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Traveling wave solutions of reaction-diffusion equations in population dynamics

Bachelor thesis

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1 Introduction

A traveling wave is a solution of Partial Differential Equation (PDE) that propagates with a constant speed, while maintaining it's shape in space. Traveling waves are a common phenomenon in biology, evidenced by the fact Chapter 13 of the authoritative work "Mathematical Biology" of (Murray) is dedicated to biological waves. There are numerous models of population dynamics that give rise to biological waves. In the context of population dynamics, the traveling wave manifests itself as a wave of change in population population density through a habitat, for instance a plague that travels trough a continent. In this thesis we will focus on two reaction diffusion equations, that exhibit these traveling waves.

The first equation models logistic growth combined with diffusion. Let r, u_∞, and D be positive parameters. Consider the PDE:

$$u_t = Du_{xx} + ru\left(1 - \frac{u}{u_\infty}\right) \tag{1}$$

Where u is the population density, x is the spatial coordinate and t time. Although this equation has only one dimension, it could, for instance, describe a population along a coastline. By scaling time, population and distance appropriately, one obtains a dimensionless equation:

$$u_t = u_{xx} + u(1 - u) \tag{2}$$

We will refer to this equation as the Fisher-KPP equation (FKPP), but in the literature (1) or (2) may also be referred to as Fisher's equation, Kolmogorov – Petrovsky – Piscounov equation or KPP equation. A family of equations that includes this one was introduced by (Kolmogorov et al.).

The second equation that is the subject of this thesis is based on the Allee Effect combined with diffusion. The dimensionless equation is:

$$u_t = u_{xx} + u(1 - u)(u - a) \tag{3}$$

Where x and t are as in equation (2), and $a \in (0, \frac{1}{2})$ is a constant . This equation is very similar to part of a more sophisticated system of equations known as the FitzHugh-Nagumo Equation, which is used to model neurons.

The Allee effect has to do with the fact that the fitness of small populations is sometimes negative, i.e. if the population density is too small, the species or group of individuals will not survive. To model this effect an extra factor is added to the reaction term of the FKPP equation. If we look at the growth rate of a population as an indication of fitness, it is apparent that equation (16) indeed models the Allee effect. To see this, consider a spatially homogeneous population. For such a population, the diffusion term is zero, and the equation becomes $u_t = u(1-u)(u-a)$. See also Figure 1. The Allee effect is covered in detail in (Courchamp et al. (2008)).

In Section 2 and 5 we prove that there are traveling wave solutions for (2) and (3) respectively, while in Section 3 the stability of traveling waves (TWs) of the Fisher-KPP equation are discussed, explaining all the steps in detail and relating them to stability analysis of fixed points of an ODE. When discussing the stability

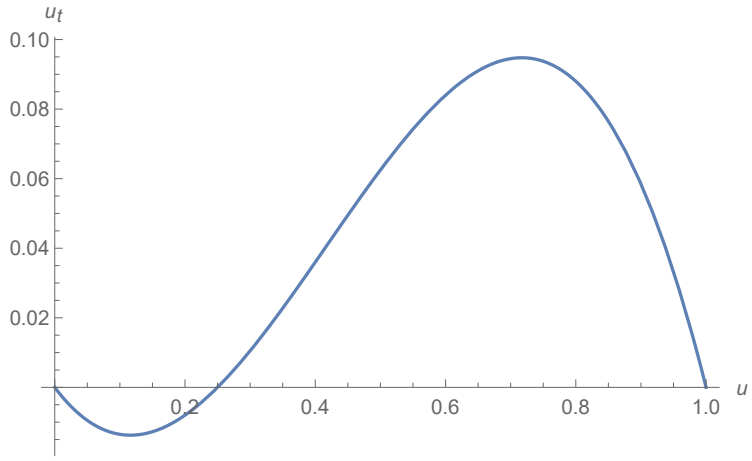

Figure 1: A plot of $u_t = u(1-u)(u-a)$ as a function of u for $a = \frac{1}{4}$

of TWs of the Fisher-KPP equation with Allee Effect (FKPPA) in Section6 these detailed explanations will be omitted.

In the Section 2 we will prove that there are TW solutions of equation (2), and in Section 4 we will show that these solutions are stable for $c > 2$, under certain conditions. The TW solutions of (3) are also stable. One could say that they are "more stable" than the solutions of equation (2), since the stability analysis will show stability in a more straightforward manner, i.e. stability can be shown, without restricting the analysis to "certain conditions". The special conditions for which the TW solutions of equation (4) are stable will be explained in detail in Section 4.

Suppose the dynamics of a population can be accurately modeled by equation (2) and (3). The existence and stability of TW solutions in equations (2) and (3) imply that one could indeed observe traveling waves of population density in this population.

The analytical results of this thesis can also be verified by numerical simulations. That is why the last section focuses on numerical results. The availability of cheap computing power makes numerical analysis an easy tool to study dynamical systems, now even more so than when the models discussed in this thesis were first introduced. The computing power available today is such, that even simulations of models in two spatial dimensions can be performed on a standard PC. Section 7.2 discusses such a simulation. Let x and y be the two spatial dimensions, then equation (2) can be modified to accommodate a 2D spatial domain by replacing the term u_{xx} with the Laplacian $u_{xx} + u_{yy}$. This yields the equation $u_t = u_{xx} + u_{yy} + u(1-u)$.

2 The existence of traveling wave solutions of the Fisher-KPP equation

We will determine whether the Fisher-KPP equation has traveling wave solutions relevant to population dynamics. The Fisher-KPP equation is:

$$u_t = u_{xx} + u(1-u) \tag{4}$$

Where u is a function $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ of $x \in \mathbb{R}$ and $t \geq 0$. This form of the Fisher-KPP equation is the dimensionless form of the equation, and is equal to equation (3) mentioned in the introduction. It has been derived by scaling time to the growth factor, distance to the diffusion length, and population size to the maximum population. Therefore, only values of u between zero and one are relevant.

