LEIDEN UNIVERSITY

MASTER THESIS

Functional representation of ordered vector spaces without order unit

Author:
Sibel KALKAN

Supervisor: Dr. Ir. O. van Gaans

A thesis submitted in fulfillment of the requirements for the degree of Master of Applied Mathematics

in the

Leiden University

July 5, 2018

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List of Symbols

A^{l}	The set of lower bounds of A
A^{u}	The set of upper bounds of A
$co(A)$	Convex hull of set A
$\mathrm{ext}(A)$	The set of extreme points of A
$C(0,1)$	The space of continuous functions on $(0,1)$
$\mathcal{C}$	Chain
δ_t	The Dirac measure supported at the point t
E^+	Positive cone of E
$\mathcal{E}(T)$	The set of positive extensions of positive operator T
$f_1 \vee f_2$	The supremum of f_1 and f_2
Σ	All positive functionals φ with $\varphi(u) = 1$
Λ	The set of extreme points of Σ
K	Cone
u	Order unit
w^*	weak*topology
$\varphi_k \xrightarrow{w^*} \varphi$	Weak convergence
x^+	The positive part of x
x^-	The negative part of x
$X^{'}$	The topological dual of X
$(X, \leq)$	Ordered vector space with order $\leq$
(X, K)	Ordered vector space with $X^+ = K$
X^δ	Dedekind completion of X
X^ρ	Riesz completion of X

Preface

In functional analysis there are many theorems that say that abstract vector spaces are isomorphic to concrete spaces of functions. For ordered vector spaces, the functional representation gives an embedding in a space of continuous functions. For this it is necessary that the space has an order unit. This condition excludes many interesting cases. In this thesis we investigate the generalization of the functional representation for spaces without an order unit.

The essential problem is the bipositivity of the embedding. To prove bipositivity a suitable Riesz homomorphism must be constructed. This appears to be possible with the aid of the extension theorem of Lipecki-Luxemburg-Schep, at least if there is an order unit. We give a generalization of the Lipecki-Luxemburg-Schep extension theorem that also works in case there is no order unit. We use this statement for a proof of the bipositivity of the functional representation. The most important results are Theorem 4.4, Theorem 4.11 and Theorem 4.12.

In the first chapter we will look at the basic properties of ordered spaces that we will need in this thesis. Chapter 2 contains a couple of theorems, lemmas and definitions. Moreover, in this chapter we prove Theorem 2.12.

Chapter 3 contains an instructive example. In this example we consider a space of continuous functions without order unit, and we go through all the steps of the functional representation.

Functional representation without order unit is described in Chapter 4. In this chapter we give (among other things) the general version of the theorem of Lipecki-Plachky and Thomsen. We use this theorem to prove (without using the 'majorizing'-assumption) that a suitable map on the space X is a Riesz homomorphism. In addition, we end this chapter by giving an example where the condition of our theorem can be verified.

In the end we can weaken the requirement of an order unit, to a weaker but somewhat technical condition.

1. Basic properties of ordered vector spaces

Most definitions and theorems in this section can be found in [1], [2], [3], [4] and [7]. First we start with the definitions of ordered vector space and positive cone.

Definition 1.1. Let X be a (real) vector space. Then a partial order $\leq$ on X is called a **vector space** order if

1. $x, y, z \in X$ and $x \leq y$ imply $x + z \leq y + z$,
2. $x \in X$, $0 \leq x$ and $\alpha \in [0, \infty)$ imply $0 \leq \alpha x$.

X is then called a partially ordered vector space or, briefly, ordered vector space.

Definition 1.2. Let $\leq$ be a vector space ordering in X. Then the set $X^+ := \{x \in X : x \geq 0\}$ of all positive elements in X is a cone in X, called the **positive cone**.

We introduce (X, K) which is an ordered vector space with $K = X^+$. In this case we can retrieve the partial order of the vector space X from K since $x \leq y$ if and only if $y - x \in K$ for all $x, y \in X$. Therefore we can write (X, K) for an ordered vector space. We are particularly interested in spaces in which there are enough positive elements to span the space.

Definition 1.3. Let (X, K) be an ordered vector space. If $X = K - K$ that is, for every $x \in X$ there exist $u, v \in K$ such that $x = u - v$ then X is called **directed**.

Definition 1.4. Let (X, K) be an ordered vector space. An element $u > 0$ is called an **order unit** if for every $x \in X$ there is a $\lambda \in (0, \infty)$ such that $x \in [-\lambda u, \lambda u]$.

Note that u is an order unit if and only if for every $x \in X$ there is $\lambda \in (0, \infty)$ such that $x \leq \lambda u$.

An ordered vector space (X, K) is called **Archimedean** if for every $x, y \in X$ such that $nx \leq y$ for all $n \in \mathbb{N}$ one has that $x \leq 0$.

Theorem 1.5. *Let X be an Archimedean ordered vector space with order unit u. Then*

$$||x||_u = \inf \{\lambda \in [0, \infty) : x \leq \lambda u\}$$

*defines a **norm** on X.*

Definition 1.6. Let X be an ordered vector space. X is called a **Riesz space** if for every $x, y \in X$ the supremum and the infimum of the set $\{x, y\}$ both exist in X. Denote

$$x \vee y := \sup \{x, y\} \text{ and } x \wedge y := \inf \{x, y\}.$$

Definition 1.7. An ordered vector space (X, K) is called a **pre-Riesz space** if for every $x, y, z \in X$ the inclusion $\{x + z, y + z\}^u \subseteq \{x, y\}^u$ implies $z \in K$.

Every Riesz space is a pre-Riesz space since the inclusion in Definition 1.7 reduces to the inequality $(x + z) \vee (y + z) \geq x \vee y$, so $(x \vee y) + z \geq x \vee y$, which implies $z \geq 0$.

Proposition 1.8. *[2, Proposition 4.2] Every pre-Riesz space is directed and every directed Archimedean ordered vector space is pre-Riesz.*

Definition 1.9. A Riesz space E is **Dedekind complete** if for every nonempty $A \subseteq E$ which is bounded above the supremum of A, denoted $\sup A$, exists.

Definition 1.10. Let (X, Y) be an ordered vector spaces. A linear map $f : X \to Y$ is called **bipositive** if for all $x \in X$ and $x \geq 0$ is equivalent to $f(x) \geq 0$.

The next definition, theorem and the proof of the theorem can be found in [4, Theorem 2.4.5., Definition 2.4.6.].

Definition 1.11. Let (X, K) be an ordered vector space and let G be a linear subspace of X.

1. G is called **majorizing** if for every $x \in X$ there is $y \in G$ such that $x \leq y$.
2. G is called **order dense** if for every $x \in X$ one has that $x = \inf\{y \in G\colon\ y \geq x\}$.

