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Invariant Distributions for a Generalization of Wright's delay equation

Master's thesis

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Contents

1 Introduction

Non-linear delayed differential equations have been extensively studied since the nineteen-fifties. In particular, the work of E.M. Wright has proved to be of significance in this field. As a particular example of his results, he considered equations of the form

$$y'(t) = -\alpha y(t-1)(1+y(t)) \tag{1.1}$$

for some $\alpha > 0$, [12]. This differential equation arises from a transformation of the logistic differential equation with delay, $x'(t) = \alpha x(t)(1 - x(t-1))$, by the translation $y(t) = x(t) - 1$. In his paper, Wright derives several properties of the solutions of equation (1.1). Most importantly, for a solution with $y(0) > -1$, it can be shown that $y(t) > -1$ for all $t \geq 0$ and that y is bounded from above. Under certain conditions, this property can be extended to a larger family of stochastic delay differential equations such as

$$dy(t) = -\alpha y(t-1)(1+y(t))dt + \sigma(y_t)dW(t). \tag{1.2}$$

Here $(W(t))_{t\geq 0}$ denotes a standard Brownian motion, y_t denotes the segment of y on $[t-1,t]$ and σ is a function from $C[-1,0]$ to $\mathbb{R}$. If one assumes that $\sigma(f) = (1+f(0))h(f)$ for all $f \in C[-1,0]$, and $h : C[-1,0] \to \mathbb{R}$ is bounded, then for all solutions y of (1.2) with $y(0) > -1$, we have $y(t) > -1$ for all $t \geq 0$ almost surely, [11]. By considering functions $F : \mathbb{R} \to \mathbb{R}$ that are bounded from below and positive on $[0,\infty)$, and $g : \mathbb{R} \to \mathbb{R}$ that vanish at $x = -1$ and are strictly positive for $x > -1$, we are able to generalize this result to differential equations of the form

$$dy(t) = -F(y(t-1))g(y(t))dt + \sigma(y_t)dW(t) + h(y_{t-})dP(t). \tag{1.3}$$

Here σ and h are from a certain family of real-valued functions with domain $D[-1,0]$ and $D[-1,0)$ respectively, and $(P(t))_{t\geq 0}$ is a jump process. We are able to show that solutions of equation (1.3) also bounded from above if the function h and the process P are of opposite sign.

Of particular interest in stochastic differential equations are so called invariant distributions. These are distributions for the initial conditions such that the distribution of the solution at time t is the same as the distribution of the initial condition, for every time $t \geq 0$. One can use the boundedness of solutions to construct invariant distributions. Namely, if all solutions y of

$$dy(t) = \left(\int_{[-1,0]} y(t+s)\mu(ds)\right)dt + G(y)(t)dW(t) \tag{1.4}$$

are bounded in probability, then there exists an invariant distribution of (1.4), [7]. Motivated by these results, it is natural to look for invariant distributions of equation (1.3). The main result is a theorem establishing the existence of an invariant distribution of the equation

$$dy(t) = -F(y(t-1))g(y(t))dt + h(y_{t-})dP(t). \tag{1.5}$$

The construction of this distribution requires the set of segments of the solutions to be tight. The proof of this uses the fact that under certain conditions, the solutions of (1.5) are bounded in probability. Furthermore, some continuity of the solution with respect to the initial value is required. This can be ensured by imposing conditions on the functions F, g, h and the jump process $(P(t))_{t\geq 0}$.

2 Boundedness of solutions

In this chapter, we study the boundedness of solutions of

$$dy(t) = -F(y(t-1))g(y(t))dt + \sigma(y_t)dW(t). \tag{2.1}$$

The existence of solutions, under certain conditions, is guaranteed by Theorem B.1. We bound solutions of equation (2.1) both from below and from above. To prove the boundedness from below, we assume that $g(-1) = 0$, and that $\sigma(y_t)$ contains a factor $g(y(t))$. To bound solutions from above, we assume that F provides a negative feedback. In other words, if y gets large enough, $F(y(t-1))$ also becomes large, which in turn makes the term $-F(y(t-1))g(y(t))$ negative. This gives us conditions to ensure boundedness from above in probability.

2.1 Bounded from below

In this section, we derive conditions that bound solutions of equation (2.1) from above. Let $F : \mathbb{R} \to \mathbb{R}$ be continuous. Equip $C[-1,0]$ with the supremum norm, and let $\sigma : C[-1,0] \times [0,\infty) \to \mathbb{R}$ be a continuous function with respect to this norm. $g : \mathbb{R} \to \mathbb{R}$ is assumed to be continuously differentiable with $g(-1) = 0$, and $g(x) > 0$ for all $x > -1$. Define the function $\mathscr{G} : (-1,\infty) \to \mathbb{R}$ by $\mathscr{G}(x) = \int_0^x g(s)^{-1}ds$. Assume that σ can be written as $\sigma(y_t, t) = g(y(t))\phi(y_t, t)$ for some continuous $\phi : C[-1,0] \times [0,\infty) \to \mathbb{R}$.

Under some mild conditions, we then are able to prove that $y(0) > -1$ implies that $y(t) > -1$ for all $t \geq 0$. This is stated in the following theorem, which is based on Lemma 2.1 in [11].

Theorem 1. *Let $(y(t))_{t\geq 0}$ satisfy equation (2.1), with $y(0) > -1$ almost surely, and assume that $\lim_{x\searrow -1} \mathscr{G}(x) = -\infty$. Furthermore, assume that ϕ is bounded. Then $y(t) > -1$ for all $t \geq 0$ almost surely.*

Proof. Define

$$\tau = \sup\{t \geq 0 : y(s) > -1 \text{ for all } s \in [0,t]\}.$$

Define $z(t) = \mathscr{G}(y(t))$ for $t \in [0, \tau)$. Then, by using Itô's formula, we see that

$$\begin{aligned} dz(t) &= g(y(t))^{-1}(-F(y(t-1))g(y(t))dt + g(y(t))\phi(y_t, t)dW(t)) \\ &\quad - \frac{1}{2}\frac{g'(y(t))}{g(y(t))^2}(g(y(t))\phi(y_t, t))^2 dt \\ &= -F(y(t-1))dt - \frac{1}{2}g'(y(t))\phi(y_t, t)^2 dt + \phi(y_t, t)dW(t). \end{aligned}$$

Hence we find

$$z(t) = z(0) - \int_0^t F(y(s-1)) + \frac{1}{2}g'(y(s))\phi(y_s, s)^2 ds + \int_0^t \phi(y_s, s)dW(s). \tag{2.2}$$

Assume that $\tau < \infty$. Then $\lim_{t\to\tau} z(t) = -\infty$ by the assumption on $\mathscr{G}$. However, since ϕ is bounded, the right-hand side of (2.2) is bounded as $t \to \tau$. Therefore $\tau = \infty$ must hold. $\square$

An immediate family of functions that satisfy these conditions is given by polynomials of the form $g(x) = (1+x)^\gamma$, for some $\gamma \geq 1$, as stated by the following corollary.

Corollary 2. *Let $\gamma \geq 1$. Assume that ϕ is bounded, and that $(y(t))_{t\geq 0}$ satisfies*

$$dy(t) = -F(y(t-1))(1+y(t))^\gamma dt + (1+y(t))^\gamma \phi(y_t, t)dW(t),$$

with $y(0) > -1$ almost surely. Then $y(t) > -1$ for all $t \geq 0$ almost surely.

Proof. This follows readily from Theorem 1 and the fact that for $\gamma \geq 1$, we know that $\lim_{x\searrow -1} \int_0^x (1+s)^{-\gamma} ds = -\infty$. $\square$

We can use this result to apply a transformation to the solution of equation (2.1). The transformation is a generalization of an approach used on solutions of y satisfying equation (1.2) [11]. After showing that $y(0) > -1$ implies that $y(t) > -1$ for all $t \geq 0$, the transformation $z(t) = \log(1 + y(t))$ is applied to y. This is done in order to show that solutions are bounded from above. A similar transformation can be applied to solutions of equation (2.1). Under the conditions stated in Theorem 1, solutions remain strictly larger than -1. Hence we can define the transformation $z(t) = \mathscr{G}(y(t))$ for all $t \geq 0$. Using Itô's formula, we see that z satisfies

$$dz(t) = -\widetilde{F}(z(t-1))dt - \frac{1}{2}\widetilde{g}'(z(t))\widetilde{\phi}(z_t, t)^2 dt + \widetilde{\phi}(z_t, t)dW(t), \tag{2.3}$$

where $\widetilde{F}(x) = F(\mathscr{G}^{-1}(x))$, $\widetilde{g}'(x) = g'(\mathscr{G}^{-1}(x))$ and $\widetilde{\phi}(\psi, t) = \phi(t \mapsto \mathscr{G}^{-1}(\psi(t)), t)$. Note that the function $\mathscr{G}$ is indeed invertible on $(-1, \infty)$, since $\mathscr{G}'(x) = g(x)^{-1} > 0$ for all $x > -1$.

We write equation (2.3) more compactly as

$$dz(t) = -F(z(t-1))dt + dU(t) \tag{2.4}$$

for a continuous stochastic process U. In this case, U satisfies

$$dU(t) = -\frac{1}{2}\widetilde{g}'(z(t))\widetilde{\phi}(z_t, t)^2 dt + \widetilde{\phi}(z_t, t)dW(t), \tag{2.5}$$

with $U(0) = 0$. Solutions of equation (2.3) and equation (2.4) are not obvious. However, one can simplify the equation by considering linear g and a constant ϕ. These simplifications are discussed in section 2.3.

2.2 Bounded from above

In this section, the boundedness from above of solution of (2.4) in probability is established. It is induced by the so called negative feedback. In equation (1.2), this is given by the term $\alpha y(t-1)$. To generalize this feedback, it is natural to consider an $F : \mathbb{R} \to \mathbb{R}$ that is positive whenever $x \geq 0$. This makes sure that the drift term becomes negative if a solution has been positive for a whole interval of length one. To make sure that a solution does not grow too much whenever it has been negative in the past interval of length one, we need boundedness from below. We therefore use the following definition of feedback.

Definition 3. A function $F : \mathbb{R} \to \mathbb{R}$ is said to have *feedback* if F is bounded from below, and $F(x) \geq 0$ for all $x \geq 0$.

The function $\widetilde{F}$ in equation (2.3) still has negative feedback, as the following proposition shows.

Proposition 4. *If $F : \mathbb{R} \to \mathbb{R}$ has feedback, then $\widetilde{F}$ also has feedback.*

Proof. Since F is bounded from below, $\widetilde{F}$ is also bounded from below. Now assume that $x \geq 0$. Then we have $0 \leq x = \int_0^{\mathscr{G}^{-1}(x)} g(s)^{-1} ds$, and since $g(s) \geq 0$ for $s > -1$, we must have $\mathscr{G}^{-1}(x) \geq 0$. Hence $\widetilde{F}(x) = F(\mathscr{G}^{-1}(x)) \geq 0$, since F has feedback. □

It turns out that this mild condition on F is enough to give a criterion for boundedness in probability of solutions. The following is based on Lemma 3.1 in [11].

