Rational points on del Pezzo surfaces of degree one

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Contents

1 Introduction

A del Pezzo surface S over a field k is a smooth, projective, geometrically integral scheme of dimension two over k, with ample anticanonical divisor $-K_S$. See Definition 3.12 for the canonical class. The degree of S is defined to be the self-intersection number $d := K_S^2$.

In this thesis we will study del Pezzo surfaces of degree 1. Such a surface is isomorphic to a smooth sextic hypersurface in the weighted projective space $\mathbb{P}(2:3:1:1)$ with coordinates X, Y, Z, W, given by an equation

$$Y^2 + c_1 XY + c_3 Y = X^3 + c_2 X^2 + c_4 X + c_6 \tag{1}$$

where $c_n = \sum_{i=0}^{n} c_{ni} Z^i W^{n-i}$ is a homogeneous polynomial of degree n. Conversely, any smooth sextic hypersurface in this weighted projective space is a del Pezzo surface of degree 1 [[Kol96], Theorem III.3.5]. Note that such a hypersurface S contains the point $O = [1:1:0:0]$. Associated to the anticanonical divisor is a rational map $S \dashrightarrow \mathbb{P}^1$, given by sending $[X:Y:Z:W]$ to $[Z:W]$. Clearly this map is well-defined at all points except for O. By blowing up this point we obtain a rational elliptic surface, that is, a smooth surface $\mathrm{Bl}_O(S)$ with a morphism $\rho\colon \mathrm{Bl}_O(S) \to \mathbb{P}^1$ such that the exceptional curve above O is a section. All but finitely many fibers of the map ρ are smooth curves of genus 1.

If S is a del Pezzo surface of degree 1 over an infinite field k, we wish to describe the set $S(k)$ of k-rational points. The main goal of this thesis is to state conditions under which $S(k)$ is Zariski dense in S. In [VA09] A. Várilly-Alvarado considers del Pezzo surfaces of degree 1 over $\mathbb{Q}$, given by the equation

$$Y^2 = X^3 + aZ^6 + bW^6$$

where a, b are nonzero integers. In [VA09], Theorem 1.1 he shows that the rational points are Zariski dense if a finiteness conjecture on Tate-Shafarevich groups holds, and if a and b satisfy some technical conditions. He also mentions in [VA09], Remark 7.4 the surface S given by

$$Y^2 = X^3 + 243Z^6 + 16W^6$$

as an example of a surface that does not satisfy these conditions. N. Elkies proved that the rational points are dense in this surface with a different method in [Elk12]. He uses the fact that the morphism $\mathrm{Bl}_O(S) \to \mathbb{P}^1$ has a fiber with a 3-torsion point to produce elliptic curves of positive rank that are not fibers of the morphism. Elkies also mentions that there might be a similar construction whenever some fiber has a rational torsion point other than the origin.

This was the starting point of this thesis, where we consider del Pezzo surfaces of degree 1 containing a k-point P that has finite order on its fiber. Our main result is the following.

Theorem 4.8 Let S be a del Pezzo surface of degree 1 over a field k with corresponding map $\rho\colon \mathrm{Bl}_O(S) \to \mathbb{P}^1$. Let $n \geq 2$ be an integer and let $P \in S(k)$ be a point. Let $F_0 \subset S$ be the image of the fiber of ρ containing P under the blowup map $\mathrm{Bl}_O(S) \to S$. Assume that F_0 is smooth and that the order of P on F_0 is equal to n. Let F_0' be the strict transform of F_0 in $S' = \mathrm{Bl}_P(S)$.

Then

(i) The Riemann-Roch space $\mathcal{L}_{S'}(mF'_0)$ has dimension 1 as a k-vector space for $0 \leq m < n$;

(ii) $\dim_k(\mathcal{L}_{S'}(nF'_0)) = 2$;

(iii) The vector space $V := \{f \in \mathcal{L}_S(-nK_S)\backslash\{0\} : \mu_P((f)_S) \geq n\} \cup \{0\}$ has dimension 2.

This theorem shows that for a surface S as above, the complete linear system associated to the divisor $-nK_S$ has a one-dimensional subsystem L consisting of curves that contain P with multiplicity at least n. In Section 4.4 we will show that a related system on $\mathrm{Bl}_P(S)$ is base point free, so we find a morphism $\phi\colon \mathrm{Bl}_P(S) \to \mathbb{P}^1$. In Section 5 we assume $k = \mathbb{Q}$, and we will combine the morphisms ρ and ϕ to obtain the following result on density of $S(\mathbb{Q})$ in S.

Theorem 5.1 Assume S and P are as in Theorem 4.8, with $k = \mathbb{Q}$, and write ϕ for the obtained morphism $\mathrm{Bl}_P(S) \to \mathbb{P}^1$. If ϕ has a smooth fiber that contains infinitely many rational points, then $S(\mathbb{Q})$ is Zariski dense in S.

Remark 1.1. Note that Theorem 5.1 as stated in Section 5 is actually a statement on density of the rational points in $\mathrm{Bl}_{O,P}(S)$ instead of S, but this is not an issue, since $\mathrm{Bl}_{O,P}(S)$ and S have isomorphic non-empty open subspaces.

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2 Density of Rational Points

In this section we prove a sufficient condition for the rational points to be dense on a del Pezzo surface. In fact, this condition can be proven more generally, namely for Noetherian irreducible topological spaces of dimension two. We will use the following lemma.

Lemma 2.1. In a Noetherian topological space X, every nonempty closed subset Y can be expressed as a finite union

$$Y = Y_1 \cup \cdots \cup Y_r$$

of irreducible closed subsets Y_i. If we require that $Y_i \not\subset Y_j$ for $i \neq j$, then the Y_i are uniquely determined.

Proof. [Har77], Proposition I.1.5. □

Since finite unions and subsets of Noetherian spaces are Noetherian, we can apply this lemma to del Pezzo surfaces and all its subsets.

Proposition 2.2. Let X be a Noetherian irreducible topological space of dimension 2. Let T be a subset of X, and assume there are infinitely many closed irreducible one-dimensional subsets of X, each of which contains infinitely many points in T. Then T is dense in X.

Proof. Let E be one of the one-dimensional subsets containing infinitely many points in T. Consider $E \cap \overline{T}$, where $\overline{T}$ is the topological closure of T. This is closed in E, so using Lemma 2.1 we can write

$$E \cap \overline{T} = Y_1 \cup \cdots \cup Y_r,$$

with $Y_i \subset E$ closed and irreducible in E. Assume $\dim Y_i = 0$ for all i. Since the Y_i are irreducible, this implies that they are just singletons. Then we get a contradiction, because $E \cap T$ contains infinitely many points. So without loss of generality, we get $\dim Y_1 = 1$. Since E is irreducible and $\dim E = 1$, this implies $E = Y_1$, so $E = E \cap \overline{T}$, which implies $E \subset \overline{T}$. So $\overline{T}$ must contain all irreducible one-dimensional subspaces with infinitely many points in T. Again by Lemma 2.1, we can write $\overline{T} = T_1 \cup \cdots \cup T_m$ with T_i irreducible and closed. Assume $\dim T_i \leq 1$ for all i. The equality

$$E = \bigcup_{i=1}^{m} (E \cap T_i)$$

yields that $E = E \cap T_i$ for some i, since E is irreducible. Since T_i is irreducible and of dimension 1, we find $E = T_i$. Since we have infinitely many such subspaces E, this again yields a contradiction. So without loss of generality, we find $\dim T_1 = 2$. Because S is irreducible, this implies $S = T_1$, so T is dense in S. □

Thus in order to conclude density of the k-points in a del Pezzo surface S over k, it suffices to show that there are infinitely many closed irreducible one-dimensional subspaces in S, each containing infinitely many k-points.

3 Finding the Right Curves

In this section we start by recalling some definitions. Then we show what curves are candidates for having infinitely many rational points, and give a way to find such curves.

3.1 Definitions

Definition 3.1. A *variety* over a field k is a geometrically integral, separated scheme of finite type over k. A *surface* is a variety of dimension 2. A *curve* is a separated scheme of finite type over k, such that all the irreducible components are of dimension 1.

Throughout this subsection, let X be a smooth variety over a field k.

Notation 3.2. We denote the structure sheaf of X by O_X, and the function field by $k(X)$.

