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An analysis of the single server queue with Erlang and Coxian service distributions.

Master Thesis

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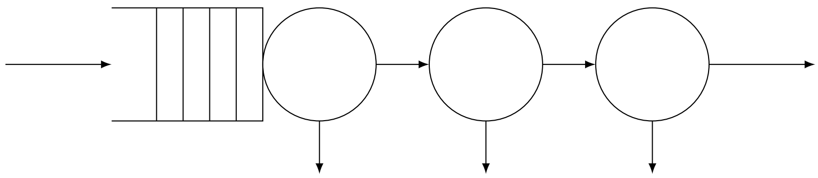

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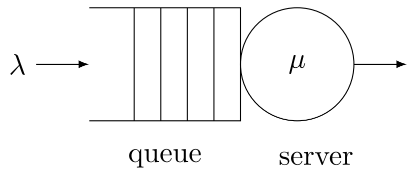
Chapter 1

Introduction

We consider a single server queue where customers arrive according to a Poisson process with arrival rate $\lambda \in \mathbb{R}$. The customers are served according to a First Come First Serve discipline at a single server. A simple and well understood model is the $M|M|1$-queue, in which the service time is exponentially distributed. Though this model does have the Markov property, it is in general not a realistic model. There are other service time distributions that give a more realistic model.



One generalization of the $M|M|1$-queue is the $M|E_r|1$-queue, in which customers are served according to an Erlang distribution. In Chapter 2 we will determine the stationary distribution π_n of this model, as in Adan and Resing [1]. An interesting observation of this distribution, is that it behaves sinusoidal for $r \geq 3$. In Section 2.3 we will discuss this behaviour, as well as prove that it disappears in the limit as $n \to \infty$.

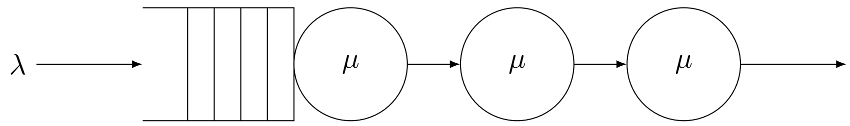
The $M|E_r|1$-queue can again be generalized to the $M|Cox|1$-queue, where the service time has a Coxian distribution. For the latter, we will determine the stationary distribution $\pi_{(i,j)}$ in Chapter 3. This distribution can be described by means of a rate matrix. In Section 3.2, we will analyse this matrix. We will prove that it is diagonalizable when $r = 2$ as well as when $r = 3$ and the phase rates are equal.

The most general and realistic model, with respect to the service time, is the $M|G|1$-queue, in which customers are served according to an arbitrary (non-negative) distribution function. However, this model does not have the Markov property. We can tackle this problem using a Coxian distribution. This distribution can be used to approximate any service time distribution arbitrarily well, while keeping the Markov property intact by means of the associated Markov process. In Chapter 4, we will construct a simple algorithm that provides such a Coxian approximation.

Chapter 2

The Erlang queue

One of the models that can be used to approximate the $M|G|1$ queue is the $M|E_r|1$-queue. In this section we will analyse the stationary distribution of this queue, following the method from Adan and Resing [1]. In the $M|E_r|1$-queue customers are being served according to an Erlang distribution that has two parameters, r and μ. This can be viewed as a series of r service phases, each exponentially distributed with parameter μ.



For this system to be stable, we require that

$$\rho = \frac{\lambda \cdot r}{\mu} < 1.$$

We can describe the state of this system by means of the number of customers in the system and the remaining number of phases of the customer in service (if there is one). The following flow diagram describes this system.

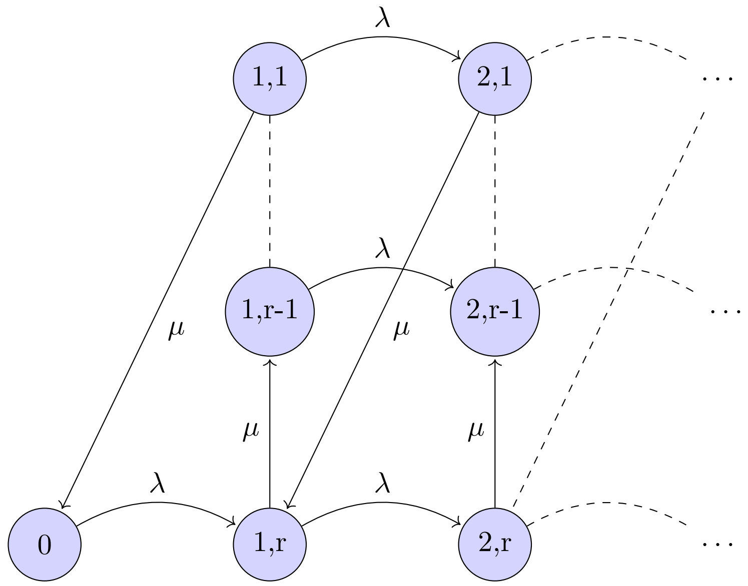


FIGURE 2.1: Diagram for the $M|E_r|q$-queueing system.

There is, however, an easier way. We can describe the state of the system by means of the total number of remaining phases of work in the system. Each customer brings r phases of work to the system. Thus, state (n, l) corresponds to $(n-1)r + l$ phases of work to be done by the server. On the other hand, $k > 0$ phases of work in the system corresponds to state $\left(\lfloor \frac{k-1}{r} \rfloor + 1,\ k - \lfloor \frac{k-1}{r} \rfloor \cdot r\right)$. The Markov process associated with the number of phases of work in the system has state space $\{0, 1, \ldots\}$, where state i corresponds to i phases of work in the system.

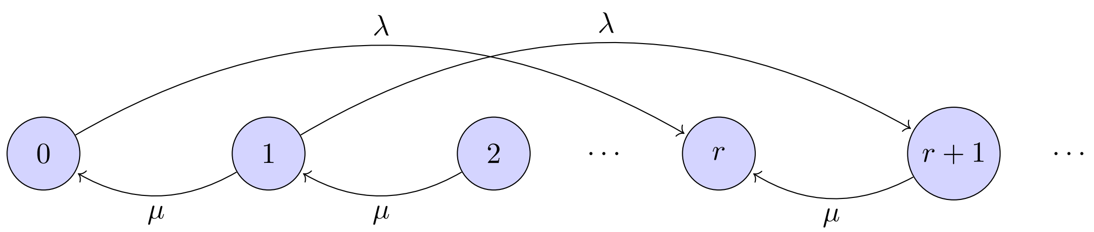


FIGURE 2.2: Diagram for the $M|E_r|q$-queueing system, with state space given by the number of phases of work in the system.

2.1 Analysing the equilibrium equations

Assuming that the system is in equilibrium, the flow into a state i must equal the flow out of state i. This gives the following equilibrium equations for the stationary distribution:

$$\lambda\pi_0 = \mu\pi_1 \tag{2.1}$$

$$\lambda\pi_n + \mu\pi_n = \mu\pi_{n+1}, \qquad n = 1, \dots, r-1, \tag{2.2}$$

$$\lambda\pi_n + \mu\pi_n = \lambda\pi_{n-r} + \mu\pi_{n+1}, \qquad n \geq r. \tag{2.3}$$

There cannot be a negative number of customers in the system, so by convention it holds that $\pi_n = 0$ for $n < 0$. Using this, we can combine Eqns. (2.2) and (2.3) into

$$\lambda\pi_n + \mu\pi_n = \lambda\pi_{n-r} + \mu\pi_{n+1}, \qquad n \geq 1. \tag{2.4}$$

This system of equations can be solved by a generating functions method, as described in Adan and Resing [1]. The approach is to first analyse this equation and some properties of its solutions. We can do this by looking for solutions in the form of

$$\pi_n = x^n, \qquad n = 0, 1, 2, \dots \tag{2.5}$$

We substitute Eqn. (2.5) into Eqn. (2.3) to get

$$\lambda x^n + \mu x^n = \lambda x^{n-r} + \mu x^{n+1}.$$

Dividing both sides by x^{n-r} gives

$$(\lambda + \mu)x^r = \lambda + \mu x^{r+1},$$

which we can rewrite to get

$$\mu x^{r+1} - (\lambda + \mu)x^r + \lambda = 0 \tag{2.6}$$

or

$$(\lambda + \mu) - \mu x = \frac{\lambda}{x^r}. \tag{2.7}$$

Both formulations will be useful as we analyse this equation and its $r + 1$ roots. In the following subsections we will show the properties of Eqn. (2.6) that are necessary to derive a formula for π. These properties are as follows:

i) Eqn. (2.6) has precisely two positive real roots and, there is an additional negative real root, if r is even. The other roots are complex.

ii) Eqn. (2.6) has r roots z with $|z| < 1$.

iii) all $r + 1$ roots of Eqn. (2.6) are distinct.

2.1.1 Real and imaginary roots

To start off, we want to get an indication of the number of real roots of Eqn. (2.6).

Lemma 2.1. *Let $r \in \mathbb{N}_+$. For odd r, Eqn. (2.6) has 2 real positive roots and $r - 1$ complex roots. For even r, it has two positive real roots, one negative real root and $r - 2$ complex roots.*

To prove this lemma, we use Descartes' rule of signs, which is as follows.

Lemma 2.2 (Descartes' rule of signs)**.** *Let $f(x) = a_0 x^{b_0} + a_1 x^{b_1} + \ldots + a_n x^{b_n}$ be a polynomial, with $a_i \in \mathbb{R} \setminus \{0\}$, for $i = 0, \ldots, n$ and $0 \leq b_0 < b_1 < \ldots < b_n \in \mathbb{N}$. The following two statements hold.*

i) Let m be the number of sign changes in $a_0, \ldots, a_n$. Let m_+ be the number of positive real roots of $f(x)$ (multiple roots counted seperately). Then it holds, that $m_+ \in \{m, m-2, m-4, \ldots\}$.

ii) Let m be the number of sign changes in $a_0(-1)^{b_0}, \ldots, a_m(-1)^{b_1}$. Let m_- be the number of negative real roots of $f(x)$ (multiple roots counted seperately). Then it holds, that $m_- \in \{m, m-2, m-4, \ldots\}$.

For the proof of Lemma 2.2 see Wang [5]. We now apply this to Eqn. (2.6).

Apply Descartes' rule of signs to $f(x) = \mu x^{r+1} - (\lambda+\mu)x^r + \lambda$, with $a_0 = 1, a_1 = -(\lambda+\mu)$ and $a_2 = \mu$. There are two sign changes. Hence, there are either two or no positive real roots. Clearly $x = 1$ is a positive real root of $f(x)$, therefore there must be two positive real roots.

