M. Heemskerk

Derived Categories of Projective Gorenstein Varieties

Master's thesis

Advisor: dr. D. Holmes

Date of graduation: June 15, 2018

Mathematisch Instituut, Universiteit Leiden

Acknowledgements

I would like to express my gratitude to my supervisor dr. David Holmes. His knowledge, humor and inquisitive nature were essential in guiding me through the work of this past year. When I had questions, he would always be available to answer them, and when I did not, he gave me the time to work things out myself.

My thanks also go to Prof.dr. Bas Edixhoven for his comments and corrections and to him and Prof.dr. Hendrik Lenstra alike for sparking my interest in this topic.

Further gratitude goes to my family and in particular to my parents, for believing in me, for encouraging me and for supporting me throughout. They truly gave me the perfect environment to thrive in.

Last but definitely not least, I would like to thank my friends. Without their assistance, I would not have been able to work as hard. They provided the necessary breaks and showed endless patience putting up with my occasional rambling. This would not have been possible without them.

Contents

0 Introduction

The theory of derived categories was first exposed by Grothendieck in the early sixties. He needed it to prove the theorem of Grothendieck duality, which can be seen as an extension of Serre duality. Derived categories are in many ways the natural environment in which to consider cohomology and derived functors in general. To illustrate this, one may recall the Grothendieck spectral sequence. If $F\colon \mathcal{A}_1 \to \mathcal{A}_2$ and $G\colon \mathcal{A}_2 \to \mathcal{A}_3$ are left-exact functors between abelian categories with enough injectives such that F takes injective objects to G-acyclic objects, one has a spectral sequence

$$E_2^{p,q} := R^pG(R^qF(A)) \Rightarrow R^{p+q}(GF)(A)$$

computing the images of the higher derived functors of the composition. On derived categories, the same assumptions give an isomorphism of functors $RG \circ RF \cong R(GF)$.

But derived categories are not just of interest to homological-algebraists. Recently, interest has grown for the geometric side of the derived category $D^b_{\mathrm{coh}}(X)$ of bounded complexes of coherent sheaves on a locally-Noetherian scheme X. Bondal and Orlov proved in 1997 that if X and Y are smooth projective varieties, such that the canonical sheaf of X is ample or anti-ample and such that $D^b_{\mathrm{coh}}(X) \cong D^b_{\mathrm{coh}}(Y)$ as triangulated categories, then $X \cong Y$. In [1], Ballard succeeds in relaxing the condition on smoothness and proves the following extension of this reconstruction theorem.

Theorem 0.1. *Let X be a projective Gorenstein variety over a field k with ample or anti-ample canonical bundle. Suppose furthermore that Y is a quasi-projective k-variety and that there is an equivalence $D^b_{\mathrm{coh}}(X) \cong D^b_{\mathrm{coh}}(Y)$ of triangulated categories. Then X is isomorphic to Y.*

We prove this theorem for Y projective in Theorem 5.1. If the canonical sheaf is not ample or antiample, then one can construct a counterexample to the reconstruction theorem. Mukai did this for smooth projective varieties in [13] and we do this for projective Gorenstein varieties in Proposition 7.11. Where Mukai defines Fourier-Mukai transforms to find an exact equivalence $D^b_{\mathrm{coh}}(A) \cong D^b_{\mathrm{coh}}(\hat{A})$ for an abelian variety A and its dual $\hat{A}$, which are generally non-isomorphic, we extend the theory of Fourier-Mukai transforms developed in [1] and [9] to arrive at the following statement, which is the main result of this work.

Proposition 0.2. *Suppose k is a field. Let A be an abelian k-variety and X a projective Gorenstein k-variety with singular locus consisting of a single point. If A is not isomorphic to its dual $\hat{A}$, then $A \times X$ and $\hat{A} \times X$ are non-isomorphic, non-smooth projective Gorenstein varieties and $D^b_{\mathrm{coh}}(A \times X) \cong D^b_{\mathrm{coh}}(\hat{A} \times X)$.*

This is Proposition 7.11. In particular, we prove the following, where a bounded complex of locally-free coherent sheaves is called perfect.

Theorem 0.3. *Let X_1, X_2, Y_1, Y_2 be projective Gorenstein varieties and let $\mathcal{E}_1 \in D^b_{\mathrm{coh}}(X_1 \times Y_1)$ and $\mathcal{E}_2 \in D^b_{\mathrm{coh}}(X_2 \times Y_2)$ be kernels for Fourier-Mukai transforms $\Phi_{\mathcal{E}_i} \colon D^b_{\mathrm{coh}}(X_i) \to D^b_{\mathrm{coh}}(Y_i)$ which send perfect complexes to perfect complexes. Write $\mathcal{E}_1 \boxtimes \mathcal{E}_2$ for the tensor product of the pullbacks of these complexes to $X_1 \times Y_1 \times X_2 \times Y_2$. If both $\Phi_{\mathcal{E}_i}$ are fully-faithful, resp. equivalences, then $\Phi_{\mathcal{E}_1 \boxtimes \mathcal{E}_2} \colon D^b_{\mathrm{coh}}(X_1 \times X_2) \to D^b_{\mathrm{coh}}(Y_1 \times Y_2)$ is fully-faithful, resp. an equivalence as well.*

The corresponding statement for smooth projective varieties was proven by Orlov in [18] in 1997. The proof uses uniqueness of Fourier-Mukai kernels for fully faithful Fourier-Mukai transforms between derived categories of smooth projective varieties, which is an incredibly strong fact. Our proof, which is found in Proposition 7.9 and Theorem 7.10, makes no use of such a uniqueness statement.

0.1 Preliminaries

For an abelian category $\mathcal{A}$, one has the derived category $D(\mathcal{A})$ of unbounded complexes. In it are the two derived categories $D^+(\mathcal{A}), D^-(\mathcal{A})$ of bounded below complexes and bounded above complexes and their intersection $D^b(\mathcal{A})$. On a scheme X, there is the category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves, which forms an abelian category. If X is locally-Noetherian, there is also the abelian category $\mathrm{Coh}(X)$ of coherent sheaves and we get all the associated derived categories. Instead of $D(\mathrm{QCoh}(X))$ we write $D(X)$ and instead of $D(\mathrm{Coh}(X))$ we write $D_{\mathrm{coh}}(X)$ and similarly for their decorations. By [14], the derived functors of $f^*, f_*, \otimes$ and $\mathcal{H}om$ are well-defined on all of $D(X)$ when X is quasi-compact and has a separating open cover of affines. That is, a cover $\mathcal{U}$ of affines such that every intersection $U_1 \cap U_2$ for $U_1, U_2 \in \mathcal{U}$ is affine. If $\mathcal{E}^\bullet$ is a complex of quasi-coherent sheaves which are acyclic for a functor F, then the image of $\mathcal{E}^\bullet$ under the derived functor of F is computed by directly applying F to the complex $\mathcal{E}^\bullet$. This is made rigorous in [20, Section 10.5] and can be used directly to show any exact functor $F \colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ for any two schemes X, Y has a derived functor $D(X) \to D(Y)$, which we also denote by F.

Suppose now that X is quasi-projective over a field. Then X is quasi-compact and has a separating open cover of affines. In particular, we get all the derived functors mentioned above. If one restricts to bounded derived categories, there is another approach to defining these derived functors. The category $\mathrm{Coh}(X)$ has enough locally-frees and the ambient category $\mathrm{QCoh}(X)$ has enough injectives, meaning every complex in $D^-_{\mathrm{coh}}(X)$ is quasi-isomorphic to a bounded above complex of locally-free coherent sheaves and every complex in $D^+_{\mathrm{coh}}(X)$ is quasi-isomorphic to a bounded below complex of injectives in $D(X)$. These are acyclic resolutions for the functors above, hence all the derived functors can be defined on the appropriate bounded derived categories through these resolutions. A word of caution is warranted here. A bounded complex of, say, coherent sheaves is an object of both $D^-_{\text{coh}}(X)$ and $D^+_{\text{coh}}(X)$, so we can consider its image under left- and right-derived functors alike. However, the necessary acyclic resolutions may not exist in the bounded derived category. Often we will replace a complex by a resolution with a certain property and accept that we may lose boundedness in one direction. Technically, this means that we work with all the flavors of derived categories as full subcategories of $D(X)$ and $D_{\text{coh}}(X)$.

Left-derived functors are written as LF and right-derived functors are written as RF. The following are a few statements about these derived functors, mimicking the non-derived versions. The proofs of these can be found in [14], see Propositions 9 and 65 there. In the reference, a quasi-compact quasi-separated scheme is called concentrated. Let $f\colon X \to Y$ be a morphism of quasi-compact schemes with separating open affine covers. Fix complexes of quasi-coherent sheaves $\mathcal{G}^\bullet, \mathcal{H}^\bullet \in D(Y)$ and $\mathcal{F}^\bullet \in D(X)$.

(i) Pullbacks commute with tensor product, that is, the natural morphism

$$Lf^*\mathcal{G}^\bullet \overset{L}{\otimes} Lf^*\mathcal{H}^\bullet \longrightarrow Lf^*(\mathcal{G}^\bullet \overset{L}{\otimes} \mathcal{H}^\bullet)$$

is an isomorphism.

(ii) (The projection formula) The natural morphism

$$Rf_*\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{G}^\bullet \longrightarrow Rf_*(\mathcal{F}^\bullet \overset{L}{\otimes} Lf^*\mathcal{G}^\bullet)$$

is an isomorphism.

(iii) There is an adjunction $Lf^* \dashv Rf_*$.

(iv) (Flat base change) If $g\colon Y' \to Y$ is a flat morphism from a quasi-compact scheme Y' with a separating open cover of affines and

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

is a cartesian diagram, then for any bounded complex $\mathcal{F}^\bullet$ on X, the base change map $g^*Rf_*\mathcal{F}^\bullet \to Rf'_*g'^*\mathcal{F}^\bullet$ is a quasi-isomorphism.

(v) If X is a quasi-projective scheme over a field, there is an adjunction $\mathcal{F}^\bullet \overset{L}{\otimes} - \dashv R\mathcal{H}om(\mathcal{F}^\bullet, -)$.

(vi) If X is quasi-projective over a field and $\mathcal{P}^\bullet$ is a bounded complex of locally-free sheaves of finite rank, then the functor $R\mathcal{H}om(\mathcal{P}^\bullet, -)$ is left-adjoint to $\mathcal{P}^\bullet \overset{L}{\otimes} -$ and isomorphic to the functor $R\mathcal{H}om(\mathcal{P}^\bullet, \mathcal{O}_X) \overset{L}{\otimes} -$.

1 Triangulated Categories

Triangulated categories are powerful objects, encoding much of what we want from (co)homology and homological algebra. They come with a collection of exact triangles, providing the analogue of exact sequences in abelian categories. While we lose the power of kernels and cokernels, we can still fit morphisms into exact triangles, analogous to fitting morphisms into exact sequence in abelian categories.

Using homological functors like $\mathrm{Hom}(A, -)$ for some object A of our category, we can still use our knowledge of homological algebra to analyse the objects we work with. In particular, the Hom-functors have the added merit of the Yoneda lemma. Before we begin, let us get some definitions straight.

When we say triangulated category, we shall mean one in which the octahedral axiom holds. Let $\mathcal{C}$ be a triangulated category with shift functor T. We will write $A[n]$ for the n-fold shift of an object A of $\mathcal{C}$ and triangles in $\mathcal{C}$ will be written as

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1].$$

A *homological functor* on $\mathcal{C}$ is an additive functor $\mathcal{C} \to \mathcal{A}$ to an abelian category sending exact triangles to long exact sequences. The derived category of an abelian category $\mathcal{A}$ comes with a natural homological functor $\mathcal{H}^0 \colon D(\mathcal{A}) \to \mathcal{A}$, which is just taking cohomology in degree 0. For each $n \in \mathbb{Z}$, we define a functor $\mathcal{H}^n$ by $\mathcal{H}^n := \mathcal{H}^0 \circ T^n$ called *n-th cohomology.* A functor $F \colon \mathcal{C} \to \mathcal{D}$ between triangulated categories is called *triangulated* or *exact* if it commutes with the shift functor and takes exact triangles to exact triangles. For the full definition of triangulated categories and a very nice introduction to their theory, we refer to [9, Chapter 1]. We shall summarize the results we will need here.

1.1 Adjoints

Exact functors between nice triangulated categories often turn out to have left- and/or right-adjoints. These adjoints are again exact, see [9, Proposition 1.41] for a proof, so in studying them we can make heavy use of homological algebra. More specifically, to study the images of objects and morphisms under these functors, we will find it useful to look instead at the functors they represent. The results that follow, together with their proofs, are taken from [9, Chapter 1].

Definition 1.1. Let $\mathcal{C}$ be a triangulated category. A set Ω of objects in $\mathcal{C}$ is called a *spanning class* if for all $B \in \mathcal{C}$ the following two conditions hold:

(i) if $\mathrm{Hom}(A, B[i]) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$,

(ii) if $\mathrm{Hom}(B[i], A) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$.

Proposition 1.2. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a triangulated functor of triangulated categories with a left- and a right-adjoint. Suppose Ω is a spanning class of $\mathcal{C}$. Then F is fully-faithful if and only if for all $A, B \in \Omega$ and all $i \in \mathbb{Z}$ the homomorphisms*

$$\mathrm{Hom}(A, B[i]) \longrightarrow \mathrm{Hom}(F(A), F(B)[i])$$

given by applying F are bijective.

Proof. See [9, Proposition 1.49]. □

We can even use this result to show a given functor is an equivalence, provided it has the necessary adjoints. For this, we will need the following well-known facts about adjunctions in general.

Lemma 1.3. *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a functor with a left-adjoint G. Then F is an equivalence if and only if F and G are both fully faithful.*

Proof. If F is an equivalence, then G is necessarily its inverse. Indeed, we have isomorphisms

$$\mathrm{Hom}(G(A), B) \cong \mathrm{Hom}(A, FB) \cong \mathrm{Hom}(F^{-1}A, B)$$

natural in both A and B, so the Yoneda lemma gives $F^{-1} \cong G$. The rest of the proof follows easily from the following lemma, which can be found in many a reference on category theory. □

Lemma 1.4. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor with a left-adjoint G. Then:*

(i) F is fully faithful if and only if the counit $\varepsilon\colon FG \Rightarrow Id$ is an isomorphism,

(ii) G is fully faithful if and only if the unit $\eta\colon Id \Rightarrow GF$ is an isomorphism.

Proof. We will only show the first statement. The second follows by dual arguments. Consider the unit $\eta\colon \mathrm{Id} \Rightarrow FG$ of the adjunction $G \dashv F$. For any two objects $A, B \in \mathcal{D}$ there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(A, B) & \longrightarrow & \mathrm{Hom}(A, FG(B)) \\ & \searrow & \downarrow \wr \\ & & \mathrm{Hom}(G(A), G(B)), \end{array}$$

where the top map is postcomposition by η_B.

If G is fully-faithful, it follows that η_A induces an isomorphism of functors

$$\mathrm{Hom}(-, B) \Rightarrow \mathrm{Hom}(-, FG(B)),$$

hence it is an isomorphism by the Yoneda lemma in $\mathcal{D}^{\mathrm{op}}$ and thus η is an isomorphism. □

1.2 Compact objects and compact generation

Definition 1.5. Let $\mathcal{C}$ be an additive category. Then an object $A \in \mathcal{C}$ is called *compact* if $\mathrm{Hom}(A, -)$ commutes with coproducts in $\mathcal{C}$, that is, for each coproduct $\coprod_{i \in I} B_i$ in $\mathcal{C}$, the natural morphism

$$\bigoplus_{i \in I} \mathrm{Hom}(A, B_i) \longrightarrow \mathrm{Hom}(A, \coprod_{i \in I} B_i)$$

is an isomorphism. We denote by $\mathcal{C}^c$ the full subcategory of $\mathcal{C}$ of compact objects.

Example 1.6. In the unbounded derived category of quasi-coherent sheaves $D(X)$ of a quasi-compact and quasi-separated scheme X, the locally-free sheaves of finite rank are compact objects. The structure sheaf $\mathcal{O}_X$ is compact because $\mathrm{Hom}(\mathcal{O}_X, -)$ coincides with $\Gamma(X, -)$, which commutes with coproducts because X is quasi-compact and quasi-separated. This extends to finite rank locally-free sheaves E because we have natural isomorphisms

$$\begin{aligned}\bigoplus_{i \in I} \mathrm{Hom}(E, A_i) &\cong \bigoplus_{i \in I} \mathrm{Hom}(\mathcal{O}_X, E^\vee \otimes A_i) \cong \mathrm{Hom}\left(\mathcal{O}_X, \coprod_{i \in I} E^\vee \otimes A_i\right) \\ &\cong \mathrm{Hom}\left(\mathcal{O}_X, E^\vee \otimes \coprod_{i \in I} A_i\right) \cong \mathrm{Hom}\left(E, \coprod_{i \in I} A_i\right).\end{aligned}$$

Just in this subsection, we will write $[-,-]$ for the functor $\mathrm{Hom}(-,-)$, simply to save on space. It should be clear in which category we are taking the morphisms. We will also suppress the subscripts on coproducts and direct sums for the same reason.

Lemma 1.7 ([1, Lemmas 2.2.3 and 2.2.4])**.** *Let $\mathcal{C}$ be a triangulated category. Then $\mathcal{C}^c$ is a thick triangulated subcategory of $\mathcal{C}$.*

Proof. Suppose $A \in \mathcal{C}$ is a compact object and $\coprod B_i$ a small coproduct in $\mathcal{C}$. Then there are natural isomorphisms

$$\left[A[1], \coprod B_i\right] \cong \left[A, \coprod (B_i[-1])\right] \cong \bigoplus [A, B_i[-1]] \cong \bigoplus [A[1], B_i],$$

hence $A[1]$ is compact as well. The functors $[-, B]$ for any object $B \in \mathcal{C}$ are cohomological, meaning that an exact triangle yields a long exact sequence. Applying the cohomological functor $[-, \coprod B_i]$ to an exact triangle with vertices A, B and C, where A and B are compact, yields a commutative diagram with exact columns

$$\begin{array}{ccc}
\bigoplus [B[-1], B_i] & \longrightarrow & [B[-1], \coprod B_i] \\
\downarrow & & \downarrow \\
\bigoplus [A[-1], B_i] & \longrightarrow & [A[-1], \coprod B_i] \\
\downarrow & & \downarrow \\
\bigoplus [C, B_i] & \longrightarrow & [C, \coprod B_i] \\
\downarrow & & \downarrow \\
\bigoplus [B, B_i] & \longrightarrow & [B, \coprod B_i] \\
\downarrow & & \downarrow \\
\bigoplus [A, B_i] & \longrightarrow & [A, \coprod B_i]
\end{array}$$

where the top and bottom two rightward maps are isomorphisms, so the middle is as well by the five lemma. Hence $\mathcal{C}^c$ is a triangulated subcategory of $\mathcal{C}$.

Next up is showing $\mathcal{C}^c$ is thick. Suppose $A \oplus C$ is compact for $A, B \in \mathcal{C}$. The natural map $\bigoplus [A \oplus C, B_i] \to [A \oplus C, \coprod B_i]$ is the product of the maps

$$\bigoplus [A, B_i] \to [A, \coprod B_i] \text{ and } \bigoplus [C, B_i] \to [C, \coprod B_i],$$

implying both of these are isomorphisms, hence A and C are both compact. So $\mathcal{C}^c$ is a thick triangulated subcategory of $\mathcal{C}$. □

Given a subcategory $\mathcal{D} \subset \mathcal{C}$ of an additive category, we define

$$\mathcal{D}^{\perp} := \{A \in \mathcal{C} \mid [B, A] = 0 \text{ for all } B \in \mathcal{D}\}.$$

This is often called the *right-orthogonal* to $\mathcal{D}$. We also write $\Omega^{\perp}$ for a set of objects $\Omega \subset \mathcal{C}$ for the right-orthogonal of the full subcategory of $\mathcal{C}$ with set of objects Ω.

Definition 1.8. An additive category $\mathcal{C}$ is called *compactly-generated* if $\mathcal{C}$ possesses all small coproducts and the right-orthogonal $(\mathcal{C}^c)^{\perp}$ is trivial. If $\Omega^{\perp} = 0$ for some subset $\Omega \subset \mathcal{C}$ which is closed under shifts and consists of compact objects, then we say Ω is a *set of compact generators*.

Example 1.9. For a quasi-projective scheme X over a field, the derived category $D(X)$ is compactly generated. Indeed, let $\mathcal{O}_X(1)$ denote a choice of an ample invertible sheaf on X. We saw before that this is a compact object in $D(X)$ and we will show here that the set $\{\mathcal{O}_X(r)[i] \mid r, i \in \mathbb{Z}\}$ is a set of compact generators. If $\mathcal{F}^{\bullet} \in D(X)$ is non-zero, then it has some non-trivial cohomology $\mathcal{H} := \mathcal{H}^i(\mathcal{F}^{\bullet})$. We have a surjection $\ker(\mathcal{F}^i \to \mathcal{F}^{i+1}) \longrightarrow \mathcal{H}$. Let C be a coherent submodule of this kernel with non-zero image in $\mathcal{H}$. There exists an $r \in \mathbb{Z}_{\geq 0}$ such that $C \otimes \mathcal{O}_X(r)$ is generated by global sections, ensuring the existence of a global section $\sigma \in C \otimes \mathcal{O}_X(r)$ which does not map to zero in $\mathcal{H}$. This defines a morphism $\mathcal{O}_X(-r) \to C$ for which the composition with $C \to \mathcal{H}$ is non-trivial. We see that the composition $\mathcal{O}_X(-r) \to C \to \mathcal{F}^i$ defines a non-zero map in $\mathrm{Hom}(\mathcal{O}_X(-r), \mathcal{F}^\bullet[i])$, because the i-th cohomology of this composition is the map $\mathcal{O}_X(-r) \to C \to \mathcal{H}$.

The following lemma highlights an argument used multiple times in [1].

Lemma 1.10. *Suppose $\mathcal{C}$ is a compactly generated triangulated category with a set of compact generators Ω. Let $f\colon A \to B$ be a morphism in $\mathcal{C}$. Then f is an isomorphism if and only if for each $D \in \Omega$ the induced map $[D, A] \to [D, B]$ is an isomorphism.*

Proof. If f is an isomorphism, then obviously each $[D, A] \to [D, B]$ is as well. So conversely, suppose each $[D, A] \to [D, B]$ is an isomorphism. Let C be the cone over f, that is, take an exact triangle

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1].$$

Applying $[D, -]$ for some compact object D gives an exact sequence

$$[D, A] \longrightarrow [D, B] \longrightarrow [D, C] \longrightarrow [D, A[1]] \longrightarrow [D, B[1]].$$

The map $[D, A[1]] \to [D, B[1]]$ is an isomorphism because $D[-1]$ is compact as well and Ω is closed under shifts, hence the map $[D, C] \to [D, A[1]]$ has trivial image and trivial kernel, showing $[D, C] = 0$. Because $\Omega^\perp = 0$, it follows that $C = 0$, hence f is an isomorphism. □

This can also be extended to isomorphisms of functors, as alluded to in the proof of [1, Lemma 4.2.5].

Lemma 1.11. *Suppose $\mathcal{C}$ is a compactly generated triangulated category and $F_1, F_2\colon \mathcal{C} \to \mathcal{S}$ are triangulated functors with right-adjoints G_1 and G_2. Suppose furthermore that $\eta\colon F_1 \Rightarrow F_2$ is a natural transformation. If $\eta_A\colon F_1A \to F_2A$ is an isomorphism for every A in a set of compact generators Ω, then η is an isomorphism of functors.*

Proof. The natural transformation η induces a natural transformation $\varepsilon\colon G_2 \Rightarrow G_1$ of the right-adjoints. For any $A \in \Omega$ and $B \in \mathcal{S}$ we have

$$\mathrm{Hom}(A, G_2B) \cong \mathrm{Hom}(F_2A, B) \cong \mathrm{Hom}(F_1A, B) \cong \mathrm{Hom}(A, G_1B)$$

via the isomorphism η_A and the adjunctions. The resulting isomorphism $\mathrm{Hom}(A, G_2B) \cong \mathrm{Hom}(A, G_1B)$ is induced by ε_B, so through the previous lemma, we find ε_B is an isomorphism. This shows ε is an isomorphism, and therefore η is as well. □

The following theorem by Thomason and Neeman [15, Lemma 2.2] is going to allow us to focus on a set of compact generators when we need to, instead of having to consider every compact object.

Theorem 1.12. *Let $\mathcal{C}$ be a compactly-generated triangulated category. If Ω is a set of compact generators, then $\mathcal{C}^c$ is the smallest thick triangulated subcategory of $\mathcal{C}$ containing Ω.*

2 Derived Categories of Coherent Sheaves

2.1 Pullbacks and supports

For $\mathcal{O}_X$-modules of finite type on a scheme X in general we have the intuitive fact that $x \in \operatorname{supp}(\mathcal{F})$ if and only if $i_x^*\mathcal{F} \neq 0$, implied by Nakayama's lemma. If X is a quasi-projective scheme over a field, it turns out that this can be extended to complexes in $D^b_{\mathrm{coh}}(X)$ by defining

$$\operatorname{supp}(\mathcal{F}^\bullet) = \bigcup_{m\in\mathbb{Z}} \operatorname{supp}(\mathcal{H}^m(\mathcal{F}^\bullet))$$

and taking the left-derived functor Li_x^* instead of i_x^*.

Lemma 2.1 ([9, Exercise 3.30])**.** *Let $i_x\colon \{x\} \hookrightarrow X$ be the inclusion of a closed point into a scheme X quasi-projective over a field. Then for any complex $\mathcal{F}^\bullet \in D^b_{\mathrm{coh}}(X)$ we have $Li_x^*\mathcal{F}^\bullet \neq 0$ if and only if $x \in \operatorname{supp}(\mathcal{F}^\bullet)$.*

Proof. Our proof is adapted from [9, Lemma 3.29]. Suppose $x \in \operatorname{supp}(\mathcal{F}^\bullet)$. Choose $m \in \mathbb{Z}$ maximal such that $x \in \operatorname{supp}(\mathcal{H}^m(\mathcal{F}^\bullet))$ and write $\mathcal{H} := \mathcal{H}^m(\mathcal{F}^\bullet)$. Let us consider the spectral sequence

$$E_2^{p,q} = \mathcal{H}^p(Li_x^*\mathcal{H}^q(\mathcal{F}^\bullet)) \Rightarrow \mathcal{H}^{p+q}(Li_x^*\mathcal{F}^\bullet).$$

Note $\mathcal{H}^0(Li_x^*\mathcal{H}^m(\mathcal{F}^\bullet)) = i_x^*\mathcal{H}^m(\mathcal{F}^\bullet) \neq 0$ by assumption. Because i_x^* is right-exact and $\mathcal{H}^m(\mathcal{F}^\bullet)$ is just a coherent sheaf, the sheaves $\mathcal{H}^p(Li_x^*\mathcal{H}^q(\mathcal{F}^\bullet))$ are trivial for all $p > 0$ and all $q \in \mathbb{Z}$. The stalk of $\mathcal{H}^q(\mathcal{F}^\bullet)$ at x is zero for all $q > m$, so

$$Li_x^*\mathcal{H}^q(\mathcal{F}^\bullet) = \mathcal{H}^q(\mathcal{F}^\bullet)_x \overset{L}{\otimes} k(x) = 0$$

for $q > m$ as well. Hence, the sheaf $\mathcal{H}^0(Li_x^*\mathcal{H}^m(\mathcal{F}^\bullet)) = i_x^*\mathcal{H}$ survives at ∞, showing $\mathcal{H}^m(Li_x^*\mathcal{F}^\bullet) = i_x^*\mathcal{H} \neq 0$, so $Li_x^*\mathcal{F}^\bullet$ is non-zero.

Conversely, if the complex $Li_x^*\mathcal{F}^\bullet$ is non-zero, it has some top cohomology $\mathcal{H}^m(Li_x^*\mathcal{F}^\bullet) \neq 0$. Here we use that Li_x^* sends bounded above complexes to bounded above complexes. Consider the entry at $(p, q) = (0, m)$ of the second page of the spectral sequence mentioned. Turning to the next page of the spectral sequence, this entry is replaced by some subquotient. Hence the entry at ∞ is a subquotient as well, so we see $\mathcal{H}^m(Li_x^*\mathcal{F}^\bullet)$ is a non-zero subquotient of

$$\mathcal{H}^0(Li_x^*\mathcal{H}^m(\mathcal{F}^\bullet)) = i_x^*\mathcal{H}^m(\mathcal{F}^\bullet),$$

showing the latter is non-zero and thus $x \in \operatorname{supp}(\mathcal{F}^\bullet)$. □

Suppose now that X is a locally Noetherian scheme. Skyscraper sheaves on closed points of X are simple to describe. For one, their support is a single closed point, so the natural morphism from their global sections to their stalk is an isomorphism. Indeed, one constructs an inverse as follows. Say $\mathcal{F}$ is supported only on x. Let $s_x \in \mathcal{F}_x$ be the stalk of a section s on a neighborhood U of x. The stalks of s away from y are all 0, so s coincides with the zero section on $U \cap X \setminus \{x\}$, hence these glue to a unique global section of $\mathcal{F}$.

