with $g^-, g^+ \in \mathbb{R}^G$ positive. For μ_g and $f \in C_c(G)$ we find

$$\langle f, \mu_g\rangle = \langle fg, \mu\rangle = \langle fg^+ - fg^-, \mu\rangle = \langle fg^+, \mu\rangle - \langle fg^-, \mu\rangle = \langle f, \mu_{g^+}\rangle - \langle f, \mu_{g^-}\rangle.$$

This suggests that $\mu_g = \mu_{g^+} - \mu_{g^-} = g^+ \cdot \mu - g^- \cdot \mu$ where μ_{g^-} and μ_{g^+} are now positive. As a result of the Lebesgue-Radon-Nikodym Theorem [8, §4.13.1, §4.15.1-§4.15.3] it follows that this is indeed the minimal decomposition into positive parts. As a result of this we find, for $f \in C_c(G)$ and g being locally μ-integrable,

$$\begin{aligned}\langle f, |\mu_g| \rangle &= \left\langle f, \mu_{g^+} + \mu_{g^-} \right\rangle \\ &= \left\langle f, \mu_{g^+} \right\rangle + \left\langle f, \mu_{g^-} \right\rangle \\ &= \left\langle f g^+, \mu \right\rangle + \left\langle f g^-, \mu \right\rangle \\ &= \left\langle f g^+ + f g^-, \mu \right\rangle \\ &= \langle f|g|, \mu \rangle \\ &= \left\langle f, \mu_{|g|} \right\rangle.\end{aligned}$$

Let γ be defined on the set of locally ν-integrable functions by sending such function f to the measure $\nu_f \in M(G)$. It is routine to verify that γ is linear. Furthermore, it is clear from the above that γ is positive. We end with a small result on the continuity of γ, which is only needed on $L^p(G) \subseteq L^1_{loc}(G)$ for $1 \leqslant p < \infty$.

2.7. Proposition. *Let $p \in [1, \infty)$. The map $\gamma' \colon L^p(G) \to M(G), f \mapsto \nu_f$ is continuous.*

Proof. Let $p \in [1, \infty)$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $(f_n)_{n\in\mathbb{N}} \subseteq L^p(G)$ be converging to $f \in L^p(G)$. It follows, using Hölders Inequality [8, §4.11.2], that for all $g \in C_c(G)$

$$\left|\int f_n g \, d\nu - \int f g \, d\nu\right| = \left|\int (f_n - f) g \, d\nu\right| \leqslant \int |f_n - f||g| \, d\nu = N_1((f_n - f)g) \leqslant N_p(f_n - f) N_q(g).$$

Since $f_n \to f$ in the sense of $L^p(G)$, it follows that $N_p(f_n - f) \to 0$ and $N_q(g) < \infty$ since g is of compact support, hence

$$\left|\int f_n g \, d\nu - \int f g \, d\nu\right| \to 0$$

and then also

$$\langle g, \nu_{f_n} \rangle = \int f_n g \, d\nu \to \int f g \, d\nu = \langle g, \nu_f \rangle$$

for every $g \in C_c(G)$. We conclude that ν_{f_n} converges vaguely to ν_f, that is, γ' is continuous. □

2.7 Vector-valued integration

In the previous parts we have set up integration of scalar-valued functions with respect to some Radon measure, this is sometimes called *scalar-valued integration*. Aside from that we will also consider a variation called *vector-valued integration* where we integrate a vector-valued function f a real-valued measure μ. The following definition gives the precise statement.

2.8. Definition [19, Definition 3.26]. Let $(Q, \mathcal{B}(Q))$ be a measurable space and let μ be a real-valued measure on Q. Let X a topological vector space such that X^* separates the points of X (that is, for $x_1, x_2 \in X$ with $x_1 \neq x_2$ there exists $x^* \in X^*$ such that $\langle x_1, x^* \rangle \neq \langle x_2, x^* \rangle$). Let $f \in X^Q$ be such that the scalar-valued functions $\langle f, x^* \rangle$ on Q are μ-integrable for every $x^* \in X^*$. Note that $\langle f, x^* \rangle$ is a real-valued function on Q defined by

$$\langle f, x^* \rangle(q) = \langle f(q), x^* \rangle,$$

with $q \in Q$. If there exists a vector $x \in X$ such that

$$\langle x, x^* \rangle = \int_Q \langle f, x^* \rangle \, d\mu = \int_Q \langle f, x^* \rangle(q) \, d\mu(q) = \int_Q \langle f(q), x^* \rangle \, d\mu(q) \tag{2}$$

for every $x^* \in X^*$, then we define

$$x = \int_Q f \, d\mu = \int_Q f(q) \, d\mu(q).$$

2.9. Remark. The existence of the vector x is proven in the next theorem only considering a specific case and is based on [19, Theorem 3.27] and its preceding remark. The uniqueness of x follows directly from the assumption that X^* separates the points of X. Suppose that there exists $x_1, x_2 \in X$ with $x_1 \neq x_2$ such that

$$\langle x_1, x^* \rangle = \int_Q \langle f, x^* \rangle \, d\mu \text{ and } \langle x_2, x^* \rangle = \int_Q \langle f, x^* \rangle \, d\mu$$

for all $x^* \in X^*$. Since X^* separates the points of X, there exists $x_s^* \in X^*$ such that

$$\langle x_1, x_s^* \rangle \neq \langle x_2, x_s^* \rangle,$$

which is a contradiction.

2.10. Theorem. *Let X be a Banach space. Let μ be a probability measure on a compact Hausdorff space Q. If $f \colon Q \to X$ is a continuous map, then the vector-valued integral*

$$x = \int_Q f \, d\mu$$

exists in X, as defined in Definition 2.8.

In our application, a well chosen Radon measure will be used instead of an arbitrary probability measure. In particular, the theorem holds for any Radon measure $\tilde{\mu}$ that is finite on Q as $\tilde{\mu}$ only differs by a scalar multiple from a probability measure. The next corollary extends the result.

2.11. Corollary. *If, combined with the assumptions of Theorem 2.10, Y is a topological vector space such that Y^* separates the points of Y and $T \colon X \to Y$ is a continuous linear map, then*

$$\int_Q Tf \, d\mu$$

exists as a vector-valued integral in Y, and

$$\int_Q Tf \, d\mu = T \int_Q f \, d\mu.$$

Proof. By Theorem 2.10 the integral

$$x = \int_Q f \, d\mu$$

exists in X such that (2) holds for every $x^* \in X^*$. Now consider $Tx \in Y$ and let $y^* \in Y^*$. Then $T^* y^* \in X^*$ since T is continuous. Then

$$\langle Tx, y^* \rangle = \langle x, T^* y^* \rangle \overset{(2)}{=} \int_Q \langle f, T^* y^* \rangle \, d\mu = \int_Q \langle Tf, y^* \rangle \, d\mu,$$

where

$$\langle Tf, y^* \rangle(q) = \langle (Tf)(q), y^* \rangle = \langle T(f(q)), y^* \rangle,$$

for $q \in Q$. Thus, by definition

$$\int_Q Tf \, d\mu$$

exists as a vector-valued integral in Y and equals Tx. □

3 Haar measure

There is one special measure on G that will play an important role throughout this thesis. In this part we will follow the approach of Nachbin in [15, §2.4]. Nachbin has actually given two detailed proofs of Haar's Theorem. In §2.8 the first proof is given. This proof, following the work of André Weil, is based on the original idea for existence of Haar but applies Haar's essential idea to integrals instead of measures. The most important argument uses Tychonoff's Theorem, which depends on the Axiom of Choice. The uniqueness is done according to von Neumann. The second proof in §2.9 is following the approach of Henri Cartan. Using Cauchy's convergence criterion Cartan establishes existence and uniqueness simultaneously, while not relying on the Axiom of Choice or the work of von Neumann.

We will need one extra definition before we are able to state this important theorem. A positive Radon measure $\mu \in M(G)$ is called *left* (resp. *right*) *invariant* if for every $f \in L^1(G,\mu)$ and all $a \in G$ the map $x \mapsto f(a^{-1}x) \in L^1(G,\mu)$ (resp. $x \mapsto f(xa^{-1}) \in L^1(G,\mu)$) and

$$\int f\big(a^{-1}x\big)\,d\mu(x) = \int f(x)\,d\mu(x)\,\left(\text{resp. } \int f\big(xa^{-1}\big)\,d\mu(x) = \int f(x)\,d\mu(x)\right).$$

3.1. Theorem [15, Theorem 2.4.1]. *On every locally compact group G there exists at least one left invariant positive Radon measure $\mu \neq 0$. If $\tilde{\mu} \neq 0$ is another left invariant positive Radon measure there exists some $c \in \mathbb{R}_{>0}$ such that $\mu = c\tilde{\mu}$.*

Following this theorem, a *left* (resp. *right*) *Haar measure* on G is a left (resp. right) invariant positive Radon measure $0 \neq \mu \in M(G)$.

If μ is a left invariant Haar measure on G we define for each $a \in G$ a positive linear functional $\tilde{\mu}$ on $C_c(G)$ by

$$\langle f, \tilde{\mu}\rangle := \int f\big(xa^{-1}\big)\,d\mu(x).$$

Then $\tilde{\mu}$ is again a left Haar measure since the multiplication of G is associative and μ is left invariant. By Theorem 3.1 there exists $\Delta(a) \in \mathbb{R}_{>0}$ such that $\tilde{\mu} = \Delta(a)\mu$. In this way we obtain a map $\Delta\colon G \to \mathbb{R}_{>0}$. By the second part of Theorem 3.1 we find that this map is independent of the left Haar measure as it would be proportional to μ, resulting in the same value for $\Delta(a)$. We call Δ the *(right) modular function* or *(right) Haar modulus* of G. Sometimes Δ is also called the *right-hand modulus* of G. The following proposition shows that Δ is actually a continuous homomorphism.[3]

3.2. Proposition [15, Proposition 2.5.7]. *The map $\Delta\colon G \to \mathbb{R}_{>0}$ is a continuous group homomorphism when $\mathbb{R}_{>0}$ is given its usual group structure. Moreover, if μ is a left Haar measure on G, for any $f \in L^1(G,\mu)$ and any $a \in G$ it holds that*

$$\int f\big(xa^{-1}\big)\,d\mu(x) = \Delta(a)\int f(x)\,d\mu(x). \tag{Δ}$$

There is one property of the Haar measure and this modular function that we will use in the next part. The proof goes via the right Haar measure and is omitted here.

[3] We follow the work and definition of André Weil, however note that there are other authors that use another definition of Δ. For example, $\Delta(a)$ in [4] is our and Weil's $\Delta\big(a^{-1}\big) = \Delta(a)^{-1}$.

3.3. Proposition [15, Proposition 2.5.8]. *If* μ *is a left Haar measure on* G *then*

$$\int f\big(x^{-1}\big)\, d\mu(x) = \int \frac{f(x)}{\Delta(x)}\, d\mu(x)$$

for every $f \in C_c(G)$*. In fact, both sides of this equality define the same right Haar measure.*

3.4. Remark [15, p. 79, 80]. The equality in the previous proposition could be formulated as

$$\int f\big(x^{-1}\big)\Delta\big(x^{-1}\big)\, d\mu(x) = \int f(x)\, d\mu(x)$$

by substitution of $f\Delta$ for f in the proposition. In [15] it is shown that this statement holds for $f \in L^1(G)$ instead of only $f \in C_c(G)$.

It depends on the group G whether or not the left Haar measure can be calculated more explicitly. In this thesis we don't need any additional information on the left Haar measure. Thus we may assume, by Theorem 3.1, that we have a left Haar measure on G; this fixed left Haar measure will be denoted by ν. Note that the measure ν is necessarily unbounded when G is not compact. When we consider the Haar measure we will write $L^p(G)$ instead of $L^p(G,\nu)$. In the same way, we use the terms *(locally) negligible*, *(locally) almost everywhere* and *(p-fold) integrable* instead. Furthermore, we will abbreviate (for example for $f \in L^1(G)$) the so called *Haar integral*

$$\int f(x)\, d\nu(x) \text{ to } \int f\, dx.$$

The group G is called *unimodular* if $\Delta(a) = 1$ for all $a \in G$. This is equivalent to ν being both left and right invariant. Both abelian and compact groups are known to be unimodular, for abelian groups this follows immediately from (Δ). For compact groups this follows since $\chi_G \in C_c(G)$, hence $\langle \chi_G, \nu\rangle = \int \chi_G\, d\nu > 0$. With ($\Delta$) it follows that $\Delta(a) = 1$.

Note that we started with a left invariant Haar measure and that we found a right-hand modulus of G. In the same way, if one starts with a right invariant Haar measure, one ends up with a left-hand modulus of G. It turns out that they are inverses of each other in the multiplicative group $\mathbb{R}_{>0}$. We will mainly consider the first case, as we already did.

4 Translation operators

In this section we follow the work of Nachbin again while investigating translations of functions. After that, we will consider translations of measures.

4.1 Translation of functions

Let $a \in G$ and define the *left* and *right translation operators* on $\mathbb{R}^G$ to be the maps

$$\lambda_a : \mathbb{R}^G \to \mathbb{R}^G, \lambda_a f(x) = f\big(a^{-1}x\big)$$

and

$$\rho_a : \mathbb{R}^G \to \mathbb{R}^G, \rho_a f(x) = f\big(xa^{-1}\big).$$

4.1. Definition. Let $f \in \mathbb{R}^G$ be given. For any $\varepsilon \in \mathbb{R}_{>0}$, we call any neighborhood V of e a *neighborhood of left* (resp. *right*) *ε-uniformity of f* if it holds for $x^{-1}y \in V$ (resp. $yx^{-1} \in V$) that $|f(x) - f(y)| \leqslant \varepsilon$. A *neighborhood of ε-uniformity* is a neighborhood that is of both left and right ε-uniformity. Then, a map $f \in \mathbb{R}^G$ is called *left* (resp. *right*) *uniformly continuous* if for every $\varepsilon \in \mathbb{R}_{>0}$ there is a neighborhood of left (resp. right) ε-uniformity. Furthermore, f is called *uniformly continuous* if it is both left and right uniformly continuous.

4.2. Proposition [15, Proposition 2.3.1]. *Every $f \in C_c(G)$ is uniformly continuous.*

In the next proposition we will show that the maps

$$G \to C_c(G), x \mapsto \lambda_x f \tag{3}$$

and

$$G \to C_c(G), x \mapsto \rho_x f \tag{4}$$

are continuous for every $f \in C_c(G)$. To prove this, we use the next small result about these translation operators. To that end, let $f \in C_c(G)$ and let $(\varepsilon_\alpha)_{\alpha\in\mathbb{A}} \subseteq \mathbb{R}_{>0}$ be a net converging to 0. Since f is uniformly continuous, there exists an ε_α-uniformity V_α of f for each $\alpha \in \mathbb{A}$. Then $(V_\alpha)_{\alpha\in\mathbb{A}} \subseteq \mathcal{N}_e$ is a net that converges to $\{e\}$ when $\varepsilon_\alpha \to 0$. Let $\alpha \in \mathbb{A}$ be given. If $y_\alpha x^{-1} \in V_\alpha$ it holds for all $z \in G$ that

$$\big|\big(\lambda_{y_\alpha x^{-1}} f - f\big)(y_\alpha z)\big| = |f(xz) - f(y_\alpha z)| \leqslant \varepsilon_\alpha$$

and if $x^{-1}y_\alpha \in V_\alpha$ that

$$\big|\big(\rho_{x^{-1}y_\alpha} f - f\big)(z y_\alpha)\big| = |f(zx) - f(z y_\alpha)| \leqslant \varepsilon_\alpha.$$

These statements are true for all $z \in G$, so we may substitute z with $y_\alpha^{-1}z$ to find in the first case

$$\big|\big(\lambda_{y_\alpha x^{-1}} f - f\big)(z)\big| \leqslant \varepsilon_\alpha$$

and we substitute z with zy_α^{-1} in the second case to find

$$\big|\big(\rho_{x^{-1}y_\alpha} f - f\big)(z)\big| \leqslant \varepsilon_\alpha.$$

Hence $\big\|\lambda_{y_\alpha x^{-1}} f - f\big\|_\infty \leqslant \varepsilon_\alpha$ if $y_\alpha x^{-1} \in V_\alpha$ and $\big\|\rho_{x^{-1}y_\alpha} f - f\big\|_\infty \leqslant \varepsilon_\alpha$ if $x^{-1}y_\alpha \in V_\alpha$. If we take the limit $\varepsilon_\alpha \to 0$ we find that $y_\alpha x^{-1} \in V_\alpha \to \{e\}$ (in the other case $x^{-1}y_\alpha \in V_\alpha \to \{e\}$), or equivalently, (in both cases) $y_\alpha \to x$ in G. It follows that $\big\|\lambda_{y_\alpha x^{-1}} f - f\big\|_\infty \to 0$ and $\big\|\rho_{x^{-1}y_\alpha} f - f\big\|_\infty \to 0$ when $y_\alpha \to x$. From now on, we will use this result for $f \in C_c(G)$ and $x \in G$ by loosely taking $y \in U$ for some neighborhood U of x and consider the limit $y \to x$.