Theorem 1.12. *Let X be an ordered vector space. The following statements are equivalent:*

1. *X is a pre-Riesz space.*
2. *There exist a Riesz space Y and a bipositive linear map $i : X \to Y$ such that $i[X]$ is order dense in Y.*
3. *There exist a Riesz space Y and a bipositive linear map $i : X \to Y$ such that $i[X]$ is order dense in Y and generates Y as a Riesz space.*

Moreover, all Riesz spaces Y as in 3 *are isomorphic as Riesz spaces.*

Definition 1.13. Let X be an ordered vector space and let Y be a Riesz space and let $i : X \to Y$ a bipositive linear map such that $i[X]$ is order dense in Y and generates Y as a Riesz space. Then the pair (Y, i) is called a **Riesz completion** of X.

The next theorem concerns existence of Dedekind completions, see [3, Chapter 2].

Theorem 1.14. *If (X, K) is an Archimedean directed order vector space, then there is a Dedekind complete Riesz space X^δ and an embedding: $J : X \to X^\delta$ such that $J[X]$ is order dense in X^δ.*

Let X be an Archimedean ordered vector space with an order unit u and let X' be the set of all $f : X \to \mathbb{R}$ such that f is linear and bounded, with respect to $||.||_u$, that is, there exists M with $|f(x)| \leq M||x||_u$ for all $x \in X$.

Definition 1.15. Let $f : X \to \mathbb{R}$ be linear. If $f(x) \geq 0$ for all $x \in K$ then f is called **positive**.

Definition 1.16. An operator $T : E \to F$ between two Riesz spaces is said to be a **Riesz** or **lattice homomorphism** whenever $T(x \vee y) = T(x) \vee T(y)$ holds for all $x, y \in E$.

For $A \subseteq X$ denote $A^{\mathrm{u}} = \{u \in X : u \geq a \text{ for all } a \in A\}$ and $A^{\mathrm{l}} = \{x \in X : x \leq a \text{ for all } a \in A\}$.

Definition 1.17. A linear map $\phi : X \to \mathbb{R}$ is a **Riesz homomorphism** if for every non-empty finite set $F \subseteq X$ holds

$$\phi(F^{\mathrm{u}})^{\mathrm{l}} \subseteq \phi(F)^{\mathrm{ul}}.$$

Definition 1.18. A linear map $\phi : X \to \mathbb{R}$ is a **Riesz* homomorphism** if for every non-empty finite set $F \subseteq X$ holds

$$\phi(F^{\mathrm{ul}}) \subseteq \phi(F)^{\mathrm{ul}}.$$

We need the following ingredients from the theory of normed spaces.

Definition 1.19. The **weak* topology** on X' is a vector space topology such that for every net $(f_i) \in X'$ and $f \in X'$ we have

$$f_i \to f \text{ in } \textit{weak}^*\textit{topology} \iff \forall x \in X : f_i(x) \to f(x).$$

Theorem 1.20. *[3, Theorem 1.8.6] (Banach-Alaoglu) Let X be a normed vector space and let X' be the dual space with the operator norm.*

The closed unit ball of X'

$$\{f \in X' : ||f||_{X'} \leq 1\}$$

is compact with respect to the weak topology.*

Definition 1.21. A vector f in a **convex** set Σ is said to be an extreme point of Σ if it follows from $f = tf_1 + (1-t)f_2$ with $f_1, f_2 \in \Sigma$ and $0 < t < 1$ that $f_1 = f_2 = f$.

Definition 1.22. Let X be a vector space and let $A \subseteq X$. The **convex hull** co(A) is the smallest convex set that includes A. Then co(A) consists of all convex combinations of A i.e.,

$$\mathrm{co}(A) := \left\{ t_1 v_1 + \ldots + t_n v_n : v_i \in V, t_i \in [0,1], \sum_{i=1}^{n} t_i = 1 \right\}.$$

The next theorem and its proof can be found in [4, Theorem 4].

Theorem 1.23. *(van Haandel) Let (X, K) be a directed Archimedean ordered vector space with Riesz completion (Y, i). For a linear functional $\varphi : X \to \mathbb{R}$ the following statements are equivalent:*

1. *ϕ is a Riesz* homomorphism.*
2. *There exists a unique Riesz homomorphism $\psi : Y \to \mathbb{R}$ with $\psi \circ i = \varphi$.*

Remark 1.24. If u is an order unit on X then $i(u)$ is an order unit on X^ρ.

2. Functional representation of an ordered space with order unit

Let us now consider an Archimedean ordered vector space (X, K) with an order unit u. Define $K' = \{x \in X' : x \geq 0\}$ and we denote $\Sigma := K' \cap \{f \in X' : f(u) = 1\}$. On X we will consider the norm $||.||_u$ induced by the order unit. We are going to look at embedding of ordered spaces in spaces of continuous functions.

Definition 2.1. [4, Definition 2.5.1] Let (X, K) be an Archimean ordered vector space with order unit u. A pair (σ, Ω), where Ω is a compact Hausdorff space and $\sigma : X \to C(\Omega)$ a bipositive linear map, is called a **functional representation** of X if

1. σ maps u to the constant 1-function,
2. $\sigma(X)$ seperates the points of Ω, i.e. for every $w_1 \neq w_2$ in Ω there is an $x \in X$ such that $\sigma(x)(w_1) \neq \sigma(x)(w_2)$.

It is often more convenient to work with spaces of functions rather than abstract, ordered vector spaces. The next theorem presents an embedding into a space of continuous functions, called the functional representation.

Theorem 2.2. *[4, Propositions 1,2] If X is an Archimedean ordered vector space with order unit u, then there is a compact Hausdorff space Ω and a bipositive linear $\Phi : X \to C(\Omega)$ with $\Phi(u) = 1$ such that $\Phi[X]$ is order dense in $C(\Omega)$.*

Lemma 2.3. *If $f \in \Sigma$, then $||f||_{X'} \leq 1$.*

Proof. Let $x \in X$, $||x||_u < 1$. Then from the definition of the norm follows that $x \leq u$ and $-x \leq u$. This gives that $-u \leq x \leq u$. Now $-f(u) \leq f(x) \leq f(u)$, because f is positive. So $|f(x)| \leq f(u) = 1$. So $||f||_{X'} \leq 1$.

□

Lemma 2.4. *Σ is a closed set of the unit ball of X'. So Σ is weak* compact with respect to the weak* topology.*

We next describe the extreme points of Σ. Denote $\Lambda = \text{ext}(\Sigma)$.

Theorem 2.5. *(Hayes)*

$$\Lambda = \{\phi : X \to \mathbb{R} : \phi(u) = 1 \text{ and } \phi \text{ is a Riesz homomorphism}\}.$$

$\bar{\Lambda}$ is the closure in $X^{'}$ with respect to the weak* topology.

Theorem 2.6. *(van Haandel)*

$$\bar{\Lambda} = \{\phi : X \to \mathbb{R} : \phi(u) = 1 \text{ and } \phi \text{ is a Riesz* homomorphism}\}.$$

The set Σ can be recovered from its extreme points Λ by means of a closed convex hull.

Theorem 2.7. *[1, Theorem 3.14](Krein-Milman) If a convex set C of a vector space X is compact for some locally convex topology τ on X, then C has an extreme point. Moreover C is the τ-closed convex hull of its extreme points.*

It follows from Theorem 2.7 that Σ is the weak* closure of Λ.