Suppose now that we have two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ with $\operatorname{supp}(\mathcal{F}) \cap \operatorname{supp}(\mathcal{G}) \subset \{x\}$ for a closed point $x \in X$. Taking the stalk of the i-th Ext-sheaf $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})$ at any point $y \in X$, we get by [8, Proposition III.6.8]

$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})_y \cong \operatorname{Ext}^i(\mathcal{F}_y, \mathcal{G}_y),$$

which is trivial if $y \neq x$, so $\operatorname{supp}(\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})) \subset \{x\}$. Hence this Ext-sheaf is a skyscraper sheaf. This implies

$$\operatorname{Ext}^i(\mathcal{F}, \mathcal{G}) = \Gamma(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G})) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{G})_x \cong \operatorname{Ext}^i(\mathcal{F}_x, \mathcal{G}_x).$$

Combined with the fact that there are natural isomorphisms $\operatorname{Ext}^i(A, B) \cong \operatorname{Hom}_{D(\mathcal{A})}(A, B[i])$ for A, B in an abelian category $\mathcal{A}$ with enough injectives, this shows that the functor of taking stalks at x induces isomorphisms

$$\operatorname{Hom}(\mathcal{F}, \mathcal{G}[i]) \longrightarrow \operatorname{Hom}(\mathcal{F}_x, \mathcal{G}_x[i])$$

for all $i \geq 0$. In particular, there are no non-trivial morphisms between shifts of coherent sheaves with disjoint supports.

The following lemma says we can extend this isomorphism to bounded complexes of coherent sheaves.

Lemma 2.2. *Let X be a locally Noetherian scheme. Suppose $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are objects of $D^b_{\mathrm{coh}}(X)$ with supports either being disjoint, or intersecting in a single point $y \in X$. Then the natural morphism*

$$\operatorname{Hom}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \xrightarrow{\sim} \operatorname{Hom}(\mathcal{F}^\bullet_x, \mathcal{G}^\bullet_x)$$

is an isomorphism for any $x \in X$ in the first case and for the single point $x = y$ in the second case.

Proof. First off, note that x is automatically a closed point, because $\operatorname{supp}(\mathcal{F}^\bullet)$ and $\operatorname{supp}(\mathcal{G}^\bullet)$ are closed. Consider for any fixed j the spectral sequences

$$E_2^{p,q} = \operatorname{Hom}(\mathcal{H}^j(\mathcal{F}^\bullet), \mathcal{H}^q(\mathcal{G}^\bullet[p])) \Rightarrow \operatorname{Hom}(\mathcal{H}^j(\mathcal{F}^\bullet), \mathcal{G}^\bullet[p+q])$$

and

$$'E_2^{p,q} = \operatorname{Hom}(\mathcal{H}^j(\mathcal{F}^\bullet_x), \mathcal{H}^q(\mathcal{G}^\bullet_x[p])) \Rightarrow \operatorname{Hom}(\mathcal{H}^j(\mathcal{F}^\bullet_x), \mathcal{G}^\bullet_x[p+q]).$$

Taking stalks is exact, so the latter spectral sequence is isomorphic to

$$\operatorname{Hom}(\mathcal{H}^j(\mathcal{F}^\bullet)_x, \mathcal{H}^q(\mathcal{G}^\bullet[p])_x) \Rightarrow \operatorname{Hom}(\mathcal{H}^j(\mathcal{F}^\bullet)_x, \mathcal{G}^\bullet_x[p+q]).$$

Hence there are natural morphisms $E_2^{p,q} \to {'E}_2^{p,q}$ and $E_\infty^{p,q} \to {'E}_\infty^{p,q}$, given by taking stalks and these are compatible. The former morphism is an isomorphism, so [20, Theorem 5.2.12] implies that the morphism $E_\infty^{p,q} \to {'E}_\infty^{p,q}$ is an isomorphism as well. In particular, for each pair p, q, the natural map

$$\operatorname{Hom}(\mathcal{H}^{-q}(\mathcal{F}^\bullet), \mathcal{G}^\bullet[p]) \longrightarrow \operatorname{Hom}(\mathcal{H}^{-q}(\mathcal{F}^\bullet)_x, \mathcal{G}^\bullet_x[p])$$

is an isomorphism. With this insight, we can repeat the argument using the other Hom spectral sequence to get

$$\operatorname{Hom}(\mathcal{F}^\bullet, \mathcal{G}^\bullet[p+q]) \xrightarrow{\sim} \operatorname{Hom}(\mathcal{F}^\bullet_x, \mathcal{G}^\bullet_x[p+q]),$$

which we evaluate at $p + q = 0$ to finish the proof. □

Lemma 2.3 ([9, Lemma 3.9])**.** *Let X be a locally Noetherian scheme. Suppose $\mathcal{F}^\bullet \in D^b_{\mathrm{coh}}(X)$ is such that* $\operatorname{supp}(\mathcal{F}^\bullet) = Z_1 \sqcup Z_2$ *is a disjoint union of two closed subsets of X. Then $\mathcal{F}^\bullet \cong \mathcal{F}^\bullet_1 \oplus \mathcal{F}^\bullet_2$ with* $\operatorname{supp}(\mathcal{F}^\bullet_i)$ *contained in Z_i for $i = 1, 2$.*

Proof. We follow the proof in the reference. We prove the statement by induction of the length of $\mathcal{F}^\bullet$. If this length is 1, then we are simply dealing with a coherent sheaf $\mathcal{F}$. Consider the open cover $X = U_1 \cup U_2$, where $U_1 = X \setminus Z_2$ and $U_2 = X \setminus Z_1$. Let j_i denote the inclusion of U_i into X. Then $j_i^*\mathcal{F}$ is a coherent sheaf on U_i, supported on the closed set $Z_i \subset X$. For $i = 1, 2$, the pushforward $\mathcal{F}_i := j_{i,*}j_i^*\mathcal{F}$ is coherent, because its restrictions to U_1 and U_2 are either 0 or the matching restriction of $\mathcal{F}$. Taking the product $\mathcal{F} \to \mathcal{F}_1 \oplus \mathcal{F}_2$ of the unit morphisms $\mathcal{F} \to j_{i,*}j_i^*\mathcal{F}$ and looking at stalks, we conclude this is an isomorphism.

Now suppose $\mathcal{F}^\bullet$ has length > 1. Let m be minimal such that $\mathcal{H} := \mathcal{H}^m(\mathcal{F}^\bullet) \neq 0$. This is a coherent sheaf with support contained in $Z_1 \sqcup Z_2$, so its support is also a disjoint union of closed subsets and we may write $\mathcal{H} \cong \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\operatorname{supp}(\mathcal{H}_i) \subset Z_i$. By the axioms of triangulated categories, we may fit the natural morphism $\mathcal{H}[-m] \to \mathcal{F}^\bullet$ in an exact triangle

$$\mathcal{H}[-m] \longrightarrow \mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet \longrightarrow \mathcal{H}[1-m].$$

Considering the long exact sequence of cohomology, we see $\mathcal{H}^q(\mathcal{H}[-m]) = \mathcal{H}^{q+1}(\mathcal{H}[-m]) = 0$ for all $q < m - 1$ and all $q > m$. This shows $\mathcal{H}^q(\mathcal{F}^\bullet) \cong \mathcal{H}^q(\mathcal{G}^\bullet)$ for these q. Around $q = m$, the long exact sequence reads

$$0 \longrightarrow \mathcal{H}^{m-1}(\mathcal{G}^\bullet) \longrightarrow \mathcal{H} \longrightarrow \mathcal{H} \longrightarrow \mathcal{H}^m(\mathcal{G}^\bullet) \longrightarrow 0,$$

where the arrow $\mathcal{H} \to \mathcal{H}$ is an isomorphism, because it is induced by the natural morphism $\mathcal{H}[-m] \to \mathcal{F}^\bullet$. This shows $\mathcal{H}^{m-1}(\mathcal{G}^\bullet)$ and $\mathcal{H}^m(\mathcal{G}^\bullet)$ are both trivial. Hence $\mathcal{G}^\bullet$ is shorter than $\mathcal{F}^\bullet$ and, like for $\mathcal{H}$, we may write $\mathcal{G}^\bullet \cong \mathcal{G}^\bullet_1 \oplus \mathcal{G}^\bullet_2$.

Consider now the spectral sequence

$$E_2^{p,q} = \operatorname{Hom}(\mathcal{H}^{-q}(\mathcal{G}^\bullet_1), \mathcal{H}_2[p]) \Rightarrow \operatorname{Hom}(\mathcal{G}^\bullet_1, \mathcal{H}_2[p+q]).$$

Every term of this is trivial, because $\mathcal{H}^{-q}(\mathcal{G}_1^\bullet)$ and $\mathcal{H}_2$ have disjoint supports. This is a consequence of the spectral sequence for the composition $\Gamma \circ \mathcal{E}xt = \mathrm{Ext}$. This shows that $\mathrm{Hom}(\mathcal{G}_1^\bullet, \mathcal{H}_2[1-m]) = 0$ in particular. Similarly, one gets $\mathrm{Hom}(\mathcal{G}_2^\bullet, \mathcal{H}_1[1-m]) = 0$. Complete the morphisms $\mathcal{G}_i^\bullet \to \mathcal{H}_i[1-m]$ to exact triangles

$$\mathcal{F}_i^\bullet \longrightarrow \mathcal{G}_i^\bullet \longrightarrow \mathcal{H}_i[1-m] \longrightarrow \mathcal{F}_i^\bullet[1].$$

Taking stalks is exact, so that $\mathcal{H}^q(\mathcal{F}_i^\bullet)_x = 0$ for all q if $x \notin Z_i$, hence $\mathrm{supp}(\mathcal{F}_i) \subset Z_i$. The direct sum of these triangles is again exact and because the two hom sets are trivial, the isomorphisms $\mathcal{G}_1^\bullet \oplus \mathcal{G}_2^\bullet \to \mathcal{G}$ and $(\mathcal{H}_1 \oplus \mathcal{H}_2)[1-m] \to \mathcal{H}[1-m]$ can be completed to an isomorphism of exact triangles, showing $\mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet \cong \mathcal{F}^\bullet$. □

2.2 External tensor product

For two schemes X and Y over a common base S we have an external tensor product

$$p^*(-) \otimes q^*(-) \colon \mathrm{Mod}_{\mathcal{O}_X} \times \mathrm{Mod}_{\mathcal{O}_Y} \longrightarrow \mathrm{Mod}_{\mathcal{O}_{X\times Y}},$$

where p and q denote the projections from $X \times Y$ to X and Y respectively. We will denote this by $- \boxtimes -$. When X, Y and $S = \mathrm{Spec}(R)$ are all affine, this is just the bifunctor $- \otimes_{\mathcal{O}_R} -$. Working over a field $R = k$, this is an exact bifunctor.

If now X and Y are arbitrary k-schemes, we can work locally to find the functor $-\boxtimes-$ is exact as well. We can extend this to a bifunctor $K(\mathrm{Mod}_{\mathcal{O}_X}) \times K(\mathrm{Mod}_{\mathcal{O}_Y}) \to K(\mathrm{Mod}_{\mathcal{O}_{X\times Y}})$, where $K(\mathrm{Mod}_{\mathcal{O}_X})$ is the homotopy category of complexes of $\mathcal{O}_X$-modules, in the same way as we do for the usual tensor product. Restricting to quasi-coherent sheaves, we can thus easily form its derived functor $- \boxtimes -$, defined on all of $D(X) \times D(Y)$.

Because the morphisms p and q are flat, their pullbacks p^* and q^* are exact, so we get derived functors for them given by directly applying p^* and q^* and we denote these by p^* and q^* as well. By the usual theorems on compositions of derived functors, we get an isomorphism

$$- \boxtimes - \cong p^*(-) \overset{L}{\otimes} q^*(-)$$

of bifunctors $D(X) \times D(Y) \to D(X \times Y)$.

Lemma 2.4 (The Künneth formula). *Let X and Y quasi-compact separated k-schemes. Then, given $\mathcal{F} \in D^b(X)$ and $\mathcal{G} \in D^b(Y)$, we have an isomorphism*

$$R\Gamma(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) \cong R\Gamma(X, \mathcal{F}) \overset{L}{\otimes}_k R\Gamma(Y, \mathcal{G})$$

in $D(\mathrm{Spec}(k(x)))$.

Proof. The main idea of this proof is from [9], located right after Remark 3.33, and is to combine flat base change and the projection formula, both statements holding in the present setting. Applying flat base change to the cartesian square

$$\begin{array}{ccc} X \times Y & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f_X} \\ Y & \xrightarrow[f_Y]{} & \mathrm{Spec}(k) \end{array}$$

we get

$$q_* p^* \mathcal{F}^\bullet \cong f_Y^* f_{X,*} \mathcal{F}^\bullet \cong R\Gamma(X, \mathcal{F}^\bullet) \overset{L}{\otimes} \mathcal{O}_Y.$$

Now, by the projection formula,

$$\begin{aligned} R\Gamma(X \times Y, p^*\mathcal{F}^\bullet \overset{L}{\otimes}_{\mathcal{O}_{X\times Y}} q^*\mathcal{G}^\bullet) &\cong R\Gamma(Y, q_*(p^*\mathcal{F}^\bullet \overset{L}{\otimes}_{\mathcal{O}_{X\times Y}} q^*\mathcal{G}^\bullet)) \\ &\cong R\Gamma(Y, q_*p^*\mathcal{F}^\bullet \overset{L}{\otimes}_{\mathcal{O}_Y} \mathcal{G}^\bullet) \\ &\cong R\Gamma(Y, (R\Gamma(X, \mathcal{F}^\bullet) \otimes \mathcal{O}_Y) \overset{L}{\otimes}_{\mathcal{O}_Y} \mathcal{G}^\bullet). \end{aligned}$$

As $R\Gamma(X, \mathcal{F}^\bullet)$ is a complex of vector spaces, the pullback $R\Gamma(X, \mathcal{F}^\bullet) \overset{L}{\otimes} \mathcal{O}_Y$ is a complex of free $\mathcal{O}_Y$-modules. This means that if we choose an injective resolution $\mathcal{G}^\bullet \overset{\sim}{\to} \mathcal{I}^\bullet$, every sheaf in the complex $(R\Gamma(X, \mathcal{F}^\bullet) \otimes \mathcal{O}_Y) \overset{L}{\otimes}_{\mathcal{O}_Y} \mathcal{I}^\bullet$ is still injective, because a direct sum of injective modules is injective. We can indeed choose such an injective resolution, because the quasi-compact quasi-separated hypothesis implies $\mathrm{QCoh}(X)$ has enough injectives. Injective modules are acyclic for left-exact functors, so the derived functor $R\Gamma(Y, -)$ acts degreewise as $\Gamma(Y, -)$ on a complex of injective modules. Hence, we find for the degree n part

$$\begin{aligned} R\Gamma(Y, (R\Gamma(X, \mathcal{F}^\bullet) \otimes \mathcal{O}_Y) \overset{L}{\otimes}_{\mathcal{O}_Y} \mathcal{I}^\bullet)^n &\cong \bigoplus_{i+j=n} \Gamma(Y, (R\Gamma(X, \mathcal{F}^\bullet)^j \otimes \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \mathcal{I}^i) \\ &\cong \bigoplus_{i+j=n} R\Gamma(X, \mathcal{F}^\bullet)^j \otimes_k \Gamma(Y, \mathcal{I}^i) \\ &\cong (R\Gamma(X, \mathcal{F}^\bullet) \otimes_k R\Gamma(Y, \mathcal{I}^\bullet))^n, \end{aligned}$$

showing

$$R\Gamma(Y, (R\Gamma(X, \mathcal{F}^\bullet) \otimes \mathcal{O}_Y) \overset{L}{\otimes}_{\mathcal{O}_Y} \mathcal{I}^\bullet) \cong R\Gamma(X, \mathcal{F}^\bullet) \otimes_k R\Gamma(Y, \mathcal{I}^\bullet)$$

and thus completing the proof. □

In the following lemma and its proof, the decorations L and R for the derived pushforward, pullback and tensor product are suppressed. This should not lead to any confusion.

Lemma 2.5. *Let k be any field and suppose $f\colon X \to X'$ and $g\colon Y \to Y'$ are morphisms of quasi-compact, separated k-schemes. Then for any $\mathcal{F}^\bullet \in D^b(X)$ and any $\mathcal{G}^\bullet \in D^b(Y)$ the natural morphism $f_*\mathcal{F}^\bullet \boxtimes g_*\mathcal{G}^\bullet \to (f \times g)_*(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet)$ is an isomorphism in $D^b(X' \times Y')$. In other words, the bifunctors $(f_*-) \boxtimes (g_*-)$ and $(f \times g)_*(- \boxtimes -)$ are isomorphic.*

Proof. Let us suppress the bullets from our notation for brevity. First, let us find the mentioned natural morphism. We have the following natural bijections:

$$\begin{aligned}\mathrm{Hom}(f_*\mathcal{F} \boxtimes g_*\mathcal{G}, (f \times g)_*(\mathcal{F} \boxtimes \mathcal{G})) &\cong \mathrm{Hom}((f \times g)^*(f_*\mathcal{F} \boxtimes g_*\mathcal{G}), \mathcal{F} \boxtimes \mathcal{G}) \\ &\cong \mathrm{Hom}(p^*f^*f_*\mathcal{F} \otimes q^*g^*g_*\mathcal{G}, \mathcal{F} \boxtimes \mathcal{G}) \\ &\cong \mathrm{Hom}(f^*f_*\mathcal{F} \boxtimes g^*g_*\mathcal{G}, \mathcal{F} \boxtimes \mathcal{G}),\end{aligned}$$

where p and q denote the projections from $X \times Y$ to X and Y respectively. Taking the tensor product of the pullbacks of the counit morphisms $f^*f_*\mathcal{F} \to \mathcal{F}$ and $g^*g_*\mathcal{G} \to \mathcal{G}$, we get a natural morphism in last set of morphisms. Tracing back the bijections above produces a natural morphism $f_*\mathcal{F} \boxtimes g_*\mathcal{G} \to (f \times g)_*(\mathcal{F} \boxtimes \mathcal{G})$.

We will show this is a quasi-isomorphism. This is a local question on X' and Y', so assume these are affine, say given by $\mathrm{Spec}(A')$ and $\mathrm{Spec}(B')$. Then the pushforwards f_*, g_* and $(f \times g)_*$ are nothing but the global sections functors and we have to show that

$$R\Gamma(X, \mathcal{F}) \otimes_k R\Gamma(Y, \mathcal{G}) \longrightarrow R\Gamma(X \times Y, \mathcal{F} \boxtimes \mathcal{G})$$

is an isomorphism. This is just the Künneth formula, so we are done. □

3 Locally-finite Duality

From this section onward, we only decorate left- and right derived functors if we wish to emphasize that they are in fact derived.

3.1 The perfect derived category

Definition 3.1. Let X be a locally-Noetherian scheme. The smallest triangulated subcategory of $D(X)$ containing the finite rank locally-free sheaves is called the *perfect derived category* of X and is denoted by $D_{\mathrm{perf}}(X)$. The complexes in $D_{\mathrm{perf}}(X)$ are called *perfect*.

In the case of X quasi-projective over a field, this subcategory will turn out to be exactly the full subcategory of compact objects $D(X)^c$. We will prove this momentarily.

Now, this definition is difficult to work with in general, so we would first like to have a more concrete description of the objects of $D_{\mathrm{perf}}(X)$. Let us write $\mathrm{Perf}(X) \subset D(X)$ for the full subcategory of complexes quasi-isomorphic to bounded complexes of finite rank locally-free sheaves. We will show that for schemes X which are quasi-projective over a field, the subcategories $\mathrm{Perf}(X)$ and $D_{\mathrm{perf}}(X)$ coincide.

Lemma 3.2. *Let X be a quasi-projective scheme over a Noetherian ring R. Then* $\mathrm{Perf}(X) \subset D(X)$ *is a thick triangulated subcategory.*

Proof. We start by showing $\mathrm{Perf}(X)$ is a triangulated subcategory. It is obviously closed under shifts, so it is enough to prove that for any exact triangle

$$\mathcal{P}^\bullet \longrightarrow \mathcal{Q}^\bullet \longrightarrow \mathcal{R}^\bullet \longrightarrow \mathcal{P}^\bullet[1]$$

with $\mathcal{P}^\bullet, \mathcal{R}^\bullet \in \mathrm{Perf}(X)$, we have $\mathcal{Q}^\bullet \in \mathrm{Perf}(X)$ as well.

Because $D^b_{\mathrm{coh}}(X)$ is triangulated, it follows immediately that $\mathcal{Q}^\bullet \in D^b_{\mathrm{coh}}(X)$. Because X is quasi-projective over a Noetherian ring, there exists a finite rank locally-free resolution $\mathcal{E}^\bullet \xrightarrow{\sim} \mathcal{Q}^\bullet$ and we replace $\mathcal{Q}^\bullet$ by this resolution. Taking the tensor product $- \otimes \mathcal{F}$ for any $\mathcal{O}_X$-module $\mathcal{F}$ yields a long exact sequence, which shows exactness of the sequences

$$\mathcal{H}^i(\mathcal{P}^\bullet \otimes \mathcal{F}) \longrightarrow \mathcal{H}^i(\mathcal{Q}^\bullet \otimes \mathcal{F}) \longrightarrow \mathcal{H}^i(\mathcal{R}^\bullet \otimes \mathcal{F})$$

for all $i \in \mathbb{Z}$, where we use the fact that locally-free sheaves are acyclic for the tensor product. Because $\mathcal{P}^\bullet$ and $\mathcal{R}^\bullet$ are bounded complexes of locally-free sheaves, there exist $a, b \in \mathbb{Z}$ such that for every $i \in \mathbb{Z}$ with $i < a$ or $i > b$ both the first and the last sheaf in the sequence for i are zero for any $\mathcal{F}$. It follows that $\mathcal{H}^i(\mathcal{Q}^\bullet \otimes \mathcal{F}) = 0$ for such i as well. We show that the canonical truncation $\tau_{\geq a}\mathcal{Q}^\bullet$ given by

$$\cdots \longrightarrow 0 \longrightarrow \mathrm{coker}(d^{a-1}) \longrightarrow \mathcal{Q}^{a+1} \longrightarrow \mathcal{Q}^{a+2} \longrightarrow \cdots,$$

where $d^\bullet$ is the differential of $\mathcal{Q}^\bullet$, consists of locally-free sheaves and is quasi-isomorphic to $\mathcal{Q}^\bullet$.

Taking $\mathcal{F} = \mathcal{O}_X$, we find $\mathcal{Q}^\bullet$ has cohomology bounded below by a, hence the canonical map $\mathcal{Q}^\bullet \to \tau_{\geq a}\mathcal{Q}^\bullet$ is a quasi-isomorphism. Furthermore, we obtain a locally-free resolution

$$\cdots \longrightarrow \mathcal{Q}^{a-2} \longrightarrow \mathcal{Q}^{a-1} \longrightarrow \mathcal{Q}^a \longrightarrow \mathrm{coker}(d^{a-1}) \longrightarrow 0.$$

This resolution computes $\mathrm{Tor}^1(\mathrm{coker}(d^{a-1}), \mathcal{F})$ for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, giving

$$\mathrm{Tor}^1(\mathrm{coker}(d^{a-1}), \mathcal{F}) \cong \frac{\ker(\mathcal{Q}^{a-1} \otimes \mathcal{F} \to \mathcal{Q}^a \otimes \mathcal{F})}{\mathrm{im}(\mathcal{Q}^{a-2} \otimes \mathcal{F} \to \mathcal{Q}^{a-1} \otimes \mathcal{F})} = \mathcal{H}^{a-1}(\mathcal{Q}^\bullet \otimes \mathcal{F}) = 0,$$

so that $\operatorname{coker}(d^{a-1})$ is flat. Note $\operatorname{coker}(d^{a-1})$ is a flat coherent $\mathcal{O}_X$-module, hence it is locally-free of finite rank, which means $\tau_{\geq a}\mathcal{Q}^\bullet$ is a bounded complex of locally-free sheaves of finite rank and therefore $\mathcal{Q}^\bullet \in \operatorname{Perf}(X)$. Thus $\operatorname{Perf}(X)$ is a triangulated subcategory of $D(X)$.

Now suppose $\mathcal{Q}^\bullet \oplus \mathcal{R}^\bullet \in D(X)$ is in $\operatorname{Perf}(X)$, so that it is quasi-isomorphic to a bounded complex $\mathcal{P}^\bullet$ of finite rank locally-free sheaves. Because the subcategory $D^b_{\mathrm{coh}}(X) \subset D(X)$ is thick, we see immediately that $\mathcal{Q}^\bullet$ and $\mathcal{R}^\bullet$ are quasi-isomorphic to bounded complexes of coherent sheaves. Like before, we replace both complexes by bounded above complexes of locally-free sheaves of finite rank. Because the tensor product is additive, we get for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ that

$$\mathcal{H}^i((\mathcal{Q}^\bullet \oplus \mathcal{R}^\bullet) \otimes \mathcal{F}) \cong \mathcal{H}^i((\mathcal{Q}^\bullet \otimes \mathcal{F}) \oplus (\mathcal{R}^\bullet \otimes \mathcal{F})) \cong \mathcal{H}^i(\mathcal{Q}^\bullet \otimes \mathcal{F}) \oplus \mathcal{H}^i(\mathcal{R}^\bullet \otimes \mathcal{F}).$$

Again like before, there exist $a, b \in \mathbb{Z}$ such that $\mathcal{H}^i(\mathcal{P}^\bullet \otimes \mathcal{F}) = 0$ for $i < a$ and $i > b$ and any $\mathcal{F}$. By the previous string of quasi-isomorphisms, these same a and b bound the cohomology sheaves of any of the tensor products $\mathcal{Q}^\bullet \otimes \mathcal{F}$ and $\mathcal{R}^\bullet \otimes \mathcal{F}$. By the same arguments as before, the canonical maps $\mathcal{Q}^\bullet \to \tau_{\geq a}\mathcal{Q}^\bullet$ and $\mathcal{R}^\bullet \to \tau_{\geq a}\mathcal{R}^\bullet$ are quasi-isomorphisms and the two truncated complexes are bounded complexes of finite rank locally-free sheaves, so we are done. □

Remark 3.3. Let $\mathcal{Q}^\bullet \in D(X)$ be any complex of quasi-coherent sheaves. If there exist $a, b \in \mathbb{Z}$ such that $\mathcal{Q}^\bullet \otimes \mathcal{F}$ has non-trivial i-th cohomology only when $a \leq i \leq b$, then we say $\mathcal{Q}^\bullet$ has *Tor amplitude in* $[a, b]$. Without specifying these a and b, we say $\mathcal{Q}^\bullet$ has *finite Tor amplitude*. Thus, for a complex $\mathcal{Q}^\bullet$ quasi-isomorphic to a bounded above complex of finite rank locally-free sheaves on a locally-Noetherian scheme X, we saw in the above proof that $\mathcal{Q}^\bullet$ is perfect if and only if it has finite Tor amplitude. We will use this fact again in the next subsection.

The following lemma is inspired by [1, Lemma 2.2.10 and Corollary 2.2.13]. Additionally, it asserts that the complexes in $D_{\mathrm{perf}}(X)$ are easy to describe.

Lemma 3.4. *Let X be a quasi-projective scheme over a Noetherian ring R. Then the subcategories* $\operatorname{Perf}(X), D_{\mathrm{perf}}(X)$ *and* $D(X)^c$ *are all equal.*

Proof. The inclusion $\operatorname{Perf}(X) \subset D_{\mathrm{perf}}(X)$ is quickly shown. Let $\mathcal{P}^\bullet \in \operatorname{Perf}(X)$ be a bounded complex of finite rank locally-free sheaves. We prove $\mathcal{P}^\bullet \in D_{\mathrm{perf}}(X)$ by induction on its length, that is, the number

$$1 + \max\{i - j \mid i, j \in \mathbb{Z} \colon \mathcal{P}^i, \mathcal{P}^j \neq 0\}.$$

If $\mathcal{P}^\bullet$ has length 1, then it is a shift of a locally-free sheaf of finite rank, so we see immediately that $\mathcal{P}^\bullet \in D_{\mathrm{perf}}(X)$. Suppose that $\mathcal{P}^\bullet$ has length $n > 1$ and that every brutal truncation of $\mathcal{P}^\bullet$ of length less than n is in $D_{\mathrm{perf}}(X)$ as well. Let $m \in \mathbb{Z}$ be maximal so that $\mathcal{P}^m \neq 0$. Then the truncations $\sigma_{\leq m-1}\mathcal{P}^\bullet$

and $\sigma_{\geq m}\mathcal{P}^\bullet$ are shorter than $\mathcal{P}^\bullet$, so they are contained in $D_{\text{perf}}(X)$. There is a canonical exact triangle

$$\sigma_{\geq m}\mathcal{P}^\bullet \longrightarrow \mathcal{P}^\bullet \longrightarrow \sigma_{\leq m-1}\mathcal{P}^\bullet \longrightarrow \sigma_{\geq m}\mathcal{P}^\bullet[1]$$

with two of its three vertices in $D_{\text{perf}}(X)$, so $\mathcal{P}^\bullet \in D_{\text{perf}}(X)$ as well.