Theorem 5. *Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and with feedback. Suppose z is a solution of*

$$dz(t) = -F(z(t-1))dt + dU(t)$$

with $z(t) = z^0(t)$ for $t \in [-1, 0]$. Then for all $M \geq 0$ and all $t \geq 0$ we have

$$z(t) \leq C + \max\{\sup_{s\in[0,t-1]} (U(t) - U(s) - (t-s+1)F(M)), \sup_{s\in[t-1,t]} (U(t) - U(s))\}$$

almost surely, with $C = A_0 + M + \sup_{r\in\mathbb{R}} -F(r)$ and

$$A_0 = \max\{z^0(0) + \sup_{t\in[0,1]} \left(\int_0^t -F(z^0(s-1))ds\right), 0\}.$$

Proof. Let τ_z be the maximal existence time of the solution z. The solution z satisfies

$$z(t) = z(t_0) - \int_{t_0}^t F(z(s-1))ds + U(t) - U(t_0)$$

for $t \in [t_0, \tau_z)$. Let $M \in \mathbb{R}_+$ and define $t_0 := 0$. Define inductively

$$\begin{aligned} s_{n+1} &:= \inf\{t \geq t_n : z(s) > M \text{ for all s } \in [t-1, t]\}, \\ t_{n+1} &:= \inf\{t \geq s_{n+1} : z(t) \leq M\}. \end{aligned}$$

If $t \leq 1$, then we have

$$\begin{aligned} z(t) &= z(0) - \int_0^t F((z(\theta-1)))d\theta + U(t) - U(t_0) \\ &\leq A_0 + U(t) - U(t_0). \end{aligned}$$

Otherwise, if $t \in [t_n, s_{n+1})$, then there is an $s \in [t-1, t]$ such that $z(s) \leq M$. Since F is bounded from below, we get

$$\begin{aligned} z(t) &= z(s) - \int_s^t F(z(\theta-1))d\theta + U(t) - U(s) \\ &\leq C + U(t) - U(s) \end{aligned}$$

with $C = A_0 + M - \inf_{r\in\mathbb{R}} F(r)$. Since z is continuous, we let $t \nearrow s_{n+1}$ and see that

$$z(s_{n+1}) \leq C + U(s_{n+1}) - U(s_{n+1} - 1).$$

If $t \in [s_{n+1}, t_{n+1})$, then $z(\theta) \geq M$ for all $\theta \in [s_{n+1} - 1, t]$, so

$$\begin{aligned} z(t) &= z(s_{n+1}) - \int_{s_{n+1}}^t F(z(\theta-1))d\theta + U(t) - U(s_{n+1}) \\ &\leq C + U(s_{n+1}) - U(s_{n+1}-1) + U(t) - U(s_{n+1}) - (t - s_{n+1})F(M) \\ &= C + U(t) - U(s_{n+1}-1) - (t - (s_{n+1}-1) + 1)F(M). \end{aligned}$$

Since $s_{n+1} - 1 \in [0, t-1]$, we get

$$z(t) \leq C + \sup_{s\in[0,t-1]} (U(t) - U(s) - (t-s+1)F(M)).$$

We conclude that for all $t \geq 1$ we have

$$z(t) \leq C + \max\{\sup_{s\in[0,t-1]} (U(t) - U(s) - (t-s+1)F(M)), \sup_{s\in[t-1,t]} (U(t) - U(s))\},$$

which concludes the proof. □

Together with Theorem 1, Theorem 5 gives us a way to derive conditions on the differential equation (2.1) to ensure boundedness in probability of solutions.

2.3 Autonomous equations

We now return to equations (2.4) and (2.5). The estimate found in Theorem 5 can be bounded by $C + \sup_{s\in[0,t]} U(t) - U(s)$ for some deterministic $C \in \mathbb{R}$. With Theorem 5 in mind, we thus know that if $(\sup_{s\in[0,t]} U(t) - U(s))_{t\geq 0}$ is bounded in probability,

$z(t)$ is bounded above in probability as well. We can therefore reduce this problem to finding $\widetilde{g}$ and $\widetilde{\phi}$ such that for U satisfying (2.5), $(\sup_{s\in[0,t]} U(t) - U(s))_{t\geq 0}$ is bounded in probability.

The easiest choice is taking g linear and ϕ constant. This makes $\widetilde{g}'$ and $\widetilde{\phi}$ constant, so U is a Brownian motion with drift. However, it turns out that in this case, the process $(\sup_{s\in[0,t]} U(t) - U(s))_{t\geq 0}$ is not bounded in probability. To show this, we first derive an expression for the joint density of Brownian motion and its running minimum. The following lemma and proof is from [9].

Lemma 6. *Let $(W(t))_{t\geq 0}$ be standard Brownian motion, and $(S(t))_{t\geq 0}$ its running minimum: $S(t) := \inf_{s\in[0,t]} W(s)$. The joint density of $(W(t), S(t))$ is then given by*

$$(x, y) \mapsto 2\phi_t'(2y - x)\mathbb{1}(y \leq x \wedge 0),$$

where ϕ_t is the density function of $W(t)$.

Proof. Let $x, y \in \mathbb{R}$ such that $y \leq \min\{x, 0\}$. Define $\tau_y = \inf\{t \geq 0 : W(t) = y\}$. Define the reflected Brownian motion by

$$W'(t) := \begin{cases} W(t) & t \leq \tau_y, \\ 2y - W(t) & t \geq \tau_y. \end{cases}$$

Since τ_y is a stopping time, $(W'(t))_{t\geq 0}$ is also a Brownian motion.

The event $\{S(t) \leq y\}$ is equal to the event that $\{t \geq \tau_y\}$. Hence we can write

$$\mathbb{P}(W(t) \geq x, S_t \leq y) = \mathbb{P}(W'(t) \leq 2y - x, t \geq \tau_y).$$

We have $2y - x \leq y$, so if $W'(t) \leq 2y - x \leq y$, then we must have $t \geq \tau_y$. Therefore

$$\begin{aligned} \mathbb{P}(W(t) \geq x, S_t \leq y) &= \mathbb{P}(W'(t) \leq 2y - x) \\ &= \mathbb{P}(W(t) \leq 2y - x). \end{aligned}$$

It now easy to see that

$$\begin{aligned} \mathbb{P}(W(t) \leq x, S(t) \leq y) &= 1 - \mathbb{P}(W(t) \geq x, S(t) \leq y) - \mathbb{P}(S(t) > y) \\ &= 1 - \mathbb{P}(W(t) \leq 2y - x) - (1 - \mathbb{P}(S(t) \leq y)) \\ &= 2\mathbb{P}(W(t) \geq -y) - \mathbb{P}(W(t) \leq 2y - x). \end{aligned}$$

For other choices of x and y, the derivation is trivial. It follows that for any $x, y \in \mathbb{R}$ we have

$$\mathbb{P}(W(t) \leq x, S(t) \leq y) = \begin{cases} 2\mathbb{P}(W(t) \geq -y) - \mathbb{P}(W(t) \leq 2y - x) & y \leq x \wedge 0, \\ \mathbb{P}(W(t) \leq x) & \text{otherwise.} \end{cases}$$

Taking the derivative with respect to x and y now gives the stated density. □

We now derive a general expression for the cumulative distribution function of $(\sup_{s\in[0,t]} U(t) - U(s))_{t\geq 0}$ if $U(t) = W(t) + f(t)$, where f is twice differentiable.

Theorem 7. *Let* $f : [0, \infty) \to \mathbb{R}$ *be twice differentiable with* $f(0) = 0$*. Define* $U(t) = W(t) + f(t)$*. Then for any* $M \geq 0$*, we have*

$$\mathbb{P}(\sup_{s \in [0,t]} U(t) - U(s) \leq M) = \exp\left(-\frac{1}{2}\left[f(t)f'(t) - \int_0^t f(s)f''(s)ds\right]\right)$$
$$\mathbb{E}\left[\exp\left(W(t)f'(t) - \int_0^t f''(s)W(s)ds\right)\mathbb{1}(\sup_{s \in [0,t]} W(t) - W(s) \leq M)\right].$$

Proof. We use Girsanov's theorem (see Theorem B.2). Define $Z : [0, \infty) \to \mathbb{R}$ by

$$Z(t) = \exp\left(\int_0^t -f'(s)dW(s) - \frac{1}{2}\int_0^t f'(s)^2 ds\right).$$

Using stochastic integration by parts, one can write

$$Z(t) = \exp\left(\frac{1}{2}\left(f(t)f'(t) - \int_0^t f(s)f''(s)ds\right)\right)$$
$$\exp\left(-U(t)f'(t) + \int_0^t f''(s)U(s)ds\right).$$

Define the measure Q on $(\Omega, \mathcal{F})$ by $Q(A) = \mathbb{E}^{\mathbb{P}}(Z(t)\mathbb{1}(A))$ for all $A \in \mathcal{F}$. Since f is deterministic, we know that $\int_0^T |f'(t)|^2 dt < \infty$ for all $T \in [0, \infty)$ and that

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T |f'(s)|^2 ds\right)\right] < \infty.$$

Therefore, by Girsanov's theorem, $(U(t))_{t \geq 0}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, Q)$. For any $t \geq 0$, we write $\Delta(t) = \sup_{s \in [0,t]}(U(t) - U(s))$. Then we see that

$$\begin{aligned}\mathbb{P}(\Delta(t) \leq M) &= \mathbb{E}^Q(Z(t)^{-1}\mathbb{1}(\Delta(t) \leq M)) \\ &= \exp\left(-\frac{1}{2}\left[f(t)f'(t) - \int_0^t f(s)f''(s)ds\right]\right) \\ &\quad \mathbb{E}^Q\left[\exp\left(U(t)f'(t) - \int_0^t f''(s)U(s)ds\right)\mathbb{1}(\Delta(t) \leq M)\right]\end{aligned}$$

Since $(W(t))_{t \geq 0}$ is a standard Bronwian motion on $(\Omega, \mathcal{F}, Q)$, we have

$$\begin{aligned}&\mathbb{E}^Q\left[\exp\left(U(t)f'(t) - \int_0^t f''(s)U(s)ds\right)\mathbb{1}(\Delta(t) \leq M)\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\exp\left(W(t)f'(t) - \int_0^t f''(s)W(s)ds\right)\mathbb{1}(\Delta(t) \leq M)\right].\end{aligned}$$

The statement now follows from combining these equations. □

The expression for the distribution function derived above can be simplified if one considers a linear f, as is shown in the following corollary.

Corollary 8. *Let $\mu \in \mathbb{R}$ and $M \geq 0$ and let $(W(t))_{t\geq 0}$ be standard Brownian motion. Define the Brownian motion with drift $(U(t))_{t\geq 0}$ by $U(t) := W(t) + \mu t$. Then, for any $M \geq 0$,*

$$\mathbb{P}(\sup_{s\in[0,t]} U(t) - U(s) \leq M) = \Phi(Mt^{-1/2} - \mu t^{1/2}) - e^{2\mu M}\Phi(-Mt^{-1/2} - \mu t^{1/2}),$$

where Φ is the distribution function of a standard normal random variable.