4.3. Proposition. *The maps in* (3) *and in* (4) *are continuous for every* $f \in C_c(G)$*, where* $C_c(G)$ *has the inductive limit topology.*

Proof. Let $f \in C_c(G)$ and let $x \in G$. If $y \in U \in \mathcal{N}_x$ we find

$$\begin{aligned}\|\lambda_y f - \lambda_x f\|_\infty &= \sup_{z\in G}|\lambda_y f(z) - \lambda_x f(z)| \\ &= \sup_{z\in G}|\lambda_x f(z) - \lambda_y f(z)| \\ &= \sup_{z\in G}\left|f\left(x^{-1}z\right) - f\left(y^{-1}z\right)\right| \\ &= \sup_{z\in G}\left|\left(\lambda_{y^{-1}x} f - f\right)\left(y^{-1}z\right)\right|.\end{aligned}$$

By the previous result, this means that $\|\lambda_y f - \lambda_x f\|_\infty \to 0$ for $y^{-1} \to x^{-1}$. Since $y^{-1} \to x^{-1}$ is equivalent with $y \to x$, we have $\lambda_y f \to \lambda_x f$ for $y \to x$. Similarly, we see that

$$\begin{aligned}\|\rho_y f - \rho_x f\|_\infty &= \sup_{z\in G}|\rho_y f(z) - \rho_x f(z)| \\ &= \sup_{z\in G}|\rho_x f(z) - \rho_y f(z)| \\ &= \sup_{z\in G}\left|f\left(zx^{-1}\right) - f\left(zy^{-1}\right)\right| \\ &= \sup_{z\in G}\left|\left(\rho_{xy^{-1}} f - f\right)\left(zy^{-1}\right)\right|.\end{aligned}$$

By the previous result, this means that $\|\rho_y f - \rho_x f\|_\infty \to 0$ for $y^{-1} \to x^{-1}$. Hence $\rho_y f \to \rho_x f$ for $y \to x$. We conclude that both maps are continuous with respect to the inductive limit topology. $\square$

We end with a small result about translations on $L^p(G)$ spaces. For $f \in L^p(G)$ with $p \in [1, \infty)$ and $a \in G$ we have

$$\|\lambda_a f\|_p^p = \int \left|f\left(a^{-1}x\right)\right|^p dx = \int |f(x)|^p dx = \|f\|_p^p$$

thus $\|\lambda_a f\|_p = \|f\|_p$ and

$$\|\rho_a f\|_p^p = \int \left|f\left(xa^{-1}\right)\right|^p dx \overset{(\Delta)}{=} \Delta(a) \int |f(x)|^p dx = \Delta(a)\|f\|_p^p$$

hence $\|\rho_a f\|_p = \Delta(a)^{\frac{1}{p}}\|f\|_p$.

4.2 Translation of measures

We will also extend the definitions of translation operators on $\mathbb{R}^G$ to translation operators on the measure space $M(G)$, denoted by the maps

$$\lambda_a \colon M(G) \to M(G), (\lambda_a \mu)(f) = \langle \lambda_{a^{-1}} f, \mu \rangle$$

and

$$\rho_a \colon M(G) \to M(G), (\rho_a \mu)(f) = \langle \Delta(a) \rho_{a^{-1}} f, \mu \rangle$$

for $f \in C_c(G)$. These are well defined, since $\lambda_{a^{-1}} f$ and $\Delta(a)\rho_{a^{-1}} f$ are elements of $C_c(G)$ for all $f \in C_c(G)$. Before we formulate and prove some other properties of these translation operators, we first show that the definition of λ_a and ρ_a on $M(G)$ is in fact an extension of the previous definition. We show that it does not matter in which order the measures and functions are translated.

4.4. Proposition. *For $a \in G$ and $f \in C_c(G)$ it holds that*

$$\lambda_a \nu_f = \nu_{\lambda_a f} \text{ and } \rho_a \nu_f = \nu_{\rho_a f}.$$

Proof. Let $a \in G$ and $f \in C_c(G)$. Then both f and $\rho_a f$ are locally ν-integrable. Let $h \in C_c(G)$, using the left invariance of ν we find that

$$\begin{aligned}
\langle h, \lambda_a \nu_f \rangle &= \langle \lambda_{a^{-1}} h, \nu_f \rangle \\
&= \langle f \lambda_{a^{-1}} h, \nu \rangle \\
&= \langle \{x \mapsto f(x) h(ax)\}, \nu \rangle \\
&= \int h(ax) f(x) \, dx \\
&= \int h(z) f\big(a^{-1} z\big) \, d\nu\big(a^{-1} z\big) \\
&= \int h(z) f\big(a^{-1} z\big) \, dz \\
&= \int h(z) \lambda_a f(z) \, dz \\
&= \langle \{z \mapsto h(z) \lambda_a f(z)\}, \nu \rangle \\
&= \langle h \lambda_a f, \nu \rangle \\
&= \langle h, \nu_{\lambda_a f} \rangle.
\end{aligned}$$

For the other equality, we proceed in the same way. Note that $\lambda_a f$ is also locally ν-integrable. For $h \in C_c(G)$ we find, using (Δ) that

$$\begin{aligned}
\langle h, \rho_a \nu_f \rangle &= \langle \Delta(a) \rho_{a^{-1}} h, \nu_f \rangle \\
&= \langle \Delta(a) f \rho_{a^{-1}} h, \nu \rangle \\
&= \langle \{x \mapsto \Delta(a) f(x) h(xa)\}, \nu \rangle \\
&= \Delta(a) \int h(xa) f(x) \, dx \\
&= \int h(x) f\big(x a^{-1}\big) \, dx && ((\Delta)) \\
&= \int h(x) \rho_a f(x) \, dx \\
&= \langle \{x \mapsto h(x) \rho_a f(x)\}, \nu \rangle \\
&= \langle h \rho_a f, \nu \rangle \\
&= \langle h, \nu_{\rho_a f} \rangle.
\end{aligned}$$

□

As before, the extended translation operators are again continuous.

4.5. Proposition. *The maps $x \mapsto \lambda_x \mu$ and $x \mapsto \rho_x \mu$ are continuous for $\mu \in M(G)$.*

Proof. Let $\mu \in M(G)$, then there are positive measures $\mu^+, \mu^- \in M(G)$ such that $\mu = \mu^+ - \mu^-$. We must show that for every $f \in C_c(G)$ the maps $x \mapsto \langle f, \lambda_x \mu \rangle$ and $x \mapsto \langle f, \rho_x \mu \rangle$ are continuous. Let $x \in G$ and let $N \in \mathcal{N}_x$ be a compact neighborhood of x, such N exists since G is locally compact. The set

$$K = x^{-1} \cdot \operatorname{supp}(f) \cup N^{-1} \cdot \operatorname{supp}(f)$$

is compact again since N^{-1} and $N^{-1} \cdot \operatorname{supp}(f)$ are both compact. We find for $y \in N$ that

$$
\begin{aligned}
|\langle f, \lambda_x\mu\rangle - \langle f, \lambda_y\mu\rangle| &= \left|\langle \lambda_{x^{-1}} f, \mu\rangle - \left\langle \lambda_{y^{-1}} f, \mu\right\rangle\right| \\
&= \left|\left\langle \lambda_{x^{-1}} f - \lambda_{y^{-1}} f, \mu\right\rangle\right| \\
&= \left|\left\langle \lambda_{x^{-1}} f - \lambda_{y^{-1}} f, \mu^+ - \mu^-\right\rangle\right| \\
&= \left|\left\langle \lambda_{x^{-1}} f - \lambda_{y^{-1}} f, \mu^+\right\rangle - \left\langle \lambda_{x^{-1}} f - \lambda_{y^{-1}} f, \mu^-\right\rangle\right| \\
&\leqslant \left|\left\langle \lambda_{x^{-1}} f - \lambda_{y^{-1}} f, \mu^+\right\rangle\right| + \left|\left\langle \lambda_{x^{-1}} f - \lambda_{y^{-1}} f, \mu^-\right\rangle\right| \\
&= \left\langle \left|\lambda_{x^{-1}} f - \lambda_{y^{-1}} f\right|, \mu^+\right\rangle + \left\langle \left|\lambda_{x^{-1}} f - \lambda_{y^{-1}} f\right|, \mu^-\right\rangle \\
&= \left\langle \left|\lambda_{x^{-1}} f - \lambda_{y^{-1}} f\right|, \mu^+ + \mu^-\right\rangle \\
&= \left\langle \left|\lambda_{x^{-1}} f - \lambda_{y^{-1}} f\right|, |\mu|\right\rangle \\
&= \int \left|\lambda_{x^{-1}} f(z) - \lambda_{y^{-1}} f(z)\right| d|\mu|(z) \\
&= \int_G |f(xz) - f(yz)|\, d|\mu|(z) \\
&= \int_K |f(xz) - f(yz)|\, d|\mu|(z) \\
&= \int_K \left|\left(\lambda_{yx^{-1}} f - f\right)(yz)\right| d|\mu|(z) \\
&\leqslant \int_K \left\|\lambda_{yx^{-1}} f - f\right\|_\infty d|\mu| \\
&= \left\|\lambda_{yx^{-1}} f - f\right\|_\infty |\mu|(K).
\end{aligned}
$$

Hence $|\langle f, \lambda_x\mu\rangle - \langle f, \lambda_y\mu\rangle| \to 0$ since $\left\|\lambda_{yx^{-1}} f - f\right\|_\infty \to 0$ for $y \to x$. In a similar reasoning, we consider $x \in G$ and $y \in N \in \mathcal{N}_x$ such that N is compact. We then find, using the compact sets

$$K_1 = \operatorname{supp}(f) \cdot x^{-1} \text{ and } K_2 = \operatorname{supp}(f) \cdot x^{-1} \cup \operatorname{supp}(f) \cdot N^{-1}$$

that

$$
\begin{aligned}
|\langle f, \rho_x\mu\rangle - \langle f, \rho_y\mu\rangle| &= \left|\langle \Delta(x)\rho_{x^{-1}} f, \mu\rangle - \left\langle \Delta(y)\rho_{y^{-1}} f, \mu\right\rangle\right| \\
&= \left|\left\langle \Delta(x)\rho_{x^{-1}} f - \Delta(y)\rho_{y^{-1}} f, \mu\right\rangle\right| \\
&= \left|\left\langle \Delta(x)\rho_{x^{-1}} f - \Delta(y)\rho_{y^{-1}} f, \mu^+ - \mu^-\right\rangle\right| \\
&= \left|\left\langle \Delta(x)\rho_{x^{-1}} f - \Delta(y)\rho_{y^{-1}} f, \mu^+\right\rangle - \left\langle \Delta(x)\rho_{x^{-1}} f - \Delta(y)\rho_{y^{-1}} f, \mu^-\right\rangle\right| \\
&\leqslant \left|\left\langle \Delta(x)\rho_{x^{-1}} f - \Delta(y)\rho_{y^{-1}} f, \mu^+\right\rangle\right| + \left|\left\langle \Delta(x)\rho_{x^{-1}} f - \Delta(y)\rho_{y^{-1}} f, \mu^-\right\rangle\right| \\
&= \left\langle \left|\Delta(x)\rho_{x^{-1}} f - \Delta(y)\rho_{y^{-1}} f\right|, \mu^+\right\rangle + \left\langle \left|\Delta(x)\rho_{x^{-1}} f - \Delta(y)\rho_{y^{-1}} f\right|, \mu^-\right\rangle \\
&= \left\langle \left|\Delta(x)\rho_{x^{-1}} f - \Delta(y)\rho_{y^{-1}} f\right|, \mu^+ + \mu^-\right\rangle \\
&= \left\langle \left|\Delta(x)\rho_{x^{-1}} f - \Delta(y)\rho_{y^{-1}} f\right|, |\mu|\right\rangle \\
&= \int \left|\Delta(x)\rho_{x^{-1}} f(z) - \Delta(y)\rho_{y^{-1}} f(z)\right| d|\mu|(z) \\
&= \int_G |\Delta(x) f(zx) - \Delta(y) f(zy)|\, d|\mu|(z) \\
&= \int_G |\Delta(x) f(zx) - \Delta(y) f(zx) + \Delta(y) f(zx) - \Delta(y) f(zy)|\, d|\mu|(z) \\
&\leqslant \int_G |\Delta(x) - \Delta(y)||f(zx)| + |\Delta(y)||f(zx) - f(zy)|\, d|\mu|(z)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_G |\Delta(x) - \Delta(y)||f(zx)|\, d|\mu|(z) + \int_G |\Delta(y)||f(zx) - f(zy)|\, d|\mu|(z) \\
&= \int_{K_1} |\Delta(x) - \Delta(y)||f(zx)|\, d|\mu|(z) + \int_{K_2} |\Delta(y)||f(zx) - f(zy)|\, d|\mu|(z) \\
&= \int_{K_1} |\Delta(x) - \Delta(y)||\rho_{x^{-1}} f(z)|\, d|\mu|(z) + \int_{K_2} |\Delta(y)|\big|\big(\rho_{x^{-1}y} f - f\big)(zy)\big|\, d|\mu|(z) \\
&\leqslant \int_{K_1} |\Delta(x) - \Delta(y)|\|\rho_{x^{-1}} f\|_\infty\, d|\mu| + \int_{K_2} |\Delta(y)|\big\|\rho_{x^{-1}y} f - f\big\|_\infty\, d|\mu| \\
&\leqslant |\Delta(x) - \Delta(y)||\mu|(K_1)\|\rho_{x^{-1}} f\|_\infty + |\mu|(K_2)|\Delta(y)|\big\|\rho_{x^{-1}y} f - f\big\|_\infty \\
&= |\Delta(x) - \Delta(y)||\mu|(K_1)\|f\|_\infty + |\mu|(K_2)|\Delta(y)|\big\|\rho_{x^{-1}y} f - f\big\|_\infty.
\end{aligned}
$$

In this estimate, all terms are positive and finite. Since Δ is continuous, $|\Delta(x) - \Delta(y)| \to 0$ for $y \to x$ and also $\big\|\rho_{x^{-1}y} f - f\big\|_\infty \to 0$ for $y \to x$. It follows that also $|\langle f, \rho_x \mu\rangle - \langle f, \rho_y \mu\rangle| \to 0$ for $y \to x$ and we conclude that $x \mapsto \rho_x \mu$ is also a continuous map. $\square$

5 Convolutions

In the main results of this thesis we use convolutions, which is a special kind of multiplicative operation on an appropriate collection of Radon measures. In this section we will give the definition and some results that we will use. Before we start with that, we need a short introduction to the general theory of product measures.

In that short introduction (and later on also with the convolutions) we will use Fubini's Theorem a couple times to interchange the order of integration. There is also the Theorem of Tonelli, with the same results of Fubini's Theorem that applies to σ-finite measures, however our Haar measure ν or any measure $\mu \in M(G)$ is not necessarily σ-finite. Folland has shown in [10, p. 44] that in our case, integrating functions with compact support, this version of Fubini's Theorem is still applicable. We won't discuss the details here as they are beyond the scope of this work. The same reasoning holds for the application of the Minkowski's Inequality for Integrals, the proof of which also depends on changing the order of integration.