Any shift of a locally-free sheaf of finite rank is compact by Example 1.6 and by Example 1.9, the set of shifts of the ample invertible sheaves $O_X(r)$ forms a set of compact generators, hence the set of shifts of the finite rank locally-free sheaves does so as well. Now Theorem 1.12 implies $D(X)^c$ is the smallest thick triangulated subcategory of $D(X)$ containing the shifts of the finite rank locally-free sheaves. This implies that $D_{\text{perf}}(X) \subset D(X)^c$. Because $\text{Perf}(X)$ contains the locally-free sheaves of finite rank, the previous lemma shows $D(X)^c \subset \text{Perf}(X)$. Hence we have a chain of inclusions

$$D_{\text{perf}}(X) \subset D(X)^c \subset \text{Perf}(X) \subset D_{\text{perf}}(X),$$

yielding equality throughout. □

3.2 Locally-finite (co)homological functors

It turns out that for quasi-projective k-schemes, there is a duality between $D^b_{\text{coh},c}(X)$ and $D_{\text{perf}}(X)$ in which either acts as functors on the other, satisfying some finiteness property. Here the derived category $D^b_{\text{coh},c}(X)$ is the derived category of coherent sheaves with proper support. This duality is the key to understanding functors between the bounded derived categories of coherent sheaves on quasi-projective varieties. While Ballard works in the more general setting of quasi-projective k-schemes in [1], we choose here to restrict to projective schemes. Many of the results that follow generalize directly, provided one replaces $D^b_{\text{coh}}(X)$ by the bounded derived category of coherent sheaves with proper support where necessary. We will not prove every result we need, but we will try to provide some intuition along the way.

Let k be a field. In the following, we write Vec_k and vec_k for the categories of all k-vector spaces and finite-dimensional k-vector spaces respectively.

Definition 3.5. Let $\mathcal{C}$ be a k-linear triangulated category. A functor $H\colon \mathcal{C}^{op} \to \text{Vec}_k$ is called *cohomological* if for each exact triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

the sequence

$$\cdots \longleftarrow H(C[i-1]) \longleftarrow H(A[i]) \longleftarrow H(B[i]) \longleftarrow H(C[i]) \longleftarrow \cdots$$

is exact. Dually, a functor $H\colon \mathcal{C} \to \text{Vec}_k$ is called *homological* if for each exact triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

the sequence

$$\cdots \longrightarrow H(A[i]) \longrightarrow H(B[i]) \longrightarrow H(C[i]) \longrightarrow H(A[i+1]) \longrightarrow \cdots$$

is exact.

Example 3.6. Like we have already used before, for any object $A \in \mathcal{C}$, the functors $\mathrm{Hom}(-, A)$ and $\mathrm{Hom}(A, -)$ are cohomological and homological respectively.

Example 3.7. Perhaps the prototypical example is of course just taking cohomology in $D(X)$ for a k-scheme X, or any k-linear derived category in general. Slightly confusingly, this is a homological functor in the above parlance.

Definition 3.8. Let $\mathcal{C}$ be a triangulated category and $\varphi\colon \mathcal{C} \to \mathrm{Vec}_k$ a functor. Then φ is called *locally-finite* if

$$\dim_k \left(\bigoplus_{j\in\mathbb{Z}} \varphi(A[j]) \right) < \infty$$

for all $A \in \mathcal{C}$.

For a k-linear triangulated category $\mathcal{C}$, we denote by ${}^\vee\mathcal{C}$ and $\mathcal{C}^\vee$ the categories of locally-finite homological and cohomological functors respectively, with morphisms given by k-linear natural transformations.

Let X be a projective scheme over a Noetherian ring R. Then we have a theorem by Serre saying all cohomology modules of any coherent sheaf are finitely-generated R-modules. We can extend this result to Ext-modules using spectral sequences. For $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in D^b_{\mathrm{coh}}(X)$ and $j \in \mathbb{Z}$, consider the spectral sequences

$$E_2^{p,q} = \mathrm{Ext}^p(\mathcal{H}^{-j}(\mathcal{F}^\bullet), \mathcal{H}^q(\mathcal{G}^\bullet)) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{H}^{-j}(\mathcal{F}^\bullet), \mathcal{G}^\bullet) \tag{3.1}$$

and

$$E_2^{p,q} = \mathrm{Ext}^p(\mathcal{H}^{-q}(\mathcal{F}^\bullet), \mathcal{G}^\bullet) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \tag{3.2}$$

from [9, Remark 3.7.i]. The limit at (p, q) of a spectral sequence is a subquotient of the term at (p, q) on any page, so the first spectral sequence tells us $\mathrm{Ext}^p(\mathcal{H}^{-q}(\mathcal{F}^\bullet), \mathcal{G}^\bullet)$ is finitely generated and the second tells us $\mathrm{Ext}^p(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ is as well, where we take some liberties with the roles of p and q along the way.

Now let $\mathcal{P}^\bullet \in D_{\mathrm{perf}}(X)$ and $\mathcal{F}^\bullet \in D^b_{\mathrm{coh}}(X)$ and consider the local version of the first spectral sequence

$$E_2^{p,q} = \mathcal{E}xt^p(\mathcal{P}^\bullet, \mathcal{H}^q(\mathcal{F}^\bullet)) \Rightarrow \mathcal{E}xt^{p+q}(\mathcal{P}^\bullet, \mathcal{F}^\bullet).$$

The term at (p, q) on the left-hand side is equal to the p-th homology module of the complex $\mathcal{H}^\bullet$ with terms $\mathcal{H}om(\mathcal{P}^i, \mathcal{H}^q(\mathcal{F}^\bullet))$ and differentials given by post-composition with the differentials of $\mathcal{P}^\bullet$. This holds because locally-free sheaves are acyclic for $\mathcal{H}om(-, \mathcal{G})$ for any sheaf of $\mathcal{O}_X$-modules $\mathcal{G}$. Now, because $\mathcal{P}^i$ and $\mathcal{H}^q(\mathcal{F}^\bullet)$ are trivial for i and q outside a bounded set, we see that $E_2^{p,q}$ is bounded in p and q. As an immediate consequence, we find that $\mathcal{E}xt^i(\mathcal{P}^\bullet, \mathcal{F}^\bullet)$ is bounded in i. Finally, consider the spectral sequence for the derived functor of the composition $\Gamma \circ \mathcal{H}om = \mathrm{Hom}$:

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{P}^\bullet, \mathcal{F}^\bullet)) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{P}^\bullet, \mathcal{F}^\bullet).$$

This has trivial terms at (p, q) for $p > \dim(X)$ by the Grothendieck Vanishing Theorem, for $p < 0$ by left-exactness of Γ and for q outside a bounded set by the previous, hence $\mathrm{Ext}^i(\mathcal{P}^\bullet, \mathcal{F}^\bullet)$ is bounded in i.

Taking $R = k$, this implies $\mathrm{Hom}(-, \mathcal{G}^\bullet)$ furnishes a locally-finite cohomological functor on $D^b_{\mathrm{coh}}(X)$ and the Yoneda lemma says the natural transformations between functors of this form come from morphisms between their representing objects. Because $D_{\mathrm{perf}}(X) \subset D^b_{\mathrm{coh}}(X)$ is a full subcategory, we can restrict the functor $\mathrm{Hom}(-, \mathcal{G}^\bullet)$ to $D_{\mathrm{perf}}(X)$ and keep the statement on natural transformations. Hence, we get a functor $D^b_{\mathrm{coh}}(X) \to D_{\mathrm{perf}}(X)^\vee$.

A converse to the previous discussion is given by the following lemma.

Lemma 3.9 ([1, Lemma 2.3.4]). *Let $\mathcal{C}$ be a compactly generated triangulated category. Then any cohomological functor $F\colon (\mathcal{C}^c)^{op} \to \mathrm{vec}_k$ is representable by an object of $\mathcal{C}$ and any natural transformation between such functors is induced by a morphism of their representing objects.*

Note that a locally-finite functor has to land in vec_k, so that this theorem applies to functors in $D_{\mathrm{perf}}(X)^\vee$. At this point, Ballard proves the following result, which is the first half of the duality between $D_{\mathrm{perf}}(X)$ and $D^b_{\mathrm{coh}}(X)$.

Theorem 3.10 ([1, Theorem 2.3.3]). *Let X be a projective scheme over a field k. Then the functor $y\colon D^b_{\mathrm{coh}}(X) \to D_{\mathrm{perf}}(X)^\vee$ is an equivalence.*

To be able to state the other half of the duality, we need a few more definitions. Let $\mathcal{C}$ be a triangulated category and $\mathcal{I}, \mathcal{J} \subset \mathcal{C}$ two subcategories. Write $\mathcal{I} * \mathcal{J}$ for the full subcategory of objects $B \in \mathcal{C}$ which fit in an exact triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

with $A \in \mathcal{I}$ and $B \in \mathcal{J}$. Let $\langle \mathcal{I} \rangle$ denote the smallest subcategory of $\mathcal{C}$ containing $\mathcal{I}$, which is closed under shifts, finite direct sums and direct summands. We will also write $\langle C \rangle$ for objects $C \in \mathcal{C}$ when we mean $\langle \mathcal{I} \rangle$ for the full subcategory $\mathcal{I} \subset \mathcal{C}$ with one object C. Finally, set $\langle \mathcal{I} \rangle_0 = 0$ and $\langle \mathcal{I} \rangle_n = \langle \langle \mathcal{I} \rangle_{n-1} * \langle \mathcal{I} \rangle \rangle$.

Definition 3.11. A triangulated category is called *strongly finitely generated* if there exist an object $C \in \mathcal{C}$ and a positive integer d such that $\mathcal{C} = \langle C \rangle_d$. The object C is then called a *strong generator*. The minimal d for which there exists such an object C is called the *dimension* of $\mathcal{C}$. If $\mathcal{C}$ is non strongly finitely generated, we say $\mathcal{C}$ has dimension ∞.

Exactly the same arguments as before show that the functor $\mathrm{Hom}(\mathcal{P}^\bullet, -)$ for a perfect complex $\mathcal{P}^\bullet$ is a locally-finite cohomological functor on $D^b_{\mathrm{coh}}(X)$, so we get a functor $D_{\mathrm{perf}}(X) \to {}^\vee D^b_{\mathrm{coh}}(X)$. The statement corresponding to Lemma 3.9 is the following corollary to [19, Theorem 4.16].

Theorem 3.12. *If $\mathcal{C}$ is a strongly finitely generated k-linear triangulated category, which is closed under direct summands, then any locally-finite cohomological functor $\mathcal{C} \to \mathrm{Vect}_k$ is representable.*

The following theorem is another result by Rouquier with an incredibly technical proof, which we will not be able to treat here.

Theorem 3.13 ([19, Theorem 7.39])**.** *Let X be a quasi-projective scheme over a perfect field k. Then $D^b_{\mathrm{coh}}(X)$ is strongly finitely generated.*

Corollary 3.14 ([1, Corollary 2.4.4])**.** *If X is a projective scheme over a perfect field k, then any locally-finite cohomological functor on $D^b_{\mathrm{coh}}(X)$ is representable by a complex in $D^b_{\mathrm{coh}}(X)$.*

Before we prove the second duality statement, we need a somewhat technical lemma. The point of this is that given a bounded complex of finitely generated modules over a Noetherian local ring R, we can not only find a free resolution $F^\bullet$, but we can find one that is *minimal*, as it is called, meaning $d^i F^i \subset \mathfrak{m} F^{i+1}$ for all $i \in \mathbb{Z}$. This resolution is useful because the differentials of the complex

$$\mathrm{Hom}^*(F^\bullet, \kappa)\colon \; \cdots \leftarrow \mathrm{Hom}(F^{i-1}, \kappa) \leftarrow \mathrm{Hom}(F^i, \kappa) \leftarrow \mathrm{Hom}(F^{i+1}, \kappa) \leftarrow \cdots$$

are trivial, where κ is the residue field, resulting in an isomorphism

$$\mathrm{Hom}_{D^b(R-\mathrm{mod})}(F^\bullet, \kappa[i]) \cong H_i \,\mathrm{Hom}^*(F^\bullet, \kappa) \cong \mathrm{Hom}(F^i, \kappa).$$

The proof of the following is just an exercise in homological algebra and provides little geometric intuition. The arguments laid out will not be used in the later sections, so the reader is free to skip the proof.

Lemma 3.15. *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring and let $M^\bullet$ be a bounded complex of finitely-generated R-modules. Then there exists a minimal complex $F^\bullet$ of free modules of finite rank and a quasi-isomorphism $F^\bullet \xrightarrow{\sim} M^\bullet$.*

Proof. The proof we present is essentially dual to the proof in [3, Proposition 7.6], with slight adaptations made to yield a minimal resolution instead of just any resolution. We construct the complex $F^\bullet$ inductively and check minimality afterwards. To be precise, take $n \in \mathbb{Z}$ and suppose that we have constructed F^i with differentials $d^i_F\colon F^i \to F^{i+1}$ and maps $f^i\colon F^i \to M^i$ for all $i > n$ so that the maps and differentials commute, the induced morphism $H^{i+1}(F^\bullet) \to H^{i+1}(M^\bullet)$ is an isomorphism and the induced morphism $\ker(d^i_F) \to H^i(M)$ is a surjection. Note that there exists some such n, because M is bounded.

The differential $d^{i-1}_M\colon M^{i-1} \to M^i$ induces a map $\mathrm{coker}(d^{i-2}_M) \to \ker(d^i_M)$ and the morphism $F^i \to M^i$ induces a map $\ker(d^i_F) \to \ker(d^i_M)$. Let P^{i-1} denote the fibre product

$$P^{i-1} := \operatorname{coker}(d_M^{i-2}) \times_{\ker(d_M^i)} \ker(d_F^i).$$

A basis for $P^{i-1}/\mathfrak{m}P^{i-1}$ yields a minimal set of generators for P^{i-1} by Nakayama's lemma, so we get a surjection $F^{i-1} \to P^{i-1}$ with F^{i-1} free of minimal rank. Free modules are projective, so the map $F^{i-1} \to P^{i-1} \to \operatorname{coker}(d_M^{i-2})$ extends along the surjection $M^{i-1} \to \operatorname{coker}(d_M^{i-2})$ to a morphism $f^{i-1} \colon F^{i-1} \to M^{i-1}$. So we have a commutative diagram

$$\begin{array}{ccccccc}
& & & \overset{d_F^{i-1}}{\frown} & & & \\
F^{i-1} & \twoheadrightarrow & P^{i-1} & \longrightarrow & \ker(d_F^i) & \hookrightarrow & F^i \\
\downarrow{\scriptstyle f^{i-1}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f^i} \\
M^{i-1} & \twoheadrightarrow & \operatorname{coker}(d_M^{i-2}) & \longrightarrow & \ker(d_M^i) & \hookrightarrow & M^i
\end{array}$$

and we wish to show the induced maps $H^i(F^\bullet) \to H^i(M^\bullet)$ and $\ker(d_F^{i-1}) \to H^{i-1}(M^\bullet)$ are bijective and surjective respectively.

The map $\ker(d_F^i) \to H^i(M^\bullet)$ was already surjective, so we just have to show it has kernel $\operatorname{im}(d_F^{i-1})$. If s is in this kernel, then $f^i(s) \in \operatorname{im}(d_M^{i-1})$, so we may take $a \in M^{i-1}$ such that $f^i(s) = d_M^{i-1}a$. If $\overline{a}$ denotes the class of a in $\operatorname{coker}(d_M^{i-2})$, then $(\overline{a}, s) \in P^{i-1}$ maps to s and therefore $s \in \operatorname{im}(d_F^{i-1})$. Conversely, consider $t \in F^{i-1}$. Its image in P^{i-1} is a pair $(\overline{a}, s)$ such that $d_M^{i-1}a = f^i(s)$, so that $f^i(s) \in \operatorname{im}(d_M^{i-1})$, showing $f^i(s)$ is trivial in $H^i(M)$ and thus s is in the kernel. We get an induced isomorphism $H^i(F^\bullet) \to H^i(M^\bullet)$.

Now consider $\ker(d_F^{i-1}) \to H^{i-1}(M^\bullet)$. Let $\overline{a} \in H^{i-1}(M^\bullet)$ be a class represented by $a \in \ker(d_M^{i-1})$. Because $d_M^{i-1}a = 0 = f^i(0)$, we see $(\overline{a}, 0) \in P^{i-1}$ and choose $e \in F^{i-1}$ mapping onto $(\overline{a}, 0)$. Then e maps to zero in F^i and to $\overline{a}$ in $H^i(M)$, showing surjectivity.

By induction, we get a complex $F^\bullet$ and a quasi-isomorphism $F^\bullet \to M^\bullet$ given by $(f^\bullet)$. Finally, we wish to show that $F^\bullet$ is minimal, so take $i \in \mathbb{Z}$. Note that the map $F^i \to P^i$ we chose becomes an isomorphism after applying $- \otimes \kappa$. This means its kernel is contained in $\mathfrak{m}F^i$. We finish up by showing $\operatorname{im}(d_F^{i-1}) \subset \ker(F^i \to P^i)$, or equivalently, that the composition $F^{i-1} \to F^i \to P^i$ is trivial. By the definition of P^i, this is trivial if and only if the compositions with the maps to $\ker(d_F^{i+1})$ and $\operatorname{coker}(d_M^{i-1})$ are both trivial. The map $F^{i-1} \to \ker(d_F^{i+1})$ is trivial because it is given by $d_F^i \circ d_F^{i-1}$ and the commutative diagram

$$\begin{array}{ccccccc}
F^{i-1} & \longrightarrow & P^{i-1} & \longrightarrow & F^i & \longrightarrow & P^i \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M^{i-1} & \longrightarrow & \operatorname{coker}(d_M^{i-2}) & \longrightarrow & M^i & \longrightarrow & \operatorname{coker}(d_M^{i-1})
\end{array}$$

shows $F^{i-1} \to \operatorname{coker}(d_M^{i-1})$ is trivial as well, so we are done. □

The main idea of the proof of the lemma below is from [1, Lemma 2.4.5]. However, because we have not quite developed as much theory as Ballard does, we need to do some more work.

Lemma 3.16 ([1, Lemma 2.4.5]). *Let X be a projective scheme over a field k and $\mathcal{F}^\bullet \in D^b_{\text{coh}}(X)$. Then $\mathcal{F}^\bullet$ furnishes a locally-finite cohomological functor on $D^b_{\text{coh}}(X)$ if and only if $\mathcal{F}^\bullet$ is perfect.*

Proof. If $\mathcal{F}^\bullet$ is perfect, then the same arguments as in the discussion on locally-finite homological functors on $D_{\text{perf}}(X)$ show $\operatorname{Hom}(\mathcal{F}^\bullet, -)$ is locally-finite. Conversely, assume $\operatorname{Hom}(\mathcal{F}^\bullet, -)$ is locally-finite. We will show that $\mathcal{F}^\bullet$ has finite Tor amplitude. For a closed point $x \in X$, the skyscraper sheaf $\mathcal{O}_x$ is supported on $\{x\}$, hence Lemma 2.2 says

$$\operatorname{Hom}(\mathcal{F}^\bullet, \mathcal{O}_x[i]) \cong \operatorname{Hom}(\mathcal{F}^\bullet_x, k(x)[i]).$$

By the previous lemma, there exists a minimal free resolution $F^\bullet \xrightarrow{\sim} \mathcal{F}^\bullet_x$. This resolution computes the groups $\operatorname{Hom}(\mathcal{F}^\bullet_x, k(x)[i])$, meaning

$$\operatorname{Hom}(\mathcal{F}^\bullet_x, k(x)[i]) \cong \operatorname{Hom}(F^\bullet, k(x)[i]) \cong \operatorname{Hom}(F^i, k(x)).$$

At this point, the complex $\mathcal{F}^\bullet$ furnishing a locally-finite functor on $D^b_{\text{coh}}(X)$ implies the $k(x)$-vector spaces $\operatorname{Hom}(F^i, k(x))$ are trivial for i outside a bounded set, thus the same holds for the $\mathcal{O}_{X,x}$-modules F^i. Hence $F^\bullet \xrightarrow{\sim} \mathcal{F}^\bullet_x$ is a quasi-isomorphism of $\mathcal{F}^\bullet_x$ with a perfect complex. This quasi-isomorphism extends to an open neighborhood of x, so by quasi-compactness and the fact that the closed points are dense in X, we get a finite open cover $X = \bigcup U_i$ such that each $\mathcal{F}^\bullet|_{U_i}$ is perfect. The finitely many complexes $\mathcal{F}^\bullet|_{U_i}$ all have finite Tor amplitude and because $\otimes$ commutes with restriction, so does $\mathcal{F}^\bullet$. By Remark 3.3, we are done. □

At this point, we have shown that the functor $D_{\text{perf}}(X) \to {}^\vee D^b_{\text{coh}}(X)$ is essentially surjective and we trust the work done by Ballard to arrive at the following theorem.

Theorem 3.17 ([1, Proposition 2.4.6]). *Let X be a projective scheme over a perfect field k. Then the functor $D_{\text{perf}}(X) \to {}^\vee D^b_{coh}(X)$ is an equivalence.*

3.3 Exceptional pullback

The following theorem is a generalization by Neeman to triangulated categories of an exceptionally powerful theorem first proved by Brown in 1962.

Theorem 3.18. *Let $\mathcal{C}$ be a compactly generated triangulated category, together with a homological functor $H\colon \mathcal{C}^{op} \to Ab$. Suppose H takes all coproducts to products, that is, each of the natural maps*

$$H\left(\coprod_{i\in I} A_i\right) \longrightarrow \prod_{i\in I} H(A_i)$$

is an isomorphism. Then H is representable.

We will not go into the proof here, as it is quite technical and requires the introduction of a few notions we will have no use for. Instead, we refer to the original work by Neeman [17, Theorem 3.1]. Our main use for this theorem is the following, the proof of which we take from [1].

Proposition 3.19 ([1, Proposition 2.2.15])**.** *Let $\mathcal{C}$ be a compactly generated triangulated category and $\mathcal{S}$ a triangulated category. Let $F\colon \mathcal{C} \to \mathcal{S}$ be a triangulated functor which commutes with coproducts. Then F has a right-adjoint $G\colon \mathcal{S} \to \mathcal{C}$.*

Proof. For each $B \in \mathcal{S}$, we have a homological functor $\operatorname{Hom}(F(-), B)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$. For any coproduct $\coprod_{i\in I} A_i$ in $\mathcal{C}$ we have

$$\operatorname{Hom}\left(F\left(\coprod_{i\in I} A_i\right), B\right) \cong \operatorname{Hom}\left(\coprod_{i\in I} F(A_i), B\right) \cong \prod_{i\in I} \operatorname{Hom}(F(A_i), B),$$

hence $\operatorname{Hom}(F(-), B)$ is representable, say by $G(B)$. A morphism $f\colon B \to B'$ provides a natural transformation of representable functors $\operatorname{Hom}(F(-), B) \to \operatorname{Hom}(F(-), B')$, which corresponds to a morphism $G(B) \to G(B')$ by the Yoneda lemma.

The fact that $G(B)$ represents $\operatorname{Hom}(F(-), B)$ provides naturality of

$$\operatorname{Hom}(F(A), B) \xrightarrow{\sim} \operatorname{Hom}(A, G(B))$$

in the first argument and the definition of the morphisms $G(B) \to G(B')$ ensures naturality in the second argument. □

Lemma 3.20 ([17, Theorem 4.1])**.** *Let $F\colon \mathcal{C} \to \mathcal{S}$ be a triangulated functor with $\mathcal{C}$ compactly generated. If F has a left-adjoint G, then F commutes with coproducts if and only if G takes compact objects to compact objects.*

Proof. Suppose G takes compact objects to compact objects. Let $\coprod_{i\in I} A_i$ be a coproduct in $\mathcal{C}$. Then for any compact $E \in \mathcal{S}$, we have

$$\begin{aligned}\operatorname{Hom}\left(E, F\left(\coprod_{i\in I} A_i\right)\right) &\cong \operatorname{Hom}\left(G(E), \coprod_{i\in I} A_i\right)\\ &\cong \bigoplus_{i\in I} \operatorname{Hom}(G(E), A_i)\\ &\cong \bigoplus_{i\in I} \operatorname{Hom}(E, F(A_i))\\ &\cong \operatorname{Hom}\left(E, \coprod_{i\in I} F(A_i)\right),\end{aligned}$$

the isomorphism from the first to the last Hom-group being induced by the natural morphism $\eta\colon F(\coprod_{i\in I} A_i) \to \coprod_{i\in I} F(A_i)$. By Lemma 1.10, we find η is an isomorphism.

For the converse, a reordering of the chain of isomorphisms above shows that for compact $E \in \mathcal{S}$, the functor $\operatorname{Hom}(G(E), -)$ commutes with arbitrary coproducts, hence $G(E)$ is compact. $\square$

Corollary 3.21 ([17, Example 4.2])**.** *If $f\colon X \to Y$ is a morphism of projective k-schemes, then the right-derived pushforward $Rf_*\colon D(X) \to D(Y)$ of f has a right-adjoint $f^!\colon D(Y) \to D(X)$.*

Proof. We have an adjunction $Lf^* \dashv Rf_*$ and Lf^* takes perfect complexes to perfect complexes. Because $D(X)^c = D_{\mathrm{perf}}(X)$, this finishes the proof. $\square$

Example 3.22. For a projective k-scheme X, let $f\colon X \to \operatorname{Spec}(k)$ denote the structure morphism. Then we get the exceptional pullback $f^!\colon D(\operatorname{Spec}(k)) \to D(X)$. In $D(\operatorname{Spec}(k))$ we have the structure sheaf $\mathcal{O}_{\operatorname{Spec}(k)}$, so let us consider one interesting property of the complex $f^!\mathcal{O}_{\operatorname{Spec}(k)}$. If $\mathcal{P}^\bullet$ is a perfect complex, then taking the tensor product with $\mathcal{P}^{\bullet\vee} := (\operatorname{Hom}(\mathcal{P}^{-i}, \mathcal{O}_X))_i$ is right-adjoint to $\mathcal{P}^\bullet \otimes -$. Hence if $\mathcal{F}^\bullet$ is a complex of quasi-coherent sheaves, then we have natural isomorphisms

$$\begin{aligned}\operatorname{Hom}(\mathcal{P}^\bullet, \mathcal{F}^\bullet)^* &\cong \operatorname{Hom}(\mathcal{O}_X, \mathcal{P}^{\bullet\vee} \otimes \mathcal{F}^\bullet)^* \cong (f_*(\mathcal{P}^{\bullet\vee} \otimes \mathcal{F}^\bullet))^*\\ &= \operatorname{Hom}(f_*(\mathcal{P}^{\bullet\vee} \otimes \mathcal{F}^\bullet), \mathcal{O}_{\operatorname{Spec}(k)}) \cong \operatorname{Hom}(\mathcal{P}^{\bullet\vee} \otimes \mathcal{F}^\bullet, f^!\mathcal{O}_{\operatorname{Spec}(k)})\\ &\cong \operatorname{Hom}(\mathcal{F}^\bullet, \mathcal{P}^\bullet \otimes f^!\mathcal{O}_{\operatorname{Spec}(k)}),\end{aligned}$$

which means $- \otimes f^!\mathcal{O}_{\operatorname{Spec}(k)}$ is what is called a *Rouquier functor* for the inclusion $D_{\mathrm{perf}}(X) \hookrightarrow D(X)$. In general, if $F\colon \mathcal{C} \to \mathcal{D}$ is a functor of k-linear categories, then a Rouquier functor for F is a functor $R_F\colon \mathcal{C} \to \mathcal{D}$ such that there are natural isomorphisms

$$\operatorname{Hom}(B, R_F(A)) \cong \operatorname{Hom}(F(A), B)^*.$$

Note that $\operatorname{Hom}(\mathcal{P}^\bullet, -)^*$ is a locally-finite cohomological functor on $D_{\mathrm{perf}}(X)$, represented by $\mathcal{P}^\bullet \otimes f^!\mathcal{O}_{\operatorname{Spec}(k)}$. This means that $\mathcal{P}^\bullet \otimes f^!\mathcal{O}_{\operatorname{Spec}(k)} \in D^b_{\mathrm{coh}}(X)$, by Theorem 3.10 and in particular, taking $\mathcal{P}^\bullet = \mathcal{O}_X$, we get $f^!\mathcal{O}_{\mathrm{Spec}(k)} \in D^b_{\mathrm{coh}}(X)$. The functor $f^!$ can also be defined for quasi-projective k-schemes, but then we can not argue that $f^!\mathcal{O}_{\mathrm{Spec}(k)}$ is a bounded complex of coherent sheaves, because $\mathrm{Hom}(\mathcal{P}^\bullet, -)$ need not be locally-finite on $D^b_{\mathrm{coh}}(X)$. For the dualizing nature of the functor $- \otimes f^!\mathcal{O}_{\mathrm{Spec}(k)}$, the complex $f^!\mathcal{O}_{\mathrm{Spec}(k)}$ is called a *dualizing complex* of X. We will not state the precise definition of this term here. When X is a smooth projective variety, then there is also the dualizing sheaf ω_X, which we will see when we consider Serre duality and Serre functors is closely related to the dualizing complex. Specifically, there is a quasi-isomorphism $f^!\mathcal{O}_{\mathrm{Spec}(k)} \cong \omega_X[d]$, where d is the dimension of X.

One of the properties of the exceptional pullback that we will need is that it behaves nicely under base change. Suppose

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array}$$

is a Cartesian diagram of projective k-schemes with u flat. Flat base change says the functors f'_*v^* and u^*f_* are isomorphic, so the counit $f_*f^! \Rightarrow \mathrm{Id}$ of the adjunction $f_* \dashv f^!$ gives a natural transformation

$$f'_*v^*f^! \cong u^*f_*f^! \Longrightarrow u^*.$$

This transposes via the adjunction $f'_* \dashv f^!$ to a natural transformation $v^*f^! \Rightarrow f'^!u^*$. A theorem proved by Lipman in [11, Corollary 4.4.3] says that this is an isomorphism of functors. The theorem is much more general than our setting, so we will not be able to state it here. However, at least the following is true.