Proof. We set $f(t) = \mu t$, so $f'(t) = \mu$ and $f''(t) = 0$. Then by Theorem 7 we have

$$\mathbb{P}(\sup_{s\in[0,t]} U(t) - U(s) \leq M) = e^{-\frac{1}{2}\mu^2 t}\mathbb{E}\left[e^{\mu W(t)}\mathbb{1}(\sup_{s\in[0,t]} W(t) - W(s) \leq M)\right].$$

Note that $\sup_{s\in[0,t]} W(t) - W(s) = W(t) - \inf_{s\in[0,t]} W(s)$. Hence

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}(e^{\mu W(t)}\mathbb{1}(\sup_{s\in[0,t]} W(t) - W(s) \leq M)) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\mu x}\mathbb{1}(x - y \leq M)2\phi_t'(2y - x)\mathbb{1}(y \leq x \wedge 0)dydx \\
&= e^{\frac{\mu^2 t}{2}}\left[\Phi(Mt^{-1/2} - \mu t^{1/2}) - e^{2\mu M}\Phi(-Mt^{-1/2} - \mu t^{1/2})\right].
\end{aligned}$$

Combining this with the expression we found earlier, we see that

$$\mathbb{P}(\sup_{s\in[0,t]} U(t) - U(s) \leq M) = \Phi(Mt^{-1/2} - \mu t^{1/2}) - e^{2\mu M}\Phi(-Mt^{-1/2} - \mu t^{1/2}),$$

which proves the statement. □

From this corollary it follows readily that $(\sup_{s\in[0,t]} U(t) - U(s))_{t\geq 0}$ is not bounded in probability.

Corollary 9. *Let $(W(t))_{t\geq 0}$ be standard Brownian motion and suppose that the process $(U(t))_{t\geq 0}$ satisfies*

$$dU(t) = \mu dt + \sigma dW(t),$$

for some $\mu \in \mathbb{R}$ and $\sigma > 0$. Then the process $(\sup_{s\in[0,t]} U(t) - U(s))_{t\geq 0}$ is not bounded in probability.

Proof. Without loss of generality, we may assume $\sigma = 1$. Fixing $M \geq 0$ and taking the limit $t \to \infty$, we see that

$$\begin{aligned}
\lim_{t\to\infty}\mathbb{P}(\sup_{s\in[0,t]} U(t) - U(s) \leq M) &= \lim_{t\to\infty} \Phi(Mt^{-1/2} - \mu t^{1/2}) - e^{2\mu M}\Phi(-Mt^{-1/2} - \mu t^{1/2}) \\
&= 0.
\end{aligned}$$

Hence $(\sup_{s\in[0,t]} U(t) - U(s))_{t\geq 0}$ cannot be bounded in probability. □

Since Theorem 5 only gives feedback that is linear in time, it is clear that it is not strong enough to show boundedness in probability in the case of Brownian motion with drift. Processes U that satisfy (2.5) and such that $(\sup_{s\in[0,t]} U(t) - U(s))_{t\geq 0}$ is bounded in probability are not straightforward to derive.

It is therefore natural to consider for which f the process

$$\sup_{s\in[0,t]} (W(t) + f(t) - W(s) - f(s))$$

is bounded in probability. We see from Theorem 7 that this reduces to evaluating

$$\mathbb{E}\left[\exp\left(W(t)f'(t) - \int_0^t f''(s)W(s)ds\right)\mathbb{1}(\sup_{s\in[0,t]} W(t) - W(s) \leq M)\right]. \tag{2.6}$$

It is clear that only in the case of linear drift this expectation is easily evaluated, since the integral $\int_0^t f''(s)W(s)ds$ does not contribute to the expectation. When f is not linear, $f''(t)$ is not identically zero. In order to calculate this expectation by elementary methods, it is necessary to derive an expression for the joint distribution of $W(t)$, $\int_0^t f''(s)W(s)ds$ and $\sup_{s\in[0,t]} -W(s)$. There is no known literature about these joint distributions in general.

It is natural to consider f for which $\int_0^t f''(s)W(s)ds$ simplifies to a workable expression. For an f that is twice-differentiable f, this is not possible. However, it indicates more general functions f for which the expectation might be more readily evaluated. In particular, one may examine f for which the second derivative vanishes almost everywhere. Two obvious candidates for f are a piecewise linear function and an approximately piecewise constant function. It is the former that we discuss in the next section.

Theorem 10. *Let $\widetilde{f} : \mathbb{N} \to \mathbb{R}$ be any function. Define*

$$f(t) = \sum_{i=0}^{\lfloor t\rfloor -1} \widetilde{f}(i) + \widetilde{f}(\lfloor t\rfloor)(t - \lfloor t\rfloor)$$

and $U(t) := W(t) - f(t)$. Then we have

$$\begin{aligned}\mathbb{P}(&\sup_{s\in[0,t]} U(t) - U(s) \leq M) \\ &= \exp\left(-\frac{1}{2}\sum_{i=0}^{\lfloor t\rfloor -1} \widetilde{f}(i)^2\right) \\ &\quad \mathbb{E}\left[\exp\left(-\sum_{i=0}^{\lfloor t\rfloor -1} \widetilde{f}(i)(W(i+1) - W(i))\right)\mathbb{1}(\sup_{s\in[0,t]} W(t) - W(s) \leq M)\right].\end{aligned} \tag{2.7}$$

Proof. This follows by applying Girsanov's theorem in the same manner as in Theorem 7. □

To evaluate the last expectation in equation (2.7), one is inclined to want to derive the joint distribution of $(W(0), W(1), \ldots, W(n), \sup_{s\in[0,n]} -W(s))$. Similar to the joint distribution of $W(t)$, $\int_0^t f''(s)W(s)ds$ and $\sup_{s\in[0,t]} -W(s)$, how to find a tractable expression is not clear.

2.4 Non-autonomous equations

As can be seen in the previous section, the autonomous case does not give an easily derivable stochastic process U that satisfies equation (2.5) such that $(\sup_{s\in[0,t]} U(t) - U(s))_{t\geq 0}$ is bounded in probability. However, we can find such a process by looking at non-autonomous equations.

Proposition 11. *Let $(W(t))_{t\geq 0}$ be standard Brownian motion and let $\widetilde{\phi} : [0,\infty) \to \mathbb{R}$ be continuously differentiable such that*

$$\sup_{s\geq 1} |\widetilde{\phi}(s)|s^{\beta} + |\widetilde{\phi}'(s)|s^{\gamma} < \infty \tag{2.8}$$

for some $\beta > \frac{1}{2}$ and $\gamma > \frac{3}{2}$. Then a process $(U(t))_{t\geq 0}$ satisfying

$$dU(t) = -\frac{1}{2}\widetilde{\phi}(t)^2 dt + \widetilde{\phi}(t)dW(t) \tag{2.9}$$

is almost surely uniformly bounded over $[0,\infty)$.

Proof. Integrating (2.9) and using stochastic integration by parts, we see that

$$U(t) - U(0) = -\frac{1}{2}\int_0^t \widetilde{\phi}(s)^2 ds + W(t)\widetilde{\phi}(t) - \int_0^t W(s)\widetilde{\phi}'(s)ds.$$

By the assumptions on $\widetilde{\phi}$, $\sup_{t\geq 0}\left|\int_0^t \widetilde{\phi}(s)^2 ds\right|$ is finite. We know that almost surely $\lim_{t\to\infty} W(t)t^{-\beta} = 0$, and therefore $\sup_{t\geq 0} |W(t)\widetilde{\phi}(t)| < \infty$ almost surely.

We have

$$\begin{aligned}\sup_{t\geq 1}\int_1^t |W(s)\widetilde{\phi}'(s)|ds &= \sup_{t\geq 1}\int_1^t |W(s)s^{-\gamma}|\,|\widetilde{\phi}'(s)s^{\gamma}|ds \\ &\leq \sup_{t\geq 1} M\int_1^t |W(s)s^{-\gamma}|ds\end{aligned}$$

almost surely, for some $M > 0$. Now note that if we choose an ϵ such that $0 < \epsilon < \gamma - \frac{3}{2}$, we get

$$\begin{aligned}\sup_{t\geq 1}\int_1^t \left|W(s)s^{-\gamma}\right| ds &= \sup_{t\geq 1}\int_1^t |W(s)s^{-\frac{1}{2}-\epsilon}s^{\frac{1}{2}-(\gamma-\epsilon)}|ds \\ &\leq M'\sup_{t\geq 1}\int_1^t s^{\frac{1}{2}-(\gamma-\epsilon)}ds,\end{aligned}$$

almost surely, for some $M' > 0$. The latter supremum is finite since $\frac{1}{2} - (\gamma - \epsilon) < -1$. We can thus conclude that

$$\sup_{t\geq 0}\left|\int_0^t W(s)\widetilde{\phi}'(s)ds\right| < \infty$$

almost surely.

Combining these results shows that

$$\sup_{t\geq 0}\left|-\frac{1}{2}\int_0^t \widetilde{\phi}(s)^2 ds + W(t)\widetilde{\phi}(t) - \int_0^t W(s)\widetilde{\phi}'(s)ds\right| < \infty$$

almost surely, which shows that $U(t)$ is almost surely uniformly bounded on $[0,\infty)$.

□

Since U is uniformly bounded, we can apply Theorem 5 to see that solutions of

$$dz(t) = -\widetilde{F}(z(t-1)) - \frac{1}{2}\widetilde{\phi}(t)^2 dt + \widetilde{\phi}(t)dW(t) \tag{2.10}$$

are bounded in probability. This equation has arisen from the transformation $z(t) = \mathscr{G}(y(t))$, with $\mathscr{G}(x) = \int_0^x g(s)^{-1}ds$. Therefore, if $\lim_{x\to\infty}\mathscr{G}(x) = \infty$, and z is bounded in probability, y is bounded in probability. We can therefore state the following theorem.

Theorem 12. *Let $F : \mathbb{R} \to \mathbb{R}$ have feedback. Let g be such that $\lim_{x\searrow -1}\mathscr{G}(x) = -\infty$ and $\lim_{x\to\infty}\mathscr{G}(x) = \infty$. Let ϕ satisfy the conditions of Proposition 11. If y satisfies*

$$dy(t) = -F(y(t-1))g(y(t))dt + g(y(t))\phi(t)dW(t), \tag{2.11}$$

then y is bounded in probability.

Proof. Theorem 1 shows that y is bounded from below. Proposition 11 shows that the process U satisfying

$$dU(t) = -\frac{1}{2}\widetilde{\phi}(t)^2 dt + \widetilde{\phi}(t)dW(t)$$

is bounded. It follows that $(\sup_{s\in[0,t]} U(t) - U(s))_{t\geq 0}$ is bounded in probability. So by Theorem 5, y must be bounded in probability. □

Although this method yields a process y that is bounded in probability, it is not possible to use the standard approach to show the existence of an invariant distribution. In most cases, it is assumed that the process is autonomous, which is not the case if ϕ satisfies condition (2.8).

3 Large time behaviour with jump processes

In this chapter, we consider a stochastic delay differential equation with jumps. The main results is the proof of the existence of an invariant distribution of the stochastic delay differential equation

$$dy(t) = -F(y(t-1))g(y(t))dt + h(y_{t-})dP(t), \tag{3.1}$$

where $(P(t))_{t\geq 0}$ is a jump process. For the construction of this distribution we need two main components: the tightness of the set of segments of the solution, and a certain continuity in the initial value of the solution. We discuss these conditions in the following sections.

We endow the space of continuous functions on $[-1,0]$, $C[-1,0]$ with the supremum norm. The space of càdlàg functions $D[-1,0]$ is equipped with the Skorokhod metric, see [3]. We now state the definition of a jump process.