We then apply the rule to

$$\begin{aligned} f(-x) &= \mu(-x)^{r+1} - (\lambda+\mu)(-x)^r + \lambda \\ &= (-1)^{r+1}\mu x^{r+1} + (-1)^{r+1}(\lambda+\mu)^r + \lambda. \end{aligned}$$

We see that for odd r there are no sign changes and for even r there is one sign change. Therefore $f(x)$ has no negative real roots for odd r and has one negative real root for even r.

Thus for odd r we have that $f(x)$ has two real positive roots and $r-1$ complex roots. Using the formulation of Eqn. (2.7) gives rise to the following picture.

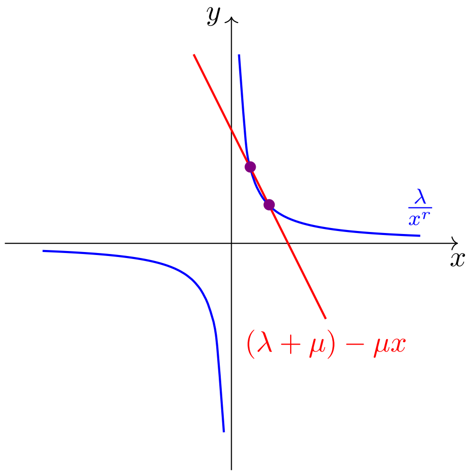


FIGURE 2.3: Plot of Eqn. (2.7) and its solutions, for odd r.

For even r we have that $f(x)$ has three real roots, two of which are positive, the other is negative. Furtheremore it has $r-2$ complex roots. Again the formulation of Eqn. (2.7) this gives rise to the following picture:

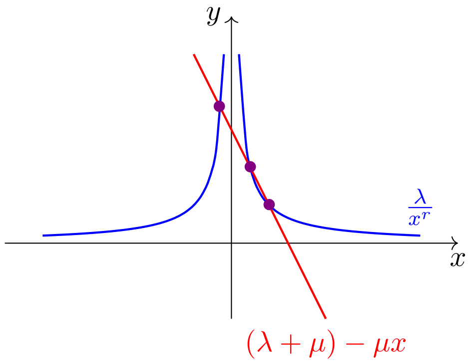


FIGURE 2.4: Plot of Eqn. (2.7) and its solutions, for even r.

This proves Lemma 2.1. Note, that in both pictures it is assumed that the two positive real solutions are distinct. We have not proven this yet, but we will do so in Subsection 2.1.3.

Note that, for any r, it holds that $f(x)$ has an even number of complex roots, that can be divided into pairs of complex conjugates.

Let n_c be the number of complex roots, i.e.

$$n_c = \begin{cases} r-1 & \text{if } r \text{ is odd,} \\ r-2 & \text{if } r \text{ is even.} \end{cases}$$

We order the roots as follows. Let $x_1, ..., x_{n_c}$ be the complex roots, such that $x_k = \bar{x}_{k+1}$, for $k = 1, 3, ..., n_c - 1$. Furthermore, let $x_{r+1} = 1$ and let $x_{n_{c+1}}, \ldots, x_r$ be the other real roots, ordered such that $x_{n_{c+1}} \leq \ldots \leq x_r$.

We can write the complex roots in the form of $x_k = \alpha_k e^{\phi_k i}$, $k = 1, 3, \ldots, n_c - 1$. We then get that

$$\begin{aligned} x_k^m + x_{k+1}^m &= x_k^m + \bar{x}_k^m \\ &= \alpha_k^m e^{m\phi_k i} + \alpha_k^m e^{-m\phi_k i} \\ &= \alpha_k^m \left(e^{m\phi_k i} + e^{-m\phi_k i}\right) \\ &= \alpha_k^m \left(\cos(m\phi_k) + i\sin(m\phi_k) + \cos(m\phi_k) - i\sin(m\phi_k)\right) \\ &= \alpha_k^m \left(\cos(m\phi_k) + \cos(m\phi_k)\right) \\ &= \alpha_k^m \left(2\cos(m\phi_k)\right), \end{aligned}$$

for $k = 1, 3, ..., n_c - 1$. We note that π_n depends on (among others) these complex roots. Given the cosine form, this gives rise to the conjecture that π_n might not be concave, let alone strictly decreasing, in n, as one would expect. In Section 2.3 we will further examine this.

2.1.2 Position of roots

Secondly, we need all roots of Eqn. (2.6) to lie in the interior of the complex unit circle.

Lemma 2.3. *Eqn. (2.6) has exactly r roots z with $|z| < 1$.*

To prove this, we need *Rouché's theorem.*

Lemma 2.4 (Rouché's theorem [1])**.** *Let the bounded region D have the contour C for its boundary. Let the functions $f(z)$ and $g(z)$ be analytic both in D and on C, and assume that $|f(z)| < |g(z)|$ on C. Then $f(z) + g(z)$, has the same number of roots as $g(z)$, in D, all roots counted according to their multiplicity.*

For the proof of Lemma 2.4 see Titchmarsh [2, p.116-117].

Let D be the disc $\{z \in \mathbb{C} : |z| \leq 1 - \varepsilon\}$, for some arbitrarily small $\varepsilon > 0$ and let C be its boundary $\{z \in \mathbb{C} : |z| = 1 - \varepsilon\}$. Let $f(z) = \mu z^{r+1}$ and let $g(z) = \lambda - (\lambda + \mu)z^r$. For $g(z)$, it holds that

$$\begin{aligned} g(z) = 0 &\iff \lambda - (\lambda + \mu)z^r = 0 \\ &\iff \frac{\lambda}{\lambda + \mu} = z^r. \end{aligned}$$

Hence, $g(z)$ has r roots in D. We can prove that $|f(z)| < |g(z)|$ for all $z \in C$ by proving the equivalent statement that $|f(z)|^2 < |g(z)|^2$ for all $z \in C$. We can write $z = (1-\varepsilon)^{i\phi}$ and find that

$$|f(z)|^2 = |\mu z^{r+1}|^2 = \mu^2 |z^{r+1}|^2 = \mu^2 (1-\varepsilon)^{2r+2}$$

and

$$\begin{aligned}
|g(z)|^2 &= |\lambda - (\lambda+\mu) z^r|^2 \\
&= |\lambda - (\lambda+\mu)(1-\varepsilon)^r (\cos(r\phi) + i\sin(r\phi))|^2 \\
&= (\lambda - (\lambda+\mu)(1-\varepsilon)^r \cos(r\phi))^2 + (-(\lambda+\mu)(1-\varepsilon)^r \sin(r\phi))^2 \\
&= \lambda^2 - 2\lambda(\lambda+\mu)(1-\varepsilon)^r \cos(r\phi) + (\lambda+\mu)^2(1-\varepsilon)^{2r}(\cos^2(r\phi) + \sin^2(r\phi)) \\
&= \lambda^2 - 2\lambda(\lambda+\mu)(1-\varepsilon)^r \cos(r\phi) + (\lambda+\mu)^2(1-\varepsilon)^{2r}.
\end{aligned}$$

We need to prove that $|f(z)|^2 < |g(z)|^2$ for all $z \in C$. This is equivalent to

$$\lambda^2 - 2\lambda(\lambda+\mu)(1-\varepsilon)^r \cos(r\phi) + (\lambda+\mu)^2(1-\varepsilon)^{2r} - \mu^2(1-\varepsilon)^{2r+2} > 0,$$

the left-hand side of which is minimum when $\cos(r\phi) = 1$. Hence, it suffices to prove that

$$h(\varepsilon) := \lambda^2 - 2\lambda(\lambda+\mu)(1-\varepsilon)^r + (\lambda+\mu)^2(1-\varepsilon)^{2r} - \mu^2(1-\varepsilon)^{2r+2} > 0.$$

Let $\varepsilon = 0$. We get that

$$\begin{aligned}
h(0) &= \lambda^2 - 2\lambda(\lambda+\mu) + (\lambda+\mu)^2 - \mu^2 \\
&= \lambda^2 - 2\lambda^2 - 2\lambda\mu + \lambda^2 + 2\lambda\mu + \mu^2 - \mu^2 \\
&= 0.
\end{aligned}$$

It now suffices to prove that $h'(0) > 0$. To this end, we calculate the derivative of h with respect to ε:

$$h'(\varepsilon) = 2r\lambda(\lambda+\mu)(1-\varepsilon)^{r-1} - 2r(\lambda+\mu)^2(1-\varepsilon)^{2r-1} + \mu^2(2r+2)(1-\varepsilon)^{2r+1}.$$

For $\varepsilon = 0$ we get

$$\begin{aligned}
h'(0) &= 2r\lambda(\lambda+\mu) - 2r(\lambda+\mu)^2 + \mu^2(2r+2) \\
&= 2r\lambda^2 + 2r\lambda\mu - 2r\lambda^2 - 4r\lambda\mu - 2r\mu^2 + 2r\mu^2 + 2\mu^2 \\
&= -2r\lambda\mu + 2\mu^2
\end{aligned}$$

We need to prove that $h'(0) > 0$, which is equivalent to

$$\begin{aligned}-2r\lambda\mu + 2\mu^2 > 0 &\iff -r\lambda\mu + \mu^2 > 0\\ &\iff \mu > r\lambda\\ &\iff 1 > \frac{\lambda r}{\mu} = \rho,\end{aligned}$$

which is true, by assumption. Hence $h'(0) > 0$, for $\varepsilon > 0$ sufficiently small, proving that $h(\varepsilon) > 0$. In turn, this which proves that $|f(z)| < |g(z)|$. As $g(z)$ has r roots in D, it follows from Lemma 2.4 that $f(z) + g(z)$ has r roots in D. Thus we have proven that Eqn. (2.6) has r roots in D, for arbitrarily small $\varepsilon > 0$. This proves Lemma 2.3.

2.1.3 Uniqueness of roots

Lastly, we need to show that the roots of Eqn. (2.6) in D are unique.