A traveling wave solution is a solution which satisfies $u(x,t) = \phi(x-ct)$ for some function $\phi : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. The function ϕ is the wave profile and c the wave velocity. We introduce the co-moving frame write $\xi = x - ct$.

When we substitute $\phi(x-ct)$ for u, in the Fisher-KPP equation (4), then the left-hand side of the equation (4) is:

$$u_t(\xi) = \frac{d\phi}{d\xi}(\xi)\frac{d\xi}{dt}(\xi) = -c\phi'(\xi)$$

and the right hand side becomes:

$$u_{xx} + u(1-u) = \phi''(\xi) + \phi(\xi)(1-\phi(\xi)).$$

This gives us an ordinary differential equation:

$$\phi''(\xi) + c\phi'(\xi) + \phi(\xi)(1-\phi(\xi)) = 0 \tag{5}$$

or:

$$\phi'' + c\phi' + \phi(1-\phi) = 0$$

However, even though explicit solutions can be found for $c = \pm\frac{5}{6}\sqrt{6}$ (Ablowitz and Zeppetella, 1979), for other values of c the exact solutions of this equation are not easily determined, since the equation is nonlinear. Therefore, to study the behavior of the solutions, we write this equation as a two-dimensional system of first order equations. We can do this by setting $\psi = \phi'$, as in paragraph 3.1 of (Braun and Golubitsky):

$$\begin{aligned} \phi' &= \psi \\ \psi' &= -c\psi - \phi(1-\phi) \end{aligned} \tag{6}$$

The nullcline for $\phi' = 0$ is the line $\psi = 0$ and the nullcline for $\psi' = 0$ is the line $\psi = \frac{1}{c}(\phi^2 - \phi)$. The system has two fixed points $(\phi, \psi) = (0,0)$ and $(\phi, \psi) = (1,0)$.

To determine the character of the fixed point, we compute the Jacobian matrix J of this system:

$$J(\phi,\psi) = \begin{pmatrix} \frac{\partial\phi'}{\partial\phi} & \frac{\partial\phi'}{\partial\psi} \\ \frac{\partial\psi'}{\partial\phi} & \frac{\partial\psi'}{\partial\psi} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2\phi - 1 & -c \end{pmatrix}.$$

We have:

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix} \quad \text{and} \quad J(0,1) = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}.$$

In $(0,0)$, the characteristic equation of the Jacobian matrix is $\lambda^2 + c\lambda + 1$. Therefore, the eigenvalues are:

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4}}{2} = -\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 - 1}.$$

In $(0,1)$ the characteristic equation of the Jacobian matrix is $\lambda^2 + c\lambda - 1$ and the eigenvalues are:

$$\lambda = \frac{-c \pm \sqrt{c^2 + 4}}{2} = -\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 + 1}.$$

Note that $u = 1$ corresponds to the maximum carrying capacity of the ecosystem. Futhermore, we will not consider negative population sizes. Thus, the domain of u is $[0,1]$ and a solution of (5) is relevant only if we have $u(x,t) = \phi(x - ct) = \phi(\xi) \in [0,1]$ for all t and x.

We will now consider two cases for c separately: the case $0 < c < 2$ and the case $c > 2$.

2.1 The case $0 < c < 2$

The eigenvalues of the linearized system in the fixed point (0,0) are $-\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 - 1}$. For $0 < c < 2$ the eigenvalues are complex with a negative real part. Therefore, (0,0) is a stable focus.

To illustrate the argument we are about to make, a phase plot is shown for $c = 1$ in Figure 2. In this figure, the nullcline for which $\phi' = 0$ is in red and the nullcline for which $\psi' = 0$ is in green. The area shaded in blue consist of points where ϕ is outside its domain of $[0,1]$, i.e. phase curves that pass through the area shaded in blue are not relevant in our context.

We will argue that for $0 < c < 2$ all phase curves pass outside of the domain of ϕ at some point. In this section our argument will strongly rely on the phase plot 2. A more rigorous analysis is performed in Appendix A. This leaves no phase curves at all that are relevant in our context. To do this, we first give the definition of a stable set:

Definition 2.1. Let $u_t = f(u)$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ a C^1-map and $u \in \mathbb{R}^n$ be a system of ordinary differential equations. The stable set of a fixed point p with respect to $u_t = f(u)$ is the set of points x such that the solution $u(t;x)$ that starts in x for $t = 0$ satisfies $\lim_{t\to\infty} u(t;x) = p$. If a stable set is a manifold, it is called a stable manifold, see (Meiss).

Consider, for example, the linear system $\dot{x} = Ax$ with $x \in \mathbb{R}^n$ and A a $n \times n$-matrix with real coefficients. Suppose A has n distinct eigenvalues of which k have negative real part. Then the stable set is a k dimensional manifold and it is equal to the product of eigenspaces associated to the negative eigenvalues.

In the plot of the phase plane (Figure 2), one can see the stable manifold of (1,0) represented by blue line. Figure 2 clearly indicates that the stable manifold of (1,0)

is a separatrix. A separatrix is a boundary between two regions of the phase space such that no orbit passes through both regions, see (Meiss, p168). In particular, one can see that the phase curves left of the stable manifold of the fixed point (1,0) tend to the fixed point in the origin. These orbits spiral towards the origin, and therefore the value of ϕ must be negative at some point. Thus, these orbits are not relevant for applications in population dynamics.