Definition 13. Let S_k, $k \in \mathbb{N}$, be i.i.d. random variables. Let Λ_j be i.i.d. random variables, $j \in \mathbb{N}$, such that $\mathbb{P}(0 < \Lambda_1 < \infty) = 1$. For all $k \in \mathbb{N}_0$, define $t_k := \sum_{i=1}^{k} \Lambda_i$. If we define the càdlàg process P by

$$P(t) = \sum_{k=1}^{\infty} S_k \mathbb{1}(t \in [t_k, \infty)),$$

then $(P(t))_{t\geq 0}$ is called a *jump process*.

A common choice for Λ_1 is taking it to be exponentially distributed for some parameter $\lambda > 0$. This makes $(P(t))_{t\geq 0}$ a compound Poisson process. If in addition to that we have $S_1 = 1$ almost surely, then $(P(t))_{t\geq 0}$ is a Poisson process.

Let a $t \geq 0$ be given. With y_{t-} we denote the segment of y on $[t-1, t)$, so

$$\begin{aligned} y_{t-} : [-1, 0) &\to \mathbb{R}, \\ \theta &\mapsto y(t+\theta). \end{aligned}$$

To make the analysis more clear, it is useful to have a finite number of jumps in a finite interval, almost surely. To this end, we prove that $\lim_{k\to\infty} t_k = \infty$ almost surely.

Lemma 14. *We have* $\mathbb{P}(\lim_{k\to\infty} t_k = \infty) = 1$.

Proof. Since $\mathbb{P}(0 < \Lambda_1 < \infty) = 1$, there exists an $\epsilon > 0$ such that $\mathbb{P}(\Lambda_1 \geq \epsilon) > 0$. Define $X_k = \epsilon \mathbb{1}(\Lambda_k \geq \epsilon)$. Then

$$\mathbb{P}(\lim_{k\to\infty} t_k = \infty) = \mathbb{P}(\sum_{k=1}^{\infty} \Lambda_k = \infty) \geq \mathbb{P}(\sum_{k=1}^{\infty} X_k = \infty).$$

We have $\mathbb{P}(\sum_{k=1}^{\infty} X_k = \infty) = \mathbb{P}(\cap_{N\in\mathbb{N}} \cup_{n\geq N} \{X_n = \epsilon\})$. Since

$$\mathbb{P}(\cap_{N\in\mathbb{N}} \cup_{n\geq N} \{X_n = \epsilon\}) \geq \limsup_{n\to\infty} \mathbb{P}(X_n = \epsilon)$$

and $\mathbb{P}(X_1 = \epsilon) > 0$, we have

$$\mathbb{P}(\lim_{k\to\infty} t_k = \infty) > 0.$$

By Kolmogorov's zero-one law, we must have $\mathbb{P}(\lim_{k\to\infty} t_k = \infty) \in \{0, 1\}$. Hence, $\mathbb{P}(\lim_{k\to\infty} t_k = \infty) = 1$. □

3.1 Dynamics of solutions

In the following section we consider the differential equation

$$dy(t) = -F(y(t-1))g(y(t))dt + g(y(t))\phi(y_t, t)dW(t) + h(y_{t-})dP(t) \tag{3.2}$$

with $F, g : \mathbb{R} \to \mathbb{R}$, $\phi : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ and $h : D[-1, 0) \to \mathbb{R}$. The existence of solutions under certain conditions is established by Theorem B.1. We state several results about the dynamical behaviour of solutions to equation (3.2) and a bound similar to Theorem 1 is derived. The following assumptions are made in this section, unless otherwise stated.

Assumptions A. Assume that all the random variables S_i, Λ_j and the Brownian motion $(W(t))_{t\geq 0}$ are independent. We assume that there exists an $M \in \mathbb{R}$ such that $|S_1| < M$ almost surely. Let $h : D[-1, 0) \to \mathbb{R}$ be a function such that $|h(y)| \leq \frac{|1+y(0-)|}{M}$ for all $y \in D[-1, 0)$. Let $F : \mathbb{R} \to \mathbb{R}$ be continuous and with feedback (recall Definition 3). Let the function $\phi : D[-1, 0] \to \mathbb{R}$ be bounded. Let $g : \mathbb{R} \to \mathbb{R}$ be continuously differentiable, with $g(-1) = 0$, $g(x) > 0$ for all $x > -1$ and $\lim_{x\searrow -1} \mathcal{G}(x) = -\infty$, where $\mathcal{G}(x) = \int_0^x g(s)^{-1} ds$.

A similar statement as Theorem 1 can be stated for processes y that satisfy equation (3.2).

Theorem 15. *Assume that* $(y(t))_{t\geq 0}$ *satisfies equation* (3.2)*. Let* τ_y *be the maximal existence time of* y*. Then, under the conditions stated above,* $y(t) > -1$ *if* $y(0) > -1$ *for all* $t \in [0, \tau_y)$ *almost surely.*

Proof. We first consider $y(0) > -1$. Define the stopping time $\kappa = \inf\{t \in [0, \tau_y) : y(t) \leq -1\}$ and assume that $\kappa < \tau_y$. By the previous lemma, we know that $K := \max\{k : t_k \leq \kappa\}$ is finite almost surely. If $t_K < \kappa$, then y satisfies

$$y(t) - y(t_K) = \int_{t_K}^{t} -F(y(s-1))g(y(s))ds + \int_{t_K}^{t} g(y(s))\phi(y_s, s)dW(s)$$

for $t \in (t_K, \kappa]$. Since the integrands are continuous almost surely, y is continuous on this interval almost surely. It follows that $\lim_{t \nearrow \kappa} y(t) = -1$. We can then apply the machinery of Theorem 1 to derive a contradiction, so $\kappa = \tau_y$.

The other possibility is that $t_K = \kappa$, with $\lim_{t \nearrow \kappa} y(t) > -1$. Since y is càdlàg, we know that $y(t_K) \leq -1$. Combining this with $y(t_K-) > -1$ and the identity

$$y(t_K) = y(t_K-) + h(y_{t_K-})S_k$$

shows that $h(y_{t_K})S_K < 0$ and $S_K \neq 0$. Since

$$0 < y(t_K-) + 1 \leq -h(y_{t_K-})S_K,$$

we can take the absolute value and divide by $|S_K|$ to see that

$$|h(y_{t_K-})| \geq \frac{|1 + y(t_K-)|}{|S_K|} > \frac{|1 + y(t_K-)|}{M}.$$

This contradicts the assumption on h, hence $\kappa = \tau_y$. □

It is indeed necessary to include the condition that S_1 is bounded, as is made clear by the following proposition.

Proposition 16. *Let $a, \sigma : D[-1, 0] \times [0, \infty) \to \mathbb{R}$ be continuous functions. Furthermore, let $h : D[-1, 0) \to \mathbb{R}$ be such that it is strictly positive (resp. negative) for all $y \in D[-1, 0)$ that are bounded away from -1 on $[-1, 0)$. Assume that S_1 is unbounded with negative (resp. positive) sign. If y satisfies*

$$dy(t) = a(y_t, t)dt + \sigma(y_t, t)dW(t) + h(y_{t-})dP(t),$$

with existence time $\tau_y = \infty$, we have $\mathbb{P}(\exists t : y(t) < -1) > 0$.

Proof. Without loss of generality, we may assume that h is strictly positive and for all $M < 0$, we have $\mathbb{P}(S_1 < M) > 0$. Note that $t_1 < \infty$ almost surely. We can write $y(t_1) = y(t_1-) + h(y_{t_1-})S_1$. Then

$$\begin{aligned}\mathbb{P}(y(t_1) < -1) &= \mathbb{P}(S_1 < \frac{-1 - y(t_1-)}{h(y_{t_1-})}) \\ &> 0,\end{aligned}$$

since S_1 is independent of $(y(t))_{t<t_1}$. □

This statement can be generalized to arbitrary many crossings of the solution around -1. We state the case for two crossings in the following proposition.

Proposition 17. *Let $a, \sigma : D[-1,0] \times [0,\infty) \to \mathbb{R}$ be continuous functions. Furthermore, let $h : D[-1,0) \to \mathbb{R}$ be such that it is strictly positive or strictly negative for all $y \in D[-1,0)$ that are bounded away from -1 on $[-1,0)$. Assume that S_1 is unbounded in both directions, i.e. for all $M > 0$, $\mathbb{P}(S_1 > M) > 0$ and $\mathbb{P}(S_1 < -M) > 0$. Suppose that $y(0) > -1$. Then if y satisfies*

$$dy(t) = a(y_t, t)dt + \sigma(y_t, t)dW(t) + h(y_{t-})dP(t),$$

with existence time $\tau_y = \infty$ a.s., we have $\mathbb{P}(\exists\, 0 < \widetilde{t}_1 < \widetilde{t}_2 : y(\widetilde{t}_1) < -1 < y(\widetilde{t}_2)) > 0$.

Proof. Without loss of generality, we may assume that h is strictly positive. Note that $t_1, t_2 < \infty$ almost surely. We can write $y(t_1) = y(t_1-) + h(y_{t_1-})S_1$ and similarly $y(t_2) = y(t_2-) + h(y_{t_2-})S_2$. Then

$$\mathbb{P}(y(t_1) < -1, y(t_2) > -1) = \mathbb{P}(S_1 < \frac{-1 - y(t_1-)}{h(y_{t_1-})}, S_2 > \frac{-1 - y(t_2-)}{h(y_{t_2-})}).$$

Write $s_1 := \frac{-1-y(t_1-)}{h(y_{t_1-})}$ and $s_2 := \frac{-1-y(t_2-)}{h(y_{t_2-})}$. We now write this probability as a sum of conditional probabilities

$$\begin{aligned}
&\mathbb{P}(S_1 < s_1, S_2 > s_2) \\
&\quad > \sum_{k\in\mathbb{Z}} \mathbb{P}(S_1 < s_1, S_2 > k+1 | s_2 \in [k, k+1))\mathbb{P}(s_2 \in [k, k+1)) \\
&\quad = \sum_{k\in\mathbb{Z}} \mathbb{P}(S_1 < s_1 | s_2 \in [k, k+1))\mathbb{P}(s_2 \in [k, k+1))\mathbb{P}(S_2 > k+1),
\end{aligned}$$

since S_2 is independent of $(y(t))_{t<t_2}$. Note that

$$\sum_{k\in\mathbb{Z}} \mathbb{P}(S_1 < s_1 | s_2 \in [k, k+1))\mathbb{P}(s_2 \in [k, k+1)) = \mathbb{P}(S_1 < s_1),$$

which is a strictly positive probability by the previous theorem. Hence there exists a $k \in \mathbb{Z}$ such that $\mathbb{P}(S_1 < s_1 | s_1 \in [k, k+1))\mathbb{P}(s_2 \in [k, k+1)) > 0$. By assumption, $\mathbb{P}(S_2 > i+1) > 0$ for all $i \in \mathbb{Z}$, so in particular for $i = k$. We conclude that $\mathbb{P}(S_1 < s_1, S_2 > s_2) > 0$, hence $\mathbb{P}(y(t_1) < -1, y(s_2) > -1) > 0$. □

It is clear that the previous proposition can be generalized to arbitrarily many crossings of $(y(t))_{t\geq 0}$ of -1. When only negative jumps are allowed and y satisfies Wright's delay equation with jumps, it diverges to $-\infty$ if $y(0) < -1$.