Definition 3.3. A *prime divisor* on X is a closed irreducible subscheme of codimension 1. A Weil divisor is an element of the free abelian group $\mathrm{Div}(X)$ generated by the prime divisors. Such a divisor is of the form $D = \sum_Y n_Y Y$, where Y runs over all the prime divisors and n_Y is an integer, called the *multiplicity of D at Y*. For all but finitely many Y we have $n_Y = 0$.

Definition 3.4. Let Y be a prime divisor on X, and let P be a point on Y. Let f be a local equation for Y at P. This exists because X is smooth, so $O_{P,X}$ is a unique factorization domain. The *multiplicity of Y at P*, denoted by $\mu_P(Y)$, is the largest integer r such that $f \in \mathfrak{m}_P^r$, where $\mathfrak{m}_P$ is the maximal ideal of $O_{P,X}$.

Definition 3.5. The multiplicity of a Weil divisor $D = \sum_Y n_Y Y$ at a point P is the sum

$$\mu_P(D) := \sum_Y n_Y \mu_P(Y).$$

Definition 3.6. A Weil divisor D is said to be *effective* if $n_Y \geq 0$ for all Y. We denote this by $D \gg 0$. The *support* of D is the union of those Y for which $n_Y \neq 0$.

Given a prime divisor Y of X the local ring $O_{Y,X}$, which is the local ring of O_X at the generic point of Y, is a discrete valuation ring, since X is smooth. We obtain a normalized valuation, which we extend to the function field

$$\mathrm{ord}_Y \colon k(X)^* \to \mathbb{Z}.$$

For a fixed $f \in k(X)^*$, there are only finitely many Y such that $\mathrm{ord}_Y(f) \neq 0$ (See [Har77], Lemma II.6.1), so we can associate a divisor to f.

Definition 3.7. Let $f \in k(X)^*$.

- The *divisor associated to f* is given by:

$$(f) := \sum_Y \mathrm{ord}_Y(f) \cdot Y.$$

  Such a divisor is called a *principal divisor*.

- Two divisors D, D' are said to be *linearly equivalent* if their difference is principal. We write $D \sim D'$.
- The *Picard group* $\operatorname{Pic} X$ is the group of Weil divisors modulo linear equivalence.
- For fixed D the *complete linear system associated to* D, denoted by $|D|$, consists of all effective divisors that are linearly equivalent to D.

Definition 3.8. A *Cartier divisor* on X is an equivalence class of collections of pairs $(U_i, f_i)_{i \in I}$ satisfying the following conditions:

- The U_i's are open sets covering X;
- The f_i's are nonzero rational functions in $k(U_i)$;
- $f_i f_j^{-1} \in O_X(U_i \cap U_j)^*$ for all $i, j \in I$.

Two Cartier divisors $(U_i, f_i)_{i \in I}$ and $(V_j, g_j)_{j \in J}$ are equivalent if we have

$$f_i g_j^{-1} \in O_X(U_i \cap V_j)^*$$

for all i, j. If $f \in k(X)^*$, then $\{(X, f)\}$ is a Cartier divisor. Such a divisor is called a *principal Cartier divisor*.

The *support* of a principal Cartier divisor is the set of poles and zeros of f on X. For a general Cartier divisor $(U_i, f_i)_{i \in I}$, the support is the union of the supports of the principal divisors $(U_i, f_i)_{i \in I}$ on U_i. A Cartier divisor is called *effective* if it can be defined with $f_i \in O(U_i)$ for all i.

We define a group structure on the set $\operatorname{CaDiv}(X)$ of Cartier divisors on X by setting $(U_i, f_i)_i + (V_j, g_j)_j = (U_i \cap V_j, f_i g_j)_{i,j}$.

Example 3.9 ([HS00], page 40)**.** If $g \colon Y \to Z$ is a morphism of varieties, and $D = (U_i, f_i)_i$ is a Cartier divisor on Z, such that $f(Y) \not\subset \operatorname{supp}(D)$, then the collection

$$g^*(D) := (g^{-1}(U_i), f_i \circ g)$$

is a Cartier divisor on Y. Clearly, if D is principal, so is $g^*(D)$. If D is effective, so is $g^*(D)$. If g is surjective, then the converse also holds: if $f_i \notin O_Z(U_i)$, then f_i has a pole somewhere on U_i, say at y. Then $f_i \circ g$ has a pole at points in $g^{-1}(\{y\})$, so $f_i \circ g \notin O_Y(g^{-1}(U_i))$.

It follows that we get a map $g^* \colon \operatorname{CaDiv}(Z) \to \operatorname{CaDiv}(Y)$ if $g(Y)$ is not contained in the support of any divisor, so if $g(Y)$ is dense in Z.

Remark 3.10. There is a map $\operatorname{CaDiv}(X) \to \operatorname{Div}(X)$. See [HS00], page 38, for this. For smooth varieties, this map is an isomorphism. See [Har77], Proposition II.6.11 for this. The map sends effective Cartier divisors to effective Weil divisors, and principal Cartier divisors to principal Weil divisors. We can apply II.6.11 since for smooth varieties the local rings are unique factorization domains. We will freely identify Weil and Cartier divisors when we work with smooth varieties. If $g \colon X \to Y$ is a morphism between smooth varieties such that $g(X)$ is dense in Y, then we also obtain a map $g^* \colon \operatorname{Div}(Y) \to \operatorname{Div}(X)$ by composing the isomorphism $\operatorname{Div}(Y) \to \operatorname{CaDiv}(Y)$, the map $\operatorname{CaDiv}(Y) \to \operatorname{CaDiv}(X)$, and the map $\operatorname{CaDiv}(X) \to \operatorname{Div}(X)$.

Remark 3.11. For effective divisors D we will often write $P \in D$ instead of $P \in \operatorname{supp}(D)$.

Definition 3.12. Set $n := \dim X$. The *canonical class* K_X is the divisor class of a divisor associated to a nonzero n-form. See for example [HS00], Example A.2.2.3. for this. Every divisor in this class is called a canonical divisor.

Definition 3.13. Let D be a Weil divisor. The *Riemann-Roch space associated to D* is the vector space over k given by

$$\mathcal{L}_X(D) = \{f \in k(X)^* : (f) + D \gg 0\} \cup \{0\}.$$

Its dimension is denoted by $\ell(D)$. There is a natural, surjective map

$$\begin{aligned} \mathcal{L}_X(D)\backslash\{0\} &\to |D|, \\ f &\mapsto (f) + D. \end{aligned}$$

We have $(f) = (g)$ if and only if $f = \lambda g$ for some $\lambda \in k$, so we find that $|D|$ is naturally parametrized by $\mathbb{P}^{\ell(D)-1}(k)$.

Next, we define the arithmetic and geometric genus. For this we will use the definition of the cohomology groups.

Definition 3.14. Let Z be a topological space, and let $\mathcal{F}$ be a sheaf on Z. We define the *cohomology functors* $H^i(Z, \mathcal{F})$ to be the right derived functors of $\Gamma(Z, \mathcal{F})$. See [Har77], Chapter III.2, for more details.

Definition 3.15. Let Y be a projective scheme of dimension n over k The *arithmetic genus* of Y is given by:

$$p_a(Y) := (-1)^n(\chi(O_Y) - 1),$$

where $\chi(O_Y) = \sum_i (-1)^i \dim_k H^i(Y, O_Y)$.

This is a rather abstract definition, but we will always use the following proposition, based on intersection numbers, to compute the arithmetic genus of curves. For the definition of the intersection number of curves or divisors we refer to [Har77], Chapter V.1.

Proposition 3.16. Let C be a curve on a smooth, projective, geometrically integral surface Y over a field k. Then we have:

$$2p_a(C) - 2 = C(C + K_Y),$$

where K_Y is any canonical divisor on Y, and $C(C + K_Y)$ is the intersection number.

Definition 3.17. The *geometric genus* of a smooth, projective, geometrically integral curve C over field k is given by

$$p_g(C) := \dim_k H^0(C, \Omega),$$

where Ω is the sheaf of holomorphic 1-forms on C. If C is singular and k is perfect, we write C' for a normalization of C, and we define $p_g(C) := p_g(C')$.

For projective nonsingular curves C the values $p_g(C)$ and $p_a(C)$ agree (See [Har77], Proposition IV.1.1), so we will often speak of the genus without specifying which one we mean. For projective, connected curves C that are not necessarily smooth we have $p_g(C) \leq p_a(C)$. This follows for example from Lemma 4.22, since $p_a(C) = \dim_k H^1(C, O_C)$ by [Har77], Exercise III.5.3 , and we have

$$p_g(C) = p_g(C') = p_a(C') = \dim_k H^1(C', O_{C'})$$

for a normalization C' of C. A deep result known as Faltings's theorem ([Fal86], Theorem II.6.7) implies that any smooth algebraic curve of genus $p_g(C) \geq 2$ over a number field contains only finitely many rational points. So in order to find curves that contain infinitely many rational points, our first goal should be to find curves C with $p_g(C) \leq 1$.

