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Gromov-Witten invariants of the classifying stack of principal $\mathbb{G}_m$-bundles

Master's thesis

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Contents

Introduction

In algebraic geometry, moduli spaces are studied to answer questions related to classification problems, such as classifying lines in the plane through the origin by the projective line. In this thesis, we are interested in

$$\widetilde{\mathcal{M}}_{g,I}(\mathbf{B}\mathbb{G}_m),$$

which is a moduli stack for Gieseker $\mathbb{G}_m$-bundles, which are line bundles satisfying certain conditions, on (modifications of) stable curves of genus g marked by an ordered set I. This is the stack to consider if we want to study line bundles on stable curves. However, it is not a Deligne-Mumford algebraic stack, but an Artin algebraic stack. In the article [6], the authors attempt to study Gromov-Witten invariants of this stack. In particular, they prove coherence of the pushforwards of certain specific K-theory classes along the forgetful map

$$F\colon \widetilde{\mathcal{M}}_{g,I}(\mathbf{B}\mathbb{G}_m) \to \overline{\mathcal{M}}_{g,I}$$

to the stack of stable curves of genus g marked by I. The aim of this thesis is to give examples to illustrate the article [6] and prove its main results in the case where the genus is 0 and we only have three marks.

Overview

In chapter one, the classifying stack $\mathbf{B}G$ of G-bundles for a group scheme G is defined, for which we introduce the concepts of stacks, group schemes, bundles and torsors. In particular we are interested in $\mathbf{B}\mathbb{G}_m$ for the multiplicative group scheme $\mathbb{G}_m$, in which case $\mathbf{B}\mathbb{G}_m$ will be the stack of line bundles, considered with isomorphisms.

In chapter two, a proof is given of the fact that for a smooth group scheme G over a field k, there is an equivalence of categories

$$\{\text{Quasi-coherent sheaves on the stack } \mathbf{B}G\} \leftrightarrow \{\text{Representations of } G\}.$$

In particular, if $G = \mathbb{G}_m$ we can classify the coherent sheaves on $\mathbf{B}\mathbb{G}_m$ by studying the finite-dimensional representations of $\mathbb{G}_m$.

In the third chapter, we introduce the moduli stack of Gieseker bundles $\widetilde{\mathcal{M}}_{g,I}(\mathbf{B}\mathbb{G}_m)$, and explain by examples why we study Gieseker $\mathbb{G}_m$-bundles instead of the stack of principal $\mathbb{G}_m$-bundles.

In chapter four, we turn to the main theorem in [6], introducing first the natural evaluation maps that we study for the Gromov-Witten invariants and then the admissible line bundles and admissible complexes of coherent sheaves. Each concept is illustrated for the case of genus 0 curves with 3 marks. Finally, we show how the pushforwards of these admissible complexes along the forgetful map

$$F\colon \widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m) \to \overline{\mathcal{M}}_{0,3}$$

where $\overline{\mathcal{M}}_{0,3}$ is a point, are actually coherent sheaves. Moreover, we compute the numerical Gromov-Witten invariant attached to the pushforward of such an admissible complex.

Preliminaries

The readers of this thesis should have a background in algebraic geometry, so they may for example be master students with an algebraic geometry specialisation. We assume that the reader is familiar with some scheme theory, and is familiar with category theory, because we shall be working with stacks.

For T a scheme, we may use the letter T for the scheme T, but also for the category $\underline{\mathrm{Sch}}_T$, although this should not lead to confusion. Schemes are generally considered with the étale topology, so a collection of maps $\{f_i\colon X_i \to X\}$ for schemes X_i, X is a covering if each f_i is étale and the f_i are jointly surjective.

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1 The stack BG

The purpose of this chapter is to define the classifying stack of G-bundles for a group scheme G, and then deduce that for the multiplicative group scheme $\mathbb{G}_m$, $\mathbf{B}\mathbb{G}_m$ can be described as the stack of line bundles.

1.1 Algebraic stacks

Firstly we need to introduce stacks and an algebraic stacks. In [1], one finds a short introduction to algebraic stacks, although we prefer to use the more thorough book [2] as reference.

To define a stack, we need the concept of descent, which we introduce here because it will also be important for studying quasi-coherent sheaves on the stack $\mathbf{B}\mathbb{G}_m$. We use the notation found in [2] paragraph 4.2.

Definition 1.1. Let C be a category and let $p\colon F \to C$ be a fibered category over C, then we write $F(Y)$ for the *fiber* (category) over an object Y in C, meaning all objects in F that map to object Y in C and all morphisms mapping to id_Y.

Definition 1.2. A *category fibered in groupoids* over a category C is a fibered category $p\colon F \to C$ such that, for each object Y in C, the fiber category $F(Y)$ is a groupoid, i.e., a category where all morphisms are isomorphisms.

Definition 1.3. Let C be a category with finite fiber products. Consider a fibered category $p\colon F \to C$, and a morphism $f\colon X \to Y$ in C and an object E in $F(X)$. A *descent datum* σ for an object E is an isomorphism $\sigma\colon \mathrm{pr}_1^*E \to \mathrm{pr}_2^*E$ in $F(X \times_Y X)$ satisfying the following compatibility or cocycle condition in $F(X \times_Y X \times_Y X)$. Write $\mathrm{pr}_i\colon X \times_Y X \to X$ for the projection to the i-th component and $\mathrm{pr}_{ij}\colon X \times_Y X \times_Y X \to X \times_Y X$ for the projection to the i-th and j-th component, then we have canonical isomorphisms $\mathrm{pr}_{12}^*\mathrm{pr}_2^*E \cong \mathrm{pr}_{23}^*\mathrm{pr}_1^*E$ and $\mathrm{pr}_{12}^*\mathrm{pr}_1^*E \cong \mathrm{pr}_{13}^*\mathrm{pr}_1^*E$ and $\mathrm{pr}_{13}^*\mathrm{pr}_2^*E \cong \mathrm{pr}_{23}^*\mathrm{pr}_2^*E$. We require

$$
\begin{array}{ccccc}
\mathrm{pr}_{12}^*\mathrm{pr}_1^*E & \xrightarrow{\mathrm{pr}_{12}^*\sigma} & \mathrm{pr}_{12}^*\mathrm{pr}_2^*E & \xrightarrow{\sim} & \mathrm{pr}_{23}^*\mathrm{pr}_1^*E \\
\downarrow{\scriptstyle\sim} & & & & \downarrow{\scriptstyle \mathrm{pr}_{23}^*\sigma} \\
\mathrm{pr}_{13}^*\mathrm{pr}_1^*E & \xrightarrow{\mathrm{pr}_{13}^*\sigma} & \mathrm{pr}_{13}^*\mathrm{pr}_2^*E & \xrightarrow{\sim} & \mathrm{pr}_{23}^*\mathrm{pr}_2^*E
\end{array}
$$

to commute.

Definition 1.4. Let C be a category with finite fiber products, $p\colon F \to C$ be a fibered category and $f\colon X \to Y$ a morphism in C. We define the category

$$F(f\colon X \to Y)$$

by having objects pairs (E, σ), where $E \in F(X)$ is an object and σ is a descent datum for E. A morphism $(E', \sigma') \to (E, \sigma)$ in $F(X \to Y)$ is a morphism $g\colon E' \to E$ in $F(X)$ such that $\sigma \circ \mathrm{pr}_1^* g = \mathrm{pr}_2^* g \circ \sigma'$.

More generally, if $\{X_i \to Y\}_{i \in I}$ is a set of morphisms in C, we define $F(\{X_i \to Y\}_{i \in I})$ to be the following category: the objects are data $(\{E_i\}_{i \in I}, \{\sigma_{ij}\}_{i,j \in I})$, where $E_i \in F(X_i)$ and $\sigma_{ij}\colon \mathrm{pr}_1^* E_i \to \mathrm{pr}_2^* E_j$ is an isomorphism in $F(X_i \times_Y X_j)$ for each $i, j \in I$ satisfying the desired compatibility conditions in $F(X_i \times_Y X_j \times_Y X_k)$. Again we refer to the set of isomorphisms $\{\sigma_{ij}\}_{i,j \in I}$ as the descent data on the $\{E_i\}_{i \in I}$. Note that for the collection of morphisms $\{f_i\colon X_i \to Y\}$, there is a functor

$$\epsilon\colon F(Y) \to F(\{X_i \to Y\}),$$

sending an object E to the data $(\{f_i^* E\}, \{\sigma_{\mathrm{can}, ij}\})$ where σ_{can} is the canonical isomorphism.

Definition 1.5. For an object $(\{E_i\}_{i \in I}, \{\sigma_{ij}\}_{i,j \in I}) \in F(\{X_i \to Y\}_{i \in I})$ as above, we say the descent data $\{\sigma_{ij}\}_{i,j \in I}$ are *effective* if $(\{E_i\}_{i \in I}, \{\sigma_{ij}\}_{i,j \in I})$ is in the essential image of $\epsilon\colon F(Y) \to F(\{X_i \to Y\})$.

Definition 1.6. A category fibered in groupoids over a site C, $p\colon F \to C$, is a *stack* if and only if

(i) For any object X in C and objects x, y in the fiber $F(X)$, the presheaf $\underline{\mathrm{Isom}}(x, y)\colon (C/X)^{\mathrm{op}} \to \underline{\mathrm{Set}}$ defined by

$$\underline{\mathrm{Isom}}(x, y)(f\colon Y \to X) := \mathrm{Isom}_{F(Y)}(f^* x, f^* y)$$

is a sheaf.

(ii) For any covering $\{X_i \to X\}_{i \in I}$ of an object X in C, any descent data with respect to this covering are effective.

Equivalently to (i) and (ii), we may require that for every object X in C and covering $\{X_i \to X\}$ the functor $\epsilon\colon F(X) \to F(\{X_i \to X\})$ is an equivalence of categories.

A morphism of stacks $p\colon F \to C, q\colon G \to C$ is simply a morphism of categories over C, that is, a functor $a\colon F \to G$ such that $q \circ a = p$.

Most stacks in this thesis will be stacks over the site $C = \underline{\mathrm{Sch}}_S$ for S a scheme with the étale topology, such as $\mathbf{B}_S G$ over $\underline{\mathrm{Sch}}_S$ in Example 1.25 or $\widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m)$ over $\underline{\mathrm{Sch}}_k$ in Definition 3.12.

Definition 1.7. Let $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Z} \to \mathcal{Y}$ be morphisms of categories fibered in groupoids over category C, then the *fiber product* $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ has as objects triples (x, z, α) where x and z are objects of $\mathcal{X}$ and $\mathcal{Z}$ respectively and $\alpha\colon f(x) \to g(z)$ is an isomorphism in $\mathcal{Y}$. A morphism $(x', z', \alpha') \to (x, z, \alpha)$ is a pair of isomorphisms $(a\colon x' \to x, b\colon z' \to z)$ is such that $g(b) \circ \alpha' = \alpha \circ f(a)$ as maps $f(x') \to g(z)$.

The fiber product of stacks is again a stack, by 4.6.4 in [2]. By 8.1.16, the fiber product of algebraic stacks, which will be defined in Definition 1.14, is also an algebraic stack.

Definition 1.8. Let S be a scheme and consider the category of S-schemes with the étale topology. Let P be a property of morphisms of schemes which is satisfied by isomorphisms and closed under composition. We say property P is *stable* with respect to the étale topology if for all morphisms $f\colon Z \to Y$ of S-schemes and covers $\{Y_i \to Y\}$, f has P if and only if all the $f_i\colon X \times_Y Y_i \to Y_i$ have P.

From Proposition 5.1.4 in [2], we get the following result.

Lemma 1.9. *The following properties are stable: surjective, proper, flat, étale and smooth.*

Recall that a functor $F\colon C^{\mathrm{op}} \to \underline{\mathrm{Set}}$ is representable if it is naturally isomorphic to $h_X\colon C^{\mathrm{op}} \to \underline{\mathrm{Set}}, Y \mapsto \mathrm{Hom}_C(Y, X)$ for some object X in C.

Definition 1.10. Let S be a scheme and $f\colon F \to G$ a morphism of sheaves on $\underline{\mathrm{Sch}}_S$ with the étale topology. Then f is *representable by schemes* if, for every S-scheme T and morphism of sheaves $h_T \to G$, the fiber product $F \times_G h_T$ is representable.

Definition 1.11. Let P be a stable property of morphisms of schemes and let $f\colon F \to G$ be a morphism of sheaves on $\underline{\mathrm{Sch}}_S$ with the étale topology. If f is representable by schemes, we say that f *has property* P if, for every S-scheme T and morphism $h_T \to G$, the morphism of schemes induced by $\mathrm{pr}_1\colon h_T \times_G F \to h_T$ has P.

Note that there is also a way to define, for example, a smooth morphism of stacks that is not representable. However, we will not need that definition.

Definition 1.12. Let S be a scheme. An *algebraic space* over S is a functor $X : \underline{\mathrm{Sch}}_S^{\mathrm{op}} \to \underline{\mathrm{Set}}$ such that

- X is a sheaf with respect to the big étale topology,
- the diagonal $\Delta\colon X \to X \times_S X$ is representable by schemes, and
- there exists an S-scheme U and a surjective étale morphism $h_U \to X$.

Morphisms of algebraic spaces over S are morphisms of functors.

Note that the second condition in the definition, as shown in Lemma 5.1.9 in [2], implies that any morphism $h_T \to X$ for T a scheme and X an algebraic space is representable by schemes. Therefore, in the third condition, $h_U \to X$ is representable and it makes sense to require it to be étale and surjective.

Definition 1.13. A morphism of stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is *representable* if, for every scheme T and morphism $T \to \mathcal{Y}$, the fiber product $\mathcal{X} \times_{\mathcal{Y}} T$ is an algebraic space.

Lemma 8.1.3 in [2] then gives that, if a morphism of stacks $\mathcal{X} \to \mathcal{Y}$ is representable, then the fiber product $\mathcal{X} \times_{\mathcal{Y}} X$ is an algebraic space, not just for schemes but for every algebraic space X and morphism $X \to \mathcal{Y}$.

Definition 1.14. A stack $\mathcal{X}$ over $\underline{\mathrm{Sch}}_S$ is an *algebraic stack* in the sense of Artin (resp. Deligne-Mumford) if

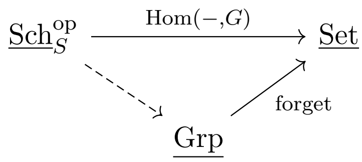
- the diagonal $\Delta\colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, and
- there is a smooth (resp. étale) and surjective morphism of stacks $\underline{\mathrm{Sch}}_T \to \mathcal{X}$ for a scheme T.

A morphism of algebraic stacks is a morphism of stacks.

An example of an Artin algebraic stack, is given by $\mathbf{B}G$ in Example 1.25 and Proposition 1.26.

1.2 Group schemes

Definition 1.15. A *group scheme* G over a scheme S is a group object in the category of schemes over S.
Equivalently, a group scheme over S is a scheme G over S together with a factorisation of the functor $h_G\colon \underline{\mathrm{Sch}}_S^{\mathrm{op}} \to \underline{\mathrm{Set}}$ over $\underline{\mathrm{Grp}}$.



Example 1.16. The *multiplicative group* $\mathbb{G}_{m,S}$ represents the functor

$$\underline{\mathrm{Sch}}_S^{\mathrm{op}} \to \underline{\mathrm{Grp}}, T \mapsto \Gamma(T, \mathcal{O}_T)^* = \mathcal{O}_T(T)^*$$

where $\Gamma(T, \mathcal{O}_T)^*$ is the multiplicative group of invertible elements in $\mathcal{O}_T(T)$. As a scheme, we have $\mathbb{G}_{m,S} = \mathrm{Spec}_S(\mathcal{O}_S[X, X^{-1}])$.

We can illustrate this when $S = \mathrm{Spec}(R)$ is affine. Then $\mathbb{G}_{m,S}$ equals $\mathrm{Spec}_S(R[X, X^{-1}]) \cong \mathbb{A}_S^1 \setminus \{0\}$ and we factorise $\mathrm{Hom}_S(-, \mathbb{G}_{m,S})$ over $\underline{\mathrm{Grp}}$ by the group structure for each scheme T over S specified via

$$\mathrm{Hom}_S(T, \mathbb{G}_{m,S}) \xrightarrow{\sim} \mathrm{Hom}_{R-\mathrm{alg}}(R[X, X^{-1}], \mathcal{O}_T(T)) \xrightarrow{\sim} \mathcal{O}_T(T)^*.$$

When working over a field k, we often write $\mathbb{G}_m$ instead of $\mathbb{G}_{m,k}$.

The following examples are relevant for representation theory for group schemes.

Example 1.17. For a field k and $n \in \mathbb{Z}_{>0}$, the *general linear group* $\mathrm{GL}_{n,k}$ is the affine group scheme

$$\mathrm{Spec}(k[X_{ij}, Y]_{i,j=1,\dots,n}/(\det(X_{ij})Y - 1)),$$

where $\det(X_{ij})$ is the determinant formula in the variables X_{ij} (seeing the X_{ij} as coefficients in an n by n matrix).

Again we explain the name by considering $T = \mathrm{Spec}(R)$ an affine k-scheme. Then we have

$$\mathrm{Hom}_{\underline{\mathrm{Sch}}_k}(T, \mathrm{GL}_{n,k}) \cong \mathrm{Hom}_{k-\mathrm{alg}}(k[X_{ij}, \det(X_{ij})^{-1}], R).$$

Hence this is equivalent to give n^2 elements of R, namely the images of the X_{ij}, which can be viewed as coefficients in an n by n matrix, in such a way that the determinant is invertible. Thus we assign to $T = \mathrm{Spec}(R)$ the multiplicative group of invertible n by n matrices.

Note that, by definition, $\mathrm{GL}_{1,k} = \mathbb{G}_{m,k} = \mathbb{G}_m$.

We can reformulate this example to the following.

Example 1.18. Let V be a finite-dimensional k-vector space. Then we naturally view V as quasi-coherent sheaf on $\underline{\mathrm{Sch}}_k$ via associating to $t\colon T \to k$ the $\mathcal{O}_T$-module $V \otimes_k \mathcal{O}_T$. We define $\mathrm{Aut}_k(V)$ as group scheme to be

$$\mathrm{Aut}_k(V)\colon \underline{\mathrm{Sch}}_k^{\mathrm{op}} \to \underline{\mathrm{Grp}},$$
$$(t\colon T \to k) \mapsto \mathrm{Aut}_T(V_T) = \mathrm{Aut}_T(t^*V) = \mathrm{Aut}_T(V \otimes_k \mathcal{O}_T),$$

where we mean automorphisms as the quasi-coherent sheaf over T. A morphism $f\colon T' \to T$ over k is mapped to $\mathrm{Aut}_T(t^*V) \to \mathrm{Aut}_{T'}(t'^*V), \sigma \mapsto f^*\sigma$.

Note that if V is infinite-dimensional, we can still consider the above functor $\mathrm{Aut}_k(V)$, but it need not be representable.

1.3 Torsors and principal bundles

Firstly we introduce torsors, as in 4.5.1. in [2].

Definition 1.19. Let C be a site and let μ be a sheaf of groups on C. A *μ-torsor* on C is a sheaf $\mathcal{P}$ on C together with a left action $\rho\colon \mu \times \mathcal{P} \to \mathcal{P}$ such that

(1) for every object X in C there exists a covering $\{X_i \to X\}$ such that $\mathcal{P}(X_i) \neq \emptyset$ for all i, and

(2) the map $\mu \times \mathcal{P} \to \mathcal{P} \times \mathcal{P}, (g, p) \mapsto (p, gp)$ is an isomorphism.

Equivalently to the second condition we may require that if $\mathcal{P}(X)$ is non-empty, then $\mu(X)$ acts on $\mathcal{P}(X)$ simply transitively.

A μ-torsor $(\mathcal{P}, \rho)$ is trivial if $\mathcal{P}$ has a global section. Then we have an isomorphism $\mu \to \mathcal{P}, g \mapsto gp$ identifying $\mathcal{P}$ and μ and ρ with left translation. A morphism of μ-torsors $(\mathcal{P}, \rho) \to (\mathcal{P}', \rho')$ is a morphism $f\colon \mathcal{P} \to \mathcal{P}'$ of sheaves such that

$$\begin{array}{ccc} \mu \times \mathcal{P} & \xrightarrow{\rho} & \mathcal{P} \\ \downarrow{\scriptstyle \mathrm{id}_\mu \times f} & & \downarrow{\scriptstyle f} \\ \mu \times \mathcal{P}' & \xrightarrow{\rho'} & \mathcal{P}' \end{array}$$

commutes.