The orbits to the right of the stable manifold of the fixed point (1,0) all have positive ϕ', which indicates unbounded growth of ϕ. Hence, these orbits are not relevant in our context either.

There are still two orbits on the stable manifold of (1,0) to consider; the separatrix itself. On the left, we have an orbit whose ϕ-coordinate becomes negative for backward time. On the right, there is an orbit whose ϕ-coordinate is always greater than one. Since the ϕ-coordinates are at some point smaller than zero or larger than one, these orbits are not relevant in our context either.

In conclusion, for $0 < c < 2$ there are no orbits $(\phi(\xi), \psi(\xi))$ in system (5), such that $\phi(x - ct)$ is a traveling wave that is relevant in the context of population dynamics.

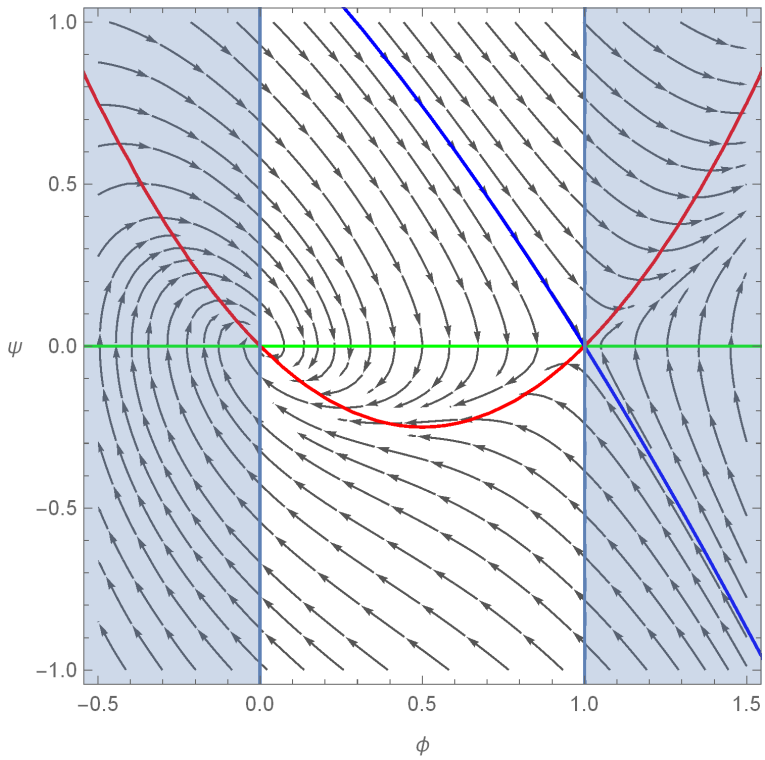

Figure 2: Phase plot of the vector field defined by system (3) for $c = 1$

2.2 The case $c \geq 2$

Now that we have concluded that there are no solutions relevant in the context of population dynamics for $0 < c < 2$, we will prove that there do exist relevant solutions when $c \geq 2$. The plot of the phase plane in Figure 3 suggests that there is

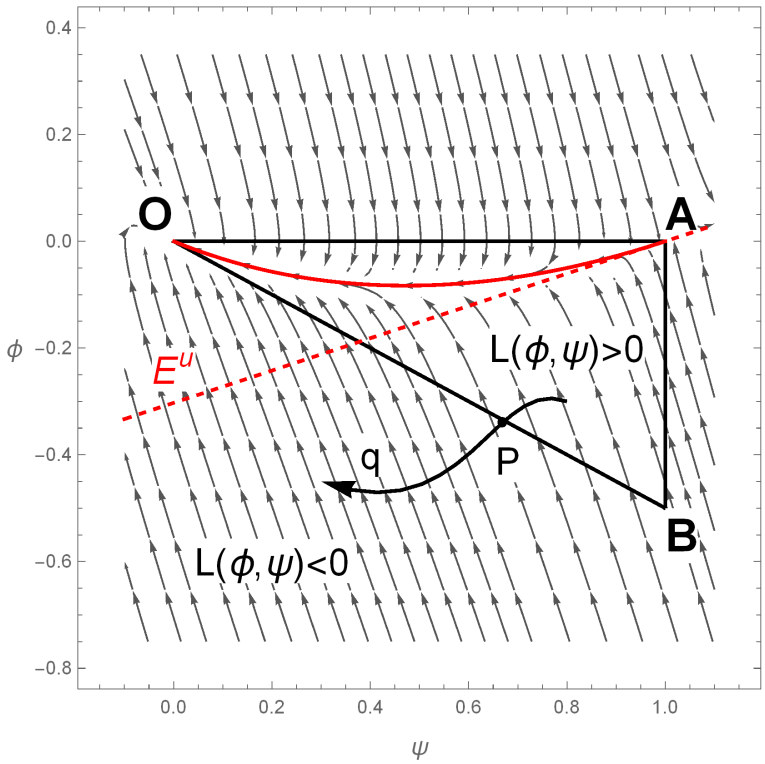

Figure 3: Phase plot of system (3) for $c = 3$ with a sketch of the triangle OAB, the heteroclinic solution in red, and the linear unstable manifold of the fixed point (1,0) dashed in red.

a heteroclinic orbit from the fixed point in (1,0) to the fixed point in (0,0). We will now prove its existence for all $c \geq 2$.

Denote the origin by O, the point (1,0) by A and the point $(1, -b)$ by B. We will prove that there is an orbit that leaves (1,0) and enters OAB and that no orbit can leave the triangle OAB in forward direction, see Figure 3. To do this we first give the definition of the unstable set of a fixed point. This definition is analogous to Definition 2.1 of the stable set of a fixed point. The only difference is that now we look at points that tend to a fixed point for $t \to -\infty$, instead of $t \to \infty$.