Theorem 3.23. *Suppose one has a Cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array}$$

such that the natural transformation $f'_*v^* \to u^*f_*$ is an isomorphism, with f proper and u flat. Then the natural transformation $v^*f^! \Rightarrow f'^!u^*$ is an isomorphism of functors.

The adjunctions yield even more. Suppose that $f\colon X \to Y$ is a morphism of projective k-schemes. Then for $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in D(Y)$, there is a natural morphism

$$f_*(f^!\mathcal{F}^\bullet \otimes f^*\mathcal{G}^\bullet) \cong f_*f^!(\mathcal{F}^\bullet) \otimes \mathcal{G}^\bullet \longrightarrow \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet,$$

where the isomorphism is the projection formula. Like before, we transpose this to get a natural morphism

$$f^!\mathcal{F}^\bullet \otimes f^*\mathcal{G}^\bullet \longrightarrow f^!(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet).$$

Taking $\mathcal{F}^\bullet = \mathcal{O}_Y$, we get a natural morphism

$$f^!O_Y \otimes f^*\mathcal{G}^\bullet \longrightarrow f^!\mathcal{G}^\bullet$$

and if we assume furthermore that f is flat and $\mathcal{G}^\bullet$ is bounded above, then again a theorem by Lipman says this is an isomorphism, see [11, Theorem 4.9.4]. Again, the following is just specialized form of the general theorem.

Theorem 3.24. *If $f\colon X \to Y$ is a proper flat morphism of Noetherian schemes, then for any $\mathcal{G}^\bullet \in D^+(Y)$, the natural morphism*

$$f^!\mathcal{O}_Y \otimes f^*\mathcal{G}^\bullet \longrightarrow f^!\mathcal{G}^\bullet$$

is an isomorphism.

3.4 Pseudo-adjoint functors

In the following, we take X and Y to be projective schemes over a field k. Where there is a construction resembling a duality, one hopes for this construction to be functorial. The problem is, the duality between $D^b_{\text{coh}}(X)$ and $D_{\text{perf}}(X)$ is not symmetric, so we have to make do with something weaker.

Definition 3.25. If $F\colon D_{\text{perf}}(X) \to D_{\text{perf}}(Y)$ is an exact functor, then a functor $F^\vee\colon D^b_{\text{coh}}(Y) \to D^b_{\text{coh}}(X)$ is called a *right pseudo-adjoint* to F if there are natural isomorphisms

$$\text{Hom}_{D^b_{\text{coh}}(Y)}(F(A), B) \cong \text{Hom}_{D^b_{\text{coh}}(X)}(A, F^\vee(B))$$

for $A \in D_{\text{perf}}(X)$ and $B \in D^b_{\text{coh}}(Y)$. Similarly, if $G\colon D^b_{\text{coh}}(X) \to D^b_{\text{coh}}(Y)$ is an exact functor, then we call a functor ${}^\vee G\colon D_{\text{perf}}(Y) \to D_{\text{perf}}(X)$ a *left pseudo-adjoint* to F if there are natural isomorphisms

$$\text{Hom}_{D^b_{\text{coh}}(Y)}(A, G(B)) \cong \text{Hom}_{D^b_{\text{coh}}(X)}({}^\vee G(A), B)$$

for $A \in D_{\text{perf}}(Y)$ and $B \in D^b_{\text{coh}}(X)$.

Example 3.26. For a morphism $f\colon X \to Y$, we have the left-derived pullback $f^*\colon D(Y) \to D(X)$ and its right-adjoint, the right-derived pushforward $f_*\colon D(X) \to D(Y)$. These restrict to functors $D_{\text{perf}}(Y) \to D_{\text{perf}}(X)$ and $D^b_{\text{coh}}(X) \to D^b_{\text{coh}}(Y)$ respectively, so they form a pseudo-adjoint pair.

Lemma 3.27 ([1, Proposition 2.5.3 and Lemma 2.5.5])**.** *A left- or right pseudo-adjoint is unique up to unique isomorphism, if it exists.*

Proof. Say $F^\vee$ is right pseudo-adjoint to an exact functor $F\colon D_{\text{perf}}(X) \to D_{\text{perf}}(Y)$. Then for any $B \in D^b_{\text{coh}}(Y)$, the object $F^\vee(B)$ represents the functor $\text{Hom}(F(-), B)$ by the definition of $F^\vee$, hence $F^\vee(B)$ is unique up to a unique isomorphism. These isomorphisms are natural because the isomorphisms $\text{Hom}(F(A), B) \cong \text{Hom}(A, F^\vee(B))$ are natural. The proof for left pseudo-adjoints is analogous. □

Note that these pseudo-adjoints are not exact a priori. However, Ballard proves in [1, Lemma 2.5.11 and Lemma 2.5.12] that any right pseudo-adjoint is exact and so is any left pseudo-adjoint.

Lemma 3.28. *Let $F\colon D_{\text{perf}}(X) \to D_{\text{perf}}(Y)$ and $G\colon D^b_{\text{coh}}(Y) \to D^b_{\text{coh}}(X)$ be functors for which there are natural isomorphisms*

$$\text{Hom}_{D^b_{\text{coh}}(Y)}(F(A), B) \cong \text{Hom}_{D^b_{\text{coh}}(X)}(A, G(B))$$

for all $A \in D_{\text{perf}}(X)$ and $B \in D^b_{\text{coh}}(Y)$. Then exactness of F implies exactness of G and vice versa. In particular, pseudo-adjoints are exact.

It is a natural question at this point to consider the left pseudo-adjoint of the right pseudo-adjoint and vice versa. Indeed, this notion of pseudo-adjunctions is symmetric, like it is for adjunctions in the usual sense. To be precise, if F has a right pseudo-adjoint G, then F is left pseudo-adjoint to G. Moreover, by Lemma 3.27, every left pseudo-adjoint is uniquely isomorphic to F.

Lemma 3.29 ([1, Corollary 2.5.6])**.** *Suppose $F\colon D_{\text{perf}}(X) \to D_{\text{perf}}(Y)$ is an exact equivalence with inverse G such that both F and G have right pseudo-adjoints $F^\vee$ and $G^\vee$, then $F^\vee$ is an exact equivalence with inverse $G^\vee$. Furthermore, the functor $G^\vee$ extends F to $D^b_{\text{coh}}(X)$.*

Proof. Pseudo-adjoints compose like usual adjoints do, hence $F^\vee G^\vee$ is right pseudo-adjoint to $GF \cong \text{id}_{D_{\text{perf}}(X)}$. Of course, the identity functor $D^b_{\text{coh}}(X) \to D^b_{\text{coh}}(X)$ is right pseudo-adjoint to $\text{id}_{D_{\text{perf}}(X)}$, so by Lemma 3.27 we see $F^\vee G^\vee$ is isomorphic to the identity functor. Similarly, the composition $G^\vee F^\vee$ is isomorphic to the identity as well, showing $F^\vee$ is an equivalence inverted by $G^\vee$. There are natural isomorphisms

$$\text{Hom}(A, F(B)) \cong \text{Hom}(G(A), B) \cong \text{Hom}(A, G^\vee(B))$$

for $A, B \in D_{\text{perf}}(Y)$, showing $F \cong G^\vee|_{D_{\text{perf}}(X)}$ through the Yoneda lemma. □

An analogous proof shows the same statement for left pseudo-adjoints.

Lemma 3.30 ([1, Lemma 2.5.7])**.** *Suppose $G\colon D^b_{\text{coh}}(X) \to D^b_{\text{coh}}(Y)$ is an exact equivalence with inverse F such that both G and F have left pseudo-adjoints ${}^\vee G$ and ${}^\vee F$, then ${}^\vee G$ is an exact equivalence with inverse ${}^\vee F$. Furthermore, the functor ${}^\vee F$ is a restriction of G to $D_{\text{perf}}(X)$.*

Of course, none of this would be of any use if we could not show the existence of these pseudo-adjoints. Luckily, in the present setting, these pseudo-adjoints exist for every exact functor.

Proposition 3.31 ([1, Proposition 2.5.3])**.** *Let $F\colon D_{\text{perf}}(X) \to D_{\text{perf}}(Y)$ be an exact functor. Then F has a right pseudo-adjoint.*

Proof. If $\Phi\colon D_{\text{perf}}(Y) \to \text{Vec}_k$ is a locally-finite cohomological functor, then so is $\Phi \circ F$, so we get an induced functor $D_{\text{perf}}(Y)^\vee \to D_{\text{perf}}(X)^\vee$. By the equivalence in Theorem 3.10, this is the same as a functor $D^b_{\text{coh}}(Y) \to D^b_{\text{coh}}(X)$ and this is the desired right pseudo-adjoint. □

Using Theorem 3.17 instead, the same arguments show the following.

Proposition 3.32 ([1, Proposition 2.5.4])**.** *If k is perfect, then any exact functor $G\colon D^b_{\text{coh}}(X) \to D^b_{\text{coh}}(Y)$ has a left pseudo-adjoint.*

Lemma 3.33 ([1, Lemma 2.5.7])**.** *If $F\colon D^b_{\text{coh}}(X) \to D^b_{\text{coh}}(Y)$ is an exact equivalence, then F restricts to an exact equivalence $D_{\text{perf}}(X) \to D_{\text{perf}}(Y)$.*

Proof. Since F is an equivalence, the functor $\text{Hom}(A, -)$ for $A \in D^b_{\text{coh}}(X)$ is locally-finite on $D^b_{\text{coh}}(X)$ if and only if $\text{Hom}(F(A), -)$ is locally-finite on $D^b_{\text{coh}}(Y)$. Thus F takes perfect complexes to perfect complexes. Likewise, an inverse G to F restricts to the perfect derived categories as well, showing the restriction is an equivalence. Note that for any two objects $A, B \in D_{\text{perf}}(X)$ we have

$$\text{Hom}(A, G(B)) \cong \text{Hom}(F(A), B),$$

so that the restriction of F to $D_{\text{perf}}(X) \to D_{\text{perf}}(Y)$ is left pseudo-adjoint to G, thus it is exact by Lemma 3.28. □

4 Derived Categories of Coherent Sheaves on Gorenstein Varieties

4.1 Gorenstein varieties

The results in this subsection form in a way a minimal subset of the theory of dualizing complexes developed in [7, Section V.2], say, containing the results we will need in the later sections. We choose not to define dualizing complexes in generality here, and provide direct proofs.

Let X be a smooth projective scheme over a field k. Then we have the ubiquitous canonical bundle ω_X, most important for the theorem of Serre duality, stating the existence of a canonical isomorphism

$$H^i(X, \mathcal{F}) \longrightarrow H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X)^*$$

for locally-free sheaves $\mathcal{F}$ of finite rank. We want to consider more general varieties, without losing the power of the canonical bundle ω_X. The point is that we consider schemes which are mildly singular, in the following sense.

Definition 4.1. A Noetherian local ring $(R, \mathfrak{m}, k)$ of Krull dimension n is called *Gorenstein* if it has finite injective dimension as a module over itself.

The following theorem is a small part of a list of equivalent conditions for a Noetherian local ring to be Gorenstein. We do not state in its full form, simply because we only need a small part of it. For the full statement and the proof, we refer to [12, Theorem 18.1].

Theorem 4.2. *Let $(R, \mathfrak{m}, \kappa)$ be an n-dimensional Noetherian local ring. Then R is a Gorenstein local ring if and only if one of the following equivalent conditions hold.*

1. *For every $i \neq n$, it holds that $\operatorname{Ext}^i_R(\kappa, R) = 0$ and $\operatorname{Ext}^n_R(\kappa, R) \cong \kappa$;*
2. *For some $i > n$ it holds that $\operatorname{Ext}^i_R(\kappa, R) = 0$.*

Definition 4.3. A locally Noetherian scheme X is called *Gorenstein* if all of its local rings are Gorenstein.

For any bounded complex of coherent sheaves $\mathcal{F}^\bullet$ we can form the dual $\mathcal{F}^{\bullet\vee} := \mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{O}_X)$, where $\mathcal{H}om^\bullet$ denotes the derived internal Hom functor

$$\mathcal{H}om^\bullet(-,-)\colon D^-(X) \times D^+(X) \longrightarrow D^+(X).$$

Because the sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is coherent for coherent sheaves $\mathcal{F}$ and $\mathcal{G}$, the above functor restricts to the appropriately bounded derived categories of coherent sheaves. For general varieties X, if $\mathcal{F}^\bullet \in D^b_{\mathrm{coh}}(X)$ is not perfect, then the dual $\mathcal{F}^{\bullet\vee}$ need not be bounded, but for Gorenstein varieties, it is bounded.

Lemma 4.4. *Let X be a projective Gorenstein k-scheme. Then the restricted functor*

$$\mathcal{H}om^\bullet(-, \mathcal{O}_X)\colon D^b(X) \longrightarrow D^+(X)$$

has image in $D^b(X)$.

Proof. Let $\mathcal{F}^\bullet$ be a bounded complex of quasi-coherent sheaves. For any point $x \in X$ we have a quasi-isomorphism

$$\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{O}_X)_x \cong R\operatorname{Hom}^\bullet(\mathcal{F}^\bullet_x, \mathcal{O}_{X,x}).$$

The local ring $\mathcal{O}_{X,x}$ is Gorenstein, so $\mathcal{O}_{X,x}$ has a finite injective resolution $\mathcal{O}_{X,x} \xrightarrow{\sim} I^\bullet$. We show that the complex $R\operatorname{Hom}^\bullet(\mathcal{F}^\bullet_x, \mathcal{I}^\bullet)$ has bounded cohomology. By definition of Ext the j-th cohomology is

$$\mathcal{H}^j(\mathcal{F}^\bullet_x, \mathcal{I}^\bullet) \cong \operatorname{Ext}^j(\mathcal{F}^\bullet_x, \mathcal{I}^\bullet)$$

and to compute this, we consider again the spectral sequence Eq. (3.2)

$$E_2^{p,q} = \mathrm{Ext}^p(\mathcal{H}^{-q}(\mathcal{F}_x^\bullet), \mathcal{I}^\bullet) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{F}_x^\bullet, \mathcal{I}^\bullet).$$

Because injectives are acyclic for $\mathrm{Hom}(\mathcal{G}, -)$ for any quasi-coherent sheaf $\mathcal{G}$, we see immediately that the term at (p, q) is the p-th cohomology of the complex with terms $\mathrm{Hom}(\mathcal{H}^{-q}(\mathcal{F}_x^\bullet), \mathcal{I}^i)$. Hence the terms $E_2^{p,q}$ are bounded in p and q and therewith so are the groups $\mathrm{Ext}^j(\mathcal{F}_x^\bullet, \mathcal{I}^\bullet)$.

Now consider the cohomology sheaves $\mathcal{H}^i := \mathcal{H}^i\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{O}_X)$. Because taking stalks is exact, the previous says that for each $x \in X$ the sheaves $\mathcal{H}_x^i$ are trivial for i outside a bounded set and this extends to an open set around x. Because X is quasi-compact, we get a finite open cover $X = \bigcup U_j$ such that for each j the sheaves $\mathcal{H}^i|_{U_j}$ are bounded in i. This implies the sheaves $\mathcal{H}^i$ are bounded in i themselves, so that $\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{O}_X) \in D^b(X)$. □

Of course, this is just the functor sending $\mathcal{F}^\bullet$ to $\mathcal{F}^{\bullet\vee}$. We will call this functor the *dualizing functor* and denote its restriction to $D^b_{\mathrm{coh}}(X)$ by $\mathbb{D}$. By the lemma above, we can take the double dual $(\mathcal{F}^{\bullet\vee})^\vee$ of a bounded complex of coherent sheaves. For any locally-free sheaf $\mathcal{F}$, the sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ is locally-free as well, hence $\mathbb{D}^2$ is the derived functor for the composition $D^2 := \mathcal{H}om(\mathcal{H}om(-, \mathcal{O}_X), \mathcal{O}_X))$ by the Composition Theorem [20, 10.8.2]. For coherent sheaves $\mathcal{F}$, there are natural bijections

$$\begin{aligned}\mathrm{Hom}(\mathcal{H}om(\mathcal{F}, \mathcal{O}_X), \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)) &\cong \mathrm{Hom}(\mathcal{F} \otimes \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)) \\ &\cong \mathrm{Hom}(\mathcal{F}, \mathcal{H}om(\mathcal{H}om(\mathcal{F}, \mathcal{O}_X))),\end{aligned}$$

so we get natural maps $\mathcal{F} \to \mathcal{H}om(\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)))$, forming a natural transformation $\mathrm{id} \Rightarrow D^2$. Taking right-derived functors, we get a natural transformation $\mathrm{id} \Rightarrow \mathbb{D}^2$.

Lemma 4.5. *Let X be a projective Gorenstein k-scheme. Then the natural transformation* $\mathrm{id} \Rightarrow \mathbb{D}^2$ *is an isomorphism of functors.*

Proof. Let $\mathcal{F}^\bullet \in D^b_{\mathrm{coh}}(X)$ be represented by a bounded above complex of locally free sheaves of finite rank. We know that locally-free sheaves are acyclic for D^2, so $\mathbb{D}^2$ is computed by directly applying D^2 to $\mathcal{F}^\bullet$ and the natural transformation $\mathrm{id} \Rightarrow \mathbb{D}^2$ goes degreewise on $\mathcal{F}^\bullet$. Now, on finite rank locally-free sheaves $\mathcal{P}$, the natural map $\mathcal{P} \to D^2(\mathcal{P})$ is an isomorphism, so $\mathcal{F}^\bullet \to \mathbb{D}^2(\mathcal{F}^\bullet)$ is an isomorphism as well. □

Corollary 4.6. *The dualizing functor* $\mathbb{D}\colon D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(X)$ *is an autoequivalence.*

One dual we want to make explicit for later use is that of the skyscraper sheaf $\mathcal{O}_x$ of a closed point $x \in X$. Because the stalk of the sheaf $\mathcal{H}om$ at y is just the Hom of the stalks at y and because taking stalks is exact, we have an isomorphism

$$\mathcal{H}om^\bullet(\mathcal{O}_x, \mathcal{O}_X)_y \cong \mathrm{Hom}^\bullet((\mathcal{O}_x)_y, \mathcal{O}_{X,y})$$

for any $y \in X$. For $y \neq x$, the module $(\mathcal{O}_x)_y$ is trivial and for $y = x$ it is just $k(x)$. This shows that the support of $\mathcal{O}_x^\vee$ is precisely $\{x\}$. By Theorem 4.2, the only non-trivial group among

$$\mathcal{H}^i \operatorname{Hom}^\bullet(k(x), \mathcal{O}_{X,x}) = \operatorname{Ext}^i(k(x), \mathcal{O}_{X,x})$$

is the one for $i = n$ and it is isomorphic to $k(x)$. Now, because for all i

$$(\mathcal{H}^i \mathcal{H}om^\bullet(\mathcal{O}_x, \mathcal{O}_X))_x \cong \mathcal{H}^i(\mathcal{H}om^\bullet(\mathcal{O}_x, \mathcal{O}_X)_x) \cong \mathcal{H}^i \operatorname{Hom}^\bullet(k(x), \mathcal{O}_{X,x}),$$

only $\mathcal{H}^n \mathcal{O}_x^\vee$ is non-trivial. Furthermore, this shows both $\mathcal{H}^n \mathcal{O}_x^\vee$ and $\mathcal{O}_x$ are skyscraper sheaves with stalk $k(x)$, showing $\mathcal{H}^n \mathcal{O}_x^\vee \cong \mathcal{O}_x$. Hence, we have

$$\mathcal{O}_x^\vee \cong \mathcal{O}_x[-n]. \tag{4.1}$$

4.2 Serre functors

Definition 4.7. Let $\mathcal{C}$ be a k-linear category. An equivalence $S\colon \mathcal{C} \to \mathcal{C}$ is called a *Serre-functor* if for any two objects $A, B \in \mathcal{C}$ we have an isomorphism

$$\operatorname{Hom}(B, S(A)) \xrightarrow{\sim} \operatorname{Hom}(A, B)^*$$

natural in A and B.

In this parlance, the statement of Serre duality takes on the following form.

Theorem 4.8 (Serre duality)**.** *Let X be a smooth projective variety over a field k. Then the composition*

$$D^b_{\mathrm{coh}}(X) \xrightarrow{-\otimes \omega_X} D^b_{\mathrm{coh}}(X) \xrightarrow{[n]} D^b_{\mathrm{coh}}(X)$$

defines a Serre functor on $D^b_{\mathrm{coh}}(X)$.

Proof. Let $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in D^b_{\mathrm{coh}}(X)$. Then both $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are perfect, so we have the following natural isomorphisms:

$$\begin{aligned}
\operatorname{Hom}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) &\cong \operatorname{Hom}(\mathcal{O}_X, \mathcal{F}^{\bullet\vee} \otimes \mathcal{G}^\bullet) \\
&= H^0(X, \mathcal{F}^{\bullet\vee} \otimes \mathcal{G}^\bullet) \\
&\cong H^n(X, \mathcal{F}^\bullet \otimes \mathcal{G}^{\bullet\vee} \otimes \omega_X)^* \\
&\cong \operatorname{Hom}(\mathcal{O}_X, \mathcal{F}^\bullet \otimes \mathcal{G}^{\bullet\vee} \otimes \omega_X[n])^* \\
&\cong \operatorname{Hom}(\mathcal{G}^\bullet, \mathcal{F}^\bullet \otimes \omega_X[n])^*
\end{aligned}$$

and taking the dual of both sides yields the desired isomorphism, because $D^b_{\mathrm{coh}}(X)$ has finite-dimensional Hom-spaces by projectivity. □

Remark 4.9. For less well-behaved varieties, one may not have the first and fourth isomorphisms. The reason being the lack of finite locally-free resolutions. The weakest condition on the variety X ensuring the existence of these resolutions is regularity. Indeed, to show that existence of finite locally-free resolutions implies regularity, let $x \in X$ be a closed point. By assumption, we get a finite locally-free resolution $\mathcal{F}^\bullet \to \mathcal{O}_x$. Taking stalks is exact, so we get a free resolution

$$0 \longrightarrow \mathcal{F}_x^{-m} \longrightarrow \ldots \longrightarrow \mathcal{F}_x^{-1} \longrightarrow k(x) \longrightarrow 0$$

of the residue field $k(x)$ of $\mathcal{O}_{X,x}$. Hence $k(x)$ has finite projective dimension, showing regularity of $\mathcal{O}_{X,x}$, see [12, Lemmas 19.1 and 19.2] for a proof. If $y \in X$ is any point, then the closure $\overline{\{y\}}$ contains a closed point x and $\mathcal{O}_{X,y}$ is a localization of $\mathcal{O}_{X,x}$. Hence $\mathcal{O}_{X,y}$ is a localization of a regular ring, and so it is regular itself.

One important property of Serre functors is the following, originally found in [4]. For a nice proof, see [9, Proposition 1.46].

Proposition 4.10 (Bondal, Kapranov)**.** *Let $S\colon \mathcal{C} \to \mathcal{C}$ be a Serre functor between k-linear triangulated categories. Then S is exact.*

This functor, when it exists, is a powerful tool. Inspired by it, one may look to study exactly the locally-Noetherian k-schemes X which have such a functor on $D^b_{\mathrm{coh}}(X)$. However, it turns out that among the projective varieties these are just the regular varieties, so not much is gained. The following result seems well-known throughout the literature, but we provide our own proof.

Lemma 4.11. *Let X be a projective variety over k. Then X is regular if and only if $D^b_{\mathrm{coh}}(X)$ has a Serre functor.*

Proof. If X is regular, then every complex $\mathcal{F}^\bullet \in D^b_{\mathrm{coh}}(X)$ is quasi-isomorphic to a bounded complex of locally-free sheaves and the proof for the smooth case still shows the existence of the Serre functor $D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(X)$.

Now suppose $D^b_{\mathrm{coh}}(X)$ has a Serre functor. We show every local ring $\mathcal{O}_{X,x}$ is regular. By the previous remark, we only have to show that for each closed point $x \in X$, the skyscraper sheaf $\mathcal{O}_x$ has a finite locally-free resolution. Equivalently, we show $\mathcal{O}_x$ is isomorphic to a perfect complex in $D^b_{\mathrm{coh}}(X)$ and by locally-finite duality, this is equivalent to showing $\mathrm{Hom}(\mathcal{O}_x, -)$ is a locally-finite functor on $D^b_{\mathrm{coh}}(X)$. Now, the Serre functor yields for every $\mathcal{F}^\bullet \in D^b_{\mathrm{coh}}(X)$ that

$$\mathrm{Hom}(\mathcal{O}_x, \mathcal{F}^\bullet[n]) \cong \mathrm{Hom}(S^{-1}(\mathcal{F}^\bullet[n]), \mathcal{O}_x)^* \cong \mathrm{Hom}(S^{-1}(\mathcal{F}^\bullet), \mathcal{O}_x[-n])^*.$$

Because the internal Hom functor $\mathcal{H}om^\bullet$ lands in the bounded-below complexes, we see both the left-most and right-most groups are trivial when n and $-n$ are small enough, respectively. This implies $\mathrm{Hom}(\mathcal{O}_x, \mathcal{F}^\bullet[n])$ is non-zero for only finitely many n. Because X is a projective variety and

$S^{-1}(\mathcal{F}^\bullet) \in D^b_{\mathrm{coh}}(X)$, each of these Hom-sets has finite dimension over k, so we get

$$\sum_{i\in\mathbb{Z}} \dim_k \mathrm{Hom}(\mathcal{O}_x, \mathcal{F}^\bullet[n]) < \infty.$$

Hence $\mathcal{O}_x$ is a perfect complex. □

Definition 4.12. A locally-Noetherian k-scheme X is called *categorically Gorenstein* if $D_{\mathrm{perf}}(X)$ possesses a Serre functor.

This differs from the usual definition of a Gorenstein scheme, but it turns out that if X is a projective variety, then X is categorically Gorenstein if and only if it is Gorenstein in the usual sense. If X is Gorenstein, then the dualizing complex is a shift of an invertible sheaf, see [7, Proposition 9.3]. The functor given by taking the tensor product with the invertible sheaf thus restricts to the perfect derived category and is invertible. As we saw in Example 3.22, this functor dualizes Hom-sets, hence furnishes a Serre functor on $D_{\mathrm{perf}}(X)$. The converse is proven in the remark below the following lemma.

Lemma 4.13 ([2, Lemma 6.6])**.** *If X is a projective k-variety of dimension n which is categorically Gorenstein, then the Serre functor $S\colon D_{\mathrm{perf}}(X) \to D_{\mathrm{perf}}(X)$ is given by $- \otimes \omega_X[n]$ for some invertible sheaf ω_X.*

Proof. Recall from Example 3.22 that the complex $f^!\mathcal{O}_{\mathrm{Spec}(k)}$ takes $D_{\mathrm{perf}}(X)$ to $D^b_{\mathrm{coh}}(X)$ and dualizes Hom-sets:

$$\mathrm{Hom}(\mathcal{P}^\bullet, \mathcal{F}^\bullet)^* \cong \mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{P}^\bullet \otimes f^!\mathcal{O}_{\mathrm{Spec}(k)}).$$

If $\mathcal{F}^\bullet$ is a perfect complex, we also get

$$\mathrm{Hom}(\mathcal{P}^\bullet, \mathcal{F}^\bullet)^* \cong \mathrm{Hom}(\mathcal{F}^\bullet, S(\mathcal{P}^\bullet)),$$

so plugging in $\mathcal{F}^\bullet = S(\mathcal{P}^\bullet)$, the identity $S(\mathcal{P}^\bullet) \to S(\mathcal{P}^\bullet)$ provides a natural morphism

$$S(\mathcal{P}^\bullet) \to \mathcal{P}^\bullet \otimes f^!\mathcal{O}_{\mathrm{Spec}(k)}.$$

This morphism induces isomorphisms

$$\mathrm{Hom}(\mathcal{Q}^\bullet, S(\mathcal{P}^\bullet)) \cong \mathrm{Hom}(\mathcal{Q}^\bullet, \mathcal{P}^\bullet \otimes f^!\mathcal{O}_{\mathrm{Spec}(k)})$$

for every perfect complex $\mathcal{Q}^\bullet$, so it is an isomorphism itself, hence S is isomorphic to $- \otimes f^!\mathcal{O}_{\mathrm{Spec}(k)}$.

This implies $f^!\mathcal{O}_{\mathrm{Spec}(k)} = S(\mathcal{O}_X) \in D_{\mathrm{perf}}(X)$, so S has a right-adjoint given by $- \otimes (f^!\mathcal{O}_{\mathrm{Spec}(k)})^\vee$, which inverts S because S is an equivalence. In particular, we get

$$f^!\mathcal{O}_{\mathrm{Spec}(k)} \otimes (f^!\mathcal{O}_{\mathrm{Spec}(k)})^\vee \cong \mathcal{O}_X,$$

so we have two perfect complexes $\mathcal{P}^\bullet$ and $\mathcal{Q}^\bullet$ for which the tensor product is quasi-isomorphic to $\mathcal{O}_X$. We will show that this implies $\mathcal{P}^\bullet$ is quasi-isomorphic to a shift of an invertible sheaf. Let $x \in X$ be any point. First, we replace the stalks $\mathcal{P}_x^\bullet$ and $\mathcal{Q}_x^\bullet$ by minimal complexes $P^\bullet$ and $Q^\bullet$ of finite rank free sheaves respectively, see Lemma 3.15. On taking the tensor product with $k(x)$, the differentials of $P^\bullet$ and $Q^\bullet$ become trivial and we get a quasi-isomorphism

$$(P^\bullet \otimes k(x)) \otimes (Q^\bullet \otimes k(x)) \cong k(x).$$

The left-hand side has trivial differentials as well and its term in degree n is

$$\bigoplus_{i+j=n} (P^i \otimes k(x)) \otimes (Q^j \otimes k(x)),$$

which is thus non-zero if and only if $n = 0$ and in this case it is isomorphic to $k(x)$. Considering dimensions in this direct sum, we find that there is exactly one i such that $P^i, Q^{-i} \neq 0$ and furthermore that both vector spaces have dimension 1. Now it follows from Nakayama's lemma that $P^\bullet$ and $Q^\bullet$ are concentrated in i and $-i$ respectively and $P^i \cong Q^{-i} \cong \mathcal{O}_{X,x}$.