In the article [6] the following definition of a principal $\mathbb{C}^*$-bundle is used: "*principal* $\mathbb{C}^*$*-bundle* on a scheme X is a scheme $\mathcal{P}$ on which $\mathbb{C}^*$ acts freely from the right and a $\mathbb{C}^*$-invariant map $p\colon \mathcal{P} \to X$ which is locally trivial: X has an open cover $\{U_i\}$ such that $U_i \times_X \mathcal{P} \cong U_i \times \mathbb{C}^*$, $\mathbb{C}^*$-equivariantly."

Slightly generalised to fit $\mathbb{G}_m$ as group scheme over k, we use the definition found in 4.5.4 in [2].

Definition 1.20. Let G be a flat locally finitely presented group scheme over scheme X. A *principal G-bundle* over X is a pair $(\pi\colon P \to X, \rho)$ where π is a flat locally finitely presented surjective morphism of schemes and $\rho\colon G \times_X P \to P$ an action such that the map

$$(\rho, \mathrm{pr}_2)\colon G \times_X P \to P \times_X P$$

is an isomorphism.

A morphism of principal G-bundles $(P, \rho) \to (P', \rho')$ is a morphism of X-schemes $f\colon P \to P'$ such that the following diagram commutes:

$$\begin{array}{ccc} G \times_X P & \xrightarrow{\mathrm{id}_G \times f} & G \times_X P' \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho'} \\ P & \xrightarrow{\quad f \quad} & P'. \end{array}$$

There is a connection between torsors and principal G-bundles.

Fix a scheme X and let G be a flat locally finitely presented X-group scheme and let $\mu = h_G = \mathrm{Hom}_X(-, G)$ be the sheaf of groups. Given a principal G-bundle (P, ρ), we can define the μ-torsor $(\mathcal{P}, \tilde{\rho})$ by setting $\mathcal{P} = \mathrm{Hom}_X(-, P)\colon \underline{\mathrm{Sch}}_X \to \underline{\mathrm{Set}}$ with action induced by ρ:

$$\mathrm{Hom}_X(T, G) \times \mathrm{Hom}_X(T, P) \to \mathrm{Hom}_X(T, P), (f, g) \mapsto \rho \circ (f \times g)$$

for a X-scheme T. This construction defines a fully faithful functor from principal G-bundles on X to μ-torsors on $\underline{\mathrm{Sch}}_X$ with the fppf topology.

In specific cases, this is also an equivalence of categories, as stated in [2] Proposition 4.5.6.

Lemma 1.21. *If the structure morphism $G \to X$ is affine, then the functor from principal G-bundles on X to μ-torsors on $\underline{Sch}_X$ associating $Hom_X(-, P)$ to a principal G-bundle (P, ρ), is an equivalence of categories.*

Remark 1.22. In the case where G is a smooth affine group scheme, such as $G = \mathbb{G}_m$, Remark 4.5.7 in [2] states that the category of principal G-bundles on X is equivalent to μ-torsors on $\underline{\mathrm{Sch}}_X$ with the étale topology.

Also, if we had defined a principal G-bundle over X as an algebraic space over X satisfying the above properties, we have an equivalence between principal G-bundles and μ-torsors.

1.4 The algebraic stack $[X/G]$

We define $[X/G]$ as in Example 8.1.12 in [2].

Definition 1.23. Let X be a scheme over S and let G be a smooth group scheme over S which acts on X from the left. We define $[X/G]$ to be the stack with objects triples $(T, \mathcal{P}, \pi)$ where

- T is an S-scheme,
- $\mathcal{P}$ is a $G_T = G \times_S T$-torsor on $\underline{\mathrm{Sch}}_T$ with the étale topology, and
- $\pi\colon \mathcal{P} \to X \times_S T$ is a G_T-equivariant morphism of sheaves on $\underline{\mathrm{Sch}}_T$, i.e., a morphism of G_T-torsors.

A morphism $(T', \mathcal{P}', \pi') \to (T, \mathcal{P}, \pi)$ is a pair (f, f^b) where $f\colon T' \to T$ is a morphism of S-schemes and $f^b\colon \mathcal{P}' \to f^*\mathcal{P}$ is an isomorphism of G_T-torsors on $\underline{\mathrm{Sch}}_{T'}$ such that the diagram

$$\begin{array}{ccc} \mathcal{P}' & \xrightarrow{\quad f^b \quad} & f^*\mathcal{P} \\ & {\scriptstyle \pi'}\searrow \quad \swarrow{\scriptstyle f^*\pi} & \\ & X \times_S T' & \end{array}$$

commutes.

Here we used the notation G_T and $X \times_S T$ also for the sheaf represented by G_T and $X \times_S T$ respectively at points 2 and 3. Also, implicitly we used that if G is a group scheme over S, then G_T is a group scheme over T. By definition of the fiber product, we have $\mathrm{Hom}_T(Y, G_T) \cong \mathrm{Hom}_S(Y, G)$ and so we give $\mathrm{Hom}_T(Y, G_T)$ the group structure inherited from $\mathrm{Hom}_S(Y, G)$.

We consider $[X/G]$ as a category fibered in groupoids over $\underline{\mathrm{Sch}}_S$ via the functor $[X/G] \to \underline{\mathrm{Sch}}_S, (T, \mathcal{P}, \pi) \mapsto T$. Because of descent theory for sheaves, we also have that $[X/G]$ is actually a stack. The proof that it is an (Artin) algebraic stack when G is smooth, can be found in the example of [2]. Here we prove it for our main example, Example 1.25.

Remark 1.24. It is important to note the following about maps from an S-scheme T to $[X/G]$: because $[X/G](T)$ is the category of triples $(T, \mathcal{P}, \pi)$ in $[X/G]$, giving a map $\underline{\mathrm{Sch}}_T \to [X/G]$ is equivalent to specifying a G_T-torsor $\mathcal{P}$ on $\underline{\mathrm{Sch}}_T$ and a G_T-equivariant $\pi\colon \mathcal{P} \to X \times_S T$.

Example 1.25. Let S be a base scheme and G a smooth group scheme over S. The map $G \times_S S \to S$ as morphism over S must be pr_2 and so the action is trivial. We describe the algebraic stack $[S/G]$, also known as $\mathbf{B}G$ or $\mathbf{B}_S G$. Let T be an S-scheme, then giving a $G \times_k T$- torsor on $\underline{\mathrm{Sch}}_T$ is equivalent to giving a G-torsor on $\underline{\mathrm{Sch}}_T$. We need not specify the morphism of torsors $\pi\colon \mathcal{P} \to S \times_S T \cong T$ over T, because it must be the trivial map. Therefore objects of $\mathbf{B}G$ are pairs $(T, \mathcal{P})$ of an S-scheme T and a G-torsor $\mathcal{P}$ on $\underline{\mathrm{Sch}}_T$. Morphisms $(T', \mathcal{P}') \to (T, \mathcal{P})$ then are pairs (f, f^b) where $f\colon T' \to T$ and $f^b\colon \mathcal{P}' \to f^*\mathcal{P}$ is an isomorphism of torsors over T'.

By Remark 1.24, maps from any S-scheme T to $\mathbf{B}G$ are specified by giving a G-torsor on T. By Remark 1.22, if we assume G is affine or if we work with principal G-bundle that are algebraic spaces, we have that giving a G-torsor on T is equivalent to giving a principal G-bundle on T.

Proposition 1.26. *Let S be a base scheme and G a smooth group scheme over S with trivial action on S. The stack $\boldsymbol{B}G$ is an Artin algebraic stack.*

Proof. Consider the morphism of stacks $f\colon S \to \mathbf{B}G$ defined by the trivial bundle sending $t\colon T \to S$ to $(T, G \times_S T)$. Given a morphism of k-schemes $t\colon T' \to T$, we get a morphism $(t, t^b)\colon (T, G \times_k T) \to (T', G \times_k T')$ because there is a natural isomorphism of G-bundles $t^b\colon G \times_k T \to f^*(G \times_k T') = G \times_k T' \times_{T'} T$. This is a morphism of stacks over the category $\underline{\mathrm{Sch}}_S$.

Let X be any S-scheme X with a map $X \to \mathbf{B}G$ specified by a principal G-bundle $\mathcal{E}$, so $F_\mathcal{E}\colon X \to \mathbf{B}G, (a\colon T \to X) \mapsto (T, a^*\mathcal{E})$. To deduce that f is representable, consider $X \times_{\mathbf{B}G} S$. By definition of the fiber product of stacks, for a k-scheme T we have

$$\begin{aligned}(X \times_{\mathbf{B}G} S)(T) &= \{(a, b, \phi) \mid a \in \underline{\mathrm{Sch}}_X(T), b \in \underline{\mathrm{Sch}}_S(T), \text{ and } \phi\colon f(b) \xrightarrow{\sim} F_\mathcal{E}(a)\} \\ &\cong \{(a, \phi) \mid a\colon T \to X, \text{ and an isomorphism } \phi\colon G \times_S T \cong a^*\mathcal{E}\}.\end{aligned}$$

Because the isomorphism ϕ must be an isomorphism of G-bundles on T, so a map that respects the G-action, it is enough to know the image of $\{e\} \times T$ where e is the unit. So equivalently, we may give a map $s\colon T \to a^*\mathcal{E} = T \times_X \mathcal{E}$ of G-bundles over T. By property of the fiber product, we conclude

$$X \times_{\mathbf{B}G} S(T) \cong \{(a, s) \mid a\colon T \to X, s\colon T \to T \times_X \mathcal{E}\} \cong \mathrm{Hom}_X(T, \mathcal{E}).$$

Therefore, $X \times_{\mathbf{B}G} S$ is representable, by $\mathcal{E}$, for all such $(X, \mathcal{E})$. So the morphism f is representable by a scheme. Also, $X \times_{\mathbf{B}G} S \to X$, or as schemes $\mathcal{E} \to X$, is indeed surjective by definition of a G-bundle and smooth because if G/k is smooth, then any principal G-bundle is smooth.

Therefore, $f\colon S \to \mathbf{B}G$ is a surjective, smooth, representable morphism. The diagonal $\Delta\colon \mathbf{B}G \to \mathbf{B}G \times_S \mathbf{B}G$ is representable if and only if for all S-schemes T and maps $F_{\mathcal{P}_1} \times F_{\mathcal{P}_2}\colon T \to \mathbf{B}G \times_S \mathbf{B}G$ specified by G-torsors $\mathcal{P}_1, \mathcal{P}_2$, the sheaf

$$\underline{\mathrm{Isom}}(\mathcal{P}_1, \mathcal{P}_2)\colon \underline{\mathrm{Sch}}_T \to \underline{\mathrm{Set}}, (t\colon T' \to T) \mapsto \mathrm{Isom}_{T'}(t^*\mathcal{P}_1, t^*\mathcal{P}_2)$$

mapping to the set of isomorphisms of torsors over T', is actually an algebraic space, because

$$\begin{array}{ccc} \underline{\mathrm{Isom}}(\mathcal{P}_1, \mathcal{P}_2) & \longrightarrow & T \\ \downarrow & & \downarrow F_{\mathcal{P}_1} \times F_{\mathcal{P}_2} \\ \mathbf{B}G & \xrightarrow{\Delta} & \mathbf{B}G \times_S \mathbf{B}G \end{array}$$

is a pullback diagram. To verify that $\underline{\mathrm{Isom}}(\mathcal{P}_1, \mathcal{P}_2)$ is an algebraic space, we may consider an étale cover $U \to T$ by a scheme U and verify that $\underline{\mathrm{Isom}}(\mathcal{P}_1, \mathcal{P}_2)$ pulled back to U is an algebraic space. Therefore we may assume that $\mathcal{P}_1, \mathcal{P}_2$ are trivial torsors, so isomorphic to G_T, and we have

$$\underline{\mathrm{Isom}}(\mathcal{P}_1, \mathcal{P}_2)(t\colon T' \to T) = \mathrm{Isom}(t^* G_T, t^* G_T) = \mathrm{Isom}(G_{T'}, G_{T'}).$$

Note that giving an isomorphism $G \times_S T' \to G \times_S T'$ over T' as G-bundles is equivalent to giving $g \in G(T')$. So $\underline{\mathrm{Isom}}(\mathcal{P}_1, \mathcal{P}_2)$ is representable by a scheme when $\mathcal{P}_1$ and $\mathcal{P}_2$ are trivial, and so it is an algebraic space and so the diagonal is representable. Hence $\mathbf{B}G$ is an Artin algebraic stack. □

This proof also shows that $\mathbf{B}G$ is the classifying stack for principal G-bundles: any principal G-bundle $p\colon \mathcal{P} \to X$ is pulled back by a unique morphism $\phi\colon X \to \mathbf{B}G$ (defined by bundle $\mathcal{P}$) from the universal principal G-bundle $f\colon S \to \mathbf{B}G$:

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{p} & X \\ \downarrow & & \downarrow \phi \\ S & \xrightarrow{f} & \mathbf{B}G. \end{array} \tag{1}$$

1.5 The stack $\mathbf{B}\mathbb{G}_m$ as stack of line bundles

We want to identify the stack $\mathbf{B}\mathbb{G}_m$ with the stack of line bundles.

Definition 1.27. A *line bundle* on a scheme X is a locally free $\mathcal{O}_X$- module of rank 1.
Equivalently, a line bundle on a scheme X is a coherent sheaf L on X such that there is a cover $\{U_i \to X\}_i$ with $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$.

Also common is to define a line bundle on X as an invertible sheaf on X, where we mean that it is an invertible element in the monoid consisting of coherent sheaves on X with the operation $\otimes_{\mathcal{O}_X}$ and unit $\mathcal{O}_X$.

Example 1.28. As a simple and useful example, we introduce the line bundles $\mathcal{O}(n)$ on $\mathbb{P}^1_k$ for a field k. The notes of [4] (chapter 14) provide an introduction to line bundles, including this example. Let $\mathbb{P}^1_k = \mathrm{Proj}(k[x_0, x_1])$ and we describe the sheaf $\mathcal{O}(n)$ by defining it to be trivial on the standard opens $U_0 = D(x_0) = \mathrm{Spec}(k[x_1/x_0])$ and $U_1 = D(x_1) = \mathrm{Spec}(k[x_0/x_1])$. We glue by the transition function $U_0 \to U_1$, which is defined by multiplying by $(x_0/x_1)^n = (x_1/x_0)^{-n}$, and $U_1 \to U_0$, which is defined by multiplying by $(x_1/x_0)^n = (x_0/x_1)^{-n}$.

These line bundles satisfy the following properties: $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$ and $\mathcal{O}(m) \otimes \mathcal{O}(n) = \mathcal{O}(m+n)$. Also, we can explicitly compute the global sections. Such a global section are polynomials $f(x_1/x_0) \in k[x_1/x_0]$ and $g(x_0/x_1) \in k[x_0/x_1]$ such that

$$f((x_0/x_1)^{-1})(x_0/x_1)^n = g(x_0/x_1)$$

and therefore there are no global sections if $n < 0$. Moreover, f must have degree lesser than or equal to n if $n \geq 0$. Therefore, we have

$$\dim H^0(\mathbb{P}^1_k, \mathcal{O}(n)) = n + 1 \text{ if } n \geq 0.$$

The invertible sheaves up to isomorphism form an abelian group under the tensor product, called the Picard group. We have $\mathrm{Pic}(\mathbb{P}^1_k) \cong \mathbb{Z}$ and every invertible sheaf on $\mathbb{P}^1_k$ is of the form $\mathcal{O}(n)$ for some $n \in \mathbb{Z}$. We define the degree of the line bundle to be this n.

Let X be a scheme and consider $C = \underline{\mathrm{Sch}}_X$ with the étale topology. Write $\mathcal{O}\colon \underline{\mathrm{Sch}}_X \to \underline{\mathrm{CRing}}, \{f\colon Y \to X\} \mapsto f^*\mathcal{O}_X(Y) = \mathcal{O}_Y(Y)$ for the natural sheaf of commutative rings, then an invertible sheaf on C is a sheaf of $\mathcal{O}$-modules L such that for any object U in $\underline{\mathrm{Sch}}_X$, there exists a covering $\{U_i \to U\}_{i \in I}$ such that the restriction of L to the localized site C/U_i, so to $\underline{\mathrm{Sch}}_{U_i}$, is isomorphic to the restriction of $\mathcal{O}$ viewed as $\mathcal{O}$-module. Recall that $\mathbb{G}_{m,X} = \mathbb{G}_m \colon \underline{\mathrm{Sch}}_X \to \underline{\mathrm{CRing}}, \{f\colon Y \to X\} \mapsto \mathcal{O}_Y(Y)^*$ and that giving a $\mathbb{G}_{m,X}$- torsor on $\underline{\mathrm{Sch}}_X$ is the same as a $\mathbb{G}_m$-torsor on $\underline{\mathrm{Sch}}_X$.

Proposition 1.29. *There is an equivalence of categories*

$$\{\mathbb{G}_m - \textit{torsors on } X\} \leftrightarrow \{\textit{line bundles on } X\},$$

where the latter category is only considered with isomorphisms.

Proof. Let X be a scheme and $\mathcal{L}$ a line bundle on X, so an invertible $\mathcal{O}_X$-module. We define the associated $\mathbb{G}_m$-torsor $\mathcal{P}_\mathcal{L}$ of $\mathcal{L}$

$$\mathcal{P}_\mathcal{L}\colon \underline{\mathrm{Sch}}_X^{\mathrm{op}} \to \underline{\mathrm{Set}}$$

by sending an object $f\colon U \to X$ in $\underline{\mathrm{Sch}}_X$ to the set of isomorphisms of $\mathcal{O}|_{\underline{\mathrm{Sch}}_U}$- modules $\mathcal{O}|_{\underline{\mathrm{Sch}}_U} \to \mathcal{L}|_{\underline{\mathrm{Sch}}_U}$. There is a natural action of $\mathbb{G}_m$ on $\mathcal{P}_\mathcal{L}$ making $\mathcal{P}_\mathcal{L}$ in a $\mathbb{G}_m$-torsor. An element of $a \in \mathcal{O}_U(U)^*$ induces an isomorphism $\tilde{a}$ of $\mathcal{O}|_{\underline{\mathrm{Sch}}_U}$ and then we define

$$\mathbb{G}_m(U) \times \mathrm{Isom}(\mathcal{O}|_{\underline{\mathrm{Sch}}_U}, \mathcal{L}|_{\underline{\mathrm{Sch}}_U}) \to \mathrm{Isom}(\mathcal{O}|_{\underline{\mathrm{Sch}}_U}, \mathcal{L}|_{\underline{\mathrm{Sch}}_U}), (a, \phi) \mapsto \phi \circ \tilde{a}.$$

Checking the two conditions for a torsor, firstly choose a cover $\{U_i \to U\}$ where $\mathcal{L}|_{U_i}$ is trivial and there we have

$$\mathcal{P}_{\mathcal{L}}(U_i) = \mathrm{Isom}(\mathcal{O}|_{\underline{\mathrm{Sch}}_{U_i}}, \mathcal{L}|_{\underline{\mathrm{Sch}}_{U_i}}) \cong \mathrm{Isom}(\mathcal{O}|_{\underline{\mathrm{Sch}}_{U_i}}, \mathcal{O}|_{\underline{\mathrm{Sch}}_{U_i}}) \cong \mathcal{O}_{U_i}(U_i)^*$$

which is non-empty for all non-empty U_i. Secondly, whenever $\mathcal{P}_{\mathcal{L}}(U)$ is non-empty, the $\mathcal{O}_X(U)^*$ action is simply transitive, because locally on trivial opens, the elements of $\mathcal{O}_X(U)^*$ precisely differ by elements of $\mathcal{O}_X(U)^*$.

Let $\mathcal{L}_1, \mathcal{L}_2$ be line bundles and let $\phi\colon \mathcal{L}_1 \to L_2$ be an isomorphism of line bundles, then we have the map $\mathcal{P}_{\mathcal{L}_1} \to \mathcal{P}_{\mathcal{L}_2}$ defined on $U \in \underline{\mathrm{Sch}}_X$ by

$$\mathrm{Isom}(\mathcal{O}|_{\underline{\mathrm{Sch}}_U}, \mathcal{L}_1|_{\underline{\mathrm{Sch}}_U}) \to \mathrm{Isom}(\mathcal{O}|_{\underline{\mathrm{Sch}}_U}, \mathcal{L}_2|_{\underline{\mathrm{Sch}}_U}), f \mapsto \phi \circ f,$$

where $\phi \circ f$ is again an isomorphism because ϕ is. This morphism indeed respects the $\mathbb{G}_m$-action by associativity of the composition.