Let us now do the same construction with the Riesz completion X^ρ of X. Denote

$$\Sigma_{X^\rho} = \{f \in (X^\rho)' \colon\ f \text{ positive, } f(i(u)) = 1\}$$

and $\Omega = \mathrm{ext}(\Sigma_{X^\rho})$.

Lemma 2.8. *Ω is weak* closed in X'.*

Proof. From Theorem 2.5 due to Hayes, follows that

$$\Omega = \{\varphi \in \Sigma_{X^\rho} \colon\ \varphi \text{ is a Riesz homomorphism, } \varphi(i(u)) = 1\}\,.$$

Let $(\varphi_i)_{i \in I}$ be a net in Ω and $\varphi \in X'$ such that $\varphi_i \to \varphi$ in the weak* topology. We want to prove that $\varphi \in \Omega$. Let $y_1, y_2 \in X^\rho$ then $\varphi(y_1 \vee y_2) = \lim_i \varphi_i(y_1 \vee y_2) = \lim_i \varphi_i(y_1) \vee \varphi_i(y_2) = \varphi(y_1) \vee \varphi(y_2)$, because $\varphi_i \in \Omega$. So φ is a Riesz homomorphism and this means that $\varphi \in \Omega$. □

We also need the following well-known result [6] for the proof of Theorem 2.11.

Theorem 2.9. *(Stone-Weierstrass) Let Ω be a non-empty compact Hausdorff space. Let D be a linear subspace of $C(\Omega)$ such that*

1. *D seperates the points of Ω: for all $\varphi_1, \varphi_2 \in \Omega, \varphi_1 \neq \varphi_2$ there exists an $f \in D$ such that $f(\varphi_1) \neq f(\varphi_2)$.*
2. *If $f_1, f_2 \in D$ then $f_1 \vee f_2 \in D$.*

Then D is norm dense in $(C(\Omega), ||.||_\infty)$.

Define $\Phi_{X^\rho}(y) := (\varphi \mapsto \varphi(y)), \varphi \in \Omega$ and $y \in X^\rho$. We will show that $\Phi_{X^\rho} : X^\rho \to C(\Omega)$ is bipositive and $\Phi_{X^\rho}[X^\rho]$ is order dense in $C(\Omega)$.

Lemma 2.10. *[4, Lemma 9] Let Ω be a compact Hausdorff space and let D be a linear subspace of $C(\Omega)$. If $D \subseteq C(\Omega)$ is norm dense, then is D order dense in $C(\Omega)$.*

Theorem 2.11. *[3, Proposition 1.8.11] Let Ω and Λ as above. The topological spaces Ω and $\bar{\Lambda}$ are homeomorphic. That means that there exists a bijection $h : \Omega \to \bar{\Lambda}$ such that h and h^{-1} are continuous.*

Proof. Recall that $\varphi \in \Lambda_{X^\rho}$. Define $h(\varphi) := (x \mapsto \varphi(i(x)))$ with $\varphi \in \Lambda_{X^\rho}$. Then is $h(\varphi) : X \to \mathbb{R}$ is linear and positive. Moreover, we have $h(\varphi)(u) = \varphi(i(u)) = 1$. So $h(\varphi) \in \Sigma$. This gives that $h : \Omega \to \Sigma$. Now we want to prove that $h(\Omega) \subseteq \bar{\Lambda}$. Let $\varphi \in \Omega$. Then $\varphi : X^\rho \to \mathbb{R}$ is a Riesz homomorphism. From this follows that $x \mapsto \varphi(i(x))$ is a Riesz* homomorphism on X. So $h(\varphi) : X \to \mathbb{R}$ is a Riesz* homomorphism and so $h(\varphi) \in \bar{\Lambda}$. So $h : \Omega \to \bar{\Lambda}$.

Next we want to show that h is bijective. Let $\varphi \in \bar{\Lambda}$. Then $\varphi : X \to \mathbb{R}$ is a Riesz* homomorphism. Then there exists a unique Riesz homomorphism $\psi : X^\rho \to \mathbb{R}$ such that $\varphi = \psi \circ i$. Then $\psi \in \Omega$ and $h(\psi) = \varphi$. So h is surjective.

If $\varphi_1, \varphi_2 \in \Omega$ are such that $h(\varphi_1) = h(\varphi_2)$ then for all $x \in X$ we get $h(\varphi_1)(x) = h(\varphi_2)(x)$ and so $\varphi_1(i(x)) = \varphi_2(i(x))$. So $\varphi_1 = \varphi_2$ on $i(X)$.

Because φ_1 is a Riesz homomorphism that extends $\varphi_1 \circ i$ and φ_2 is a Riesz homomorphism that extends $\varphi_2 \circ i$ and $\varphi_1 \circ i = \varphi_2 \circ i$ it follows from the uniqueness of the extension that $\varphi_1 = \varphi_2$. So h is injective. And this gives that h is bijective.

Next we show that h is continuous.

If $\varphi_k \stackrel{w^*}{\to} \varphi$ in Ω, then for all $y \in X^\rho$ we have $\varphi_k(y) \to \varphi(y)$. Then for all $x \in X$ $\varphi_k(i(x)) \to \varphi(i(x))$ and so $h(\varphi_k) \stackrel{w^*}{\to} h(\varphi)$. So h is continuous. Because Ω is compact and $\bar{\Lambda}$ is Hausdorff it follows that h is a homeomorphism (see [3, Proposition 1.8.11]). Therefore, h^{-1} is continuous. □

Now we are ready to show the first part of Theorem 2.2.

Theorem 2.12. *If (X, K) is an Archimedean directed partial ordered vector space with order unit u. Then there exists a space of continuous functions $C(\Omega)$ and an injective, bipositive, linear map $\Phi : X \to C(\Omega)$.*

Proof. Recall that

$$X' = \{f : X \to \mathbb{R} \text{ such that } \exists M|f(x)| \leq M||x||_u \text{ where } f \text{ is linear and bounded}\}.$$

Let $K' = \{f \in X' : f \geq 0\} \subseteq X'$.
Define $\Phi : X \to C(\bar{\Lambda})$ by $(\Phi(x))(\lambda) = \lambda(x)$, $\lambda \in \bar{\Lambda}, x \in X$. Consider the closure $\bar{\Lambda}$ of Λ with respect to the weak* topology. Then $\bar{\Lambda} \subseteq \Sigma$, so $\bar{\Lambda}$ is weak* compact. Consider $C(\bar{\Lambda})$. We want to show that $\Phi : X \to C(\bar{\Lambda})$ is injective and bipositive.