5.1 Product of measures

Throughout this paragraph, let S and T be locally compact Hausdorff spaces and let $P = S \times T$ denote their product (again locally compact and Hausdorff). Let μ_1 and μ_2 be positive Radon measures on S and T, respectively. Let $f \in C_c(P)$ be given and let $s \in S$. Then it is clear that the map $f_s \colon T \to \mathbb{R}, t \mapsto f(s,t)$ is an element of $C_c(T)$, hence

$$\mu_2(f_s) = \int f(s,t)\, d\mu_2(t)$$

is well defined. Furthermore, the map $S \to \mathbb{R}, s \mapsto \mu_2(f_s)$ is an element of $C_c(S)$, thus the integral

$$\int \mu_2(f_s)\, d\mu_1(s) = \iint f(s,t)\, d\mu_2(t)\, d\mu_1(s)$$

is also well defined. Since both μ_1 and μ_2 are positive, these integrals are also positive whenever f is positive. We conclude that the map

$$\pi_S \colon C_c(P) \to \mathbb{R}, f \to \iint f(s,t)\, d\mu_2(t)\, d\mu_1(s)$$

is a positive Radon measure on P. In this construction we used an iterated integral and we do not yet know whether or not the order is important. By a similar reasoning, we find another positive Radon measure on P defined as

$$\pi_T \colon C_c(P) \to \mathbb{R}, f \to \iint f(s,t)\, d\mu_1(s)\, d\mu_2(t).$$

According to the Theorem of Fubini, the order of integration does not matter, so the Radon measures π_S and π_T are equal. We will not consider the details of the proof, but give a reference instead. The proof can be found in Edwards [7, §11, Theorem 11]. The measure π_T is the *product measure of* μ_1 *and* μ_2 and is, from now on, denoted by $\mu_1 \otimes \mu_2$.

5.2 Convolution of measures

Now we define what will be the convolution of two measures. Let μ_1, μ_2 be positive Radon measures, both on G. For $f \in C_c(G)$ define f' to be the map $G \times G \to \mathbb{R}, (a,b) \mapsto f(ab)$. The *convolution* $\mu_1 * \mu_2$ *of* μ_1 *and* μ_2 is then said to exist if and only if f' is $\mu_1 \otimes \mu_2$-integrable for all $f \in C_c(G)$. That is, if and only if $f' \in L^1(G \times G, \mu_1 \otimes \mu_2)$ for each $f \in C_c(G)$. In that case, define the convolution by

$$\langle f, \mu_1 * \mu_2 \rangle \coloneqq \langle f', \mu_1 \otimes \mu_2 \rangle = \int f' \, d\mu_1 \otimes \mu_2$$

for all $f \in C_c(G)$. The convolution $\mu_1 * \mu_2$ is by construction a Radon measure, a real-valued linear functional on $C_c(G)$. The convolution $\mu_1 * \mu_2$ is a positive Radon measure whenever f' and both μ_1 and μ_2 are positive. In that case, using the Riesz Representation Theorem, $\mu_1 * \mu_2$ is a positive regular Borel measure. This construction is, unlike what follows, available for any pair of positive Radon measures. From this construction it is clear that $\mu_1 * \mu_2$ is a bounded measure when both μ_1 and μ_2 are.

If μ, μ' are Radon measures with minimal decompositions $\mu = \mu_1 - \mu_2$ and $\mu' = \mu'_1 - \mu'_2$, the convolution $\mu * \mu'$ is said to exist if and only if each of the convolutions $\mu_i * \mu'_j$ exists for $i, j = 1, 2$. In that case $\mu * \mu'$ is defined to be the measure $\mu * \mu' = \mu_1 * \mu'_1 - \mu_1 * \mu'_2 - \mu_2 * \mu'_1 + \mu_2 * \mu'_2$.

Let δ_x be the *Dirac measure* at $x \in G$. If multiplication is given by convolution, then the unit of $M_b(G)$ is δ_e and we conclude that $M_b(G)$ is a Banach algebra known as the *measure algebra* of G.

5.3 Convolution of a measure and a function

In this part we consider the convolution of a measure with a function, working toward an expression for the left convolution[4] $\mu * f$. Let f be locally ν-integrable and let $\mu_1 = \mu$ be a positive Radon measure and $\mu_2 = \nu_f = f \cdot \nu$. By [8, Proposition 4.19.4:1], the convolution $\mu_1 * \mu_2 = \mu * (f \cdot \nu)$ exists if and only if $|\mu_1| * |\mu_2| = |\mu| * (|f| \cdot \nu)$ exists. By [8, Proposition 4.19.2:2] this is the case if and only if

$$|\mu|^*(\{t \mapsto |\mu_2|^*(\chi_{t^{-1}U})\}) = \mu^*(\{t \mapsto |\mu_2|^*(\chi_{t^{-1}U})\}) < \infty$$

for all relatively compact open subsets $U \subseteq G$. Let $t \in G$ be given, so $t^{-1}U \subseteq G$ is also a relatively compact open subset, hence $\chi_{t^{-1}U} \in \Phi(G)$ is $|\mu_2|$-integrable. The second equality follows from the construction of μ_2^* as an extension of μ'_2. The result now follows from the integrability of $\chi_{t^{-1}U}$ and an application of [8, Lemma 4.6.2].

$$\begin{aligned}
|\mu_2|^*(\chi_{t^{-1}U}) &= (|f| \cdot \nu)^*(\chi_{t^{-1}U}) \\
&= (|f| \cdot \nu)'(\chi_{t^{-1}U}) \\
&= (|f| \cdot \nu)(\chi_{t^{-1}U}) \\
&= \nu(\chi_{t^{-1}U}|f|) \\
&= \langle (\chi_{t^{-1}U}|f|), \nu \rangle \\
&= \int \chi_{t^{-1}U}(t')|f(t')| \, dt' \\
&= \int \chi_U(tt')|f(t')| \, dt'.
\end{aligned}$$

[4] In the same way, one may find a similar expression for the right convolution $f * \mu$. Consult [8, §4.19.11] for details.

Thus, $\mu * (f \cdot \nu)$ exists if and only if (by the left invariance of ν)

$$\mu^*\left(\left\{t \mapsto \int \chi_U(tt')|f(t')|\,dt'\right\}\right) = \mu^*\left(\left\{t \mapsto \int \chi_U(s)\left|f\left(t^{-1}s\right)\right|ds\right\}\right) < \infty \tag{5}$$

for all relatively compact open subsets $U \subseteq G$. Now, assume that the convolution exists. We then find for $h \in C_c(G)$ that

$$\begin{aligned}
\langle h, \mu_1 * \mu_2 \rangle &= \langle h, \mu * (f \cdot \nu) \rangle \\
&= \langle \{t' \mapsto \langle \{t \mapsto h(tt')\}, \mu \rangle\}, f \cdot \nu \rangle \\
&= \left\langle \left\{t' \mapsto \int h(tt')\,d\mu(t)\right\}, f \cdot \nu \right\rangle \\
&= \left\langle \left\{t' \mapsto \int f(t')h(tt')\,d\mu(t)\right\}, \nu \right\rangle \\
&= \iint f(t')h(tt')\,d\mu(t)\,dt' \\
&= \iint f(t')h(tt')\,dt'\,d\mu(t) && \text{(Fubini)} \\
&= \iint f\left(t^{-1}s\right)h(s)\,d\nu\left(t^{-1}s\right)\,d\mu(t) \\
&= \iint f\left(t^{-1}s\right)h(s)\,ds\,d\mu(t) && \text{(Left invariance of } \nu\text{)} \\
&= \iint f\left(t^{-1}s\right)h(s)\,d\mu(t)\,ds. && \text{(Fubini)}
\end{aligned}$$

Thus, by assumption, the map $t \mapsto f\left(t^{-1}s\right)$ is μ-integrable for ν-a.e. value of s. Now let $\psi\colon G \to \mathbb{R}$ be defined ν-a.e. by

$$\psi(s) = \int f\left(t^{-1}s\right)d\mu(t) = \int \lambda_t f(s)\,d\mu(t). \tag{6}$$

Then ψ is locally integrable for ν and we find that (with $h \in C_c(G)$)

$$\langle h, \mu_1 * \mu_2 \rangle = \int \psi(s)h(s)\,ds = \langle h, \psi \cdot \nu \rangle.$$

Thus, $\mu * (f \cdot \nu) = \psi \cdot \nu$. This leads to the following result.

5.1. Theorem. *Let μ be a positive Radon measure on G and let f be a locally ν-integrable function. If* (5) *holds, the convolution of μ and f exists and is denoted by $\mu * f$. It can be seen as the function ψ being defined ν-a.e by the pointwise expression in* (6).

Alternatively, if the convolution exists, we will write

$$\mu * f(s) = \int \lambda_s \check{f}(t)\,d\mu(t).$$

We will continue by showing some properties of this convolution.

5.2. Proposition. *Let $\mu \in M(G)$ be a positive Radon measure on G and let $f \in C_c(G)$. The convolution $\mu * f$ exists and it is defined everywhere on G by the pointwise expression in* (6), *and besides that, $\mu * f$ is also a continuous function on G. Moreover, $\|\mu * f\|_\infty < \infty$ if μ is also bounded.*

Proof. We must first show that the convolution $\mu * f$ exists for $\mu \in M(G)$ positive and $f \in C_c(G)$. Any $f \in C_c(G)$ is a ν-integrable function, hence locally ν-integrable. We will show that (5) holds. To that end, let $U \subseteq G$ be a relatively compact set and let $K = \operatorname{supp}(f)$. Then, for $t \in G$,

$$\int \chi_U(tt')|f(t')|\, dt' \leqslant \chi_{U\cdot K^{-1}}(t) \int |f(t')|\, dt' = \langle |f|, \nu\rangle \chi_{U\cdot K^{-1}}(t),$$

since $|f| \in C_c(G)$. Then, by the sublinearity of μ^*,

$$\mu^*\left(\left\{t \mapsto \int \chi_U(tt')|f(t')|\, dt'\right\}\right) \leqslant \langle |f|, \nu\rangle \mu^*(\chi_{U\cdot K^{-1}}).$$

Since K^{-1} is compact and U is open, $U \cdot K^{-1}$ is open in G. The set $K' = \overline{U} \cdot K^{-1}$ is compact and contains $U \cdot K^{-1}$. For each $x \in K'$, let U_x be an open set containing x. Then $V = \cup_{x\in K'} U_x$ is an open set containing K'. Apply [18, 2.12 Urysohn's Lemma] to find a function $h \in C_c(G)$ such that

$$\chi_{K'} \leqslant h \leqslant \chi_V.$$

Then, since $\chi_{U\cdot K^{-1}} \leqslant \chi_{K'} \leqslant h$ and μ^* is a sublinear function

$$\mu^*(\chi_{U\cdot K^{-1}}) \leqslant \mu^*(h) = \langle h, \mu\rangle.$$

Hence

$$\mu^*\left(\left\{t \mapsto \int \chi_U(tt')|f(t')|\, dt'\right\}\right) \leqslant \langle |f|, \nu\rangle \langle h, \mu\rangle,$$

which is finite using the definition of a Radon measure in (1) on both ν and μ. We conclude that the convolution $\mu * f$ exists. Since f is defined everywhere on G, the convolution is defined everywhere by the pointwise expression in (6).

Let $x \in G$ and let $K = \operatorname{supp}(f)$ be compact, then xK^{-1} is compact and we see that

$$\begin{aligned}
|\mu * f(x)| &= \left|\int_G f(y^{-1}x)\, d\mu(y)\right| \\
&\leqslant \int_G \left|f(y^{-1}x)\right| d\mu(y) \\
&= \int_{xK^{-1}} \left|f(y^{-1}x)\right| d\mu(y) \\
&\leqslant \int_{xK^{-1}} \|f\|_\infty\, d\mu(y) \\
&\leqslant \|f\|_\infty \mu\big(xK^{-1}\big) < \infty.
\end{aligned}$$

Thus $\mu * f$ is everywhere defined and finite. We now show that $\mu * f \in C(G)$. Let $x \in G$ and let $N \in \mathcal{N}_x$ be a compact neighborhood of x, such N exists since G is locally compact. The set

$$K = x \cdot \operatorname{supp}(f)^{-1} \cup N \cdot \operatorname{supp}(f)^{-1}$$

is compact again since $\operatorname{supp}(f)^{-1}$ and $N \cdot \operatorname{supp}(f)^{-1}$ are both compact. We find for $y \in N$ that

$$\begin{aligned}
|\mu * f(x) - \mu * f(y)| &= \left|\int_G f\big(z^{-1}x\big)\, d\mu(z) - \int_G f\big(z^{-1}y\big)\, d\mu(z)\right| \\
&= \left|\int_G f\big(z^{-1}x\big) - f\big(z^{-1}y\big)\, d\mu(z)\right|
\end{aligned}$$

$$\begin{aligned}
&\leqslant \int_K \big|f\big(z^{-1}x\big) - f\big(z^{-1}y\big)\big| \, d\mu(z) \\
&= \int_K \big|\big(\rho_{x^{-1}y} f - f\big)\big(z^{-1}y\big)\big| \, d\mu(z) \\
&\leqslant \int_K \big\|\rho_{x^{-1}y} f - f\big\|_\infty \, d\mu(z) \\
&= \big\|\rho_{x^{-1}y} f - f\big\|_\infty \mu(K).
\end{aligned}$$

For $y \to x$ we have that $\|\rho_{x^{-1}y} f - f\|_\infty \to 0$ so we conclude that $\mu * f$ is indeed a continuous function on G.

Suppose that μ is also bounded, then

$$\begin{aligned}
\|\mu * f\|_\infty &= \sup_{x \in G} |\mu * f(x)| \\
&\leqslant \sup_{x \in G} \|f\|_\infty \mu\big(xK^{-1}\big) \\
&\leqslant \sup_{x \in G} \|f\|_\infty \mu(G) \\
&= \|f\|_\infty \|\mu\| < \infty.
\end{aligned}$$

□

5.3. Proposition. *Let $\mu \in M_b(G)$ be a positive Radon measure and let $p \in [1, \infty)$. Then $\mu * f$ exists for every $f \in L^p(G)$ and it is defined ν-a.e. on G by the pointwise expression in* (6)*. Furthermore, $\mu * f \in L^p(G)$ with $\|\mu * f\|_p \leqslant \|\mu\| \|f\|_p$ for all $f \in L^p(G)$.*

Proof. Let $\mu \in M_b(G)$ be a positive Radon measure. Let $p \in [1, \infty)$ and let $f \in L^p(G)$. Since f is p-fold ν-integrable, it is also locally ν-integrable using Hölder's Inequality [8, §4.11.2]. Moreover, f is bounded ν-a.e. on G, say by $M \geqslant 0$. We start by showing that $\mu * f$ exists. We will show that (5) holds. To that end, let $U \subseteq G$ be a relatively compact set. It holds for ν-a.e. $t \in G$ that

$$\int \chi_U(s) \big|f\big(t^{-1}s\big)\big| \, ds \leqslant \int \chi_U(s) M \, ds \leqslant \nu(U) M.$$

It is shown in [8, §4.6] that $\mu^*(h_1) \leqslant \mu^*(h_2)$ for positive h_1, h_2 on G such that $h_1(x) \leqslant h_2(x)$ for ν-a.e. $x \in G$. Then, together with the sublinearity of μ^* it follows that

$$\mu^*\left(\left\{t \mapsto \int \chi_U(s) \big|f\big(t^{-1}s\big)\big| \, ds\right\}\right) \leqslant \mu^*(\nu(U) M) = \nu(U) M \mu^*(\chi_G).$$

Since $\chi_G \in C(G) \subseteq \Phi(G)$ and $\mu^* = \mu'$ on $\Phi(G)$ it follows that $\mu^*(\chi_G) = \mu'(\chi_G)$. By definition of μ' and the assumption that $\mu \in M_b(G)$,

$$\begin{aligned}
\mu'(\chi_G) &= \sup\{\langle h, \mu \rangle \colon h \in C_c(G), h \leqslant \chi_G\} \\
&= \sup\left\{\int h \, d\mu \colon h \in C_c(G), h \leqslant \chi_G\right\} \\
&\leqslant \sup\{\mu(G) \colon h \in C_c(G), h \leqslant \chi_G\} \\
&= \mu(G) < \infty.
\end{aligned}$$

We conclude that the convolution $\mu * f$ exists and is defined ν-a.e. on G by the pointwise expression in (6).