Lemma 2.5. *The $r+1$ roots of Eqn. (2.6) are all distinct.*

To prove this lemma, we assume that $\frac{1}{x}$ is a root of Eqn. (2.6) in D, for any x such that $|x| > 1$. Note that this is well-defined, because Eqn. (2.6) does not have a root equal to zero. We observe,

$$\begin{aligned}\mu\left(\frac{1}{x}\right)^{r+1} - (\lambda+\mu)\left(\frac{1}{x}\right)^{r} + \lambda = 0 &\iff \mu - (\lambda+\mu)x + \lambda x^{r+1} = 0\\ &\iff \mu(1-x) - \lambda x(1-x^r) = 0\\ &\iff \mu - \lambda x\frac{1-x^r}{1-x} = 0\\ &\iff 1 - \frac{\lambda x}{\mu}\cdot\frac{1-x^r}{1-x} = 0.\end{aligned}$$

Let

$$f(x) = 1 - \frac{\lambda x}{\mu}\cdot\frac{1-x^r}{1-x} = 0.$$

It holds that x with $|x| > 1$ is a root of $f(x)$ if and only if $\frac{1}{x}$ is a root of Eqn. (2.6). It follows from Lemma 2.3 that $f(x)$ has r roots with $|x| > 1$. We can rewrite $f(x)$ as

$$\begin{aligned}f(x) = 1 - \frac{\lambda x}{\mu}\cdot\frac{1-x^r}{1-x} &= 1 - \frac{\lambda}{\mu}\cdot x\cdot(1 + x + x^2 + \ldots + x^{r-1})\\ &= 1 - \frac{\rho}{r}\cdot(x + x^2 + x^3 + \ldots + x^r)\\ &= 1 - \rho\cdot\frac{x + x^2 + x^3 + \ldots + x^r}{r}.\end{aligned} \tag{2.8}$$

Proving that $f(x)$ has r distinct roots, proves that Eqn. (2.6) has r distinct roots in D. To show the first, it suffices to prove that if x is a root of $f(x)$, then x is not a root of

$f'(x)$. Because $f(x)$ has r roots x with $|x| > 1$, it suffices to prove that $f'(x)$ has no roots x with $|x| > 1$. To this end, we calculate the derivative of $f(x)$:

$$f'(x) = -\frac{\rho}{r} \cdot \left(1 + 2x + 3x^2 + \ldots + rx^{r-1}\right).$$

This function equals zero if and only if

$$1 + 2x + 3x^2 + \ldots + rx^{r-1} = 0.$$

Note that $x = 1$ is not a solution to this equation. We can therefore multiply both sides of the equation by $(1 - x)$, without increasing the algabraic multiplicity of the original roots. This gives

$$1 + x + x^2 + \ldots + x^{r-1} - rx^r = 0, \tag{2.9}$$

which is equivalent to

$$1 + x + x^2 + \ldots + x^{r-1} = rx^r.$$

Suppose x is a root of this equation with $|x| > 1$. It then follows that

$$\begin{aligned} |1 + x + x^2 + \ldots + x^{r-1}| &\leq |1| + |x| + |x^2| + \ldots + |x^{r-1}| \\ &\leq 1 + (r-1)|x^r| \\ &< r|x^r|. \end{aligned}$$

This is in contradiction with Eqn. (2.9). It therefore follows that $|x| \leq 1$. The conclusion is that Eqn. (2.6) has r distinct roots in D. Taking into account the root $x = 1$, which is not in D, implies that Eqn. (2.6) has $r + 1$ distinct roots. This proves Lemma 2.5.

The result is that all roots of Eqn. (2.6), except $x = 1$, are located in the interior of the complex unit circle and that all roots are distinct. Hence, we can update the pictures, and get the following:

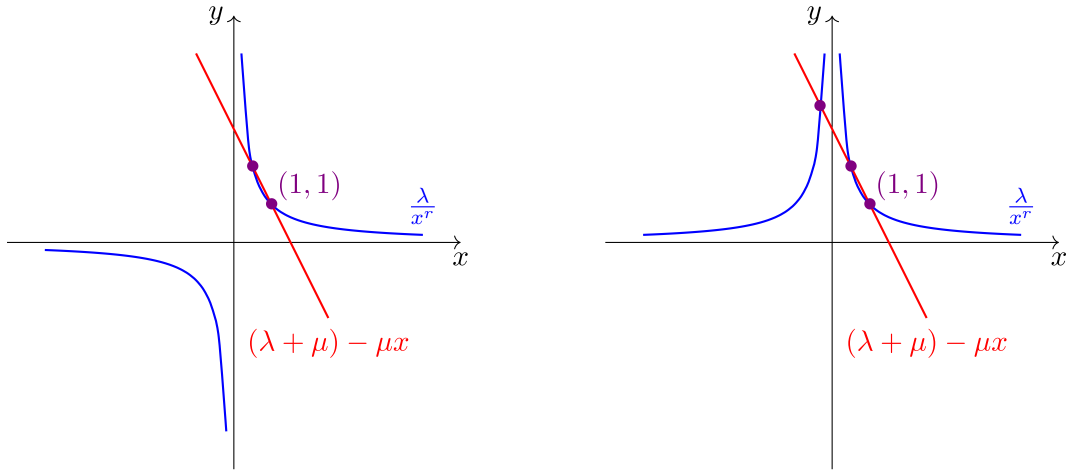


FIGURE 2.5: Updated plots of Eqn. (2.7) and its solutions, for odd r (left) and even r (right).

2.2 The stationary distribution

We return to the system of Eqns. (2.1) and (2.4). We can solve this system by means of generating functions. The generating function of the stationary distribution is defined as

$$f(z) = \sum_{n=0}^{\infty} \pi_n z^n, \tag{2.10}$$

for $|z| < 1$.

Lemma 2.6. *The stationary distribution of the Erlang queue is*

$$\pi_n = (1-\rho) \sum_{i=1}^{r} A_i x_i^n, \tag{2.11}$$

for $n > 0$, where x_i is a root of Eqn. (2.6) and $A_i = \prod_{j \neq i} \frac{1}{1-\frac{x_j}{x_i}}$.

By multiplying Eqn. (2.4) by z^n and summing over all $n \geq 1$ we get the following:

$$(\lambda + \mu) \sum_{n=1}^{\infty} \pi_n z^n = \lambda \sum_{n=1}^{\infty} \pi_{n-r} z^n + \mu \sum_{n=1}^{\infty} \pi_{n+1} z^n.$$

Therefore,

$$(\lambda + \mu) \left(\sum_{n=0}^{\infty} \pi_n z^n - \pi_0 \right) = \lambda z^r \sum_{n=1}^{\infty} \pi_{n-r} z^{n-r} + \mu z^{-1} \sum_{n=1}^{\infty} \pi_{n+1} z^{n+1},$$

so that

$$(\lambda+\mu)\left(\sum_{n=0}^{\infty}\pi_n z^n-\pi_0\right)=\lambda z^r\sum_{n=0}^{\infty}\pi_n z^n+\mu z^{-1}\left(\sum_{n=0}^{\infty}\pi_n z^n-\pi_1 z-\pi_0\right).$$

Substituting $f(z)=\sum_{n=0}^{\infty}\pi_n z_n$ gives

$$(\lambda+\mu)\left(f(z)-\pi_0\right)=\lambda z^r f(z)+\mu z^{-1}\left(f(z)-\pi_1 z-\pi_0\right),$$

which gives

$$(\lambda+\mu)f(z)-(\lambda+\mu)\pi_0=\lambda z^r f(z)+\mu z^{-1}f(z)-\mu z^{-1}\pi_1 z-\mu z^{-1}\pi_0.$$

Hence,

$$(\lambda+\mu)f(z)-\lambda z^r f(z)-\mu z^{-1}f(z)=(\lambda+\mu)\pi_0-\mu\pi_1-\mu z^{-1}\pi_0,$$

yielding

$$f(z)\left(\lambda+\mu-\lambda z^r-\mu z^{-1}\right)=(\lambda+\mu)\pi_0-\mu\pi_1-\mu z^{-1}\pi_0.$$

This gives

$$f(z)=\frac{(\lambda+\mu)\pi_0-\mu\pi_1-\mu z^{-1}\pi_0}{\lambda+\mu-\lambda z^r-\mu z^{-1}}.$$

Dividing both sides by $-z$ gives

$$f(z)=\frac{-z(\lambda+\mu)\pi_0+z\mu\pi_1+\mu\pi_0}{-z\lambda-z\mu+\lambda z^{r+1}+\mu}.$$

We substitute the boundary condition, Eqn. (2.1) into $f(z)$ to get

$$f(z)=\frac{-z(\lambda+\mu)\pi_0+z\lambda\pi_0+\mu\pi_0}{-z\lambda-z\mu+\lambda z^{r+1}+\mu}.$$

This gives

$$
\begin{aligned}
f(z) = \frac{\mu\pi_0(1-z)}{\lambda z(z^r-1)+\mu(1-z)} &= \frac{\mu\pi_0(1-z)}{\mu(1-z)-\lambda z(z^r-1)} \\
&= \frac{\mu\pi_0}{\mu-\lambda z\frac{1-z^r}{1-z}} \\
&= \frac{\mu\pi_0}{\mu-\lambda z(1+z+\ldots+z^{r-1})} \\
&= \frac{\pi_0}{1-\frac{\rho}{r}(z+z^2+\ldots+z^r)} \\
&= \frac{1-\rho}{1-\rho\cdot\frac{z+z^2+\ldots+z^r}{r}},
\end{aligned}
$$

where we inserted $\rho = \frac{\lambda r}{\mu}$ and $\pi_0 = 1-\rho$.

From Lemma 2.5 it follows that the denominator of $f(z)$ has r distinct roots z_i with $|z_i| > 1$. Note, that each root z_i corresponds to a root $\frac{1}{z_i} = x_i$ of Eqn. (2.6). We can thus write $f(z)$ as

$$
f(z) = \frac{1-\rho}{\left(1-\frac{z}{z_1}\right)\cdot\ldots\cdot\left(1-\frac{z}{z_r}\right)}.
$$

Using partial fraction decomposition this can be written as

$$
f(z) = \frac{1-\rho}{\left(1-\frac{z}{z_1}\right)\cdot\ldots\cdot\left(1-\frac{z}{z_r}\right)} = (1-\rho)\left(\frac{A_1}{1-\frac{z}{z_1}}+\ldots+\frac{A_r}{1-\frac{z}{z_r}}\right),
$$

with

$$
A_i = \left(\prod_{j\neq i}\left(1-\frac{z_i}{z_j}\right)\right)^{-1}, \tag{2.12}
$$

for $i = 1, 2, \ldots, r$. Note that A_i is well-defined, since all z_i are distinct. To prove the validity of Eqn. (2.12) it suffices to prove for

$$
g(z) = \left(1-\frac{z}{z_1}\right)\cdot\ldots\cdot\left(1-\frac{z}{z_r}\right)\cdot\sum_{i=1}^{r}\frac{A_i}{1-\frac{z}{z_i}} - 1,
$$

that $g(z) \equiv 0$.

Note, that

$$
\begin{aligned}
\left(1-\frac{z}{z_1}\right)\cdot\ldots\cdot\left(1-\frac{z}{z_r}\right)\cdot\sum_{i=1}^{r}\frac{A_i}{1-\frac{z}{z_i}} &= \sum_{i=1}^{r}\prod_{j=1}^{r}\left(1-\frac{z}{z_j}\right)\cdot\frac{A_i}{1-\frac{z}{z_i}} \\
&= \sum_{i=1}^{r}A_i\prod_{j\neq i}\left(1-\frac{z}{z_j}\right),
\end{aligned}
$$

so that

$$g(z) = \sum_{i=1}^{r} A_i \prod_{j \neq i} \left(1 - \frac{z}{z_j}\right) - 1.$$

By substituting Eqn. (2.12) into $g(z)$ we get

$$g(z) = \sum_{i=1}^{r} \frac{\prod_{j \neq i} \left(1 - \frac{z}{z_j}\right)}{\prod_{j \neq i} \left(1 - \frac{z_i}{z_j}\right)} - 1.$$

Recall, that $z_1, \ldots, z_r$ are all distinct. Plugging in $z = z_k$ yields $g(z_k) = 0$. Note, that g is a polynomial of degree $r - 1$. Since $g(z_i) = 0$, $i = 1, \ldots, r$, and z_i are all distinct, g has at least r roots. Hence $g \equiv 0$. This proves the statement.