Definition 2.2. Let $u_t = f(u)$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ a C^1-map and $u \in \mathbb{R}^n$ be a system of ordinary differential equations. The unstable set of a fixed point p with respect to $u_t = f(u)$ is the set of points x such that the solution $u(t; x)$ that starts in x for $t = 0$ satisfies $\lim_{t\to-\infty} u(t; x) = p$. If an unstable set is a manifold, it is called an unstable manifold, see (Meiss)

Definition 2.3. Let $u_t = f(u)$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ a C^1-map and $u \in \mathbb{R}^n$ be a system of ordinary differential equations. Let p be a fixed point of this system. Let $u_t = g(u)$ be the linearization of $u_t = g(u)$ in p. The linear unstable manifold of p is the unstable manifold of p in $u_t = g(u)$.

A heteroclinic orbit from (1,0) to (0,0) must lie on the unstable manifold of (1,0). The unstable manifold is tangent to the linear unstable manifold. The corresponding positive eigenvalue of $J(1, 0)$ is $-\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + 1}$ and the associated eigenspace is spanned by the eigenvector $(\frac{c}{2}+\sqrt{c^2+4}, 1)^\mathsf{T}$. Therefore, the linear unstable manifold is:

$$E^u = \left\{(1, 0) \, + \, t \, \left(\frac{c}{2} + \sqrt{c^2 + 4}\,, 1\right) \; : \; t \in \mathbb{R}\right\}.$$

Note that $\psi_e < 0$ implies $\phi_e < 1$ for points $(\phi_e, \psi_e) \in E^u$. Hence there is a point on stable manifold inside OAB. Now we must prove that the part of the unstable manifold of (1,0) that lies to the left of (1,0) connects to the origin.

On the line OA we have $\phi \in [0, 1]$ and $\psi = 0$. Therefore, we have $\psi' = -c \cdot 0 + \phi(\phi - 1) = \phi(\phi - 1)$. Since $\psi'(\phi, 0) = \phi(\phi - 1)$ has no zeros on the interval $(0, 1)$ a simple check implies that we have $\psi' \leq 0$ on the line OA. Therefore, any solution must pass this line in a downwards direction, which implies that a solution cannot leave the triangle OAB through the line OA.

On the line AB we have $\psi \in [-b, 0]$. Therefore, we have $\phi' = \psi \leq 0$. Thus, the direction of the flow again points towards the interior of the triangle, indicating that a solution cannot leave the triangle OAB through the line AB.

Lastly, we must show that an orbit cannot leave the triangle OAB through the line OB. To show this we define the function L of ϕ and ψ as $L : (\phi, \psi) \mapsto b\phi + \psi$. For all the points (ϕ, ψ) left of the line OB it holds that $L(\phi, \psi) < 0$ and for all the points (ϕ, ψ) to the right of the line OB we have $L(\phi, \psi) > 0$ (see Figure 3). Therefore, in order for an orbit $q : (\phi(t), \psi(t))$ to exit the triangle OAB through the line OB, we must have $\frac{d}{dt}L(\phi(t), \psi(t)) < 0$ on a point P where q passes through OB, see Figure 3. We will show that for every c there is a value of b (and thus a point B) for which this cannot happen:

$$\frac{d}{dt}L(\phi(t), \psi(t)) = b\phi'(t) + \psi'(t) = b\psi - c\psi + \phi(\phi - 1) = (b - c)\psi + \phi(\phi - 1).$$

Since P lies on OB we have $\psi = -b\phi$ and therefore:

$$\frac{d}{dt}L(\phi(t),\psi(t)) = (b-c)(-b\phi) + \phi(\phi-1) = -b^2\phi + bc\phi + \phi^2 - \phi$$

$$= \phi(b(c-b) + \phi - 1).$$

Now we choose $b = c/2$:

$$= \phi\left(\frac{c}{2}\left(c - \frac{c}{2}\right) + \phi - 1\right) = \phi\left(\frac{c^2}{4} + \phi - 1\right).$$

Note that since we are considering the case $c \geq 2$, we have $c^2/4 - 1 \geq 0$. This implies $\frac{c^2}{4} + \phi - 1 > 0$, since $\phi > 0$. Therefore, we have $\frac{d}{dt}L(\phi(t),\psi(t)) = \phi\left(\frac{c^2}{4} + \phi - 1\right) > 0$, and no orbit can exit the triangle OAB through OB.

Let ω be the ω-limit set of the orbit on the unstable manifold of $(1,0)$. (See Meiss (2007) Paragraph 4.9 for the definition of an ω-limit set). The orbit on the part of unstable manifold left of of (1,0) cannot exit OAB. Therefore, ω is contained in the bounded set OAB, and, thus, not empty, compact, and connected. Therefore, the Poincaré-Bendixson theorem (Teschl (2012) Thm 7.16) applies.

The Poincaré-Bendixon theorem implies that ω must either be a fixed point, a periodic orbit, or a set of heteroclinic orbits, homoclinic orbits, and fixed points. Let (ϕ^*, ψ^*) be a solution that lies on the part of the unstable manifold of $(1,0)$ that lies strictly left of $(1,0)$. Since we have $\phi' > 0$ inside OAB, ϕ^* is monotone. The first coordinate of the solution ϕ^* is also bounded by OAB, thus, as a consequence of the monotone convergence theorem, it has a limit. By the Poincaré-Bendixon theorem this limit must be the first coordinate of a fixed point. The limit is not equal to one, and, since there are only two fixed points, it is equal to zero.