It follows from Minkowski's Inequality for Integrals [9, Inequality 6.19.a] that

$$\begin{aligned}
\|\mu * f\|_p &= \left\| x \mapsto \int f\left(y^{-1}x\right) d\mu(y) \right\|_p \\
&= \left(\int \left| \int f\left(y^{-1}x\right) d\mu(y) \right|^p dx \right)^{\frac{1}{p}} \\
&\leqslant \left(\int \left(\int \left| f\left(y^{-1}x\right) \right| d\mu(y) \right)^p dx \right)^{\frac{1}{p}} \\
&\leqslant \int \left(\int \left| f\left(y^{-1}x\right) \right|^p dx \right)^{\frac{1}{p}} d\mu(y) && \text{(Minkowski)} \\
&= \int \left(\int |f(x)|^p \, dx \right)^{\frac{1}{p}} d\mu(y) \\
&= \int \|f\|_p \, d\mu(y) \\
&= \|f\|_p \mu(G) \\
&= \|f\|_p \|\mu\| < \infty.
\end{aligned}$$

We conclude that $\mu * f \in L^p(G)$ and $\|\mu * f\|_p \leqslant \|f\|_p \|\mu\|$. □

This implies that, under the conditions of the previous Proposition, $\nu_{\mu * f}$ is a Radon measure on G. We end with the definition of a special set of Radon measures.

5.4. Definition. Let $1 \leqslant p, q < \infty$ and let $M^{p,q}(G)$ denote the set of measures $\mu \in M(G)$ that allow for a constant $c \in \mathbb{R}_{>0}$ to exist such that

$$\|\mu * f\|_q \leqslant c \|f\|_p$$

for $f \in C_c(G)$.

From the previous results it is clear that $M_b(G) \subseteq M^{p,p}(G)$.

5.4 Convolution of functions

We will now use the definition to formulate the definition of convolution of functions which will be denoted by $f * g$. This construction uses our fixed left Haar measure ν. Let f and g be locally ν-integrable and consider the case where $\mu_1 = \nu_f = f \cdot \nu$ and $\mu_2 = \nu_g = g \cdot \nu$. Then, by [8, Proposition 4.19.4:1], the convolution $\mu_1 * \mu_2 = (f \cdot \nu) * (g \cdot \nu)$ only exists if and only if $|\mu_1| * |\mu_2| = (|f| \cdot \nu) * (|g| \cdot \nu)$ exists. By [8, Proposition 4.19.2:2] the convolution $\mu_1 * \mu_2$ exists if and only if

$$|\mu_1|^*(\{t \mapsto |\mu_2|^*(\chi_{t^{-1}U})\}) < \infty$$

for all relatively compact open subsets $U \subseteq G$. Let $t \in G$ be given, then also $t^{-1}U \subseteq G$ is a relatively compact open subset, hence $\chi_{t^{-1}U} \in \Phi(G)$. This means that $\operatorname{supp}(\chi_{t^{-1}U}) = \overline{U}$ is compact, hence ν-σ-finite. By [8, Proposition 4.13.2:1] it follows that

$$|\mu_2|^*(\chi_{t^{-1}U}) = (|g| \cdot \nu)^*(\chi_{t^{-1}U}) = \nu^*(|g| \chi_{t^{-1}U}).$$

Hence the maps $t \mapsto |\mu_2|^*(\chi_{t^{-1}U})$ and $t \mapsto \nu^*(|g|\chi_{t^{-1}U})$ are equal. We conclude that the convolution $(f \cdot \nu) * (g \cdot \nu)$ exists if and only if

$$|\mu_1|^*(\{t \mapsto \nu^*(|g|\chi_{t^{-1}U})\}) = (|f| \cdot \nu)^*(\{t \mapsto \nu^*(|g|\chi_{t^{-1}U})\}) < \infty \tag{7}$$

for all relatively compact open subsets $U \subseteq G$. Now, assume that the convolution exists. We then find for $h \in C_c(G)$ that

$$\begin{aligned}
\langle h, \mu_1 * \mu_2 \rangle &= \big\langle \big\{t' \mapsto \big\langle \big\{t \mapsto h(tt')\big\}, \mu_1 \big\rangle\big\}, \mu_2 \big\rangle \\
&= \big\langle \big\{t' \mapsto \big\langle \big\{t \mapsto h(tt')\big\}, f \cdot \nu \big\rangle\big\}, g \cdot \nu \big\rangle \\
&= \big\langle \big\{t' \mapsto \big\langle \big\{t \mapsto h(tt')f(t)\big\}, \nu \big\rangle\big\}, g \cdot \nu \big\rangle \\
&= \left\langle \left\{t' \mapsto \int h(tt')f(t)\, dt\right\}, g \cdot \nu \right\rangle \\
&= \left\langle \left\{t' \mapsto \int h(tt')f(t)g(t')\, dt\right\}, \nu \right\rangle \\
&= \iint h(tt')f(t)g(t')\, dt\, dt' \\
&= \iint h(tt')f(t)g(t')\, dt'\, dt && \text{(Fubini)} \\
&= \iint h(s)f(t)g\big(t^{-1}s\big)\, d\nu\big(t^{-1}s\big)\, dt \\
&= \iint h(s)f(t)g\big(t^{-1}s\big)\, ds\, dt && \text{(Left invariance of } \nu) \\
&= \iint h(s)f(t)g\big(t^{-1}s\big)\, dt\, ds. && \text{(Fubini)}
\end{aligned}$$

Thus, by assumption, the map $t \mapsto f(t)g\big(t^{-1}s\big)$ is ν-integrable for ν-a.e. value of s. Now let $\varphi\colon G \to \mathbb{R}$ be defined ν-a.e. by

$$\varphi(s) = \int f(t)g\big(t^{-1}s\big)\, dt = \int f(t)\lambda_t g(s)\, dt. \tag{8}$$

Then φ is locally integrable for ν and we find that (with $h \in C_c(G)$)

$$\langle h, \mu_1 * \mu_2 \rangle = \int \varphi(s)h(s)\, ds = \langle h, \varphi \cdot \nu \rangle.$$

Thus, $(f \cdot \nu) * (g \cdot \nu) = \varphi \cdot \nu$. This leads to the following result.

5.5. Theorem. *Let f and g be locally ν-integrable functions. If* (7) *holds, the convolution of f and g exists is denoted by $f * g$. It can be seen as the function φ being defined ν-a.e by the pointwise expression in* (8).

Using a couple of substitutions and again the left invariance of ν and Remark 3.4 we find the following alternative formulas for $f * g$, in case it exists

$$\begin{aligned}
f * g(x) &= \int f(y)g\big(y^{-1}x\big)\, dy \\
&= \int f(xy)g\big(y^{-1}\big)\, d\nu(xy)
\end{aligned}$$

$$
\begin{aligned}
&= \int f(xy)g\big(y^{-1}\big)\,dy \\
&= \int \Delta\big(y^{-1}\big)f\big(xy^{-1}\big)g(y)\,dy \\
&= \int \Delta\big(y^{-1}\big)\rho_y f(x)g(y)\,dy.
\end{aligned}
$$

Furthermore, still for $f, g \in \mathbb{R}^G$ locally ν-integrable, it follows from these equalities that

$$
f * \check{g}(x) = \int f(y)\check{g}\big(y^{-1}x\big)\,dy = \int f(y)g\big(x^{-1}y\big)\,dy = \int f(xy)g(y)\,dy.
$$

We will continue by showing some properties of this convolution.

5.6. Proposition. *Let $f, g \in C_c(G)$. The convolution $f * g$ exists and it is everywhere defined on G by the pointwise expression in* (8)*. Moreover, $f * g$ is a continuous function with compact support.*

Proof. We must first show that the convolution $f * g$ exists for all $f, g \in C_c(G)$. Any $f \in C_c(G)$ is a ν-integrable function, hence locally ν-integrable. We will show that (7) holds. To that end, let $U \subseteq G$ be a relatively compact set. For $g \in C_c(G)$, also $|g| \in C_c(G)$ and $|g|\chi_{t^{-1}U} \leqslant |g|$. Since ν^* is a sublinear extension of the linear functional ν we have

$$
\nu^*(|g|\chi_{t^{-1}U}) \leqslant \nu^*(|g|) = \langle |g|, \nu\rangle,
$$

where $\langle |g|, \nu\rangle \in C(G) \subseteq \Phi(G)$. By the sublinearity of $(|f| \cdot \nu)^*$ and [8, Proposition 4.13.2:1] we find

$$
\begin{aligned}
(|f| \cdot \nu)^*(\{t \mapsto \nu^*(|g|\chi_{t^{-1}U})\}) &\leqslant (|f| \cdot \nu)^*(\langle |g|, \nu\rangle) \\
&\leqslant \nu^*(|f|\langle |g|, \nu\rangle) \\
&= \langle |g|, \nu\rangle \nu^*(|f|) \\
&= \langle |g|, \nu\rangle\langle |f|, \nu\rangle.
\end{aligned}
$$

Since ν is a positive Radon measure and $|f|, |g| \in C_c(G)$ are positive, both $\langle |f|, \nu\rangle$ and $\langle |g|, \nu\rangle$ finite by the definition of a Radon measure in (1). Hence (7) is finite and we conclude that the convolution $f * g$ exists. Since f and g are defined everywhere on G, the convolution is defined everywhere by the pointwise expression in (8).

Let $x \in G$ and let

$$
K = \operatorname{supp}(f) \cap x \cdot \operatorname{supp}(g)^{-1}
$$

which is clearly compact again. We see that

$$
\begin{aligned}
|f * g(x)| &= \left| \int_G f(z)g\big(z^{-1}x\big)\,dz \right| \\
&\leqslant \int_G |f(z)|\big|g\big(z^{-1}x\big)\big|\,dz \\
&= \int_K |f(z)|\big|g\big(z^{-1}x\big)\big|\,dz \\
&\leqslant \int_K \|f\|_\infty \|g\|_\infty\,dz \\
&\leqslant \|f\|_\infty \|g\|_\infty \nu(K) < \infty.
\end{aligned}
$$

Thus $f * g$ is everywhere defined and finite. We will now show that $f * g$ is continuous. Let $x \in G$ and let $N \in \mathcal{N}_x$ be a compact neighborhood of x, such N exists since G is locally compact. The set

$$K = x \cdot \operatorname{supp}(g)^{-1} \cup N \cdot \operatorname{supp}(g)^{-1}$$

is compact again since $\operatorname{supp}(g)^{-1}$ and $N \cdot \operatorname{supp}(g)^{-1}$ are both compact. We find for $y \in N$ that

$$\begin{aligned}
|f * g(x) - f * g(y)| &= \left| \int_G f(z) g\big(z^{-1}x\big)\, dz - \int f(z) g\big(z^{-1}y\big)\, dz \right| \\
&\leqslant \int_G |f(z)| \big|g\big(z^{-1}x\big) - g\big(z^{-1}y\big)\big|\, dz \\
&\leqslant \|f\|_\infty \int_G \big|g\big(z^{-1}x\big) - g\big(z^{-1}y\big)\big|\, dz \\
&= \|f\|_\infty \int_K \big|g\big(z^{-1}x\big) - g\big(z^{-1}y\big)\big|\, dz \\
&= \|f\|_\infty \int_K \big|\big(\rho_{x^{-1}y} g - g\big)\big(z^{-1}y\big)\big|\, dz \\
&\leqslant \|f\|_\infty \int_K \big\|\rho_{x^{-1}y} g - g\big\|_\infty\, dz \\
&= \|f\|_\infty \|\rho_{x^{-1}y} g - g\|_\infty \nu(K).
\end{aligned}$$

For $y \to x$ we have that $\|\rho_{x^{-1}y} g - g\|_\infty \to 0$ so we conclude that $f * g$ is indeed a continuous function on G.

Furthermore, suppose that $x \in G$ such that $f * g(x) \neq 0$. Then there exists some $y \in G$ such that $f(y) g\big(y^{-1}x\big) \neq 0$. Thus, it must hold for y that $f(y) \neq 0$ and $g\big(y^{-1}x\big) \neq 0$. That is, y must be an element of $\operatorname{supp}(f)$ and $y^{-1}x$ must be in $\operatorname{supp}(g)$. We conclude that $x \in y \cdot \operatorname{supp}(g)$ and after taking closures that

$$\operatorname{supp}(f * g) \subseteq \operatorname{supp}(f) \cdot \operatorname{supp}(g)$$

which is compact. We conclude that $f * g \in C_c(G)$ whenever f and g are. In particular, this means that $f * g$ is a bounded function. □

In addition to the above, we may and will view this convolution of functions as a vector-valued integral.

5.7. Theorem. *Let $f, g \in C_c(G)$ and let $Q = \operatorname{supp}(g)$. The convolution*

$$f * g = \int_Q \Delta\big(q^{-1}\big) g(q) \rho_q f \, d\nu|_Q(q)$$

exists as a vector-valued integral in $C_c(G)$ with the inductive limit topology.

Proof. Let $f, g \in C_c(G)$ be given and let $Q = \operatorname{supp}(g)$. Consider the space $X = C_c(K, G)$ where

$$K = \operatorname{supp}(f) \cdot \operatorname{supp}(g).$$

Then X is a Banach space using the supremum norm such that X^* separates the points of X. Let $\nu|_Q$ be our Haar measure ν restricted to the compact set Q, which is then a finite measure. Define the map $h\colon Q \to X$ by

$$q \mapsto \Delta\big(q^{-1}\big) g(q) \rho_q f.$$

We will first show that h is continuous. Let $q_1, q_2 \in Q$ be such that $q_2 \to q_1$. We then find that

$$
\begin{aligned}
&\|h(q_1) - h(q_2)\|_\infty \\
=&\|\Delta\big(q_1^{-1}\big)g(q_1)\rho_{q_1}f - \Delta\big(q_2^{-1}\big)g(q_2)\rho_{q_2}f\|_\infty \\
=&\|\{y \mapsto \Delta\big(q_1^{-1}\big)g(q_1)\rho_{q_1}f(y) - \Delta\big(q_2^{-1}\big)g(q_2)\rho_{q_2}f(y)\}\|_\infty \\
=&\sup_{y\in K}\big|\Delta\big(q_1^{-1}\big)g(q_1)\rho_{q_1}f(y) - \Delta\big(q_2^{-1}\big)g(q_2)\rho_{q_2}f(y)\big| \\
=&\sup_{y\in K}\big|\Delta\big(q_1^{-1}\big)g(q_1)\rho_{q_1}f(y) - \Delta\big(q_2^{-1}\big)g(q_2)\rho_{q_1}f(y) + \Delta\big(q_2^{-1}\big)g(q_2)\rho_{q_1}f(y) - \Delta\big(q_2^{-1}\big)g(q_2)\rho_{q_2}f(y)\big| \\
\leqslant&\sup_{y\in K}\big|\Delta\big(q_1^{-1}\big)g(q_1) - \Delta\big(q_2^{-1}\big)g(q_2)\big||\rho_{q_1}f(y)| + \big|\Delta\big(q_2^{-1}\big)g(q_2)\big||\rho_{q_1}f(y) - \rho_{q_2}f(y)| \\
=&\sup_{y\in K}\big|\Delta\big(q_1^{-1}\big)g(q_1) - \Delta\big(q_2^{-1}\big)g(q_2)\big||\rho_{q_1}f(y)| + \big|\Delta\big(q_2^{-1}\big)g(q_2)\big|\big|f\big(yq_1^{-1}\big) - f\big(yq_2^{-1}\big)\big| \\
=&\sup_{y\in K}\big|\Delta\big(q_1^{-1}\big)g(q_1) - \Delta\big(q_2^{-1}\big)g(q_2)\big||\rho_{q_1}f(y)| + \big|\Delta\big(q_2^{-1}\big)g(q_2)\big|\Big|\Big(\rho_{q_1q_2^{-1}}f - f\Big)\big(yq_2^{-1}\big)\Big| \\
\leqslant&\sup_{y\in K}\big|\Delta\big(q_1^{-1}\big)g(q_1) - \Delta\big(q_2^{-1}\big)g(q_2)\big|\|\rho_{q_1}f\|_\infty + \big|\Delta\big(q_2^{-1}\big)g(q_2)\big|\Big\|\rho_{q_1q_2^{-1}}f - f\Big\|_\infty \\
=&\big|\Delta\big(q_1^{-1}\big)g(q_1) - \Delta\big(q_2^{-1}\big)g(q_2)\big|\|\rho_{q_1}f\|_\infty + \big|\Delta\big(q_2^{-1}\big)g(q_2)\big|\Big\|\rho_{q_1q_2^{-1}}f - f\Big\|_\infty \\
=&\big|\Delta\big(q_1^{-1}\big)g(q_1) - \Delta\big(q_2^{-1}\big)g(q_2)\big|\|f\|_\infty + \big|\Delta\big(q_2^{-1}\big)g(q_2)\big|\Big\|\rho_{q_1q_2^{-1}}f - f\Big\|_\infty.
\end{aligned}
$$

In this estimate all terms are positive and finite. Note that

$$\big|\Delta\big(q_2^{-1}\big)g(q_2)\big| \leqslant \max_{q\in Q^{-1}} \Delta(q)\|g\|_\infty$$

since Q^{-1} is compact and Δ is continuous. Since both Δ and g are continuous maps on G, their product is continuous again, hence $\big|\Delta\big(q_1^{-1}\big)g(q_1) - \Delta\big(q_2^{-1}\big)g(q_2)\big| \to 0$ when $q_2 \to q_1$. We have also seen that

$$\Big\|\rho_{q_1q_2^{-1}}f - f\Big\|_\infty \to 0$$

when $q_2 \to q_1$ since f is uniformly continuous on G. We conclude that $\|h(q_1) - h(q_2)\|_\infty \to 0$ for $q_2 \to q_1$, hence h is continuous on Q. We apply Theorem 2.10 to find that

$$f * g = \int_Q \Delta\big(q^{-1}\big)g(q)\rho_q f \, d\nu|_Q(q)$$

exists as a vector-valued integral in X. We know that $X \subseteq C_c(G)$ by construction, hence the inclusion map $\subseteq\colon X \to C_c(G)$ is continuous. It is clear that $C_c(G)^* = M(G)$ separates the points of $C_c(G)$, since $\mathrm{ev}_x \in M(G)$ for all $x \in G$. By an application of Corollary 2.11, $f * g$ exists as a vector-valued integral in $C_c(G)$. □

5.5 Convolution and translation

We end with a small result about the translation of a convolution.