To construct an inverse, let X be a scheme and $\mathcal{P}\colon \underline{\mathrm{Sch}}_X^{\mathrm{op}} \to \underline{\mathrm{Set}}$ be a $\mathbb{G}_m$-torsor. Define a $\mathbb{G}_m$-action on the product $\mathcal{O}_X \times \mathcal{P}$ by

$$\mathcal{O}_X(U)^* \times (\mathcal{O}_X(U) \times \mathcal{P}(U)) \to \mathcal{O}_X(U) \times \mathcal{P}(U), (a, (s, t)) \mapsto (sa^{-1}, at).$$

Consider the quotient $\mathcal{O}_X^* \setminus \mathcal{O}_X \times \mathcal{P}$: as presheaves we simply mean

$$\mathcal{O}_X^* \setminus \mathcal{O}_X \times \mathcal{P}(U) = \mathcal{O}_X^*(U) \setminus (\mathcal{O}_X(U) \times \mathcal{P}(U))$$

and then we sheafify to obtain $\mathcal{L}_{\mathcal{P}}$. Because $\mathcal{P}$ is a $\mathbb{G}_m$-torsor we can choose a U (in an appropiate étale cover of X) such that $T(U)$ is non-empty, and we can choose $t \in T(U)$. Then we know that because of the simply transitive action that $\mathcal{O}_X^*(U) \xrightarrow{\sim} T(U), a \mapsto at$ is an isomorphism. Then we have $\mathcal{O}_X^*(U) \setminus (\mathcal{O}_X(U) \times \mathcal{P}(U)) \cong \mathcal{O}_X(U)$. Therefore, as a presheaf, $\mathcal{L}_{\mathcal{P}}$ is indeed étale locally trivial. Using that a $\mathbb{G}_m$-torsor for the étale topology is trivial also Zariski locally (this is known for GL_n-torsors, see for example [14]), we can also deduce that $\mathcal{L}_{\mathcal{P}}$ is trivial Zariski locally as a presheaf. However the isomorphism also gives an isomorphism on stalks and so also the sheaf $\mathcal{L}_{\mathcal{P}}$ is locally isomorphic to $\mathcal{O}_X$. A morphism of torsors $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ is sent to the map induced by $\mathrm{id} \times \phi$ on $\mathcal{O}_X \times \mathcal{P}_1$.

These constructions give an equivalence of categories.

□

Hence, given a $\mathbb{G}_m$-torsor $\mathcal{P}$ on a scheme X, there is an associated line bundle, and given a line bundle $\mathcal{L}$ on X, there is an associated $\mathbb{G}_m$-torsor. The trivial principal $\mathbb{G}_m$-bundle $\mathbb{G}_m \times_k T$ corresponds to the $\mathbb{G}_m$-torsor $\mathrm{Hom}_T(-, \mathbb{G}_m \times_k T) \cong \mathrm{Hom}_k(-, \mathbb{G}_m) \cong \mathcal{O}_T^*$, which in turn via the proposition corresponds to the trivial line bundle

$$\mathcal{O}_T^* \setminus \mathcal{O}_T \times \mathcal{O}_T^* \cong \mathcal{O}_T.$$

2 Quasi-coherent sheaves on BG

Schemes in this section are considered with the étale topology. In chapter 4, we study pullbacks of coherent sheaves on $\mathbf{B}G$, and so it is useful to classify quasi-coherent sheaves on $\mathbf{B}G$. We will prove the following.

Theorem 2.1. *Let G be a smooth group scheme over k. There is an equivalence of categories*

$$\{\textit{Quasi-coherent sheaves on } \boldsymbol{B}G\} \leftrightarrow \{\textit{Representations of } G\}.$$

2.1 Representations and G-equivariant quasi-coherent sheaves

Firstly, we define a representation of a group scheme over field k, using Definition 1.18 of $\mathrm{Aut}_k(V)$.

Definition 2.2. A *representation* of a group scheme G over a field k is an morphism of functors $G \to \mathrm{Aut}_k(V)$ for V a k-vector space.

The category of representations of group scheme G over k, denoted as $\underline{\mathrm{Repr}}_k(G)$, has as objects representations, so pairs (V, ϕ) of a k-vector space V and a representation $\phi\colon G \to \mathrm{Aut}_k(V)$. Morphisms $(V, \phi) \to (W, \psi)$ are G-equivariant maps $h\colon V \to W$, i.e., a k-linear map such that for all $g \in G(T)$, the diagram

$$\begin{array}{ccc} V_T & \xrightarrow{\phi(g)} & V_T \\ \downarrow{\scriptstyle h_T} & & \downarrow{\scriptstyle h_T} \\ W_T & \xrightarrow{\psi(g)} & W_T \end{array}$$

commutes, where h_T is the pullback of h via $T \to \mathrm{pt}$.

We use the following category as intermediate step to prove the equivalence in theorem 2.1.

Definition 2.3. Let S be a scheme and G a smooth group scheme over S and X a scheme over S on which G acts via $\rho\colon G \times_S X \to X$. A *$G$-equivariant quasi-coherent sheaf* on X is a pair (F, σ) where F is a quasi-coherent sheaf on X and $\sigma\colon \rho^* F \xrightarrow{\sim} \mathrm{pr}_2^* F$ is an isomorphism such that for any S-scheme T and $g, g' \in G(T)$, the diagram of quasi-coherent sheaves on $T \times_S X$

$$\begin{array}{ccc} \rho_g^* \rho_{g'}^* (\mathrm{pr}_2^* F) & \xrightarrow{\rho_g^* \sigma_{g'}} & \rho_g^* (\mathrm{pr}_2^* F) \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sigma_g} \\ \rho_{g'g}^* (\mathrm{pr}_2^* F) & \xrightarrow{\sigma_{g'g}} & \mathrm{pr}_2^* F \end{array}$$

commutes, where

- $\mathrm{pr}_2 \colon T \times_S X \to X$ is the projection,
- ρ_g is $\mathrm{pr}_1 \times (\rho \circ (g \times \mathrm{id}_X)) \colon T \times_S X \to T \times_S X$ the map induced by the action, and
- $\sigma_g \colon \rho_g^* \mathrm{pr}_2^* F \to \mathrm{pr}_2^* F$ is the pullback of σ via map $g \times \mathrm{id}_X$.

Lemma 2.4. *Let X, Y be S-schemes with G-actions $a_X \colon G \times_S X \to X$ and $a_Y \colon G \times_S Y \to Y$ respectively and let $f \colon Y \to X$ be a G-equivariant morphism of schemes, that is, $f \circ a_Y = a_X \circ (\mathrm{id}_G \times f)$. Let (F, σ) be a G-equivariant quasi-coherent sheaf on X, then $(f^*F, (\mathrm{id}_G \times f)^*\sigma)$ is a G-equivariant quasi-coherent sheaf on Y.*

Proof. Firstly, f^*F is a quasi-coherent sheaf on Y. Then $(\mathrm{id}_G \times f)^*\sigma$ is an isomorphism from

$$(\mathrm{id}_G \times f)^* a_X^* F \cong (a_X \circ (\mathrm{id}_G \times f))^* F = (f \circ a_Y)^* F \cong a_Y^*(f^* F)$$

to

$$(\mathrm{id}_G \times f)^* \mathrm{pr}_X^* F \cong (\mathrm{pr}_X \circ (\mathrm{id}_G \times f))^* F = (f \circ \mathrm{pr}_Y)^* F \cong \mathrm{pr}_Y^*(f^* F).$$

For any S-scheme T and $g, g' \in G(T)$ the commutativity of the diagram of quasi-coherent sheaves on $T \times_S Y$ follows directly from pulling back the commutative diagram on $T \times_S X$ via $(\mathrm{id}_T \times f)$. □

Denote the category with objects quasi-coherent G-equivariant sheaves on a scheme X by $\underline{\mathrm{Qcoh}}^G(X)$. A morphism $(F, \sigma) \to (E, \tau)$ is a morphism $f \colon F \to E$ of quasi-coherent sheaves on X such that

$$\begin{array}{ccc} \rho^* F & \xrightarrow{\sigma} & \mathrm{pr}^* F \\ \downarrow{\scriptstyle \rho^* f} & & \downarrow{\scriptstyle \mathrm{pr}^* f} \\ \rho^* E & \xrightarrow{\tau} & \mathrm{pr}^* E \end{array}$$

commutes. The next proposition is the reason for introducing G-equivariant quasi-coherent sheaves.

Proposition 2.5. *Let G be a smooth group scheme over k and let $\mathrm{pt} = \underline{Sch}_k$. There is an equivalence of categories*

$$\underline{Qcoh}^G(\mathrm{pt}) \leftrightarrow \underline{Repr}_k(G).$$

Proof. Write $p\colon G \to \mathrm{pt}$ for the trivial map. Let (F, σ) be an object of $\underline{\mathrm{Qcoh}}^G(\mathrm{pt})$, then we can see F as a k-vector space and $\sigma \in \mathrm{Aut}_G(p^*F)$ defines a scheme morphism $\rho_\sigma\colon G \to \mathrm{Aut}_k(F)$ sending a $g \in G(T)$ to $g^*\sigma \in \mathrm{Aut}_T(g^*p^*F) \cong \mathrm{Aut}_T(t^*F)$ where T is $t\colon T \to k$. This is a group homomorphism: G-equivariance structure gives that $\sigma_{g'g} = \sigma_g\sigma_{g'}$ and also σ_e is idempotent and an isomorphism, so the identity. Then $\rho_\sigma\colon G \to \mathrm{Aut}_k(F)$ is a representation.

Given a representation $\rho\colon G \to \mathrm{Aut}_k(V)$, the k-vector space V naturally gives a quasi-coherent sheaf on pt. Also, $\tilde{\rho} := \rho(\mathrm{id}_G)$ is an element of $\mathrm{Aut}_G(p^*V)$ such that $G(T) \to \mathrm{Aut}_T(V_T), g \mapsto \rho(g) = g^*\tilde{\rho}$ is a group homomorphism: $\tilde{\rho}$ satisfies $(g'g)^*\tilde{\rho} = g^*\tilde{\rho}g'^*\tilde{\rho}$ for $g, g' \in G(T)$. Therefore $(V, \tilde{\rho})$ is a G-equivariant sheaf on pt.

A morphism in $\underline{\mathrm{Repr}}_k(G)$ also corresponds to a morphism in $\underline{\mathrm{Qcoh}}^G(\mathrm{pt})$; let $f\colon (F, \sigma) \to (E, \tau)$ be a morphism in $\underline{\mathrm{Qcoh}}^G(\mathrm{pt})$, and consider the representations $\rho_\sigma\colon G \to \mathrm{Aut}_k(F)$ and $\rho_\tau\colon G \to \mathrm{Aut}_k(E)$. The induced linear map $f\colon F \to E$ by definition satisfies $\tau \circ p^*f = p^*f \circ \sigma$, and then for all k-schemes T and for all $g \in G(T)$ we have by pulling back along g that $\tau_g \circ f_T = f_T \circ \sigma_g$, and so we indeed have a morphism of representations.

For the other direction, given a map $h\colon V \to W$ between representations $\phi\colon G \to \mathrm{Aut}_k(V)$ and $\psi\colon G \to \mathrm{Aut}_k(W)$, it induces a morphism $(V, \phi(\mathrm{id}_G)) \to (W, \psi(\mathrm{id}_G))$. Because h satisfies $h_T \circ \phi(g) = \psi(g) \circ h_T$ for all $g \in G(T)$, for $g = \mathrm{id}_G$ we get the desired $p^*h \circ \phi(\mathrm{id}_G) = \psi(\mathrm{id}_G) \circ p^*h$.

The constructions are inverses, so $\underline{\mathrm{Qcoh}}^G(\mathrm{pt})$ is equivalent to $\underline{\mathrm{Repr}}_k(G)$. □

2.2 Quasi-coherent sheaves on BG

At this point it is important to specify what we mean by a quasi-coherent sheaf on a stack. We follow the construction of Olsson, [2].

Definition 2.6. Let $\mathcal{X}/S$ be an algebraic stack, that is a stack $\mathcal{X}$ over the category $\underline{\mathrm{Sch}}_S$. Define $\underline{Sch}/\mathcal{X}$ as the category containing as objects pairs (T, t) where T is a scheme over S and $t\colon T \to \mathcal{X}$ is a morphism of stacks over S. The morphisms $(T', t') \to (T, t)$ in $\underline{\mathrm{Sch}}/\mathcal{X}$ are pairs (f, f^b) where $f\colon T' \to T$ is a morphism of S-schemes and $f^b\colon t' \to t \circ f$ is an isomorphism of functors $T' \to \mathcal{X}$.

Consider the site Lis-Ét$(\mathcal{X})$, *the lisse-étale site on* $\mathcal{X}$, which is the full subcategory of $\underline{\mathrm{Sch}}/\mathcal{X}$ consisting of the pairs (T, t) where $t\colon T \to \mathcal{X}$ is a smooth morphism, and a covering is a collection $\{(f_i, f_i^b)\colon (T_i, t_i) \to (T, t)\}$ such that $\{f_i\colon T_i \to T\}$ is an étale cover of T.

Remark 2.7. Note that for each $(T,t) \in \text{Lis-Ét}(\mathcal{X})$, we have an inclusion from the étale site on T to the lisse-étale site on $\mathcal{X}$, $\text{Ét}(T) \to \text{Lis-Ét}(\mathcal{X})$ given by sending $h\colon T' \to T$ to $(T', t \circ h)$. Then a functor $\mathcal{F}\colon\ \text{Lis-Ét}(\mathcal{X})^{\text{op}} \to \underline{\text{Set}}$ is a sheaf if and only if for all (T,t) the restriction of $\mathcal{F}$ to $\text{Ét}(T)$, denoted by $\mathcal{F}_{(T,t)}$, is a sheaf.

Definition 2.8. We define the sheaf $\mathcal{O}_{\mathcal{X}}$ on $\text{Lis-Ét}(\mathcal{X})$ by associating $\mathcal{O}_T(T)$ to (T,t).

Remark 2.9. By construction 9.1.10 in Olsson [2], we may also describe a sheaf of $\mathcal{O}_{\mathcal{X}}$-modules on $\text{Lis-Ét}(\mathcal{X})$ by the following data: for each object $(T,t) \in \text{Lis-Ét}(\mathcal{X})$, an étale sheaf $F_{(T,t)}$ of $\mathcal{O}$-modules on T, and for each morphism $(f, f^b)\colon (T',t') \to (T,t)$ in $\text{Lis-Ét}(\mathcal{X})$ a morphism of sheaves $\phi_{(f,f^b)}\colon f^*F_{(T,t)} \to F_{(T',t')}$ satisfying two conditions. Firstly, the $\phi_{(f,f^b)}$ need to be compatible with composition, that is $\phi_{(f,f^b)\circ(g,g^b)} = \phi_{(g,g^b)} \circ g^*\phi_{(f,f^b)}$. Secondly, if we have (f, f^b) with f étale, the map $\phi_{(f,f^b)}$ has to be an isomorphism.

We may describe a morphism of sheaves

$$\{F_{(T,t)}, \phi_{(f,f^b)}\} \to \{E_{(T,t)}, \psi_{(f,f^b)}\}$$

as a collection $\gamma_{(T,t)}\colon F_{(T,t)} \to E_{(T,t)}$ for each (T,t) such that the diagram

$$\begin{array}{ccc} f^*F_{(T,t)} & \xrightarrow{f^*\gamma_{(T,t)}} & f^*E_{(T,t)} \\ \downarrow{\scriptstyle \phi_{(f,f^b)}} & & \downarrow{\scriptstyle \psi_{(f,f^b)}} \\ F_{(T',t')} & \xrightarrow{\gamma_{(T',t')}} & E_{(T',t')} \end{array}$$

commutes.

Definition 2.10. Let $\mathcal{X}$ be an algebraic stack. A sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathcal{X}}$-modules on $\text{Lis-Ét}(\mathcal{X})$ is *cartesian* if for every morphism $(f, f^b)\colon (T',t') \to (T,t)$ the map of $\mathcal{O}'_T$-modules $\phi_{(f,f^b)}\colon f^*\mathcal{F}_{(T,t)} \to \mathcal{F}_{(T',t')}$ is an isomorphism. A sheaf on $\text{Lis-Ét}(\mathcal{X})$ of $\mathcal{O}_X$-modules $\mathcal{F}$ is *quasi-coherent* if $\mathcal{F}$ is cartesian and for every (T,t) the sheaf $\mathcal{F}_{(T,t)}$ is a quasi-coherent sheaf on T. If $\mathcal{X}$ is locally Noetherian, then for any $(T,t) \in \text{Lis-Ét}(\mathcal{X})$ the scheme T is locally Noetherian, and we define a quasi-coherent $\mathcal{F}$ on $\mathcal{X}$ to be *coherent* if each $\mathcal{F}_{(T,t)}$ is coherent.

Proof of 2.1. Let $\text{pt} = \underline{\text{Sch}}_k$. By Proposition 2.5, it is enough to construct an equivalence

$$\{\text{Quasi-coherent sheaves on } \mathbf{B}G\} \leftrightarrow \underline{\text{Qcoh}}^G(\text{pt}).$$

Step 1: Let (F, σ) be an object of $\underline{\mathrm{Qcoh}}^G(\mathrm{pt})$, then we will construct a quasi-coherent sheaf $\mathcal{F}$ on $\mathbf{B}G$ by giving data $\{\mathcal{F}_{(T,t)}, \phi_{(f,f^b)}\}$ (Remark 2.9).

Step 1.1: Let (T, t) be an object in Lis-Ét($\mathbf{B}G$), and $\mathcal{P}$ the G-bundle on T defining $t\colon T \to \mathbf{B}G$. We define $\mathcal{F}_{(T,t)}$ or $\mathcal{F}_{(T,\mathcal{P})}$ as follows. Let $f\colon \mathrm{pt} \to \mathbf{B}G$ be the morphism defined by the trivial bundle, then we have the pullback square

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & T \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle t} \\ \mathrm{pt} & \xrightarrow{f} & \mathbf{B}G. \end{array}$$

By Lemma 2.4, the pullback $(b^*F, (\mathrm{id}_G \times b)^*\sigma)$ is again a G-equivariant quasi-coherent sheaf on $\mathcal{P}$. We will descend it along $\mathcal{P} \to T$, see Definition 1.3. Write $a\colon G \times_k \mathcal{P} \to \mathcal{P}$ for the G-action on G-bundle $\mathcal{P}$, then the map $a \times \mathrm{pr}_2\colon G \times_k \mathcal{P} \to \mathcal{P} \times_k \mathcal{P}$ is an isomorphism. Therefore, the map τ defined as $\tau := ((a \times \mathrm{pr}_2)^{-1})^*(\mathrm{id}_G \times b)^*\sigma$ is an isomorphism $\mathrm{pr}_1^* b^*F \to \mathrm{pr}_2^* b^*F$. The G-equivariance structure on b^*F ensures that τ satisfies the cocycle condition and thus specifies a descent datum (b^*F, τ). The map $\mathcal{P} \to T$ is an fppf morphism, and so this descent datum is effective, Theorem 4.3.12 in [2], so we obtain a quasi-coherent sheaf $\mathcal{F}_{(T,\mathcal{P})}$ on T.

Step 1.2: Let $(f, f^b)\colon (T', t') \to (T, t)$ be a morphism in Lis-Ét($\mathbf{B}G$). The morphism (f, f^b) is the composition

$$(T', \mathcal{P}') \overset{(\mathrm{id}_{T'}, f^b)}{\to} (T', f^*\mathcal{P}) \overset{(f,\mathrm{id})}{\to} (T, \mathcal{P}),$$

and so we will construct the isomorphism $\phi_{(f,f^b)}\colon f^*F_{(T,t)} \to F_{(T,t)}$ via $\phi_{(\mathrm{id}_{T'}, f^b)} \circ f^*\phi_{(f,\mathrm{id})}$.