Note that $\lim_{t\to-\infty}(\phi(t),\psi(t)) = (1,0)$ and $\lim_{t\to\infty}(\phi(t),\psi(t)) = (0,0)$. Therefore, the wave profile function ϕ^* the traveling wave satisfies $\lim_{\xi\to\infty}\phi^*(\xi) = 0$ and $\lim_{\xi\to-\infty}\phi^*(\xi) = 1$.

3 Linear stability analysis of TW solutions of the Fisher-KPP equation

In this section, we examine the stability of the traveling wave solutions found in Section 2. Stability of a solution is important, since unstable solutions never occur in practice. Imagine a pyramid standing on its apex. If we analyze a mathematical model of the dynamics of a pyramid in a gravitational field, we will find a stationary solution where the pyramid is standing on its apex, which would suggest that a pyramid can stand on its apex. Although in theory such a solution exists, it is not observed in practice. The problem is that such a solution is unstable, as is the pyramid standing on its apex. Therefore, we must examine the stability of the traveling wave solutions we found earlier. Suppose the population dynamics can be reliably modeled with the FKPP equation. If a traveling wave solution is stable, one would expect to see the dynamics of the population in the wild closely resemble the traveling wave, if the initial population density resembles an initial condition that gives rise to a traveling wave in the FKPP equation.

In the context of population dynamics, stability of traveling wave solutions means that a plague of an invasive species will not change, if, for instance, a small group of individuals is caught.

The stability analysis of the traveling wave can be compared to the stability analysis of a fixed point of an ordinary differential equation (ODE). A fixed point of an ODE is a stationary solution of an ODE. Similarly, one can do stability analysis on stationary solutions of PDEs, which do not depend on time.

However, traveling waves are not stationary solutions. Therefore, we apply a coordinate transformation to a reference frame that moves along with the wave. We write $\phi(\xi, \tau) = u(x - ct, t)$ and then substitute this into $u_t = u_{xx} + u(1 - u)$. We first calculate the derivative ϕ_t:

$$\phi_t = \frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial \phi}{\partial \tau} = -c\phi_\xi + \phi_\tau$$

and thus:

$$\phi_\tau = \phi_{\xi\xi} + c\phi_\xi + \phi(1 - \phi)$$

Since there is no longer a need to distinguish between τ and t we will use only t:

$$\phi_t = \phi_{\xi\xi} + c\phi_\xi + \phi(1 - \phi) \tag{7}$$

Let ϕ^* be a traveling wave solution of $u_t = u_{xx} + u(1 - u)$. The new reference frame is moving along with the traveling wave solution. Therefore, we have $(\phi^*)_t = 0$, i.e. ϕ^* is stationary and we have:

$$\phi^*_{\xi\xi} + c\phi^*_\xi + \phi^*(1 - \phi^*) = 0 \tag{8}$$

To analyze the stability of traveling waves with must define what stability is. Stability can be defined analogously to Lyapunov stability of a fixed point of an ODE.

Definition 3.1. Let $u_t = f(u)$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ a C^1 map and $u \in \mathbb{R}^n$ be a system of ordinary differential equations and let $x \in \mathbb{R}^n$ be a fixed point of this system. Then x is Lyapunov stable if for every neighborhood $M \subset \mathbb{R}^n$ of x there is a neighborhood $N \subset \mathbb{R}^n$ of x such that any solution that with an initial condition in N remains in M for all $t \geq 0$

If we could define neighborhoods of a stationary solution, then we could modify this definition to suit stationary solutions of a traveling wave. For ODEs, this is straightforward, because the neighborhoods of a point in $\mathbb{R}^n$ can be induced by a norm, and since all norms in $\mathbb{R}^n$ are equivalent, the definition of Lyapunov stability does not depend on the choice of norm.

However, the situation for spaces of functions is different. Not all norms on spaces of functions are equivalent. Therefore, if we generalize the definition of Lyapunov stability to stationary solutions of PDEs, the definition of stability will depend on the choice of the norm. Therefore, there is no unambiguous definition of stability for stationary solutions of PDEs.

Thus, the first step in stability analysis is to choose a norm. We will use a standard norm, namely the supremum norm. The supremum norm is defined on the

space of bounded and continuous functions $B_0(\mathbb{R})$. Let $f \in B_0(\mathbb{R})$. Then the norm of f is $\|f\| = \sup_{x\in\mathbb{R}} f(x)$. Conversely, a norm could define a function space as the set of all functions with finite norm.

To examine the stability of traveling wave ϕ^* we analyze solutions of the form:

$$\phi(\xi, t) = \phi^*(\xi) + \overline{v}(\xi, t)$$

with:

$$\overline{v}(\xi, t) = v(\xi)e^{\lambda t} \text{ with } \lambda \in \mathbb{C} \text{ and } v \in B_0(\mathbb{R})$$

We choose these functions to perturb the traveling waves, because these functions form an orthonormal basis for the L^2 space. This is similar to stability analysis of ODEs. Let v_i be the eigenvectors and λ_i be the eigenvalues of the matrix a linearized ODE. Then the space of solutions of the linearized ODE is spanned by the solutions $v_i e^{\lambda_i t}$. Indeed, we call λ an eigenvalue and v an eigenfunction of a linear differential operator L, that we will define later. The stability of the traveling wave ϕ^* is determined by the spectrum of the operator L. In the context of this thesis, the spectrum of L is equal to the set of all eigenvalues of L, as we will argue later.