3.2 Computations on the Blowup

Throughout this subsection, let S be a del Pezzo surface of degree 1 over a field k, and let $P \in S$ be a closed point not equal to O. Let $F_\infty \in |-K_S|$ be such that $P \notin F_\infty$. This is possible since O is the unique base point of $|-K_S|$. Let $F_0 \in |-K_S|$ be such that $P \in F_0$. Note that F_0 is unique with this property, since its strict transform is a fiber of the morphism $\rho\colon \mathrm{Bl}_O(S) \to \mathbb{P}^1$, and fibers are disjoint.

For the definition of the *blowing-up of a surface at a point* we refer to [Har77], Chapter V.3. We denote the blowing-up of S at P by S', together with the map $\pi\colon S' \to S$. The restriction of this map induces an isomorphism

$$S' \backslash \{\pi^{-1}(P)\} \to \; S \backslash \{P\},$$

and $E := \pi^{-1}(\{P\})$ is a curve, called the *exceptional curve*.

The map π is an isomorphism on $\pi^{-1}(S - \{P\}) = S' - E$, so S and S' are birational, and we obtain an isomorphism $\pi^*\colon k(S) \to k(S')$, given by composition with π. By [Har77], Proposition V.3.3, a canonical divisor $K_{S'}$ on S' is linearly equivalent to $\pi^*(K_S) - E$.

Proposition 3.18. The natural maps $\pi^*\colon \operatorname{Pic} S \to \operatorname{Pic} S'$ and $\mathbb{Z} \to \operatorname{Pic} S'$ defined by $1 \mapsto 1 \cdot E$ give rise to an isomorphism $\operatorname{Pic} S' \cong \operatorname{Pic} S \oplus \mathbb{Z}$. On S' the following rules on intersection hold:

(1) If $C, D \in \operatorname{Div}(S)$, then $\pi^*(C) \cdot \pi^*(D) = C \cdot D$;

(2) If $C \in \operatorname{Div}(S)$, then $\pi^*(C) \cdot E = 0$;

(3) For the self-intersection of E we have $E^2 = -1$.

Proof. This follows from [Har77], Proposition V.3.2. $\square$

Definition 3.19. Let $Y \subset S$ be an irreducible curve. The *strict transform of* Y is the closure Y' of $\pi^{-1}(Y - \{P\})$ in S'. It is an irreducible curve in S'. For a divisor $D = \sum_Y n_Y Y \in \operatorname{Div}(S)$ the *strict transform of D* is

$$D' := \sum_Y n_Y Y' \in \operatorname{Div}(S').$$

Proposition 3.20. The map $\pi^*\colon \operatorname{Div}(S) \to \operatorname{Div}(S')$ is given by

$$D \mapsto D' + \mu_P(D) \cdot E,$$

where $\mu_P(D)$ is the multiplicity of D at P.

Proof. [Har77], Proposition V.3.6. □

Proposition 3.21. For $D \in \operatorname{Div}(S)$, the restriction of π^* to $\mathcal{L}_S(D)$ is an isomorphism

$$\mathcal{L}_S(D) \to \mathcal{L}_{S'}(\pi^*(D)).$$

Proof. We will use the following commutative diagram

$$\begin{array}{ccc} k(S)^* & \xrightarrow{\pi^*} & k(S')^* \\ \downarrow & & \downarrow \\ \operatorname{Div}(S) & \xrightarrow{\pi^*} & \operatorname{Div}(S') \end{array}$$

If $f \in \mathcal{L}_S(D)$ for a divisor D, then $(f)_S + D \gg 0$, so

$$(\pi^*(f))_{S'} + \pi^*(D) = \pi^*((f)_S + D) \gg 0.$$

This implies that $\pi^*(f) \in \mathcal{L}_{S'}(\pi^*(D))$. So $\pi^*\colon k(S)^* \to k(S')^*$ restricts to a map

$$\pi^*\colon \mathcal{L}_S(D) \to \mathcal{L}_{S'}(\pi^*(D)).$$

It is still injective as it is the restriction of an isomorphism. It is also surjective: for $g \in \mathcal{L}_{S'}(\pi^*(D))$ there is an $f \in k(S)$ such that $\pi^*(f) = g$. We have that

$$\pi^*((f)_S + D) = (g)_{S'} + \pi^*(D) \gg 0,$$

so since π is surjective we get, using Example 3.9 that $(f)_S + D$ is effective, and thus $f \in \mathcal{L}_S(D)$. □

Definition 3.22. For $n, m \in \mathbb{Z}_{\geq 1}$, we define the following vector space

$$V_{n,m} := \{f \in \mathcal{L}_S(nF_\infty)\backslash\{0\} : \mu_P((f)_S) \geq m\} \cup \{0\}.$$

We denote by $L_{n,m}$ the image of $V_{n,m}\backslash\{0\}$ in $|-nK_S|$ under the map in Definition 3.13, so

$$L_{n,m} = \{D \in |-nK_S| : \mu_p(D) \geq m\}.$$

The map sends f to $D = (f)s + n \cdot F_\infty$. Because $\mu_P(f) = \mu_P((f)_S)$ and $\mu_P(F_\infty) = 0$, we get $\mu_P(D) \geq m$, so the set mentioned above is indeed the image of $V_{n,m}$. We will link $V_{n,m}$ to a Riemann-Roch space of the blowup S'. Note that $\pi^*(nF_\infty) = nF'_\infty$, since $\mu_P(nF_\infty) = 0$.

Lemma 3.23. For all $n, m \geq 1$, the restriction of π^* to $V_{n,m}$ is an isomorphism

$$V_{n,m} \xrightarrow{\sim} \mathcal{L}_{S'}(nF'_\infty - mE).$$

Proof. Let $f \in V_{n,m}$ such that $f \neq 0$. We have $\pi^*((f)_S) = (f)'_S + \mu_P((f)_S)E$. So we get

$$\begin{aligned}(\pi^*(f))_{S'} - mE + nF'_\infty &= (f)'_S + \mu_P((f)_S)E - mE + nF'_\infty \\ &= (\mu_P((f)_S) - m)E + (f)'_S + nF'_\infty\end{aligned}$$

Now $(\mu_P((f)_S) - m)E \gg 0$ since $\mu_P((f)_S) \geq m$, and we have

$$(f)'_S + nF'_\infty = ((f)_S + nF_\infty)' \gg 0,$$

since $f \in \mathcal{L}_S(nF_\infty)$. So $\pi^*(f) \in \mathcal{L}_{S'}(nF'_\infty - mE)$. So π^* restricts to a map $V_{n,m} \to \mathcal{L}_{S'}(nF'_\infty - mE)$. It is still injective, as it is the restriction of an isomorphism. For surjectivity, let $g \in \mathcal{L}_{S'}(nF'_\infty - mE) \subset \mathcal{L}_{S'}(nF'_\infty)$. There is $f \in \mathcal{L}(nF_\infty)$ such that $\pi^*(f) = g$. Using Lemma 3.20, we get $\mu_P((f)_S) = \mathrm{ord}_E(g) \geq m$, so $f \in V_{n,m}$. □

We get the following corollary.

Corollary 3.24. There is an $s \in k(S')^*$ such that for all $n \geq 1$ the following map is an isomorphism

$$\begin{aligned} V_{n,n} &\to \mathcal{L}_{S'}(nF'_0), \\ f &\mapsto \pi^*(f)/s^n. \end{aligned}$$

Proof. The map $\rho\colon S' \to \mathbb{P}^1$ induces a map $\rho^*\colon k(\mathbb{P}^1) \to k(S')$. Let $s \in k(\mathbb{P}^1)$ such that s has a zero at the image of F_0 and a unique, simple pole at the image of F_∞. Note that such an s exists because P is a k-point, so it has the same degree as the image of F_∞. Identify s with its image $\rho^*(s)$ in $k(S') \cong k(S)$. Then $(s)_S = F_0 - F_\infty$ and $(s)_{S'} = F'_0 + E - F'_\infty$. Now multiplication by $s^n \in k(S')$ is an isomorphism

$$\mathcal{L}_{S'}(nF'_0) \to \mathcal{L}_{S'}(nF'_\infty - nE).$$

Combining this with Lemma 3.23, applied to the case $m = n$, we get the desired result. □

Remark 3.25. In particular, since $\mathcal{L}_{S'}(nF'_0)$ contains the constants, $V_{n,n}$ contains the element $s^n \in k(S)$. This corresponds to the element $nF_0 \in L_{n,n}$.