The quasi-isomorphism $\mathcal{P}_x^\bullet \cong \mathcal{O}_{X,x}[-i]$ extends to an open neighborhood of x, showing that $\mathcal{P}^\bullet$ is locally quasi-isomorphic to a shift of the structure sheaf. The support $\operatorname{supp}(\mathcal{H}^i(\mathcal{P}^\bullet))$ of $\mathcal{H}^i(\mathcal{P}^\bullet)$ is open by the previous and it is closed because $\mathcal{H}^i(\mathcal{P}^\bullet)$ is coherent. We have also seen that it is non-empty, so that $\operatorname{supp}(\mathcal{H}^i(\mathcal{P}^\bullet)) = X$ by connectedness. Hence $\mathcal{P}^\bullet \cong \mathcal{H}^i(\mathcal{P}^\bullet)[-i]$ and this sheaf is invertible because $\mathcal{P}^\bullet$ is locally quasi-isomorphic to a shift of $\mathcal{O}_X$.

At this point we have $f^!\mathcal{O}_{\operatorname{Spec}(k)}$ quasi-isomorphic to a shift $\omega_X[i]$ of an invertible sheaf. Note that for $x \in X$ a closed point, we have $\operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_x[j]) = k(x)$ for $j = 0$ and is trivial for $j \neq 0$, so that

$$\operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_x[j])^* \cong \operatorname{Hom}(\mathcal{O}_x[j], f^!\mathcal{O}_{\operatorname{Spec}(k)}) \cong \operatorname{Hom}(\mathcal{O}_x, \omega_X[i-j])$$

is non-trivial only for $j = 0$. The right-hand side is isomorphic to

$$\operatorname{Hom}(k(x), \mathcal{O}_{X,x}[i-j]) \cong \operatorname{Ext}^{i-j}(k(x), \mathcal{O}_{X,x})$$

by Lemma 2.2, so that the equivalence of the conditions in Theorem 4.2 shows $i = n$. □

Remark 4.14. Of course, our use of Theorem 4.2 above also shows the local rings $\mathcal{O}_{X,x}$ of X at closed points are Gorenstein local rings. A localization of a Gorenstein local ring R at a prime $\mathfrak{p} \in \operatorname{Spec}(R)$ is again Gorenstein, because if

$$0 \longrightarrow R \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^n \longrightarrow 0$$

is an injective resolution, then so is

$$0 \longrightarrow R_\mathfrak{p} \longrightarrow I^0_\mathfrak{p} \longrightarrow I^1_\mathfrak{p} \longrightarrow \cdots \longrightarrow I^n_\mathfrak{p} \longrightarrow 0,$$

which shows $R_\mathfrak{p}$ has finite injective dimension. Hence X is a projective Gorenstein variety.

Subsequently, when we talk about projective Gorenstein varieties, we assume both the data of a Serre functor S and of a dualizing sheaf ω_X so that $S \cong - \otimes \omega_X[n]$.

5 Reconstructing Varieties from the Derived Category

The goal of this section is to prove an adaptation to Gorenstein varieties of the following theorem, showcasing much of the power of derived categories along the way. In this, we closely follow Matthew Ballard's PhD-thesis [1], filling in details and providing some examples along the way.

Theorem 5.1 (Bondal-Orlov Reconstruction Theorem)**.** *Let X, Y be two smooth projective varieties over a field k with ω_X ample. If there exists an exact equivalence $D^b_{\text{coh}}(X) \to D^b_{\text{coh}}(Y)$, then X and Y are isomorphic.*

In general, one calls two varieties *derived equivalent* if their derived categories are equivalent as k-linear triangulated categories, for an appropriate realization of "derived category". For the many different derived categories in use, this terminology needs to be specified further. In the following, we shall take it to mean the perfect derived categories are equivalent. As one may suspect from the previous chapters, one of the generalizations will be to work with a variety X which is merely Gorenstein instead of smooth. The assumptions on Y can be weakened greatly: we need only assume Y is a projective variety over k and any requirement on smoothness can be dropped. Again, it is even enough to only take Y quasi-projective, see [1, Proposition 3.1.2], but we choose to just work with projective varieties instead. So, the result we will prove is the following.

Theorem 5.2. *Let X be a projective Gorenstein variety over a field k such that the canonical sheaf ω_X is ample or anti-ample. If Y is a projective variety derived equivalent to X, then Y is isomorphic to X.*

The proof uses the duality between $D_{\text{perf}}(X)$ and $D^b_{\text{coh}}(X)$ in an essential way. The most straightforward route to reconstruct a variety from some associated category is by first identifying the points, then constructing the topology and finally constructing the structure sheaf. Here, we take a different route. First, we identify the closed points of X as objects in $D^b_{\text{coh}}(X)$. These let us identify the locally free sheaves in $D_{\text{perf}}(X)$, among which in particular the canonical sheaf. Finally, we construct an isomorphism of canonical algebras, wrapping up the proof. In particular, we use the projectivity of X and Y in an essential way.

We already know that on a projective Gorenstein variety X, the Serre functor $D_{\text{perf}}(X) \to D_{\text{perf}}(X)$ is given by $- \otimes \omega_X[n]$. As ω_X is an invertible sheaf, this functor extends to an auto-equivalence $S' \colon D^b_{\text{coh}}(X) \to D^b_{\text{coh}}(X)$ given by the same formula.

Definition 5.3. Let X be a projective Gorenstein variety over a field k. An object $\mathcal{P}^\bullet \in D^b_{\text{coh}}(X)$ is called a *closed point functor of codimension d* if there exists a finite extension $k(\mathcal{P}^\bullet)$ of k such that the following hold:

(i) $S'(\mathcal{P}^\bullet) \cong \mathcal{P}^\bullet[d]$,

(ii) $\mathrm{Hom}(\mathcal{P}^\bullet, \mathcal{P}^\bullet[i]) = 0$ for $i < 0$, and

(iii) $\mathrm{Hom}(\mathcal{P}^\bullet, \mathcal{P}^\bullet) \cong k(\mathcal{P}^\bullet)$ as k-algebras.

Remark 5.4. Our definition of closed point functors differs slightly from the one found in [1]. The Serre functor S_X has a right-pseudo adjoint

$$S^\vee \colon D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(X),$$

which is necessarily given by $-\otimes\omega_X^\vee[-n]$ and Ballard uses this functor instead. However, it is sufficient for the results we prove to just use the extension of S_X to $D^b_{\mathrm{coh}}(X)$ instead. This has the added merit of shortening our proofs, if only slightly.

Remark 5.5. When X is smooth, every bounded complex of coherent sheaves is perfect, so $D_{\mathrm{perf}}(X) \cong D^b_{\mathrm{coh}}(X)$, and the locally-finite duality trivializes. This means we can ignore the interpretation of an object $\mathcal{P}^\bullet \in D^b_{\mathrm{coh}}(X)$ as a functor. The extension S' overlaps with the Serre functor S_X and we may replace the first condition by $S_X(\mathcal{P}^\bullet) \cong \mathcal{P}^\bullet[d]$. In this setting, the closed point functors are referred to as *closed point objects*.

Example 5.6. Suppose X is an elliptic curve. In this case, the canonical sheaf is isomorphic to $\mathcal{O}_X$, which is neither ample nor anti-ample. We show that the structure sheaf $\mathcal{O}_X$ is a closed point object of dimension 1. Considering the hom-sets, we see

$$\mathrm{Hom}_{D^b_{\mathrm{coh}}(X)}(\mathcal{O}_X, \mathcal{O}_X[i]) = \mathrm{Ext}^i_{\mathrm{Coh}(X)}(\mathcal{O}_X, \mathcal{O}_X) \cong \begin{cases} k & \text{if } i = 0 \\ 0 & \text{if } i < 0, \end{cases}$$

because $\mathrm{Hom}_{\mathrm{Coh}(X)}(\mathcal{O}_X, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) \cong k$. The first condition is vacuous, because $\omega_X = \mathcal{O}_X$, so that $S_X(\mathcal{P}^\bullet) = \mathcal{P}^\bullet[1]$ by definition. In the same way, any coherent sheaf $\mathcal{P} \in \mathrm{Coh}(X)$ with $\mathrm{End}_{\mathrm{Coh}(X)}(\mathcal{P}) \cong k$ is a closed point functor of codimension 1 in $D^b_{\mathrm{coh}}(X)$. Coherent sheaves with this property are called *simple*.

Example 5.7. Now let X be a projective Gorenstein variety. Again, not necessarily with ample canonical sheaf. For any closed point $i_x \colon \{x\} \to X$ we have the skyscraper sheaf $\mathcal{O}_x = i_{x,*}\mathcal{O}_{\mathrm{Spec}(k(x))}$ on X, which is coherent. We would not define the closed point functors as we did if these coherent sheaves were not among them. The first condition is satisfied as seen through a simple application of the projection formula. Writing $\mathcal{O} := \mathcal{O}_{\mathrm{Spec}(k(x))}$ for brevity, we have $i_x^*\omega_X \cong \mathcal{O}$, because ω_X is locally free of rank 1. We compute

$$S'(\mathcal{O}_x) = i_{x,*}\mathcal{O} \otimes \omega_X[n] \cong i_{x,*}(\mathcal{O} \otimes i_x^*\omega_X)[n] \cong i_{x,*}(\mathcal{O} \otimes \mathcal{O})[n] \cong \mathcal{O}_x[n],$$

where $n = \dim(X)$, so condition (i) holds for $d = n$. The second condition, like in the previous example, is just vanishing of negative Ext groups in an abelian category. For condition (iii), consider the following commutative diagram for the adjunction $i_x^* \dashv i_{x,*}$:

$$
\begin{array}{ccc}
\mathrm{Hom}(\mathcal{O}_{\mathrm{Spec}(k(x))}, \mathcal{O}_{\mathrm{Spec}(k(x))}) & \xrightarrow{i_{x,*}} & \mathrm{Hom}(\mathcal{O}_x, \mathcal{O}_x) \\
 & \searrow^{-\circ e} & \downarrow \wr \\
 & & \mathrm{Hom}(i_x^* i_{x,*}\mathcal{O}_{\mathrm{Spec}(k(x))}, \mathcal{O}_{\mathrm{Spec}(k(x))}),
\end{array}
$$

where e is the counit morphism

$$i_x^* i_{x,*}\mathcal{O}_{\mathrm{Spec}(k(x))} \longrightarrow \mathcal{O}_{\mathrm{Spec}(k(x))}.$$

Because i_x is the inclusion of a closed subscheme, this morphism is an isomorphism and so $i_{x,*}$ is an isomorphism of k-algebras. Furthermore,

$$
\begin{aligned}
\mathrm{Hom}(\mathcal{O}_{\mathrm{Spec}(k(x))}, \mathcal{O}_{\mathrm{Spec}(k(x))}) &\cong \mathrm{Hom}_{\mathrm{Coh}(\mathrm{Spec}(k(x)))}(\mathcal{O}_{\mathrm{Spec}(k(x))}, \mathcal{O}_{\mathrm{Spec}(k(x))}) \\
&\cong \mathrm{Hom}_{k(x)}(k(x), k(x)) \cong k(x)
\end{aligned}
$$

also as k-algebras, showing condition (iii).

The first example shows that this notion of closed point functors is not strong enough in general. We get too many of them. This is the first point in the proof of Theorem 5.1 where we need ampleness of the canonical sheaf.

Proposition 5.8 ([1, Lemma 3.1.8])**.** *Let X be a projective Gorenstein variety with ample canonical sheaf. Then the closed point functors on $D_{\mathrm{perf}}(X)$ are exactly the complexes $\mathcal{O}_x[i]$, for closed points $x \in X$ and integers $i \in \mathbb{Z}$.*

Proof. By Example 5.7, at least the sheaves $\mathcal{O}_x$ are closed point functors and because S' commutes with translation and translation is an equivalence, we see right away that each $\mathcal{O}_x[i]$ for $i \in \mathbb{Z}$ is such as well. Conversely, suppose $\mathcal{P}^\bullet$ is a closed point functor of codimension d. By condition (iii), the complex $\mathcal{P}^\bullet$ is non-trivial, so $\mathcal{H} := \mathcal{H}^i(\mathcal{P}^\bullet) \neq 0$ for some i. Condition (i) yields

$$\mathcal{P}^\bullet \otimes \omega_X[n] = S'(\mathcal{P}^\bullet) \cong \mathcal{P}^\bullet[d].$$

By exactness of the functor $- \otimes \omega_X$, taking cohomology, we see $d = n$ and $\mathcal{H} \otimes \omega_X \cong \mathcal{H}$. Replacing ω_X by some large (possible negative) tensor power, we may assume ω_X is very ample. Let $i\colon X \hookrightarrow \mathbb{P}^m_k$ be the corresponding embedding into projective space, so $\omega_X \cong i^*\mathcal{O}(1)$. Note that this is even an isomorphism of ω_X with $Li^*\mathcal{O}(1)$ in the derived category, because $\mathcal{O}(1)$ is locally-free. The coherent sheaf $i_*\mathcal{H}$ on $\mathbb{P}^m_k$ has support $i(\mathrm{supp}(\mathcal{H}))$ and

$$Ri_*\mathcal{H} \otimes \mathcal{O}(1) \cong Ri_*(\mathcal{H} \otimes i^*\mathcal{O}(1)) \cong Ri_*(\mathcal{H} \otimes \omega_X) \cong Ri_*\mathcal{H}$$

in $D(\mathbb{P}^m_k)$ implies $i_*\mathcal{H} \otimes \mathcal{O}(1) \cong i_*\mathcal{H}$ as coherent sheaves after taking 0th cohomology. It follows that the Hilbert function $l \mapsto \dim(\Gamma(\mathbb{P}^m_k, i_*\mathcal{H} \otimes \mathcal{O}(l)))$ of $i_*\mathcal{H}$ is constant, so its Hilbert polynomial has degree 0. This degree equals the dimension of the support, hence $i(\mathrm{supp}(\mathcal{H}))$ and therewith $\mathrm{supp}(\mathcal{H})$ are of dimension 0.

Now, this holds for each of the cohomology sheaves of $\mathcal{P}^\bullet$, so the whole complex has zero-dimensional support. Write $Z = \text{supp}(\mathcal{P}^\bullet)$. Suppose Z is not connected. Then we can write Z as a disjoint union $Z_1 \sqcup Z_2$. By Lemma 2.3, our complex splits as

$$\mathcal{P}^\bullet = \mathcal{P}_1^\bullet \oplus \mathcal{P}_2^\bullet.$$

However, in this case, the endomorphism ring of $\mathcal{P}^\bullet$ has a non-trivial idempotent given by $\text{id} \oplus 0$, contradicting the fact that it is a field. We conclude that Z is a closed connected zero-dimensional subset of the projective variety X, so it is a single point.

Suppose $\mathcal{P}^\bullet$ is supported on the closed point $x \in X$. Write $\mathcal{H} := \mathcal{H}^{m_0}(\mathcal{P}^\bullet)$ and $\mathcal{H}' := \mathcal{H}^{m_1}(\mathcal{P}^\bullet)$ for the bottom and top cohomology sheaves of $\mathcal{P}^\bullet$. Because the supports of both $\mathcal{H}$ and $\mathcal{H}'$ are precisely the closed point x, a morphism (of $\mathcal{O}_X$-modules) between them is the same as a morphism between their stalks at this point. The stalks are non-trivial, finitely generated $\mathcal{O}_{X,x}$-modules supported on $\{\mathfrak{m}_x\}$, where $\mathfrak{m}_x$ is the maximal ideal. By Nakayama's lemma, the $k(x)$-vector space $\mathcal{H}'_x/\mathfrak{m}_x\mathcal{H}'_x$ is non-trivial, so there exists a surjection $\mathcal{H}'_x/\mathfrak{m}_x\mathcal{H}'_x \to k(x)$, which we compose with the quotient map $\mathcal{H}'_x \to \mathcal{H}'_x/\mathfrak{m}_x\mathcal{H}'_x$ to get a surjection $\mathcal{H}'_x \to k(x)$. Let us now show there exists an injection $k(x) \to \mathcal{H}_x$. The family $\mathcal{M} := \{\text{ann}(h) \mid h \in \mathcal{H}_x \setminus \{0\}\}$ of ideals is non-empty because $\mathcal{H}_x$ is non-zero. As $\mathcal{O}_{X,x}$ is Noetherian, this family contains a maximal element, say $\mathfrak{p} := \text{ann}(h)$. This ideal is a prime ideal. Indeed, if $ab \in \text{ann}(h)$ and $a \notin \text{ann}(h)$, then $\text{ann}(ah) \in \mathcal{M}$ and for any $c \in \text{ann}(h)$ it holds that $c(ah) = a(ch) = 0$, so that $c \in \text{ann}(ah)$ and hence $\text{ann}(h) \subset \text{ann}(ah)$. By maximality of $\text{ann}(h)$, this shows $\text{ann}(ah) = \text{ann}(h)$ and therefore $b \in \text{ann}(ah) = \text{ann}(h)$. Because h is non-zero in $(\mathcal{H}_x)_\mathfrak{p}$, we see $\mathfrak{p} \in \text{supp}(\mathcal{H}_x) = \{\mathfrak{m}_x\}$, so $\text{ann}(h) = \mathfrak{p} = \mathfrak{m}_x$. The morphism $\mathcal{O}_{X,x} \to \mathcal{H}_x$ given by $a \mapsto ah$ has kernel $\text{ann}(h)$, so it yields an injection $k(x) \to \mathcal{H}_x$.

The composition $\mathcal{H}'_x \to \mathcal{H}_x$ of these morphisms is non-trivial, so we obtain a non-trivial morphism $\mathcal{H}' \to \mathcal{H}$. The complex $\mathcal{P}^\bullet$ is quasi-isomorphic to the complex truncated to have non-zero terms only between m_0 and m_1, so let us replace $\mathcal{P}^\bullet$ by this. We can form the composition

$$\mathcal{P}^\bullet[m_1] \longrightarrow \mathcal{H}' \longrightarrow \mathcal{H} \longrightarrow \mathcal{P}^\bullet[m_0],$$

which is seen to be non-trivial by taking 0th cohomology. Hence

$$\text{Hom}(\mathcal{P}^\bullet[m_1], \mathcal{P}^\bullet[m_0]) \cong \text{Hom}(\mathcal{P}^\bullet, \mathcal{P}^\bullet[m_0 - m_1])$$

is non-zero, which by condition (ii) together with $m_0 \leq m_1$ implies $m_0 = m_1$. Hence $\mathcal{P}^\bullet \cong \mathcal{H}[-m_0]$ is simply a shift of a sheaf.

Finally, to show $\mathcal{H}$ is isomorphic to $\mathcal{O}_x$, suppose it is not. Because $\mathcal{O}_x$ has support $\{x\}$ and stalk $k(x)$, we get an epimorphism $\mathcal{H} \to \mathcal{O}_x$ and a monomorphism $\mathcal{O}_x \to \mathcal{H}$ like before. Neither is an isomorphism, so the composition $\mathcal{H} \to \mathcal{O}_x \to \mathcal{H}$ is not invertible, contradicting condition (iii). $\square$

Let us emphasize one of the steps we worked out in the proof above. It will turn out to be useful again later.

Lemma 5.9. *Let* $(R, \mathfrak{m}, \kappa)$ *be a Noetherian local ring, with* M *a finitely generated* R*-module with* $\operatorname{supp}(M) = \{\mathfrak{m}\}$*. Then there exist an injection* $\kappa \to M$ *and a surjection* $M \to \kappa$*.*

Proof. Read the proof above with $\mathcal{O}_{X,x}$ replaced by R, the residue field $k(x)$ replaced by κ and $\mathcal{H}_x$ and $\mathcal{H}'_x$ replaced by M. □

Definition 5.10. Let X be a variety over a field k. A complex $\mathcal{Q}^\bullet \in D_{\text{perf}}(X)$ is called a *locally free object* if there exist integers n, m such that for any closed point object $\mathcal{P}^\bullet$ of $D^b_{\text{coh}}(X)$ it holds that

$$\operatorname{Hom}(\mathcal{Q}^\bullet, \mathcal{P}^\bullet[i]) \cong \begin{cases} k(\mathcal{P}^\bullet)^m & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

If $m = 1$, then $\mathcal{Q}^\bullet$ is called an *invertible object.* The integer m is called the *rank* of $\mathcal{Q}^\bullet$ and we say $\mathcal{Q}^\bullet$ is *concentrated in degree* n.

Intuitively, the condition just says $\mathcal{Q}^\bullet$ is concentrated in degree n and locally free of rank m. When we were looking at closed point functors, we found that in some cases there are too many. Thus, one may suspect that this notion of locally free object may be too strong in these cases, and this is true.

Example 5.11. Let X be an elliptic curve. In Example 5.6, we saw $\mathcal{O}_X$ is a closed point object of $D^b_{\text{coh}}(X)$. Take any non-effective divisor D of degree zero on X. Then $\mathcal{O}_X(D)$ has no global sections and by Serre duality

$$H^1(X, \mathcal{O}_X(D)) \cong H^0(X, \mathcal{O}_X(-D))$$

is zero as well. Hence $\mathcal{O}_X(D)$ is a locally free sheaf with vanishing cohomology. By Serre duality in $D^b_{\text{coh}}(X)$, we get for the closed point object $\mathcal{O}_X$

$$\operatorname{Hom}(\mathcal{O}_X(D), \mathcal{O}_X[i]) \cong \operatorname{Hom}(\mathcal{O}_X[i], \mathcal{O}_X(D)[1]) \cong \operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_X(D)[1-i])$$

and this is isomorphic to $R^{1-i}\Gamma(X, \mathcal{O}_X(D)) \cong H^{1-i}(X, \mathcal{O}_X(D))$, which is 0 for all i. This shows $\mathcal{O}_X(D)$ is not a locally free object of $D^b_{\text{coh}}(X)$, even though it is locally free as a sheaf on X.

Example 5.12. If the closed point functors on $D_{\text{perf}}(X)$ are precisely the shifts of the sheaves $\mathcal{O}_x$ for closed points $x \in X$, then the free $\mathcal{O}_X$-module $\mathcal{O}_X^m$ is a locally free object of rank m. Indeed, a straightforward computation yields

$$\operatorname{Hom}(\mathcal{O}_X^m, \mathcal{O}_x) \cong \operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_x[i])^m = R^i\Gamma(X, \mathcal{O}_x)^m = \begin{cases} k(x)^m & \text{if } i = 0 \\ 0 & \text{otherwise,} \end{cases}$$

the final equality following because $\mathcal{O}_x$ is flasque. In particular, this holds on a projective Gorenstein variety with (anti-)ample canonical sheaf.

To show that the locally free objects are the shifts of locally free sheaves we want them to be in nice cases, we will need the following lemma.

Lemma 5.13 ([1, Lemma 3.1.11]). *Let X be a reduced quasi-projective scheme over a field k. Let $\mathcal{F}^\bullet$ be a bounded complex of locally free coherent sheaves. If there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\mathcal{F}^\bullet \otimes \mathcal{O}_x$ is quasi-isomorphic to $\mathcal{O}_x^n$ for all closed points $x \in X$, then $\mathcal{F}^\bullet$ is quasi-isomorphic to a locally free sheaf of rank n.*

Proof. Because X is locally Noetherian, we know the support of any non-trivial coherent sheaf $\mathcal{F}$ contains a closed point. Then, because X is reduced, there exists a closed point $x \in X$ such that $\mathcal{F} \otimes \mathcal{O}_x \neq 0$.

Because locally free sheaves are acyclic for the tensor product, it holds that $\mathcal{F}^\bullet \otimes \mathcal{O}_x$ is just the complex

$$\cdots \longrightarrow \mathcal{F}^{-1} \otimes \mathcal{O}_x \longrightarrow \mathcal{F}^0 \otimes \mathcal{O}_x \longrightarrow \mathcal{F}^1 \otimes \mathcal{O}_x \longrightarrow \cdots$$

and there is a natural morphism $\mathcal{H}^i(\mathcal{F}^\bullet) \otimes \mathcal{O}_x \to \mathcal{H}^i(\mathcal{F}^\bullet \otimes \mathcal{O}_x)$ for any $i \in \mathbb{Z}$. If we consider its only non-trivial stalk, we get a map

$$\mathcal{H}^i(\mathcal{F}^\bullet)_x \otimes k(x) \longrightarrow \mathcal{H}^i(\mathcal{F}^\bullet_x \otimes k(x)),$$

which turns out to be injective. Indeed, if $\overline{s} \otimes \lambda \mapsto 0$, then $s \otimes \lambda$ lies in the image of $d_x^{i-1} \otimes k(x)$, say $s \otimes \lambda = d_x^{i-1} t \otimes \lambda'$. It follows that

$$\left(s - \frac{\lambda'}{\lambda} d_x^{i-1} t\right) \otimes \lambda = s \otimes \lambda - d_x^{i-1} t \otimes \lambda' = 0,$$

so that $s - \frac{\lambda'}{\lambda} d_x^{i-1} t \in \mathfrak{m}_x \ker(d_x^i)$ and thus

$$\overline{s} \otimes \lambda = \overline{\frac{\lambda'}{\lambda} d_x^{i-1} t} \otimes \lambda + \overline{s - \frac{\lambda'}{\lambda} d_x^{i-1} t} \otimes \lambda = 0 + 0 = 0.$$

By assumption, for every closed point $x \in X$, it holds that the sheaf $\mathcal{H}^i(\mathcal{F}^\bullet \otimes \mathcal{O}_x)$ is non-zero only if $r = 0$, hence $\mathcal{H}^r(\mathcal{F}^\bullet) \otimes \mathcal{O}_x$ is trivial for $r \neq 0$. By the argument at the start of this proof, this shows $\mathcal{H}^r(\mathcal{F}^\bullet) \neq 0$ only if $r = 0$, so $\mathcal{F}^\bullet$ is quasi-isomorphic to the coherent sheaf $\mathcal{F} := \mathcal{H}^0(\mathcal{F}^\bullet)$.

Now, the quasi-isomorphism of complexes $\mathcal{F} \otimes \mathcal{O}_x \cong \mathcal{F}^\bullet \otimes \mathcal{O}_x \cong \mathcal{O}_x^n$ yields an isomorphism $\mathcal{F} \otimes \mathcal{O}_x \cong \mathcal{O}_x^n$, where $\otimes$ is the usual tensor product of sheaves. Considering stalks of this isomorphism for all closed points x, we find the dimension function $\varphi \colon x \mapsto \dim_{k(x)}(\mathcal{F}_x \otimes k(x))$ is constant on closed points. Because X is quasi-projective over k and φ is upper semi-continuous, this function is constant everywhere and therefore $\mathcal{F}$ is locally free of rank n. □

Lemma 5.14 ([1, Lemma 3.1.10]). *Let X be a projective variety and $\mathcal{F}^\bullet \in D_{\text{perf}}(X)$. Then $\mathcal{F}^\bullet$ is quasi-isomorphic to a shift of a locally-free sheaf if and* *only if there exist integers $m, n \in \mathbb{Z}$ with $m \geq 0$ such that for each closed point $x \in X$ it holds that*

$$\mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{O}_x[j]) \cong \begin{cases} k(x)^m & \text{if } j = n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $\mathcal{F}^\bullet \cong \mathcal{F}[n]$ is a shift of a locally free sheaf of rank m, then $\mathcal{F}^\vee$ has rank m as well and thus $\mathcal{F}^\vee \otimes \mathcal{O}_x \cong \mathcal{O}_x^m$. It follows that

$$\mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{O}_x[j]) \cong \mathrm{Hom}(\mathcal{O}_X, \mathcal{F}^\vee \otimes \mathcal{O}_x[j-n]) \cong \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_x^m[j-n]),$$

for all closed points $x \in X$. The latter group is non-trivial only when $j = n$ and in this case it is isomorphic to $k(x)^m$.