We substitute $\phi(\xi, t) = \phi^*(\xi) + \overline{v}(\xi)e^{\lambda t}$ into (7). For the left hand side we find:

$$\phi_t = (\phi^*)_t + \overline{v}_t = 0 + \lambda v(\xi)e^{\lambda t}$$

And for the right hand side we find:

$$(\phi^*)_{\xi\xi} + \overline{v}_{\xi\xi} + c(\phi^*)_\xi + c\overline{v}_\xi + (\phi^* + \overline{v})(1 - \phi^* - \overline{v})$$

$$= (\phi^*)_{\xi\xi} + v_{\xi\xi}e^{\lambda t} + c(\phi^*)_\xi + cv_\xi e^{\lambda t} + \phi^*(1 - \phi^*) - \phi^*\overline{v} + \overline{v}(1 - \phi^* - \overline{v})$$

$$= (\phi^*)_{\xi\xi} + c(\phi^*)_\xi + \phi^*(1 - \phi^*) + v_{\xi\xi}e^{\lambda t} + cv_\xi e^{\lambda t} + \overline{v}(1 - \phi^* - \overline{v}) - \phi^*\overline{v} = (*)$$

Now we use (8):

$$(*) = v_{\xi\xi}e^{\lambda t} + cv_\xi e^{\lambda t} + \overline{v}(1 - \phi^* - \overline{v}) - \phi^*\overline{v}$$

$$= v_{\xi\xi}e^{\lambda t} + cv_\xi e^{\lambda t} + \overline{v} - 2\overline{v}\phi^* - \overline{v}^2$$

$$= v_{\xi\xi}e^{\lambda t} + cv_\xi e^{\lambda t} + ve^{\lambda t} - 2v\phi^* e^{\lambda t} - v^2 e^{2\lambda t}$$

$$= e^{\lambda t}(v_{\xi\xi} + cv_\xi + v(1 - 2\phi^*)) - v^2 e^{2\lambda t}$$

Analogous to stability analysis of ODEs, the perturbations are small. Therefore, the square of the perturbation $v^2 e^{2\lambda t}$ will be even smaller. Thus, we ignore this nonlinear term $-v^2 e^{2\lambda t}$, and we obtain a linear ODE:

$$\lambda v = v_{\xi\xi} + cv_\xi + (1 - 2\phi^*)v$$

The next step is again analogous to ODE stability. In ODE stability, we would proceed to analyze a linear map, specifically, the map induced by the Jacobian matrix. In our stability analysis of the traveling wave (TW), we did not find a matrix, but λ in our problem is still an eigenvalue of the linear differential operator L which we define as:

$$Lv := v_{\xi\xi} + cv_\xi + (1 - 2\phi^*)v,$$

and corresponding to the eigenvalue λ, we label v an eigenfunction. As a side note, L is linear, but not bounded as an operator on $B_0(\mathbb{R})$. Now that we have introduced the operator L we can precisely define what stability of ϕ^* is. Before we do that, we explain why this definition is not a direct generalization of Definition 2.1.

Note that, in theory, the eigenvalues of L, would imply stability in the same way as the eigenvalues of a linear operator on a finite dimensional space, that one would find when analyzing the stability of an ODE. That is, if all the eigenvalues have negative real part, then one finds stability. However, we will see that zero is always an eigenvalue of L. Therefore, we cannot draw a conclusion about stability directly form the spectrum of L. For this reason, we will consider a wave that shifts in space under perturbation stable. This type of stability is called orbital stability (Sandstede, 2002, page 4).

In Section 4, we will analyze the stability of TW solutions for perturbations for a family of spaces $B_\mu(\mathbb{R})$. These spaces are defined by a norm $\|\cdot\|_{B_\mu(\mathbb{R})}$, which we define here:

Definition 3.2. Let $\mu > 0$. The norm $\|\cdot\|_{B_\mu(\mathbb{R})}$ is defined by:

$$\|v\|_{B_\mu(\mathbb{R})} := \sup_{\xi\in\mathbb{R}} |v(\xi)|e^{\mu\xi}$$

Then, we define family of spaces $B_\mu(\mathbb{R})$ induced by $\|\cdot\|_{B_\mu(\mathbb{R})}$, and prove the $B_\mu(\mathbb{R})$ is a Banach space for all $\mu > 0$.

Definition 3.3. Let $\mu > 0$.

$$B_\mu(\mathbb{R}) := \{v \in B_0(\mathbb{R}) : \|v\|_{B_\mu(\mathbb{R})} < \infty\} = \{v \in B_0(\mathbb{R}) : \sup_{\xi\in\mathbb{R}} |v(\xi)e^{\mu\xi}| < \infty\}$$

Proposition 3.1. The normed space $\left(B_\mu(\mathbb{R}), \|\cdot\|_{B_\mu(\mathbb{R})}\right)$ is a Banach space, i.e. every Cauchy sequence $(v_n)_{n\geq 0}$ in this normed space has a limit $v \in B_\mu(\mathbb{R})$.

Proof. By Theorem 2.11(b) of (Rynne and Youngson, 2000) we have $\|v\|_{B_\mu(\mathbb{R})} = \lim_{n\to\infty} \|v_n\|_{B_\mu(\mathbb{R})}$. The sequence $(v_n)_{n\geq 0}$ is Cauchy, thus we have

$$\forall\epsilon > 0 \;\; \exists N > 0 \;\; \forall n, m > N \quad \|v_n - v_m\|_{B_\mu(\mathbb{R})} < \epsilon$$

Theorem 2.11(a) of (Rynne and Youngson) implies $\left|\|v_n\|_{B_\mu(\mathbb{R})} - \|v_m\|_{B_\mu(\mathbb{R})}\right| \leq \|v_n - v_m\|_{B_\mu(\mathbb{R})}$. Therefore, the sequence $\left(\|v_n\|_{B_\mu(\mathbb{R})}\right)_{n\geq 0}$ is Cauchy in $(\mathbb{R}, |\cdot|)$, and thus it has a limit in $\mathbb{R}$, from which we conclude $\|v\|_{B_\mu(\mathbb{R})} = \lim_{n\to\infty} \|v_n\|_{B_\mu(\mathbb{R})} < \infty$ and $v \in B_\mu(\mathbb{R})$. □

A last remark before we define stability, is that we will use this defintion for solutions of the FKPP equation with Allee effect (equation (15)) as well. Therefore, the definition also refers to equation (20), which is the equivalent of equation (15) in a co-moving frame.