Proposition 18. *Let $h : D[-1,0) \to \mathbb{R}$ be non-positive (resp. non-negative) and S_1 be non-negative (resp. non-positive). Suppose that $y(0) < -1$. If y satisfies*

$$dy(t) = -\alpha y(t-1)(1 + y(t))dt + h(y_{t-})dP(t),$$

then $\lim_{t\to\infty} y(t) = -\infty$ almost surely.

Proof. Let $(x(t))_{t\geq 0}$ be the process that satisfies $dx(t) = -\alpha x(t-1)(1 + x(t))dt$ for $t \geq 0$, with $x(s) = y(s)$ for $s \in [-1, 0]$. Then $x(0) = y(0) < -1$, hence we know that from [12, Theorem 1] that $\lim_{t\to\infty} x(t) = -\infty$. Define $\tau := \inf\{t \geq 0 : y(t) > x(t)\}$, so it suffices to show that $\tau = \infty$. Hence, we assume that $\tau < \infty$. It immediately follows that $t_1 \leq \tau$, and without loss of generality we may assume that $h(y_{t_1-})S_1 < 0$. We have

$$\begin{aligned} x(\tau) - y(\tau) =& -\alpha \int_0^\tau x(s-1)(1+x(s))ds + \alpha \int_0^\tau y(s-1)(1+y(s))ds \\ &- \sum_{k:0<t_k\leq\tau} h(y_{t_k-})S_k \\ &> \alpha \int_0^\tau y(s-1)(1+y(s)) - x(s-1)(1+x(s))ds. \end{aligned}$$

For $s \in [-1,\tau]$, $y(s) \leq x(s) < -1$, hence $y(s-1)(1+y(s)) - x(s-1)(1+x(s)) \geq 0$. We can thus conclude that $x(\tau) > y(\tau)$, which is not possible, since both processes are càdlàg. Hence $\tau = \infty$, which completes the proof. □

The following theorem bounds the solutions from above.

Theorem 19. *Let $h : D[-1,0) \to \mathbb{R}$ be non-positive (resp. non-negative). Let S_1 be non-negative (resp. non-positive). Let $F : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be such that $\int_0^\infty g(s)^{-1}ds > -\inf_{s\geq -1} F(s)$. If y satisfies*

$$dy(t) = -F(y(t-1))g(y(t))dt + h(y_{t-})dP(t),$$

with $y(0) > -1$, then y is bounded almost surely. If there exists a $C > 0$ such that $-1 < y(s) \leq C$ for all $s \in [-1,0]$ almost surely, then there exists an $M > 0$ such that $\|y\|_\infty < M$ almost surely.

Proof. We follow the same proof as outlined in [12, Theorem 1]. By Theorem 15 we know that y is bounded from below by -1, so it suffices to show that it is bounded from above.

First assume that there exists a $T > 0$ such that y is of constant sign for all $t \geq T$. If y is negative, then it immediately follows that y is bounded. If y is positive, then for all $t > T+1$, $-F(y(t-1))g(y(t)) \leq 0$. Hence

$$\begin{aligned} y(t) &= y(T+1) + \int_{T+1}^t -F(y(s-1))g(y(s))ds + \sum_{k:T+1<t_k\leq t} h(y_{t_k-})S_k \\ &\leq y(T+1). \end{aligned}$$

Since y is càdlàg, it is also bounded on $[0, T+1]$. Hence y is bounded on $[0,\infty)$.

Now assume that y changes sign infinitely often. Let $s_0 < s_1$ be such that $y(t) \geq 0$ for all $t \in [s_0, s_1)$. Set $z(t) = \int_0^{y(t)} g(s)^{-1}ds$. Using Itô's formula, we see that for all $t \in [s_0, s_1]$ we have

$$z(t) - z(s_0) = \int_{s_0}^t -F(y(s-1))ds + \sum_{k:s_0<t_k\leq t} \int_{y(t_k-)}^{y(t_k)} g(s)^{-1}ds.$$

Since $F(x) \geq 0$ for $x \geq 0$ and F is continuous, $F(x)$ is bounded from below for all $x \geq -1$. Since $y(t_k) \leq y(t_{k-})$, $y > -1$ and $z(s_0) = 0$, we have

$$z(t) \leq z(s_0) + \int_{s_0}^t -F(y(s-1))ds \leq z(s_0) - (t-s_0)\inf_{s\geq -1} F(s). \tag{3.3}$$

For all $t \in [s_0+1, s_1]$ we have

$$\begin{aligned} z(t) &= z(s_0+1) + \int_{s_0+1}^{t} -F(y(s-1))ds + \sum_{k: s_0+1 < t_k \le t} \int_{y_{t_k-}}^{y_{t_k}} g(s)^{-1} ds \\ &\le z(s_0+1), \end{aligned}$$

since $y(s) \ge 0$ for $s \in [s_0, t-1]$. Hence we may assume that the supremum of z on $[s_0, s_1]$ is achieved on $[s_0, (s_0+1) \wedge s_1]$. Combination with equation (3.3) now yields that

$$\sup_{t \in [s_0, s_1]} \int_0^{y(t)} g(s)^{-1} ds \le z(s_0) - \inf_{s \ge -1} F(s). \tag{3.4}$$

Set s_0 such that $y(s_0) = 0$. We conclude from the assumptions made on g that y is bounded.

Now suppose we have a $C > 0$ such that $-1 < y(s) \le C$ for all $s \in [-1, 0]$. We know from the previous argument that there is a $K > 0$ such that if $y(s) \le 0$, then $y(t) \le K$ for all $t \ge s$.

Suppose that $y(0) \ge 0$. Define $s_1 := \inf\{t \ge 0 : y(t) \le 0\}$. We know from equation (3.4) that

$$\begin{aligned} \sup_{t \in [0, s_1]} \int_0^{y(t)} g(s)^{-1} ds &\le z(0) - \inf_{s \ge -1} F(s) \\ &\le \int_0^{C} g(s)^{-1} ds - \inf_{s \ge -1} F(s). \end{aligned}$$

Hence y is also bounded on the interval where $y(s) \ge 0$. This concludes the proof.

□

3.2 Tightness of solutions

Consider again equation (3.1) in the setting of assumptions A. One of the two main components of the derivation of an invariant measure is that that the set $\{y_t : t \ge 0\}$ is tight. In this section, we show that under some conditions on the jump process, this set is indeed tight. Recall that a family of random variables $\{X^t : t \ge 0\}$ with values in a metric space E is called tight if for all $\epsilon > 0$, there exists a compact set $K_\epsilon \subseteq E$ such that for all $t \ge 0$, $\mathbb{P}(X^t \in K_\epsilon) \ge 1 - \epsilon$. We show that the set of segments of the solution is tight. In order to do this, we need a notion of the absence of structure in the jump process. This is made concrete in the following definition of periodicity, see [4].

Definition 20. A random variable X is called *periodic*, if there exists a $D \in \mathbb{R}$ such that $\mathbb{P}(X \in \{0, D, 2D, \ldots\}) = 1$.

To show that the set of segments of the solution is tight, we need to show that the probability of jumping at one particular interval goes to zero as the interval length goes to zero. The non-periodicity of the jumping times enables us to use Blackwell's Theorem (see Theorem B.4) which establishes this fact.

Theorem 21. *Let $h : D[-1,0) \to \mathbb{R}$ be non-positive (resp. non-negative) and S_1 be non-negative (resp. non-positive). Let $F : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be such that $\int_0^\infty g(s)^{-1}ds > -\inf_{s\geq -1} F(s)$. Assume that $\lim_{n\to\infty} \mathbb{P}(\Lambda_1 \geq \frac{1}{n})^n = 1$ and Λ_1 is not periodic. If y satisfies*

$$dy(t) = -F(y(t-1))g(y(t))dt + h(y_{t-})dP(t),$$

and there is an $M > 0$ such that $-1 < y(t) < M$ for all $t \in [-1,0]$ almost surely, then

$$\{\theta \mapsto y(t+\theta) : [0,1] \to \mathbb{R} : t \geq 1\}$$

is tight in the Skorokhod metric.

Proof. We show that the continuous part of y and the jump process part of y are both tight. Then the sum, which is y, is also be tight. We make use of Theorem B.3.

The continuous part of y can be written as

$$\{\theta \mapsto \int_t^{t+\theta} -F(y(s-1))g(y(s))ds : [0,1] \to \mathbb{R} : t \geq 1\}.$$

We know that by Theorem 19, y is bounded. Therefore, for all $t \geq 0$, we have

$$\sup_{\theta\in[0,1]} |\int_t^{t+\theta} -F(y(s-1))g(y(s))ds| \leq \sup_{s\in[-1,\|y\|_\infty]} |F(s)| \sup_{s\in[-1,\|y\|_\infty]} |g(s)|.$$

Hence, we have

$$\lim_{K\to\infty} \limsup_{t\to\infty} \mathbb{P}(\sup_{\theta\in[0,1]} |\int_t^{t+\theta} -F(y(s-1))g(y(s))ds| > K) = 0.$$

Let $\mathcal{P}_\delta$ denote the set of partitions of $[0,1]$ with mesh at least δ. We then have, for every $\eta > 0$,

$$\begin{aligned}
&\lim_{\delta\to 0} \limsup_{t\to\infty} \mathbb{P}(\inf\{\max_{i\leq r} \sup_{\theta_1,\theta_2\in[s_{i-1},s_i)} |\int_{t+\theta_1}^{t+\theta_2} -F(y(s-1))g(y(s))ds| : (s_i)_{i=1}^r \in \mathcal{P}_\delta\} > \eta) \\
&\leq \lim_{\delta\to 0} \mathbb{P}(\delta \sup_{s\in[-1,\|y\|_\infty]} F(s) \sup_{s\in[-1,\|y\|_\infty]} g(s) > \eta) \\
&= 0.
\end{aligned}$$

So by Theorem B.3 we are able to conclude that the set

$$\{\theta \mapsto \int_t^{t+\theta} -F(y(s-1))g(y(s))ds : [0,1] \to \mathbb{R} : t \geq 1\}$$

is tight.

It now suffices to show that the set

$$\{\theta \mapsto \sum_{k:t_k\in(t,t+\theta]} h(y_{t_k-})S_k : [0,1] \to \mathbb{R} : t \geq 1\}$$

is tight.