Conversely, suppose the conditions on the Hom-sets hold. Let i denote the inclusion of x into X. Because $\mathcal{F}^\bullet$ is perfect, we get

$$\mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{O}_x[j]) \cong \mathrm{Hom}(i^*\mathcal{F}^\bullet, \mathcal{O}_{\mathrm{Spec}(k(x))}[j]) \cong \mathrm{Hom}(\mathcal{O}_{\mathrm{Spec}(k(x))}, i^*\mathcal{F}^{\bullet,\vee}[j]).$$

The latter group is $R^j\Gamma(\mathrm{Spec}(k(x)), i^*\mathcal{F}^{\bullet,\vee})$, so $\Gamma(\mathrm{Spec}(k(x)), i^*\mathcal{F}^{\bullet,\vee})$ is concentrated in degree n. The global sections functor is an exact equivalence on affine schemes, with inverse given by taking the associated sheaf of modules. This means $\Gamma\colon D^b_{\mathrm{coh}}(\mathrm{Spec}(k(x))) \to D^b(\mathrm{vec}_{k(x)})$ is an equivalence as well. Now $\Gamma(\mathrm{Spec}(k(x)), i^*\mathcal{F}^{\bullet,\vee})$ is quasi-isomorphic to $k(x)^m$ concentrated in degree n and taking the associated sheaf of modules shows $i^*\mathcal{F}^{\bullet,\vee}$ is quasi-isomorphic to $\mathcal{O}^m_{\mathrm{Spec}(k(x))}[-n]$. Finally, by the projection formula,

$$\mathcal{F}^{\bullet,\vee} \otimes \mathcal{O}_x \cong i_*(i^*\mathcal{F}^{\bullet,\vee} \otimes \mathcal{O}_{\mathrm{Spec}(k(x))}) \cong i_*i^*\mathcal{F}^{\bullet,\vee} \cong \mathcal{O}_x^m[-n],$$

so that the previous lemma applies to $\mathcal{F}^{\bullet,\vee}[n]$, showing $\mathcal{F}^{\bullet,\vee}$ is quasi-isomorphic to a locally free sheaf in degree $-n$. □

Corollary 5.15 ([1, Lemma 3.1.10])**.** *If the closed point functors on $D_{\mathrm{perf}}(X)$ are precisely the shifts of the sheaves $\mathcal{O}_x$ for closed points $x \in X$, then the locally free objects of $D_{\mathrm{perf}}(X)$ are precisely the shifts of the locally free sheaves on X.*

We are now ready to begin the proof of the reconstruction theorem. The proof will be clearly divided into multiple steps.

Proof of Theorem 5.2. Let $\Phi\colon D_{\mathrm{perf}}(X) \to D_{\mathrm{perf}}(Y)$ be an equivalence. First we show that the closed point functors on $D_{\mathrm{perf}}(Y)$ are precisely the shifts of the sheaves $\mathcal{O}_y$ for closed points $y \in Y$. For a scheme Z for which $D_{\mathrm{perf}}(Z)$ has a Serre functor, we write $P(Z) \subset D^b_{\mathrm{coh}}(X)$ for the set of closed point functors on $D_{\mathrm{perf}}(Z)$. By Proposition 5.8, the set $P(X)$ is in bijection with the set of shifts of sheaves $\mathcal{O}_x$ for closed points $x \in X$. In particular, for any two $\mathcal{P}, \mathcal{Q} \in P(X)$, it holds that either $\mathrm{Hom}(\mathcal{P}, \mathcal{Q}[i]) = 0$ for all i, or $\mathcal{P} = \mathcal{Q}[j]$ for some j.

If Φ^{-1} denotes an inverse to Φ, then the functor $\Phi \circ S_X \circ \Phi^{-1}$ is a Serre functor on $D_{\mathrm{perf}}(Y)$. Because the Serre functor commutes with k-linear equivalences, the notion of closed point functors is invariant under our equivalence, meaning Φ bijects $P(X)$ with $P(Y)$. This also means that for two closed point functors $\mathcal{P}, \mathcal{Q} \in P(Y)$ either all Ext-groups between them are trivial, or one is a shift of the other. Take $\mathcal{P} \in P(Y)$, and let $y \in Y$ be a closed point in the support of the top cohomology group of $\mathcal{P}$, which lies in degree m, say. There is a morphism $\mathcal{P} \to \mathcal{O}_y[-m]$, so it immediately follows that $\mathcal{P}$ is a shift of $\mathcal{O}_y$, as desired.

Now we will prove that the map induced by Φ on the underlying topological spaces of closed points is an homeomorphism. The varieties X and Y are determined by their subsets of closed points, so this will show X and Y are homeomorphic as topological spaces. Let $\mathcal{L}_0$ denote a choice of invertible sheaf on X, which is a locally-free object of rank 1 by Corollary 5.15. We write $P(\mathcal{L}_0)$ for the set of closed point functors $\mathcal{P}$ such that $\mathrm{Hom}(\mathcal{L}_0, \mathcal{P}) \neq 0$. Note that this set bijects with the closed points of X, because $\mathrm{Hom}(\mathcal{L}_0, \mathcal{P}[j]) = 0$ for all $\mathcal{P} \in P(\mathcal{L}_0)$ and all $j \neq 0$. Applying Φ, we see $P(\mathcal{L}_0)$ is sent bijectively to the set of closed point functors $\mathcal{P}$ for which $\mathrm{Hom}(\Phi(\mathcal{L}_0), \mathcal{P}) \neq 0$, that is, to $P(\Phi(\mathcal{L}_0))$. Because Φ sends closed point functors to closed points functors, it also sends locally-free objects to locally-free objects, so the $P(\Phi(\mathcal{L}_0))$ bijects with the closed points of Y and we get a bijection between the closed points of X and the closed points of Y.

Let $F(\mathcal{L}_0)$ denote the set of locally-free objects $\mathcal{F}$ in $D_{\mathrm{perf}}(X)$ such that $\mathrm{Hom}(\mathcal{F}, \mathcal{P}) \neq 0$ for all $\mathcal{P} \in P(\mathcal{L}_0)$. For $\mathcal{F}, \mathcal{F}' \in F(\mathcal{L}_0)$ and $\alpha \in \mathrm{Hom}(\mathcal{F}, \mathcal{F}')$, we define Z_α to be the subset of closed points $x \in X$ for which the induced map

$$\mathrm{Hom}(\mathcal{F}', \mathcal{P}) \longrightarrow \mathrm{Hom}(\mathcal{F}, \mathcal{P})$$

is zero. Because X has enough finite locally-free sheaves, it turns out that every closed subset of X is among the Z_α. Indeed, let Z be a closed subset of X, cut out by an ideal sheaf $\mathcal{I}$. Then $\mathcal{I}$ is coherent, because X is a variety and there exists an epimorphism $\mathcal{F} \to \mathcal{I}$ with $\mathcal{F}$ finite locally-free. Composing with the inclusion $\mathcal{I} \hookrightarrow \mathcal{O}_X$, we get a morphism $\alpha \colon \mathcal{F} \to \mathcal{O}_X$.

Let us show $Z = Z_\alpha$. Recall that for any closed point $x \in X$ and all complexes $\mathcal{F} \in D^b_{\mathrm{coh}}(X)$ we have $\mathrm{Hom}(\mathcal{F}, \mathcal{O}_x) = \mathrm{Hom}(\mathcal{F}_x, k(x))$. Hence, this is a local question, so suppose $\mathrm{Spec}(A) \subset X$ is some affine open subset trivializing $\mathcal{F}$. We are set with a closed subset $Z \subset \mathrm{Spec}(A)$ cut out by a finitely generated ideal $I \subset A$ and an epimorphism $A^n \to I$. The epimorphism $A^n \to I$ really is a collection $a_1, \dots a_n$ of generators of I. The induced morphism of Hom-sets for $A^n \to A$ becomes

$$A/\mathfrak{m}_x \cong \mathrm{Hom}(A, A/\mathfrak{m}_x) \to \mathrm{Hom}(A^n, A/\mathfrak{m}_x) \cong (A/\mathfrak{m}_x)^n,$$

where $\mathfrak{m}_x$ is the maximal ideal corresponding to the closed point x and this map is given by $\overline{1} \mapsto (\overline{a_1}, \dots, \overline{a_n})$. Clearly, this is trivial if and only if each a_i

is contained in $\mathfrak{m}_x$, that is, if and only if $(a_1, \ldots, a_n) \subset \mathfrak{m}_x$. So, this is exactly the closed subset $V((a_1, \ldots, a_n))$ cut out by the ideal generated by the a_i. But this ideal is just I, so we are done.

On Y, we have the analogous subset $F(\Phi(\mathcal{L}_0))$, and Φ induces a bijection $F(\mathcal{L}_0) \to F(\Phi(\mathcal{L}_0))$. As Φ induces bijections on Hom-sets, we see that our bijection on closed points sends Z_α precisely to $Z_{\Phi(\alpha)}$, from which it follows that it is a homeomorphism. In particular, the dimensions of X and Y are equal, given by $d \in \mathbb{Z}_{\geq 0}$.

If V is a projective variety together with a line bundle $\mathcal{L}$, then $\mathcal{L}$ is ample if and only if the subsets Z_α for $\alpha \in \mathrm{Hom}(\mathcal{L}^{\otimes i}, \mathcal{L}^{\otimes j})$ with $i \geq j$ form a basis of closed sets for the topology on the set of closed points of V. The proof of this fact can be found below. We can not directly use this fact, because Φ does not commute with tensor products in general, so we need something more. Because Φ sends locally-free objects to locally-free objects, and also preserves rank, it sends $\mathcal{O}_X$ to a shift of a line bundle. We know that the restriction of Φ to $D_{\mathrm{perf}}(X)$ commutes with the Serre functors S_X and S_Y, which are given by tensoring by the dualizing complex. Twisting Φ by $\Phi(\mathcal{O}_X)^\vee$, we may assume Φ sends $\mathcal{O}_X$ to $\mathcal{O}_Y$. We see that

$$\Phi(\omega_X[d]) = \Phi(S_X(O_X)) \cong S_Y(\Phi(\mathcal{O}_X)) = S_Y(\mathcal{O}_Y) = f_Y^! \mathcal{O}_{\mathrm{Spec}(k)},$$

hence the dualizing complex $f_Y^! \mathcal{O}_{\mathrm{Spec}(k)}$ of Y is a shift of an invertible sheaf, so Y is Gorenstein. As Y also has dimension d, its dualizing complex is concentrated in degree $-d$, so say $\omega_X[d]$ is sent to $\omega_Y[d]$. Then furthermore, because $\omega_X = S_X(\mathcal{O}_X)[-d]$, we may compute

$$\Phi(\omega_X^{\otimes i}) = \Phi(([-d] \circ S_X)^i(\mathcal{O}_X)) \cong ([-d] \circ S_Y)^i(\Phi(\mathcal{O}_X)) = \omega_Y^{\otimes i}$$

and we see that ω_Y is ample or anti-ample, matching ω_X.

We are done if we can show that the two canonical algebras $\bigoplus_{n \in \mathbb{Z}} H^0(X, \omega_X^{\otimes n})$ and $\bigoplus_{n \in \mathbb{Z}} H^0(Y, \omega_Y^{\otimes n})$ are isomorphic. Indeed, by projectivity, both X and Y are isomorphic to the Proj of the top or bottom half of the respective algebras, depending on whether the canonical bundle is ample or anti-ample. Note that

$$H^0(X, \omega_X^{\otimes n}) \cong \mathrm{Hom}(\mathcal{O}_X, \omega_X^{\otimes n}) \cong \mathrm{Hom}(\mathcal{O}_Y, \omega_Y^{\otimes n}) \cong H^0(X, \omega_Y^{\otimes n}),$$

where the middle isomorphism is applying Φ. For $m, n \in \mathbb{Z}$, we can interpret the product of two sections $s \in H^0(X, \omega_X^{\otimes n})$ and $t \in H^0(X, \omega_X^{\otimes m})$ as a composition $s \circ t$, where we consider s to be the unique morphism $\omega_X^{\otimes m} \to \omega_X^{\otimes n+m}$ it defines. In this way, we see that Φ induces an isomorphism

$$\bigoplus_{n \in \mathbb{Z}} H^0(X, \omega_X^{\otimes n}) \longrightarrow \bigoplus_{n \in \mathbb{Z}} H^0(Y, \omega_Y^{\otimes n})$$

not just of graded k-vector spaces, but of graded k-algebras. □

If we leave out the assumption on projectivity in the following, then we have stated a result found in [10], but we have no need for the general statement.

Lemma 5.16. *Let X be a projective k-variety, together with a line bundle $\mathcal{L}$ on X. Then $\mathcal{L}$ is ample if and only if the sets Z_α defined above for $\alpha \in \mathrm{Hom}(\mathcal{L}^{\otimes i}, \mathcal{L}^{\otimes j})$ with $i \geq j$ form a basis of closed sets for the Zariski topology on the set of closed points of X.*

Proof. Before we begin the proof, note that a morphism $\alpha\colon \mathcal{L}^{\otimes i} \to \mathcal{L}^{\otimes j}$ is given by multiplication by a global section $s \in \Gamma(X, \mathcal{L}^{\otimes j-i})$. Considered this way, for a closed point $x \in X$, the morphism

$$\mathrm{Hom}(\mathcal{L}^{\otimes j}, \mathcal{O}_x) \longrightarrow \mathrm{Hom}(\mathcal{L}^{\otimes i}, \mathcal{O}_x)$$

is simply given by pre-multiplying by the residue class $\overline{s_x}$ of the stalk. Because there are natural isomorphisms $\mathrm{Hom}(\mathcal{L}', \mathcal{O}_x) \cong \mathrm{Hom}(\mathcal{O}_{X,x}, \mathcal{O}_x) \cong k(x)$ for any invertible sheaf $\mathcal{L}'$, the induced map is just multiplication by $\overline{\alpha_x} \in k(x)$, so we find Z_α is precisely the vanishing locus of the section s.

First suppose $\mathcal{L}$ is ample. Let Z be a closed subset of X, cut out by an ideal sheaf $\mathcal{I}$. Then $\mathcal{I}$ is coherent, because X is a variety, hence $\mathcal{I} \otimes \mathcal{L}^{\otimes r}$ is generated by global sections for some r. This yields an epimorphism $\mathcal{O}_X^n \to \mathcal{I} \otimes \mathcal{L}^{\otimes r}$, which we twist to an epimorphism $(\mathcal{L}^{\otimes -r})^n \to \mathcal{I}$. Composing with the inclusion $\mathcal{I} \hookrightarrow \mathcal{O}_X$, we get a morphism $\alpha\colon (\mathcal{L}^{\otimes r})^n \to \mathcal{O}_X$. Like in the proof above, one shows $Z = Z_\alpha$. To finish up, note that α is a direct sum of n maps $\alpha_i\colon \mathcal{L}^{\otimes r} \to \mathcal{I}$ and subsequently that Z_α is the intersection of the Z_{α_i}.

Conversely, say the Z_α form a basis of closed sets. We show that for every coherent sheaf $\mathcal{F}$ and any $r \gg 0$, the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes r}$ is generated by global sections. Pick a trivializing open cover $X = \bigcup_i U_i$ for $\mathcal{L}$ of affines U_i. By assumption on $\mathcal{L}$, there exist morphisms α_{ij} between tensor powers of $\mathcal{L}$ such that $U_i = \bigcup_j U_{ij}$, where $U_{ij} = Z_{\alpha_{ij}}$. By the discussion at the start of this proof, the U_{ij} are the non-vanishing loci in X of sections s_{ij}. Because U_{ij} is contained in U, it is just the non-vanishing locus of $s_{ij}|_U$ in U, which is affine. The open sets U_{ij} cover X, so we may choose a finite subcover. The result of all this is a finite trivializing cover $X = \bigcup_i U_i$ for $\mathcal{L}$ of open affines U_i, each of which is the non-vanishing locus of a global section s_i of an invertible sheaf $\mathcal{L}^{\otimes r_i}$. Replacing s_i by its $r_i^{-1} \prod_j r_j$-th tensor power, we may assume the powers r_i are all equal to $r := \prod_j r_j$.

Each module $\mathcal{F}(U_i)$ is finitely generated, say by elements a_{ij}. The usual arguments on extending local sections on non-vanishing loci show that there exist integers e_{ij} such that $s_i^{\otimes e_{ij}} \otimes a_{ij}$ extends to a global section of $\mathcal{L}^{\otimes e_{ij} r} \otimes \mathcal{F}$, see for example [8, Lemma II.5.14]. For any $e \geq \max_{ij}\{e_{ij}\}$, we see $s_i^e \otimes a_{ij}$ all extend to global sections of $\mathcal{L}^{\otimes er} \otimes \mathcal{F}$. Note that for each i, the sections $s_i^e \otimes a_{ij}$ still form a generating set. Indeed, because s_i does not vanish on U_i, it is an invertible element of $\mathcal{L}(U_i)$. This shows $\mathcal{L}^{\otimes er} \otimes \mathcal{F}$ is generated by global sections for $e \gg 0$, hence $\mathcal{L}^{\otimes r}$ is ample and therewith $\mathcal{L}$ is ample itself. $\square$

6 Relatively Perfect Complexes and Fourier-Mukai Transforms for Projective Gorenstein Schemes

Throughout this section, we will leave out the superscript $\bullet$ from the objects of the derived categories we consider. This should not lead to any confusion, because we do not use the more concrete description of an object $\mathcal{F} \in D(X)$ as a complex.

6.1 Relatively perfect complexes

In this whole subsection $f\colon X \to Y$ will be morphism of projective schemes over a field k. If $\mathcal{C}$ is a triangulated category, we will write $[-,-]$ for $\mathrm{Hom}_{\mathcal{C}}(-,-)$ and leave the subscript $\mathcal{C}$ implicit. We will also leave the subscripts on coproducts and direct sums implicit, when no confusion is possible. We only prove the results we need in the sequel and refer to [1] for a more extensive account of the theory of relative perfection.

Definition 6.1. A complex $\mathcal{E} \in D(X)$ is called *f-perfect* or *perfect relative to f* if $f_*(\mathcal{E} \otimes -)\colon D(X) \to D(Y)$ sends perfect objects to perfect objects.

Example 6.2. The perfect complexes of $D(X)$ are precisely the id_X-perfect objects. Any tensor product of perfect complexes is perfect and conversely, we easily see $\mathcal{E} \cong \mathrm{id}_{X,*}(\mathcal{E} \otimes \mathcal{O}_X)$ is perfect.

Example 6.3. If $\mathcal{E} \in D(X)$ is f-perfect, then for any perfect complex $\mathcal{G}$, the tensor product $\mathcal{E} \otimes \mathcal{G}$ is also f-perfect. This follows from the fact that a tensor product of perfect complexes is perfect.

In the next section, we are going to need a way to prove some complexes are perfect relative to some morphism f. Simply checking $f_*(\mathcal{E} \otimes -)$ is perfect for all perfect complexes can be too difficult and we would like to only have to check complexes of a certain nice form. The following lemma, which summarizes an argument used in the proof of [1, Lemma 4.2.17], lightens our burden greatly.

Lemma 6.4. *Suppose $\Omega \subset D_{\mathrm{perf}}(X)$ generates $D_{\mathrm{perf}}(X)$. Then an object $\mathcal{E} \in D(X)$ is f-perfect if and only if $f_*(\mathcal{E} \otimes -)$ sends every element of Ω to a perfect complex.*

Proof. The only if part is obvious, so assume $f_*(\mathcal{E} \otimes -)$ sends the objects in Ω to perfect objects. Let $\mathcal{D} \subset D_{\mathrm{perf}}(X)$ be the full subcategory of objects $\mathcal{F}$ such that $f_*(\mathcal{E} \otimes \mathcal{F})$ is perfect. By assumption we have $\Omega \subset \mathcal{D}$, so we just have to show $\mathcal{D}$ is a thick triangulated subcategory, so that we can use Theorem 1.12. Because $f_*(\mathcal{E} \otimes -)$ is a triangulated functor and $D_{\mathrm{perf}}(Y) \subset D(Y)$ is a triangulated subcategory, it follows immediately that $\mathcal{D}$ is triangulated as well. To show $\mathcal{D}$ is thick, suppose $\mathcal{A} \oplus \mathcal{B} \in \mathcal{D}$. The pushforward f_* commutes with direct sums, so

$$f_*(\mathcal{E} \otimes (\mathcal{A} \oplus \mathcal{B})) \cong f_*(\mathcal{E} \otimes \mathcal{A} \oplus \mathcal{E} \otimes \mathcal{B}) \cong f_*(\mathcal{E} \otimes \mathcal{A}) \oplus f_*(\mathcal{E} \otimes \mathcal{B}).$$

Because $D_{\mathrm{perf}}(Y)$ is thick, it contains both summands, hence $\mathcal{A}, \mathcal{B} \in \mathcal{D}$ and therewith $\mathcal{D}$ is thick. □

Remark 6.5. Note that this proof generalizes directly to show the following statement. Let $F: \mathcal{C} \to \mathcal{C}'$ be a triangulated functor which commutes with finite direct sums. Suppose $\Omega \subset \mathcal{C}$ is a set of objects and $\mathcal{D} \subset \mathcal{C}$ is the smallest thick triangulated subcategory containing Ω. Furthermore, suppose $\mathcal{D}' \subset \mathcal{C}'$ is any thick triangulated subcategory. Then F sends $\mathcal{D}$ into $\mathcal{D}'$ if and only if it sends Ω into $\mathcal{D}'$.

Lemma 6.6 ([1, Lemma 4.2.3])**.** *Let $F: \mathcal{C} \to \mathcal{S}$ be a functor of triangulated categories, where $\mathcal{C}$ is compactly generated. Suppose F commutes with coproducts and let $G: \mathcal{S} \to \mathcal{C}$ be its right-adjoint. Then G commutes with coproducts if and only if F takes a generating set of compact objects to compact objects.*

Proof. First, assume G commutes with coproducts. If $A \in \mathcal{C}$ is compact, then for any coproduct $\coprod B_i$ in $\mathcal{S}$, we have natural isomorphisms

$$\begin{aligned}\bigoplus [F(A), B_i] \cong \bigoplus [A, G(B_i)] &\cong [A, \coprod G(B_i)] \\ &\cong [A, G(\coprod B_i)] \cong [F(A), \coprod B_i],\end{aligned}$$

showing $F(A)$ is compact. For the reverse implication, suppose Ω is a set of compact generators and that $F(A)$ is compact for each $A \in \Omega$. For any $A \in \Omega$ and any coproduct $\coprod B_i \in \mathcal{S}$, we can reorder the above chain of isomorphisms to get

$$[A, \coprod G(B_i)] \cong [A, G(\coprod B_i)]$$

and this shows the natural map $\coprod G(B_i) \to G(\coprod B_i)$ is an isomorphism by Lemma 1.10. □

Corollary 6.7 ([1, Lemma 4.2.2])**.** *A complex $\mathcal{E} \in D(X)$ is f-perfect if and only if $\mathcal{H}om^\bullet(\mathcal{E}, f^! -)$ commutes with coproducts.*

Proof. Because $\mathcal{H}om^\bullet(\mathcal{E}, -)$ is right-adjoint to $\mathcal{E} \otimes -$, the functor $\mathcal{H}om^\bullet(\mathcal{E}, f^! -)$ is right-adjoint to $f_*(\mathcal{E} \otimes -)$. The corollary is now just a combination of the previous two lemmas and the fact that the compact objects of $D(X)$ are precisely the perfect complexes. □

Let us inspect the adjunction $f_*(\mathcal{E} \otimes -) \dashv \mathcal{H}om^\bullet(\mathcal{E}, f^! -)$ a bit more thoroughly. The counit yields a natural map

$$f_*(\mathcal{E} \otimes \mathcal{H}om^\bullet(\mathcal{E}, f^! \mathcal{G})) \longrightarrow \mathcal{G}$$

for every $\mathcal{G} \in D(Y)$. Tensoring this with $\mathcal{H} \in D(Y)$, the left-hand side becomes isomorphic to

$$f_*(\mathcal{E} \otimes \mathcal{H}om^\bullet(\mathcal{E}, f^!\mathcal{G})) \otimes \mathcal{H} \cong f_*(\mathcal{E} \otimes \mathcal{H}om^\bullet(\mathcal{E}, f^!\mathcal{G}) \otimes f^*\mathcal{H})$$

and the resulting morphism

$$f_*(\mathcal{E} \otimes \mathcal{H}om^\bullet(\mathcal{E}, f^!\mathcal{G}) \otimes f^*\mathcal{H}) \longrightarrow \mathcal{G} \otimes \mathcal{H}$$

transposes to a natural morphism

$$\mathcal{H}om^\bullet(\mathcal{E}, f^!\mathcal{G}) \otimes f^*\mathcal{H} \longrightarrow \mathcal{H}om^\bullet(\mathcal{E}, f^!(\mathcal{G} \otimes \mathcal{H})).$$

Lemma 6.8 ([1, Lemma 4.2.5])**.** *If $E \in D(X)$ is f-perfect, then the natural map*

$$\mathcal{H}om^\bullet(\mathcal{E}, f^!\mathcal{G}) \otimes f^*\mathcal{H} \longrightarrow \mathcal{H}om^\bullet(\mathcal{E}, f^!(\mathcal{G} \otimes \mathcal{H}))$$

is an isomorphism for any $\mathcal{G}, \mathcal{H} \in D(Y)$.

Proof. We show that for every $\mathcal{G} \in D(Y)$, these natural maps form an isomorphism of functors

$$\mathcal{H}om^\bullet(\mathcal{E}, f^!\mathcal{G}) \otimes f^*- \Longrightarrow \mathcal{H}om^\bullet(\mathcal{E}, f^!(\mathcal{G} \otimes -)).$$

As $\otimes$ and f^* are right-exact functors, they commute with arbitrary colimits and indeed with coproducts. By the corollary above, the functor $\mathcal{H}om^\bullet(\mathcal{E}, f^!-)$ commutes with coproducts as well, hence the two functors above both commute with coproducts. By Proposition 3.19, both functors have right-adjoints and thus Lemma 1.11 applies, hence we are done if we can show that

$$\mathcal{H}om^\bullet(\mathcal{E}, f^!\mathcal{G}) \otimes f^*\mathcal{H} \longrightarrow \mathcal{H}om^\bullet(\mathcal{E}, f^!(\mathcal{G} \otimes \mathcal{H}))$$

is an isomorphism for $\mathcal{H}$ perfect. First, note that for arbitrary $\mathcal{F} \in D(X)$ we have natural isomorphisms

$$\begin{aligned} [(f^*\mathcal{H})^\vee, \mathcal{F}] &\cong [\mathcal{O}_X, f^*\mathcal{H} \otimes \mathcal{F}] \cong [\mathcal{O}_Y, f_*(\mathcal{F} \otimes f^*\mathcal{H})] \\ &\cong [\mathcal{H}^\vee, f_*\mathcal{F}] \cong [f^*(\mathcal{H}^\vee), \mathcal{F}], \end{aligned}$$

so that $(f^*\mathcal{H})^\vee$ is naturally isomorphic to $f^*(\mathcal{H}^\vee)$.

For arbitrary $\mathcal{F} \in D(X)$, there are natural isomorphisms

$$\begin{aligned} [\mathcal{F}, \mathcal{H}om^\bullet(\mathcal{E}, f^!\mathcal{G}) \otimes f^*\mathcal{H}] &\cong [\mathcal{F} \otimes (f^*\mathcal{H})^\vee, \mathcal{H}om^\bullet(E, f^!\mathcal{G})] \\ &\cong [f_*(\mathcal{E} \otimes \mathcal{F} \otimes f^*(\mathcal{H}^\vee), \mathcal{G}] \\ &\cong [f_*(\mathcal{E} \otimes \mathcal{F}) \otimes \mathcal{H}^\vee, \mathcal{G}] \\ &\cong [f_*(\mathcal{E} \otimes \mathcal{F}), \mathcal{G} \otimes \mathcal{H}] \\ &\cong [\mathcal{F}, \mathcal{H}om^\bullet(\mathcal{E}, f^!(\mathcal{G} \otimes \mathcal{H}))]. \end{aligned}$$

The map from the first group to the last is induced by the morphism we wanted to show is an isomorphism, so the proof is complete by the Yoneda lemma. □

Lemma 6.9 ([1, Lemma 4.2.6])**.** *If $\mathcal{E}$ is f-perfect, then the two functors $f_*\mathcal{H}om^\bullet(E, f^!-)$ and $\mathcal{H}om^\bullet(f_*E, -)$ are isomorphic.*

Proof. For any $\mathcal{F}, \mathcal{G} \in D(Y)$, we have natural isomorphisms

$$\begin{aligned}[][\mathcal{F}, f_*\mathcal{H}om^\bullet(\mathcal{E}, f^!\mathcal{G})] &\cong [f^*\mathcal{F}, \mathcal{H}om^\bullet(\mathcal{E}, f^!\mathcal{G})] \cong [f_*(\mathcal{E} \otimes f^*\mathcal{F}), \mathcal{G}] \\ &\cong [f_*\mathcal{E} \otimes \mathcal{F}, \mathcal{G}] \cong [\mathcal{F}, \mathcal{H}om^\bullet(f_*\mathcal{E}, \mathcal{G})],\end{aligned}$$

so the Yoneda lemma completes the proof. □

Lemma 6.10 ([1, Lemma 4.2.7])**.** *If $\mathcal{E}$ is f-perfect, then so is $\mathcal{H}om^\bullet(\mathcal{E}, f^!\mathcal{F})$ for any $\mathcal{F} \in D_{\mathrm{perf}}(Y)$.*

Proof. We have to show $f_*(\mathcal{H}om^\bullet(\mathcal{E}, f^!\mathcal{F}) \otimes -)$ takes perfect objects to perfect objects. If $\mathcal{G}$ is perfect, then so is $\mathcal{H}om^\bullet(\mathcal{G}, \mathcal{O}_X)$, so $\mathcal{E} \otimes \mathcal{H}om^\bullet(\mathcal{G}, \mathcal{O}_X)$ is still f-perfect. The above lemma yields

$$\begin{aligned} f_*(\mathcal{H}om^\bullet(\mathcal{E}, f^!\mathcal{F}) \otimes \mathcal{G}) &\cong f_*\mathcal{H}om^\bullet(\mathcal{E} \otimes \mathcal{H}om^\bullet(\mathcal{G}, \mathcal{O}_X), f^!\mathcal{F}) \\ &\cong \mathcal{H}om^\bullet(f_*(\mathcal{E} \otimes \mathcal{H}om^\bullet(\mathcal{G}, \mathcal{O}_X)), \mathcal{F}).\end{aligned}$$

Because $\mathcal{E}$ is f-perfect, the complex $f_*(\mathcal{E} \otimes \mathcal{H}om^\bullet(\mathcal{G}, \mathcal{O}_X))$ is perfect, hence so is the last complex in the chain of isomorphisms. □

Hence the functor $\mathcal{H}om^\bullet(-, f^!\mathcal{O}_Y)$ restricts to an endofunctor on the full subcategory of f-perfect objects in $D(X)$. It turns out that this is an involution.