Definition 3.4. Let $\mu \geq 0$. A solution ϕ^* of (7) or (20) is stable for perturbations from the space $B_\mu(\mathbb{R})$, if there exists an $\epsilon > 0$ such that for any solution $\phi(\xi, t)$ of (7) or (20) with initial condition:

$$\phi(\xi, 0) = \phi^*(\xi) + v(\xi)$$

with $v \in B_\mu(\mathbb{R})$ satisfying $\|v\|_{B_\mu(\mathbb{R})} \leq \epsilon$, there exists $\theta \in \mathbb{R}$ and $K, \nu > 0$ such that

$$||\phi(\,\cdot\,, t) - \phi^*(\,\cdot\, + \theta)||_{B_\mu(\mathbb{R})} = \sup_{\xi \in \mathbb{R}} |\phi(\xi, t) - \phi^*(\xi + \theta)| e^{\mu\xi} \leq K e^{-\nu t}$$

To work with this definition we give a proposition:

Proposition 3.2. Let $P : \mathbb{R} \to \mathbb{R}$ be a polynomial function. Let L be the differential operator of the eigenvalue problem associated the stability analysis of the stationary solution ϕ^* and suppose L satisfies:

$$Lv = v_{\xi\xi} + cv_\xi + P'(\phi^*)v$$

Then, ϕ^* is stable for perturbations from $B_\mu(\mathbb{R})$, if L satisfies the following conditions:

1. L has a simple eigenvalue 0
2. $\sigma(L) \setminus \{0\}$ is contained in $\{\lambda \in \mathbb{C} \mid (\mathrm{Im}(\lambda)^2 + \mathrm{Re}(\lambda) + \alpha < 0)\}$ for some $\alpha > 0$.

Proof. Theorem 4.1 of Sattinger (1976) directly proves this proposition. All we need to do is to check the conditions of this theorem. The first condition is that L must satisfy the conditions of Lemma 3.4 of the same article. Condition (i) of Lemma 3.4 is exactly the hypothesis of the proposition we are proving, and it can be proved that L satisfies condition (ii) by using Theorem 5.6 of the article. The conditions of Theorem 5.6 requires that the coefficient of the term v_ξ, which is assumed to be dependent on ξ in the article, approaches limits for $\xi \to \pm\infty$ sufficiently fast. This condition is trivially satisfied in our case, since the coefficient of the term v_ξ is constant. □

In the next subsection, we determine whether L satisfies the conditions of this proposition.

3.1 The spectrum of L

To analyze the spectrum of L, we must find solutions of the ODE $Lv = \lambda v$, as for every solution $v \in B_0(\mathbb{R})$, there is a (not necessarily unique) $\lambda \in \sigma(L)$. As we did before we write $Lv := v_{\xi\xi} + cv_\xi + (2\phi^*(\xi) - 1) = \lambda v$ as a system of equations:

$$\begin{aligned} v_\xi &= q \\ q_\xi &= -cq + (2\phi^*(\xi) - 1 + \lambda)v \end{aligned}$$

We write this system using a matrix:

$$\begin{pmatrix} v_\xi \\ q_\xi \end{pmatrix} = A(\xi, \lambda) \begin{pmatrix} v \\ q \end{pmatrix} \quad \text{with} \quad A(\xi, \lambda) = \begin{pmatrix} 0 & 1 \\ 2\phi^*(\xi) - 1 + \lambda & -c \end{pmatrix} \tag{9}$$

From Section 2 we know that the wave profile function ϕ^* of a traveling wave solution satisfies $\lim_{\xi\to-\infty}\phi^*(\xi)=1$ and $\lim_{\xi\to\infty}\phi^*(\xi)=0$ for positive c. We use this fact to study system (9). The results of the analysis of the system for $\xi\to-\infty$ and $\xi\to\infty$ can be used, with Theorem 3.3 of (Sandstede) to determine a part of the spectrum of L called the essential spectrum $\sigma_{ess}(L)$ of L. Thus, we write $A_{-\infty}$ for $\lim_{\xi\to-\infty}A(\xi,\lambda)$ and A_∞ for $\lim_{\xi\to\infty}A(\xi,\lambda)$, where:

$$A_{-\infty}=\begin{pmatrix}0 & 1\\ 1+\lambda & -c\end{pmatrix}\quad\text{and}\quad A_\infty=\begin{pmatrix}0 & 1\\ -1+\lambda & -c\end{pmatrix}$$

The eigenvalues of $A_{-\infty}$ are $\gamma^{\pm}_{-\infty}=-\frac{c}{2}\pm\sqrt{\left(\frac{c}{2}\right)^2+\lambda+1}$ and the eigenvalues of A_∞ are $\gamma^{\pm}_{\infty}=-\frac{c}{2}\pm\sqrt{\left(\frac{c}{2}\right)^2+\lambda-1}$. Note that we are now dealing with three sets of eigenvalues, eigenvalues λ of L, and the eigenvalues $\gamma^{\pm}_{\pm\infty}$ of $A_{\pm\infty}$. We will therefore write "γ-eigenvalues" when referring to the eigenvalues of $A_{-\infty}$ and A_∞, and "λ-eigenvalues" when referring to the elements of $\sigma(L)$.