Lemma 3.26. Assume $n \geq 2$ and let $C \in L_{n,n}$ be a connected curve. Then the arithmetic genus $p_a(C')$ of the strict transform of C is equal to 1, and $\mu_P(C) = n$.

Proof. Write $K_{S'}$ for a canonical divisor on S'. Set $\mu := \mu_P(C) \geq n$. Since $K_{S'} \sim \pi^*(K_S) + E$, we get

$$C' + \mu E = \pi^*(C) \sim \pi^*(-nK_S) \sim -n(K_{S'} - E) = -nK_{S'} + nE.$$

Rewriting yields $C' \sim -nK_{S'} + (n-\mu)E$. Thus computing the arithmetic genus using Proposition 3.16 yields

$$\begin{aligned}2p_a(C') - 2 &= C'(C' + K_{S'}) \\ &= (-nK_{S'} + (n-\mu)E) \cdot ((1-n)K_{S'} + (n-\mu)E) \\ &= n(n-1)K_{S'}^2 + (n-\mu)(1-2n)E \cdot K_{S'} + (n-\mu)^2E^2.\end{aligned}$$

We have

$$K_{S'}^2 = (\pi^*(-F_\infty) + E)^2 = (F_\infty)^2 + 2\pi^*(-F_\infty) \cdot E + E^2.$$

Since $F_\infty^2 = 1$, and $\pi^*(-F_\infty) \cdot E = 0$, this yields

$$K_{S'}^2 = 1 + 2 \cdot 0 - 1 = 0.$$

Using Lemma 3.18 and the fact that $K_{S'} \cong \pi^*(K_S) + E$, we get:

$$K_{S'} \cdot E = (\pi^*(K_S) + E) \cdot E = 0 + E^2 = -1.$$

Now, writing $c = \mu - n \geq 0$, we get

$$\begin{aligned} 2p_a(C') - 2 &= (n - \mu)(1 - 2n) \cdot (-1) - (n - \mu)^2 \\ &= -(c^2 + (2n - 1)c). \end{aligned}$$

Assume $c > 0$. Then $2p_a(C') - 2 = -(c^2 + (2n-1)c) \leq -4$, since $n \geq 2$, which implies $p_a(C') \leq -1$. This yields a contradiction, since $p_a(C') \geq p_g(C') \geq 0$. We conclude $c = 0$, so $\mu = n$, and $p_a(C') = 1$. □

Note that $L_{n,n}$ is parametrized by a projective space of dimension $\dim V_{n,n} - 1$. The connected curves in $L_{n,n}$ are birational to curves on S' of arithmetic genus 1. So if we have $\dim V_{n,n} \geq 2$ we can find such curves, and these are candidates for having infinitely many rational points.

4 Existence of a Linear System

In this section we will study the space $\mathcal{L}_{S'}(nF_0')$ defined in the previous section, and we will give a sufficient condition for it to have dimension at least 2.

4.1 An Exact Sequence of Sheaves

In order to get a better understanding of the space $\mathcal{L}_{S'}(nF_0')$, we will use an exact sequence of sheaves on S'. We start by defining some sheaves on general schemes. Throughout this section, let X be an integral scheme. We follow [Har77], Section II.6.

Definition 4.1. Let $f\colon Y \to X$ be a morphism of schemes.

- We define the *direct image sheaf* f_*O_Y on X by setting

  $$f_*O_Y(U) \;=\; O_Y(f^{-1}(U))$$

  for any $U \subset X$ open.

- The map f is called a *closed immersion* if it induces a homeomorphism $Y \to f(Y)$, where $f(Y)$ has the topology induced from X, and the morphism $f^{\#}\colon O_X \to f_*O_Y$ is surjective.

Lemma 4.2. Let $j : Y \to X$ be a closed immersion, and let $\mathcal{F}$ be a sheaf of abelian groups on Y. Then $H^i(Y, \mathcal{F}) = H^i(X, j_*\mathcal{F})$ for all i.

Proof. [Har77], Lemma II.2.10. □

Definition 4.3. Assume $D = \{(U_j, f_j)\}_j$ is an effective Cartier divisor on X. We define the associated subscheme Y to be the closed subscheme defined by the f_i. We write $i\colon Y \to X$ for the closed immersion, and we define the ideal sheaf $\mathscr{I}_Y$ to be the kernel of the morphism $i^{\#}\colon O_X \to i_*O_Y$.

Lemma 4.4. Let D be an effective Cartier divisor on X, with associated subscheme Y. We have an exact sequence of sheaves on X

$$0 \to \mathscr{I}_Y \to O_X \to i_*O_Y \to 0.$$

Proof. This follows immediately from the definitions. □

Definition 4.5. Let $D = \{(U_j, f_j)\}_j$ be a Cartier divisor on X. Given an open set U of X, consider the set

$$O_X(D)(U) = \{f \in k(X) : (f \cdot f_j)|_{U_j \cap U} \in O_X(U_j \cap U) \text{ for all } j\}$$

This yields a sheaf $O_X(D)$ on X, called the *sheaf associated to* D.

Lemma 4.6. Let D, D' be Cartier divisors on X. Then $O_X(D)$ is a locally free O_X-module of rank 1, and $O_X(D + D') = O_X(D) \otimes O_X(D')$.

Proof. [Har77], Proposition II.6.13. □

Lemma 4.7. Let D be an effective Cartier divisor on X, and let Y be the corresponding closed subscheme. Then we have an isomorphism of sheaves

$$\mathscr{I}_Y \cong O_X(-D).$$

Proof. [Har77], Proposition II.6.18. □

4.2 The Sequence for del Pezzo Surfaces

In this subsection we will apply the lemmas in the previous subsection to the case of del Pezzo surfaces of degree 1. Recall that for such a surface S we have a morphism $\rho\colon \mathrm{Bl}_O(S) \to \mathbb{P}^1$ associated to $|-K_S|$, and almost all fibers under this morphism are smooth curves of genus 1, so they are elliptic curves, each with the unique point on the exceptional curve above O. We will prove the following theorem.

Theorem 4.8. Let S be a del Pezzo surface of degree 1 over a field k with corresponding map $\rho\colon \mathrm{Bl}_O(S) \to \mathbb{P}^1$. Let $n \geq 2$ be an integer and let $P \in S(k)$ be a point. Let $F_0 \in |-K_S|$ be such that $P \in F_0$. Assume that F_0 is smooth and that the order of P on F_0 is equal to n. Let F_0' be the strict transform of F_0 in $S' = \mathrm{Bl}_P(S)$. Then

(i) $\dim_k(\mathcal{L}_{S'}(mF_0')) = 1$ for $0 \leq m < n$;

(ii) $\dim_k(\mathcal{L}_{S'}(nF_0')) = 2$;

(iii) $V_{m,m}$ has dimension 1 for $0 \leq m < n$, and $V_{n,n}$ has dimension 2.

Before we give the proof we state some results on elliptic curves and del Pezzo surfaces.

Remark 4.9. A Weil divisor on an elliptic curve E is a formal sum, an element of the free abelian group generated by the closed points on E. On E we also have a group law, an actual addition of points. To avoid confusion, we write $\sum_Q n_Q(Q)$ for a divisor, and $\sum_Q n_Q Q$ for a sum of points on E.

Lemma 4.10. Let (E, O) be an elliptic curve, and let $D = \sum_Q n_Q(Q)$ be a Weil divisor on E. Assume that all the points in the support of D are k-points. Then D is principal if and only if

$$\sum_Q n_Q = 0 \qquad \text{and} \qquad \sum_Q n_Q Q = O \text{ on } E.$$

Proof. This is [Sil09], Corollary III.3.5.

□

For an elliptic curve E and a canonical divisor K_E we have $\deg K_E = 0$ by [Sil09], Corollary II.5.5. Note that $\ell(K_E) = p_g(E) = 1$, so $K_E \sim 0$.

Lemma 4.11. Let X be a del Pezzo surface. Then $H^1(X, O_X) = 0$.

Proof. [Kol96], Lemma III.3.2.1. □

Lemma 4.12. Let X be a surface over a field k, and let $\pi\colon X' \to X$ be the blowup of a point P. Then $p_a(X') = p_a(X)$.