Lemma 6.11 ([1, Lemma 4.2.9])**.** *If $\mathcal{E}$ is f-perfect, then the natural morphism*

$$\nu \colon \mathcal{E} \longrightarrow \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{E}, f^!\mathcal{O}_Y), f^!\mathcal{O}_Y)$$

is an isomorphism.

Proof. For any $\mathcal{F}$, the complex $\mathcal{E} \otimes \mathcal{F}$ is still f-perfect, so

$$\begin{aligned} f_*\mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{E}, f^!\mathcal{O}_Y), f^!\mathcal{O}_Y) \otimes \mathcal{F} &\cong f_*\mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{E} \otimes \mathcal{F}, f^!\mathcal{O}_Y), f^!\mathcal{O}_Y) \\ &\cong \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(f_*(\mathcal{E} \otimes \mathcal{F}), \mathcal{O}_Y), \mathcal{O}_Y)\end{aligned}$$

by a twofold application of Lemma 6.9 together with the previous lemma. Because $f_*(\mathcal{E} \otimes \mathcal{F})$ is perfect, the natural morphism from it to the complex at the end above is an isomorphism. This shows that $f_*(\nu \otimes \mathcal{F})$ is an isomorphism for every perfect $\mathcal{F}$.

Let $\mathcal{K}$ be the third vertex of a completion of ν to an exact triangle. Then $f_*(\mathcal{K} \otimes \mathcal{F})$ is zero for every perfect object $\mathcal{F}$. Applying $[\mathcal{O}_Y, -]$, we find

$$0 = [\mathcal{O}_Y, f_*(\mathcal{K} \otimes \mathcal{F})] \cong [\mathcal{O}_X, \mathcal{K} \otimes \mathcal{F}] \cong [\mathcal{H}om^\bullet(\mathcal{F}, \mathcal{O}_X), \mathcal{K}]$$

and because $\mathcal{H}om^\bullet(-, \mathcal{O}_X)$ is an involution on $D_{\mathrm{perf}}(X)$, we see $[\mathcal{G}, \mathcal{K}] = 0$ for every perfect object $\mathcal{G}$, hence $\mathcal{K} = 0$ and therewith ν is an isomorphism. □

Lemma 6.12 ([1, Lemma 4.2.10])**.** *If $\mathcal{E}$ is f-perfect, then $f_*(\mathcal{H}om^\bullet(\mathcal{E}, f^!\mathcal{O}_Y)\otimes -)$ is left-adjoint to $\mathcal{E} \otimes f^*-$.*

Proof. A simple computation yields for every $\mathcal{F} \in D(X)$ and every $\mathcal{G} \in D(Y)$:

$$\begin{aligned}[\mathcal{F}, \mathcal{E} \otimes f^*\mathcal{G}] &\cong [\mathcal{F}, \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{E}, f^!\mathcal{O}_Y), f^!\mathcal{O}_Y) \otimes f^{\mathcal{G}}] \\ &\cong [\mathcal{F}, \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{E}, f^!\mathcal{O}_Y), f^!\mathcal{G})] \\ &\cong [f_*(\mathcal{H}om^\bullet(\mathcal{E}, f^!\mathcal{O}_Y) \otimes \mathcal{F}), \mathcal{G}].\end{aligned}$$

□

6.2 Fourier-Mukai Transforms

Again, throughout this subsection, we will mean X and Y to be projective k-schemes. Let p, q denote the projections from $X \times Y$ to X and Y respectively.

Definition 6.13. Let $\mathcal{E} \in D(X \times Y)$. The functor

$$\begin{aligned}\Phi_{\mathcal{E}}^{X\to Y} \colon D(X) &\to D(Y) \\ \mathcal{F} &\mapsto q_*(\mathcal{E} \otimes p^*\mathcal{F})\end{aligned}$$

is called the *Fourier-Mukai transform with kernel $\mathcal{E}$*. When no confusion is possible, the superscript $X \to Y$ is left implicit.

Some authors choose instead to call these functors integral transforms and reserve the name Fourier-Mukai transform for an integral transform which is also an equivalence. We simply call these Fourier-Mukai transforms which are also equivalences, or sometimes Fourier-Mukai equivalences.

Example 6.14. Let $f \colon X \to Y$ be any morphism. The graph Γ of f is the scheme theoretic image of the morphism $(\mathrm{id}, f) \colon X \to X \times Y$. Its structure sheaf $\mathcal{O}_\Gamma = (\mathrm{id}, f)_*\mathcal{O}_X$ is a coherent $\mathcal{O}_{X\times Y}$-module, so we may consider the Fourier-Mukai transform $D(X) \to D(Y)$ with kernel $\mathcal{O}_\Gamma$. This functor sends $\mathcal{F} \in D(X)$ to

$$q_*(\mathcal{O}_\Gamma \otimes p^*\mathcal{F}) \cong q_*((\mathrm{id}, f)_*\mathcal{O}_X \otimes p^*\mathcal{F}) \cong q_*(\mathrm{id}, f)_*(\mathcal{O}_X \otimes (\mathrm{id}, f)^*p^*\mathcal{F}),$$

which is isomorphic to

$$f_*(\mathcal{O}_X \otimes \mathcal{F}) \cong f_*\mathcal{F}.$$

So the Fourier-Mukai transform $\Phi_{\mathcal{O}_\Gamma}^{X\to Y}$ is just f_*. The same kernel also defines a Fourier-Mukai transform $D(Y) \to D(X)$ and one finds this is just f^*, through the same computation as above. In particular, for $f = \mathrm{id} \colon X \to X$, one obtains the diagonal $\Gamma = \Delta$ and finds that the identity functor $\mathrm{Id} \colon D(X) \to D(X)$ is a Fourier-Mukai transform with kernel $\mathcal{O}_\Delta$.

Example 6.15. Suppose $\mathcal{L}$ is a line bundle on X. Then for any $i \in \mathbb{Z}$, we have an automorphism of $D(X)$ given by twisting by $\mathcal{L}[i]$, which also turns out to be a Fourier-Mukai transform. Indeed, let $\iota\colon X \to X \times X$ denote the diagonal embedding and let Φ be the Fourier-Mukai transform with kernel $\iota_*\mathcal{L}[i]$. Again, the same computation as above shows $\Phi \cong - \otimes \mathcal{L}[i]$.

Example 6.16. The raison d'être of the Fourier-Mukai transform is the derived equivalence between $D^b_{\text{coh}}(A)$ and $D^b_{\text{coh}}(\hat{A})$ for an abelian variety A. The Fourier-Mukai transform was first defined by Mukai in [13] in 1981 and this specific example of which he calls a Fourier functor, hence the now conventional name. He noted that the normalized Poincaré bundle $\mathcal{P}$ on $A \times \hat{A}$ supplies a Fourier-Mukai transform which is an equivalence $D^b_{\text{coh}}(A) \to D^b_{\text{coh}}(\hat{A})$. This is proved in [13, Theorem 2.2]. In contrast to the reconstruction theorem we proved in Section 5, this is an example of two varieties which are derived equivalent, but not isomorphic, at least if the dimension of A is greater than 1.

The categories we are really interested in are $D^b_{\text{coh}}(X)$ and $D_{\text{perf}}(X)$, but we can not be sure that a Fourier-Mukai transform Φ restricts to these categories in general. Of course, if we take the kernel to be in $D^b_{\text{coh}}(X)$, then because any pushforward restricts to the bounded derived category of coherent sheaves, we find Φ does so as well. However, if we want Φ to be an equivalence, then by Lemma 3.30, we see Φ would also have to restrict to the perfect derived categories, which again, does not always work. The answer to these questions on restrictions turns out to be a natural application of the theory developed in the previous subsection. The following fact about the external tensor products of ample invertible sheaves, which seems to be well-known throughout the literature, will be useful.

Lemma 6.17. *Let X and Y be quasi-projective k-schemes. Suppose $\mathcal{L}_X$ and $\mathcal{L}_Y$ are ample invertible sheaves on X and Y respectively. Then $\mathcal{L}_X \boxtimes \mathcal{L}_Y$ is an ample invertible sheaf on $X \times Y$. The same holds if we replace ample by very ample.*

Proof. Choose $r \gg 0$ such that $\mathcal{L}_X^{\otimes r}$ and $\mathcal{L}_Y^{\otimes r}$ are both very ample. We get closed immersions $i_X\colon X \hookrightarrow \mathbb{P}^n$ and $i_Y\colon Y \hookrightarrow \mathbb{P}^m$ such that $i_X^*\mathcal{O}_X(1) \cong \mathcal{L}_X^{\otimes r}$ and $i_X^*\mathcal{O}_X(1) \cong \mathcal{L}_X^{\otimes r}$. As $\mathcal{O}(1,1) \cong \mathcal{O}_{\mathbb{P}^n}(1) \boxtimes \mathcal{O}_{\mathbb{P}^m}(1)$, we get

$$\mathcal{L}_X^{\otimes r} \boxtimes \mathcal{L}_Y^{\otimes r} \cong i_X^*\mathcal{O}_{\mathbb{P}^n}(1) \boxtimes i_Y^*\mathcal{O}_{\mathbb{P}^m}(1) \cong (i_X \times i_Y)^*\mathcal{O}(1,1)$$

and $\mathcal{O}(1,1)$ is very ample, so $\mathcal{L}_X^{\otimes r} \boxtimes \mathcal{L}_Y^{\otimes r}$ is as well. This shows $\mathcal{L}_X \boxtimes \mathcal{L}_Y$ is ample. For the statement on very ampleness, we just take $r = 1$. ☐

We will need a technical lemma, the proof of which is found in [1, Lemma 4.2.19].

Lemma 6.18. *Let X be a quasi-projective k-scheme and $\mathcal{G} \in D(X)$ and $\mathcal{O}_X(1)$ a choice of a very ample line bundle on X. Then there exists an $N \geq 0$ such* *that $\mathcal{G}$ is bounded if and only if for all $0 \le j \le N$, the groups* $\mathrm{Hom}(\mathcal{O}_X(j), \mathcal{G}[l])$ *are bounded in l.*

Recall that p and q denote the projections of $X \times Y$ to X and Y respectively.

Lemma 6.19 ([1, Lemma 4.2.18])**.** *Suppose $\mathcal{E} \in D(X \times Y)$ is a complex of coherent sheaves. Then $\Phi_{\mathcal{E}}$ takes $D^b_{\mathrm{coh}}(X)$ to $D^b_{\mathrm{coh}}(Y)$ if and only if $\mathcal{E}$ is p-perfect.*

Proof. If $\mathcal{E}$ is p-perfect, then $p_*(\mathcal{E} \otimes -)$ takes perfect complexes to perfect complexes. Its right pseudo-adjoint takes $D^b_{\mathrm{coh}}(X)$ to $D^b_{\mathrm{coh}}(X \times Y)$ and it coincides with the right-adjoint $\mathcal{H}om^\bullet(\mathcal{E}, p^! -)$. In particular, the complex $\mathcal{H}om^\bullet(\mathcal{E}, p^! \mathcal{O}_X)$ is a bounded complex of coherent sheaves. Similarly, the complex $\mathcal{H}om^\bullet(\mathcal{H}om^\bullet(\mathcal{E}, p^!\mathcal{O}_X), p^!\mathcal{O}_X)$ is in $D^b_{\mathrm{coh}}(X \times Y)$ as well, but this is just $\mathcal{E}$. Hence $\Phi_{\mathcal{E}}$ takes $D^b_{\mathrm{coh}}(X)$ to $D^b_{\mathrm{coh}}(Y)$.

Conversely, if $\Phi_{\mathcal{E}}$ takes $D^b_{\mathrm{coh}}(X)$ to $D^b_{\mathrm{coh}}(Y)$, then the lemma above shows $\mathcal{E} \otimes p^* -$ takes $D^b_{\mathrm{coh}}(X)$ to $D^b_{\mathrm{coh}}(X \times Y)$. Indeed, if $\mathcal{O}_X(1)$ and $\mathcal{O}_Y(1)$ are choices of very ample line bundles on X and Y, then the external tensor product $\mathcal{O}_X(1) \boxtimes \mathcal{O}_Y(1)$ is very ample and we compute for $\mathcal{G} \in D^b_{\mathrm{coh}}(X)$:

$$\begin{aligned}\mathrm{Hom}(\mathcal{O}_X(j) \boxtimes \mathcal{O}_Y(j), \mathcal{E} \otimes p^*\mathcal{G}[l]) &\cong \mathrm{Hom}(q^*\mathcal{O}_Y(j), \mathcal{E} \otimes p^*\mathcal{G}(-j)[l]) \\ &\cong \mathrm{Hom}(\mathcal{O}_Y(j), q_*(\mathcal{E} \otimes p^*\mathcal{G}(-j))[l]).\end{aligned}$$

As $q_*(\mathcal{E} \otimes p^*\mathcal{G}(-j)) = \Phi_{\mathcal{E}}(\mathcal{G}(-j))$ is bounded by assumption, it follows that $\mathcal{E} \otimes p^*\mathcal{G}$ is bounded.

In particular, we find for every $\mathcal{F} \in D^b_{\mathrm{coh}}(X \times Y)$ and every $\mathcal{G} \in D^b_{\mathrm{coh}}(X)$ that

$$p_*(\mathcal{E} \otimes \mathcal{F} \otimes p^*\mathcal{G}) \cong p_*(\mathcal{E} \otimes \mathcal{F}) \otimes \mathcal{G}$$

has bounded cohomology. We wish to show $\mathcal{P} := p_*(\mathcal{E} \otimes \mathcal{F})$ is perfect. For any closed point $x \in X$, the tensor product $\mathcal{P} \otimes \mathcal{O}_x$ has bounded cohomology. Replacing $\mathcal{P}_x$ by a minimal free resolution, we see that $\mathcal{P}_x \otimes k(x)$ has trivial differentials, so that $\mathcal{P}_x$ has to be bounded. Hence $\mathcal{P}$ is locally quasi-isomorphic to a perfect complex and the usual arguments show $\mathcal{P}$ is perfect, finishing the proof. □

Lemma 6.20 ([1, Lemma 4.2.17])**.** *Let $\mathcal{E} \in D(X \times Y)$. Then $\Phi_{\mathcal{E}}$ takes $D_{\mathrm{perf}}(X)$ to $D_{\mathrm{perf}}(Y)$ if and only if $\mathcal{E}$ is q-perfect.*

Proof. If $\mathcal{E}$ is q-perfect, then $q_*(\mathcal{E} \otimes \mathcal{F})$ is perfect for any perfect $\mathcal{F}$. As p^* takes perfect complexes to perfect complexes, we see $\Phi_{\mathcal{E}}$ takes $D_{\mathrm{perf}}(X)$ to $D_{\mathrm{perf}}(Y)$.

Conversely, suppose $\Phi_{\mathcal{E}}$ takes perfect complexes to perfect complexes. As $\mathcal{O}_X(1) \boxtimes \mathcal{O}_Y(1)$ is an ample invertible sheaf on $X \times Y$, the set $\{\mathcal{O}_X(j) \boxtimes \mathcal{O}_Y(j)[l]\}_{j,l \in \mathbb{Z}}$ forms a generating set. See Example 1.9 and Theorem 1.12. Keeping in mind Lemma 6.4, we are done if we show that

$$q_*(\mathcal{E} \otimes (\mathcal{O}_X(j) \boxtimes \mathcal{O}_Y(j)))$$

is perfect for each j, where we note that every functor in sight commutes with translations. By the projection formula, we get

$$q_*(\mathcal{E} \otimes (\mathcal{O}_X(j) \boxtimes \mathcal{O}_Y(j))) \cong q_*(\mathcal{E} \otimes p^*\mathcal{O}_X(j)) \otimes \mathcal{O}_Y(j) = \Phi_{\mathcal{E}}(\mathcal{O}_X(j)) \otimes \mathcal{O}_Y(j),$$

which is a tensor product of two perfect complexes, hence it is perfect and the proof is complete. □

In the smooth case, a Fourier-Mukai transform $D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$ always has a left- and a right-adjoint. If $\mathcal{E}$ is the kernel, then the adjoints have kernels

$$\mathcal{E}_L := \mathcal{E}^\vee \otimes q^*\omega_Y[\dim(Y)] \text{ and } \mathcal{E}_R := \mathcal{E}^\vee \otimes p^*\omega_X[\dim(X)],$$

respectively. Indeed, if $\mathcal{F} \in D^b_{\mathrm{coh}}(Y)$ and $\mathcal{G} \in D^b_{\mathrm{coh}}(X)$, then

$$\begin{aligned}\mathrm{Hom}(\mathcal{F}, q_*(\mathcal{E} \otimes p^*\mathcal{G})) &\cong \mathrm{Hom}(q^*\mathcal{F}, \mathcal{E} \otimes p^*\mathcal{G}) \cong \mathrm{Hom}(q^*\mathcal{F} \otimes \mathcal{E}^\vee, p^*\mathcal{G}) \\ &\cong \mathrm{Hom}(p^*\mathcal{G}, q^*\mathcal{F} \otimes \mathcal{E}^\vee \otimes \omega_X[n] \boxtimes \omega_Y[m])^* \\ &\cong \mathrm{Hom}(\mathcal{G}, p_*(q^*\mathcal{F} \otimes \mathcal{E}^\vee \otimes q^*\omega_Y[m]) \otimes \omega_X[n])^* \\ &\cong \mathrm{Hom}(p_*(q^*\mathcal{F} \otimes \mathcal{E}^\vee \otimes q^*\omega_Y[m]), \mathcal{G})\end{aligned}$$

and the right-adjointness is showed similarly. The crux of this computation is the fact that $\mathcal{E}^\vee \otimes -$ is left-adjoint to $\mathcal{E} \otimes -$, which does not hold up when $\mathcal{E}$ is not perfect.

As it turns out, when $\mathcal{E}$ is p-perfect and q-perfect, then $\Phi_{\mathcal{E}}$ still has left- and right-adjoints. Note that the objects $\mathcal{E}_L$ and $\mathcal{E}_R$ are relative duals to $\mathcal{E}$, in a sense. In the smooth case, they are isomorphic to

$$\mathcal{E}_L \cong \mathcal{H}om^\bullet(\mathcal{E}, p^!\mathcal{O}_X) \text{ and } \mathcal{E}_R \cong \mathcal{H}om^\bullet(\mathcal{E}, q^!\mathcal{O}_Y)$$

and these are the kernels we should consider in the non-smooth case. These final three results summarize the parts we need from [1, Lemmas 4.2.20 - 4.2.23].

Lemma 6.21. *If $\mathcal{E} \in D(X \times Y)$ is p-perfect, then $\Phi_{\mathcal{E}_L}$ is left-adjoint to $\Phi_{\mathcal{E}} \colon D(X) \to D(Y)$.*

Proof. There are natural isomorphisms

$$\begin{aligned}\mathrm{Hom}(\mathcal{F}, \Phi_{\mathcal{E}}(\mathcal{G})) = \mathrm{Hom}(\mathcal{F}, q_*(\mathcal{E} \otimes p^*\mathcal{G})) &\cong \mathrm{Hom}(q^*\mathcal{F}, \mathcal{E} \otimes p^*\mathcal{G}) \\ &\cong \mathrm{Hom}(\mathcal{H}om^\bullet(\mathcal{E}, p^!\mathcal{O}_X) \otimes q^*\mathcal{F}, p^*\mathcal{G}) \\ &\cong \mathrm{Hom}(p_*(\mathcal{H}om^\bullet(\mathcal{E}, p^!\mathcal{O}_X) \otimes q^*\mathcal{F}), \mathcal{G}) \cong \mathrm{Hom}(\Phi_{\mathcal{E}_L}(\mathcal{F}), \mathcal{G}),\end{aligned}$$

where we use Lemmas 6.10 and 6.11 for the second isomorphism. □

Lemma 6.22. *If $\mathcal{E} \in D(X \times Y)$ is q-perfect, then $\Phi_{\mathcal{E}_R}$ is right-adjoint to $\Phi_{\mathcal{E}} \colon D(X) \to D(Y)$.*

Proof. There are natural isomorphisms

$$\begin{aligned}\mathrm{Hom}(\Phi_{\mathcal{E}}(\mathcal{F}),\mathcal{G}) &= \mathrm{Hom}(q_*(\mathcal{E}\otimes p^*\mathcal{F}),\mathcal{G}) \cong \mathrm{Hom}(p^*\mathcal{F},\mathcal{H}om^\bullet(\mathcal{E},q^!\mathcal{G}))\\ &\cong \mathrm{Hom}(p^*\mathcal{F},\mathcal{H}om^\bullet(\mathcal{E},q^!\mathcal{O}_Y)\otimes q^*\mathcal{G})\\ &\cong \mathrm{Hom}(\mathcal{F},p_*(\mathcal{H}om^\bullet(\mathcal{E},q^!\mathcal{O}_Y)\otimes q^*\mathcal{G}))\cong \mathrm{Hom}(\mathcal{F},\Phi_{\mathcal{E}_R}(\mathcal{G})),\end{aligned}$$

this time using Lemma 6.8 for the second isomorphism. □

Lemma 6.23. *If $\mathcal{E}\in D(X\times Y)$ is both p-perfect and q-perfect, then the adjoints $\Phi_{\mathcal{E}_L}$ and $\Phi_{\mathcal{E}_R}$ to $\Phi_{\mathcal{E}}$ both take $D_{\mathrm{perf}}(Y)$ into $D_{\mathrm{perf}}(X)$ and $D^b_{\mathrm{coh}}(Y)$ into $D^b_{\mathrm{coh}}(X)$.*

Proof. As $\mathcal{E}$ is q-perfect, the functor $\Phi_{\mathcal{E}}$ takes $D_{\mathrm{perf}}(X)$ into $D_{\mathrm{perf}}(Y)$, by Lemma 6.20. The kernel $\mathcal{E}_L$ is p-perfect because $\mathcal{E}$ is, so that $\Phi_{\mathcal{E}_L}$ takes $D_{\mathrm{perf}}(Y)$ to $D_{\mathrm{perf}}(X)$ as well. Because the restriction of $\Phi_{\mathcal{E}_L}$ to $D^b_{\mathrm{coh}}(Y)$ is left pseudo-adjoint to an exact functor taking $D_{\mathrm{perf}}(X)$ into $D_{\mathrm{perf}}(Y)$, it also takes $D^b_{\mathrm{coh}}(Y)$ into $D^b_{\mathrm{coh}}(X)$.

The proof of Lemma 6.19 shows $\Phi_{\mathcal{E}}$ takes $D^b_{\mathrm{coh}}(X)$ into $D^b_{\mathrm{coh}}(Y)$ whenever $\mathcal{E}$ is p-perfect. The rest of the proof is analogous to the above, so we are done. □

7 Equivalence Results for Fourier-Mukai Transforms

The main goal of this section is to prove some conditions under which specific Fourier-Mukai transforms are fully faithful, or even equivalences. One corollary of these results is going to be an example of two derived-equivalent Gorenstein varieties which are non-isomorphic, extending the fact that for smooth varieties, the derived category does not quite determine the variety.

7.1 Fully-faithful functors and equivalences

Suppose X and Y are projective Gorenstein varieties and that $\mathcal{E}\in D(X\times Y)$ is a kernel. Then the kernel $\mathcal{E}_L$ for the potential left-adjoint can be simplified as

$$\begin{aligned}\mathcal{H}om^\bullet(\mathcal{E},p^!\mathcal{O}_X) &\cong \mathcal{H}om^\bullet(\mathcal{E},p^!f_X^*\mathcal{O}_{\mathrm{Spec}(k)})\\ &\cong \mathcal{H}om^\bullet(\mathcal{E},q^*f_Y^!\mathcal{O}_{\mathrm{Spec}(k)})\cong \mathcal{H}om^\bullet(\mathcal{E},q^*\omega_Y[n])\\ &\cong \mathcal{E}^\vee\otimes q^*\omega_Y[n],\end{aligned}$$

because $\omega_Y[n]$ is perfect, where n is the dimension of Y. A similar argument shows $\mathcal{E}_R\cong\mathcal{E}^\vee\otimes p^*\omega_X[m]$, where m is the dimension of X.

Theorem 7.1 ([9, Proposition 7.6])**.** *Suppose X and Y are projective Gorenstein varieties and let $\mathcal{E} \in D^b_{\mathrm{coh}}(X \times Y)$ be such that the Fourier-Mukai transform $\Phi_{\mathcal{E}} \colon D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$ is full and faithful. If*

$$\dim(X) = \dim(Y) \ \text{ and } \ \mathcal{E} \otimes p^*\omega_X \cong \mathcal{E} \otimes q^*\omega_Y,$$

where p and q are the projections to X and Y respectively, then $\Phi_{\mathcal{E}}$ is an equivalence.

Proof. Dualizing both sides of the isomorphism and twisting by $p^*\omega_X \otimes q^*\omega_Y$, we get

$$\mathcal{E}^\vee \otimes q^*\omega_Y \cong \mathcal{E}^\vee \otimes p^*\omega_X$$

and if we shift this by the common dimension of X and Y, we get $\mathcal{E}_R \cong \mathcal{E}_L$. This shows the left- and right-adjoints of Φ are isomorphic. We are done by the following proposition, where we know $D^b_{\mathrm{coh}}(X)$ is indecomposable if and only if X is connected, by [9, Proposition 3.10]. □

Proposition 7.2. *Let $F \colon \mathcal{C} \to \mathcal{C}'$ be a fully-faithful exact functor between triangulated categories. Suppose $\mathcal{C}$ contains an object not isomorphic to 0 and that $\mathcal{C}'$ is indecomposable. Then F is an equivalence of categories if and only if F has a left-adjoint G and a right-adjoint H such that for any object $B \in \mathcal{C}'$ one has $G(B) \cong 0$ if $H(B) \cong 0$.*

Proof. See [9, Proposition 1.54]. □

Keeping in mind Proposition 1.2, we will find it useful to identify a well-behaved spanning class for $D^b_{\mathrm{coh}}(X)$. Here, again, the closed points turn out useful. The proof of the following is adapted from [9], accounting for our lack of a Serre functor on $D^b_{\mathrm{coh}}(X)$.

Proposition 7.3. *Let X be a projective Gorenstein variety. Then the objects $\mathcal{O}_x$ where $x \in X$ ranges over the closed points form a spanning class for $D^b_{\mathrm{coh}}(X)$.*

Proof. We have to show that for any non-trivial $\mathcal{F}^\bullet \in D^b_{\mathrm{coh}}(X)$ there exist closed points $x_1, x_2 \in X$ and integers i_1, i_2 such that

$$\mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{O}_{x_1}[i_1]) \neq 0, \quad \mathrm{Hom}(\mathcal{O}_{x_2}[i_2], \mathcal{F}^\bullet) \neq 0.$$

Let us first check the former. Choose $m \in \mathbb{Z}$ maximal such that $\mathcal{H} := \mathcal{H}^m(\mathcal{F}^\bullet) \neq 0$ and let $x \in \mathrm{supp}(\mathcal{H})$ be a closed point. There exists an epimorphism $i^*\mathcal{H} \to \mathcal{O}_{\mathrm{Spec}(k(x))}$ where $i \colon \{x\} \to X$ is the inclusion and this yields a non-trivial morphism $\mathcal{H} \to \mathcal{O}_x$ by adjunction. Because the cohomology of $\mathcal{F}^\bullet$ is non-trivial only in degrees $\leq m$, the natural map $\tau_{\leq m}\mathcal{F}^\bullet \to \mathcal{F}^\bullet$ is an isomorphism. This gives us a morphism $\mathcal{F}^\bullet \to \tau_{\leq m}\mathcal{F}^\bullet$. The latter complex has a natural map to (the shift by $-m$ of) its m-th cohomology, which is just $\mathcal{H}$. The resulting composition

$$\mathcal{F}^\bullet \longrightarrow \tau_{\leq m}\mathcal{F}^\bullet \longrightarrow \mathcal{H}[-m] \longrightarrow \mathcal{O}_x[-m]$$

gives back the map $\mathcal{H} \to \mathcal{O}_x$ on m-th cohomology, so it is non-trivial.

To show the latter, remember that by Lemma 4.5, we have an isomorphism

$$\mathrm{Hom}(\mathcal{O}_x[i_2], \mathcal{F}^\bullet) \cong \mathrm{Hom}(\mathcal{F}^{\bullet\vee}, (\mathcal{O}_x[i_2])^\vee).$$

By Eq. (4.1), the dual of $\mathcal{O}_x[i_2]$ is $\mathcal{O}_x[-n-i_2]$, so we have reduced to finding i_2' and $x \in X$ such that

$$\mathrm{Hom}(\mathcal{F}^{\bullet\vee}, \mathcal{O}_x[i_2']) \neq 0.$$

If no such x and i_2' exist, then $\mathcal{F}^{\bullet\vee}$ is zero by the first part of this proof. Because $\mathcal{F}^\bullet$ is reflexive, this shows $\mathcal{F}^\bullet \cong 0$, contradicting non-triviality of $\mathcal{F}^\bullet$. □

Lemma 7.4. *Let X and Y be projective Gorenstein varieties and suppose $F\colon D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$ is a triangulated functor with a left-adjoint and a right-adjoint. Then F is fully faithful if and only if for every closed point $x \in X$ the homomorphism*

$$k(x) \cong \mathrm{Hom}(\mathcal{O}_x, \mathcal{O}_x) \longrightarrow \mathrm{Hom}(F(\mathcal{O}_x), F(\mathcal{O}_x))$$

is bijective and for any two closed points $x, y \in X$ and any $i \in \mathbb{Z}$, the group $\mathrm{Hom}(F(\mathcal{O}_x), F(\mathcal{O}_y)[i])$ is trivial if either $x \neq y$ or $i \neq 0$.