Furthermore, we can make use of the fact that

$$\sum_{n=1}^{\infty} z^n = \frac{1}{1 - z},$$

for $z \in (0, 1)$. This allows to rewrite the last part of Eqn. (2.12) as

$$\begin{aligned} f(z) &= (1 - \rho) \left(A_1 \cdot \sum_{n=0}^{\infty} \left(\frac{z}{z_1}\right)^n + \ldots + A_r \cdot \sum_{n=0}^{\infty} \left(\frac{z}{z_r}\right)^n \right) \\ &= (1 - \rho) \sum_{n=0}^{\infty} \left(\sum_{i=1}^{r} A_i \cdot \left(\frac{1}{z_i}\right)^n \right) z^n. \end{aligned}$$

From this and Eqn. (2.10) it follows that the stationary distribution of the Erlang queue is

$$\pi_n = (1 - \rho) \sum_{i=1}^{r} A_i \left(\frac{1}{z_i}\right)^n,$$

for $n > 0$. As $\frac{1}{z_i} = x_i$, it follows that

$$\pi_n = (1 - \rho) \sum_{i=1}^{r} A_i x_i^n.$$

This proves Lemma 2.6. We can translate the stationary distribution back to the original model. Let q_i denote the stationary probability that there are i customers in the system.

We get

$$\begin{aligned}
q_i &= \sum_{n=(i-1)r+1}^{ir} \pi_n \\
&= \sum_{n=(i-1)r+1}^{ir} (1-\rho) \sum_{j=1}^{r} A_j x_j^n \\
&= (1-\rho) \sum_{j=1}^{r} A_j \sum_{n=(i-1)r+1}^{ir} x_j^n \\
&= (1-\rho) \sum_{j=1}^{r} A_j \cdot x_j^{(i-1)r+1} \cdot (x_j^0 + \ldots + x_j^{r-1}) \\
&= (1-\rho) \sum_{j=1}^{r} A_j \cdot x_j^{(i-1)r+1} \cdot \frac{x_j^r - 1}{x_i - 1} \\
&= (1-\rho) \sum_{j=1}^{r} A_j \cdot \frac{x_j^{ir+1} - x_j^{(i-1)r+1}}{x_i - 1}.
\end{aligned}$$

Example: two phases

Let $\lambda = 1$, $r = 2$ and $\mu = 6$. Substituting these values in Eqn. (2.6) gives

$$6x^3 - 7x^2 + 1 = 0.$$

The roots are $x_1 = \frac{1}{2}$, $x_2 = -\frac{1}{3}$ and $x_3 = 1$, which gives that $A_1 = \frac{2}{5}$ and $A_2 = \frac{4}{15}$. Hence the stationary distribution is

$$\pi_n = \frac{2}{5}\left(\frac{1}{2}\right)^n + \frac{4}{15}\left(-\frac{1}{3}\right)^n, \qquad n \geq 0.$$

We get the following equilibrium probabilities:

n	π_n
0	$\frac{2}{3}$
1	$\frac{1}{9}$
2	$\frac{7}{54}$
3	$\frac{13}{324}$
4	$\frac{55}{1944}$
5	$\frac{133}{11664}$
6	$\frac{463}{69984}$
7	$\frac{1261}{419904}$

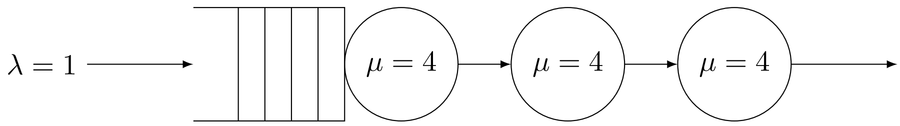
TABLE 2.1: Values of π_n.

2.3 Sinusoidal behaviour of the stationary distribution

In Subsection 2.1.1 that Eqn. (2.6) has complex roots, for $r \geq 3$, and that these roots can be written in terms of cosines. As Eqn. (2.11) shows, the equilibrium probabilites π_n depend on these complex roots. This suggests that π_n has a sinusoidal character. We first consider an example.

2.3.1 Example

Let us consider the $M|E_r|1$ queue with $\lambda = 1$, $r = 3$ and $\mu = 4$. This system has an occupation rate $\rho = \frac{\lambda r}{\mu} = \frac{3}{4}$ and is therefore stable.



We use Eqn. (2.6) to get

$$4x^{3+1} - (1+4)x^3 + 1 = 0$$

has the following solutions:

- $x_1 \approx -0.30944 + 0.43815i$,
- $x_2 \approx -0.39044 - 0.43815i$,
- $x_3 \approx 0.86888$,
- $x_4 \approx 1$.

Note, that we ordered the solutions according to the order given in Subsection 2.1.1.

Let

$$c_k = (1-\rho)A_i = \frac{1-\rho}{\prod_{j\neq k}\left(1 - \frac{x_j}{x_k}\right)},$$

for $k = 1, \ldots, r$. We get that

- $c_1 = \frac{1-\frac{3}{4}}{\left(1-\frac{x_2}{x_1}\right)\left(1-\frac{x_3}{x_1}\right)} = 0.0652884 + 0.0009800i$,
- $c_2 = \frac{1-\frac{3}{4}}{\left(1-\frac{x_1}{x_2}\right)\left(1-\frac{x_3}{x_2}\right)} = 0.0652884 - 0.0009800i$,

- $c_3 = \frac{1-\frac{3}{4}}{\left(1-\frac{x_3}{x_1}\right)\left(1-\frac{x_3}{x_2}\right)} = 0.1194232.$

Note that $c_1 = \bar{c_2}$, corresponding to the fact that $z_1 = \bar{z_2}$. Using that

$$\pi_n = \sum_{k=1}^{r} c_k x_k^n,$$

for $n = 0, 1, 2, \ldots$, we get the following equilibrium probabilities:

n	π_n
0	0.25
1	0.0625
2	0.078126
3	0.0976571
4	0.0595712
5	0.0588391
6	0.0540172
7	0.0431072

TABLE 2.2: Values of π_n.

Plotting these values shows the sinusoidal character for low values of n.

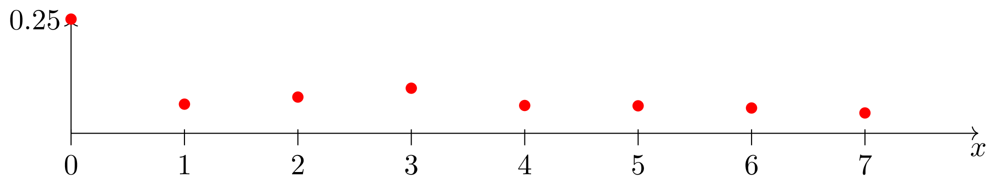


FIGURE 2.6: Plot of π_n.

2.3.2 Fading of the sinusoidal behaviour

In the example above we observed that the largest root (that does not equal 1) of the equation is a positive real one. We can prove that this causes the sinusoidal behaviour to disappear as $n \to \infty$.

Lemma 2.7. *The sinusoidal behaviour of the π_n disappears as $n \to \infty$.*

To prove that Lemma 2.7 holds, it is necessary to prove the following lemma.

Lemma 2.8. *The positive real root x with $x \neq 1$ is the largest root of Eqn. (2.6) not equal to 1.*

We use the notation described in Subsection 2.1.1. Let x_r be the unique positive real root of equation (2.6), with $x_r \neq 1$. This corresponds to the unique positive real root $z_r = \frac{1}{x_r}$ of

$$1 - \rho \cdot \frac{z + z^2 + z^3 + \ldots + z^r}{r},$$

which we rewrite as

$$1 - \frac{\lambda}{\mu} \cdot (z + z^2 + z^3 + \ldots + z^r). \tag{2.13}$$

Let $\varepsilon \in \mathbb{R}_+$ be arbitrarily small, let $D_{z_r-\varepsilon}$ be the disk $\{z \in \mathbb{C} : |z| \leq z_r - \varepsilon\}$, let $C_{z_r-\varepsilon}$ be $\{z \in \mathbb{C} : |z| = z_r - \varepsilon\}$ and let C_{z_r} be $\{z \in \mathbb{C} : |z| = z_r\}$. To prove, that x_r is the largest root of Eqn. (2.6) that does not equal 1, it suffices to prove that z_r is the smallest root of Eqn. (2.13). In other words, to prove that x_r strictly dominates the other elements of the linear combination (2.11), it is sufficient to prove that Eqn. (2.13) has no roots z with $|z| < z_r$ in $C_{z_r}, C_{z_r-\varepsilon}$ or $D_{z_r-\varepsilon}$, for ε arbitrarily small.

We again use Rouché's theorem. Let $f(z) = -\frac{\lambda}{\mu} \cdot (z + z^2 + z^3 + \ldots + z^r)$ and let $g(z) = 1$. Then for $z \in C_{z_r-\varepsilon}$ it holds that

$$\begin{aligned}
|f(z)| &< \frac{\lambda}{\mu} \cdot (|z| + |z|^2 + \ldots + |z|^r) \\
&= \frac{\lambda}{\mu} \cdot ((z_r - \varepsilon) + (z_r - \varepsilon)^2 + \ldots + (z_r - \varepsilon)^r) \\
&< \frac{\lambda}{\mu} \cdot (z_r + z_r^2 + \ldots + z_r^r) \\
&= 1 \\
&= |g(z)|.
\end{aligned}$$

From Rouché's theorem it now follows that $f(z) + g(z) = 1 - \frac{\lambda}{\mu} \cdot (z + z^2 + z^3 + \ldots + z^r)$ has the same number of zeroes in $D_{z_r-\varepsilon}$ as $g(z) = 1$, which is 0.

It now remains to be proven that Eqn. (2.13) has no solutions z in C_{z_r} with $z \neq z_r$. Suppose $-z_r$, the only other real possibility, is a solution. Then it follows that

$$\frac{\lambda}{\mu} \cdot (-z_r + z_r^2 - z_r^3 + \ldots + (-1)^r z_r^r) = 1,$$

as well as

$$\frac{\lambda}{\mu} \cdot (z_r + z_r^2 + z_r^3 + \ldots + z_r^r) = 1.$$

Subtracting the latter from the former gives

$$\frac{\lambda}{\mu} \cdot (-2z_r - 2z_r^3 + \ldots + -2z_r^r) = 0,$$

for r odd and

$$\frac{\lambda}{\mu} \cdot (-2z_r - 2z_r^3 + \ldots + -2z_r^{r-1}) = 0,$$

for r even. In both cases this is a contradiction, because all terms are strictly negative. Hence it follows that $-z_r$ is not a solution of Eqn. (2.13).