Proof. [Har77], Corollary V.3.5. proves this for algebraically closed fields. The extension of scalars from k to $\bar{k}$ is an exact functor that commutes with taking the Čech complex of a finite affine cover, so the dimensions of the cohomology groups agree. □

Lemma 4.13. Let C and D be two curves on a surface X, having no common irreducible component. Then

$$C \cdot D = \sum_{P \in C \cap D} (C \cdot D)_P,$$

where $(C \cdot P)_P$ is the intersection multiplicity at P.

Proof. See [Har77], Proposition V.1.4 for this, and the definition of the intersection multiplicity at a point. □

We are now ready to prove Theorem 4.8.

Proof of Theorem 4.8. Recall that $\mathcal{L}_{S'}(nF_0')$ is the vector space of 0 and all elements $f \in k(S')^*$ satisfying

$$(f)_{S'} + nF_0' \gg 0.$$

So $\mathcal{L}_{S'}(nF_0')$ is equal to the vector space of global sections of the sheaf $O_{S'}(nF_0')$. In other words

$$\mathcal{L}_{S'}(nF_0') = \Gamma(S', O_{S'}(nF_0')) = H^0(S', O_{S'}(nF_0')).$$

We will show that $\dim H^0(S', O_{S'}(nF_0')) = 2$. For the sake of legibility, we will write $H^i(X, D)$ instead of $H^i(X, O_X(D))$ from now on.

We have that F_0' is effective, so combining Lemma 4.4 and Lemma 4.7, we get the following exact sequence of sheaves on S'

$$0 \to O_{S'}(-F_0') \to O_{S'} \to i_* O_{F_0'} \to 0,$$

where $i\colon F_0' \to S'$ denotes the inclusion. By Lemma 4.6, $O_{S'}(mF_0')$ is locally free of rank 1, so tensoring with it is exact. We obtain the exact sequence

$$0 \to O_{S'}((m-1)F_0') \to O_{S'}(mF_0') \to i_* O_{F_0'} \otimes O_{S'}(mF_0') \to 0.$$

Here we used that $O_{S'}(D) \otimes O_{S'}(D') \cong O_{S'}(D + D')$, and $O_{S'} = O_{S'}(0)$. As before, let $F_\infty \in |-K_S|$ be such that $P \notin F_\infty$. Recall that $F_0' \sim \pi^*(F_\infty) - E$, so $O_{S'}(mF_0') \cong O_{S'}(m\pi^*(F_\infty) - mE)$. Write $D := m(\pi^*(F_\infty) - E)$. Since $F_0' \not\subset \operatorname{supp}(D)$, the inclusion $F_0' \to S'$ yields a divisor $i^*(D)$ on F_0'. We get an isomorphism

$$i_* O_{F_0'} \otimes O_{S'}(mF_0') \cong i_* O_{F_0'}(i^*(D)).$$

We will now show that O is the only common point of F_0 and F_∞. First we note that they have no common irreducible component: since F_0 is irreducible, this would imply that F_0 is this common component. In particular, we would get $F_0 \subset F_\infty$, but since $P \in F_0$ and $P \notin F_\infty$, this is a contradiction. Now by Lemma 4.13, we get

$$F_0 \cdot F_\infty = \sum_{Q \in F_0 \cap F_\infty} (F_0 \cdot F_\infty)_Q.$$

Since $F_0, F_\infty \in |-K_S|$, the left-hand side equals $(-K_S)^2 = 1$. So we get

$$\begin{aligned} 1 &= F_0 \cdot F_\infty \\ &= \sum_{Q \in F_0 \cap F_\infty} (F_0 \cdot F_\infty)_Q \\ &\geq (F_0 \cdot F_\infty)_O \\ &\geq 1, \end{aligned}$$

so we get equality throughout. In particular, O is the only common point of F_0 and F_∞, with multiplicity 1. We get $F_0' \cap m\pi^*(F_\infty) = m(O)$. Writing j for the composition of the map $F_0 \to F_0'$ with i, we get

$$i_* O_{F_0'}(i^*(D)) \cong j_* O_{F_0}(m(O) - m(P)).$$

In conclusion, we have the following exact sequence of sheaves on S'

$$0 \to O_{S'}((m-1)F_0') \to O_{S'}(mF_0') \to j_* O_{F_0}(m(O) - m(P)) \to 0.$$

Here we get the term $-m(P)$, since P is the image of E on S. This is by definition, since E was defined as the fiber above P. Taking global sections, we obtain a long exact sequence of k-vector spaces for all $m \geq 1$, from which we only need the first few terms. We use Lemma 4.2 to replace $H^i(S', j_* O_{F_0}(m(O) - m(P)))$ by $H^i(F_0, m(O) - m(P))$.

$$\begin{tikzcd} 0 \arrow[r] & H^0(S', (m-1)F_0') \arrow[r] & H^0(S', mF_0') \arrow[r] & H^0(F_0, m(O) - m(P)) \arrow[dll, rounded corners, "\delta_m"] \\ & H^1(S', (m-1)F_0') \arrow[r] & H^1(S', mF_0') \arrow[r] & H^1(F_0, m(O) - m(P)) \end{tikzcd}$$

Note the following: for $1 \leq m < n$ we have $H^0(F_0, m(O) - m(P)) = 0$. This follows from Lemma 4.10: since $mO - mP \neq O$ on F_0, the divisor $m(O) - m(P)$ is not principal, and we get $\dim H^0(F_0, m(O) - m(P)) = 0$. Since $nP = O$ on F_0, we get $\dim H^0(F_0, n(P) - n(O)) = 1$. By induction, we can now prove that $\dim H^0(S', mF_0') = 1$ for $0 \leq m < n$. For $m = 0$ this holds since $H^0(S', 0) = H^0(S', O_{S'}) \cong k$. For the induction step we use the first part of the long exact sequence, which now reads

$$0 \to H^0(S', (m-1)F_0') \to H^0(S', mF_0') \to 0,$$

for $1 \leq m < n$. So we get $H^0(S', (m-1)F_0') \cong H^0(S', mF_0')$, which proves (i).

By Serre duality ([Har77], Corollary III.7.7 and Remark III.7.12.1), using the fact that $K_E \sim 0$, we have

$$\dim_k H^1(F_0, m(O) - m(P)) = \dim_k H^0(F_0, m(P) - m(O)).$$

By the same arguments as before, we get

$$\dim H^1(F_0, m(O) - \ m(P)) = \begin{cases} 0, & 1 \le m < n, \\ 1, & m = n. \end{cases}$$

Using induction again, we can now prove that $H^1(S', mF_0') = 0$ for $0 \le m < n$. The case $m = 0$ follows from Lemmas 4.11 and 4.12, and for the induction step we note that part of the long exact sequence now yields

$$0 \to H^1(S', (m-1)F_0') \to H^1(S', mF_0') \to 0,$$

for $1 \le m < n$. So $H^1(S', mF_0') \cong H^1(S', (m-1)F_0')$, which is 0 by hypothesis. Combining all this, we note that for $m = n$ the long exact sequence now yields a short exact sequence

$$0 \to H^0(S', (n-1)F_0') \to H^0(S', nF_0') \to H^0(F_0, n(O) - n(P)) \to 0.$$

In particular, we get $\dim H^0(S', nF_0') = 1 + 1 = 2$, which proves (ii).

By Corollary 3.24, we know that $\dim_k V_{m,m} = \dim_k \mathcal{L}_{S'}(mF_0')$ for all m, so (iii) follows immediately from (i) and (ii). $\square$

4.3 Example for Small n

We have shown that the existence of a point P of order n implies that a certain vector space has dimension 2. This in turn yields a one-dimensional linear subsystem $L := L_{n,n}$ of $|-nK_S|$ (see Definition 3.22), given by curves that contain P with multiplicity n. For small n, the equations for these curves can easily be computed. In this subsection, we compute these for $n = 2$ and $n = 3$.

Lemma 4.14. Let S be a del Pezzo surface of degree 1 defined over a field k with corresponding map $\rho\colon \mathrm{Bl}_O(S) \to \mathbb{P}^1$, and let P be a point on S.

(i) If P has order 2 on its fiber under ρ, then S can be written as a sextic hypersurface in $\mathbb{P}_k(2 : 3 : 1 : 1)$ given by (1) such that the coefficients satisfy

$$c_{30} = c_{60} = c_{61} = c_{62} = 0,$$

where P corresponds to the point $[0 : 0 : 0 : 1]$.