Consider $(\mathrm{id}, f^b)\colon (T', \mathcal{P}') \to (T', \mathcal{P})$. Because G-bundles are locally trivial, we consider an étale cover $U \to T'$, where both bundles are trivial. An isomorphism of trivial G-bundles is given by multiplication m_g by a certain element $g \in G(U)$. (To be precise, $m_g\colon G \times_k U \to G \times_k U$ is the map $(a \circ (\mathrm{id}_G \times g)) \times \mathrm{pr}_U$ for a the action on G.) In the diagram

$$\begin{array}{ccccc} G \times_k U & \xrightarrow{m_g} & G \times_k U & \underset{e}{\overset{\longleftarrow}{\longrightarrow}} & U \\ & & \downarrow{\scriptstyle b} & & \downarrow \\ & & \mathrm{pt} & \longrightarrow & \mathbf{B}G \end{array}$$

the square is a pullback square and $e\colon U \to G \times_k U$ is the section given by the unit in G. The sheaves $\mathcal{F}_{(U,\mathcal{P})}$ and $\mathcal{F}_{(U,\mathcal{P}')}$ equal e^*b^*F and $e^*m_g^*b^*F$ respectively and we define isomorphism $\phi_{(\mathrm{id}_U, m_g)}$ as

$$e^*m_g^*b^*F \cong (b \circ m_g \circ e)^*F \cong \rho_g^* F_U \overset{\sigma_g}{\to} F_U \cong e^*b^*F.$$

By descent for an étale cover, see e.g. notes [12], this also defines $\phi_{(\mathrm{id}_{T'}, f^b)}$.

Consider $(f, \mathrm{id})\colon (T', f^*\mathcal{P}) \to (T, \mathcal{P})$. We have the diagram

$$\begin{array}{ccc} f^*\mathcal{P} & \longrightarrow & T' \\ \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle f} \\ \mathcal{P} & \longrightarrow & T \\ \downarrow{\scriptstyle b} & & \downarrow \\ \mathrm{pt} & \longrightarrow & \mathbf{B}G \end{array}$$

where all squares are pullback squares. By definition of $\mathcal{F}_{(T', f^*\mathcal{P})}$, its pullback to $f^*\mathcal{P}$ is defined by $(b \circ b')^* F \cong b'^* b^* F$, but also the pullback of $f^*\mathcal{F}_{(T,\mathcal{P})}$ to $f^*\mathcal{P}$ is by commutativity of the upper square the pullback along b' of b^*F. Define $\phi_{(f,\mathrm{id})}$ to be the natural isomorphism between $\mathcal{F}_{(T', f^*\mathcal{P})}$ and $f^*\mathcal{F}_{(T,\mathcal{P})}$.

Step 1.3: Because we have mapped all morphisms to isomorphisms, $\mathcal{F}$ is cartesian and, as all $\mathcal{F}_{(T,\mathcal{P})}$ are quasi-coherent, this data yields a quasi-coherent sheaf $\mathcal{F}$ on $\mathbf{B}G$.

Step 2: Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathbf{B}G$ that is given by data $\{\mathcal{F}_{(T,t)}, \phi_{(f,f^b)}\}$. Let $\widetilde{G}$ be the trivial G-bundle on pt, and consider the quasi-coherent sheaf $F := \mathcal{F}_{(\mathrm{pt},\widetilde{G})}$ on pt. Let $p\colon G \to \mathrm{pt}$ be the trivial map. We will construct an automorphism σ of p^*F such that $(F, \sigma) \in \underline{\mathrm{Qcoh}}^G(\mathrm{pt})$.

Consider $G \times_k G$ as G-bundle on G via $\mathrm{pr}_2\colon G \times_k G \to G$. 'Multiplication by G' defined as $m\colon G \times_k G \to G \times_k G, (g,h) \mapsto (gh, h)$ is a morphism of G-bundles over G, giving an isomorphism $\phi_{(\mathrm{id},m)}$ of $\mathcal{F}_{(G, G\times_k G)}$. Also, there is a morphism $(p, p^b)\colon (G, G \times_k G) \to (\mathrm{pt}, \widetilde{G})$, where p^b is an isomorphism induced by $(p \circ t)^* \cong t^* p^*$, giving an isomorphism $\phi_{(p,p^b)}$. Define an isomorphism $\sigma \in \mathrm{Aut}_G(p^*F)$ as the composition

$$p^*\mathcal{F}_{(\mathrm{pt},\widetilde{G})} \xrightarrow{\phi_{(p,p^b)}} \mathcal{F}_{(G,G\times_k G)} \xrightarrow{\phi_{(\mathrm{id},m)}} \mathcal{F}_{(G,G\times_k G)} \xrightarrow{\phi^{-1}_{(p,p^b)}} p^*\mathcal{F}_{(\mathrm{pt},\widetilde{G})}.$$

For a k-scheme T and $g, g' \in G(T)$, we have for $\sigma_g = g^*\sigma$ that $\sigma_g \circ \sigma'_g = \sigma_{g'g}$, because $g^*\phi_{(\mathrm{id},m)}$ is the map induced by m_g. Hence, σ gives G-equivariance structure for F.

Step 3: For a morphism $\gamma_{(T,t)}$ as in Remark 2.9 between quasi-coherent sheaves on $\mathbf{B}G$ defined by $\{\mathcal{F}_{(T,t)}, \phi_{(f,f^b)}\}$ and $\{\mathcal{E}_{(T,t)}, \psi_{(f,f^b)}\}$, we take the $(\mathrm{pt}, \tilde{G})$ component to define the morphism $\gamma_{(\mathrm{pt},\tilde{G})}\colon \mathcal{F}_{(\mathrm{pt},\tilde{G})} \to \mathcal{E}_{(\mathrm{pt},\tilde{G})}$ of G-equivariant quasi-coherent sheaves. Given a morphism $h\colon (F,\sigma) \to (E,\tau)$ of G-equivariant quasi-coherent sheaves, we built $\gamma_{(T,\mathcal{P})}$ by descending b^*h between quasi-coherent sheaves on $\mathcal{P}$.

These constructions give an equivalence. □

Remark 2.11. Note that under this equivalence, coherent sheaves correspond to finite-dimensional vector spaces.

2.3 Quasi-coherent sheaves on $\mathbf{B}\mathbb{G}_m$

We specifically want to study coherent sheaves on $\mathbf{B}\mathbb{G}_m$, and so by Theorem 2.1, we want to study representations of the group scheme $\mathbb{G}_m$ over a field k. To classify the representations of $\mathbb{G}_m$, we need the following notation.

Definition 2.12. Let $G = \mathrm{Spec}(A)$ be an affine group scheme over k. A *right A-comodule* is a k-linear map $r\colon V \to V \otimes_k A$ for a k-vector space V such that

$$\begin{array}{ccc} V & \xrightarrow{\quad r \quad} & V \otimes_k A \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle \mathrm{id}_V \otimes \Delta} \\ V \otimes_k A & \xrightarrow{r \otimes \mathrm{id}_A} & V \otimes_k A \otimes_k A \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\quad r \quad} & V \otimes_k A \\ & \searrow^{\mathrm{id}_V} & \downarrow{\scriptstyle \mathrm{id}_V \otimes \epsilon} \\ & & V \otimes_k k \cong V \end{array}$$

commute, where $\Delta\colon A \to A \otimes_k A$ is the diagonal map and $\epsilon\colon A \to k$ is the coidentity. The map r is called the *co-action.*
Let (V, r) be an A-comodule. An *A-subcomodule* is a k-subspace $W \subset V$ such that $r(W) \subset W \otimes_k A$. Then $(W, r|_W)$ is again an A-comodule.

Lemma 2.13. *If $G = Spec(A)$ is an affine group scheme over k and V is a finite-dimensional k-vector space, then there is a one to one correspondence between A-comodules (V, r) and representations $\rho\colon G \to Aut_k(V)$.*

Proof. Let $\dim V = n$, then a choice of basis $\{e_1, ..., e_n\}$ for V gives an isomorphism $\mathrm{Aut}_k(V) \to \mathrm{GL}_{n,k}$ and also $A^n \cong V \otimes_k A$.

Let $\rho\colon G \to \mathrm{GL}_n$ be a representation of G. Then the map id_G gives an $a \in \mathrm{GL}_n(k)$, that is a map

$$k[X_{ij}, \det(X_{ij})^{-1}] \to A$$

and so an A-linear endomorphism $r(a)$ of $A^n \cong V \otimes_k A$. This $r(a)$ is uniquely determined by its restriction to a k-linear $r\colon V \to V \otimes_k A$, and this gives a A-comodule (V, r). (Because given a restriction $r\colon V \to V \otimes_k A$, we obtain $r(a) : V \otimes_k A \to V \otimes_k A$ via $r \otimes \mathrm{id}_A$ postcomposed with $\mathrm{id}_V \otimes m$, where $m\colon A \otimes_k A \to A$ is multiplication.) The conditions of a comodule map follow from the fact that ρ is a group homomorphism.

Let $r\colon V \to V \otimes_k A$ be a comodule given by $e_j \mapsto \sum_{i=1}^n e_i \otimes a_{ij}$ where $(a_{ij})_{1 \leq i,j \leq n} \in \mathrm{GL}_n(A)$ is an invertible n by n matrix. It corresponds to the representation $G \to \mathrm{GL}_n$ sending $g \in G(T)$ so $\bar{g}\colon A \to \mathcal{O}_T(T)$ to $(\bar{g}(a_{ij}))_{1\le i,j\le n}$ by which we mean:

$$V \otimes_k \mathcal{O}_T \xrightarrow{\rho\otimes \mathrm{id}_{\mathcal{O}_T}} V \otimes_k A \otimes_k \mathcal{O}_T \xrightarrow{\mathrm{id}_V \otimes \bar{g} \otimes \mathrm{id}_{\mathcal{O}_T}} V \otimes_k \mathcal{O}_T \otimes_k \mathcal{O}_T \xrightarrow{\mathrm{id}_V \otimes m} V \otimes_k \mathcal{O}_T.$$

□

For $\mathbb{G}_m$ we may classify the representations on vector spaces V by their weight.

Definition 2.14. For $m \in \mathbb{Z}$, a $k[X, X^{-1}]$-comodule (V, r) has *weight* m if it equals

$$r\colon V \to V \otimes_k k[X, X^{-1}], v \mapsto v \otimes X^m.$$

A *weight* m *representation* of $\mathbb{G}_m$ is a representation induced by a $k[X, X^{-1}]$-comodule of weight m, via Lemma 2.13.

The following proposition is 4.4 in [13].

Proposition 2.15. *Every* $k[X, X^{-1}]$*-comodule* (V, r) *is a direct sum* $\bigoplus_{m\in\mathbb{Z}} V_m$ *of weight* m *subcomodules* V_m.

Proof. For each $m \in \mathbb{Z}$, define

$$V_m = \{v \in V : r(v) = v \otimes X^m\},$$

which is by definition weight m and a subcomodule. To show $V = \bigoplus_{m\in\mathbb{Z}} V_m$, let $v \in V$ and because $\{X^n \mid n \in \mathbb{Z}\}$ form a k-basis of $k[X, X^{-1}]$, we write $r(v) \in V \otimes_k k[X, X^{-1}]$ as

$$r(v) = \sum_{m\in\mathbb{Z}} v_m \otimes X^m$$

for $v_m \in V$. Using the properties of r as comodule: we have

$$(\mathrm{id}_V \otimes \Delta) \circ r(v) = (r \otimes \mathrm{id}_A) \circ r(v),$$

so

$$\sum_{m\in\mathbb{Z}} v_m \otimes X^m \otimes X^m = \sum_{m\in\mathbb{Z}} r(v_m) \otimes X^m.$$

Then we have $r(v_m) = v_m \otimes X^m$ and so $v_m \in V_m$. Also, we have

$$(\mathrm{id}_V \otimes \epsilon) \circ r = \mathrm{id}_V,$$

so

$$v = \mathrm{id}(v) = (\mathrm{id}_V \otimes \epsilon) \circ r(v) = \sum_{m\in\mathbb{Z}} v_m \otimes \epsilon(X^m),$$

where $\epsilon(X^m)$ are just scalars in k so under $V \otimes_k k \cong V$ this corresponds to $\sum_{m\in\mathbb{Z}} \epsilon(X^m) v_m \in \bigoplus_{m\in\mathbb{Z}} V_m$. Hence we have $V = \bigoplus_{m\in\mathbb{Z}} V_m$. □

A subrepresentation of a representation $\rho\colon G \to \mathrm{Aut}_k(V)$, is a linear subspace $W \subset V$ such that $\rho|_W\colon G \to \mathrm{Aut}_k(W)$ defined by $\rho|_W(g) := \rho(g)|_W$ is again a representation.

Corollary 2.16. *Every representation $\rho\colon \mathbb{G}_m \to Aut_k(V)$ is the direct sum of weight m subrepresentations for $m \in \mathbb{Z}$.*

Because all weight m representations are given by diagonal matrices, all finite-dimensional representations of $\mathbb{G}_m$ are direct sums of the one-dimensional representations. Also by Proposition 2.15, up to isomorphism, there are $\mathbb{Z}$-many one-dimensional representations of $\mathbb{G}_m$. Hence all finite-dimensional representations of $\mathbb{G}_m$ may be expressed as direct sums, tensor products and duals of the weight 1, one-dimensional representation, called the standard representation.

3 Moduli stack of Gieseker bundles

3.1 Stable curves

A reference for the stack of stable curves is [7] which mainly discusses stable curves of genus zero, and [8] which is the more classic reference for the coarse moduli space $\overline{\mathcal{M}}_{g,n}$ for stable n-pointed curves of arithmetic genus g.

Definition 3.1. For $n \in \mathbb{Z}_{\geq 0}$, a *prestable n-pointed curve* $(C, p_1, ..., p_n)$ over an algebraically closed field $\bar{k}$ is a connected curve C over $\bar{k}$ whose only singularities are nodal singularities (i.e., ordinary double points), with a choice of n distinct smooth points $p_1, ..., p_n \in C$ called *marks.* By a *special point* we mean a node or a mark. A *stable n-pointed curve* is a prestable n-pointed curve satisfying the following stability condition: every genus-0 component of the normalisation must have at least 3 special points and every genus-1 component must have at least one special point.

Let B be a scheme over a field k. A *(pre)stable n-pointed curve* over B is a flat and proper map, locally of finite presentation, $\pi: C \to B$ with n disjoint sections, whose geometric fibers $\pi^{-1}(b)$ are (pre)stable n-pointed curves.

The condition that a genus-1 component must have at least one special point ensures that the space $\overline{\mathcal{M}}_{1,0}$ of stable curves with zero points and genus 1 is empty. Given an ordered set I, we may also consider I-marked curves with a choice of distinct smooth sections $p_i \in C$ for $i \in I$.

Example 3.2. Over an algebraically closed field and in genus zero, a stable n-pointed rational curve may be described as follows. A *tree of projective lines* is a connected curve such that each irreducible component is isomorphic to a projective line, the points of intersection of the components are ordinary double points, and there are no closed circuits: if a node is removed the curve becomes disconnected. The word *twig* is used for the irreducible components of a tree. A stable n-pointed rational curve then is a tree C of projective lines with n distinct marks, which are smooth points of C, such that every twig has at least three special points.

Definition 3.3. An *isomorphism* of two n-pointed curves $(C, p_1, ..., p_n)$ and $(C', p'_1, ..., p'_n)$ over scheme B is an isomorphism of curves $\phi: C \xrightarrow{\sim} C'$ over B such that $\phi(p_i) = p'_i$. Thus an automorphism of $(C, p_1, ..., p_n)$ is an automorphism of C fixing each mark.

Example 3.4. In genus zero, the stability condition is equivalent to saying that there are no non-trivial automorphisms. An automorphism must map a twig with a mark onto itself, and so by induction each twig onto itself. Hence an automorphism is a gluing of automorphisms on each twig and it must fix all marks and nodes. Then the condition that there are three special points on each twig is equivalent to there being no non-trivial automorphisms. Note that stable n-pointed curves of genus $g > 0$ can have non-trivial automorphisms, but only finitely many.

A theorem by Knudsen and Mumford, Theorem 2.7 in [8], is that for each $n \geq 3$, there is a smooth projective variety $\overline{\mathcal{M}}_{0,n}$ which is a fine moduli space for stable n-pointed rational curves.

Lemma 3.5. *Given a stable n-pointed curve $(C, p_1, ..., p_n)$ over an algebraically closed field and an arbitrary point $q \in C$, there is a canonical way to get a stable $(n+1)$-pointed curve.*

Proof. If q is not a special point, we can simply put $p_{n+1} = q$ so that $(C, p_1, ..., p_{n+1})$ is a stable $(n+1)$-pointed curve. If q is a node, then we can pull the two components apart and add a projective line joining those two points to obtain a new curve and putting the new mark p_{n+1} anywhere on this line but not on the nodes. If q coincides with one of the marks p_i, then we can glue a projective line to the original curve at this point, and place marks p_i and p_{n+1} anywhere on this new line but not on the node. For any two choices, you get a unique isomorphism of the resulting stable $(n+1)$-pointed curves. The situations are sketched in Figure 1. □

Definition 3.6. This process to obtain a new $(n+1)$-pointed curve from the data $(C, p_1, ..., p_n, q)$ is called *stabilisation*.

Definition 2.3 in [8] tells us that stabilisation also works for general curves.

Proposition 3.7. *Given a stable n-pointed curve $(\pi \colon C \to B, \sigma_1, ..., \sigma_n)$ and let $d \colon B \to C$ be an arbitrary extra section. Then there is a stable $(n+1)$-pointed curve $(C' \to B, \sigma'_1, ..., \sigma'_n, \sigma'_{n+1})$ and a morphism $\phi \colon C' \to C$ of B-schemes such that $\phi^{-1}(C \setminus d) \tilde{\to} C \setminus d$ is an isomorphism, $\phi \circ \sigma'_{n+1} = d$, and $\phi \circ \sigma'_i = \sigma_i$ for $i = 1, ..., n$. Up to isomorphism this curve is unique and is called the* stabilisation *of $(C \to B, \sigma_1, ..., \sigma_n, \delta)$. Futhermore, stabilisation commutes with fiber products.*

Inverse to stabilisation is a process called either *contraction* or *forgetting a section.* Given a stable $(n+1)$-pointed curve $(C, p_1, ..., p_n, p_{n+1})$ over an algebraically closed field, we forget p_{n+1} and obtain a stable n-pointed curve by firstly removing p_{n+1} and then contracting any unstable twig if

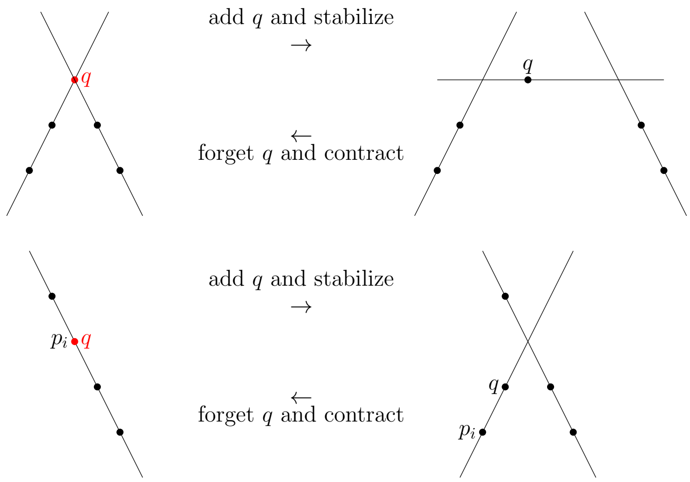


Figure 1: Stabilisation and contraction in genus 0.

such appears. This is illustrated in Figure 1. Forgetting a section again also works for curves over a scheme B.

Given a prestable marked curve C, there is a stabilisation morphism $\mathrm{st}\colon C \to C^{st}$ from C to its stabilisation, that blows down every unstable rational curve in C.

3.2 Stack of principal bundles

The reason we do not want to work with the stack $\mathrm{Bun}_{\mathbb{G}_m}(g, I)$ of principal $\mathbb{G}_m$-bundles on stable marked curves of type (g, I), is that it does not satisfy the existence part of the valuative criterion for properness, [9, tag 0CLY], this is stated in Remark 1.5 in [6] (where the term valuative criterion for completeness is used). We illustrate this in the example below for $g = 0$ and $\#I = 4$, using that $\mathbb{G}_m$-bundles are equivalent to line bundles from Proposition 1.29.

Example 3.8. Let k be an algebraically closed field. Consider inside $\mathbb{P}^2_k$ the variety Y that on an affine open is given by $V(xy) \subset U \subset \mathbb{P}^2_k$, which is a reducible variety, namely two lines meeting at the origin. Let $R := k[b]$ and $B = \mathrm{Spec}(R)$, and consider the $\Sigma = B \times Y$ over B by the natural projection morphism. The coordinate ring of Σ on an open inside $B \times \mathbb{P}^2$ is given by $A := R[x, y]/(xy)$ and the fiber over each point in $b_0 \in B$ is isomorphic to the pullback of Y to the residue field of b_0.