Suppose we have an arbitrary solution y to the differential equation, some $t \geq 0$ and some $\delta > 0$. To ensure that the second condition of Theorem B.3 holds, we prove that we can choose the points of the partition on the jumping times with a high probability. The probability that we can find a partition of $[t, t+1]$, $t = t_0 < \ldots < t_r = t+1$ with grid size at least δ and such that $t_1 - t_0 \geq \delta$ and $1 - t_{r-1} \geq \delta$ with renewals only on the boundary of the subintervals, is bounded above by the probability that there is no renewal in $[t, t+\delta]$, no renewal in $[t+1-\delta, t+1]$ and no renewal in $[t, t+1]$ with holding time less that δ. Therefore, we have the following bound:

$$\begin{aligned}
&\mathbb{P}(\inf\{\max_{i \leq r} \sup_{\theta_1, \theta_2 \in [s_{i-1}, s_i)} |y_t(\theta_1) - y_t(\theta_2)| : (s_i)_{i=1}^r \in \mathcal{P}_\delta\} > \eta) \\
&\quad \leq [\mathbb{P}(\exists k \in \mathbb{N} : t_k \in [t, t+\delta]) + \mathbb{P}(\exists k \in \mathbb{N} : t_k \in [t+1-\delta, t+1]) \\
&\quad + \mathbb{P}(\exists k \in \mathbb{N} : t_k, t_{k+1} \in [t, t+1] : t_{k+1} - t_k < \delta)] \,.
\end{aligned}$$

We can bound the first and second probability with Markov's inequality and Theorem B.4:

$$\limsup_{t \to \infty} \mathbb{P}(\exists k \in \mathbb{N} : t_k \in [t, t+\delta]) \leq \frac{\delta}{\mathbb{E}(\Lambda_1)}$$

and

$$\limsup_{t \to \infty} \mathbb{P}(\exists k \in \mathbb{N} : t_k \in [t+1-\delta, t+1]) \leq \frac{\delta}{\mathbb{E}(\Lambda_1)}.$$

The probability that there is a renewal in the interval $[t, t+1]$ with holding time $\tau_n < \delta$ is lower bounded by the probability that we have $\lceil \delta^{-1} \rceil$ consecutive holding times with at least one holding time of less than δ. Hence

$$\begin{aligned}
&\limsup_{t \to \infty} \mathbb{P}(\exists k \in \mathbb{N} : t_k, t_{k+1} \in [t, t+1] : t_{k+1} - t_k < \delta) \\
&\quad \leq 1 - \mathbb{P}(\Lambda_1 \geq \delta, \ldots, \Lambda_{\lceil \delta^{-1} \rceil} \geq \delta) \\
&\quad = 1 - \mathbb{P}(\Lambda_1 \geq \delta)^{\lceil \delta^{-1} \rceil}.
\end{aligned}$$

By the assumption on Λ_1, we have $\lim_{\delta \searrow 0} 1 - \mathbb{P}(\Lambda_1 \geq \delta)^{\lceil \delta^{-1} \rceil} = 0$. We conclude that

$$\lim_{\delta \searrow 0} \limsup_{t \to \infty} \mathbb{P}(\inf\{\max_{i \leq r} \sup_{\theta_1, \theta_2 \in [s_i - s_{i-1}]} |y_t(\theta_1) - y_t(\theta_2)| : (s_i)_{i=1}^r \in \mathcal{P}_\delta\} > \eta) = 0.$$

Hence the second requirement of Theorem B.3 is satisfied, so the jump process part of y is also tight.

Lastly, the set $\{\theta \mapsto y(t) : [0,1] \to \mathbb{R} : t \geq 1\}$ is tight, since y is bounded by Theorem 19. The solution y is the sum of these processes:

$$y(t+\theta) = y(t) + \int_t^{t+\theta} -F(y(s-1))g(y(s))ds + \sum_{k : t_k \in (t, t+\theta]} h(y_{t_k -}) S_k.$$

Therefore, the set

$$\{\theta \mapsto y(t+\theta) : [0,1] \to \mathbb{R} : t \geq 1\}$$

is tight in the Skorokhod topology. □

The random variables Λ_1 with a density function near 0 that satisfy the condition $\lim_{n\to\infty} \mathbb{P}(\Lambda_1 \geq \frac{1}{n})^n = 1$ are characterised by the following lemma.

Lemma 22. *Suppose Λ_1 has Lebesgue density f in a neighbourhood of 0. If f is such that $\lim_{x\searrow 0} f(x) = 0$, then $\lim_{n\to\infty} \mathbb{P}(\Lambda_1 \geq \frac{1}{n})^n = 1$. If $\liminf_{x\searrow 0} f(x) > 0$, then $\limsup_{n\to\infty} \mathbb{P}(\Lambda_1 \geq \frac{1}{n})^n < 1$.*

Proof. We only prove the first statement. Let $\epsilon > 0$ and let $t > 0$ be such that for all $0 < y < x$ we have $f(y) < \epsilon$. Then

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}(\Lambda_1 \geq \tfrac{1}{n})^n &= \lim_{n\to\infty} (1 - \int_0^{\frac{1}{n}} f(y)dy)^n \\ &\geq \lim_{n\to\infty} (1 - \int_0^{\frac{1}{n}} \epsilon dy)^n \\ &= e^{-\epsilon}.\end{aligned}$$

This holds for all $\epsilon > 0$, so $\lim_{n\to\infty} \mathbb{P}(\Lambda_1 \geq \frac{1}{n})^n = 1$. The second statement follows from a similar argument. □

3.3 Continuity in initial value

The method to show the existence of an invariant distribution of equation (3.1) also requires that there is some continuity of the distribution of the solution segments at any time $t \geq 1$ in terms of the initial segments. The main theorems in this section, namely Theorem 24 and Theorem 25, shows that this holds under some restrictions on the coefficients of F, h, g and the underlying jump process. We need a small lemma relating convergence in the Skorkhod metric to convergence in L^1.

Lemma 23. *Let $(x^n)_{n\in\mathbb{N}} \subset D[-1,0]$ and $x \in D[-1,0]$ be such that $\lim_{n\to\infty} x^n = x$ in $(D[-1,0], d_s)$. Then $\lim_{n\to\infty} \int_{-1}^0 |x^n(s) - x(s)|ds = 0$.*

Proof. Let $\alpha_n : [-1,0] \to [-1,0]$ be increasing homeomorphisms such that

$$\begin{aligned}&\lim_{n\to\infty} \sup_{s\in[-1,0]} |x^n(s) - x(\alpha_n(s))| = 0 \quad \text{and} \\ &\lim_{n\to\infty} \|\alpha_n - \mathrm{id}\|_\infty = 0.\end{aligned}$$

We write

$$\int_{-1}^0 |x^n(s) - x(s)|ds \leq \int_{-1}^0 |x^n(s) - x(\alpha_n(s))|ds + \int_{-1}^0 |x(\alpha_n(s)) - x(s)|ds.$$

The first integral at the right hand side converges to 0 by the assumptions on α_n. For the second integral, note that $x \in D[-1,0]$, so x is continuous almost everywhere on $[-1,0]$. Hence $\lim_{n\to\infty} x(\alpha_n(s)) = x(s)$ almost everywhere. The function x is bounded since it is càdlàg, so by the Dominated Convergence theorem, we have

$$\lim_{n\to\infty} \int_{-1}^0 |x(\alpha_n(s)) - x(s)|ds = 0.$$

We conclude that $\lim_{n\to\infty}\int_{-1}^{0}|x^n(s)-x(s)|ds = 0$. □

We now have enough to prove the continuity in the initial segments.

Theorem 24. *Let F and g be bounded and Lipschitz with Lipschitz constants L_F and L_g. Let the distribution of Λ_1 be such that there is a K such that $|\{k : t_k \in (0,1]\}| \leq K$ almost surely. Let $h(y_t) = h(y(t-))$ be Lipschitz with constant L_h. Assume that $1 - 3\|F\|_\infty L_g - ML_h K > 0$. Let $(y^n)_{n\in\mathbb{N}}$ be $D[-1,0]$-valued random variables such that $y_0^n \to y_0^0$ in $(D[-1,0], d_s)$ uniformly over Ω. Let y^n and y be solutions of*

$$dy(t) = -F(y(t-1))g(y(t))dt + h(y_{t-})dP(t) \tag{3.5}$$

with initial segments y_n^0 and y^0 respectively. Then $y^n \to y$ uniformly over $[0,1]$ and Ω.

Proof. Let $\delta > 0$ and let $\epsilon \in (0,\delta)$ such that if $s,t \in [-1,0]$ are such that $|s-t| < \delta$, then $|F(s) - F(t)| < \epsilon$. Let $N \in \mathbb{N}$ be such that for all $n \geq N$, $d_s(y_0^0, y_0^n) < \epsilon$ and such that $\int_{-1}^{0}|y_0^n(t) - y_0^0(t)|dt < \delta$, which is possible by Lemma 23. Then using the triangle inequality, we see that

$$\begin{aligned}\sup_{t\in[0,1]}|y^n(t)-y(t)| \leq & \sup_{t\in[0,1]}|y^n(0)-y(0)| \\ & + \left|\int_0^t -F(y^n(s-1))g(y^n(s))ds - \int_0^t -F(y(s-1))g(y(s))ds\right| \\ & + \left|\sum_{k:t_k\in(0,t]} h(y^n_{t_k-})S_k - \sum_{k:t_k\in(0,t]} h(y(t_k-))S_k\right|.\end{aligned}$$

We have $|y^n(0) - y(0)| < \epsilon$. We also have

$$\begin{aligned}&\sup_{t\in[0,1]}\left|\int_0^t -F(y^n(s-1))g(y^n(s))ds - \int_0^t -F(y(s-1))g(y(s))ds\right| \\ &\quad \leq \int_0^1 |-F(y^n(s-1))g(y^n(s))ds + F(y(s-1))g(y(s))|ds.\end{aligned}$$

Note that

$$\begin{aligned}&\int_0^1 |F(y^n(s-1))g(y^n(s)) - F(y(s-1))g(y(s))|ds \\ &\quad\leq \int_0^1 |F(y(s-1))(g(y^n(s)) - g(y(s)))|ds \\ &\quad+ \int_0^1 |(F(y^n(s-1)) - F(y(s-1)))g(y(s))|ds \\ &\quad+ \int_0^1 |(g(y^n(s)) - g(y(s)))(F(y^n(s-1)) - F(y(s-1)))|ds.\end{aligned}$$

Again, since F and g are bounded, we have

$$\begin{aligned}
&\int_0^1 |F(y^n(s-1))g(y^n(s)) - F(y(s-1))g(y(s))|ds \\
&\leq \|F\|_\infty \int_0^1 |g(y^n(s)) - g(y(s))|ds + \|g\|_\infty \int_0^1 |F(y^n(s-1)) - F(y(s-1))|ds \\
&\quad + \int_0^1 |(g(y^n(s)) - g(y(s)))(F(y^n(s-1)) - F(y(s-1)))|ds \\
&\leq 3\|F\|_\infty \int_0^1 |g(y^n(s)) - g(y(s))|ds + \|g\|_\infty \int_0^1 F(y^n(s-1)) - F(y(s-1))ds \\
&\leq 3\|F\|_\infty L_g \sup_{s\in[0,1]} |y^n(s) - y(s)| + \|g\|_\infty L_F \delta.
\end{aligned}$$

Similarly, we can bound

$$\begin{aligned}
&\sup_{t\in[0,1]} |\sum_{k:t_k\in(0,t]} h(y^n(t_k-))S_k - \sum_{k:t_k\in(0,t]} h(y(t_k-))S_k| \\
&\quad \leq M \sum_{k:t_k\in(0,1]} |h(y^n(t_k-)) - h(y(t_k-))| \\
&\quad \leq ML_h K \sup_{s\in[0,1]} |y^n(s) - y(s)|.
\end{aligned}$$

Hence for all $\delta > 0$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\begin{aligned}
&\sup_{t\in[0,1]} |y^n(t) - y(t)| \\
&\quad \leq \delta + 3\|F\|_\infty L_g \sup_{s\in[0,1]} |y^n(s) - y(s)| + \|g\|_\infty L_F \int_{-1}^0 |y^n(s) - y(s)|ds \\
&\quad + ML_h K \sup_{s\in[0,1]} |y^n(s) - y(s)|.
\end{aligned}$$

We conclude that

$$\sup_{s\in[0,1]} |y^n(t) - y(t)| \leq \left[1 - 3\|F\|_\infty L_g - ML_h K\right]^{-1} \left(1 + \|g\|_\infty L_f\right) \delta,$$

so y^n converges uniformly over $[0,1]$ and Ω to y. This concludes the proof.