(ii) If P has order 3 on its fiber under ρ, then S can be written as a sextic hypersurface in $\mathbb{P}_k(2 : 3 : 1 : 1)$ given by (1) such that the coefficients satisfy

$$c_{20} = c_{40} = c_{41} = c_{60} = c_{61} = c_{62} = c_{63} = 0,$$

where P corresponds to the point $[0 : 0 : 0 : 1]$.

Proof. Embed S in $\mathbb{P}_k(2 : 3 : 1 : 1)$. Since P has order two or three on its fiber, it is not equal to $[1 : 1 : 0 : 0]$, so by changing variables we can assume $P = [0 : 0 : 0 : 1]$. Consider the affine patch $W \neq 0$. Here S can be given by the equation

$$y^2 + a_1(s)xy + a_3(s)y = x^3 + a_2(s)x^2 + a_4(s)x + a_6(s),$$

where $y = \frac{Y}{W^3}, x = \frac{X}{W^2}, s = \frac{Z}{W}$ and $a_n = \sum_{i=0}^{n} a_{ni}s^i$ is a polynomial of degree at most n. Since $P = (0,0,0)$ lies on the surface, we find $a_{60} = 0$.

Now assume P has order two on its fiber F. Then the tangent line to F at P is given inside F by the equation $x = 0$. Since the line is given by $a_{30}y = a_{40}x$, we get $a_{30} = 0, a_{40} \neq 0$. We can now change coordinates, with the new coordinate $x' := x + \frac{a_{61}}{a_{40}}s$ instead of x. We get the following equation for S, where we omit the primes at x for the sake of legibility.

$$y^2 + b_1(s)xy + b_3(s)y = x^3 + b_2(s)x^2 + b_4(s)x + b_6(s)$$

where $b_n = \sum_{i=0}^{n} b_{ni}s^i$, and $b_{30} = b_{60} = b_{61} = 0, b_{40} \neq 0$. We again change coordinates, setting $x' := x + \frac{b_{62}}{b_{40}}s^2$. We get an equation for S given by

$$y^2 + c_1(s)xy + c_3(s)y = x^3 + c_2(s)x^2 + c_4(s)x + c_6(s),$$

where, again, the primes are left out. Here we have $c_n = \sum_{i=0}^{n} c_{ni}s^i$ with $c_{30} = c_{60} = c_{61} = c_{62} = 0$, as desired.

Next, assume P is of order three on its fiber. Then we immediately get $a_{30} \neq 0$, since otherwise the tangent line to F at P would be vertical, which would imply P had order two. We change coordinates, setting $y' := y - \frac{a_{40}}{a_{30}}x - \frac{a_{61}}{a_{30}}s$, to get an equation

$$y^2 + b_1(s)xy + b_3(s)y = x^3 + b_2(s)x^2 + b_4(s)x + b_6(s),$$

where, again, the primes are left out. Here we have $b_n = \sum_{i=0}^{n} b_{ni}s^i$, and $b_{40} = b_{60} = b_{61} = 0, b_{30} \neq 0$. Since P has order three, we get $2P = -P$, so the tangent line to F at P only intersects F at P. This tangent line is given by $y = 0$, so we get that the following equation can only have one solution

$$x^3 + b_{20}x^2 + b_{40}x + b_{60} = 0.$$

Since we have $b_{40} = b_{60} = 0$, this implies $b_{20} = 0$. As our next step, we change coordinates, setting $y' := y - \frac{b_{62}}{b_{30}}s^2 - \frac{b_{41}}{b_{30}}sx$ to get an equation

$$y^2 + d_1(s)xy + d_3(s)y = x^3 + d_2(s)x^2 + d_4(s)x + d_6(s),$$

where $d_n = \sum_{i=0}^{n} d_{ni}s^i$, and $d_{20} = d_{40} = d_{41} = d_{60} = d_{61} = d_{62} = 0, d_{30} \neq 0$. Again, the primes are left out. Finally, setting $y' := y - \frac{d_{63}}{d_{30}}s^3$, we get an equation in coefficients $c_n = \sum_{i=0}^{n} c_{ni}s^i$, such that we also have $c_{63} = 0$. $\square$

In the case where P has order 2, after assuming S is in the form of Lemma 4.14(i), we have $V_{2,2} = \langle x, s^2 \rangle$. The linear system $L_{2,2}$ consists of the curves of the form $x - \lambda s^2 = 0$, for $\lambda \in k$, which is indeed of dimension 1. If we substitute $x = \lambda s^2$ in the equation for S, we obtain the following equation for the curve

$$y^2 + \lambda s^2 \cdot c_1(s)y + c_3(s)y = s^2 \cdot f(s),$$

for certain $f \in k[s]$ with $\deg(f) \leq 4$. The multiplicity of this curve at the point $P = (0,0,0)$ is at least 2, since $c_{30} = 0$.
In the case where P has order 3, after assuming S is in the form of Lemma

4.14(ii), something similar happens. We get $V_{3,3} = \langle y, s^3 \rangle$, and $L_{3,3}$ consists of the curves $y - \lambda s^3 = 0$, for $\lambda \in k$. Substituting $y = \lambda s^3$ yields the following equation

$$\lambda^2 s^6 + \lambda s^3 \cdot c_1(s) \cdot x + \lambda s^3 \cdot c_3(s) = x^3 + c_2(s)x^2 + c_4(s)x + c_6(s),$$

which indeed has multiplicity at least 3 at $P = (0,0,0)$.

Remark 4.15. To get a better understanding of what these curves look like, one can apply the following coordinate change

$$\begin{cases} t = \frac{1}{s}, \\ x' = \frac{x}{s^2}, \\ y' = \frac{y}{s^3}, \end{cases}$$

and then multiply the resulting equations by t^6. In the case where $n = 2$, this yields the following equations

$$y^2 + \lambda(c_{10}t + c_{11})y + (c_{31}t^2 + c_{32}t + c_{33})y = t^4 \cdot f(1/t).$$

So the resulting curves are double covers of $\mathbb{P}^1$, ramified at four points. For $n = 3$, one gets cubic equations.

4.4 Properties of the Linear System

Throughout this section, we let everything be as defined in Theorem 4.8, with the extra assumption that k is a perfect field. So far, we have proved that the linear system $|-nK_S|$ has a one-dimensional subsystem L, consisting of those curves that have multiplicity at least n in P. We are interested in the nature of the corresponding curves. In this section we will show that all the strict transforms of these curves on the blowup S' are connected, and that an open dense subset of the curves is smooth. First, we will prove that the base locus of L consists solely of the point P, and that the base locus of $|nF_0'|$ is empty. Then we will apply the following theorem.

Theorem 4.16. Let X be a smooth projective variety over an algebraically closed field k of characteristic 0. Let M be a linear system with base locus Σ. Then there is an open dense subset of curves in M that are smooth at all points in $X - \Sigma$.

Proof. This is Bertini's theorem. See [Har77], Corollary III.10.9 and Remark III.10.9.2. □

Proposition 4.17. Write L for the linear system obtained in Theorem 4.8. Then we have:

(1). The base locus Σ of L is just the point P;

(2). If $D_1, D_2 \in L$, then $D_1' \cdot D_2' = 0$;

(3). The base locus of $|nF_0'|$ is empty.

Proof. First of all, note that P is contained in the base locus of L. Now assume that $\dim \Sigma \geq 1$. We know that nF_0 is one of the element of L, so $\Sigma \subset F_0$, which implies $\dim \Sigma = 1$. Since F_0 is irreducible, we get $\Sigma = F_0$. For $D \in L$, this now implies that $D - F_0 \gg 0$, since $F_0 \subset \operatorname{supp} D$. Thus we get an injective map

$$\begin{aligned} L &\to M, \\ D &\mapsto D - F_0, \end{aligned}$$

where $M := \{D \in |-nK_S - F_0| : \mu_P(D) \geq n-1\}$. For $D \in L$ we have $\mu_P(D - F_0) = \mu_P(D) - \mu_P(F_0) \geq n-1$, so indeed the image is contained in M. We get $\dim M \geq \dim L = 1$. Since $F_0 \in |-K_S|$, we have $M = \{D \in |-(n-1)K_S| : \mu_P(D) \geq n\}$. This set is contained in $\{D \in |-(n-1)K_S| : \mu_P(D) \geq n-1\}$, and this set is the projective space associated to the vector space $V_{n-1,n-1}$. By Theorem 4.8, this vector space has dimension 1, so $\dim M = 0$, and we arrive at a contradiction. We conclude that $\dim \Sigma = 0$.