If $x \geq 0$ then for $f \in \bar{\Lambda} \subseteq \Sigma$ we have f is positive, so $f(x) \geq 0$. So $\Phi(x) \geq 0$. If $\Phi(x) \geq 0$ then for all $f \in \bar{\Lambda}$ we have $f(x) \geq 0$. By Theorem 2.7 for all $f \in \Sigma$ and $\alpha \geq 0$ we have $(\alpha f)(x) \geq 0$. Then for all $f \in K'$ it follows that $f(x) \geq 0$. Therefore, we find that $x \geq 0$. So Φ is bipositive and so injective. □

The proof of Theorem 2.2 is complete by the next result, where we use the same notations as in the proof of Theorem 2.12.

Theorem 2.13. *$\Phi[X]$ is order dense in $C(\bar{\Lambda})$.*

Proof. Let Ω the extreme points of

$$\Sigma_{X^\rho} = \{\varphi : X^\rho \to \mathbb{R} \text{ positive linear}, \varphi(i(u)) = 1\}.$$

Define $D := \Phi_{X^\rho}[X^\rho] \subseteq C(\Omega)$. Let $\varphi_1, \varphi_2 \in \Omega$, $\varphi_1 \neq \varphi_2$ two different Riesz homomorphisms on X^ρ. So there exists $y \in X^\rho$ such that $\varphi_1(y) \neq \varphi_2(y)$. Then holds $\Phi_{X^\rho}(y)(\varphi_1) \neq \Phi_{X^\rho}(y)(\varphi_2)$. So $f := \Phi_{X^\rho}(y) \in D$ and $f(y_1) \neq f(y_2)$.

Let $f_1, f_2 \in D$. Then there are y_1, y_2 such that $f_1 = \Phi_{X^\rho}(y_1)$ and $f_2 = \Phi_{X^\rho}(y_2)$ and for all $\varphi \in \Omega$,

$$(f_1 \vee f_2)(\varphi) = f_1(\varphi) \vee f_2(\varphi) = \varphi(y_1) \vee \varphi(y_2) = \varphi(y_1 \vee y_2) = \Phi_{X^\rho}(y_1 \vee y_2)(\varphi).$$

So $f_1 \vee f_2 = \Phi_{X^\rho}(y_1 \vee y_2) \in D$. So from the Theorem of Stone-Weierstrass, 2.9, we get that D is norm dense in $C(\Omega)$. From Lemma 2.10 follows that $D = \Phi_{X^\rho}[X^\rho]$ is order dense in $C(\Omega)$.

Now this gives us the following:

1. $i : X \to X^\rho$ is bipositive and $i[X^\rho]$ is order dense in X^ρ.
2. $\Phi_{X^\rho} : X^\rho \to C(\Omega)$ is bipositive and $\Phi_{X^\rho}[X^\rho]$ order dense in $C(\Omega)$.

So this gives that:

$$\Phi_{X^\rho} \circ i : X \to C(\Omega) \text{ is bipositive and } \Phi_{X^\rho}[i[X]] \text{ order dense in } C(\Omega)$$

Since by Theorem 2.11 the spaces Ω and $\bar{\Lambda}$ are homeomorphic, we can replace $C(\bar{\Lambda})$ by $C(\Omega)$ for all purposes. □

Let us now complete the proof of Theorem 2.2.

Proof. Let X be an Archimedean ordered vector space with order unit u. From Theorem 2.12 we conclude that there exists a space of continuous functions $C(\Omega)$ and a map $\Phi : X \to C(\Omega)$ which is injective, bipositive and linear. From Theorem 2.11 it now follows that Ω and $\bar{\Lambda}$ are homeomorphic, which gives us that $\Phi : X \to C(\bar{\Lambda})$. Theorem 2.13 now gives us that $\Phi[X]$ is order dense in $C(\Lambda)$. Therefore, the subspace $\Phi[X]$ is order dense in $C(\Omega)$. □

Corollary 2.14. *If X is an Archimedean ordered vector space with order unit u and $\bar{\Lambda}$ is as defined above and $\Phi : X \to C(\bar{\Lambda})$ is given by $\Lambda(x)(\varphi) = \varphi(x), \varphi \in \bar{\Lambda}, x \in X$, then Φ is a bipositive linear map and $\Phi(X)$ is order dense in $C(\bar{\Lambda})$. Consequently, the Riesz subspace of $C(\bar{\Lambda})$ generated by X is the Riesz completion of X.*

3. An instructive example

We denote by $C(0,1)$ the space of continuous functions on the open interval $(0,1)$. This is a space of functions which does not have an order unit. What happens when we apply the functional representation method to this space? It turns out that we obtain $(0,1)$ as underlying set and the identity map as embedding map. First denote $\delta_t : C(0,1) \to \mathbb{R}$ where $\delta_t(g) = g(t)$ for all $g \in C(0,1)$.

Theorem 3.1. *If $C(0,1)$ and $h : C(0,1) \to \mathbb{R}$ is a Riesz homomorphism then there exists $t \in (0,1)$ and $\alpha \geq 0$ such that $h(f) = \alpha f(t)$ for all $f \in C(0,1)$.*

Proof. For $g \in C[0,1]$ we have $g|_{(0,1)} \in C(0,1)$. Define $\tilde{h}(g) := h(g|_{(0,1)})$. This gives $\tilde{h} : C[0,1] \to \mathbb{R}$. We want to prove that $\tilde{h}$ is a Riesz homomorphism on $C[0,1]$.

First we show that $\tilde{h}$ is linear. Let $f, g \in C[0,1]$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}
\tilde{h}(\alpha(f+g)) &= h(\alpha(f+g)|_{(0,1)}) \\
&= h(\alpha f|_{(0,1)} + \alpha g|_{(0,1)}) \\
&= h(\alpha f|_{(0,1)}) + h(\alpha g|_{(0,1)}) \\
&= \tilde{h}(\alpha f) + \tilde{h}(\alpha g).
\end{aligned}$$

So $\tilde{h}$ is linear.

For the supremum of f and g we have $(f \vee g)(x) = \max\{f(x), g(x)\}$ for every $x \in [0,1]$. Hence

$$\begin{aligned}
\tilde{h}(f \vee g) &= h((f \vee g))|_{(0,1)}) \\
&= h(f|_{(0,1)} \vee g|_{(0,1)}) \\
&= h(f|_{0,1}) \vee h(g|_{(0,1)}) \\
&= \tilde{h}(f) \vee \tilde{h}(g).
\end{aligned}$$

So $\tilde{h}$ is a Riesz homomorphism. Then due to [1, Theorem 2.33], there exists $t \in [0,1]$ and $\alpha \geq 0$ such that $\tilde{h}(g) = \alpha g(t)$ for all $g \in C\,[0,1]$.

Now we want to show that $t \neq 0$ and $t \neq 1$. It suffices to consider the case $\alpha > 0$. Suppose that $t = 0$, then we define $g_n(x) = \frac{1}{x}$ where $\frac{1}{n} < x \leq 1$ and $g_n(x) = n$ for $0 \leq x \leq \frac{1}{n}$. This gives

$$\begin{aligned}
h(g_{n|(0,1)}) &= \tilde{h}(g_n) \\
&= \alpha g_n(0) \\
&= \alpha n.
\end{aligned}$$

Let $f(x) = \frac{1}{x}$ for $x \in (0,1)$. Then $f \geq g_n|_{(0,1)}$ for all n. So this gives that $h(f) \geq h(g_n|_{(0,1)}) = n$ for all n, which is a contradiction. In the same way we get a contradiction when we suppose that $t = 1$. So this gives that $t \neq 0$ and $t \neq 1$. In order words $t \in (0,1)$.