5.8. Proposition. *Let $f \in \mathbb{R}^G$ be locally ν-integrable and let $\mu \in M(G)$ be positive. Suppose that the convolution $\mu * f$ exists. The convolutions $(\lambda_a\mu) * f$ and $\mu * (\rho_a f)$ then exist for all $a \in G$. It follows that*

$$\lambda_a(\mu * f) = (\lambda_a\mu) * f \text{ and } \rho_a(\mu * f) = \mu * (\rho_a f).$$

Proof. Let $f \in \mathbb{R}^G$ be locally ν-integrable and let $\mu \in M(G)$ be positive. If the convolution $\mu * f$ exists on G, we can apply a translation. Then we find for $x, a \in G$ that

$$\begin{aligned}
\lambda_a(\mu * f)(x) &= \mu * f\big(a^{-1}x\big) \\
&= \int \lambda_y f\big(a^{-1}x\big)\, d\mu(y) \\
&= \int f\big(y^{-1}a^{-1}x\big)\, d\mu(y) \\
&= \int f\big((ay)^{-1}x\big)\, d\mu(y) \\
&= \int \lambda_{ay} f(x)\, d\mu(y) \\
&= \int \lambda_{a^{-1}} \lambda_y f(x)\, d\mu(y) \\
&= \int \lambda_y f(x)\, d(\lambda_a \mu)(y) \\
&= (\lambda_a \mu) * f(x).
\end{aligned}$$

and

$$\begin{aligned}
\rho_a(\mu * f)(x) &= \mu * f\big(xa^{-1}\big) \\
&= \int \lambda_y f\big(xa^{-1}\big)\, d\mu(y) \\
&= \int f\big(y^{-1}xa^{-1}\big)\, d\mu(y) \\
&= \int \rho_a f(y^{-1}x)\, d\mu(y) \\
&= \int \lambda_y(\rho_a f)(x)\, d\mu(y) \\
&= \mu * (\rho_a f)(x).
\end{aligned}$$

□

6 Positive operators between $L^p(G)$-spaces that commute with translations

In this section we will, in a general sense without any other assumptions on our group G other than G being locally compact and Hausdorff, consider the first main problem of this work, considering a positive linear operator $T\colon L^p(G) \to L^q(G)$ that commutes with the ρ_a for all $a \in G$. We will start with a widely applicable proposition.

6.1. Proposition. *Let $(N_\alpha)_{\alpha\in\mathbb{A}} \subseteq \mathcal{N}_e$ be a base of relatively compact neighborhoods and assume that all N_α lie within some fixed compact neighborhood N_0 of e. For $\alpha \in \mathbb{A}$, fix $\varphi_\alpha \in C_c(G)$ such that $\varphi_\alpha \geqslant 0$, $\operatorname{supp}(\varphi_\alpha) \subseteq N_\alpha$ and $\int \varphi_\alpha \, d\nu = 1$. Let $W \subseteq M(G)$ be a right translation invariant vector subspace with the subspace topology. Let T be a continuous linear map*

$$T\colon C_c(G) \to W$$

such that $\rho_a T = T\rho_a$ for all $a \in G$. Define $\mu_\alpha = T\varphi_\alpha \in W \subseteq M(G)$. Furthermore, suppose that $\mu_\alpha = \mu_\alpha^+ - \mu_\alpha^-$ with $\mu_\alpha^+, \mu_\alpha^- \in M(G)$ positive. In that case, the set of measures

$$\big\{ \big(\mu_\alpha^+ * f\big) \cdot \nu - \big(\mu_\alpha^- * f\big) \cdot \nu \colon \alpha \in \mathbb{A} \big\}$$

is vaguely bounded in $M(G)$ for every $f \in C_c(G)$ and

$$Tf = \lim_\alpha \big[\big(\mu_\alpha^+ * f\big) \cdot \nu - \big(\mu_\alpha^- * f\big) \cdot \nu \big] \tag{9}$$

for each $f \in C_c(G)$.

6.2. Remark. In the statement of the proposition we used the concept of a *right translation invariant vector subspace*. In this case this just means that W must satisfy

$$\rho_a(W) = \{\rho_a \mu \colon \mu \in W\} \subseteq W$$

for all $a \in G$. We don't actually use this in the proof, but it is required to be able to assume that ρ_a commutes with the operator T. Since for $a \in G$ it is true that $\rho_a(C_c(G)) = C_c(G)$ we find that

$$\rho_a \circ T \colon C_c(G) \xrightarrow{T} W \xrightarrow{\rho_a} \rho_a(W)$$

and

$$T \circ \rho_a \colon C_c(G) \xrightarrow{\rho_a} C_c(G) \xrightarrow{T} W.$$

6.3. Proof of Proposition 6.1. Let $\alpha \in \mathbb{A}$ and let $f \in C_c(G)$ be given. We have seen before that in this case $\varphi_\alpha * f$ exists in $C_c(G)$ and is a continuous function on G and that

$$\varphi_\alpha * f = \int_Q \Delta\big(q^{-1}\big) f(q) \rho_q \varphi_\alpha \, d\nu|_Q(q)$$

exists as a vector-valued integral in $C_c(K, G) \subseteq C_c(G)$ where both $K = \operatorname{supp}(\varphi_\alpha) \cdot \operatorname{supp}(f)$ and $Q = \operatorname{supp}(f)$ are compact. Note that W is a subspace of $M(G)$ with the weak*-topology , which is the dual of $C_c(G)$, so the dual W^* separates the points of W. By assumption T is continuous and linear on $C_c(G)$, so we apply Corollary 2.11 and use the fact that T commutes with the right translation operators to find that

$$T(\varphi_\alpha * f) = \int_Q T\big(\Delta\big(q^{-1}\big) f(q) \rho_q \varphi_\alpha\big) \, d\nu|_Q(q)$$

$$
\begin{aligned}
&= \int_Q \Delta\big(q^{-1}\big) f(q) T(\rho_q \varphi_\alpha)\, d\nu|_Q(q) \\
&= \int_Q \Delta\big(q^{-1}\big) f(q) \rho_q T \varphi_\alpha\, d\nu|_Q(q) \\
&= \int_Q \Delta\big(q^{-1}\big) f(q) \rho_q \mu_\alpha\, d\nu|_Q(q).
\end{aligned}
$$

Note that this final vector-valued integral exists in $W \subseteq M(G)$ as a result of Corollary 2.11. Let $h \in C_c(G)$, then $\mathrm{ev}_h \in C_c(G)^{**} = M(G)^* \subseteq W^*$ and we have

$$
\begin{aligned}
\langle T(\varphi_\alpha * f), \mathrm{ev}_h\rangle &= \left\langle \int_Q \Delta\big(q^{-1}\big) f(q) \rho_q \mu_\alpha\, d\nu|_Q(q), \mathrm{ev}_h \right\rangle \\
&= \int_Q \Delta\big(q^{-1}\big) f(q) \langle \rho_q \mu_\alpha, \mathrm{ev}_h\rangle\, d\nu|_Q(q) \\
&= \int_Q \Delta\big(q^{-1}\big) f(q) \Big\langle \mu_\alpha, \mathrm{ev}_{\Delta(q)\rho_{q^{-1}} h} \Big\rangle\, d\nu|_Q(q) \\
&= \int_Q f(q) \Big\langle \mu_\alpha, \mathrm{ev}_{\rho_{q^{-1}} h} \Big\rangle\, d\nu|_Q(q).
\end{aligned}
$$

At this stage, the measure μ_α is not yet positive, so suppose that $\mu_\alpha = \mu_\alpha^+ - \mu_\alpha^-$ with $\mu_\alpha^+, \mu_\alpha^- \in M(G)$ positive. Then

$$
\begin{aligned}
\langle T(\varphi_\alpha * f), \mathrm{ev}_h\rangle &= \int_Q f(q) \Big\langle \mu_\alpha^+, \mathrm{ev}_{\rho_{q^{-1}} h} \Big\rangle - \Big\langle \mu_\alpha^-, \mathrm{ev}_{\rho_{q^{-1}} h} \Big\rangle\, d\nu|_Q(q) \\
&= \int_Q f(q) \Big\langle \mu_\alpha^+, \mathrm{ev}_{\rho_{q^{-1}} h} \Big\rangle\, d\nu|_Q(q) - \int_Q f(q) \Big\langle \mu_\alpha^-, \mathrm{ev}_{\rho_{q^{-1}} h} \Big\rangle\, d\nu|_Q(q).
\end{aligned}
$$

We will first rewrite the first one of these integrals, the second is done in the same way. Note that, at this moment, we are considering regular integrals and may alter measures and the order of integration. It then follows from the left invariance of our (restricted) Haar measure that

$$
\begin{aligned}
\int_Q f(q) \Big\langle \mu_\alpha^+, \mathrm{ev}_{\rho_{q^{-1}} h} \Big\rangle\, d\nu|_Q(q) &= \int_Q \int f(q) h(zq)\, d\mu_\alpha^+(z)\, d\nu|_Q(q) \\
&= \int \int_Q f(q) h(zq)\, d\nu|_Q(q)\, d\mu_\alpha^+(z) && \text{(Fubini)} \\
&= \int \int_G f(q) h(zq)\, dq\, d\mu_\alpha^+(z) && \text{(Since } Q = \mathrm{supp}(f)) \\
&= \iint f\big(z^{-1}x\big) h(x)\, d\nu\big(z^{-1}x\big)\, d\mu_\alpha^+(z) \\
&= \iint \lambda_z f(x) h(x)\, dx\, d\mu_\alpha^+(z) \\
&= \int \lambda_z f(x) h(x)\, d\mu_\alpha^+(z)\, dx && \text{(Fubini)} \\
&= \int \mu_\alpha^+ * f(x) h(x)\, dx \\
&= \big\langle \big(\mu_\alpha^+ * f\big) \cdot \nu, \mathrm{ev}_h \big\rangle.
\end{aligned}
$$

We conclude that

$$
\langle T(\varphi_\alpha * f), \mathrm{ev}_h\rangle = \big\langle \big(\mu_\alpha^+ * f\big) \cdot \nu - \big(\mu_\alpha^- * f\big) \cdot \nu, \mathrm{ev}_h \big\rangle,
$$

hence

$$T(\varphi_\alpha * f) = \big(\mu_\alpha^+ * f\big) \cdot \nu - \big(\mu_\alpha^- * f\big) \cdot \nu \in M(G) \tag{10}$$

for all $\alpha \in \mathbb{A}$ and all $f \in C_c(G)$.

Let $f \in C_c(G)$ and let

$$S = \{T(\varphi_\alpha * f) \colon \alpha \in \mathbb{A}\} \subseteq W \subseteq M(G).$$

We have to show that S is vaguely bounded in $M(G)$, that is, we must show that

$$\sup_{\alpha\in\mathbb{A}} |\langle h, T(\varphi_\alpha * f)\rangle| < \infty$$

for each $h \in C_c(G)$. It follows from the proof of Proposition 5.6 and our assumptions on φ_α that

$$\operatorname{supp}(\varphi_\alpha * f) \subseteq \operatorname{supp}(\varphi_\alpha) \cdot \operatorname{supp}(f) \subseteq N_\alpha \cdot \operatorname{supp}(f) \subseteq N_0 \cdot \operatorname{supp}(f)$$

for every $\alpha \in \mathbb{A}$. Let $K = N_0 \cdot \operatorname{supp}(f)$, then K is compact and $\varphi_\alpha * f \in C_c(K, G)$ for every $\alpha \in \mathbb{A}$. Furthermore, by our assumption on φ_α,

$$\|\varphi_\alpha * f\|_\infty = \sup_{x\in G}\left|\int \varphi_\alpha(y) f\big(y^{-1}x\big)\, dy\right| \leqslant \sup_{x\in G} \int \varphi_\alpha(y)\big|f\big(y^{-1}x\big)\big|\, dy \leqslant \sup_{x\in G} \int \varphi_\alpha(y)\|f\|_\infty\, dy = \|f\|_\infty$$

for every $\alpha \in \mathbb{A}$. Let $h \in C_c(G)$ be given. The map

$$C_c(K, G) \xrightarrow{\subseteq} C_c(G) \xrightarrow{T} W \xrightarrow{\subseteq} M(G) \xrightarrow{\mathrm{ev}_h} \mathbb{R}$$

is the composition of continuous maps, that is, it is a continuous linear functional between normed spaces. Hence, there exists a constant $C \in \mathbb{R}_{>0}$ such that

$$|\langle h, T(\varphi_\alpha * f)\rangle| = |\langle T(\varphi_\alpha * f), \mathrm{ev}_h\rangle| \leqslant C\|\varphi_\alpha * f\|_\infty \leqslant C\|f\|_\infty$$

for every $\alpha \in \mathbb{A}$. Thus,

$$\sup_{\alpha\in\mathbb{A}} |\langle h, T(\varphi_\alpha * f)\rangle| \leqslant C\|f\|_\infty < \infty$$

for every $h \in C_c(G)$. We conclude that S is vaguely bounded in $M(G)$, proving the first statement.

The assumptions on the family $(\varphi_\alpha)_{\alpha\in\mathbb{A}}$ turn it into a left approximate identity. We find, using Minkowski's Inequality for Integrals [9, Inequality 6.19.b] and the uniform continuity of $f \in C_c(G)$ that $\|\varphi_\alpha * f - f\|_\infty \to 0$, that is,

$$\lim_\alpha \varphi_\alpha * f = f$$

in $C_c(G)$ with the inductive limit topology. We conclude that, by the continuity of T and (10),

$$Tf = T\Big(\lim_\alpha \varphi_\alpha * f\Big) = \lim_\alpha T(\varphi_\alpha * f) = \lim_\alpha \big[\big(\mu_\alpha^+ * f\big) \cdot \nu - \big(\mu_\alpha^- * f\big) \cdot \nu\big]$$

for $f \in C_c(G)$. □

6.4. Theorem. *Let $T \colon C_c(G) \to M(G)$ be a positive linear map that commutes with the ρ_a for all $a \in G$. Then there exists a positive measure $\mu \in M(G)$ such that*

$$Tf = (\mu * f) \cdot \nu \tag{11}$$

for every $f \in C_c(G)$. Moreover, such a measure μ is unique.