□

It is now a matter of applying the previous theorem multiple times to show that the solution is continuous in the initial value in probability, at all times $t \geq 0$.

Theorem 25. *Let $(y^n)_{n\in\mathbb{N}} \subset D[-1,0]$ such that $y_0^n \to y_0^0$ in $(D[-1,0], d_s)$. Let y^n and y be solutions of*

$$dy(t) = -F(y(t-1))g(y(t))dt + h(y_{t-})dP(t)$$

with initial segments y_n^0 and y^0 respectively. Under the assumptions of Theorem 24, y^n converges uniformly to y on $[t, t+1]$ for all $t \geq 0$.

Proof. Since uniform convergence implies convergence in the Skorokhod metric, we can apply Theorem 24 iteratively. □

With Theorem 25 in mind, we can finally prove that there exists an invariant distribution.

3.4 Existence of an invariant distribution

In this section, we prove the existence of an invariant distribution of equation (3.1). We use the Krylov-Bogoliubov method, similar to the approach used in [7]. Let $B_b(D[-1,0])$ be the set of all real-valued, bounded Borel-measurable functions on $D[-1,0]$. Let $C_b(D[-1,0]) \subset B_b(D[-1,0])$ be the subset of continuous functions thereof. Define $\langle \cdot, \cdot \rangle : \mathcal{P}(D[-1,0]) \times B_b(D[-1,0]) \to \mathbb{R}$ by $\langle \xi, f \rangle = \int f d\xi$. Let $\phi \in D[-1,0]$ be any càdlàg function on $[-1,0]$, and let $(X^\phi(t))_{t\geq 0}$ be the solution of equation (3.1) with $X^\phi = \phi$ on the interval $[-1,0]$. For all $t \geq 0$, define $P_t : B_b(D[-1,0]) \to B_b(D[-1,0])$ by $(P_t f)(\phi) = \mathbb{E}(f(X_t^\phi))$. By Theorem 25, P_t maps $C_b(D[-1,0])$ into $C_b(D[-1,0])$. We can identify $B_b(D[-1,0])^*$ by all finite, signed Borel measures on $D[-1,0]$. Define $P_t^* : B_b(D[-1,0])^* \to B_b(D[-1,0])^*$ by

$$(P_t^* \xi) f = \langle \xi, P_t f \rangle$$

for $f \in B_b(D[-1,0])$. Since $P_t^* \xi$ defines a linear functional on $B_b(D[-1,0])$, it can be written as the integral of g with respect to $P_t^* \xi$, so

$$\begin{aligned} P_t^* \xi &= f \mapsto \langle \xi, P_t f \rangle \\ &= f \mapsto \int f d(P_t^* \xi). \end{aligned}$$

Hence for all $f \in B_b(D[-1,0])$ and $\xi \in \mathcal{P}(D[-1,0])$, we see that $\langle P_t^* \xi, f \rangle = \langle \xi, P_t f \rangle$.

Note that if ξ is the distribution of the initial segment Φ of a solution X satisfying equation (3.5), then $P_t^* \xi$ is the distribution of the segment X_t:

$$\begin{aligned} \int f d(P_t^* \xi) &= \langle P_t^* \xi, f \rangle \\ &= \langle \xi, P_t f \rangle \\ &= \int \mathbb{E}(f(X_t^\phi)) d\xi(\phi). \end{aligned}$$

We have

$$\begin{aligned} \int \mathbb{E}(f(X_t^\phi)) d\xi(\phi) &= \mathbb{E}(\mathbb{E}(f(X_t^\Phi) | \mathcal{F}_0)) \\ &= \mathbb{E}(f(X_t^\Phi)). \end{aligned}$$

Hence $\int f d(P_t^* \xi) = \mathbb{E}(f(X_t^\Phi))$. Since this holds for all $f \in B_b(D[-1,0])$, $P_t^* \xi$ has the same distribution as X_t^Φ. Therefore, a measure $\xi \in \mathcal{P}(D[-1,0])$ is called an invariant measure if $P_t^* \xi = \xi$ for all $t \geq 0$.

Note that $(P_t)_{t\geq 0}$ is a Markov semigroup. In particular, for $t, s \geq 0$, we have $P_{t+s}f = P_s(P_t f)$. This follows from the fact that the differential equation is autonomous:

$$\begin{aligned}(P_{t+s}f)(\phi) &= \mathbb{E}(f(X^{\phi}_{t+s}))\\ &= \mathbb{E}(\mathbb{E}(f(X^{\phi}_{t+s})|X^{\phi}_t))\\ &= \mathbb{E}((P_t f)(X^{\phi}_s))\\ &= (P_s(P_t f))(\phi).\end{aligned}$$

We also have $P^*_{s+t} = P^*_s P^*_t$, since for all $\xi \in \mathcal{P}(D[-1,0])$ and $f \in B_b(D[-1,0])$ we have

$$\begin{aligned}(P^*_{s+t}\xi)(f) &= \langle \xi, P_{s+t}f\rangle\\ &= \langle \xi, P_s P_t f\rangle\\ &= \langle P^*_t P^*_s \xi, f\rangle\\ &= (P^*_t P^*_s \xi) f.\end{aligned}$$

Let $\xi \in \mathcal{P}(D[-1,0])$ the distribution of the initial segment $\phi : [-1,0] \to \mathbb{R}$ of X. We want to define the integral $\int_0^T \langle \xi, P_s f\rangle ds$. Hence we need to show that $t \mapsto \langle \xi, P_t f\rangle$ is measurable for all $f \in C_b(D[-1,0])$. We show that this maps is continuous. Note that

$$\lim_{s\to t}\langle \xi, P_t f\rangle = \lim_{s\to t} \int \mathbb{E}(f(X^{\phi}_s)) d\xi(\phi).$$

By using the dominated convergence, it suffices to show that $\lim_{s\to t} \mathbb{E}(f(X^{\phi}_s)) = \mathbb{E}(f(X^{\phi}_t))$ ξ-almost surely. For this it suffices to show that $\lim_{s\to t} f(X^{\phi}_s) = f(X^{\phi}_t)$ for all ϕ, which follows if $\lim_{s\to t} X^{\phi}_s = X^{\phi}_t$ in the Skorokhod metric, almost surely.

The following theorem states the existence of an invariant distribution.

Theorem 26. *Assume that the set $\{P^*_t \zeta : t \geq 1\}$ is tight for some $\zeta \in P(D[-1,0])$. Then there exists an $\eta \in \mathcal{P}(D[-1,0])$ such that $P^*_t \eta = \eta$ for all $t \geq 0$. Furthermore, η is contained in the closed convex hull of $\{P^*_t \xi : t \geq 1\}$.*

Proof. Let $T \in (0,\infty)$ and define $\psi_T : B_b(D[-1,0]) \to \mathbb{R}$ by $\psi_T(f) := \frac{1}{T}\int_0^T \langle \xi, P_s f\rangle ds$. Note that ψ_T is linear and $\|\psi_T\| \leq 1$. Let $\epsilon > 0$. Since $\{P^*_t \xi : s \in [1, T]\}$ is tight, there is a compact set $K_\epsilon \subset D[-1,0]$ such that $P^*_s(\xi)(K_\epsilon) \geq 1 - \epsilon$ for all $s \in [1, T]$. Let $f \in C_b(D[-1,0])$ such that $|f| \leq 1$ and $f = 0$ on K_ϵ. We then have

$$\begin{aligned}|\psi_T(f)| &\leq \frac{1}{T}\int_0^T |\langle P^*_s \xi, f\rangle| ds\\ &\leq \frac{1}{T}\int_0^T \int |f| dP^*_s \xi ds\\ &= \frac{1}{T}\int_0^T \int_{D[-1,0]\setminus K_\epsilon} |f| dP^*_s \xi ds\\ &\leq \frac{1}{T}\int_0^T P^*_s \xi(D[-1,0] \setminus K_\epsilon) ds\\ &\leq \epsilon.\end{aligned}$$

We endowed $D[-1,0]$ by the Skorokhod metric, thus $(D[-1,0], d_S)$ is regular. So by the Riesz-Bourbaki representation theorem (see Theorem B.6), there exists a signed Borel measure θ_T on $D[-1,0]$ in the closed convex hull of $\{P_t^* \xi : t \in [1,T]\}$ such that for all $f \in C_b(D[-1,0])$ we have

$$\psi_T(f) = \int_{D[-1,0]} f d\theta_T.$$

Hence there exists a θ_T in the closed convex hull of $\{P_T^* \xi : t \geq 1\}$ such that

$$\langle \theta_T, f \rangle = \frac{1}{T} \int_0^T \langle P_s^* \xi, f \rangle ds$$

for all $f \in C_b(D[-1,0])$. We assumed that the set $\{P_t^* \xi : t \geq 1\}$ is tight. The convex hull of a tight set is tight. So by Prokhorov's theorem (Theorem B.5) the convex hull of $\{P_t^* \xi : t \geq 1\}$ is relatively compact. Since $(\theta_t)_{t \geq 0}$ is contained in this closed convex hull, we can find a subsequence $(\theta_{t_n})_{n \in \mathbb{N}} \subset (\theta_t)_{t \geq 0}$ such that $\lim_{n \to \infty} \theta_{t_n} = \eta \in \mathcal{P}(D[-1,0])$.

We have

$$\begin{aligned}
&\lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \langle P_{(t+s)}^* \xi, f \rangle ds \\
&= \lim_{n \to \infty} \frac{1}{t_n} \int_t^{t_n+t} \langle P_s^* \xi, f \rangle ds \\
&= \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \langle P_s^* \xi, f \rangle ds + \int_{t_n}^{t_n+t} \langle P_s^* \xi, f \rangle ds - \int_0^t \langle P_s^* \xi, f \rangle ds.
\end{aligned}$$

Note that

$$\begin{aligned}
|\int_{t_n}^{t_n+t} \langle P_s^* \xi, f \rangle ds| &= |\int_{t_n}^{t_n+t} \langle \xi, P_s f \rangle ds| \\
&\leq \int_{t_n}^{t_n+t} \int \mathbb{E}(|f(X_s^\phi)|) d\xi(\phi) ds \\
&\leq \int_{t_n}^{t_n+t} \|f\|_\infty ds \\
&= \|f\|_\infty t.
\end{aligned}$$

Hence $\lim_{n \to \infty} \frac{1}{t_n} \int_{t_n}^{t_n+t} \langle P_s^* \xi, f \rangle ds = 0$. We also have $\lim_{n \to \infty} \frac{1}{t_n} \int_0^t \langle P_s^* \xi, f \rangle ds = 0$. Hence

$$\begin{aligned}
\lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \langle P_{(t+s)}^* \xi, f \rangle ds &= \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \langle P_s^* \xi, f \rangle ds \\
&= \lim_{n \to \infty} \langle \theta_{t_n}, f \rangle \\
&= \langle \eta, f \rangle.
\end{aligned}$$

We also have

$$\begin{aligned}
\lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \langle P_{(t+s)}^* \xi, f \rangle ds &= \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \langle P_s^* \xi, P_t f \rangle ds \\
&= \lim_{n \to \infty} \langle \theta_{t_n}, P_t f \rangle \\
&= \langle \eta, P_t f \rangle.
\end{aligned}$$

We conclude that $\langle \eta, P_t f \rangle = \langle \eta, f \rangle$, so $\langle P_t^* \eta, f \rangle = \langle \eta, f \rangle$ for all $f \in B_b(D[-1,0])$. Therefore $\int f d(P_t^* \eta - \eta) = 0$ for all $f \in B_b(D[-1,0])$, so it follows that $P_t^* \eta - \eta$ is the zero-measure. Therefore $P_t^* \eta = \eta$, so η is an invariant distribution.