□

Proposition 3.2. *$t \mapsto \delta_t : (0,1) \to \Lambda$ is bijective.*

Proof. We have by Theorem 2.5 the following equality:

$$\Lambda = \{f : C(0,1) \to \mathbb{R} \colon \, f \text{ Riesz homomorphism and } f(1) = 1\}\,.$$

From Theorem 3.1 it follows that $t \mapsto \delta_t : (0,1) \to \Lambda$ is surjective. Suppose now $t_1, t_2 \in (0,1)$ such that $\delta_{t_1} = \delta_{t_2}$. Then for all $f \in C(0,1)$ $\delta_{t_1}(f) = \delta_{t_2}(f)$. In order words $f(t_1) = f(t_2)$. So the map is injective. Therefore the map $t \mapsto \delta_t$ is bijective.

□

Proposition 3.3. *$t \mapsto \delta_t : (0,1) \to \Lambda$ is a homeomorphism.*

Proof. First we prove that $t \mapsto \delta_t$ is continuous. Let (t_n) in $(0,1)$ and $t \in (0,1)$ be such that $t_n \to t$. We want to prove that for all $f \in C(0,1)$ we have that $\delta_{t_n}(f) \to \delta_t(f)$. Note that $f(t_n) = \delta_{t_n}(f) \to \delta_t(f) = f(t)$. f is continuous so $f(t_n) \to f(t)$. This gives that $t \mapsto \delta_t$ is continuous.

Now we prove that the inverse map is continuous. Let (t_i) be a net in $(0,1)$ and t in $(0,1)$ such that $\delta_{t_i} \to \delta_t$. Suppose that $t_i \to t$ does not hold. Then there is $\epsilon > 0$ and a subnet $(s_j)_{j\in J}$ of $(t_i)_{i\in I}$ such that $|j - t| \geq \varepsilon$ for every $j \in J$. Due to Urysohn's lemma, there is a $g \in [0,1]$ such that $g(t) = 1$ and $g = 0$ on $[0,1] \setminus (t-\varepsilon, t+\varepsilon)$. Take $f := g|_{(0,1)}$. Then $f \in C(0,1)$, $f(t) = g(t) = 1$ and for every $j \in J$ we have $f(s_j) = g(s_j) = 0$. Hence, $f(s_j) \not\to f(t)$, so that $f(t_i) \not\to f(t)$. This contradicts $\delta_{t_i}(f) \to \delta_t(f)$. Consequently, $t_i \to t$. Therefore the inverse of $t \mapsto \delta_t$ is also continuous. □

Theorem 3.4. *Let $J : C(\Lambda) \to C(0,1)$ be defined by $J(f)(t) = f(\delta_t)$ where $f \in C(\Lambda)$. Then J is a Riesz isomorphism.*

Proof. We want to prove that J is linear, bijective and continuous.

First we want to show that J is linear. For $f, g \in C(\Lambda)$ and $t \in (0,1)$ we have

$$\begin{aligned} J(f+g)(t) &= (f+g)(\delta_t) \\ &= f(\delta_t) + g(\delta_t) \\ &= J(f)(t) + J(g)(t). \end{aligned}$$

So J is linear.

Now we want to prove that J is bijective. Suppose that $f_1, f_2 \in C(\Lambda), f_1 \neq f_2$. This gives that there exists an $x \in \Lambda$ such that $f_1(x) \neq f_2(x)$. Then $J(f_1)(t) = f_1(\delta_t)$ and $J(f_2)(t) = f_2(\delta_t)$. Since $x \in \Lambda$ due to Theorem 3.1 there exists $t \in (0,1)$ such that $x = \delta_t$. So

$$f_1(\delta_t) = f_1(x) \neq f_2(x) = f_2(\delta_t).$$

So $J(f_1) \neq J(f_2)$ and J is injective.

Next we show that J is surjective. Define $g(\delta_t) := f(t)$ with $t \in (0,1)$. We want to show that $g \in C(\Lambda)$ and $J(g) = f$. Let (x_i) be a net in Λ and $x \in \Lambda$ be such that $x_i \to x$. Take (t_i) in $(0,1)$ and $t \in (0,1)$ such that $x_i = \delta_{t_i}$ and $x = \delta_t$. As $\delta_{t_i} \to \delta_t$, we have $\delta_{t_i}(f) \to \delta_t(f)$. So $g(x_i) = f(t_i) \to f(t) = g(x)$. So g is continuous.

Since $J(g)(t) = g(\delta_t) = f(t)$, so $J(g) = f$, we obtain that J is surjective.

At last we want to show that J is a Riesz homomorphism.

For $f, g \in C(\Lambda)$ and $t \in (0,1)$,

$$\begin{aligned} J(f \vee g)(t) &= (f \vee g)(\delta_t) \\ &= f(\delta_t) \vee g(\delta_t) \\ &= J(f)(t) \vee J(g)(t). \end{aligned}$$

So the map $J : C(\Lambda) \to C(0,1)$ is a Riesz isomorphism. $\square$

4. Functional representation without order unit

Let X be an Archimedean directed ordered vector space with

$$\Lambda = \{\varphi : X \to \mathbb{R} \colon \varphi \text{ is a Riesz* homomorphism}\}.$$

On Λ we consider the topology of pointwise convergence in other words $\varphi_i \to \varphi$ if and only if $\varphi_i(x') \to \varphi(x')$ for all $x' \in X$. Define $J(x)(\varphi) = \varphi(x)$, $\varphi \in \Lambda$.

Lemma 4.1. *For every $x \in X$, the map $J(x) : \Lambda \to \mathbb{R}$ is continuous.*

Proof. Let $x \in X$. If $\varphi_i \to \varphi$, then $J(x)(\varphi_i) = \varphi_i(x) \to \varphi(x) = J(x)(\varphi)$. Hence $J(x)$ is continuous. □

4.1 Positive embedding

Lemma 4.2. *$J : X \to C(\Lambda)$ is linear and positive.*

Proof. First we want to show that J is linear. Let $x, y \in X$. For every $\varphi \in \Lambda$ we have

$$\begin{aligned} J(x+y)(\varphi) &= \varphi(x+y) \\ &= \varphi(x) + \varphi(y) \\ &= J(x)(\varphi) + J(y)(\varphi). \end{aligned}$$

So J is linear.

Now we want to show that J is positive.

Let $x \in X$ with $x \geq 0$, then $J(x)(\varphi) = \varphi(x) \geq 0$ for every $\varphi \in \Lambda$. □

4.2 Bipositivity of the embedding

We would like to show that J is bipositive. We need an extra condition on X.