Proof. We will use the previous Proposition 6.1 using $W = M(G)$, which has the vague topology of measures. To that end, we must first show that T is continuous. Such map T is continuous if and only if T is continuous in 0. That is, it suffices to show that $Tf_n \to 0$ vaguely for $f_n \twoheadrightarrow 0$.

Let $K \subseteq G$ be a compact set and let $(f_n)_{n\in\mathbb{N}} \subseteq C_c(G)$ be such that $\operatorname{supp}(f_n) \subseteq K$ for all $n \in \mathbb{N}$ and assume that this sequence converges uniformly to 0. Let $g \in C_c(G)$ be such that

$$0 \leqslant \chi_K(x) \leqslant g(x)$$

for all $x \in G$. Since g is positive on K, put

$$0 \leqslant a_n = \sup_{x\in K} \frac{|f_n(x)|}{g(x)},$$

then $|f_n| \leqslant a_n g$. Since f_n converges uniformly to 0, the sequence $(a_n)_{n\in\mathbb{N}}$ will also converge to 0. It follows from the positivity and linearity of T that

$$|Tf_n| = |T(f_n^+ - f_n^-)| = |Tf_n^+ - Tf_n^-| \leqslant |Tf_n^+| + |Tf_n^-| = Tf_n^+ + Tf_n^- = T|f_n| \leqslant a_n Tg.$$

Then it holds for $h \in C_c(G)$ that

$$|\langle Tf_n, \mathrm{ev}_h\rangle| \leqslant \big\langle |Tf_n|, \mathrm{ev}_{|h|}\big\rangle \leqslant \big\langle a_n Tg, \mathrm{ev}_{|h|}\big\rangle = a_n \big\langle Tg, \mathrm{ev}_{|h|}\big\rangle \to 0.$$

We conclude that (Tf_n) converges vaguely to 0 and we conclude that T is indeed a continuous map.

Using the same notations of Proposition 6.1, we now know that the measures μ_α are in fact positive elements of $M(G)$. By the first statement the set $S = \{(\mu_\alpha * f)\cdot\nu \colon \alpha \in \mathbb{A}\}$ is vaguely bounded in $M(G)$ for every $f \in C_c(G)$. This means, in particular for all positive $f, h \in C_c(G)$, that

$$\begin{aligned}
0 \leqslant \sup_{\alpha\in\mathbb{A}} \langle (\mu_\alpha * f)\cdot\nu, \mathrm{ev}_h\rangle &= \sup_{\alpha\in\mathbb{A}} \int h(z)(\mu_\alpha * f)(z)\, dz \\
&= \sup_{\alpha\in\mathbb{A}} \iint h(z) f\big(x^{-1}z\big)\, d\mu_\alpha(x)\, dz \\
&= \sup_{\alpha\in\mathbb{A}} \iint h(z) f\big(x^{-1}z\big)\, dz\, d\mu_\alpha(x) && \text{(Fubini)} \\
&= \sup_{\alpha\in\mathbb{A}} \iint h(xy) f(y)\, d\nu(xy)\, d\mu_\alpha(x) \\
&= \sup_{\alpha\in\mathbb{A}} \iint h(xy) f(y)\, dy\, d\mu_\alpha(x) \\
&= \sup_{\alpha\in\mathbb{A}} \int h * \check{f}(x)\, d\mu_\alpha(x) < \infty.
\end{aligned}$$

We continue by showing that the set $V = \{\mu_\alpha \colon \alpha \in \mathbb{A}\}$ is also vaguely bounded. To that end, let $f, h \in C_c(G)$ be positive and consider $k = h * \check{f} \in C_c(G)$. We then find, using that the μ_α are positive, that

$$0 \leqslant \sup_{\alpha\in\mathbb{A}} \langle \mu_\alpha, \mathrm{ev}_k\rangle = \sup_{\alpha\in\mathbb{A}} \int k(x)\, d\mu_\alpha(x) < \infty. \tag{12}$$

For any compact $K \subseteq G$, we are able to find positive functions f and h with compact support such that k satisfies

$$0 \leqslant \chi_K(x) \leqslant k(x)$$

for $x \in K$. For example, let $\psi \in C_c(G)$ such that $\psi|_K = 2$. Then $\varphi_\alpha * \psi \to \psi$ in $C_c(G)$. Fix $\alpha_0 \in \mathbb{A}$ such that $\|\varphi_{\alpha_0} * \psi - \psi\|_\infty < \frac{1}{2}$. Then $k = \varphi_{\alpha_0} * \psi$ suffices, since $\varphi_{\alpha_0} * \psi \geqslant \frac{3}{2}$ on K. Then, by (12) there exists $M_K \in \mathbb{R}_{>0}$ such that for any $\alpha \in \mathbb{A}$

$$0 \leqslant \mu_\alpha(K) \leqslant \langle \mu_\alpha, \mathrm{ev}_k \rangle \leqslant M_K.$$

Recall that the μ_α are positive, so we find for $h' \in C_c(G)$ that

$$0 \leqslant \sup_{\alpha\in\mathbb{A}} |\langle \mu_\alpha, \mathrm{ev}_{h'} \rangle| \leqslant \sup_{\alpha\in\mathbb{A}} \int_{\mathrm{supp}(h')} |h'(x)| \, d\mu_\alpha(x) \leqslant \sup_{\alpha\in\mathbb{A}} \|h'\|_\infty M_{\mathrm{supp}(h')} = \|h'\|_\infty M_{\mathrm{supp}(h')} < \infty.$$

We conclude that the set $V = \{\mu_\alpha \colon \alpha \in \mathbb{A}\} \subseteq W$ is vaguely bounded.

According to [3, Proposition III.1.9.15] or, as an application of [19, Theorem 3.15], this means for the weak*-topology on $M(G)$ that V is relatively compact in W, so there is a convergent subnet of $(\mu_\alpha)_{\alpha\in\mathbb{A}}$ in $\overline{V}$ converging to $\mu \in M(G)$, the vague limiting point of $(\mu_\alpha)_{\alpha\in\mathbb{A}}$. Note that μ is positive since all μ_α are. Hence, the net $(\mu_\alpha - \mu)_{\alpha\in\mathbb{A}}$ has a limiting point, the zero measure.

We continue by showing that for $f \in C_c(G)$

$$Tf \overset{(9)}{=} \lim_\alpha (\mu_\alpha * f) \cdot \nu = (\mu * f) \cdot \nu.$$

Then, since $h * \check{f} \in C_c(G)$ for $h, f \in C_c(G)$, it follows from (9) that,

$$\begin{aligned}
\langle Tf - (\mu * f) \cdot \nu, \mathrm{ev}_h \rangle &= \left\langle \lim_\alpha ((\mu_\alpha * f) \cdot \nu), \mathrm{ev}_h \right\rangle - \langle ((\mu * f) \cdot \nu), \mathrm{ev}_h \rangle \\
&= \lim_\alpha \int h(y) \mu_\alpha * f(y) \, dy - \int h(y) \mu * f(y) \, dy \\
&= \lim_\alpha \iint h(y) f\big(x^{-1}y\big) \, d\mu_\alpha(x) \, dy - \iint h(y) f\big(x^{-1}y\big) \, d\mu(x) \, dy \\
&= \lim_\alpha \iint h(y) f\big(x^{-1}y\big) \, dy \, d\mu_\alpha(x) - \iint h(y) f\big(x^{-1}y\big) \, dy \, d\mu(x) \qquad \text{(Fubini)} \\
&= \lim_\alpha \int h * \check{f}(x) \, d\mu_\alpha(x) - \int h * \check{f}(x) \, d\mu(x) \\
&= \lim_\alpha \Big\langle \mu_\alpha, \mathrm{ev}_{h*\check{f}} \Big\rangle - \Big\langle \mu, \mathrm{ev}_{h*\check{f}} \Big\rangle \\
&= \lim_\alpha \Big\langle \mu_\alpha - \mu, \mathrm{ev}_{h*\check{f}} \Big\rangle = 0.
\end{aligned}$$

We conclude that for $f \in C_c(G)$ it now holds that

$$Tf \overset{(9)}{=} \lim_\alpha (\mu_\alpha * f) \cdot \nu = (\mu * f) \cdot \nu.$$

We finish this proof by showing that μ is uniquely determined. Suppose that μ_1, μ_2 both satisfy (11), that is, assume that $(\mu_1 * f) \cdot \nu = (\mu_2 * f) \cdot \nu$ for all $f \in C_c(G)$. Both $\mu_1 * f$ and $\mu_2 * f$ are continuous functions and everywhere defined on G and the Haar measure is positive hence $\mu_1 * f = \mu_2 * f$ pointwise. In particular, at the point $x = e$

$$\int f(y^{-1}e) \, d\mu_1(y) = \mu_1 * f(e) = \mu_2 * f(e) = \int f(y^{-1}e) \, d\mu_2(y)$$

for all $f \in C_c(G)$. It follows from the Riesz Representation Theorem that $\mu_1 = \mu_2$. ☐

In the next theorem we extend this result to hold for positive linear maps $T\colon L^p(G)\to L^q(G)$ such that $\rho_a T = T\rho_a$ for every $a\in G$. The result is actually a bit weaker than one may expect, as we will point out first before stating and proving the actual result. In general we have assumed G to be only locally compact, in particular, G is not necessarily unimodular.

6.5. Proposition. *Suppose that G is not unimodular and $1\leqslant p,q<\infty$ with $p\neq q$. For any continuous linear map $T\colon L^p(G)\to L^q(G)$ such that $\rho_a T = T\rho_a$ for all $a\in G$ it holds that $T=0$.*

Proof. We already know (in general) for $p\in[1,\infty)$ and $f\in L^p(G)$ that

$$\|\rho_a f\|_p = \Delta(a)^{\frac{1}{p}}\|f\|_p.$$

Let $T\neq 0$ be a continuous linear map from $L^p(G)$ to $L^q(G)$ that commutes with the ρ_a for all $a\in G$. For $0\neq f\in L^p(G)$ we find that $0\neq Tf=g\in L^q(G)$. Using the aforementioned equality it holds that

$$\Delta(a)^{\frac{1}{q}}\|g\|_q = \|\rho_a g\|_q = \|\rho_a Tf\|_q = \|T\rho_a f\|_q \leqslant \|T\|\|\rho_a f\|_p = \Delta(a)^{\frac{1}{p}}\|T\|\|f\|_p,$$

hence it holds for all $a\in G$ that

$$\Delta(a)^{\frac{1}{q}-\frac{1}{p}} \leqslant \frac{\|T\|\|f\|_p}{\|g\|_q}.$$

Observe that the map

$$G\to\mathbb{R}_{>0}, a\mapsto \Delta(a)^{\frac{1}{q}-\frac{1}{p}}$$

is a group homomorphism. If G is not unimodular and $p\neq q$ then there exists $a'\in G$ such that $\Delta(a')\neq 1$. That is, the image of G is a non-trivial multiplicative subgroup of $\mathbb{R}_{>0}$, which is then unbounded. This is a contradiction with the finite upper bound $\frac{\|T\|\|f\|_p}{\|g\|_q}$, there is no $a'\in G$ such that $\Delta(a')\neq 1$. Hence $T=0$ when $p\neq q$. □

In any case $0\in M^{p,q}(G)$ and this Proposition shows that $M^{p,q}(G)=\{0\}$ for $p\neq q$ if G is not compact, abelian or at least unimodular. We will now formulate the conclusive result of this section without any other restrictions on p and q then $1\leqslant p,q<\infty$.

6.6. Theorem [2, Theorem 3.1], [4, Theorem 3.5]. *Let $T\colon L^p(G)\to L^q(G)$ be a positive operator for $1\leqslant p,q<\infty$ such that $\rho_a T = T\rho_a$ for every $a\in G$. Then there exists a unique positive measure $\mu\in M^{p,q}(G)$ such that*

$$Tf = \mu * f \tag{13}$$

holds ν-a.e. on G for every $f\in C_c(G)$.

Proof. To prove this Theorem, we will use Theorem 6.4. In the proof of this theorem we first proved that such map is continuous. However, in this application, we can easily deduce that the map $\tilde{T}$, where we will apply the Theorem to, is already continuous.

Consider the map

$$\tilde{T}\colon C_c(G) \overset{T|_{C_c(G)}}{\longrightarrow} L^q(G) \overset{\gamma'}{\longrightarrow} M(G),$$

which maps $f\in C_c(G)$ to $\nu_{Tf}\in M(G)$. We have seen before that each of the maps in this composition is positive and linear, hence $\tilde{T}$ is.

Next, we show that $\tilde{T}$ is a continuous map. To see this, let $f \in C_c(G)$ and $(f_n)_{n\in\mathbb{N}} \subseteq C_c(G)$ be such that $f_n \twoheadrightarrow f$, then surely $f_n \to f$ in $L^p(G)$ via the Dominated Convergence Theorem [8, Theorem 4.8.2] since the f_n are uniformly bounded and of compact support. A positive map $T\colon L^p(G) \to L^q(G)$ is in general continuous, see [1, Theorem 4.3] for more details. It follows from the continuity of T that $Tf_n \to Tf$ in $L^q(G)$. By Proposition 2.7, it follows that γ' is continuous on $L^q(G)$, and we see that $\nu_{Tf_n} \to \nu_{Tf}$ vaguely. We conclude that $\tilde{T}$ is continuous.

Let $a \in G$, we will now verify that $\tilde{T}$ also commutes with the ρ_a. To that end, let $f \in C_c(G)$ to find that

$$\rho_a \tilde{T} f = \rho_a \circ \gamma' \circ T|_{C_c(G)}(f) = \gamma' \circ \rho_a \circ T|_{C_c(G)}(f) = \gamma' \circ T|_{C_c(G)} \circ \rho_a(f) = \tilde{T}\rho_a f.$$

We conclude that $\tilde{T}$ commutes with all the ρ_a.

We now use Theorem 6.4 to find a unique positive measure $\mu \in M(G)$ such that $\tilde{T}f = (\mu * f) \cdot \nu$ for every $f \in C_c(G)$. That is,

$$T|_{C_c(G)} f = \mu * f$$

holds ν-a.e. on G for all $f \in C_c(G)$. Because T is a map from $L^p(G)$ to $L^q(G)$ it must hold that $\mu * f \in L^q(G)$, hence $\mu \in M^{p,q}(G)$. □

We continue with two results on pointwise expressions for T as in Theorem 6.6.

6.7. Lemma. *Let $T\colon L^p(G) \to L^q(G)$ be a positive operator for $1 \leqslant p, q < \infty$ such that $\rho_a T = T\rho_a$ for every $a \in G$. Then there exists a unique positive measure $\mu \in M^{p,q}(G)$ such that*

$$T\chi_K(x) = \int \chi_K\big(y^{-1}x\big)\, d\mu(y)$$

for ν-a.e. $x \in G$ and every compact $K \subseteq G$.

Proof. By Theorem 6.6, we have a unique positive measure $\mu \in M^{p,q}(G)$ such that

$$Tf(x) = \mu * f(x) = \int f\big(y^{-1}x\big)\, d\mu(y)$$

for ν-a.e. $x \in G$ and every $f \in C_c(G)$.

Let $K \subseteq G$ be compact and let $U \subseteq G$ be relatively compact such that $K \subseteq U$. Then, by a Corollary of Lusin's Theorem [18, Theorem 2.24], there is a sequence $(g_n)_{n\in\mathbb{N}} \subseteq C_c(U, G) \subseteq C_c(G)$ such that $|g_n| \leqslant \chi_U$ and

$$\chi_K(x) = \lim_{n\to\infty} g_n(x)$$

for ν-a.e. $x \in G$. Since $|g_n| \leqslant \chi_U$, we have

$$|g_n - \chi_K|^p \leqslant (\chi_U + \chi_K)^p \leqslant 2^p \chi_U \in L^1(G).$$

It follows from the Dominated Convergence Theorem [18, Theorem 1.34] that

$$\int |g_n - \chi_K|^p \, d\nu \to 0,$$

hence $g_n \to \chi_K$ in $L^p(G)$. By continuity of T, then also $Tg_n \to T\chi_K$ in $L^q(G)$. Using [18, Theorem 3.12], there is a subsequence $(Tg_m)_{m\in\mathbb{N}}$ of $(Tg_n)_{n\in\mathbb{N}}$ such that $Tg_m(x)$ converges to $T\chi_K(x)$ for ν-a.e. $x \in G$. Then, for all such $x \in G$,

$$T\chi_K(x) = \lim_{m\to\infty} Tg_m(x) = \lim_{m\to\infty} \int g_m\big(y^{-1}x\big)\, d\mu(y) = \int \chi_K\big(y^{-1}x\big)\, d\mu(y),$$

by the Dominated Convergence Theorem [18, Theorem 1.34]. □

6.8. Lemma. *Let $T\colon L^p(G) \to L^q(G)$ be a positive operator for $1 \leqslant p, q < \infty$ such that $\rho_a T = T\rho_a$ for every $a \in G$. Then there exists a unique positive measure $\mu \in M^{p,q}(G)$ such that*

$$T\chi_A(x) = \int \chi_A\big(y^{-1}x\big)\, d\mu(y)$$

for ν-a.e. $x \in G$ and every $A \subseteq G$ such that $\mu(A) < \infty$.