Proof. This is just a result of combining the proposition above with Proposition 1.2. □

7.2 Products of Fourier-Mukai transforms

Throughout this subsection, we let X_1, X_2, Y_1, Y_2 be projective Gorenstein varieties, and consider two Fourier-Mukai transforms $\Phi_{\mathcal{P}_1}\colon D(X_1) \to D(Y_1)$ and $\Phi_{\mathcal{P}_2}\colon D(X_2) \to D(Y_2)$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are both perfect over both factors. Then $\mathcal{P}_1$ and $\mathcal{P}_2$ can both be represented by complexes of coherent sheaves, so that then $\mathcal{P}_1 \boxtimes \mathcal{P}_2$ is a complex of coherent sheaves as well. We can define a Fourier-Mukai transform

$$\Phi_{\mathcal{E}}\colon D(X_1 \times X_2) \to D(Y_1 \times Y_2)$$

in a natural way by taking $\mathcal{E} = \mathcal{P}_1 \boxtimes \mathcal{P}_2 \in D_{\mathrm{coh}}((X_1 \times X_2) \times (Y_1 \times Y_2))$. It turns out that this functor acts very nicely on objects of the form $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ for $\mathcal{F}_1 \in D(X_1)$ and $\mathcal{F}_2 \in D(X_2)$. To be precise, it holds that

$$\Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2}(\mathcal{F}_1 \boxtimes \mathcal{F}_2) = \Phi_{\mathcal{P}_1}(\mathcal{F}_1) \boxtimes \Phi_{\mathcal{P}_2}(\mathcal{F}_2). \tag{7.1}$$

The proof of this is short, but it can be difficult to keep track of all the morphisms involved. Let p_i, q_i denote the first and second projections of $X_i \times Y_i$ and let x_1, x_2 and y_1, y_2 denote the first and second projections of $X_1 \times X_2$ and $Y_1 \times Y_2$, respectively. For legibility, we write $r_1 := x_1 \times y_1$ and $r_2 := x_2 \times y_2$. For $i = 1, 2$, both compositions $x_i \circ (p_1 \times p_2)$ and $p_i \circ r_i = p_i \circ (x_i \times y_i)$ are equal to the projection $X_1 \times X_2 \times Y_1 \times Y_2 \to X_i$, hence

$$\begin{aligned}
\Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2}(\mathcal{F}_1 \boxtimes \mathcal{F}_2) &= (q_1 \times q_2)_*((p_1 \times p_2)^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \otimes (\mathcal{P}_1 \boxtimes \mathcal{P}_2)) \\
&\cong (q_1 \times q_2)_*((p_1 \times p_2)^*(x_1^*\mathcal{F}_1 \otimes x_2^*\mathcal{F}_2) \otimes (r_1^*\mathcal{P}_1 \otimes r_2^*\mathcal{P}_2)) \\
&\cong (q_1 \times q_2)_*(r_1^*p_1^*\mathcal{F}_1 \otimes r_2^*p_2^*\mathcal{F}_2 \otimes r_1^*\mathcal{P}_1 \otimes r_2^*\mathcal{P}_2) \\
&\cong (q_1 \times q_2)_*(r_1^*(p_1^*\mathcal{F}_1 \otimes \mathcal{P}_1) \otimes r_2^*(p_2^*\mathcal{F}_2 \otimes \mathcal{P}_2)) \\
&\cong (q_1 \times q_2)_*((p_1^*\mathcal{F}_1 \otimes \mathcal{P}_1) \boxtimes (p_2^*\mathcal{F}_2 \otimes \mathcal{P}_2)) \\
(*) &\cong q_{1,*}(p_1^*\mathcal{F}_1 \otimes \mathcal{P}_1) \boxtimes q_{2,*}(p_2^*\mathcal{F}_2 \otimes \mathcal{P}_2) \\
&\cong \Phi_{\mathcal{P}_1}(\mathcal{F}_1) \boxtimes \Phi_{\mathcal{P}_2}(\mathcal{F}_2),
\end{aligned}$$

where we use Lemma 2.5 for the isomorphism $(*)$.

Hence, the product Fourier-Mukai transform acts in a straightforward manner on complexes of the form $\mathcal{F} \boxtimes \mathcal{G}$. Next on our list is to show that $\Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2}$ restricts to a functor

$$D^b_{\mathrm{coh}}(X_1 \times X_2) \to D^b_{\mathrm{coh}}(Y_1 \times Y_2)$$

and further to a functor

$$D_{\mathrm{perf}}(X_1 \times X_2) \to D_{\mathrm{perf}}(Y_1 \times Y_2).$$

For this, it is enough to show $\mathcal{P} := \mathcal{P}_1 \boxtimes \mathcal{P}_2$ is perfect relative to both $p_1 \times p_2$ and $q_1 \times q_2$, where p_i, q_i denote the projections of $X_i \times Y_i$ onto the first and second factor respectively. The following lemma makes quick work of this.

Lemma 7.5. *Let $f_1 \colon X_1 \to Y_1$ and $g_1 \colon X_2 \to Y_2$ be morphisms of projective schemes over a field k, together with f_i-perfect complexes $\mathcal{E}_i$ for $i = 1, 2$. Then $\mathcal{E}_1 \boxtimes \mathcal{E}_2$ is $f_1 \times f_2$-perfect.*

Proof. Let $\mathcal{O}_{X_1}(1), \mathcal{O}_{X_2}(1)$ denote choices of very ample sheaves on X_1 and X_2. Because $\mathcal{O}_{X_1}(1) \boxtimes \mathcal{O}_{X_2}(1)$ is a very ample invertible sheaf, the set of tensor powers $\{\mathcal{O}_{X_1}(j) \boxtimes \mathcal{O}_{X_2}(j)[l]\}_{j,l \in \mathbb{Z}}$ is a set of compact generators for $D(X_1 \times X_2)$, see Example 1.9 and Theorem 1.12. By Lemma 6.4, we are done if we can show that

$$(f_1 \times f_2)_*((\mathcal{E}_1 \boxtimes \mathcal{E}_2) \otimes (\mathcal{O}_{X_1}(j) \boxtimes \mathcal{O}_{X_2}(j)))$$

is perfect for each j. By Eq. (7.1), this is straightforward enough. We compute

$$\begin{aligned}
&(f_1 \times f_2)_*((\mathcal{E}_1 \boxtimes \mathcal{E}_2) \otimes (\mathcal{O}_{X_1}(j) \boxtimes \mathcal{O}_{X_2}(j))) \\
\cong\ &(f_1 \times f_2)_*((\mathcal{E}_1 \otimes \mathcal{O}_{X_1}(j)) \boxtimes (\mathcal{E}_2 \otimes \mathcal{O}_{X_2}(j))) \\
\cong\ &f_{1,*}(\mathcal{E}_1 \otimes \mathcal{O}_{X_1}(j)) \boxtimes f_{2,*}(\mathcal{E}_2 \otimes \mathcal{O}_{X_2}(j))
\end{aligned}$$

which is an external tensor product of perfect complexes by f_i-perfection of $\mathcal{E}_i$, hence it is perfect. □

Before we can apply our developed machinery, we need to know that the product of two projective varieties which are categorically Gorenstein is again categorically Gorenstein. One can show this algebraically, using the fact that for projective varieties, being Gorenstein and being categorically Gorenstein are equivalent. However, we choose to take the categorical approach here, which is much more enlightening.

Lemma 7.6. *Let X and Y be projective k-schemes. If $f_Z\colon Z \to \mathrm{Spec}(k)$ denotes the structure morphism for a k-scheme Z, then the dualizing complex of $X \times Y$ is isomorphic to*

$$f^!_{X\times Y}\mathcal{O}_{\mathrm{Spec}(k)} \cong f^!_X\mathcal{O}_{\mathrm{Spec}(k)} \boxtimes f^!_Y\mathcal{O}_{\mathrm{Spec}(k)}.$$

Proof. We have a Cartesian square

$$\begin{array}{ccc} X\times Y & \xrightarrow{p} & X \\ \Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle f_X} \\ Y & \xrightarrow[f_Y]{} & \mathrm{Spec}(k) \end{array}$$

where f_Y is flat because k is a field and f_X is proper because X is projective. Hence Theorem 3.23 says $q^! f^*_Y \cong p^* f^!_X$. Adjoints compose, so that $f^!_{X\times Y} \cong p^! f^!_X$. The base change q is also proper, hence Theorem 3.24 applies and we can compute

$$\begin{aligned} f^!_{X\times Y}\mathcal{O}_{\mathrm{Spec}(k)} \cong q^! f^!_Y\mathcal{O}_{\mathrm{Spec}(k)} &\cong q^!\mathcal{O}_Y \otimes q^* f^!_Y\mathcal{O}_{\mathrm{Spec}(k)} \\ &\cong q^! f^*_Y\mathcal{O}_{\mathrm{Spec}(k)} \otimes q^* f^!_Y\mathcal{O}_{\mathrm{Spec}(k)} \\ &\cong p^* f^!_X\mathcal{O}_{\mathrm{Spec}(k)} \otimes q^* f^!_Y\mathcal{O}_{\mathrm{Spec}(k)} \\ &\cong f^!_X\mathcal{O}_{\mathrm{Spec}(k)} \boxtimes f^!_Y\mathcal{O}_{\mathrm{Spec}(k)}, \end{aligned}$$

completing the proof. □

Corollary 7.7. *If X and Y are projective k-varieties which are categorically Gorenstein, then $X \times Y$ is categorically Gorenstein as well. If ω_X and ω_Y are the corresponding dualizing sheaves, then $\omega_X \boxtimes \omega_Y$ is the dualizing sheaf for $X \times Y$.*

Proof. Say X and Y have dimensions m and n respectively. Recall that the dualizing complexes $f^!_X\mathcal{O}_{\mathrm{Spec}(k)} \cong \omega_X[m]$ and $f^!_Y\mathcal{O}_{\mathrm{Spec}(k)} \cong \omega_Y[n]$ of X and Y are shifts of line bundles. Thus

$$f^!_{X\times Y}\mathcal{O}_{\mathrm{Spec}(k)} \cong (\omega_X \boxtimes \omega_Y)[m+n]$$

is a shift of a line bundle as well, showing that

$$- \otimes f^!_{X \times Y} \mathcal{O}_{\mathrm{Spec}(k)}$$

restricts to the perfect derived categories and induces a Serre functor

$$D_{\mathrm{perf}}(X \times Y) \to D_{\mathrm{perf}}(X \times Y).$$

□

We wish to understand the product $X \times Y$ by considering the closed points of its factors X and Y. Unfortunately, if the base field k is not algebraically closed, there are no points like (x, y) in $X \times Y$ in general, where $x \in X$ and $y \in Y$. In particular, the sheaves $\mathcal{O}_z$ for closed points $z \in X \times Y$ do not seem useful right away. We do have the sheaves $\mathcal{O}_x \boxtimes \mathcal{O}_y$, however, and it turns out that these form a spanning class, as we will show shortly. We will find it useful to first consider the supports of these coherent sheaves. Let us write p, q for the projections to X and Y respectively. Noting that for any $z \in X \times Y$ it holds that

$$(\mathcal{O}_x \boxtimes \mathcal{O}_y)_z \cong (p^*\mathcal{O}_x)_z \otimes (q^*\mathcal{O}_y)_z$$

and considering

$$(p^*\mathcal{O}_x)_z \cong (\mathcal{O}_x)_{p(z)} \otimes_{\mathcal{O}_{X,p(z)}} \mathcal{O}_{X \times Y, z}$$

and similar for $(q^*\mathcal{O}_y)_z$, we can conclude immediately that the support of $\mathcal{O}_x \boxtimes \mathcal{O}_y$ consists of the points $z \in X \times Y$ such that $p(z) = x$ and $q(z) = y$. This means it is the intersection, at least as a topological space, of the fibers $\{x\} \times Y$ and $X \times \{y\}$ in $X \times Y$, which is just $\{x\} \times \{y\}$. In turn, this is isomorphic to

$$\mathrm{Spec}(k(x)) \times \mathrm{Spec}(k(y)) \cong \mathrm{Spec}(k(x) \otimes k(y)).$$

As x and y are closed points of k-varieties, the residue fields $k(x)$ and $k(y)$ are finite extensions of k, thus the ring $k(x) \otimes k(y)$ is naturally a finite-dimensional k-vector space. Because an ideal $I \subset k(x) \otimes k(y)$ is also a linear subspace, any strict chain of ideals is finite, and it follows that $k(x) \otimes k(y)$ is Artinian. All in all, this implies

$$\mathrm{supp}(\mathcal{O}_x \boxtimes \mathcal{O}_y) \cong \mathrm{Spec}(k(x) \otimes k(y)) \tag{7.2}$$

is finite discrete.

Lemma 7.8. *Let X and Y be projective Gorenstein varieties over a field k. Then the collection of objects $\mathcal{O}_x \boxtimes \mathcal{O}_y \in D^b_{\mathrm{coh}}(X \times Y)$ for closed points $x \in X$ and $y \in Y$ forms a spanning class.*

Proof. The proof of this lemma is similar to the proof of Proposition 7.3. Let $\mathcal{F}^\bullet$ be a bounded complex of coherent sheaves on $X \times Y$. We have to find closed points $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ and integers $i_1, i_2 \in \mathbb{Z}$ such that

$$\mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{O}_{x_1} \boxtimes \mathcal{O}_{y_1}[i_1]) \neq 0, \quad \mathrm{Hom}(\mathcal{O}_{x_2} \boxtimes \mathcal{O}_{y_2}[i_2], \mathcal{F}^\bullet) \neq 0.$$

If we take $m \in \mathbb{Z}$ maximal such that the coherent sheaf $\mathcal{H}^m(\mathcal{F}^\bullet)$ is non-zero, and choose z in its support, then we get a non-trivial morphism $\mathcal{H}^m(\mathcal{F}^\bullet) \to \mathcal{O}_z$. Let x, y denote the images of z in X and Y respectively. Because X and Y are projective, the projections are closed, so that x and y are closed points. By the discussion above, the point z is an isolated point of the support Z of $\mathcal{O}_x \boxtimes \mathcal{O}_y$, hence $Z \setminus \{z\}$ is still closed in $X \times Y$. The subsets

$$U = (X \times Y) \setminus (Z \setminus \{z\}) \text{ and } V = (X \times Y) \setminus \{z\}$$

form an open cover of $X \times Y$. Restricting to U, both $\mathcal{O}_x \boxtimes \mathcal{O}_y|_U$ and $\mathcal{O}_z|_U$ are supported precisely on z, so Lemma 5.9 yields a monomorphism $\mathcal{O}_z|_U \to \mathcal{O}_x \boxtimes \mathcal{O}_y|_U$. Restricting to V, the sheaf $\mathcal{O}_z|_V$ is zero, so here we have a monomorphism $\mathcal{O}_z|_V \to \mathcal{O}_x \boxtimes \mathcal{O}_y|_V$ as well. Both morphisms are trivial on the intersection $U \cap V$, simply because $\mathcal{O}_z$ is zero there, which means they glue to a monomorphism $\mathcal{O}_z \to \mathcal{O}_x \boxtimes \mathcal{O}_y$ on X. We compose the morphism $\mathcal{H}^m(\mathcal{F}^\bullet) \to \mathcal{O}_z$ we found earlier with this monomorphism and get a non-trivial morphism

$$\mathcal{H}^m(\mathcal{F}^\bullet) \to \mathcal{O}_x \boxtimes \mathcal{O}_y.$$

Like in the proof of Proposition 7.3, we can truncate $\mathcal{F}^\bullet$ and form the composition

$$\mathcal{F}^\bullet \longrightarrow \tau_{\leq m}\mathcal{F}^\bullet \longrightarrow \mathcal{H}^m(\mathcal{F}^\bullet)[-m] \longrightarrow \mathcal{O}_x \boxtimes \mathcal{O}_y[-m]$$

to get

$$\mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{O}_x \boxtimes \mathcal{O}_y[-m]) \neq 0,$$

finishing the first part of this proof.

We can apply the same reasoning to $\mathcal{F}^{\bullet\vee}$ to find closed points $x \in X$ and $y \in Y$ and an integer m such that

$$\mathrm{Hom}(\mathcal{F}^{\bullet\vee}, \mathcal{O}_x \boxtimes \mathcal{O}_y[-m]) \neq 0.$$

In Corollary 7.7, we showed $X \times Y$ is (categorically) Gorenstein, so we get an isomorphism

$$\mathrm{Hom}((\mathcal{O}_x \boxtimes \mathcal{O}_y[-m])^\vee, \mathcal{F}^\bullet) \cong \mathrm{Hom}(\mathcal{F}^{\bullet\vee}, \mathcal{O}_x \boxtimes \mathcal{O}_y[-m])$$

by Lemma 4.5. Finally, because

$$(\mathcal{O}_x \boxtimes \mathcal{O}_y[-m])^\vee \cong (\mathcal{O}_x \boxtimes \mathcal{O}_y)^\vee[m] \cong \mathcal{O}_x^\vee \boxtimes \mathcal{O}_y^\vee[m] \cong \mathcal{O}_x \boxtimes \mathcal{O}_y[m - d_X - d_Y]$$

by Eq. (4.1), where d_X and d_Y are the dimensions of X and Y respectively, we get

$$\mathrm{Hom}(\mathcal{O}_x \boxtimes \mathcal{O}_y[m - d_X - d_Y], \mathcal{F}^\bullet) \neq 0,$$

so we are done. □

From this, we can prove the following proposition. Recall that we had two Fourier-Mukai transforms $\Phi_{\mathcal{P}_1} \colon D^b_{\mathrm{coh}}(X_1) \to D^b_{\mathrm{coh}}(Y_1)$ and $\Phi_{\mathcal{P}_2} \colon D^b_{\mathrm{coh}}(X_2) \to D^b_{\mathrm{coh}}(Y_2)$, where each $\mathcal{P}_i$ is perfect over both X_i and Y_i.

Proposition 7.9. *If the Fourier-Mukai transforms $\Phi_{\mathcal{P}_1}$ and $\Phi_{\mathcal{P}_2}$ are fully faithful, then so is $\Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2}$.*

Proof. We will use Proposition 1.2. The existence of left- and right-adjoints is ensured by Lemmas 6.21 and 6.22. By Eq. (7.1), we get $\Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2}(\mathcal{O}_{x_1} \boxtimes \mathcal{O}_{x_2}) \cong \Phi_{\mathcal{P}_1}(\mathcal{O}_{x_1}) \boxtimes \Phi_{\mathcal{P}_2}(\mathcal{O}_{x_2})$ for any two closed points $x_1 \in X_1$ and $x_2 \in X_2$. An application of the Künneth formula Lemma 2.4 yields

$$\mathrm{Hom}(\mathcal{O}_{x_1} \boxtimes \mathcal{O}_{x_2}, \mathcal{O}_{y_1} \boxtimes \mathcal{O}_{y_2}) \cong \bigoplus_{i+j=0} \mathrm{Hom}(\mathcal{O}_{x_1}, \mathcal{O}_{y_1}[i]) \otimes \mathrm{Hom}(\mathcal{O}_{x_2}, \mathcal{O}_{y_2}[j]),$$

the isomorphism from right to left being given by $(\varphi_i \otimes \psi_j) \mapsto \sum \varphi_i \boxtimes \psi_j$. In the following we write $[A, B] := \mathrm{Hom}(A, B)$ and $\Phi_x := \Phi(\mathcal{O}_x)$ for any point x, with kernels for Φ as expected. The isomorphisms in the computation around Eq. (7.1) are all natural and Φ is additive, hence the square

$$\begin{array}{ccc} \bigoplus_{i+j=0} [\mathcal{O}_{x_1}, \mathcal{O}_{y_1}[i]] \otimes [\mathcal{O}_{x_2}, \mathcal{O}_{y_2}[j]] & \xrightarrow{\sim} & [\mathcal{O}_{x_1} \boxtimes \mathcal{O}_{x_2}, \mathcal{O}_{y_1} \boxtimes \mathcal{O}_{y_2}] \\ \downarrow \Phi & & \downarrow \Phi \\ \bigoplus_{i+j=0} [\Phi_{x_1}, \Phi_{y_1}[i]] \otimes [\Phi_{x_2}, \Phi_{y_2}[j]] & \xrightarrow{\sim} & [\Phi_{x_1} \boxtimes \Phi_{x_2}, \Phi_{y_1} \boxtimes \Phi_{y_2}] \end{array}$$

commutes. As both $\Phi_{\mathcal{P}_1}$ and $\Phi_{\mathcal{P}_2}$ are fully-faithful, the left-most arrow is an isomorphism, ensuring the right-most is as well. Hence $\Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2}$ induces bijections

$$[\mathcal{O}_{x_1} \boxtimes \mathcal{O}_{x_2}, \mathcal{O}_{y_1} \boxtimes \mathcal{O}_{y_2}] \cong [\Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2}(\mathcal{O}_{x_1} \boxtimes \mathcal{O}_{x_2}), \Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2}(\mathcal{O}_{y_1} \boxtimes \mathcal{O}_{y_2})],$$

finishing the proof. □

Theorem 7.10. *If the Fourier-Mukai transforms $\Phi_{\mathcal{P}_1}$ and $\Phi_{\mathcal{P}_2}$ are equivalences, then so is $\Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2}$.*

Proof. We have already shown $\Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2}$ is fully faithful. By Lemma 1.3, we only have to show the left-adjoint to $\Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2}$ is fully faithful as well. If we can show that this left-adjoint is the Fourier-Mukai transform $\Phi_{\mathcal{Q}_1 \boxtimes \mathcal{Q}_2}$, where $\mathcal{Q}_i$ is the kernel for the left-adjoint to $\Phi_{\mathcal{P}_i}$ for $i = 1, 2$, then we are done. Indeed, these left-adjoints are fully faithful because the $\Phi_{\mathcal{P}_i}$ are equivalences, hence the product $\Phi_{\mathcal{Q}_1 \boxtimes \mathcal{Q}_2}$ is fully faithful also.

Recall that the kernel for the left-adjoint of a Fourier-Mukai functor $\Phi_{\mathcal{P}}$ between projective Gorenstein varieties X and Y is $\mathcal{P}^\vee \otimes q^*\omega_Y[n]$, where n is the dimension of Y. We compute

$$\begin{aligned}\mathcal{Q}_1 \boxtimes \mathcal{Q}_2 &= (\mathcal{P}_1^\vee \otimes q_1^*\omega_{Y_1}[n_1]) \boxtimes (\mathcal{P}_2^\vee \otimes q_2^*\omega_{Y_2}[n_2]) \\ &\cong r_1^*(\mathcal{P}_1^\vee \otimes q_1^*\omega_{Y_1}[n_1]) \otimes r_2^*(\mathcal{P}_2^\vee \otimes q_2^*\omega_{Y_2}[n_2]) \\ &\cong r_1^*\mathcal{P}_1^\vee \otimes r_1^*q_1^*\omega_{Y_1}[n_1] \otimes r_2^*\mathcal{P}_2^\vee \otimes r_2^*q_2^*\omega_{Y_2}[n_2] \\ &\cong r_1^*\mathcal{P}_1^\vee \otimes (q_1 \times q_2)^*y_1^*\omega_{Y_1}[n_1] \otimes r_2^*\mathcal{P}_2^\vee \otimes (q_1 \times q_2)^*y_2^*\omega_{Y_2}[n_2] \\ &\cong r_1^*\mathcal{P}_1^\vee \otimes r_2^*\mathcal{P}_2^\vee \otimes (q_1 \times q_2)^*\omega_{Y_1 \times Y_2}[n_1+n_2] \\ &\cong \mathcal{P}_1^\vee \boxtimes \mathcal{P}_2^\vee \otimes (q_1 \times q_2)^*\omega_{Y_1 \times Y_2}[n_1+n_2] \\ &\cong (\mathcal{P}_1 \boxtimes \mathcal{P}_2)^\vee \otimes (q_1 \times q_2)^*\omega_{Y_1 \times Y_2}[n_1+n_2],\end{aligned}$$

where the p_i and q_i are the projections from $X_i \times Y_i$ to X_i and Y_i respectively, the r_i are the projections $X_1 \times X_2 \times Y_1 \times Y_2 \to X_i \times Y_i$ and the y_i are the projections $Y_1 \times Y_2 \to Y_i$. Now, the final complex in this chain of isomorphisms is precisely the kernel for the left-adjoint of $\Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2}$, so we are done. □

We close off this section by highlighting a family of examples of two non-smooth non-isomorphic projective Gorenstein varieties which are derived equivalent, which will be almost trivial with the machinery developed prior.

Proposition 7.11. *Suppose k is a field. Let A be an abelian k-variety and X a projective Gorenstein k-variety with singular locus consisting of a single rational point. If A is not isomorphic to its dual $\hat{A}$, then $A \times X$ and $\hat{A} \times X$ are non-isomorphic projective Gorenstein varieties and $D^b_{\mathrm{coh}}(A \times X) \cong D^b_{\mathrm{coh}}(\hat{A} \times X)$.*

Proof. The "non-isomorphic projective Gorenstein varieties"-part is quickly proved. The singular loci of isomorphic varieties are isomorphic and in this case they are $A \times X_{sing} \cong A$ and $\hat{A} \times X_{sing} \cong \hat{A}$.

The Poincaré bundle $\mathcal{P}$ on $A \times \hat{A}$ supplies a derived equivalence $D^b_{\mathrm{coh}}(A) \to D^b_{\mathrm{coh}}(\hat{A})$, see [18, Proposition 2.4]. The structure sheaf $\mathcal{O}_\Delta$ of the diagonal $\Delta \subset X \times X$ gives the Fourier-Mukai transform Φ_Δ, which is isomorphic to the identity functor. Hence, by Theorem 7.10, the functor

$$\Phi_{\mathcal{P} \boxtimes \mathcal{O}_\Delta} \colon D^b_{\mathrm{coh}}(A \times X) \longrightarrow D^b_{\mathrm{coh}}(\hat{A} \times X)$$

is an equivalence. □

References

[1] M.R. Ballard, *Derived categories of sheaves of quasi-projective schemes* ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–University of Washington. MR 2712236

[2] M.R. Ballard, *Derived categories of sheaves on singular schemes with an application to reconstruction*, Advances in Mathematics 226(2) (2011), 895-919.

[3] I. Birger, *Lecture Notes on Local Rings*, World Scientific Publishing Co, Singapore (2014).

[4] A. Bondal, M. Kapranov, *Representable functors, Serre functors, and mutations*, Math. USSR Izvestiya 35 (1990), 519–541.

[5] A. Bondal, D. Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalences*, Compos. Math. 125 (2001), 327-344.

[6] T. Bridgeland, *Equivalences of triangulated categories and Fourier-Mukai transforms*, Bull. London Math. Soc. 31 (1999), 25-34.

[7] R. Hartshorne, *Residues and Duality*, Lecture Notes in Mathematics 20, Springer-Verlag, Berlin (1966).

[8] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics 52. Springer-Verlag, New York (1977).

[9] D. Huybrechts, *Fourier-Mukai Transforms in Algebraic Geometry*, Oxford University Press, Oxford (2006).

[10] L. Illusie, *Existence de résolutions globales*, In *Théorie des intersections et théorème de Riemann-Roch*, pages xii+700. Springer-Verlag, Berlin (1971), Seminaire de Geometrie Algebrique du Bois-Marie 1966-1967 (SGA 6).

[11] J. Lipman, *Notes on Derived Functors and Grothendieck Duality*, In *Foundations of Grothendieck Duality for Diagrams of Schemes*, Lecture Notes in Math., no. 1960, Springer-Verlag, New York, 2009, 1-259.

[12] H. Matsumara, *Commutative Ring Theory*, Second edition, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge (1989).

[13] S. Mukai, *Duality between $D(X)$ and $D(\hat{X})$ with its application to picard sheaves*, Nagoya Math. J. Vol. 81 (1981), 153-175.

[14] D. Murfet, *Derived Categories of Quasi-coherent Sheaves*, Notes, October 2006. Available from `http://www.therisingsea.org/notes`.

[15] A. Neeman, *The connection between the if-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel*, Ann. Sci. Ecole Norm. Sup. (4) 25(5) (1992), 547-566.

[16] A. Neeman, *Triangulated categories*, volume 148 of Annals of Mathematics Studies. Princeton University Press, Princeton, New Jersey (2001).

[17] A. Neeman, *The Grothendieck duality theorem via Bousfield's techniques and Brown representability*, J. Amer. Math. Soc. 9 (1996), 205-236.

[18] D. Orlov, *Derived categories of coherent sheaves on abelian varieties and equivalences between them*, Izv. Math. 66 (2002), 569–594.

[19] R. Rouquier, *Dimensions of triangulated categories*, Journal of K-theory 1 (2008), 193-256.

[20] C.A. Weibel, *An Introduction to Homological Algebra*, Cambridge University Press, New York (1994).