Consider the sheaf of ideals $J = (y, x - b)$. We restrict the ideal sheaf to the fiber over b_0, so considering $J \otimes_R R/(b - b_0)$ over $A \otimes_R R/(b - b_0) \cong$

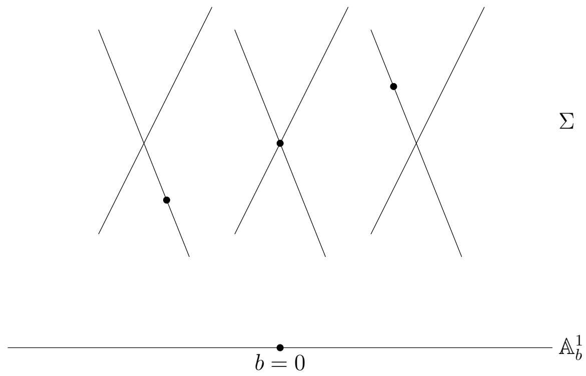


Figure 2: Sketch of space Σ and points defined by $(y, x - b)$.

$k[x,y]/(xy)$. On each of the fibers outside $b_0 = 0$ this is a smooth point and so it is an invertible sheaf. However, over $b_0 = 0$ the point is not smooth, as illustrated in Figure 2, and the sheaf is actually not invertible.

Consider Σ as a stable marked curve by adding marks $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, two on each twig. Everywhere outside $b = 0$, $\Sigma \to B$ together with the line bundle associated J specifies an object in the stack of principal $\mathbb{G}_m$-bundles, but not at $b = 0$.

3.3 Gieseker bundles

To create a space that does satisfy this valuative criterion, we will enlarge the classification problem by looking at Gieseker bundles. Here we use the introduction given in [6]. Firstly, instead of stable curves Σ, unstable twigs may be inserted at the nodes of a stable curve.

Definition 3.9. A morphism $m\colon C \to \Sigma$ of prestable marked curves over an algebraically closed field $\bar{k}$ is a *modification* if m is an isomorphism away from the preimage of the nodes of Σ, and the preimage under m of every node in Σ is either a node or a $\mathbb{P}^1$ with two special points.

A modification of a prestable marked curve $f\colon \Sigma \to B$ is a morphism $m\colon C \to \Sigma$ such that for each geometric fiber $b \in B$, $m_b\colon C_b \to \Sigma_b$ is a modification.

An example of a modification is given in Figure 3. Note that at marked points, a modification is an isomorphism and so marked points lift uniquely to the modification. Also note that, although there generally are different modifications C of a stable marked curve Σ, the modification map will always be the stabilisation map.

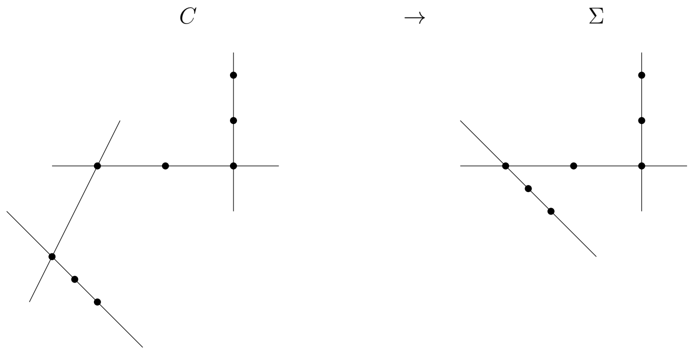


Figure 3: Sketch of a modification in genus 0.

Definition 3.10. Let (Σ, σ_i) be a stable marked curve over an algebraically closed field k. A *Gieseker* $\mathbb{G}_m$*-bundle on* (Σ, σ_i) is a pair $(m, \mathcal{P})$ consisting of a modification $m\colon (C, \sigma_i) \to (\Sigma, \sigma_i)$ and a principal $\mathbb{G}_m$-bundle $p\colon \mathcal{P} \to C$ satisfying the *Gieseker condition*: the restriction of $\mathcal{P}$ to an unstable $\mathbb{P}^1$ in C has degree 1.

If $C \to B$ is a prestable marked curve and $\Sigma \to B$ its stabilisation, and $\mathcal{P}$ a bundle on C satisfying the Gieseker condition on all geometric fibers, then $(C, \sigma_i, \mathcal{P})$ is a Gieseker bundle on Σ.

Considering Gieseker bundles will solve the issue with the existence part of the valuative criterion, as is shown in Proposition 2.14 in [6]. We will illustrate this by showing how Example 3.8 may be extended to a Gieseker bundle over $\Sigma \to B$.

Example 3.11. Continue with the notation as in Example 3.8. The goal is to write down a Gieseker bundle on Σ over B which restricts to the line bundle given by the sheaf of ideals $J = (x, y - b)$ when $b \neq 0$. The objective is to make a modification $C \to \Sigma$ which leaves all fibers Σ_{b_0} intact for $b_0 \neq 0$ and inserts a $\mathbb{P}^1$ at the node in the fiber Σ_0. Inspired by the proof in Knudsen [8] on the existence of stabilisation morphisms, we will explicitly construct the modification as a blow-up.

Firstly consider $R' = k[a, b]$ and $A' = R'[x, y]/(xy - ab)$ and consider $\mathrm{Spec}(A') \to \mathrm{Spec}(R')$. (If $a = 0$, we have the curve $\Sigma \to B$.) Consider the sheaf of ideals $J = (x - b, y - a)$, which defines a section $\mathrm{Spec}(R') \to \mathrm{Spec}(A')$ by $A' \to R', x \mapsto b, y \mapsto a$. Then we consider the graded A-algebra

$$\tilde{A}' = A'[t_1, t_2]/(xt_2 - bt_1, yt_1 - at_2)$$

where $\deg t_i = 1$ and the morphism $\pi\colon \mathrm{Proj}_A \tilde{A}' \to \mathrm{Spec} A'$. Note that as $\tilde{A}'$ is finitely generated, the map π is projective. Also the map is birational: on the affine patch $t_2 \neq 0$ we have an inverse on function fields by

$$t_1 \mapsto x/b, t_2 \mapsto yt_1/a = (yx)/(ab).$$

As it is a projective birational map to an affine Noetherian scheme, it actually is a blow-up along a closed subscheme. Also the section defined by the sheaf of ideals lifts by $\tilde{A}'_{t_i=1} \to R, x \mapsto b, y \mapsto a, t_j \mapsto 1$ where we denote the patch $t_i \neq 0$ by $\tilde{A}'_{t_i=1}$ and this section passes through the smooth locus of the fiber. Therefore, up to setting $a = 0$, this is the modification map we are looking for.

To obtain $a = 0$, we do a base change $\times_{R'} R'/a$. We have $A' \times_{R'} R'/a = A$ and

$$\tilde{A} := \tilde{A}' \times_{R'} R'/a = k[b, x, y, t_1, t_2](xy, xt_2 - bt_1, yt_1)$$

and the map $p\colon \mathrm{Proj}(\tilde{A}) \to \mathrm{Spec}(A)$ induced by π. A projective map stays projective under base change, however a birational map need not stay birational. We will show that it is an isomorphism away from (b, x, y).

We cover $\mathrm{Proj}(\tilde{A})$ by

$$D_+(t_1) \cong \mathrm{Spec}\left(\frac{k[b, x, y, t_2/t_1]}{(x(t_2/t_1) - b, y, xy)}\right) \cong \mathrm{Spec}\left(\frac{k[b, x, t_2/t_1]}{(x(t_2/t_1) - b)}\right)$$

where the A-algebra structure on $\tilde{A}_1 = k[b, x, t_2/t_1]/(x(t_2/t_1) - b)$ is given by $A \to \tilde{A}_1, b \mapsto b, x \mapsto x, y \mapsto 0$, and

$$D_+(t_2) \cong \mathrm{Spec}\left(\frac{k[b, x, y, t_1/t_2]}{(x - b(t_1/t_2), y(t_1/t_2), xy)}\right) \cong \mathrm{Spec}\left(\frac{k[b, y, t_1/t_2]}{(y \cdot t_1/t_2)}\right)$$

where the A-algebra structure on $\tilde{A}_2 = k[b, y, t_1/t_2]/(y \cdot t_1/t_2)$ is given by $A \to \tilde{A}_2, b \mapsto b, x \mapsto b \cdot t_1/t_2, y \mapsto y$. Now we will check that the fiber of p over the point (x, y, b) is a copy of $\mathbb{P}^1$. The fiber over the point corresponding to (b, x, y) is, on the affine patch $\mathrm{Spec}(\tilde{A}_1)$, given by the Spec of

$$\tilde{A}_1 \otimes_A A/(b, x, y) = k[b, x, t_2/t_1]/(x(t_2/t_1) - b) \times_A k[b, x, y]/(b, x, y) \cong k[t_2/t_1]$$

and on the patch $\mathrm{Spec}(\tilde{A}_2)$ the fiber is the Spec of

$$\tilde{A}_2 \otimes_A A/(b, x, y) = k[b, y, t_1/t_2]/(y(t_1/t_2)) \times_A k[b, x, y]/(b, x, y) \cong k[t_1/t_2].$$

Therefore the fiber equals a $\mathbb{P}^1$.

To show that it is an isomorphism away from the preimage of (x, y, b), so where either $b \neq 0$ or $x \neq 0$ or $y \neq 0$, we will show that it is an isomorphism when we can invert one of b, x and y.

Firstly, when $b \neq 0$, the structure map $A \to \tilde{A}_2$ inducing p will be

$$k[b, x, y, b^{-1}]/(xy) \to k[b, y, t_1/t_2, b^{-1}]/(y \cdot t_1/t_2),$$

where $c \mapsto b \cdot t_1/t_2$, which is an isomorphism as the inverse is given by $b \mapsto b, b^{-1} \mapsto b^{-1}, y \mapsto y, t_1/t_2 \mapsto xb^{-1}$. Hence this map is open. The structure map $A \to \tilde{A}_1$ when inverting b is given by

$$k[b, x, y, b^{-1}]/(xy) \to k[b, x, t_2/t_1, b^{-1}]/(x \cdot t_1/t_2 - b)$$

where $y \mapsto 0$. This map is actually inverting x as we have an isomorphism

$$k[b, x, t_2/t_1, b^{-1}]/(x \cdot t_1/t_2 - b) \xrightarrow{\sim} k[b, x, y, b^{-1}, x^{-1}]/(xy)$$

by $t \mapsto bx^{-1}$ and $0 \leftarrow\!\shortmid y$ and $tb^{-1} \leftarrow\!\shortmid x^{-1}$. Therefore, the image of the map $\mathrm{Spec}(\tilde{A}_1) \to \mathrm{Spec}(A)$ is $D(x)$ and so it is an open map. Also the two maps are jointly surjective, and both open, so the map p where $b \neq 0$ is étale. Then it is also finite locally free and flat, so the degree is locally constant and by connectedness there is a single degree. By computing a fiber, the degree is 1 and so it is an isomorphism where $b \neq 0$. We can similarly check this when $x \neq 0$ and $y \neq 0$.

Therefore, $p \colon \mathrm{Proj}(\tilde{A}) \to \mathrm{Spec}(A)$ is a modification of Σ over B. Let $C := \mathrm{Proj}(\tilde{A})$. We give a line bundle by taking $\mathcal{O}(-D)$ for $D = (x - b, y)$. The section defined by this sheaf of ideals can be extended to one through the smooth locus of C. Because on each fiber it is the line bundle associated to a point, i.e., a divisor of degree 1, the degree of the bundle, also at the unstable $\mathbb{P}^1$ is 1, so the Gieseker condition is satisfied. Hence this specifies the Gieseker bundles solving the existence part of the valuative criterion for Example 3.8.

3.4 Stack of Gieseker $\mathbb{G}_m$-bundles

We can define the stack of Gieseker bundles, as given in [6].

Definition 3.12. The *stack* $\widetilde{\mathfrak{M}}_{g,I}(\boldsymbol{B}\mathbb{G}_m)$ *of Gieseker* $\mathbb{G}_m$*-bundles* on stable genus g, I-marked curves is the fibered category over k-schemes with objects $(B, \pi \colon C \to B, \sigma_i, \mathcal{P})$ where

- B is a k-scheme, and
- $\pi \colon C \to B$ is a pre-stable genus g curve with marked points $\sigma_i \colon B \to C$, and
- $\mathcal{P}$ is a principal $\mathbb{G}_m$-bundle $p \colon \mathcal{P} \to C$ defining a Gieseker bundle on the stabilisation $C \to C^{st}$.

Morphisms in the category are triples $(b\colon B' \to B, f\colon C' \to C, \tilde{f}\colon \mathcal{P}' \to \mathcal{P})$ such that $C' = B' \times_B C$, the diagram

$$\begin{array}{ccc} \mathcal{P}' & \xrightarrow{\tilde{f}} & \mathcal{P} \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ C' & \xrightarrow{f} & C \\ {\scriptstyle \sigma_i'}\uparrow\downarrow{\scriptstyle \pi'} & & {\scriptstyle \pi}\downarrow\uparrow{\scriptstyle \sigma_i} \\ B' & \xrightarrow{b} & B \end{array}$$

commutes, and $\tilde{f}$ is $\mathbb{G}_m$-equivariant.

In the article [6] Corollary 2.10, it is proven that it is in fact an Artin stack. Also, some basic facts about the geometry of the stack are mentioned, such as that it is of dimension $(g-1) + 3(g-1) + \#I$.

Before we carefully discuss $\widetilde{\mathcal{M}}_{g,I}(\mathbf{B}\mathbb{G}_m)$ in genus 0 with $\#I = 3$ marks, we need the following lemma. Consider the k-scheme T and the section $\sigma_\infty\colon T \to \mathbb{P}^1_T$; the image of the section is an effective relative Cartier divisor of degree 1 on $\mathbb{P}^1_T$. As σ_∞ defines a smooth point in each fiber, the sheaf $\mathcal{O}_{\mathbb{P}^1_T}(-\sigma_\infty)$ defined by the ideal sheaf given by σ_∞, is actually a line bundle. We may also consider $\mathcal{O}(D\sigma_\infty) = \mathcal{O}(-\sigma_\infty)^{\otimes -D}$ on $\mathbb{P}^1_T$, and these sheaves satisfy the following.

Lemma 3.13. *Let $f\colon T' \to T$ be a morphism of k-schemes, $\sigma_\infty\colon T \to \mathbb{P}^1_T$ and $\sigma'_\infty\colon T' \to \mathbb{P}^1_{T'}$ sections induced by the point ∞ in $\mathbb{P}^1_k$, and $\bar{f}\colon \mathbb{P}^1_{T'} \to \mathbb{P}^1_T$ the projection map. Then we have $\bar{f}^*\mathcal{O}(D\sigma_\infty) = \mathcal{O}(D\sigma'_\infty)$ as submodules of $\mathcal{O}_{\mathbb{P}^1_{T'}} = \bar{f}^*\mathcal{O}_{\mathbb{P}^1_T}$.*

Proof. It is enough to show that $\bar{f}^*\mathcal{O}(-\sigma_\infty) = \mathcal{O}(-\sigma'_\infty)$. Reducing to the affine case, let $g\colon X = \mathrm{Spec}(B) \to Y = \mathrm{Spec}(A)$ and Cartier divisors σ_X of X and σ_Y of Y be given, such that

$$\begin{array}{ccc} \sigma_X & \longrightarrow & \sigma_Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & Y \end{array}$$

is a pullback diagram. Let $f^{\mathrm{op}}\colon A \to B$, and I the ideal of A corresponding to σ_Y, then the ideal of B corresponding to σ_X is IB because the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A/I & \longrightarrow & B/IB \end{array}$$

is a pushout diagram. Then $g^*I = I \otimes_A B$ and we have natural surjection $I \otimes_A B \to IB$. Both ideals yield an invertible sheaf, that is a line bundle, and a surjection of line bundles is an isomorphism, so we have $I \otimes_A B \cong IB$ and $g^*\mathcal{O}(-\sigma_Y) = \mathcal{O}(-\sigma_X)$. □

Remark 3.14. The proof in Lemma 3.13 actually proves something more general: given a closed subscheme locally cut out by a regular element, and its pullback is also locally cut out by a regular element, then the pullback of the ideal sheaf is the ideal sheaf of the pullback.

Example 3.15. The easiest example of the stack $\widetilde{\mathcal{M}}_{g,I}(\mathbf{B}\mathbb{G}_m)$ is where $\#I = 3$ and the curves are genus 0, so $\widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m)$. There is, up to unique isomorphism, only one stable genus zero curve with 3 marked points over k, namely simply $(\mathbb{P}^1_k, 0, 1, \infty)$, because we have only three marks to distribute. Because this curve has no nodes, the only modifications of this curve are automorphisms of $\mathbb{P}^1_k$. Because this automorphism of $\mathbb{P}^1_k$ has to preserve the three marked points, the only possible modification is the trivial modification $m = \mathrm{id}$. We have also established in Example 1.28 that there is only one line bundle of set degree D on $\mathbb{P}^1_k$, namely $\mathcal{O}_{\mathbb{P}^1_k}(D)$. Hence we may describe the stack explicitly in Proposition 3.18.

Definition 3.16. Define a functor

$$\Psi\colon \widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m) \to \bigsqcup_{D\in\mathbb{Z}} \mathbf{B}\mathbb{G}_m$$

as follows. Write $i_D\colon \mathbf{B}\mathbb{G}_m \to \bigsqcup_{D\in\mathbb{Z}} \mathbf{B}\mathbb{G}_m$ for the inclusion for the degree D component. Given an object $(T, C_T \cong \mathbb{P}^1_T, \sigma_i, \mathcal{P}) \in \widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m)(T)$ with $\mathcal{P}$ of degree D, then we define

$$(T, C_T, \sigma_i, \mathcal{P}) \mapsto i_D(T, \sigma_1^*\mathcal{P}).$$

If $\mathcal{P}$ is not of one degree, then the degree of $\mathcal{P}$ is a locally constant function $d\colon T \to \mathbb{Z}$ and $T = \bigsqcup_{a\in\mathbb{Z}} d^{-1}(a)$. Let $T_a := d^{-1}(a)$, then by the universal property of the disjoint union, a map from $T = \sqcup_a T_a$ is defined by maps from each T_a. Then we map $(T, C_T, \sigma_i, \mathcal{P})$ to the element associated to the collection $i_a(T_a, \sigma_1|_{T_a}^*\mathcal{P}|_{T_a})$ for $a \in \mathbb{Z}$.

A morphism $\phi\colon (T', C_{T'}, \sigma_i', \mathcal{P}') \to (T, C_T, \sigma_i, \mathcal{P})$ as defined in 3.12 where $C_{T'} = C_T \times_T T'$ and $\mathcal{P}, \mathcal{P}'$ are assumed to have degree D, specifies an isomorphism $f^b\colon \mathcal{P}' \to f^*\mathcal{P}$. Then we map ϕ to the morphism from $(T', \sigma_1'^*\mathcal{P}')$ to $(T, \sigma_1^*\mathcal{P})$ given by the pair $(t, \sigma_1'^* f^b)$, because we have

$$\sigma_1'^* f^b\colon \sigma_1'^*\mathcal{P}' \to \sigma_1'^* f^*\mathcal{P} \cong (f \circ \sigma_i')^*\mathcal{P} \cong (\sigma_1 \circ t)^* \cong t^*\sigma_1^*\mathcal{P}.$$

Definition 3.17. Define a functor

$$\Xi\colon \bigsqcup_{D\in\mathbb{Z}} \mathbf{B}\mathbb{G}_m \to \widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m)$$

by defining $\Xi \circ i_D$ for any D. On objects we define

$$(T, \mathcal{L}) \mapsto (T, \pi_T\colon \mathbb{P}^1_T \to T, \sigma_0, \sigma_1, \sigma_\infty, \mathcal{P}_{\mathcal{L}} := \mathcal{O}(D\sigma_\infty) \otimes_{\mathcal{O}_{C_T}} \pi_T^*\mathcal{L}).$$

For a morphism $(f, f^b)\colon (T', \mathcal{L}') \to (T, \mathcal{L})$, so $f\colon T' \to T$ and $f^b\colon f^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}'$, we define an isomorphism of line bundles

$$\bar{f}^*\left(\mathcal{O}(D\sigma_\infty) \otimes_{\mathcal{O}_{C_T}} \pi_T^*\mathcal{L}\right) \xrightarrow{\sim} \mathcal{O}(D\sigma'_\infty) \otimes_{\mathcal{O}_{C_T}} \pi_{T'}^*\mathcal{L}'$$

by $\bar{f}^*\mathcal{O}(D\sigma_\infty) = \mathcal{O}(D\sigma'_\infty)$ from Lemma 3.13 and f^b.