Proof. By Lemma 6.7, we have a unique positive measure $\mu \in M^{p,q}(G)$ such that

$$T\chi_K(x) = \int \chi_K\big(y^{-1}x\big)\, d\mu(y)$$

for ν-a.e. $x \in G$ and every compact $K \subseteq G$.

Let $A \subseteq G$ be such that $\mu(A) < \infty$. Then, by [9, Proposition 7.5],

$$\mu(A) = \sup\{\mu(K)\colon K \subseteq A \text{ and } K \text{ compact}\}.$$

Choose a sequence of compact sets $\tilde{K}_n \subseteq A$ such that $\mu(\tilde{K}_n) \to \mu(A)$. Let $K_1 = \tilde{K}_1$ and for each $n \geqslant 1$, let

$$K_n = \cup_{i=1}^n \tilde{K}_i,$$

which is again compact. Then $K_n \subseteq K_{n+1}$ for each $n \geqslant 1$ and $\mu(\tilde{K}_n)$ is then an increasing sequence that converges to $\mu(A)$. Clearly, by construction of the K_n, $\chi_{K_n} \uparrow \chi_A$ on G. Then, by the same reasoning as before, $\chi_{K_n} \to \chi_A$ in $L^p(G)$ by the Dominated Convergence Theorem [18, Theorem 1.34]. By [18, Theorem 3.12], there is a subsequence $(\chi_{K_m})_{m\in\mathbb{N}}$ of $(\chi_{K_n})_{n\in\mathbb{N}}$ such that $\chi_{K_m}(x)$ converges to $\chi_A(x)$ for ν-a.e. $x \in G$. Since also $\chi_{K_m} \uparrow \chi_A$ on G, for all such $x \in G$,

$$T\chi_A(x) = \lim_{m\to\infty} T\chi_{K_m}(x) = \lim_{m\to\infty} \int \chi_{K_m}\big(y^{-1}x\big)\, d\mu(y) = \int \chi_A\big(y^{-1}x\big)\, d\mu(y)$$

by continuity of T, Lemma 6.7 and the Monotone Convergence Theorem [18, Theorem 1.26]. $\square$

We are now able to extend the result of Theorem 6.6 by giving a pointwise expression for T.

6.9. Theorem. *Let $T\colon L^p(G) \to L^q(G)$ be a positive operator for $1 \leqslant p, q < \infty$ such that $\rho_a T = T\rho_a$ for every $a \in G$. Then there exists a unique positive measure $\mu \in M^{p,q}(G)$ such that*

$$Tf(x) = \int f\big(y^{-1}x\big)\, d\mu(y)$$

for ν-a.e. $x \in G$ for every $f \in L^p(G)$.

Proof. For $f \in C_c(G) \subseteq L^p(G)$, the result follows directly from Theorem 6.6. Furthermore, by Lemma 6.8, we have a unique positive measure $\mu \in M^{p,q}(G)$ such that

$$T\chi_A(x) = \int \chi_A\big(y^{-1}x\big)\, d\mu(y)$$

for ν-a.e. $x \in G$ and every $A \subseteq G$ such that $\mu(A) < \infty$.

If $f \in L^p(G)$ is positive, let $(s_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive simple functions such that $s_n \uparrow f$. Then

$$|f - s_n|^p \leqslant (|f| + |s_n|)^p \leqslant 2^p|f|^p \in L^1(G)$$

It follows from the Dominated Convergence Theorem [18, Theorem 1.34] that

$$\int |f - s_n|^p \, d\nu \to 0$$

hence $s_n \to f$ in $L^p(G)$. By continuity of T, then also $Ts_n \to Tf$ in $L^p(G)$. Using [18, Theorem 3.12] there is a subsequence $(Ts_m)_{m\in\mathbb{N}}$ of $(Ts_n)_{n\in\mathbb{N}}$ such that $Ts_m(x)$ converges to $Tf(x)$ for ν-a.e. $x \in G$.

A simple function s_m is a finite linear combination of indicator functions χ_A with $\mu(A) < \infty$, hence

$$\mu(\{x \in G \colon s_m(x) \neq 0\}) = \mu(\cup_{i=1}^n A_i) < \infty.$$

Now, use Lemma 6.8 and the Monotone Convergence Theorem [18, Theorem 1.26] to find, for all $x \in G$ such that $Ts_m(x)$ converges to $Tf(x)$,

$$Tf(x) = \lim_{m\to\infty} T\chi_{s_m}(x) = \lim_{m\to\infty} \int s_m\big(y^{-1}x\big)\, d\mu(y) = \int f\big(y^{-1}x\big)\, d\mu(y).$$

If $f \in L^p(G)$ is any function, let $f^+, f^- \in L^p(G)$ be positive such that $f = f^+ - f^-$. Then, since T is linear on $L^p(G)$,

$$Tf(x) = Tf^+(x) - Tf^-(x) = \int f^+\big(y^{-1}x\big)\, d\mu(y) - \int f^-\big(y^{-1}x\big)\, d\mu(y) = \int f\big(y^{-1}x\big)\, d\mu(y),$$

for ν-a.e. $x \in G$. □

7 Convolution actions of bounded measures on $L^p(G)$

The result of Theorem 6.6 still left some room for improvement in case $p, q = 1$. In the proof of that theorem we have shown that there is a measure $\mu \in M^{p,q}(G)$ such that (13) holds. We continue with the following proposition to show that μ must in fact be bounded.

7.1. Proposition. *Let $p, q = 1$. The measure $\mu \in M^{1,1}(G)$ obtained in Theorem 6.6 is bounded.*

Proof. We know from Theorem 6.6 that $\mu \in M^{1,1}(G)$ is positive. Let $K \subseteq G$ be a compact set and consider $f = \chi_K \in L^1(G)$. It follows from the left invariance of ν that

$$\begin{aligned}
\|T\chi_K\|_1 &= \left\| \left\{ x \mapsto \int \chi_K\left(y^{-1}x\right) d\mu(y) \right\} \right\|_1 \\
&= \iint \chi_K\left(y^{-1}x\right) d\mu(y)\, d\nu(x) \\
&= \iint \chi_K\left(y^{-1}x\right) d\nu(x)\, d\mu(y) && \text{(Fubini)} \\
&= \iint \chi_K(z)\, d\nu(yz)\, d\mu(y) \\
&= \iint \chi_K(z)\, d\nu(z)\, d\mu(y) \\
&= \int \nu(K)\, d\mu(y) \\
&= \nu(K)\mu(G).
\end{aligned}$$

By assumption $T\chi_K \in L^1(G)$, hence $\|T\chi_K\|_1 < \infty$ and we know that ν is finite on compact sets. Hence $\mu(G)$ must also be finite, that is, we conclude that μ must be bounded. □

We will now compute the operator norm of the operator T in Theorem 6.6.

7.2. Proposition. *Let $p, q = 1$. The operator T of Theorem 6.6 is isometric.*

Proof. Using the pointwise expression from Theorem 6.9, it follows that

$$\begin{aligned}
\|T(f)\|_1 &= \left\| \left\{ x \mapsto \left| \int f\left(y^{-1}x\right) d\mu(y) \right| \right\} \right\|_1 \\
&= \int \left| \int f\left(y^{-1}x\right) d\mu(y) \right| dx \\
&\leqslant \iint \left| f\left(y^{-1}x\right) \right| d\mu(y)\, dx \\
&= \iint \left| f\left(y^{-1}x\right) \right| dx\, d\mu(y) && \text{(Fubini)} \\
&= \iint |f(z)|\, d\nu(yz)\, d\mu(y) \\
&= \iint |f(z)|\, dz\, d\mu(y) \\
&= \|f\|_1 \|\mu\|.
\end{aligned}$$

If $f \in L^p(G)$ is arbitrary such that $\|f\|_1 \leqslant 1$, it follows from this estimate that

$$\begin{aligned} \|T\| &= \sup\{\|T(f)\|_1 \colon f \in L^1(G) \text{ such that } \|f\|_1 \leqslant 1\} \\ &\leqslant \sup\{\|f\|_1 \|\mu\| \colon f \in L^1(G) \text{ such that } \|f\|_1 \leqslant 1\} \\ &\leqslant \|\mu\|. \end{aligned}$$

Thus, $\|T\| \leqslant \|\mu\|$. Let $g \in L^1(G)$ be positive such that $\|g\|_1 = 1$. Then, by the positivity of μ and g, we can ignore the absolute value in the estimate above and have equality everywhere. Then

$$\|T(g)\|_1 = \|g\|_1 \|\mu\| = \|\mu\|.$$

We conclude that $\|T\| = \|\mu\|$ for all positive measures $\mu \in M_b(G)$. □

7.1 The convolution operator

Let $p, q \in [1, \infty)$ be given and let $\mu \in M^{p,q}(G)$. Let $c \in \mathbb{R}_{>0}$ be such that $\|\mu * f\|_q \leqslant c\|f\|_p$ for all $f \in C_c(G)$. The map

$$T_\mu^q \colon C_c(G) \to L^q(G), T_\mu^q f = \mu * f$$

is well-defined, we have seen before that $\mu * f$ exists and is defined everywhere on G for all $\mu \in M(G)$ and $f \in C_c(G)$. Furthermore, T_μ^q is linear on $C_c(G)$ and

$$\|T_\mu^q(f)\|_q = \|\mu * f\|_q \leqslant c\|f\|_p,$$

hence $\|T_\mu^q\| \leqslant c$. Thus T_μ^q is a continuous bounded linear operator on $C_c(G)$. Since $C_c(G)$ is dense in $L^p(G)$, there exists a unique continuous extension of T_μ^q to the *convolution operator*

$$T_\mu^{p,q} \colon L^p(G) \to L^q(G).$$

If we now combine the result of Theorem 6.6 with the results of Proposition 7.1 and Proposition 7.2, we have shown that there exists a positive Radon measure $\mu \in M_b(G)$ such that $T = T_\mu^{1,1}$ in case $p, q = 1$ such that $\|T\| = \left\|T_\mu^{1,1}\right\| = \|\mu\|$.

7.2 Amenability

We need to introduce another property of the group G called amenability to find similar results for $1 < p < \infty$. We follow the work of Greenleaf [13] while summarizing the definitions and some equivalent statements.

7.3. Definition. Let $B(G) \subseteq \mathbb{R}^G$ be the space of all bounded real-valued functions on G with the supremum norm $\|\cdot\|_\infty$. Let $X \subseteq B(G)$ be a closed subspace that includes all constant functions. A *mean on* X is a linear functional m on X such that

$$\inf_{x \in G} f(x) \leqslant m(f) \leqslant \sup_{x \in G} f(x). \tag{14}$$

for all $f \in X$. A mean is *left invariant* if $m(\lambda_x f) = m(f)$ for all $x \in G$ and $f \in X$.

It holds for any mean that $m(f) \geqslant 0$ for all $f \geqslant 0$, that is, m must be positive and $m(\chi_G) = 1$. This is in fact equivalent to (14). It follows that $\|m\| = 1$ for any mean on X. Usually, a mean on X is not unique.

7.4. Definition. A locally compact Hausdorff group G is *left amenable* if there exists a left invariant mean on $X = L^\infty(G)$.

In this case, (14) becomes

$$\operatorname*{ess\,inf}_{x\in G} f(x) \leqslant m(f) \leqslant \operatorname*{ess\,sup}_{x\in G} f(x).$$

This is one of the more natural definitions of amenability of G. There are numerous equivalent statements describing *left amenability of* G, see for example [13, Theorem 2.2.1]. These are still quite similar to our definition and the statement is almost the same: there exists a mean on a certain function space. As an example, compact groups are left amenable; the normalized Haar measure is a left invariant mean. Also abelian and finite groups are left amenable. From now on, we will write amenable instead of left amenable as we will only consider left amenable groups.

7.5. Definition. A continuous real-valued function f on G is said to be *positive definite* if

$$\iint f\big(y^{-1}x\big)g(y)g(x)\,dy\,dx \geqslant 0$$

for every $g \in C_c(G)$.

We will now formulate another condition on G, a uniform approximation property of the group. At first sight, this property may appear to be completely unrelated to the amenability of G.

7.6. Definition. The group G has property (R) if every continuous positive-definite function on G can be approximated uniformly on compact sets by functions of the form $f * \tilde{f}$, where $f \in C_c(G)$.

We will now use property (R) to state the following Theorem by Gilbert [11].

7.7. Theorem [2, Theorem 3.2], [11, Theorem A]. *Let $1 < p < \infty$. The following are equivalent:*

1. *G has property (R),*
2. *$\|T_\mu^{p,p}\| = \|\mu\|$ for every positive $\mu \in M_b(G)$,*
3. *If μ is a positive measure on G such that $\mu * f \in L^p(G)$ and $\|\mu * f\|_p \leqslant c\|f\|_p$ for a positive constant c and every $f \in C_c(G)$, then μ is bounded.*

This theorem extends the results of Proposition 7.1 and Proposition 7.2 to the case where $p \in (1, \infty)$ and $q = p$. We end this section with a proof that Property (R) is actually equivalent to G being amenable.

7.8. Proposition. *A locally compact group G is amenable if and only if G has property (R).*

Proof. We follow [17, Definition 8.3.1] for the first part of this proof. A locally compact group is said to have Reiter's property (P_p), with $1 \leqslant p < \infty$, if it satisfies the following condition: for any compact set $K \subseteq G$ and every $\varepsilon \in \mathbb{R}_{>0}$ there is a positive function $f \in L^p(G)$, depending on K and ε, such that $\|f\|_p = 1$ and $\|\lambda_x f - f\|_p < \varepsilon$ or all $x \in K$.

By [13, Theorem 2.2.1] and [16] we find that G is amenable if and only if property (P_1) holds. By [17, Theorem 8.3.2] the properties (P_p) are all equivalent to property (P_1). Hence, G is amenable if and only if property (P_2) holds.

Let property (P') be the condition on G that the constant function χ_G can be approximated uniformly on compact sets by functions of the form $f * \tilde{f}$, with $f \in C_c(G)$. In [17, Theorem 8.3.18] it is shown that property (P_2) and property (P') are equivalent. In the proof of this Theorem the magic happens, where [12, Theorem 17] is applied to relate positive-definite functions in $C_c(G)$ to functions of the form $f * \tilde{f}$, with $f \in C_c(G)$.

We conclude with [6, Theorem 18.3.6] where it is shown that, together with other equivalent statements, property (P') is equivalent with property (R). $\square$

8 Isometric lattice isomorphisms

In this section we will combine the previous results to answer our main question, in what case does the action of a Radon measure on $L^p(G)$ give rise to an lattice isomorphism. In order to prove this, we need to introduce a few new spaces. For this section, let $p, q \in [1, \infty)$ be given.

First of all, let $\mathcal{L}_r(L^p(G))$ be the vector space consisting of all regular operators $L^p(G) \to L^p(G)$, that is, operators that can be written as the difference of two positive operators. The space $L^p(G)$ is a Dedekind complete Riesz space. By the Riesz-Kantorovich Theorem [1, Theorem 1.18] and the Remark following the proof, $\mathcal{L}_r(L^p(G))$ is a Dedekind complete Riesz space with a lattice structure such that

$$(S \vee T)(f) = \sup\{S(g) + T(h) \colon g, h \in L^p(G) \text{ positive such that } g + h = f\}$$

for $S, T \in \mathcal{L}_r(L^p(G))$ and $f \in L^p(G)$ positive. Each $T \in \mathcal{L}_r(L^p(G))$ is a regular operator, so T has a modulus, $|T| = T^+ + T^-$. The *regular norm* on $\mathcal{L}_r(L^p(G))$ is then given by

$$\|T\|_r \coloneqq \||T|\| = \sup\{\||T|f\| \colon f \in L^p(G) \text{ such that } \|f\| \leqslant 1\}.$$

When endowed with the regular norm and composition as multiplication, $\mathcal{L}_r(L^p(G))$ becomes a Banach algebra. It follows from [1, Theorem 4.74] that $\mathcal{L}_r(L^p(G))$ is a Dedekind complete Banach lattice with respect to the regular norm.