Now let $D \in L$ such that $F_0 \not\subset D$. Then nF_0 and D are two elements of L that have no common irreducible component, so we get

$$n^2 = (-nK_S)^2 = nF_0 \cdot D \geq \sum_{Q \in nF_0 \cap D} \mu_Q(nF_0) \cdot \mu_Q(D) \geq \mu_P(nF_0) \cdot \mu_P(D) = n^2.$$

We conclude that $nF_0 \cap D = \{P\}$, so the base locus of L, which is contained in this intersection, consists only of the point P. Note that $\pi^*(D) = D + nE$ by Lemma 3.26 and Lemma 3.20, so we get, using Lemma 3.18

$$\begin{aligned} nF_0' \cdot D' &= (\pi^*(nF_0) - nE) \cdot (\pi^*(D) - nE) \\ &= nF_0 \cdot D - nE(\pi^*(nF_0) + \pi^*(D)) + n^2 E^2 \\ &= n^2 - 0 - n^2 = 0. \end{aligned}$$

Now, since the base locus of $\{\pi^*(D) : D \in L\}$ is equal to the inverse image of the base locus of L, we find that it equals E. Thus, the base locus B of $\{\pi^*(D) - nE : D \in L\}$ must be contained in E. If its dimension is 1, it must be equal to E, but since E is not contained in $\pi^*(nF_0) - nE$, we find $\dim B = 0$, so it is a finite set of points. Since nF_0 and D have no common irreducible component, and E is not a component of nF_0', the strict transforms nF_0' and D' have no common component. Above we showed $nF_0' \cdot D' = 0$, so they have no common points, and the base locus is empty. □

Combining the last results, we immediately obtain the following corollary.

Corollary 4.18. There is an open dense subset of curves in $|nF_0'|$ that are smooth.

Our next goal is to show that the strict transforms are connected. Because the linear system $|nF_0'|$ has no base points, it corresponds to a projective morphism $f\colon S' \to \mathbb{P}^1$, such that the fibers of f are the elements of $|nF_0'|$. By Stein factorization (see [Har77], Corollary III.11.5), the morphism factors as $f = g \circ f'$, where $C = \mathbf{Spec} f_* O_{S'}$, the morphism $f'\colon S' \to C$ is projective with connected fibers, and $g\colon C \to \mathbb{P}^1$ is a finite morphism. We fix C, f' and g for the remainder of this section. Our first step will be to show that C is isomorphic to $\mathbb{P}^1$. Then we will show that g is in fact a map of degree 1 to conclude that f has connected fibers. To prove the first step, we will use the following lemma.

Lemma 4.19. Assume we have the following commutative diagram of abelian groups

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"] & B \arrow[r, "\beta"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] & C'. & \end{tikzcd}$$

Assume the rows are exact, f is surjective, and g is injective. Then h is injective.

Proof. Let $c \in C$ be such that $h(c) = 0$. By exactness, there is $b \in B$ such that $\beta(b) = c$. Now $\beta'(g(b)) = h(\beta(b)) = 0$, so $g(b) \in \ker \beta' = \operatorname{im} \alpha'$. Let $a' \in A'$ map to $g(b)$. Since f is surjective, we obtain $a \in A$ with $f(a) = a'$. Now $g(\alpha(a)) = \alpha'(f(a)) = g(b)$. By injectivity of g, this yields $\alpha(a) = b$. By exactness of the upper row, we get $c = \beta(\alpha(a)) = 0$, so h is injective. □

We will use this lemma to prove that $H^1(C, O_C) = 0$, but first we state a theorem by Serre.

Theorem 4.20. Let X be a noetherian scheme. Then the following conditions are equivalent:

(i) X is affine;

(ii) $H^i(X, \mathcal{F}) = 0$ for all $\mathcal{F}$ quasi-coherent and $i > 0$;

(iii) $H^1(X, \mathcal{I}) = 0$ for all coherent sheaves of ideals $\mathcal{I}$.

Proof. [Har77], Theorem III.3.7. □

Proposition 4.21. Let $\phi\colon X \to Z$ be a morphism from a surface to a variety of dimension 1. Then there is an injection $H^1(Z, \phi_* O_X) \to H^1(X, O_X)$.

Proof. Write $\mathcal{F} = \phi_* O_X$. Take $U, V \subset Z$ open affine such that $Z = U \cup V$. Define $U' = \phi^{-1}(U)$ and $V' = \phi^{-1}(V)$. We obtain the commutative diagram below. Here the rows are part of the Mayer-Vietoris sequence, for which we refer to [Dan96], I.4.4.

$$\begin{tikzcd} H^0(U, \mathcal{F}|_U) \oplus H^0(V, \mathcal{F}|_V) \arrow[r] \arrow[d] & H^0(U \cap V, \mathcal{F}|_{U\cap V}) \arrow[r] \arrow[d] & H^1(Z, \mathcal{F}) \arrow[r] \arrow[d] & 0 \\ H^0(U', O_X|_{U'}) \oplus H^0(V', O_X|_{V'}) \arrow[r] & H^0(U' \cap V', O_X|_{U'\cap V'}) \arrow[r] & H^1(X, O_X) & \end{tikzcd}$$

Because U, V are affine, and $\mathcal{F}$ is quasi-coherent, we have

$$H^1(U, \mathcal{F}) \ = \ 0 \ = \ H^1(V, \mathcal{F}).$$

So the upper row, which would in general end in $H^1(U, \mathcal{F}|_U) \oplus H^1(V, \mathcal{F}|_V)$, ends in a 0 by the previous theorem. By definition of the direct image, the left two vertical maps are isomorphisms. Applying Lemma 4.19 to the diagram, we get that the map $H^1(Z, \mathcal{F}) \to H^1(X, O_X)$ is injective. □

Because $C = \mathbf{Spec} f_* O_{S'}$, we have $O_C \cong f_* O_{S'}$. Because the composition f is surjective, so is g, so $\dim C = \dim \mathbb{P}^1 = 1$. Now Lemma 4.21 yields an injection $H^1(C, O_C) \to H^1(S', O_{S'})$. Because S' is the blowup of a del Pezzo surface, Lemma 4.12 and Lemma 4.11 yield that $H^1(S', O_{S'}) = 0$, so $H^1(C, O_C) = 0$. To conclude that $C \cong \mathbb{P}^1$, we are left to show that C is smooth. We will show that it is equal to its normalization. For this we use two lemmas from the Stacks Project.

Lemma 4.22. Let k be a field. Let X be a proper scheme of dimension at most 1 over k. If $X' \to X$ is a birational proper morphism, then

$$\dim_k H^1(X, O_X) \geq \dim_k H^1(X', O_{X'}).$$

If X is reduced, $H^0(X, O_X) \to H^0(X', O_{X'})$ is surjective, and equality holds, then $X' = X$.

Proof. [Sta18, Tag 0CE0,] Lemma 49.18.4. □

Lemma 4.23. Let k be a field, and let Z be a proper curve over k. Set $\kappa := H^0(Z, O_Z)$. Then

$$[\kappa : k]_s \dim_\kappa H^1(Z, O_Z) \geq p_g(Z).$$

Proof. [Sta18, Tag 0CE0,] Lemma 49.18.5. □

Proposition 4.24. The curve C is isomorphic to $\mathbb{P}^1$. The finite morphism $g \colon C \to \mathbb{P}^1$ is of degree 1.

Proof. Again we use the normalization C' of C, which is smooth because k is perfect. We have a birational proper morphism $C' \to C$, so by Lemma 4.22 we get $H^1(C', O_{C'}) = 0$, since we have $H^1(C, O_C) = 0$. Since S' is normal, and C' is the normalization of C, the morphism f' factors through C'. Because S' contains k-points, so does C'. By Lemma 4.23, we get $p_g(C') = 0$, and since C' is smooth and has a k-point, this implies $C' \cong \mathbb{P}^1$. Now we note that C is reduced as it is the image of a reduced scheme, $H^0(C, O_C) \to H^0(C', O_{C'})$ is surjective, and $H^1(C, O_C) = 0 = H^1(C', O_{C'})$, so we apply Lemma 4.22 again to conclude $C = C' \cong \mathbb{P}^1$. Recall that the map g is finite, and that f' has connected fibers. Write m for the degree of g. Since f' is a morphism to C, it corresponds with a linear system M without base points. Let F be a fiber of f'. Because all fibers are pullbacks of points on C, these fibers are all linearly equivalent. Any fiber of f is a union of m fibers of f', and because all these fibers are linearly equivalent, a fiber of f is linearly equivalent to mF. In particular we have $nF'_0 \sim mF$. Recall that $\operatorname{Pic} S' \cong \operatorname{Pic} S \oplus \mathbb{Z}$ by Lemma

3.18, and F_0' corresponds to $(F_0, -1)$ in this group, so it is a primitive element. Thus the fact that mF and nF_0' are linearly equivalent implies that $F \sim dF_0'$ for $d = \frac{n}{m}$. But as shown in Theorem 4.8, the system $|dF_0'|$ has dimension 0 for $d < n$. We conclude that $d = n$, and thus $m = 1$. $\square$

Combining the results in this section, we can now prove the following theorem.