Now suppose there is a complex solution z in C_{z_r}. We can write this solution as $z = z_r e^{i\phi}$, for some $\phi \in \mathbb{R}$. Then there is also the complex conjugate solution $\bar{z} = z_r e^{-i\phi}$. For these solutions it must hold that

$$z_r e^{i\phi} + z_r^2 e^{2i\phi} + z_r^3 e^{3i\phi} + \ldots + z_r^r e^{ri\phi} = \frac{\mu}{\lambda}$$

and

$$z_r e^{-i\phi} + z_r^2 e^{-2i\phi} + z_r^3 e^{-3i\phi} + \ldots + z_r^r e^{-ri\phi} = \frac{\mu}{\lambda},$$

respectively. We can rewrite these equations using that $z_r e^{i\phi} = z_r(\cos(\phi) + i\sin(\phi))$. Because the right sides of both equations are real, it follows that both imaginary parts are equal to 0. Thus we get that

$$z_r \sin(\phi) + z_r^2 \sin(2\phi) + z_r^3 \sin(3\phi) + \ldots + z_r^r \sin(r\phi) = 0.$$

From this it follows that

$$z_r \cos(\phi) + z_r^2 \cos(2\phi) + \ldots + z_r^r \cos(r\phi) = \frac{\mu}{\lambda}. \tag{2.14}$$

We assumed that z is a complex solution, hence z is not z_r, neither is it $-z_r$. Therefore, we know that $\phi \in (0, \pi) \cup (\pi, 2\pi)$. For such ϕ it holds that $|\cos(\phi)| < 1$. From this it follows that

$$\begin{aligned}
z_r \cos(\phi) + \ldots + z_r^r \cos(r\phi) &\leq |z_r \cos(\phi) + \ldots + z_r^r \cos(r\phi)| \\
&\leq |z_r \cos(\phi)| + \ldots + |z_r^r \cos(r\phi)| \\
&< |z_r| + |z_r^2 \cos(2\phi) + \ldots + |z_r^r \cos(r\phi)| \\
&\leq |z_r| + |z_r^2| + \ldots + |z_r^r| \\
&= z_r + z_r^2 + \ldots + z_r^r \\
&= \frac{\mu}{\lambda}.
\end{aligned}$$

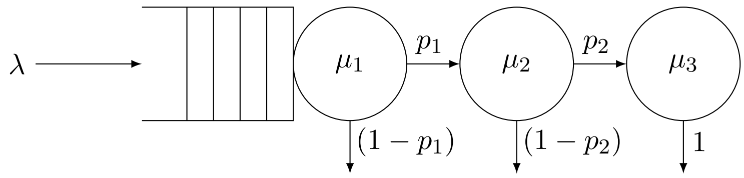
This contradicts Eqn. (2.14). We have thus proven that z_r is the only solution of Eqn. (2.13) that is in C_{z_r}. Hence, it is the smallest root of Eqn. (2.13). Therefore, x_r is the largest root of Eqn. (2.6) that does not equal 1. This proves Lemma 2.8.

If x is the largest root of Eqn. (2.6) with $x \neq 1$, then this corresponds to $z = \frac{1}{x}$ being the smallest root of Eqn. (2.8). Using Lemma 2.6, we see that $\frac{1}{z} = x$ is the largest element in the linear combination representing π_n. Hence, x dominates the other roots, as $n \to \infty$. As x is positive and real, the imaginary terms, as well as the sinusoidal behaviour that they cause, disappear as $n \to \infty$. This proves Lemma 2.7.

Chapter 3

The Coxian queue

To better approximate the $M|G|1$-queue, we can use the $M|Cox|1$-queue. Here customers arrive according to a Poisson proces and are served according to a Coxian distribution instead of an Erlang distribution. The Coxian distribution $Cox(x; \mu_1, \ldots, \mu_m, p_1, \ldots, p_{m-1})$ is defined as a continous probability distribution on the interval $[0, \infty)$, with parameters $\mu_j \in \mathbb{R}_+$, for $j = 1, \ldots, m$, $p_j \in (0, 1]$ for $j = 1, \ldots, m-1$. The service time requires maximally m phases, where each phase i is exponentially distributed with parameter μ_j. After each phase $j < m$, the customer needs a subsequent phase of service with probability p_j and finishes service with probability $1 - p_j$. For $r = 3$, we have the following picture.



A customer will always require at least one phase. From the above description, it follows that the probability that a customer requires at least $j \geq 2$ phases of service is $p_1 \cdot \ldots \cdot p_{j-1}$. The time spent in phase j is exponentially distributed with parameter μ_j, therefore the average time spent in that phase is $\frac{1}{\mu_j}$. We can thus conclude that the average customer service time $\mathbb{E}[S]$ is

$$\mathbb{E}[S] = \frac{1}{\mu_1} + \frac{p_1}{\mu_2} + \frac{p_1 \cdot p_2}{\mu_3} + \cdots + \frac{p_1 \cdot \ldots \cdot p_{m-1}}{\mu_m} = \sum_{j=1}^{m} \frac{\prod_{k=1}^{m-1} p_k}{\mu_j}.$$

For this system to be stable, we therefore assume that

$$\rho = \lambda \mathbb{E}[S] = \lambda \cdot \left(\sum_{j=1}^{m} \frac{\prod_{k=1}^{j-1} p_k}{\mu_j} \right) < 1.$$

We can associate a Markov process with the $M|Cox|1$-queue. First define the state space $S = \{(i,j) \ : \ i \in \mathbb{N}_+, \ j \in \{1,\dots,m\}\} \cup \{0\}$. Define $X_t = (X_{1,t}, X_{2,t}) = (i,j)$, if there are i customers in the system and if the customer in service is in the j'th service phase. Furthermore we will denote the state with zero customers by 0. Then $(X_t)_t$ is a Markov process. Its transition rate matrix Q is illustrated in Figure 3.1.

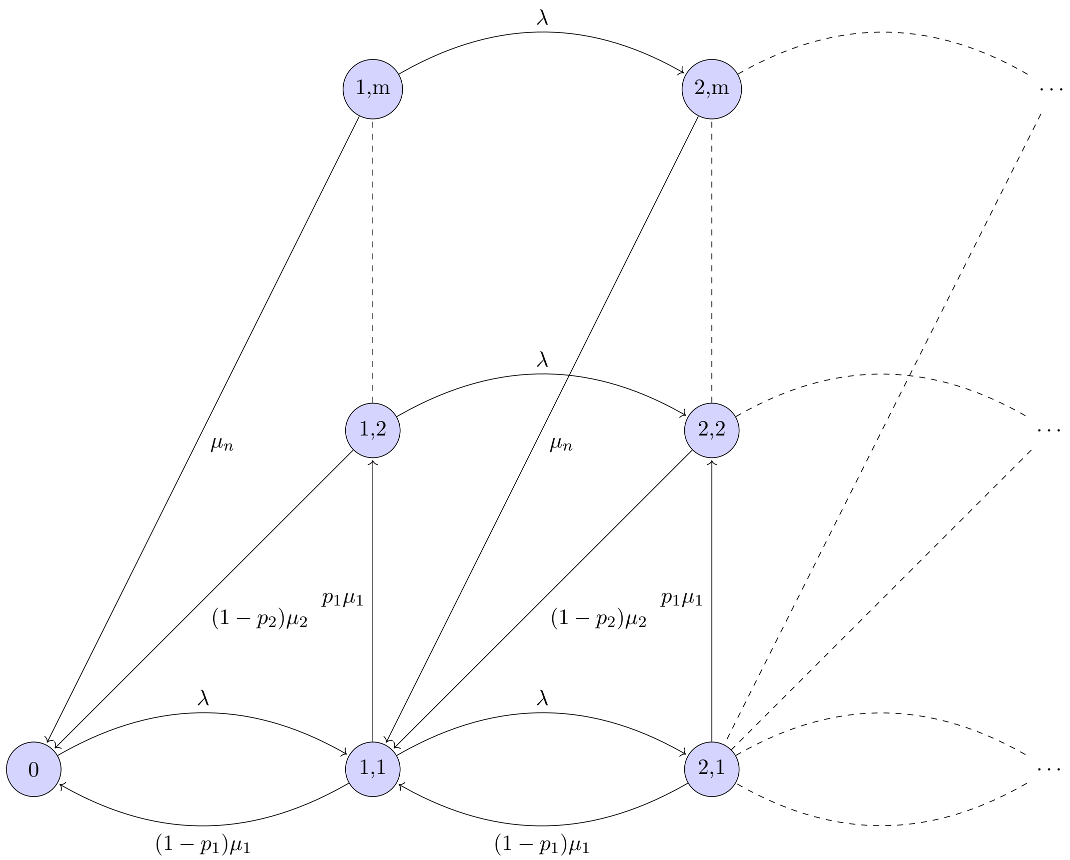


FIGURE 3.1: Flow diagram of the $M|Cox|1$ Markov process.

Let π be the stationary distribution of the system, given by $\pi_{(i,j)}$, for state (i,j) and π_0 for state 0.

3.1 The stationary distribution

Suppose, that the system is in equilibrium. To start off, as a result of the $M|G|1$-queue, we know (see Adan and Resing [1]) that

$$\pi_0 = 1 - \rho,$$

We will determine the other probabilities in two steps:

i) calculate $\pi_{(1,1)}, \dots, \pi_{(1,m)}$,

ii) calculate $\pi_{(n,1)}, \dots, \pi_{(n,m)}$ from $\pi_{(n-1,1)}, \dots, \pi_{(n-1,m)}$, for $n \geq 1$.