□

We combine all the previous theorems into one statement showing the existence of an invariant distribution for equation (3.1).

Theorem 27. *Let Assumptions A hold. Let $h : D[-1,0) \to \mathbb{R}$ be non-positive (resp. non-negative) and S_1 be non-negative (resp. non-positive). Let F and g be bounded and Lipschitz with constants L_F and L_g and such that $\int_0^\infty g(s)^{-1} ds > -\inf_{s \geq -1} F(s)$. Assume that there is an $\epsilon > 0$ such that $\mathbb{P}(\Lambda_1 \geq \epsilon) = 1$ and that Λ_1 is non-periodic. Assume that $1 - 3\|F\|_\infty L_g - M L_h \lceil \epsilon^{-1} \rceil > 0$. Let $\zeta \in D[-1,0]$ be such that $\inf_{s \in [-1,0]} \zeta(s) > -1$. If y satisfies*

$$dy(t) = -F(y(t-1))g(y(t)) + h(y_{t-})dP(t),$$

then there exists an $\eta > 0$ in the closed convex hull of $\{P_t^* \zeta : t \geq 1\}$ such that $P_t^* \eta = \eta$ for all $t \geq 0$.

Proof. Combine the statements of Theorem 15, 19, 21, 25 and 26. □

Although Theorem 27 shows the existence of an invariant distribution for equation (3.1), it is not clear that this distribution is non-trivial. Since the distribution $\zeta \equiv -1$ is invariant, one wants to exclude this from the closed convex hull of $\{P_t^* \xi : t \geq 0\}$ for some $\xi \in P(D[-1,0])$. How to do this in general remains unclear.

Now that it is clear that there exist invariant distributions for the generalized Wright equation, it is interesting to see if the conditions for its existence can be weakened. Of particular interest are the conditions for the continuity in the initial value of the solution, which are quite restrictive. It stands to reason that there are more general requirements that allow for this property to hold.

Further research may be directed at allowing for a more general jump process. Theorem 21 only allows for non-periodic processes. One may wonder if a jump process with periodicity also allows tightness of the set of segments.

Appendices

A Preliminaries

Let $y : [-1, \infty) \to \mathbb{R}$ be any function. For any $t \in [0, \infty)$, we denote with y_t the segment of y on the interval $[t-1, t]$, i.e.

$$\begin{aligned} y_t : [-1, 0] &\to \mathbb{R}, \\ \theta &\mapsto y(t+\theta). \end{aligned}$$

With y_{t-} we mean the segment of y on $[t-1, t)$, so

$$\begin{aligned} y_{t-} : [-1, 0) &\to \mathbb{R}, \\ \theta &\mapsto y(t+\theta). \end{aligned}$$

With $(W(t))_{t \geq 0}$ we denote a standard Brownian motion. Let $\mu, \sigma : D[-1,0] \times [0, \infty) \to \mathbb{R}$ and $h : D[-1,0) \times [0, \infty) \to \mathbb{R}$ be any measurable functions. Let $(P(t))_{t \geq 0}$ be a jump process (see Definition 13). In general, we consider differential equations of the form

$$dy(t) = \mu(y_t, t)dt + \sigma(y_t, t)dW(t) + h(y_{t-}, t)dP(t). \tag{A.1}$$

Let $\widetilde{y}_0 \in D[-1, 0]$. The function $y : [-1, \infty) \to \mathbb{R}$ is said to be a solution of equation (A.1) with initial solution $\widetilde{y}_0$ if $y_0 = \widetilde{y}_0$ and for any $t \geq 0$

$$y(t) = y(0) + \int_0^t \mu(y_s, s)ds + \int_0^t \sigma(y_s, s)dW(s) + \int_0^t h(y_{s-}, s)dP(s).$$

Here, the last integral is interpreted as

$$\int_0^t h(y_{s-}, s)dP(s) = \sum_{0 < t_k \leq t} h(y_{t_k-}, t_k)S_k.$$

B Important theorems

In this section we state important theorems used throughout this work. The existence of solutions of the general differential equation is given by the following theorem. A proof could be given following standard arguments as in [6], [10], and [2]. We omit the details.

Theorem B.1. *Let $f, g, h : [0,\infty) \times D[-1,0] \to \mathbb{R}$. Assume that f and g are continuous, and such that for all $T > 0$ and $r > 0$ there is an $L_{T,r}$ such that for all $x, y \in D[-1,0]$ with $\|x\|_\infty \leq r$ and $\|y\|_\infty \leq r$ en $t \in [0,T]$ we have*

$$|f(t,x) - f(t,y)| \leq L_{T,r},$$

and similarly for g and h. Let $(W(t))_{t\geq 0}$ be a Brownian motion, $(P(t))_{t\geq 0}$ be a jump process and let y_0 a independent random variable with values in $D[-1,0]$. Let $\{\mathcal{F}_t\}$ the filtration generated by Y_0, $(W(s))_{s\in[0,t]}$ and $(P(s))_{s\in[0,t]}$. Then there is a stochastic time $\tau > 0$ and a $(\mathcal{F})_t$-adapted stochastic process $(y(t))_{t\in[0,\tau)}$ with values in $D[-1,0]$, such that

1. *the map $[0,\tau(\omega)) \times \Omega \to \mathbb{R}$ given by $(s,\omega) \mapsto \|y(s)\|^2_\infty$ is Borel measurable;*
2. *the map $[-1,\tau(\omega)) \to \mathbb{R}$ given by $t \mapsto y(t,\omega)$ is càdlàg almost surely;*
3. *$y(t,\omega) = y_0(t,\omega)$ for all $t \in [-1,0]$ almost surely;*
4. *for all $t \in [0,\tau)$, $\int_0^t \mathbb{E}\|y^\tau(s)\|^2_\infty ds < \infty$, where $y^\tau(s) = y(s)$ if $s \in [0,\tau)$ and $y^\tau(s) = 0$ if $s \geq \tau$;*
5. *y satisfies*

$$y(t) = y_0(0) + \int_0^t f(s,y_s)ds + \int_0^t g(s,y_s)dW(s) + \int_0^t h(s,y_s)dP(s).$$

One can choose τ such that if $\tau < \infty$, then $\limsup_{t\mapsto\tau} |y(t)| = \infty$.

We use an adapted version of Girsanov's theorem, as stated in [8, Theorem 8.10] (see also in [5, Corollary 16.25]).

Theorem B.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with complete filtration $\{\mathcal{F}_t\}$ and $W(t)$ a 1-dimensinoal Brownian motion with respect to the filtration $\{\mathcal{F}_t\}$. Let H be an $\mathbb{R}$-valued adapted, measurable process such that $\int_0^T |H(t,\omega)|^2 dt < \infty$ for all $T \in [0,\infty)$, $\mathbb{P}$-almost surely. Let $T > 0$. Assume that that*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T |H(s)|^2 ds\right)\right] < \infty.$$

Let $Z(t) := \exp\left(\int_0^t H(s)dW(s) - \frac{1}{2}\int_0^t |H(s)|^2 ds\right)$. *Define* $U(t) = W(t) - \int_0^t H(s)ds$ *and the probability measure* Q_T *on* $\mathcal{F}_T$ *by* $Q_t(A) = \mathbb{E}^{\mathbb{P}}[\mathbb{1}(A)Z(t)]$ *for* $A \in \mathcal{F}_t$. *Then on the probability space* $(\Omega, \mathcal{F}_T, Q_T)$ *the process* $\{W(t) : t \in [0, T]\}$ *is a 1-dimensional standard Brownian motion relative to the filtration* $\{\mathcal{F}_t\}_{t\in[0,T]}$.

The following theorem is establishes an equivalent condition for tightness of a sequence of segments. It is stated in [3, Theorem 3.11].

Theorem B.3. *A sequence* $(X^n)_{n\in\mathbb{N}}$ *is tight in the Skorkhod metric if and only if*

1. *for all* $N \in \mathbb{N}^*$, $\epsilon > 0$, *there are* $n_0 \in \mathbb{N}^*$ *and* $K \in \mathbb{R}_+$ *such that if* $n \geq n_0$,

$$\mathbb{P}\left(\sup_{t\leq N} |X_t^n| > K\right) \leq \epsilon; \tag{B.1}$$

2. *for all* $\epsilon > 0$, $\eta > 0$, *there is a* $\delta > 0$ *such that there exists an* $N \in \mathbb{N}$ *such that for all* $n \geq N$ *one has*

$$\mathbb{P}\left(\inf\{\max_{i\leq r} \sup_{\theta_1,\theta_2\in[s_{i-1},s_i)} |X^n(\theta_1) - X^n(\theta_2)| : (s_i)_{i=1}^r \in \mathcal{P}_\delta\} \geq \eta\right) \leq \epsilon, \tag{B.2}$$

   where $\mathcal{P}_\delta$ *is the set of partitions of* $[-1, 0]$ *with mesh at least* δ.

We give a version of Blackwell's Theorem as stated in [4, Theorem 3.8]. It gives the asymptotic number of expected renewals in an interval of fixed length.

Theorem B.4. *Let* $(P(t))_{t\geq 0}$ *be a jump process and let* $m : [0, \infty) \to \mathbb{R}$ *be the expected number of renewals up to any time t, so* $m(t) = \mathbb{E}\left[\sum_{i=1}^\infty \mathbb{1}(t_i \leq t)\right]$. *If* Λ_1 *is not periodic, then for every* $a \geq 0$,

$$\lim_{t\to\infty} m(t + a) - m(t) = \frac{a}{\mathbb{E}(\Lambda_1)}.$$

We state a part of Prokhorov's theorem as stated in [5, Theorem 14.3].

Theorem B.5. *Let S be a metric space and let* $\xi = \{\xi_i\}_{i\in\mathbb{N}}$ *be a family of random variables in S. If* ξ *is tight, then it is relatively compact in distribution, i.e. the closure of* ξ *is compact.*

The Riesz-Bourbaki representation theorem is an adaptation from [1, 5.2 Proposition 5]).

Theorem B.6. *Let X be a completely regular space space and let* ϕ *be a continuous, positive linear functional on the normed space* $C_b(X)$. *Then there exists a tight, finite Borel measure* θ *on X such that for all* $f \in C_b(X)$,

$$\phi(f) = \int f d\theta,$$

if and only if, for every $\epsilon > 0$ *there exists a compact* $K \subseteq X$ *such that for all* $f \in C_b(X)$ *with* $f = 0$ *on K and* $\|f\|_\infty \leq 1$, *one has* $|\phi(f)| \leq \epsilon$.

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