Proposition 4.3. *J is bipositive if and only if for all $x \in X$ and $x \ngeq 0$ there exists a Riesz* homomorphism $\varphi : X \to \mathbb{R}$ with $\varphi(x) < 0$.*

Proof. First assume that J is bipositive. Let $x \in X$ be such that $x \ngeq 0$. Then $J(x) \ngeq 0$, so there exists $\varphi \in \Lambda$ such that $J(x)(\varphi) < 0$. Then $\varphi(x) < 0$. For a proof of the converse implication, let $x \in X$ with $J(x) \geq 0$. Suppose that $x \ngeq 0$. Then there exists a Riesz* homomorphism φ with $\varphi(x) < 0$ and then $J(x)(\varphi) = \varphi(x) < 0$, but this gives us a contradiction. So $x \geq 0$.

Thus, J is bipositive if and only if for all $x \in X$, $x \ngeq 0$ there exists a Riesz* homomorphism $\varphi : X \to \mathbb{R}$ with $\varphi(x) < 0$. □

Therefore, bipositivity of J comes down to existence of a Riesz* homomorphism as in Proposition 4.3. Under an extra condition, we will show existence of such a Riesz* homomorphism by constructing it on a subspace and then use a suitable extension theorem. Our analysis yields the following main theorem.

Theorem 4.4. *Let X be a Pre-Riesz space with Riesz completion X^ρ such that for every $x_0 \in X$, $x_0 \ngeq 0$ and $G := span\{x_0^+, x_0^-\}$ and $\varphi : G \to \mathbb{R}$ with $\varphi(\alpha x_0^+ + \beta x_0^-) = \beta$ there exists a majorizing subspace H of X^ρ and a positive linear $\psi : H \to \mathbb{R}$ such that $\psi = \varphi$ on G and for all $x \in H$*

$$\inf\{\inf\{\psi(z) : z \in H \text{ and } -z \leq x - y \leq z\} : y \in G\} = 0.$$

Let Λ and J be given by

- *$\Lambda = \{\varphi : X \to \mathbb{R} : \varphi$ a Riesz* homomorphism$\}$ with the topology of pointwise convergence*
- *$J(x)(\varphi) = \varphi(x)$, $\varphi \in \Lambda$, $x \in X$.*

Then $J : X \to C(\Lambda)$ is bipositive and linear.

We will start by explaining how we will use an extension theorem for Riesz homomorphisms to obtain a Riesz* homomorphism as needed in Proposition 4.3.

Let us for a moment consider the case that X is a Riesz space. If $x_0 \in X$ and $x_0 \ngeq 0$, then we can construct a Riesz homomorphism φ on $G := \text{span}\{x_0^-, x_0^+\}$ with $\varphi(x_0) < 0$. In case G would be majorizing, we could extend φ to a Riesz homomorphism on all of X by means of a theorem due to Lipecki, Luxemburg and Schep.

Theorem 4.5. *[1, Theorem 2.29] (Lipecki-Luxemburg-Schep) Let E and F be two Riesz spaces with F Dedekind complete. If G is a majorizing Riesz subspace of E and $T : G \to F$ is a Riesz homomorphism, then T extends to all of E as a Riesz homomorphism.*

The next proposition yields existence of a Riesz homomorphism $\varphi : G \to \mathbb{R}$ with $\varphi(x_0) < 0$.

Proposition 4.6. *If E is a Riesz space and $x_0 \in E \setminus E_+$, then there exists a Riesz homomorphism $\varphi : G \to \mathbb{R}$ with $\varphi(x) < 0$, where $G = span\{x_0^+, x_0^-\}$.*

Proof. Consider G :=span$\{x_0^+, x_0^-\}$. Then is G a Riesz subspace of E. Define $\varphi : G \to \mathbb{R}$ by $\varphi(\alpha x_0^+ + \beta x_0^-) = \beta$, where $\alpha, \beta \in \mathbb{R}$. Then $\varphi : G \to \mathbb{R}$ is linear, $\varphi(x_0) = \varphi(x_0^+ - x_0^-) = -1 < 0$. In addition, φ is a Riesz homomorphism. Indeed for $\alpha, \beta \in \mathbb{R}$ we have

$$(\alpha x_0^+ + \beta x_0^-)^+ = \begin{cases} \alpha x_0^+ + \beta x_0^- & \alpha, \beta \geq 0 \\ \beta x_0^- & \alpha < 0, \beta \geq 0 \\ \alpha x_0^+ & \alpha \geq 0, \beta < 0 \\ 0 & \alpha < 0, \beta < 0 \end{cases}$$

so that $\varphi((\alpha x_0^+ + \beta x_0^-)^+) = \begin{cases} \beta & \beta \geq 0 \\ 0 & \beta < 0 \end{cases} = \beta^+$.

In other words, $\varphi(x^+) = (\varphi(x))^+$ for all $x \in G$. So $\varphi : G \to \mathbb{R}$ is a Rieszhomomorphism. □

In many situations, the subspace G will not be majorizing. In order to obtain a suitable Riesz homomorphism on X^ρ we need an extension theorem for Riesz homomorphisms without assuming that the subspace is majorizing.

4.3 Extension of Riesz homomorphisms

First we will give a brief discussion of the Lipecki-Luxemburg-Schep extension theorem stated in Theorem 4.5. Let E and F be Riesz spaces, let G be a subspace of E, and let $T : G \to F$ be a positive operator. Define $\mathcal{E} = \mathcal{E}(T)$ to be the set of positive operators from E to F extending T.

Theorem 4.7. *[1, Theorem 1.33] (Lipecki) Let E and F be two Riesz spaces with F Dedekind complete. If G is a majorizing vector subspace of E and $T : G \to F$ is a positive operator, then the nonempty convex set $\mathcal{E}(T)$ has an extreme point.*

Theorem 4.8. *[1, Theorem 1.31] (Lipecki-Plachky-Thomsen). Let E and F be two Riesz spaces with F Dedekind complete. If G is a vector subspace of E and $T : G \to F$ is a positive operator, then for an operator $S \in \mathcal{E}(T)$ the following statements are equivalent:*

- *S is an extreme point of $\mathcal{E}(T)$.*
- *For each $x \in E$ we have that $\inf \{S(|x-y|)\colon\ y \in G\} = 0$.*

The proof of Theorem 4.5 is given in [1, Proof of Theorem 2.29] and is based on Theorem 4.7 and Theorem 4.8.

The following proof is based on the proof of [1, Theorem 2.51].