3.1.1 Step (i): the initial probabilities

The balance equation for π_0 is given by

$$\begin{aligned} \lambda\pi_0 &= (1-p_1)\mu_1\pi_{(1,1)} + \ldots + (1-p_{m-1})\mu_{m-1}\pi_{(1,m-1)} + \mu_m\pi_{(1,m)} \\ &= (1-p_1)\mu_1\pi_{(1,1)} + \sum_{j=2}^{m-1}(1-p_j)\mu_j\pi_{(1,j)} + \mu_m\pi_{(1,m)}. \end{aligned} \tag{3.1}$$

Furthermore, we know that state $\pi_{(1,j)}$ can only be entered via state $\pi_{(1,j-1)}$, for $j \in \{2, \dots, m\}$. From this we can derive $m-1$ additional balance equations. These are

$$(\lambda + \mu_j)\pi_{(1,j)} = p_{j-1}\mu_{j-1}\pi_{(1,j-1)}, \qquad j = 2, \dots, m.$$

Therefore,

$$\pi_{(1,j)} = \frac{p_{j-1}\mu_{j-1}}{\lambda + \mu_j}\pi_{(1,j-1)}, \qquad j = 2, \dots, m.$$

Using this equation recursively gives

$$\pi_{(1,j)} = \frac{\prod_{k=1}^{j-1} p_k\mu_k}{\prod_{k=2}^{j} \lambda + \mu_k}\pi_{(1,1)}, \qquad j = 2, \dots, m. \tag{3.2}$$

Plugging this into Eqn. (3.1) yields

$$\lambda(1-\rho) = \pi_{(1,1)}\left((1-p_1)\mu_1 + \sum_{j=2}^{m-1}\frac{(1-p_j)\mu_j\prod_{k=1}^{j-1}p_k\mu_k}{\prod_{k=2}^{j}\lambda+\mu_k} + \frac{\mu_m\prod_{k=1}^{m-1}p_k\mu_k}{\prod_{k=2}^{m}\lambda+\mu_k}\right),$$

which gives

$$\pi_{(1,1)} = \lambda(1-\rho) \cdot \left((1-p_1)\mu_1 + \sum_{j=2}^{m} \frac{(1-p_j)\mu_j \prod_{k=1}^{j-1} p_k \mu_k}{\prod_{k=2}^{j} \lambda + \mu_k} \right)^{-1}, \tag{3.3}$$

with $p_m = 0$, for notational convenience. Substituting this into Eqn. (3.2) yields

$$\pi_{(1,j)} = \frac{\prod_{k=1}^{j-1} p_k \mu_k}{\prod_{k=2}^{j} \lambda + \mu_k} \cdot \lambda(1-\rho) \cdot \left((1-p_1)\mu_1 + \sum_{j=2}^{m} \frac{(1-p_j)\mu_j \prod_{k=1}^{j-1} p_k \mu_k}{\prod_{k=2}^{j} \lambda + \mu_k} \right)^{-1}, \tag{3.4}$$

for $j = 2, \ldots, m$. We have thus calculated $\pi_{(1,1)}, \ldots, \pi_{(1,m)}$.

3.1.2 Step (ii): recursion

We will derive $\pi_{(n,1)}, \ldots, \pi_{(n,m)}$ from $\pi_{(n-1,1)}, \ldots, \pi_{(n-1,m)}$, for $n \geq 2$. For this we will use *Taboo decomposition*. Taboo decomposition is based on a so called *taboo set*, say A. Let T_k^A be the total time that the Markov process spends in state $k \notin A$ before returning to any state in A. The following lemma holds (see Verschuure [4, p.4-6]).

Lemma 3.1 (Taboo Decomposition). *For $A \subset S$, $A \neq \emptyset$ and $k \in S \setminus A$, it holds that*

$$\pi_k = \sum_{a \in A} \pi_a \sum_{r \notin A} q_{a,r} E_r[T_k^A], \tag{3.5}$$

where $E_r[T_k^A]$ is the expected time that the Markov process spends in state k, after starting in state $r \in A$ and before returning to any state in A.

It is our goal to calculate $E_r[T_k^A]$, thereby expressing π_k in π_a, $a \in A$, which are supposedly known. We will do this step by step. Suppose that we have calculated $\pi_{(i,j)}$ for all $i \leq n-1$ and $j \in \{1, \ldots, m\}$. Then we define

$$A = \{(i,j) \in S : i \leq n-1\}.$$

Using Eqn. 3.5, we get

$$\pi_{(n,j)} = \sum_{a \in A} \pi_a \sum_{r \notin A} q_{a,r} E_r[T_{(n,j)}^A]. \tag{3.6}$$

Note, that the system can only leave A when it is in a state with $n-1$ customers. Hence, $q_{a,r}$ is positive if and only if $a \in \{(n-1,1), \ldots, (n-1,m)\}$. Therefore, we can rewrite Eqn. (3.6) as

$$\pi_{(n,j)} = \sum_{a \in A'} \pi_a \sum_{r \notin A} q_{a,r} E_r[T^A_{(n,j)}], \tag{3.7}$$

with $A' = \{(i,j) \in S \mid i = n-1\}$.

If the system is in state $(n-1,j) \in A'$, $j \in \{1,\dots,m\}$, then it can only move to states $(n-2,1)$, $(n-1,j+1)$ and (n,j). The first and second do not contribute to the sum. This leaves state (n,j). Denoting a state $a \in A$ as $a = (a_1,a_2)$, we can rewrite Eqn. (3.7) as

$$\begin{aligned}\pi_{(n,j)} &= \sum_{(a_1,a_2)\in A'} \pi_{(a_1,a_2)} \sum_{(r_1,r_2)\notin A} q_{(a_1,a_2),(r_1,r_2)} E_{(r_1,r_2)}[T^A_{(n,j)}] \\ &= \sum_{k=1}^{n} \pi_{(n-1,k)} q_{(n-1,k),(n,k)} E_{(n,k)}[T^A_{(n,j)}].\end{aligned} \tag{3.8}$$

We know that arrivals occur according to a Poisson proces with rate λ, independent of the state of the system. It therefore holds that $q_{(m-1,k),(m,k)} = \lambda$ and so Eqn. (3.8) can be rewritten as

$$\begin{aligned}\pi_{(n,j)} &= \sum_{k=1}^{n} \pi_{(n-1,k)} \lambda E_{(n,k)}[T^A_{(n,j)}] \\ &= \lambda \sum_{k=1}^{n} \pi_{(n-1,k)} E_{(n,k)}[T^A_{(n,j)}].\end{aligned} \tag{3.9}$$

To calculate $E_{(n,k)}[T^A_{(n,j)}]$, we can consider the Markov process restricted to states with n customers, before getting absorbed in A. Due to an arrival the number of customers increases, but the Markov process always returns to states with n customers at state $(n,1)$. Let B be the matrix Q reduced to states $(n-1,1),\dots,(n-1,m)$. Then we have

$$B = \begin{pmatrix} -\mu_1 & p_1\mu_1 & & & \\ \lambda & -(\lambda+\mu_2) & p_2\mu_2 & & \\ \vdots & & \ddots & \ddots & \\ \lambda & & & -(\lambda+\mu_{m-1}) & p_{m-1}\mu_{m-1} \\ \lambda & & & & -(\lambda+\mu_m) \end{pmatrix}. \tag{3.10}$$

Note, that B is independent of n. The following lemma holds.

Lemma 3.2. *B is invertible and $-B^{-1}_{i,j} = E_{(n,i)}[T^A_{(n,j)}]$.*

For the proof of this lemma, see Verschuure [4, p.6].

Using vector- and matrix notation, we can write Eqn. (3.9) as

$$\begin{pmatrix} \pi_{(n,1)} \\ \vdots \\ \pi_{(n,m)} \end{pmatrix} = \lambda \cdot \left[\begin{pmatrix} \pi_{(n-1,1)} \\ \vdots \\ \pi_{(n-1,m)} \end{pmatrix}^T \cdot - \begin{pmatrix} B_{(n,1),(n,1)} & \cdots & B_{(n,1),(n,m)} \\ \vdots & \ddots & \vdots \\ B_{(n,m),(n,1)} & \cdots & B_{(n,m),(n,m)} \end{pmatrix}^{-1} \right]^T$$
$$= \lambda \left(-B^T\right)^{-1} \cdot \begin{pmatrix} \pi_{(n-1,1)} \\ \vdots \\ \pi_{(n-1,m)} \end{pmatrix}.$$

Here λ can either be seen as a scalar, or as the $(m \times m)$ diagonal matrix with λ's on the diagonal. We can use this equation recursively, which gives us

$$\begin{pmatrix} \pi_{(n,1)} \\ \vdots \\ \pi_{(n,m)} \end{pmatrix} = \lambda^{n-1} \left((-B^T)^{-1}\right)^{n-1} \begin{pmatrix} \pi_{(1,1)} \\ \vdots \\ \pi_{(1,m)} \end{pmatrix}. \tag{3.11}$$

In the previous subsection we found that the initial states, that is $\pi_{(1,1)}, \ldots, \pi_{(1,m)}$ and π_0 can be calculated for any n. Therefore, using Eqn. (3.11), $\pi_{(i,j)}$ can be calculated explicitly, for any state $(i,j) \in S$.

Example: Erlang queue with two phases

We revise the example of Subsection 2.2. We can model this as a Coxian queue with $\lambda = 1$, $r = 2$, $\mu_1 = \mu_2 = \mu = 6$ and $p_1 = 1$. Substituting these values into Eqns. (3.3) and (3.4) gives $\pi_{(1,1)} = \frac{7}{54}$ and $\pi_{(1,2)} = \frac{1}{9}$. As rate matrix we get

$$B = \begin{pmatrix} -6 & 6 \\ 1 & -7 \end{pmatrix}.$$

Substituting these values into Eqn. (3.11) B yields

$$\begin{pmatrix} \pi_{(i,1)} \\ \pi_{(i,2)} \end{pmatrix} = \left(\frac{1}{36}\right)^{i-1} \begin{pmatrix} -1 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}^{i-1} \begin{pmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} \frac{7}{54} \\ \frac{1}{9} \end{pmatrix}$$
$$= \left(\frac{1}{36}\right)^{i-1} \begin{pmatrix} \frac{4^i}{135} + \frac{9^{i-1}}{10} \\ -\frac{4^i}{45} + \frac{9^{i-1}}{5} \end{pmatrix}.$$

We get the following equilibrium probabilities:

(i,j)	$\pi_{(i,j)}$
$(1,2)$	$\frac{1}{9}$
$(1,1)$	$\frac{7}{54}$
$(2,2)$	$\frac{13}{324}$
$(2,1)$	$\frac{55}{1944}$
$(3,2)$	$\frac{133}{11664}$
$(3,1)$	$\frac{463}{69984}$
$(4,2)$	$\frac{1261}{419904}$

TABLE 3.1: Values of $\pi_{(i,j)}$.

Note, that $\pi_{(i,1)}$ and $\pi_{(i,2)}$ correspond to $2i$ and $2i-1$ phases of work respectively. These values clearly correspond to the values found in Subsection 2.2.

3.2 Diagonalizability

It would be convenient if $(-B^T)^{-1}$ were diagonalizable. If $(-B^T)^{-1} = PDP^{-1}$, for some invertable matrix P and diagonal matrix D, then $\pi_{n,j}$ can be expressed as a linear combination of the n'th powers of the eigenvalues of $(-B^T)^{-1}$. We will investigate whether or not, and under which circumstances, this is the case. The following two lemma shows that it is equivalent to examine B.