8.1. Proposition. *The convolution operator $T_\mu^{p,q}$ is a regular operator for every $\mu \in M(G)$.*

Proof. First of all, note that if $f \in C_c(G)$ is positive and μ is positive, the convolution $\mu * f$ is also positive, hence $T_\mu^{p,q}$ is positive. If μ is not positive, then there exist positive measures μ^+, μ^- such that $\mu = \mu^+ - \mu^-$. If $f \in C_c(G)$ is positive, the convolutions $\mu^+ * f$ and $\mu^- * f$ are positive. Moreover,

$$T_\mu^{p,q}(f) = \mu * f = \mu^+ * f - \mu^- * f = T_{\mu^+}^{p,q}(f) - T_{\mu^-}^{p,q}(f)$$

which is then the difference of positive operators. We conclude that $T_\mu^{p,q}$ is a regular operator. □

As a result of Proposition 6.5, we will only consider the operator $T_\mu^{p,p}$ which is then an element of $\mathcal{L}_r(L^p(G))$.

8.2. Proposition. *The map*

$$\tau_p' \colon M_b(G) \to \mathcal{L}_r(L^p(G)), \mu \mapsto T_\mu^{p,p}$$

is an injective Banach algebra homomorphism for every $p \in [1, \infty)$.

Proof. Convolution on $M_b(G)$ is associative, from which it easily follows that

$$\tau_p'(\mu_1 * \mu_2) = T_{\mu_1 * \mu_2}^{p,p} = T_{\mu_1}^{p,p} \circ T_{\mu_2}^{p,p} = \tau_p'(\mu_1) \circ \tau_p'(\mu_2)$$

It is left to show that τ_p' is injective. We will omit the proof since we have already seen it while proving that the μ in Theorem 6.4 is unique. We conclude that τ_p' is an injective Banach algebra homomorphism. □

Next, we shift our attention to lattices, in order to prove our main theorem. First, we will show some important properties of the *regular commutant of the right translation operators*, the set

$$\mathcal{L} = \{T \in \mathcal{L}_r(L^p(G)) \colon \rho_a T = T\rho_a \text{ for all } a \in G\}.$$

To that end, let $a \in G$ and consider the map

$$\psi_a \colon \mathcal{L}_r(L^p(G)) \to \mathcal{L}_r(L^p(G)), T \mapsto \rho_a \circ T \circ \rho_a^{-1}.$$

Furthermore, consider the following vector subspaces of $\mathcal{L}_r(L^p(G))$

$$\mathrm{Fix}(\psi_a) \coloneqq \{T \in \mathcal{L}_r(L^p(G)) \colon \psi_a(T) = T\}$$

and

$$\cap_{a\in G}\mathrm{Fix}(\psi_a).$$

8.3. Proposition. *The following statements are true:*

1. *the map ψ_a is a lattice automorphism of $\mathcal{L}_r(L^p(G))$ for all $a \in G$,*
2. *$\mathcal{L} = \cap_{a\in G}\mathrm{Fix}(\psi_a)$,*
3. *the vector subspaces $\mathrm{Fix}(\psi_a) \subseteq \mathcal{L}_r(L^p(G))$ and $\mathcal{L} \subseteq \mathcal{L}_r(L^p(G))$ are sublattices of $\mathcal{L}_r(L^p(G))$,*
4. *$T_\mu^{p,p} \in \mathcal{L}$ for each positive $\mu \in M_b(G)$.*

Proof. Let $a \in G$. To show that ψ_a is a lattice automorphism of $\mathcal{L}_r(L^p(G))$, we will first show that ψ_a is injective and surjective. Let $S, T \in \mathcal{L}_r(L^p(G))$ be such that $\psi_a(S) = \psi_a(T)$, then

$$\rho_a \circ S \circ \rho_a^{-1} = \rho_a \circ T \circ \rho_a^{-1},$$

and after composition with ρ_a^{-1},

$$S \circ \rho_a^{-1} = T \circ \rho_a^{-1}.$$

Let $f \in L^p(G)$, then $S(\rho_a^{-1} f) = T(\rho_a^{-1} f)$ ν-a.e. on G. Hence $S = T$, which proves that ψ_a is injective. Suppose that $T \in \mathcal{L}_r(L^p(G))$. The translation operators ρ_a and $\rho_a^{-1} = \rho_{a^{-1}}$ are positive on $L^p(G)$, hence regular. The composition $\rho_a^{-1} \circ T \circ \rho_a$ is again a regular map. Then

$$\psi_a\big(\rho_a^{-1} \circ T \circ \rho_a\big) = \rho_a \circ \big(\rho_a^{-1} \circ T \circ \rho_a\big) \circ \rho_a^{-1} = T.$$

We conclude that ψ_a is a bijection and has an inverse given by

$$\psi_a^{-1} \colon \mathcal{L}_r(L^p(G)) \to \mathcal{L}_r(L^p(G)), T \mapsto \rho_a^{-1} \circ T \circ \rho_a.$$

Both ψ_a and ψ_a^{-1} are compositions of the positive translation operators, and if $T \in \mathcal{L}_r(L^p(G))$ is positive then $\psi_a(T)$ and $\psi_a^{-1}(T)$ are compositions of positive operators. We conclude that ψ_a and ψ_a^{-1} are positive. By [1, Theorem 2.15], ψ_a is a lattice isomorphism for each $a \in G$, hence a lattice automorphism of $\mathcal{L}_r(L^p(G))$ for each $a \in G$.

To show the desired equality, let $T \in \mathcal{L}_r(L^p(G))$ be a fixed point of ψ_a for some $a \in G$. That is, $\psi_a(T) = T$, which is by definition of ψ_a equivalent to $\rho_a \circ T \circ \rho_a^{-1} = T$. After an application of ρ_a from the right side, this is equivalent to $\rho_a \circ T = T \circ \rho_a$. If $T \in \cap_{a\in G}\mathrm{Fix}(\psi_a) \subseteq \mathcal{L}_r(L^p(G))$, this is equivalent to $\rho_a \circ T = T \circ \rho_a$ for all $a \in G$, hence $T \in \mathcal{L}$.

To show that $\text{Fix}(\psi_a)$ is a sublattice of $\mathcal{L}_r(L^p(G))$ for each $a \in G$, it suffices to show that it is closed under the lattice operations. Let $a \in G$ and let $S, T \in \text{Fix}(\psi_a)$, then

$$\psi_a(S \vee T) = \psi_a(S) \vee \psi_a(T) = S \vee T.$$

Hence $S \vee T \in \text{Fix}(\psi_a)$. Thus $\text{Fix}(\psi_a)$ is a sublattice. The intersection of sublattices is again a sublattice and we conclude that $\mathcal{L}$ is also a sublattice of $\mathcal{L}_r(L^p(G))$.

Let $\mu \in M_b(G)$ be positive and let $f \in L^p(G)$. Let $a \in G$, the convolutions $\mu * f$ and $\mu * \rho_a f$ exist and since $L^p(G) \subseteq L^1_{loc}(G)$ it follows from Proposition 5.8 that

$$\rho_a T^{p,p}_\mu f = \rho_a(\mu * f) = \mu * \rho_a f = T^{p,p}_\mu(\rho_a f) = T^{p,p}_\mu \rho_a f.$$

Hence $\rho_a T^{p,p}_\mu = T^{p,p}_\mu \rho_a$ for all $a \in G$. We conclude that $T^{p,p}_\mu \in \mathcal{L}$ since $T^{p,p}_\mu$ is a regular operator. □

Since $\text{Fix}(\psi_a) \subseteq \mathcal{L}_r(L^p(G))$ and $\cap_{a\in G}\text{Fix}(\psi_a) \subseteq \mathcal{L}_r(L^p(G))$ are sublattices, they are Banach lattices in their own right.

The Banach algebra $M_b(G)$ is also a Banach lattice. To see this, let $\mu \in M_b(G)$ and let A be a measurable set. The lattice structure is then derived from

$$|\mu| = \sup\left\{\sum_{i=1}^{\infty}|\mu(A_i)| \colon \text{ all partitions } (A_i)_{i\in\mathcal{I}} \text{ of } E\right\}$$

using the lattice identity

$$\mu_1 \vee \mu_2 = \frac{\mu_1 + \mu_2 + |\mu_1 + \mu_2|}{2},$$

for $\mu_1, \mu_2 \in M_b(G)$. If μ_1, μ_2 are positive, then

$$\mu_1 \vee \mu_2(A) = \sup\{\mu_1(B) + \mu_2(A \setminus B) \colon \text{ where } B \subseteq A \text{ is measurable}\}$$

for each $A \subseteq G$ measurable. Since $M_b(G)$ is a Banach algebra and the norm is also a lattice norm, we conclude that $M_b(G)$ is a Banach algebra lattice.

8.4. Theorem [2, Theorem 3.3]. *Let $p \in [1, \infty)$ and consider the map*

$$\tau_p \colon M_b(G) \to \mathcal{L}, \mu \mapsto T^{p,p}_\mu.$$

If $p = 1$ or if G is amenable, then τ_p is an isometric algebra and lattice isomorphism.

Proof. By Proposition 8.3 this map is well-defined and by Proposition 8.2 τ_p is an injective Banach algebra homomorphism. We start by showing that every positive operator in $T \in \mathcal{L}$ is a convolution operator. The operator T then satisfies the conditions of Theorem 6.6. Thus, there exists a positive Radon measure $\mu \in M^{p,p}(G)$ such that $T = T^{p,p}_\mu$.

If $p = 1$, it follows from Proposition 7.1 that μ must be bounded. If $p \in (1, \infty)$, assume that G is amenable. By Theorem 7.7 and Proposition 7.8 amenability and property (R) are equivalent and it follows from the results of Theorem 6.6 that μ is a bounded measure. In both cases we find that

$$\tau_p(\mu) = T^{p,p}_\mu = T$$

for $p \in [1, \infty)$.

Let $T \in \mathcal{L}$ be any operator, there exist positive $T_1, T_2 \in \mathcal{L}$ such that $T = T_1 - T_2$ since $\mathcal{L}$ is a sublattice of the regular operators. By the previous reasoning, there exist $\mu_1, \mu_2 \in M_b(G)$ such that

$$T = T_1 - T_2 = T^{p,p}_{\mu_1} - T^{p,p}_{\mu_2}.$$

That is,

$$T(f) = T^{p,p}_{\mu_1}(f) - T^{p,p}_{\mu_2}(f) = \mu_1 * f - \mu_2 * f = (\mu_1 - \mu_2) * f = T^{p,p}_{\mu_1-\mu_2}(f) = \tau_p(\mu_1 - \mu_2)(f)$$

for $f \in L^p(G)$. Hence

$$T = T^{p,p}_{\mu_1-\mu_2} = \tau_p(\mu_1 - \mu_2)$$

and we conclude that every $T \in \mathcal{L}$ is a convolution operator associated with a bounded Radon measure on G. We have also shown that τ_p is surjective.

We have shown that τ_p is a bijection between the Banach lattices $M_b(G)$ and $\mathcal{L}$ and it follows from the proof that τ_p is surjective that the inverse of τ_p is given by

$$\tau_p^{-1} \colon \mathcal{L} \to M_b(G), T^{p,p}_{\mu} \to \mu.$$

We have seen that the operator $T^{p,p}_{\mu}$ is positive when μ is positive, hence τ_p is positive. The converse is also true, if $T \in \mathcal{L}$ is positive, then it is a convolution operator with respect to a positive measure, hence τ_p^{-1} is also positive. We conclude that the operator τ_p is a bipositive bijection between Riesz spaces, thus by [1, Theorem 2.15] we find that τ_p is a lattice isomorphism.

We finish this proof by showing that τ_p is isometric. Let $\mu \in M_b(G)$ be a bounded Radon measure on G. Then $|\mu|$ is a positive measure and if $p = 1$, it follows from Proposition 7.2 that $\|\tau_p(|\mu|)\| = \||\mu|\|$. If $p \in (1, \infty)$, assume that G is amenable. By Theorem 7.7 and Proposition 7.8 amenability and property (R) are equivalent and it follows that $\|\tau_p(\mu)\| = \|T^{p,p}_{\mu}\| = \|\mu\|$ for positive $\mu \in M_b(G)$. That is,

$$\|\tau_p(\mu)\| = \|T^{p,p}_{\mu}\| = \|\mu\|$$

for $p \in [1, \infty)$ and $\mu \in M_b(G)$ positive. Then, since τ_p is a lattice isomorphism,

$$\|\tau_p(\mu)\|_r = \||\tau_p(\mu)|\| = \|\tau_p(|\mu|)\| = \||\mu|\| = \|\mu\|.$$

We conclude that τ_p is an isometric algebra and lattice isomorphism. □

A General facts

In this appendix we formulate some facts that are well known in functional analysis or direct consequences of such facts.

A.1 Locally convex spaces

Let X be a vector space with a family of seminorms on it. The weak topology on X is the coarsest (weakest or smallest) topology on X such that all these seminorms are continuous. This is a *locally convex topology.* Thus any vector space has a locally convex topology.

If a vector space X is normed, simply take the family of seminorms on X to consist of only one element: the map $x \mapsto \|x\|$. If X is not normed, consider any collection $\mathcal{F}$ of linear functionals on X and consider the family of seminorms associated to $\mathcal{F}$: the seminorms are $\rho_f(x) = |f(x)|$ for all $f \in \mathcal{F}$.

Any normed space is Hausdorff locally convex and every Banach space is complete Hausdorff locally convex.

In a locally convex space, each point has a local base consisting of convex sets. This is equivalent to the previous definition via Minkowski functionals.

A.2 Inductive limit spaces

To find a topology on $C_c(G)$ we previously used the theory of inductive limit spaces. We will here give some more details about this construction, following the work of Edwards [8, page 430].

Let X be a topological vector space (a vector space with a topology such that the vector space operations are continuous) and let $(Y_i)_i$, with i in some index set I, be a family of locally convex topological vector spaces. Let ψ_i be a linear map $Y_i \to X$ such that

$$\bigcup_{i \in I} \psi_i(Y_i) = X.$$

There is a strongest locally convex topology τ on X such that all ψ_i are continuous. We call τ the *inductive limit* of the topologies on the Y_i relative to the maps ψ_i. The space (X, τ) is the inductive limit of the topological vector spaces Y_i relative to the maps ψ_i.

In the special case where the Y_i themselves are actually vector subspaces of X and for each $i \in I$ the ψ_i the inclusion map of Y_i in X, we call τ and (X, τ) the *internal inductive limit.*

We will now show that $C_c(G)$ is an internal inductive limit. To that end, let $S \subseteq G$ be relatively compact. Define the set

$$C_c(S, G) = \{f \in C_c(G) \colon \operatorname{supp}(f) \subseteq S\}$$

and note that this is a vector subspace of $C_c(G)$. It follows that each $C_c(S, G)$ is a Banach space when equipped with the supremum norm as before.

Let $(S_i)_{i \in I}$ be the collection of all relatively compact subsets of G. It follows immediately that

$$C_c(G) = \bigcup_{i \in I} C_c(S_i, G).$$

Then we can indeed endow $C_c(G)$ with the internal inductive limit topology by means of the vector subspaces $C_c(S_i, G)$ and the inclusion maps $\psi_i \colon C_c(S_i, G) \to C_c(G)$.

As a final remark, note that this inductive limit topology on $C_c(G)$ is independent of the collection $(S_i)_{i\in I}$. In case $S \subseteq S'$, the topology of $C_c(S', G)$ induces on $C_c(S, G)$ the same topology that $C_c(S, G)$ was already endowed with. Hence, we may and will take $(S_i)_{i\in I}$ to be all compact subsets of G.

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