Proposition 3.18. *The functor* Ψ *gives an isomorphism of stacks*

$$\widetilde{\mathcal{M}}_{0,3}(\boldsymbol{B}\mathbb{G}_m) \cong \bigsqcup_{D\in\mathbb{Z}} \boldsymbol{B}\mathbb{G}_m.$$

Proof. We show that Ψ and Ξ are inverses. Let $\mathcal{L}$ be any line bundle on T, then $\sigma_1^*\mathcal{P}_{\mathcal{L}}$ is naturally isomorphic to $\mathcal{L}$, because σ_1 and σ_∞ are disjoint sections and so

$$\sigma_1^*\mathcal{P}_{\mathcal{L}} = \sigma_1^*\mathcal{O}(D\sigma_\infty) \otimes_{\mathcal{O}_T} \sigma_1^*\pi_T^*\mathcal{L} \cong \mathcal{O}_T \otimes_{\mathcal{O}_T} \mathcal{L} \cong \mathcal{L}.$$

Secondly, consider an object $(T, C_T, \sigma_i, \mathcal{P})$ in $\widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m)$ where $\mathcal{P}$ is of degree $D \in \mathbb{Z}$ and $(C_T, \sigma_i) \cong (\mathbb{P}^1_T, \sigma_0, \sigma_1, \sigma_\infty)$. We want to prove that there is a natural isomorphism

$$\mathcal{P} \cong \mathcal{O}(D\sigma_\infty) \otimes_{\mathcal{O}_{C_T}} \pi_T^*\sigma_1^*\mathcal{P}.$$

For this, we consider the sheaf

$$F = \mathcal{O}(D\sigma_\infty) \otimes_{\mathcal{O}_{C_T}} \pi_T^*\sigma_1^*\mathcal{P} \otimes_{\mathcal{O}_{C_T}} \mathcal{P}^\vee$$

on $C_T = \mathbb{P}^1_T$ and we will show it is $\mathcal{O}_{\mathbb{P}^1_T}$. Note that F trivialises along σ_1 and so it is enough to trivialise locally on T. Therefore, without loss of generality, assume T is local. On the fiber over the closed point of T, F is trivial, because it has degree 0 on $\mathbb{P}^1_k$. This trivialisation is given by some global section α. We use [5] Theorem III.12.11 (Cohomology and Base Change) for $\pi_T : \mathbb{P}^1_T \to T$ and F on $\mathbb{P}^1_T$ and y the closed point of T. Because F_y is trivial, we know that $\dim H^1((\mathbb{P}^1_T)_y, F_y) = 0$ and $\dim H^0((\mathbb{P}^1_T)_y, F_y) = 1$.

Part (a) of the theorem for $i = 1$ implies that for the fibre we have $R^1\pi_{T*}(F) \otimes k(y) = 0$, as ϕ^1 is clearly surjective and so an isomorphism.

Then also $R^1\pi_{T*}(F) = 0$ by Nakayama's lemma and finite generation of cohomology (which follows from properness of π_T).

Then part (b) of the theorem implies that ϕ^0 is surjective, and we may choose an $\alpha' \in R^0\pi_{T*}(F) \otimes k(y)$ mapping to α. Then α' gives a global section and so defines a map $\mathcal{O}_{\mathbb{P}^1_T} \to F$, which restricts to alpha over the closed point. Over the closed point, the map is surjective, and then by Nakayama's lemma the map $\mathcal{O}_{\mathbb{P}^1_T} \to F$ is a surjective map of line bundles. Because a surjective map of line bundles is an isomorphism, we indeed have $F \cong \mathcal{O}_{\mathbb{P}^1_T}$ and $\mathcal{P} \cong \mathcal{O}(D\sigma_\infty) \otimes_{\mathcal{O}_{C_T}} \pi_T^*\sigma_1^*\mathcal{P}$.

Then we indeed have constructed an explicit inverse, and we conclude

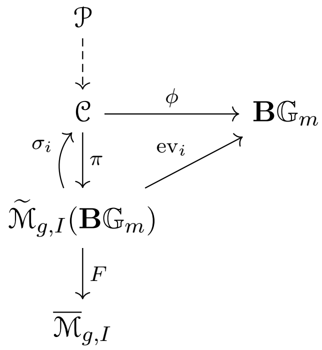
$$\widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m) \cong \bigsqcup_{D\in\mathbb{Z}} \mathbf{B}\mathbb{G}_m.$$

□

4 Gromov-Witten invariants

4.1 Evaluation maps

In this section, we define the evaluation maps on the stack $\widetilde{\mathcal{M}} := \widetilde{\mathcal{M}}_{g,I}(\mathbf{B}\mathbb{G}_m)$, and equip $\widetilde{\mathcal{M}}$ with a natural curve, a line bundle, and sections. That is, we construct the following diagram and examine them for $\widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m)$.



(2)

These constructions are standard, and if a reader is familiar with these and does not wish to see the details, he or she may continue reading at section 4.2.

If $\widetilde{\mathcal{M}}$ were a scheme, then we could by Yoneda's lemma associate to $\mathrm{id}_{\widetilde{\mathcal{M}}} \in \mathrm{Hom}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{M}})$ an element in $\widetilde{\mathcal{M}}(\widetilde{\mathcal{M}})$ and so a triple

$$(C_{\widetilde{\mathcal{M}}}, \sigma_i, \mathcal{P}),$$

which would be the natural curve, sections and line bundle. In this case we have the following stack as natural curve.

Definition 4.1. Let $\widetilde{\mathcal{M}} := \widetilde{\mathcal{M}}_{g,I}(\mathbf{B}\mathbb{G}_m)$, and define $\mathcal{C} \to \widetilde{\mathcal{M}}$ as the fibered category given by

$$\mathcal{C}(T) = \{(C_T \to T, \sigma_i, \mathcal{P}, c) \mid (C_T, \sigma_i, \mathcal{P}) \in \widetilde{\mathcal{M}}(T), c \in C_T(T)\}$$

for T a k-scheme. This means we consider $\widetilde{\mathcal{M}}$ with an extra section which need not be disjoint from the σ_i. Morphisms are commutative diagrams like morphisms in $\widetilde{\mathcal{M}}$, see Definition 3.12, where also the final section commutes. The map $\mathcal{C} \to \widetilde{\mathcal{M}}$ is the natural forgetful morphism

$$\pi \colon \mathcal{C} \to \widetilde{\mathcal{M}}, (T, C_T, \sigma_i, \mathcal{P}, c) \mapsto (T, C_T, \sigma_i, \mathcal{P})$$

forgetting the final section.

Lemma 4.2. *The map $\pi \colon \mathcal{C} \to \widetilde{\mathcal{M}}$ is representable.*

Proof. Let $T \to \widetilde{\mathcal{M}}$ be given, specified by $(C_T, \sigma_i, \mathcal{P})$, and consider any map $f\colon T' \to T$. By properties of the fiber product we have

$$C_{T'}(T') \cong C_T(T'), \text{ and}$$
$$\mathcal{C} \times_{\widetilde{\mathcal{M}}} T'(T') \cong \mathcal{C} \times_{\widetilde{\mathcal{M}}} T \times_T T'(T') \cong \mathcal{C} \times_{\widetilde{\mathcal{M}}} T(T').$$

The composition $T' \to T \to \widetilde{\mathcal{M}}$ is given by $(C_{T'} = C_T \times_T T', \sigma_i \circ f, f^*\mathcal{P})$, and then the definition of $\mathcal{C}$ gives that

$$\mathcal{C} \times_{\widetilde{\mathcal{M}}} T'(T') = \{x \in \mathcal{C}(T'), \pi(x) \xrightarrow{\sim} (C_{T'}, \sigma_i \circ f, f^*\mathcal{P})\}$$

and so we only add a section of $C_{T'}$. We conclude

$$\mathcal{C} \times_{\widetilde{\mathcal{M}}} T(T') \cong \mathcal{C} \times_{\widetilde{\mathcal{M}}} T'(T') \cong C_{T'}(T') \cong C_T(T').$$

Hence $\pi\colon \mathcal{C} \to \widetilde{\mathcal{M}}$ is representable by schemes. □

This also explains why we would call this a natural 'curve' over $\widetilde{\mathcal{M}}$; for all $T \to \widetilde{\mathcal{M}}$ the fiber product $\mathcal{C} \times_{\widetilde{\mathcal{M}}} T$ is the curve C_T over T.

We also define for $j \in I$ the section maps σ_j to $\mathcal{C}$ by

$$\sigma_j\colon \widetilde{\mathcal{M}} \to \mathcal{C}, (T, C_T, \sigma_i, \mathcal{P}) \mapsto (T, C_T, \sigma_i, \mathcal{P}, \sigma_j).$$

Moreover, we have a natural classifying map

$$\phi\colon \mathcal{C} \to \mathbf{B}\mathbb{G}_m$$

sending an object $(T, C_T, \sigma_i, \mathcal{P}, c)$ to $(T, c^*\mathcal{P})$. Suppose we have a morphism from $(T', C_{T'} = C_T \times_T T', \sigma_i', \mathcal{P}', c')$ to $(T, C_T, \sigma_i, \mathcal{P}, c)$, so a commutative diagram as in 3.12 with $f\colon T' \to T, \bar{f}\colon C_{T'} \to C_T$ and $\tilde{f}\colon \bar{f}^*\mathcal{P} \to \mathcal{P}'$. Then we map it to $(f, c'^*\tilde{f})\colon (T', c'^*\mathcal{P}') \to (T, c^*\mathcal{P})$ where $c^*\tilde{f}$ is indeed an isomorphism of $c'^*\bar{f}^*\mathcal{P} \cong f^*c^*\mathcal{P}$ and $c'^*\mathcal{P}'$ as $\tilde{f} \circ c' = c \circ f$.

This map specifies a line bundle $\mathcal{P}$ on $\mathcal{C}$ given as sheaf on the stack $\mathcal{C}$ by sending an object (T, t) in Lis-Ét$(\mathcal{C})$ to the $\mathcal{O}_T$-module and line bundle $c^*\mathcal{P}$, when $t\colon T \to \mathcal{C}$ is specified by $(C_T, \sigma_i, \mathcal{P}, c)$. By the natural triple for $\widetilde{\mathcal{M}}$ we then mean $(\mathcal{C}, \sigma_i, \mathcal{P})$.

Now define the evaluation maps $\mathrm{ev}_i = \phi \circ \sigma_i$

$$\mathrm{ev}_i\colon \widetilde{\mathcal{M}}_{g,I}(\mathbf{B}\mathbb{G}_m) \to \mathbf{B}\mathbb{G}_m$$

by sending $(T, C_T, \sigma_i, \mathcal{P})$ to $(T, \sigma_i^*\mathcal{P})$.

Finally, important to note is the forgetful map

$$F\colon \widetilde{\mathcal{M}}_{g,I}(\mathbf{B}\mathbb{G}_m) \to \overline{\mathcal{M}}_{g,I}$$

to the stack of stable genus g, I-marked curves, forgetting $\mathcal{P}$ and sending (C, σ_i) to its stabilisation. Thus we have defined diagram (2).

Lemma 4.3. *The curve* $\mathcal{C}$ *over* $\widetilde{\mathcal{M}}_{0,3}(\boldsymbol{B}\mathbb{G}_m)$ *is given by*

$$\mathbb{P}^1_{\widetilde{\mathcal{M}}} := \widetilde{\mathcal{M}}_{0,3}(\boldsymbol{B}\mathbb{G}_m) \times_{\overline{\mathcal{M}}_{0,3}} \mathbb{P}^1_{\overline{\mathcal{M}}_{0,3}}.$$

Proof. Firstly, we prove that the universal curve over $\overline{\mathcal{M}}_{0,3}$ is $\mathbb{P}^1_k$. We have the isomorphism $\overline{\mathcal{M}}_{0,3} \xrightarrow{\sim} \underline{\mathrm{Sch}}_k, (T, C_T, \sigma_i) \mapsto T$ with an inverse given by $T \mapsto (T, \mathbb{P}^1_T, \sigma_0, \sigma_1, \sigma_\infty)$. The curve $\bar{C}$ over $\overline{\mathcal{M}}_{0,3}$ on k-scheme T is given by

$$\bar{C}(T) = \{(C_T, \sigma_i, c \in C_T(T)\}$$

and $\bar{C} \to \overline{\mathcal{M}}_{0,3}$ is given by forgetting the last section. The diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,3} & \xrightarrow{\sim} & k \\ \uparrow & & \uparrow \\ \bar{C} & & \mathbb{P}^1_k \end{array}$$

may be completed with the map $\bar{C} \to \mathbb{P}^1_k$ sending (C_T, σ_i, c) to

$$T \xrightarrow{c} C_T \xrightarrow{\sim} \mathbb{P}^1_T \to \mathbb{P}^1_k.$$

For the fiber product $\mathbb{P}^1_T = \mathbb{P}^1_k \times_k T$, giving $s \colon T \to \mathbb{P}^1_k$ is equivalent to giving a map $t \colon T \to \mathbb{P}^1_T$, so an inverse $\mathbb{P}^1_k \to \bar{C}$ is given by

$$s \mapsto (\mathbb{P}^1_T, \sigma_0, \sigma_1, \sigma_\infty, t).$$

This isomorphism $\bar{C} \xrightarrow{\sim} \mathbb{P}^1_k$ makes the above diagram commute. Note that in this case, one can see that the universal curve $\bar{C} \cong \mathbb{P}^1_k$ and its sections is indeed the element of $\overline{\mathcal{M}}_{0,3}(k)$ corresponding to $\mathrm{id}_{\overline{\mathcal{M}}_{0,3}}$.

The conclusion for $\mathcal{C}$ over $\widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m)$ follows from the pullback square

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F_c} & \bar{C} \cong \mathbb{P}^1_k \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m) & \xrightarrow{F} & \overline{\mathcal{M}}_{0,3} \end{array}$$

where both F and F_c are forgetting the bundle. □

Remark 4.4. We can restrict the $(\mathcal{C}, \sigma_i, \mathcal{P})$ to degree D components of $\widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m) \cong \bigsqcup_{D\in\mathbb{Z}} \mathbf{B}\mathbb{G}_m$. We have inclusion $i_D \colon \mathbf{B}\mathbb{G}_m \to \bigsqcup_{D\in\mathbb{Z}} \mathbf{B}\mathbb{G}_m$ and by postcomposing with the isomorphism to $\widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m)$ we get

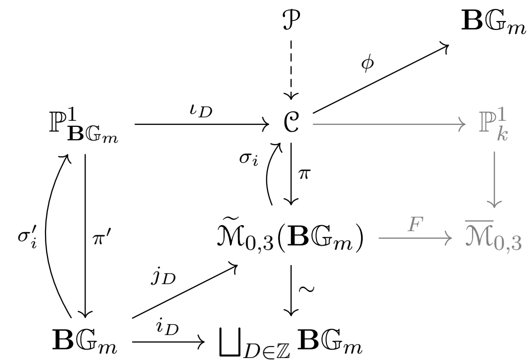
$$j_D \colon \mathbf{B}\mathbb{G}_m \to \widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m),$$

$$(T, \mathcal{L}) \mapsto (T, \pi_T \colon \mathbb{P}^1_T \to T, \sigma_0, \sigma_1, \sigma_\infty, \mathcal{O}(D\sigma_\infty) \otimes_{\mathcal{O}_{\mathbb{P}^1_T}} \pi_T^*\mathcal{L}).$$

Then we have

$$\mathbf{B}\mathbb{G}_m \times_{\widetilde{\mathcal{M}}} \mathcal{C} \cong \mathbb{P}^1_{\mathbf{B}\mathbb{G}_m}$$

with objects triples $(T, \mathcal{L}, t)$ of a k-scheme T, a line bundle $\mathcal{L}$ on T and an extra map $s\colon T \to \mathbb{P}^1_k$ (or $t\colon T \to \mathbb{P}^1_T$). Then we have the following maps



where π' is the map induced by the pullback and $\sigma'_i\colon \mathbf{B}\mathbb{G}_m \to \mathbb{P}^1_{\mathbf{B}\mathbb{G}_m}$ are given by $(T, \mathcal{L}) \mapsto (T, \mathcal{L}, \sigma_i\colon T \to \mathbb{P}^1_T)$ for $i = 0, 1, \infty$.

We can also restrict the natural bundle $\mathcal{P}$ to $\mathbb{P}^1_{\mathbf{B}\mathbb{G}_m}$. Note that the map ι_D is defined by

$$(T, \mathcal{L}, t\colon T \to \mathbb{P}^1_T) \mapsto (T, \mathbb{P}^1_T, \sigma_0, \sigma_1, \sigma_\infty, \mathcal{O}(D\sigma_\infty) \otimes_{\mathcal{O}_{\mathbb{P}^1_T}} \pi_T^*\mathcal{L}, t\colon T \to \mathbb{P}^1_T).$$

Then we have on objects that

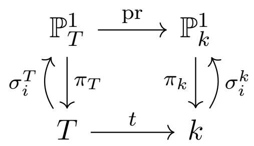
$$\mathcal{P}(\iota_D((T, \mathcal{L}, t\colon T \to \mathbb{P}^1_T))) = t^*(\mathcal{O}(D\sigma_\infty) \otimes_{\mathcal{O}_{\mathbb{P}^1_T}} \pi_T^*\mathcal{L})$$

and as ι_D is an open immersion, this defines $\iota_D^*\mathcal{P}$.

Note that an evaluation map $\mathrm{ev}_i = \phi \circ \sigma_i$ restricted to a degree D component is actually simply the identity map, because we have

$$\begin{aligned}
\mathrm{ev}_i \circ j_D(T, \mathcal{L}) &= \mathrm{ev}_i(T, \mathbb{P}^1_T, \sigma_0, \sigma_1, \sigma_\infty, \mathcal{O}(D\sigma_\infty) \otimes_{\mathcal{O}_{\mathbb{P}^1_T}} \pi_T^*\mathcal{L}) \\
&= \phi(T, \mathbb{P}^1_T, \sigma_0, \sigma_1, \sigma_\infty, \mathcal{O}(D\sigma_\infty) \otimes_{\mathcal{O}_{\mathbb{P}^1_T}} \pi_T^*\mathcal{L}, \sigma_i^T : T \to \mathbb{P}^1_T) \\
&= (T, \sigma_i^{T*}(\mathcal{O}(D\sigma_\infty) \otimes_{\mathcal{O}_{\mathbb{P}^1_T}} \pi_T^*\mathcal{L})).
\end{aligned}$$

Note that $\pi_T \circ \sigma_i^T = \mathrm{id}_T$ and so canonically $\sigma_i^{T*}\pi_T^*\mathcal{L} \cong \mathcal{L}$. Lemma 3.13 gives for



that $\mathrm{pr}^*\mathcal{O}(D\sigma_\infty^k) = \mathcal{O}(D\sigma_\infty^T)$, and so

$$\sigma_i^{T*}\mathcal{O}(D\sigma_\infty^T) \cong \sigma_i^{T*}\mathrm{pr}^*\mathcal{O}(D\sigma_\infty^k) \cong t^*\sigma_i^{k*}\mathcal{O}(D\sigma_\infty^k) \cong \mathcal{O}_T,$$

because $\sigma_i^{k*}\mathcal{O}(D\sigma_\infty^k)$ is a line bundle on k and so is trivial. Hence we have

$$\mathrm{ev}_i \circ j_D(T, \mathcal{L}) = (T, \sigma_i^{T*}(\mathcal{O}(D\sigma_\infty) \otimes_{\mathcal{O}_{\mathbb{P}^1_T}} \pi_T^*\mathcal{L})) \cong (T, \mathcal{O}_T \otimes_{O_T} \mathcal{L}) \cong (T, \mathcal{L}).$$

Similarly the map is trivial on morphisms, and we have $\mathrm{ev}_i \circ j_D = \mathrm{id}_{\mathbf{B}\mathbb{G}_m}$.