Lemma 3.3. *Let A be an invertible matrix. If A is diagonalizable, then $(A^T)^{-1}$ also diagonalizable.*

Proof. We can write A as

$$A = PDP^{-1},$$

with P an invertible matrix and D a diagonal matrix. It follows that we can write $(A^T)^{-1}$ as

$$\begin{aligned}(A^T)^{-1} &= ((PDP^{-1})^T)^{-1}\\ &= ((P^{-1})^T D^T P^T)^{-1}\\ &= (P^T)^{-1}(D^T)^{-1}((P^{-1})^T)^{-1}\\ &= (P^T)^{-1}(D^T)^{-1}P^T.\end{aligned}$$

Hence, $(A^T)^{-1}$ is also a diagonalizable matrix.

□

3.2.1 Coxian rate matrix

Consider B (cf. Eqn. (3.10)). A sufficient condition for B being diagonalizable is that its m eigenvalues are distinct. To this end, we calculate its characteristic polynomial, $P_B(\alpha) = \det(\alpha I - B)$ as a function of α. Consider

$$\alpha I - B = \begin{pmatrix} \alpha+\mu_1 & -p_1\mu_1 & & & \\ -\lambda & \alpha+\lambda+\mu_2 & -p_2\mu_2 & & \\ \vdots & & \ddots & \ddots & \\ -\lambda & & & \alpha+\lambda+\mu_{m-1} & -p_{m-1}\mu_{m-1} \\ -\lambda & & & & \alpha+\lambda+\mu_m \end{pmatrix},$$

its determinant $P_B(\alpha) = \det(\alpha I - B)$ equals

$$(\alpha+\mu_1)\cdot\prod_{j=2}^{m}(\alpha+\lambda+\mu_j) + \sum_{j=2}^{m}(-1)^j\lambda\left[\prod_{l=1}^{j-1}(-p_l\mu_l)\cdot\prod_{l=j+1}^{m}(\alpha+\lambda+\mu_l)\right]. \tag{3.12}$$

We can define $P_B(\alpha)$ by means of a recurrence relation in the number of phases, m. We do so as follows. Consider $P_B^{m+1}(\alpha)$ and $P_B^m(\alpha)$, where the superscript m denotes the number of phases. The following holds:

$$\begin{aligned} P_B^{m+1}(\alpha) &= P_B^m(\alpha)(\lambda+\mu_{m+1}+\alpha) + (-1)^m\cdot -\lambda\prod_{j=1}^{m} -p_j\mu_j \\ &= P_B^m(\alpha)(\lambda+\mu_{m+1}+\alpha) + (-1)^{m+1}\lambda\prod_{j=1}^{m} -1\cdot\prod_{j=1}^{m} p_j\mu_j \\ &= P_B^m(\alpha)(\lambda+\mu_{m+1}+\alpha) + (-1)^{m+1}\lambda(-1)^m\cdot\prod_{j=1}^{m} p_j\mu_j \\ &= P_B^m(\alpha)(\lambda+\mu_{m+1}+\alpha) - \lambda\cdot\prod_{j=1}^{m} p_j\mu_j. \end{aligned}$$

Suppose the determinant equals 0, then we get the following:

$$\begin{aligned} P_B^{m+1}(\alpha) = 0 &\iff P_B^m(\alpha)(\lambda+\mu_{m+1}+\alpha) - \lambda\cdot\prod_{i=1}^{m} p_i\mu_i = 0 \\ &\iff P_B^m(\alpha)(\lambda+\mu_{m+1}+\alpha) = \lambda\cdot\prod_{i=1}^{m} p_i\mu_i \\ &\iff P_B^m(\alpha) = \frac{\lambda\cdot\prod_{i=1}^{m} p_i\mu_i}{\lambda+\mu_{m+1}+\alpha}. \end{aligned}$$

Thus we have an equivalent statement for the determinant to equal 0.

3.2.2 Case: two phases

Suppose a Coxian queue has two phases, i.e. $m = 2$. Let B be the corresponding rate matrix. Then for the determinant of B it holds that

$$P_B^2(\alpha) = \begin{pmatrix} \alpha + \mu_1 & -p_1\mu_1 \\ -\lambda & \alpha + \lambda + \mu_2 \end{pmatrix} = 0,$$

if and only if

$$(\alpha + \mu_1)(\alpha + \lambda + \mu_2) = \lambda \cdot p_1\mu_1,$$

which is equivalent to

$$\alpha^2 + (\lambda + \mu_1 + \mu_2)\alpha + \lambda\mu_1 + \mu_1\mu_2 - \lambda\mu_1 p_1 = 0.$$

Solving this for α gives that

$$\alpha = \frac{-(\lambda + \mu_1 + \mu_2) \pm \sqrt{(\lambda + \mu_1 + \mu_2)^2 - 4(\lambda\mu_1 + \mu_1\mu_2 - \lambda\mu_1 p_1)}}{2}. \tag{3.13}$$

This equation has two distinct solutions if and only if

$$(\lambda + \mu_1 + \mu_2)^2 - 4(\lambda\mu_1 + \mu_1\mu_2 - \lambda\mu_1 p_1) > 0.$$

We analyse this expression as a function of p_1. Let

$$g(p_1) = \lambda^2 + 2\lambda\mu_1 + 2\lambda\mu_2 + \mu_1^2 + \mu_2^2 - 2\mu_1\mu_2 + 4\lambda\mu_1(p_1 - 1) > 0.$$

The above expression is minimum when $p_1 = 0$, and so

$$\begin{aligned} g(p_1) &> g(0) \\ &= \lambda^2 - 2\lambda\mu_1 + 2\lambda\mu_2 + \mu_1^2 + \mu_2^2 - 2\mu_1\mu_2 \\ &= (\lambda - \mu_1 + \mu_2)^2 \\ &\geq 0. \end{aligned}$$

Clearly, $g(0)$ is non-negative for all $(\lambda, \mu_1, \mu_2) \in \mathbb{R}^3$ and equals 0 if and only if

$$\lambda = \mu_1 - \mu_2.$$

Hence, $g(p_1) > 0$, so Eqn. (3.13) has two distinct solutions for α. It follows, that B is diagonalizable for $m = 2$.

3.2.3 Case: equal service rates

Suppose a Coxian queue has equal service rates for all phases, i.e. $\mu := \mu_1 = ... = \mu_m$. Let $x := \alpha + \lambda + \mu$, then we can rewrite Eqn. (3.12) as

$$\begin{aligned}
P_B(\alpha) &= (x-\lambda)\cdot\prod_{j=2}^{m} x + \sum_{j=2}^{m}(-1)^j\lambda\left[\prod_{k=1}^{j-1}(-p_k\mu)\cdot\prod_{k=j+1}^{m} x\right] \\
&= (x-\lambda)\cdot x^{m-1} + \sum_{j=2}^{m}(-1)^j\lambda\mu^{j-1}\left[\prod_{k=1}^{j-1}(-p_k)\cdot x^{m-j}\right] \\
&= x^m - \lambda x^{m-1} + \sum_{j=2}^{m}(-1)^j\lambda\mu^{j-1}(-1)^{j-1}\prod_{k=1}^{j-1}p_k\cdot x^{m-j} \\
&= x^m - \lambda x^{m-1} + \sum_{j=2}^{m}(-1)^{2j-1}\lambda\mu^{j-1}\prod_{k=1}^{j-1}p_k\cdot x^{m-j} \\
&= x^m - \lambda x^{m-1} - \sum_{j=2}^{m}\lambda\mu^{j-1}\prod_{k=1}^{j-1}p_k\cdot x^{m-j},
\end{aligned}$$

which yields

$$P_B(\alpha) = x^m - \sum_{j=1}^{m}\lambda\mu^{j-1}\prod_{k=1}^{j-1}p_k\cdot x^{m-j}.$$

Suppose that $m = 3$. Then for the determinant of B it holds that

$$P_B^3(\alpha) = x^3 - \lambda x^2 - \lambda\mu p_1 x - \lambda\mu^2 p_1 p_2 = 0$$

if and only if

$$y^3 - \frac{\lambda}{\mu}\cdot y^2 - \frac{\lambda}{\mu}\cdot y\cdot p_1 - \frac{\lambda}{\mu}\cdot p_1 p_2 = 0,$$

with $y = \frac{x}{\mu}$. This is equivalent to

$$-\frac{\mu}{\lambda}y^3 + y^2 + p_1 y + p_1 p_2 = 0. \tag{3.14}$$

This cubic polynomial has discriminant

$$\begin{aligned}
D :&= p_1^2 + 4\frac{\mu}{\lambda}p_1^3 - 4p_1p_2 - 27\frac{\mu^2}{\lambda^2}p_1^2p_2^2 - 18\frac{\mu}{\lambda}p_1^2p_2 \\
&= p_1 + 4\frac{\mu}{\lambda}p_1^2 - 4p_2 - 27\frac{\mu^2}{\lambda^2}p_1p_2^2 - 18\frac{\mu}{\lambda}p_1p_2 = 0.
\end{aligned}$$

The discriminant equals zero if and only if Eqn. (3.14) has at least two equal roots. Hence, $D \neq 0$ is a sufficient and necessary condition for Eqn. (3.14) to have distinct roots, which is a sufficient condition for B to be diagonalizable. If we solve $D = 0$ for p_2, we get that

$$p_2 = -\frac{9\frac{\mu}{\lambda}p_1 \pm \sqrt{(3\frac{\mu}{\lambda}p_1 + 1)^3} + 2}{27\frac{\mu^2}{\lambda^2}p_1}.$$

As p_2 is non-negative, only

$$p_2 = -\frac{9\frac{\mu}{\lambda}p_1 - \sqrt{(3\frac{\mu}{\lambda}p_1 + 1)^3} + 2}{27\frac{\mu^2}{\lambda^2}p_1} \tag{3.15}$$

can be a solution of D for p_2. Therefore, if Eqn. (3.15) does not hold, then $D \neq 0$, and then B is diagonalizable. This is a sufficient condition.

Chapter 4

Function approximation method

Let $X \geq 0$, $X \stackrel{d}{=} F^*$ model a service time. Write $F^*(x) = P(X < x)$. This is not standard, but it is notationally convenient. Schassberger [3] proved the existence of an approximation by a Coxian distribution function with an unbounded number of equal phase rates. However, the approximation does not leave the mean of the distribution function intact. We will provide an algorithm for an alternative approximation, by means of a Coxian distribution function, that does leave the mean intact.

The algorithm consists of three steps:

i) Construct a discrete finite ordered set of points A_{F^*} and, given this set, construct a convex combination F_m of degenerate distributions yielding a discrete approximation X_m, $X_m \stackrel{d}{=} F_m$ of X with expectation $E(X_m) = E(X)$.

ii) Construct a representation of the discrete distribution by a sequential stochastic network of subsequent degenerate phases and transition probabilities in between those phases, based on the earlier obtained convex combination of degenerate distributions.

iii) Approximate the degenerate distributions with Erlang distributions.