Theorem 4.25. The linear system $|nF_0'|$ induces a morphism $f : \mathrm{Bl}_P(S) \to \mathbb{P}^1$ with connected fibers of arithmetic genus 1. In characteristic zero, an open dense subset of the fibers is smooth.

Proof. We know that the morphism f' has connected fibers, and $f = g \circ f'$. Since g is of degree 1, every fiber of $g \circ f'$ is just a fiber of f', hence connected. We conclude that f has connected fibers. Recall that Lemma 3.26 implies that the genus of these curves equals 1. Corollary 4.18 implies that in characteristic zero an open dense subset of the fibers of f is smooth. $\square$

So in characteristic zero almost all elements of $|nF_0'|$ are irreducible curves of genus 1, of which those containing a k-point are elliptic curves over k.

5 A Result on Density of the Rational Points

Throughout this section let S be a del Pezzo surface of degree 1 over $\mathbb{Q}$, and let $\rho\colon \mathrm{Bl}_O(S) \to \mathbb{P}^1$ be the morphism associated to the anticanonical divisor. Set $n \geq 2$, and assume $P \in S(\mathbb{Q})$ is a point that has order n on its fiber under ρ. Let $\phi\colon \mathrm{Bl}_P(S) \to \mathbb{P}^1$ be the morphism associated to the linear system $|nF_0'|$ in Theorem 4.8. We consider the blowup $\mathrm{Bl}_{O,P}(S)$ in both points, and abusing notation, we write ρ and ϕ for the obtained morphisms from this blowup to $\mathbb{P}^1$. We will prove the following theorem.

Theorem 5.1. If ϕ has a smooth fiber that contains infinitely many rational points, then the rational points lie dense in $\mathrm{Bl}_{O,P}(S)$.

Before we prove this we state some preliminary results on elliptic curves.

Lemma 5.2. Let E be an elliptic curve over a field k, and let $m \in \mathbb{Z}$. Write $E[m]$ for the set of m-torsion points of E over an algebraic closure of k. If $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p$ and $p \nmid m$, then $E[m] \cong (\mathbb{Z}/m\mathbb{Z})^2$.

Proof. [Sil09], Corollary III.6.4. $\square$

Theorem 5.3. Let E be an elliptic curve over $\mathbb{Q}$. Then the torsion subgroup of $E(\mathbb{Q})$ is isomorphic to one of the following groups

$$\mathbb{Z}/N\mathbb{Z} \text{ with } 1 \leq N \leq 10 \text{ or } N = 12,$$
$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2N\mathbb{Z} \text{ with } 1 \leq N \leq 4.$$

Proof. This is Mazur's theorem. See [Sil09], Theorem VIII.7.5. $\square$

Recall that the affine part of S can be given by an equation

$$y^2 + c_1(s)xy + c_3(s)y = x^3 + c_2(s)x^2 + c_4(s)x + c_6(s).$$

Consider the generic fiber of the morphism ρ in $\mathrm{Bl}_{O,P}(S)$, which is the fiber of ρ, defined over the function field $\mathbb{Q}(s)$ of $\mathbb{P}^1$. Following the notation of [Sil09], Exercise III.3,7, we obtain polynomials $\phi_m, \psi_m, \omega_m \in \mathbb{Q}[s, x, y]$ such that for any $s_0 \in \mathbb{Q}$ such that its fiber is smooth, and any point $Q = (x_Q, y_Q, s_0)$ the multiple $m[Q]$ on its fiber under ρ is given by

$$\left(\frac{\phi_m(Q)}{\psi_m(Q)^2}, \frac{\omega_m(Q)}{\psi_m(Q)^3}\right).$$

Thus the point Q is m-torsion on its fiber under ρ if and only if $m[Q]$ is the point at infinity, so if and only if $\psi_m(Q) = 0$. As mentioned in the exercise, the polynomial ψ_m is independent of y, so we may actually view it as an element in $\mathbb{Q}[s, x]$.

Definition 5.4. For $m \in \mathbb{Z}_{\geq 1}$, we define $Z_m \subset \mathrm{Bl}_{O,P}(S)$ to be the zero locus of the polynomial ψ_m.

Now for every $s_0 \in \mathbb{Q}$ the x-coordinates of the m-torsion points on the fiber above s_0 are precisely the roots of $\psi_m(s_0) \in \mathbb{Q}[x]$. Accordingly, we find that for a point $Q = (x_Q, y_Q, s_Q) \in \mathrm{Bl}_{O,P}(S)$ on a smooth fiber, Q is m-torsion if and only if it is a root of $\psi_m \in \mathbb{Q}[s, x]$.

Remark 5.5. If we intersect Z_m with a smooth fiber of ρ, we obtain exactly the m-torsion points on this fiber. Since the group of m-torsion points is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^2$, we find that Z_m is a closed curve of degree m^2 over $\mathbb{P}^1$, which possibly contains components of singular fibers of ρ.

We are now ready to prove the theorem.

Proof of Theorem 5.1. Write F for the fiber of ϕ that contains infinitely many rational points. Note that each fiber R of ρ can contain only finitely many of these points: assume $R \cap F$ has infinitely many points. Then we must have $\dim(R \cap F) = 1$, so they have an irreducible component in common. Since F is smooth and connected, it is irreducible, so we have $F \subset R$. But then the image of F on $\mathrm{Bl}_O(S)$ is contained in the image of R. In particular, P is contained in the image of R with multiplicity at least n. Since P lies on a smooth fiber, this is a contradiction. Since the singular fibers of ρ form a closed strict subset, and each of those fibers contains at most finitely many rational point, we can conclude that infinitely many of the rational points on F lie on a smooth fiber of ρ.

Now we have infinitely many rational points on F that lie on smooth fibers of ρ, but cannot lie in a finite union of fibers of ρ. Using Lemma 2.2, it suffices to show that only finitely many of these points have finite order on their fiber. By Mazur's theorem it suffices to show that for $m \in \{1, \ldots, 10\} \cup \{12\}$ the curves Z_m and F have only finitely many points in common. This is immediate, unless the curves have a common irreducible component. As mentioned above F is irreducible, so this would imply that $F \subset Z_m$. We will show that this can not happen. Since F is one of the fibers of ϕ, its image on $\mathrm{Bl}_O(S)$ intersects P with multiplicity $n \geq 2$. The image of Z_m on $\mathrm{Bl}_O(S)$ intersects a smooth fiber of ρ precisely in the m-torsion points, and there are m^2 such points on a fiber. Because of this, the multiplicity on Z_m of each of these points must be 1. In particular, since P lies on a smooth fiber, its multiplicity on the image of Z_m is at most 1. Since P has multiplicity n on the image of F, we conclude that $F \not\subset Z_m$, so F only contains finitely many m-torsion points. Combining this for all m, we conclude that F contains only finitely many points that are torsion on their fiber of ρ, which is what we wanted to show. $\square$

Remark 5.6. To conclude density of the rational points on a del Pezzo surface over $\mathbb{Q}$ of degree 1 with a $\mathbb{Q}$-torsion point P, one can apply Theorem 5.1 as follows. Search for any other rational point Q, not equal to O. If Q lies on a smooth fiber of ϕ in $\mathrm{Bl}_{O,P}(S)$, then this fiber is an elliptic curve over $\mathbb{Q}$ with Q as its zero, and if it has positive rank, we are already done. Besides Q, the fiber contains another $\mathbb{Q}$-point, namely the sum of points on the fiber that lie above P. If this point has infinite order, this proves the density. If the rational points are indeed Zariski dense, one has to get rather unlucky for these points not to satisfy these conditions, since the singular fibers are contained in some closed strict subset of the blowup in O and P, and all the torsion points are contained in a finite union of the curves Z_m.

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