4.2 Gromov-Witten invariants

Gromov-Witten invariants can be computed using cohomology classes and the Chow group, for example in the article [7], where the stack of stable genus g, I-marked curves and the stack of stable maps is studied. However, as is motivated in [6], because $\widetilde{\mathcal{M}}_{g,I}(\mathbf{B}\mathbb{G}_m)$ is an Artin stack rather than a Deligne-Mumford stack, it is not likely we may integrate cohomology classes along the forgetful morphism F. Instead of cohomology classes, we will push forward K-theory classes of coherent sheaves, which are defined as follows.

Definition 4.5. For an algebraic stack X, the category $\underline{\mathrm{Coh}}(X)$ of coherent sheaves of $\mathcal{O}_X$-modules is a full additive subcategory of an abelian category $\mathcal{O}_X - \underline{\mathrm{Mod}}$. For $C := \underline{\mathrm{Coh}}(X)$, consider $F(C)$ the free abelian group on the objects of C up to isomorphism. Let $(E) : 0 \to M' \to M \to M'' \to 0$ be an exact sequence in C, and write $Q(E) = [M]-[M'']-[M'] \in F(C)$. Let $H(C)$ be the subgroup of $F(C)$ generated by the $Q(E)$ for all short exact sequences E. Then the *Grothendieck group* $K_0(X)$ of $\underline{\mathrm{Coh}}(X)$ is $K_0(X) = F(C)/H(C)$.

On $\widetilde{\mathcal{M}}_{g,I}(\mathbf{B}\mathbb{G}_m)$, we have the following (classes of) coherent sheaves that are of interest. For any finite-dimensional representation $\mathbb{G}_m \to \mathrm{Aut}_k(V)$ of $\mathbb{G}_m$, we have the coherent sheaf ϕ^*V on $\mathcal{C}$. In particular, for the standard representation $\mathbb{G}_{m1}$ given by $\mathrm{id}\colon \mathbb{G}_m \to \mathbb{G}_m$, we have $\phi^*(\mathbb{G}_{m1})$ is the line bundle attached to the torsor $\mathcal{P}$.

Definition 4.6. For a finite-dimensional representation V of $\mathbb{G}_m$, we call

$$\mathrm{ev}_i^*[V] = \sigma_i^*\phi^*V$$

an *evaluation bundle* on $\widetilde{\mathcal{M}}_{g,I}(\mathbf{B}\mathbb{G}_m)$. Its *descendant bundles* are the classes $\mathrm{ev}_i^*[V] \otimes [T_i^{\otimes j_i}]$, where $T_i = \sigma_i^*\Omega^\vee_{\mathcal{C}/\widetilde{\mathcal{M}}}$ is the relative tangent line to $\mathcal{C}$ at σ_i and j_i is an integer.

Remark 4.7. Note that every coherent sheaf on $\mathbf{B}\mathbb{G}_m$ is locally free as this may be checked on the smooth cover by a point. Thus each $\mathrm{ev}_i^*[V_a]$ is locally free and so flat. Then, we may take the product of classes $\mathrm{ev}_i^*[V]$ to be the tensor product (in that case the quotiented exact sequences are preserved by tensoring).

To define what we, in this thesis, mean by a Gromov-Witten invariant, we firstly define the K-theoretic pushforward.

Definition 4.8. For a morphism $f : X \to Y$, *the K-theoretic pushforward* of a class of a coherent sheaf $\mathcal{F}$ on X is defined as

$$f_*[\mathcal{F}] = \sum_i (-1)^i [R^i f_* \mathcal{F}],$$

if this exists in $K_0(Y)$.

Remark 4.9. Two problems may occur for the existence of this pushforward in $K_0(Y)$. Firstly, $R^i f_* \mathcal{F}$ may not be coherent, although, if f is proper, it is. Secondly, the sum may have infinitely many non-zero terms.

Definition 4.10. By a *Gromov-Witten invariant* for $\widetilde{\mathcal{M}}_{g,I}(\mathbf{B}\mathbb{G}_m)$ we mean the following. Let $F : \widetilde{\mathcal{M}}_{g,I}(\mathbf{B}\mathbb{G}_m) \to \overline{\mathcal{M}}_{g,I}$ be as above and $p\colon \overline{\mathcal{M}}_{g,I} \to \mathrm{pt}$. A Gromov-Witten invariant is, if it is defined, the K-theoretic pushforward p_*F_* of the tensor product of a number of evaluation bundles, descendant bundles, an admissible line bundle $\mathcal{L}$, as will be defined in Definition 4.19, and complexes $R^i\pi_*\phi^*V$ as will be defined in Construction 4.13.
A Gromov-Witten invariant then is a class in $K_0(\mathrm{pt})$. Note that $K_0(\mathrm{pt})$ is naturally isomorphic to $\mathbb{Z}$ by taking the dimension of a k-vector space. Thus, we can associate to a Gromov-Witten invariant a number, which we refer to as a numerical Gromov-Witten invariant.

The map $p\colon \overline{\mathcal{M}}_{g,I} \to \mathrm{pt}$ is proper, so the existence of this pushforward is less technical. However, the existence of a pushforward along F to $\overline{\mathcal{M}}_{g,I}$, is not straightforward as $\widetilde{\mathcal{M}}_{g,I}(\mathbf{B}\mathbb{G}_m)$ is not proper, see Section 2.3 Limits of Bundles in [6].

Example 4.11. For $\#I = 3$ and $\widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m)$, the map

$$F\colon \widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m) \cong \bigsqcup_{D\in\mathbb{Z}} \mathbf{B}\mathbb{G}_m \to \overline{\mathcal{M}}_{0,3}$$

is not proper, because we have infinitely many copies of $\mathbf{B}\mathbb{G}_m$ and so the map is not quasi-compact. Since $\overline{M}_{0,3}$ is a point, there is no need to push forward along p.

The main theorem in [6] states that for certain specified admissible complexes α of coherent sheaves, the derived pushforward $RF_*\alpha$ is a bounded complex of coherent sheaves. Then the K-theoretic pushforward is actually defined, and thus we have Gromov-Witten invariants. In the next sections, we shall try to illustrate this main theorem in the special case of genus 0 curves with 3 marks.

4.3 Admissible line bundles

We want to push forward bundles ϕ^*V on $\mathcal{C}$ along the map $\pi\colon \mathcal{C} \to \widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m)$ so in the complex $R\pi_*\phi^*V$. Basics of the higher direct image or derived pushforward for maps of schemes may be found in Liu, [3], Chapter 5.2.3. We mainly use the following property, Proposition 2.28 in [3].

Lemma 4.12. *Let $\pi\colon X \to Spec(k)$ be a map of schemes, and let $\mathcal{L}$ be any line bundle on X, then $R^i\pi_*(\mathcal{L})$ is naturally isomorphic to $H^i(X,\mathcal{L})^\sim$, the $\mathcal{O}_k$-module associated to $H^i(X,\mathcal{L})$.*

Note that $\pi\colon \mathcal{C} \to \widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m)$ is a map of stacks, not schemes. Therefore, we will explicitly construct the sheaf $R\pi_*\phi^*V$. Because $\widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m)$ is isomorphic to a disjoint union, we can define the sheaf $R\pi_*\phi^*V$ by the restriction to its components, as in Remark 4.4.

Construction 4.13. Write $\mathcal{F}$ for the coherent sheaf $\iota_D^*\phi^*V$ on $\mathbb{P}^1_{\mathbf{B}\mathbb{G}_m}$. We shall define $(E,\sigma) \in \underline{\mathrm{Coh}}^{\mathbb{G}_m}(\mathrm{pt})$ giving a coherent sheaf $R^i\pi'_*\mathcal{F}$ on $\mathbf{B}\mathbb{G}_m$ via Theorem 2.1 and its proof. Using the cover map pt $\to \mathbf{B}\mathbb{G}_m$ defined by the trivial line bundle, we have the pullback diagram

$$\begin{array}{ccc} \mathbb{P}^1_k & \xrightarrow{\tilde{\pi}} & \mathrm{pt} \\ \downarrow{\scriptstyle f} & & \downarrow \\ \mathbb{P}^1_{\mathbf{B}\mathbb{G}_m} & \xrightarrow{\pi'} & \mathbf{B}\mathbb{G}_m. \end{array} \tag{3}$$

where the map f is given by $(s\colon T \to \mathbb{P}^1_k) \mapsto (T, \mathcal{O}_T, s)$. We consider the coherent sheaf on pt given by

$$E := R^i\tilde{\pi}_*f^*\mathcal{F} = H^i(\mathbb{P}^1_k, f^*\mathcal{F}),$$

via Lemma 4.12. To define the automorphism $\sigma\colon p^*E \to p^*E$ where p is the map $p\colon \mathbb{G}_m \to \mathrm{pt}$, consider the pullback square

$$\begin{array}{ccc} \mathbb{P}^1_k \times_k \mathbb{G}_m & \xrightarrow{q} & \mathbb{P}^1_k \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \tilde{\pi}} \\ \mathbb{G}_m & \xrightarrow{p} & \mathrm{pt}\,. \end{array}$$

Because $\tilde{\pi}$ is flat, flat base change gives a canonical isomorphism

$$p^*E \cong R^i\pi_{2*}q^*(f^*\mathcal{F}),$$

so we can compare their automorphisms. An automorphism of $q^*(f^*\mathcal{F})$ will induce an automorphism of $R^i\pi_{2*}q^*(f^*\mathcal{F})$, and an automorphism of $q^*(f^*\mathcal{F})$ can be given by an element of $\mathbb{G}_m(\mathbb{P}^1 \times \mathbb{G}_m)$. Naively, let $\alpha\colon \mathbb{P}^1 \times \mathbb{G}_m \to \mathbb{G}_m$ be given, then for each $a \in \mathbb{P}^1 \times \mathbb{G}_m(k)$, we get an $\alpha(a) \in \mathbb{G}_m(k) = k^*$. Then $a^*(q^*(f^*\mathcal{F}))$ is a k-vector space and we get an automorphism by multiplying by $\alpha(a)$.

Hence, the automorphism σ of p^*E can be given by an element of $\mathbb{G}_m(\mathbb{P}^1 \times \mathbb{G}_m)$.

Example 4.14. Consider $\mathcal{F} = \iota_D^*\phi^*\mathbb{G}_{m1} = \iota_D^*\mathcal{P}$. For $s\colon T \to \mathbb{P}^1_k$, and $t\colon T \to \mathbb{P}^1_T = \mathbb{P}^1_k \times_k T$ such that $\mathrm{pr} \circ t = s$, we have

$$\mathcal{F}(f(s\colon T \to \mathbb{P}^1_k)) = t^*\left(\mathcal{O}(D\sigma^T_\infty) \otimes_{\mathcal{O}_{\mathbb{P}^1_T}} \pi_T^*\mathcal{O}_T\right) = t^*(\mathcal{O}(D\sigma^T_\infty)) \otimes_{\mathcal{O}_T} t^*\pi_T^*\mathcal{O}_T.$$

$$\cong t^*(\mathcal{O}(D\sigma^T_\infty)) \otimes_{\mathcal{O}_T} \mathcal{O}_T \cong t^*(\mathcal{O}(D\sigma^T_\infty)).$$

Then Lemma 3.13 gives for $\mathrm{pr}\colon \mathbb{P}^1_T \to \mathbb{P}^1_k$ that $\mathcal{O}(D\sigma^T_\infty) \cong \mathrm{pr}^*\mathcal{O}(D\sigma^k_\infty)$, and so naturally

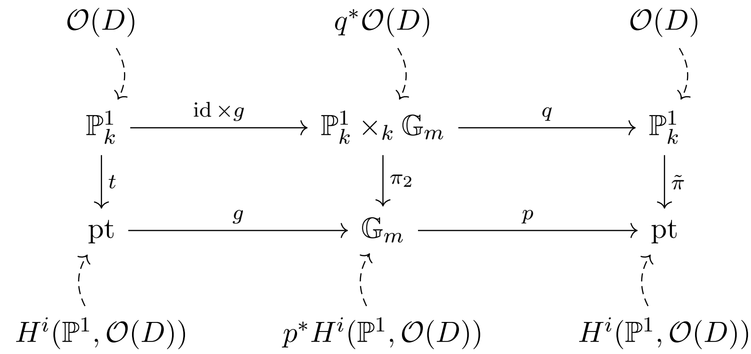
$$t^*\mathcal{O}(D\sigma^T_\infty) \cong t^*\mathrm{pr}^*\mathcal{O}(D\sigma^k_\infty) \cong s^*\mathcal{O}(D\sigma^k_\infty).$$

Then the sheaf $f^*\mathcal{F}$ on $\underline{\mathrm{Sch}}_k$ is canonically isomorphic to the line bundle $\mathcal{O}(D\sigma^k_\infty)$ on $\mathbb{P}^1_k$, which is isomorphic to the line bundle $\mathcal{O}(D)$ from Example 1.28, where D is the degree.

Via Riemann-Roch or directly as in Example 1.28 for H^0, we compute

$$\dim H^i(\mathbb{P}^1_k, \mathcal{O}(D)) = \begin{cases} 0 & \text{if} & & i < 0 \\ D+1 & \text{if} & D > -2, & i = 0 \\ 0 & \text{if} & D < 0, & i = 0 \\ 0 & \text{if} & D > -2, & i = 1 \\ -(D+1) & \text{if} & D < 0, & i = 1 \\ 0 & \text{if} & & i > 1 \end{cases} \tag{4}$$

which equals $\dim H^i(\mathbb{P}^1_k, f^*\mathcal{F}) = \dim E = \dim R^i\tilde{\pi}_*f^*\mathcal{F}$. The structure on E of a $\mathbb{G}_m$-equivariant sheaf will be induced by the map $\pi_2\colon \mathbb{P}^1 \times \mathbb{G}_m \to \mathbb{G}_m$. Let $g \in \mathbb{G}_m(k)$, then we check what the induced automorphism σ of p^*E does by studying the following diagram, where the dashed arrows indicate bundles.



Studying the automorphism of $q^*\mathcal{O}(D)$, the construction described the automorphism as multiplying $(\mathrm{id}\times g)^* q^*\mathcal{O}(D) = \mathcal{O}(D)$ on $\mathbb{P}^1_k$ by the unit given by $\pi_2(\mathrm{id}\times g)$ which equals $g \circ t$ and so it is 'multiplication by g'. The pushforward along t of this multiplication by g on $\mathcal{O}(D)$ on $\mathbb{P}^1_k$, to $g^*p^*H^i(\mathbb{P}^1_k, \mathcal{O}(D)) \cong H^i(\mathbb{P}^1_k, \mathcal{O}(D))$ on pt is again multiplication by g.

We will deduce the representation associated to the $\mathbb{G}_m$-equivariant coherent sheaf $E = H^i(\mathbb{P}^1_k, \mathcal{O}(D))$ on pt. The representation namely sends $g \in \mathbb{G}_m(k)$ to the automorphism simply multiplying by g in $\mathrm{Aut}(g^*E)$. Thus the representation of $\mathbb{G}_m$, classified in Section 2.3, is simply the weight one representation of a dimension $\dim H^i(\mathbb{P}^1, \mathcal{O}(D))$ vector space over k. This is the direct sum of $\dim H^i(\mathbb{P}^1, \mathcal{O}(D))$ copies of the standard representation.

Remark 4.15. Given a general finite-dimensional representation V of $\mathbb{G}_m$, then V equals $\bigoplus_{j\in J} {\mathbb{G}_m}_1^{\otimes n_j}$ for a certain index set J and $n_j \in \mathbb{Z}$. Now the above construction and example also tell us what representations belong to $R^i\tilde{\pi}_*\phi^*V$. We have

$$\iota_D^*\phi^*V = \bigoplus_{j\in J} \iota_D^*\phi^* {\mathbb{G}_m}_1^{\otimes n_j} \cong \bigoplus_{j\in J} \iota_D^* \mathcal{P}^{\otimes n_j}$$

on $\mathbb{P}^1_{\mathbf{B}\mathbb{G}_m}$. For $f\colon \mathrm{pt} \to \mathbf{B}\mathbb{G}_m$ defined by the trivial line bundle, $f^*\iota_D^*\phi^*V$ is isomorphic to

$$\bigoplus_{j\in J} \mathcal{O}(D)^{\otimes n_j}$$

on $\mathbb{P}^1_k$. Via Example 1.28, this is isomorphic to $\bigoplus_{j\in J} \mathcal{O}(n_j \cdot D)$. We defined the sheaf E on pt as

$$H^i(\mathbb{P}^1_k, \bigoplus_{j\in J} \mathcal{O}(n_j \cdot D)) \cong \bigoplus_{j\in J} H^i(\mathbb{P}^1_k, \mathcal{O}(n_j \cdot D))$$

which all vanish when $i \neq 0, 1$. We want to find the $\mathbb{G}_m$-equivariance action on each $H^i(\mathbb{P}^1_k, \mathcal{O}(n_j \cdot D))$. Similarly to Example 4.14, but now for a n_j-fold tensor product of the standard $\mathcal{O}(D)$, we get weight n_j representations.

Thus we get that on the degree D component, $R^i\tilde{\pi}_*\phi^*V$ is given by the direct sum over $j \in J$ of dimension $\dim H^i(\mathbb{P}^1, \mathcal{O}(n_j \cdot D))$ weight n_j representations.

In order to define admissible line bundles, we need the concept of the determinant; this definition may be found in Stacks Project [9], tag 0B37, or in [10].

Definition 4.16. Let $(X, \mathcal{O}_X)$ be a locally ringed space and $\mathcal{E}$ a finite locally free $\mathcal{O}_X$-module. Because the rank is locally constant, we obtain a decomposition $X = \bigsqcup_{n\in\mathbb{Z}} X_n$ where the X_n are open and closed, such that $\mathcal{E}$ is finite locally free of rank n on X_n. We define the *determinant* $\det(\mathcal{E})$ as the invertible sheaf on X which is equal to the exterior power $\bigwedge^n(\mathcal{E}|_{X_n})$ on X_n for all $n \geq 0$. Because $\det(\mathcal{E})$ is finite locally free of rank 1, it is a line bundle and so $\det(\mathcal{E}) \in \mathrm{Pic}(X)$.

Definition 4.17. Let $f\colon X \to Y$ be a proper morphism of schemes over S. For every coherent sheaf $\mathcal{F}$ on X that is flat over S, we have the invertible $\mathcal{O}_Y$-module

$$\det Rf_*\mathcal{F}$$

defined as follows. If a finite complex of locally free $\mathcal{O}_Y$-modules of finite rank $E^* = (0 \to E^0 \to E^1 \to ... \to E^n \to 0)$ exists such that there is a quasi-isomorphism $E^* \to Rf_*\mathcal{F}$ of complexes, then we have

$$\det Rf_*\mathcal{F} = \bigotimes_{i=0}^{n} (\det E^i)^{\otimes(-1)^i}.$$

When in particular all $\mathcal{O}_Y$-modules $R^if_*\mathcal{F}$ for $i \geq 0$ are locally free of finite rank, then we have

$$\bigotimes_i \det\left(R^i\pi_*L\right)^{\otimes(-1)^i}. \tag{5}$$

The determinant of the zero sheaf is canonically isomorphic to $\mathcal{O}_Y$.

Remark 4.18. For the proper morphism of stacks $\pi\colon \mathcal{C} \to \widetilde{\mathcal{M}}$, we have defined the coherent sheaf $R^i\pi_*\phi^*V$ via $R^i\tilde{\pi}_*\phi V$ in Construction 4.13. We define $\det R\pi_*\phi^*V$ on each D-th component of $\widetilde{\mathcal{M}} \cong \bigsqcup_{D\in\mathbb{Z}} \mathbf{B}\mathbb{G}_m$ using (5).