Proposition 4.9. *Let E, F be Riesz spaces, F Dedekind complete. Let G be a subspace of E and let $T : G \to F$ be a Riesz homomorphism. If $S \in \mathcal{E}(T)$ is an extreme point, then S is a Riesz homomorphism.*

Proof. By Theorem 4.8 we have that for each $x \in E$ the equality $\inf \{S(|x-y|)\colon\ y \in G\} = 0$ holds. For any $x \in E$ and $y \in G$ we have $S|y| = T|y| = |Ty| = |Sy|$ (since T is a Riesz homomorphism) and thus

$$\begin{aligned} S|x| &\leq S|x-y| + S|y| \\ &= S|x-y| + |Sy| \\ &\leq S|x-y| + |Sy - Sx| + |Sx| \\ &\leq 2S|x-y| + S|x|. \end{aligned}$$

So for any $x \in E$, taking the infimum over all $y \in G$ now yields $S|x| \leq |Sx| \leq S|x|$. Hence $S|x| = S|x|$ for all $x \in E$ and this proves that S is a Riesz homomorphism. □

As discussed in Section 4.2, we need an extension theorem for Riesz homomorphisms without the condition that the subspace is majorizing. We have the following generalization of Theorem 4.7. Its proof relies on Zorn's lemma, which we state first. The statement can be found in many textbooks, for example in [5].

Lemma 4.10. *Let $(P, \leq)$ be a partially ordered set such that every chain has an upper bound in P. Then $(P, \leq)$ has a maximal element.*

Let E, F be Riesz spaces with F Dedekind complete and let H be a majorizing subspace of E. Consider a positive operator $S : H \to F$. Then we define $P_{H,S} : E \to F$ by

$$P_{H,S}(x) = \inf \{S(y)\colon\ y \in H \text{ and } x \leq y\}.$$

Theorem 4.11. *Let E, F be Riesz spaces, F Dedekind complete. Let G be a Riesz subspace of E and $T : G \to F$ a positive operator. If there exists a majorizing subspace H of E with $G \subseteq H$ and a positive operator $S : H \to F$ such that $S = T$ on G and for every $x \in H$*

$$\inf \{P_{H,S}(|x-y|)\colon\ y \in G\} = 0 \text{ in } F,$$

then $\mathcal{E}(T)$ has an extreme point.

Proof. We adapt the proof given in [1, Theorem 1.33] to our setting. From Theorem 4.8 it follows that it suffices to prove the existence of an $S \in \mathcal{E}(T)$ satisfying

$$\inf \{S(|x-y|)\colon\ y \in G\} = 0$$

for all $x \in E$.

We consider the pairs (H, S) where H is a majorizing vector subspace of E and where S is a positive operator from H to F. It is clear that $P_{H,S}$ is a sublinear mapping satisfying $P_{H,S}(y) = S(y)$ for every $y \in H$. In addition, if (H_1, S_1) and (H_2, S_2) satisfy $H_1 \supseteq H_2$ and $S_2 = S_1$ on H_1, then $P_{H_1,S_1}(x) \leq P_{H_2,S_2}(x)$ holds for all $x \in E$.

Let $\mathcal{C}$ be the collection of all pairs (H, S), where H is a subspace of E and $S : H \to F$ a positive operator such that

- $H \supseteq G$ and H is majorizing
- $S = T$ on G
- For every $x \in H$ we have $\inf \{P_{H,S}(|x - y|) \colon\ y \in G\} = 0$ in F.

Note that the condition on the existence of a majorizing $H \supseteq G$ yields that $\mathcal{C}$ is non-empty.

If we define a binary relation $\geq$ on $\mathcal{C}$ by taking $(H_2, S_2) \geq (H_1, S_1)$ whenever $H_2 \supseteq H_1$ and $S_2 = S_1$ on H_1, then $\geq$ is an order relation on $\mathcal{C}$.

We want to show that every chain of $\mathcal{C}$ has an upper bound in $\mathcal{C}$.

Recall that a chain in $\mathcal{C}$ is a subset $\mathcal{A} \subseteq \mathcal{C}$ which is not empty and for all $(H_1, S_1), (H_2, S_2) \in \mathcal{A}$ holds $(H_1, S_1) \leq (H_2, S_2)$ or $(H_1, S_1) \geq (H_2, S_2)$.

Let $\mathcal{A}$ be a chain in $\mathcal{C}$. Take

$$\hat{H} := \bigcup \{H \colon\ (H, S) \in \mathcal{A}\}\,.$$

For $x \in \hat{H}$ there is a $(H, S) \in \mathcal{A}$ with $x \in H$ and then we define $\hat{S}x := Sx$. We claim that we get the following properties:

- $\hat{H}$ is a linear subspace of E,
- $\hat{S} : \hat{H} \to F$ is positive and linear,
- $\hat{S}x = Tx$ for $x \in G$,
- for every $x \in \hat{H} : \inf \left\{P_{\hat{H},\hat{S}}(|x - y|) \colon\ y \in G\right\} = 0$.

From this we get that $(\hat{H}, \hat{S}) \in \mathcal{C}$ and $(\hat{H}, \hat{S}) \geq (H, S)$ for all $(H, S) \in \mathcal{A}$. So $(\hat{H}, \mathcal{S})$ is an upper bound of $\mathcal{A}$ in $\mathcal{C}$.

We can easily prove the first three properties by straightforward verification. The last property seems more difficult and we give a proof of that now.

For $x \in \hat{H}$ there is a $(H, S) \in \mathcal{A}$ with $x \in H$. By definition, for $z \in E$ we have

$$P_{\hat{H},\hat{S}}(z) = \inf \left\{\inf \hat{S}(y) \colon\ y \in \hat{H} \text{ and } z \leq y\right\}.$$

For $(H, S) \in \mathcal{A}$ holds that $\hat{H} \supseteq H$ and $\tilde{S}(y) = S(y)$ for all $y \in H$. This gives

$$\begin{aligned}
P_{\hat{H},\hat{S}}(z) &= \inf \left\{\inf \hat{S}(y) \colon\ y \in \hat{H} \text{ and } z \leq y\right\} \\
&\leq \inf \left\{\inf \hat{S}(y) \colon\ y \in H \text{ and } z \leq y\right\} \\
&= \inf \left\{\inf S(y) \colon\ y \in H \text{ and } z \leq y\right\} \\
&= P_{H,S}(z).
\end{aligned}$$

So, for every $y \in G$,

$$P_{\hat{H},\hat{S}}(|x - y|) \leq P_{H,S}(|x - y|).$$

So

$$0 \le \inf \left\{ P_{\hat{H},\hat{S}}(|x-y|)\colon\ y \in G \right\} \le \inf \left\{ P_{H,S}(|x-y|)\colon\ y \in G \right\} = 0.$$

Therefore, $\inf \left\{ P_{\hat{H},\hat{S}}(|x-y|)\colon\ y \in G \right\} = 0$. So $(\hat{H},\hat{S}) \in \mathcal{C}$.