Step (i): Constructing a discrete approximation

We begin by partitioning $\mathbb{R}_{\geq 0}$ in sets by an n-step procedure. For every set we calculate the conditional expectation of X on that set. We use that for an approximation of F^*.

Algorithm 1: Constructing a discrete approximation F_m of F^*

Input : F^*, n
Output: discrete approximation of F^*
m-step partition of $\mathbb{R}_{\geq 0}$:
Let $P := \{[0, \infty)\}$;
Let $T_0 := \emptyset$;
for $i = 1, \ldots, n$ **do**
  Let $T_i := T_{i-1}$;
  for *all* $[j, k) \in P$ **do**
    if $F^*(k) - F^*(j) \neq 0$ **then**
      Let $t_{|T_i|+1} := \frac{\int_j^k x dF^*(x)}{F^*(j) - F^*(k)}$;
    end
    if $t_{|T_i|+1} \notin T_i$ **then**
      Let $T_i := T_i \cup \{t_{|T_i|+1}\}$;
    end
  end
  Order the elements $t_j \in T_i$ such that $t_1 < \ldots < t_{|T_i|}$;
  Let $P := \{[0, t_1), [t_1, t_2), \ldots, [t_{|T_i|-1}, t_{|T_i|}), [t_{|T_i|}, \infty)\}$;
end
Construction of a distribution function of discrete random variables:
Let $m := |T_n| + 1$, $t_0 = 0$, $t_m = \infty$;
for $i = 1, \ldots, m$ **do**
  Let $b_i := F^*(t_i) - F^*(t_{i-1})$;
  Let $\alpha_i := \frac{\int_{t_{i-1}}^{t_i} x dF^*(x)}{b_i}$;
end
for $i = 1, \ldots, m$ **do**
  Let $D_{\alpha_i} := 1_{[\alpha_i, \infty)}(t)$, $t \geq 0$;
end
Let $F_m(t) := b_1 D_{\alpha_1}(t) + b_2 D_{\alpha_2}(t) + \ldots + b_m D_{\alpha_m}(t) = \sum_{i=1}^m b_i D_{\alpha_i}(t)$;

Step (ii): Constructing a stochastic network

With the discrete distribution F_m we can associate a stochastic network consisting of $m + 1$ nodes, $\{1, \ldots, m\} \cup \{\Delta\}$, as follows. A visit to node i is called "phase i". Put $\alpha_0 = 0$. Start in the stochastic network in phase 1, where we spend a deterministic time $\alpha_1 - \alpha_0 = \alpha_1$. Then a transition to phase 2 takes places with probability p_1 and to Δ with probability $(1 - p_1)$. This procedure is iterated: after spending a deterministic time $\alpha_i - \alpha_{i-1}$ in phase i, we transition to phase $i + 1$ with probability p_i and to Δ with probability $(1 - p_i)$, for $i \in \{1, \ldots, m - 1\}$. After phase m, we move to Δ. Denote by S the distribution function of the time to reach Δ, from phase 1. Then S can be written as

$$\begin{aligned} S(t) &= (1-p_1)D_{\alpha_1}(t) + p_1(1-p_2)D_{\alpha_2}(t) + \ldots + p_1p_2\cdots p_{m-1}(1-p_m)D_{\alpha_m}(t) \\ &= \sum_{i=1}^{m}\left[\left(\prod_{j=1}^{i-1} p_j\right)(1-p_i)D_{\alpha_i}(t)\right]. \end{aligned}$$

We want to determine the probabilities $p_1, \ldots, p_{m-1}$, such that $S(t) = F_m(t)$, $t \geq 0$.

By equating with $S(t) = F_m(t)$, we get

$$b_i = \left(\prod_{j=1}^{i-1} p_j\right)(1-p_i),$$

for $i = 1, \ldots, m$. From Step (i), we know the values of the b_i's. With these, we can calculate the values of the p_i's.

Algorithm 2: Calculating $p_1, \ldots, p_m$.
Input : $b_1, \ldots, b_m$
Output: $p_1, \ldots, p_m$
for $i = 1, \ldots, m$ **do**
  Let $p_i := 1 - \frac{b_i}{1-\sum_{j=1}^{i-1} b_j}$;
end

Lemma 4.1. *Let* $p_i = 1 - \frac{b_i}{1-\sum_{j=1}^{i-1} b_j}$, *for* $i = 1, \ldots, m$, *then it holds that*

$$S(t) = F_m(t),\ t \geq 0.$$

Proof. We do this by induction. First observe that

$$b_1 = 1 - p_1 \Longrightarrow p_1 = 1 - b_1.$$

Furthermore we have that

$$b_2 = p_1(1-p_2) = (1-b_1)(1-p_2),$$

which gives that

$$\frac{b_2}{1-b_1} = 1 - p_2,$$

so that

$$p_2 = 1 - \frac{b_2}{1-b_1}.$$

It follows that

$$p_1 p_2 = 1 - b_1.$$

Next assume that $p_1 \cdots p_{i-1} = 1 - b_1 - \ldots - b_{i-1} = 1 - \sum_{j=1}^{i-1} b_j$, for some i. Then it follows that

$$b_i = p_1 \cdots p_{i-1}(1 - p_i) = \left(1 - \sum_{j=1}^{i-1} b_j\right)(1 - p_i),$$

which gives that

$$\frac{b_i}{1 - b_1 - \ldots - b_{i-1}} = 1 - p_i.$$

Hence,

$$p_i = 1 - \frac{b_i}{1 - b_1 - \ldots - b_{i-1}}.$$

Multiplying by $p_1 \cdots p_{i-1}$ gives

$$\begin{aligned} p_1 \cdots p_{i-1} \cdot p_i &= (1 - b_1 - \ldots b_{i-1}) \cdot \left(1 - \frac{b_i}{1 - b_1 - \ldots - b_{i-1}}\right) \\ &= 1 - b_1 - \ldots - b_i = 1 - \sum_{j=1}^{i} b_j. \end{aligned}$$

This completes the induction step.

□

Step (iii): Approximating degenerate functions

We have now almost modelled the approximation F_m with a Coxian distribution. We have constructed a stochastic network with m phases and transition probabilites $p_1, \ldots,$ p_{m-1}. The only difference with a Coxian distribution is that the phases of the stochastic network have a deterministic distribution. We can approximate these as follows.

Let i be a deterministic phase with time $\delta_i = \alpha_i - \alpha_{i-1}$. Let $\mu_i = \frac{1}{\delta_i}$. We split this phase up into k_i distinct subphases, say phase $i_1, \ldots, i_{k_i}$. Let each of these subphases be exponentially distributed with parameter $k_i \mu_i$. Phase i now has an Erlang distributed service time with mean $k_i \frac{1}{k_i \mu_i} = \frac{1}{\mu_i} = \delta_i$.

As each of these exponential subphases has variance $\left(\frac{1}{k_i \mu_i}\right)^2$, the variance of the sum of these phases is $k_i \left(\frac{1}{k_i \mu_i}\right)^2 = \frac{1}{k_i \mu_i^2}$. If $k_i \to \infty$, the variance tends to 0, while the mean is kept at $\frac{1}{\mu_i}$. Hence, by choosing k_i large enough, we can approximate a deterministic random variable with duration $\frac{1}{\mu_i} = \delta_i$ arbitrarily closely, by an $Erlang(k_i, \mu_i)$ variable.

We can now apply this to the stochastic network from Chapter 3. The result is a series of Erlang distributed phases, where the i'th Erlang phase has k_i exponential subphases with rate $k_i\mu_i$. The transition probability from the last subphase of phase i to the first subphase of phase $i+1$ is p_i, for $i = 0, \ldots, m-1$. The other transition rates equal 1. Now the time to get to Δ from phase 1 has a Coxian distribution with $\sum_{i=1}^{m} k_i$ phases, which is as follows. Phases $\sum_{j=1}^{i-1} k_j + 1, \ldots, \sum_{j=1}^{i-1} k_j + k_i$ have rate $k_i\mu_i = \frac{k_i}{\delta_i}$, for $i \in \{1, \ldots, m\}$. Transition probabilites from phase $\sum_{j=1}^{i} k_i$ to phase $\sum_{j=1}^{i} k_i + 1$ are p_i, for $i \in \{1, \ldots, m-1\}$, the other transition probabilities equal 1.

Special case: rational deterministic phase times

Suppose all deterministic phase times are rational. Let us rewrite these as follows:

$$\frac{1}{\mu_1} = \frac{\tau_1}{\nu_1}, \ \ldots, \ \frac{1}{\mu_m} = \frac{\tau_m}{\nu_m},$$

such that the greatest common divisor $GCD(\tau_i, \nu_i) = 1$, for $i = 1, \ldots, m$.

We use the following two facts:

1. $$GCD\left(\frac{\tau}{\nu}, \frac{\tau'}{\nu'}\right) = \frac{GCD(\tau, \tau')}{LCM(\nu, \nu')},$$

2. $$GCD(a, b, c) = GCD(\ GCD(a, b), c\ ),$$

where the greatest common divisor is a rational number. With this, we can find the greatest common divisor of $\frac{1}{\mu_1}, \ldots, \frac{1}{\mu_m}$, that is

$$\begin{aligned} g &= GCD\left(\ GCD\left(\ \ldots\ GCD\left(GCD\left(\ \frac{\tau_1}{\nu_1}, \frac{\tau_2}{\nu_2}\ \right),\ \frac{\tau_3}{\nu_3}\right)\ \ldots,\ \frac{\tau_{m-1}}{\nu_{m-1}}\ \right),\ \frac{\tau_m}{\nu_m}\ \right) \\ &= \frac{GCD(\ldots GCD\,(GCD(\tau_1, \tau_2), \tau_3), \ldots, \tau_m)}{LCM(\ldots LCM\,(LCM(\nu_1, \nu_2), \nu_3), \ldots, \nu_m)}. \end{aligned}$$

Hence, g is the greatest rational number such that we can split up all deterministic phases in a number of exponential phases with expectation g. For phase i this number is

$$k_i = \frac{1}{\mu_i g} = \frac{\delta_i}{g}$$

phases. We can also split them up in exponential phases with expectation $\frac{g}{N}$, for any $N \in \mathbb{N}_+$, thereby multiplying the number of exponential phases by N. For phase i this number is

$$k_i = \frac{N}{\mu_i g} = \frac{N\delta_i}{g}.$$

Bibliography

[1] Ivo Adan and Jacques Resing. Queueing systems. 3 2015.

[2] Titchmarsh E.C. *The Theory of Functions.* Oxford University Press, 2ed. edition, 1939.

[3] Schassberger R. *Warteschlangen.* Springer-Verlag, 1973.

[4] Josefien Verschuure. Analyse van een server farm met server-opstarttijden. Bachelor thesis, Leiden University, 07 2017.

[5] Xiaoshen Wang. A simple proof of descartes's rule of signs. 111, 06 2004.