Definition 4.19. Write $\mathbb{G}_{m1}$ for the standard representation of $\mathbb{G}_m$. The *admissible line bundles* $\mathcal{L}$ are the non-zero powers of the determinant

$$\mathcal{L} = (\det R\pi_*\phi^*\mathbb{G}_{m1})^{\otimes -n}, \text{ for } n \in \mathbb{Z} \setminus \{0\}.$$

Example 4.20. We compute $\det R\pi_*\phi^*\mathbb{G}_{m1}$ for $\widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m)$. For degree D and $\pi'\colon \mathbb{P}^1_{\mathbf{B}\mathbb{G}_m} \to \mathbf{B}\mathbb{G}_m$, because all $R^i\pi'_*\iota_D^*\mathcal{P}$ are coherent as $\iota_D^*\mathcal{P}$ is coherent and π' is proper (because $\tilde{\pi}$ is proper in diagram (3)), we may compute the determinant via

$$\bigotimes_i \det\left(R^i \pi'_* \iota_D^* \mathcal{P}\right)^{\otimes(-1)^i}.$$

The $R^i \pi'_* \iota_D^* \mathcal{P}$ vanish for $i \neq 0, 1$, and so we have

$$\begin{aligned} \det R\pi'_* \iota_D^* \mathcal{P} &= \det(H^0(\mathbb{P}^1, \mathcal{O}(D))) \otimes \det(H^1(\mathbb{P}^1, \mathcal{O}(D)))^{\otimes -1} \\ &= \begin{cases} \mathcal{O}_{\mathbf{B}\mathbb{G}_m} & \text{if } \ D = -1 \\ \det(H^0(\mathbb{P}^1, \mathcal{O}(D))) & \text{if } \ D \geq 0 \\ \det(H^1(\mathbb{P}^1, \mathcal{O}(D)))^{\otimes -1} & \text{if } \ D \leq -2 \end{cases} \end{aligned}$$

As the $H^0(\mathbb{P}^1, \mathcal{O}(D))$ and $H^1(\mathbb{P}^1, \mathcal{O}(D))^\vee$ are of dimension $(D+1)$ respectively $-(D+1)$ over k, we get a $(D+1)$-th respectively $-(D+1)$-th exterior power as determinant. Taking this exterior power, and the dual if $D \leq -2$, gives a weight $(D+1)$ one-dimensional representation of $\mathbb{G}_m$.

From this example and the fact that tensoring representations means adding weights, we get the following result.

Lemma 4.21. *Let $n \in \mathbb{Z} \setminus \{0\}$ and consider $\mathcal{L} = (\det R\pi_* \phi^* \mathbb{G}_{m1})^{\otimes -n}$ on $\widetilde{\mathcal{M}}_{0,3}(\boldsymbol{B}\mathbb{G}_m)$. Then on the degree D component of $\widetilde{\mathcal{M}}_{0,3}(\boldsymbol{B}\mathbb{G}_m) \cong \sqcup_{D \in \mathbb{Z}} \boldsymbol{B}\mathbb{G}_m$, $\mathcal{L}$ is given by a one-dimensional weight $n(D+1)$ representation.*

4.4 Coherent pushforward of an admissible complex

Definition 4.22. An *admissible complex* α on $\widetilde{\mathcal{M}}_{g,I}(\mathbf{B}\mathbb{G}_m)$ is the tensor product of an admissible line bundle $\mathcal{L}$ with any finite number of Dolbeault indexes and evaluation or descendant bundles:

$$\alpha = \mathcal{L} \bigotimes \left(\otimes_{a \in A} R\pi_* \phi^* V_a\right) \bigotimes \left(\otimes_{b \in B} \operatorname{ev}_i^* [W_b] \otimes [T_i^{\otimes n_i}]\right)$$

for A, B finite sets of indices, n_i integers, and V_a, W_b finite-dimensional representations of $\mathbb{G}_m$.

We prove the following special case of the Main Theorem 6.1 in [6].

Theorem 4.23. *The derived pushforward $RF_* \alpha$ of any admissible complex α on $\widetilde{\mathcal{M}}_{0,3}(\boldsymbol{B}\mathbb{G}_m)$ along $F \colon \widetilde{\mathcal{M}}_{0,3}(\boldsymbol{B}\mathbb{G}_m) \to \overline{\mathcal{M}}_{0,3} \cong \mathrm{pt}$ is coherent.*

The strategy to prove the theorem is to show that admissible complexes are bounded in the sense that the pushforward gives the zero class on infinitely many components of $\bigsqcup_{D \in \mathbb{Z}} \mathbf{B}\mathbb{G}_m \cong \widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m)$. For just the D-th component, so for $F_D := F \circ j_D \colon \mathbf{B}\mathbb{G}_m \to \widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m) \to \mathrm{pt}$, we will show that the pushforward does preserve coherence.

Remark 4.24. Note that, using the definitions as in the Stacks Project [9], this is not because F_D is proper [9, tag 0CL5]: F_D is not seperated as defined in [9, tag 04YV] because the map $\mathbb{G}_m \to \mathrm{pt}$ and so also Δ_{F_D} is not proper.

However, F_{D*} does preserve quasi-coherent sheaves, by [9, tag 0704]: F_D is a quasi-compact and quasi-separated morphism of algebraic stacks by the following. Firstly $\mathbb{G}_m \to \mathrm{pt}$ is affine and so quasi-seperated and quasi-compact, and so Δ_{F_D} is, and so F_D is quasi-seperated. Recall that $\mathbf{B}\mathbb{G}_m$ has the smooth surjective map $\mathrm{pt} \to \mathbf{B}\mathbb{G}_m$ where pt is quasi-compact. Also, $\mathbf{B}\mathbb{G}_m$ is quasi-compact [9, tag 04CC] and pt is quasi-seperated, and so by Lemma 93.7.8(1) [9, tag 050S], F_D is quasi-compact. Then F_{D*} is via Theorem 2.1 a functor

$$\underline{\mathrm{Repr}}_k(\mathbb{G}_m) \to \underline{\mathrm{Qcoh}}(\mathrm{pt}).$$

We will compare F_D to the invariants functor, defined as follows.

Definition 4.25. The *invariants functor* on $\underline{\mathrm{Repr}}_k(\mathbb{G}_m)$ is given by the sheafification of the presheaf

$$(-)^{\mathbb{G}_m} \colon \underline{\mathrm{Repr}}_k(\mathbb{G}_m) \to \underline{\mathrm{Qcoh}}(\mathrm{pt})$$
$$(V, \rho) \mapsto \{(t \colon T \to k) \mapsto (t^*V)^{\mathbb{G}_m(T)}\}.$$

Note that if k is an algebraically closed field, then the presheaf is a quasi-coherent sheaf, because $(t^V)^{\mathbb{G}_m(T)} = t^*(V^{\mathbb{G}_m(k)})$ holds because of the following. We reduce this to weight m representations and if the weight is 0, then $(t^V)^{\mathbb{G}_m(T)} = t^*V = t^*(V^{\mathbb{G}_m(k)})$ as everything remains invariant. If the weight in non-zero, the only 0 remains invariant under multiplication by all units and so $(t^V)^{\mathbb{G}_m(T)} = 0 = t^*0 = t^*(V^{\mathbb{G}_m(k)})$.

Proposition 4.26. *The functor $(-)^{\mathbb{G}_m}$ is right adjoint to F_D^*, and therefore $(-)^{\mathbb{G}_m} = F_{D*}$.*

Proof. Firstly $F_D^* \colon \underline{\mathrm{Qcoh}}(\mathrm{pt}) \to \underline{\mathrm{Repr}}_k(\mathbb{G}_m)$ is the trivial functor, sending $F \colon \underline{\mathrm{Sch}}_k^{\mathrm{op}} \to \mathcal{O}-\underline{\mathrm{Mod}}$ to the vector space $V = F(\mathrm{id}_k)$ with the trivial weight 0 representation, so $\rho_{\mathrm{triv}} \colon \mathbb{G}_m \to \mathrm{Aut}_k(W), g \mapsto \mathrm{id}$. We will show that for all k-vector spaces V and representations $(W, \rho \colon \mathbb{G}_m \to \mathrm{Aut}_k(W))$ of $\mathbb{G}_m$ that

$$\mathrm{Hom}_{\underline{\mathrm{Repr}}_k}(F_D^*V, (W, \rho)) \cong \mathrm{Hom}_{\underline{\mathrm{Qcoh}}(\mathrm{pt})}(V, (W, \rho)^{\mathbb{G}_m}).$$

On the right-hand side we have a natural transformation and so for each k-scheme $t \colon T \to k$ a morphism $\phi_T \colon t^*V \to (t^*W)^{\mathbb{G}_m(T)}$.

On the left-hand side we have k-linear maps $h \colon V \to W$ such that for all $g \in \mathbb{G}_m(T)$ the diagram

$$\begin{array}{ccc} t^*V & \xrightarrow{\rho_{\mathrm{triv}}} & t^*V \\ \downarrow{\scriptstyle t^*h} & & \downarrow{\scriptstyle t^*h} \\ t^*W & \xrightarrow{\rho(g)} & t^*W \end{array}$$

commutes. Therefore, it satisfies $\rho(g) \circ (t^*h) = t^*h$ for all $g \in \mathbb{G}_m(T)$, meaning that t^*h should map into $(t^*W)^{\mathbb{G}_m(T)}$.

Given $\phi = \{\phi_{t\colon T \to k}\} \in \mathrm{Hom}_{\underline{\mathrm{Qcoh}}(\mathrm{pt})}(V, (W, \rho)^{\mathbb{G}_m})$, we can send it to $\phi_{\mathrm{id}_k} \in \mathrm{Hom}_{\underline{\mathrm{Repr}}_k}(F_D^* V, (W, \rho))$. The inverse is given by sending a morphism $(h\colon V \to W) \in \mathrm{Hom}_{\underline{\mathrm{Repr}}_k}(F_D^* V, (W, \rho))$ to $\{t^*h\}_{t\colon T \to k}$. Thus $(-)^{\mathbb{G}_m}$ is right adjoint to F_D^*. Adjoints, if they exist, are unique up to isomorphism, and F_{D*} is right adjoint to F_D^*, so we conclude $(-)^{\mathbb{G}_m} \cong F_{D*}$. □

As $(-)^{\mathbb{G}_m}$ sends finite-dimensional representations to coherent sheaves, we also have that F_{D*} restricts to

$$F_{D*}\colon \underline{\mathrm{FRepr}}_k(\mathbb{G}_m) \to \underline{\mathrm{Coh}}(\mathrm{pt}),$$

so F_{D*} preserves coherence.

By definition of group homology, see [11] chapter 16, as right derived functors of the invariants functor, we now have that $R^i F_{D*}$ is $H^i(\mathbb{G}_m, -)$. We use the following vanishing result for the group cohomology.

Lemma 4.27. *For all representations (V, ρ) of $\mathbb{G}_m$ of weight n, the group cohomology is given by*

$$H^i(\mathbb{G}_m, V) = \begin{cases} V & \text{for } n = 0 \text{ and } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose V is a weight n representation. For $i = 0$, we have

$$H^0(\mathbb{G}_m, V) = V(k)^{\mathbb{G}_m(k)} = V^{k^*}.$$

If $\mathbb{G}_m$ acts trivially, so if $n = 0$, this results in V and else this is the 0 vector space. For general representations V we then have $H^0(\mathbb{G}_m, V) = V_0$ the weight 0 subrepresentation of V.

For $i \geq 0$, we use the classification in Corollary 2.16 to conclude that $\mathbb{G}_m$ is reductive, and we may use Proposition 16.16 in [11] giving $H^i(\mathbb{G}_m, V) = 0$ for all $i > 0$ and all representations V. □

Proof of 4.23. Let

$$\alpha = \mathcal{L} \bigotimes \left(\otimes_{a \in A} R\pi_* \phi^* V_a\right) \bigotimes \left(\otimes_{b \in B} \mathrm{ev}_i^*[W_b] \otimes [T_i^{\otimes n_i}]\right)$$

be an admissible complex. Firstly, by Lemma 4.21, $\mathcal{L} = (\det R\pi_*\phi^*\mathbb{G}_{m1})^{\otimes n}$ for $n \in \mathbb{Z} \setminus \{0\}$ has weight $w(\mathcal{L}) = n(D+1)$, which tends to $\pm\infty$ for $D \to \pm\infty$. Hence for all $\mathcal{L}$, $R^iF_{D*}\mathcal{L}$ will vanish for $D \to \pm\infty$ by Lemma 4.27. If the other complexes, evaluation bundles and descendant bundles that are tensored with L in an admissible complex are bounded in weight (from above and below), then we still have that the weight of α will go to $\pm\infty$ (dependent on growth of the weight of $\mathcal{L}$) for $D \to \pm\infty$. An example of the distribution of weights for $D \in \mathbb{Z}$ is illistrated in Figure 4.4.

For a finite-dimensional representation $V = \bigoplus_{j \in J} \mathbb{G}_{m1}^{\otimes n_j}$ of $\mathbb{G}_m$, consider the complex $R\pi_*\phi^*V$. For each $R^i\pi_*\phi^*V$, we have by Remark 4.15 that we get direct sums of weight n_j representations when $i = 0, 1$. So in particular weights for such a complex are bounded.

Let W be a weight $w(W)$ representation. For evaluation bundles $\mathrm{ev}_i^*[W]$, by remark 4.4, we have $\mathrm{ev}_i \circ j_D = \mathrm{id}_{\mathbf{B}\mathbb{G}_m}$ and so on the degree D component $\mathrm{ev}_i^*[W]$ has weight $w(W)$.

For descendant bundles, we can also tensor with $[T_i^{n_i}]$ for some integer n_i. Consider $\sigma_i^*\Omega_{\mathcal{C}/\widetilde{\mathcal{M}}}$. Because we have the following pullback diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{q} & \mathbb{P}^1_k \\ {\scriptstyle \sigma_i'}\uparrow\downarrow & & {\scriptstyle \sigma_i}\uparrow\downarrow \\ \widetilde{\mathcal{M}} & \xrightarrow{F} & k, \end{array}$$

we have for the relative differentials that $\Omega_{\mathcal{C}/\widetilde{\mathcal{M}}} \cong q^*\Omega_{\mathbb{P}^1_k/k}$ and so

$$\sigma_i'^*\Omega_{\mathcal{C}/\widetilde{\mathcal{M}}} \cong \sigma_i'^* q^*\Omega_{\mathbb{P}^1_k/k} \cong F^*\sigma_i^*\Omega_{\mathbb{P}^1_k/k}.$$

As $\sigma_i^*\Omega_{\mathbb{P}^1_k/k}$ is a line bundle on the field k, this gives the trivial sheaf, and so $\sigma_i'^*\Omega_{\mathcal{C}/\widetilde{\mathcal{M}}} = \mathcal{O}_{\widetilde{\mathcal{M}}}$. Hence the classes $[T_i^{n_i}]$ always have weight 0 on each D.

Hence, $w(\alpha)$ goes to ∞ or $-\infty$ for $D \mapsto \pm\infty$, and so by by Lemma 4.27, the pushforward along F_D will all vanish but for finitely many D. Therefore, the pushforward $RF_*\alpha$ is coherent. $\square$

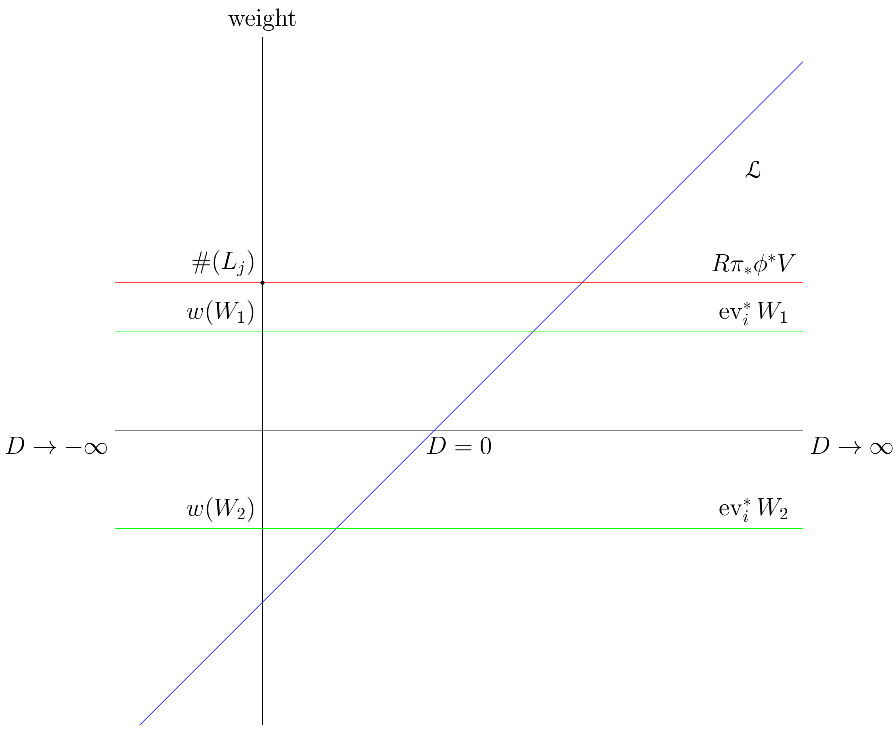


Figure 4: Example distribution of weights of components of an admissible complex.

4.5 Gromov-Witten invariants for $\widetilde{\mathcal{M}}_{0,3}(\mathbf{B}\mathbb{G}_m)$

Theorem 4.23 shows that the K-theoretic pushforward exists, but we may even calculate the numerical Gromov-Witten invariant for an admissible complex, given the following data.

Let $n \in \mathbb{Z} \setminus \{0\}$ be given and consider $\mathcal{L} = (\det R\pi_*\phi^*\mathbb{G}_{m1})^{\otimes -n}$. Because any representation is a direct sum of one-dimensional representations, and the following constructions respect direct sums, let V_a, W_b be one-dimensional weight v_a resp. w_b representations of $\mathbb{G}_m$. Finally, let $n_j \in \mathbb{Z}$ be given, and consider the complex

$$\alpha = \mathcal{L} \bigotimes (\otimes_{a\in A} R\pi_*\phi^*V_a) \bigotimes \left(\otimes_{b\in B} \mathrm{ev}_j^*[W_b] \otimes [T_j^{\otimes n_j}]\right).$$

By definition of the K-theoretic pushforward, we compute

$$F_*[\alpha] = \sum_i (-1)^i [R^i F_* \alpha].$$

For $D \in \mathbb{Z}$, we know $R^i F_{D*}$ is only non-zero if $i = 0$ and we have a weight

0 representation, by lemma 4.27. Thus we want to compute

$$R^0 F_*(\alpha) = \bigoplus_D R^0 F_{D*}(\alpha).$$

Recall that on the degree D component, the components of α are the following representations:

- The admissible line bundle $\mathcal{L}$ is a one-dimensional weight $n(D+1)$ representation of $\mathbb{G}_m$.
- If $v_a \cdot D > -1$, then $R^0\pi_*\phi^*V_a$ is a $\dim H^0(\mathbb{P}^1_k, \mathcal{O}(v_a D))$-dimensional weight v_a representation. If $v_a \cdot D < -1$, then $R^1\pi_*\phi^*V_a$ is a weight v_a representation of dimension $\dim H^1(\mathbb{P}^1_k, \mathcal{O}(v_a D))$. In other cases, $R^i\pi_*\phi^*V_a$ is a zero-dimensional representation. For convenience, write $h_0(v_a D) = \dim H^0(\mathbb{P}^1_k, \mathcal{O}(v_a D))$ and $h_1(v_a D) = \dim H^0(\mathbb{P}^1_k, \mathcal{O}(v_a D))$; we computed these dimensions explicitely in (4) in Example 4.14. Then
$$\sum_i (-1)^i [R^i\pi_*\phi^*V_a] = [R^0\pi_*\phi^*V_a] - [R^1\pi_*\phi^*V_a]$$
is a $h_0(v_a D)$-dimensional representation minus a $h_1(v_a D)$-dimensional representation.
- $\mathrm{ev}_i^*[W_b]$ is a one-dimensional weight w_b representation, and
- $[T_i^{\otimes n_j}]$ is a one-dimensional weight 0 representation.

Only for D such that the sum of these weights

$$n(D+1) + \sum_{a : v_a D \neq -1} v_a + \sum_b w_b$$

equals 0, F_{D*} can possibly not give 0. Such D exists if we have

$$D_0 := \frac{-n - \sum_{a : v_a D \neq -1} v_a - \sum_b w_b}{n} \in \mathbb{Z}.$$

Because $R^0 F_{D_0 *}$ sends a weight 0 representation V to $H^0(\mathbb{G}_m, V) = V$, and the isomorphism from $K_0(\mathrm{pt})$ to $\mathbb{Z}$ sends a vector space to its dimension, we consider the dimension of the above representations. For k-vector spaces V, W, the tensor product satisfies $\dim(V \otimes W) = \dim(V)\dim(W)$, and so when tensoring we multiply dimensions. Applying this to the above list of representations, we obtain the following result.

Theorem 4.28. *The numerical Gromov-Witten invariant of α is given by*

$$\prod_{a : v_a D > -1} h_0(v_a D_0) \cdot \prod_{a : v_a D < -1} -h_1(v_a D_0).$$

if $D_0 \in \mathbb{Z}$, and the numerical Gromov-Witten invariant equals 0 otherwise.

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