By Zorn's lemma the collection $\mathcal{C}$ has a maximal element, which we denote by (M,R).
Now we want to show that $M = E$. If this is done, then for every $x \in E = M$ we have $P_{M,R}(x) = R(x)$, which by Theorem 4.8 shows that R must be an extreme point of $\mathcal{E}(T)$.
Let us show that $M = E$. We assume by way of contradiction that there exists some vector x that does not belong to M. Consider

$$H = \{u + \lambda x\colon\ u \in M \text{ and } \lambda \in \mathbb{R}\},$$

and define $S : H \to F$ by

$$S(u + \lambda x) = R(u) + \lambda P_{M,R}(x).$$

Clearly, M is a proper subspace of H, $S = R$ holds on M, and $S : H \to F$ is a positive operator. (For the positivity of S let $u + \lambda x \ge 0$ with $u \in M$. For $\lambda > 0$ the inequality $x \ge -\frac{u}{\lambda}$ implies $p_{M,R}(x) \ge -R(\frac{u}{\lambda})$, and consequently $S(u + \lambda x) = R(u) + \lambda P_{M,R}(x) \ge 0$. The case $\lambda < 0$ is similar, while the case $\lambda = 0$ is trivial.)
At last, we verify that (H,S) satisfies $\inf \{P_{H,S}(|u-y|)\colon\ y \in G\} = 0$ for every $u \in H$. First we observe that by the sublinearity of $P_{H,S}$ the set

$$V = \{y \in E\colon\ \inf \{P_{H,S}(|y-z|)\colon\ z \in M\} = 0\}$$

is a vector subspace of E satisfying $M \subseteq V$. Since for $z \in M$ we have $z - x \in H$, we obtain $P_{H,S}(z-x) = S(z-x) = R(z) - P_{M,R}(x)$. Hence

$$\begin{aligned} 0 &\le \inf \{P_{H,S}(|x-z|)\colon\ z \in M\} \\ &\le \inf \{P_{H,S}(z-x)\colon\ z \in M \text{ and } x \le z\} \\ &= \inf \{R(z) - P_{M,R}(x)\colon\ z \in M \text{ and } x \le z\} \\ &= \inf \{R(z)\colon\ z \in M \text{ and } x \le z\} - P_{M,R}(x) = P_{M,R}(x) - P_{M,R}(x) = 0. \end{aligned}$$

Therefore $x \in V$, and thus we find that $H \subseteq V$. Notice that for any $u \in H$, $z \in M$, and $v \in G$ we have

$$\begin{aligned} P_{H,S}(|u-v|) &\le P_{H,S}(|u-z|) + P_{H,S}(|v-z|) \\ &\le P_{H,S}(|u-z|) + P_{M,R}(|v-z|). \end{aligned}$$

So it follows from $(M,R) \in \mathcal{C}$ and $H \subseteq V$ that for all $u \in H$ we have

$$\inf \{p_{H,S}(|u-v|)\colon\ v \in G\} = 0.$$

So, $(H,S) \in \mathcal{C}$. However, $(H,S) \ge (M,R)$ and $(H,S) \ne (M,R)$, which contradicts the maximality of (M,R). Therefore, $M = E$ must be true, as required.

Now we want to show that R is an extreme point.

For $x \in E$ and $y \in G$ we have $|x-y| \in E = M$, so $R(|x-y|) = P_{M,R}(|x-y|)$, hence

$$\inf \{R(|x-y|)\colon\ y \in G\} = \inf \{P_{M,R}(|x-y|)\colon\ y \in G\} = 0.$$

□

Theorem 4.12. *Let E, F be Riesz spaces, with F Dedekind complete. Let G be a Riesz subspace of E and $T : G \to F$ a Riesz homomorphism. Suppose that there exists a majorizing subspace H of E with $G \subseteq H$ and a positive operator $S : H \to F$ such that $S = T$ on G and* *for all* $x \in H$

$$\inf \{P_{H,S}(|x-y|) : y \in G\} = 0 \text{ in } F.$$

Then T *extends to a Riesz homomorphism* $R : E \to F$.

Proof. From Theorem 4.11 it follows that there exists an extreme point R of $\mathcal{E}(T)$. Proposition 4.9 shows that R is in fact a Riesz homomorphism. □

4.4 Proof of main theorem

We are now in position to prove Theorem 4.4.

Proof. Consider $x_0 \in X, x_0 \ngeq 0$. Let X^ρ be the Riesz completion of X. Then $x_0 \notin (X^\rho)_+$. Consider the Riesz subspace $G := \text{span}\left\{x_0^+, x_0^-\right\}$ of X^ρ. From Proposition 4.6 it follows that there exists a Riesz homomorphism $\varphi : G \to \mathbb{R}$ with $\varphi(x_0) < 0$. Now we want a Riesz* homomorphism on X extending φ. We can apply Theorem 4.12 to obtain a Riesz homomorphism $\psi : X^\rho \to \mathbb{R}$ extending φ. Then $\psi|_X : X \to \mathbb{R}$ is a Riesz* homomorphism on X extending φ. Moreover, $\psi|_X(x_0) = \varphi(x_0) < 0$. Now follows from Proposition 4.3 that J is bipositive.

□

Note that if in Theorem 4.12 G is majorizing, then $H := G$ is majorizing and for all $x \in H$,

$$\inf \{P_{H,S}(|x-y|) : y \in G\} = 0 \text{ on F}.$$

Thus, Theorem 4.5 (Lipecki-Luxemburg-Schep) is a special case of our Theorem 4.12.

4.5 An example

Next we give an example where we can verify the condition of Theorem 4.4.

Example 4.13. Let $X = \mathbb{R}^\mathbb{N}$ be the vector space of all sequences with entrywise order. Then X has no order unit. Let $x_0 \in X$, $x_0 \ngeq 0$, $\varphi(\alpha x_0^+ + \beta x_0^-) = \beta$, $\alpha, \beta \in \mathbb{R}$. There exists $k_0 \in \mathbb{N}$ such that $x_0(k_0) < 0$. Define $\gamma := -x_0(k_0) > 0$.

Let $\psi(x) := \frac{1}{\gamma} x(k_0)$ where $x \in X$. Then $\psi : X \to \mathbb{R}$ is positive and linear. Moreover, $H := X$ is a Riesz space, $X = X^\rho$ and trivially X is majorizing in X.

Then for $x \in H = X$, with $G = \text{span}\left\{x_0^+, x_0^-\right\}$ we have

$$\begin{aligned} &\inf \left\{\inf \left\{\psi(z) : z \in \mathbb{R}^\mathbb{N} \text{ and } |x-y| \leq z\right\} : y \in G\right\} \\ &= \inf \left\{\inf \left\{\frac{1}{\gamma} z(k_0) : z \in \mathbb{R}^\mathbb{N} \text{ and } |x-y| \leq z\right\} : y \in G\right\}. \end{aligned}$$

Choose $\beta = \frac{x(h_0)}{\gamma}$ and $y = \beta x_0^-$, then

$$\begin{aligned} \inf \left\{\frac{1}{\gamma}|x(k_0) - y(k_0)| : y \in G\right\} &\leq \inf \left\{\frac{1}{\gamma}|x(k_0) - \beta x_0^-(k_0)|\right\} \\ &= \inf \left\{\frac{1}{\gamma}|x(k_0) - \beta\gamma|\right\} \\ &= \inf \left\{\frac{1}{\gamma}|x(k_0) - x(k_0)|\right\} = 0. \end{aligned}$$

Hence

$$\inf \{\inf \{\psi(z) : z \in H \text{ and } |x-y| \leq z\} : y \in G\} = 0$$

for every $x \in H$.

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