We would like to determine the sign of the real part of the γ-eigenvalues as a function of λ. To do this, we compute the values of λ for which we have $\mathrm{Re}(\gamma^{\pm}_{\infty})=0$, to see where the signs change. For any γ-eigenvalue, $\mathrm{Re}(\gamma)=0$, implies $\gamma=\zeta i$ with $\zeta\in\mathbb{R}$. Thus, we solve $\left|A_{-\infty}-\zeta iI\right|=0$ for λ. We have:

$$\left|A_{-\infty}-\zeta iI\right|=\begin{vmatrix}-i\zeta & 1\\ 1+\lambda & -c-i\zeta\end{vmatrix}=-\zeta^2+ic\zeta-\lambda-1,$$

which is zero for $\lambda=-1-\zeta^2+ic\zeta$. Thus, the set of values of $\lambda\in\mathbb{C}$, for which the real part of the eigenvalues of $A_{-\infty}$ is zero, is a single parameter family of values:

$$\{\lambda\in\mathbb{C}\mid\exists\zeta\in\mathbb{R}\ (\lambda=-1-\zeta^2+ic\zeta)\}. \tag{10}$$

See also Figure 4.

We do the same for the eigenvalues γ_∞ of A_∞. We have:

$$\left|A_{\infty}-\zeta iI\right|=\begin{vmatrix}-i\zeta & 1\\ -1+\lambda & -c-i\zeta\end{vmatrix}=-\zeta^2+ic\zeta-\lambda+1$$

Which yields, by equating to zero, $\lambda=1-\zeta^2+ic\zeta$. As before, we find a single parameter family of values:

$$\{\lambda\in\mathbb{C}\mid\exists\zeta\in\mathbb{R}\ (\lambda=1-\zeta^2+ic\zeta)\}. \tag{11}$$

for which $\mathrm{Re}(\lambda^{\pm}_{\infty})=0$. We have plotted both families of values of λ in Figure 4.

Since the γ-eigenvalues of $A_{\pm\infty}$ are continuous functions of λ, the real part of the γ-eigenvalues can only change sign on the curves (10) and (11). Therefore, these curves divide the complex plane into three regions, labeled I, II, and III in Figure 4, such that in each region the signs of the real parts of the γ-eigenvalues are the same. Thus, all we need to do to find the signs of the real parts of the γ-eigenvalues of all points in one region, is to find the real parts of the γ-eigenvalues in one point in the region.

In the region II, which is the region between the to curves (10) and (11), we choose $\lambda=0$ and we find $\gamma^{\pm}_{-\infty}=-\frac{c}{2}\pm\sqrt{\left(\frac{c}{2}\right)^2-1}$. Note that $c\geq 2$ and that $\mathrm{Re}(\sqrt{a})$ is either

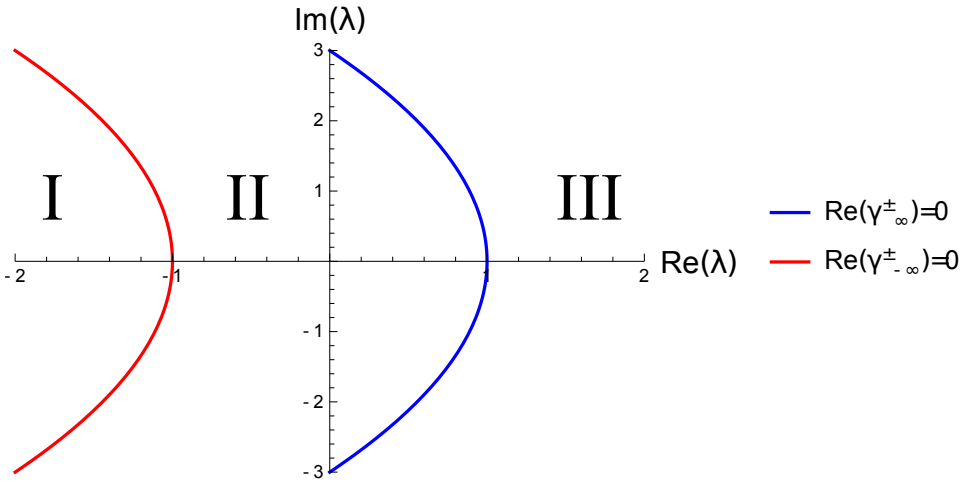

Figure 4: Two curves, one representing to values of λ for which we have $\mathrm{Re}(\gamma^{\pm}_{\infty}) = 0$ (in blue), and one representing the values for which we have $\mathrm{Re}(\gamma^{\pm}_{-\infty}) = 0$ (in red), for $c = 3$.

zero or equal to $\sqrt{a}$ for all $a \in \mathbb{R}$. Therefore, we find $\mathrm{Re}\left(\sqrt{\left(\frac{c}{2}\right)^2 - 1}\right) < \sqrt{\left(\frac{c}{2}\right)^2} = \frac{c}{2}$. Now we can conclude that the real parts of both eigenvalues of A_∞ are negative, since the square root term is less the the first term $-\frac{c}{2}$. The eigenvalues of $A_{-\infty}$ for $\lambda = 0$ are $\gamma^{\pm}_{-\infty} = -\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 + 1}$. We find $\sqrt{\left(\frac{c}{2}\right)^2 + 1} > \sqrt{\left(\frac{c}{2}\right)^2} = \frac{c}{2}$ and therefore there is one positive eigenvalue and one negative eigenvalue.

In conclusion, for all values of λ in region II, $A_{-\infty}$ has exactly one eigenvalue with positive real part and A_∞ has zero eigenvalues with positive real part. In the notation of (Sandstede), the Morse index of $A_{-\infty}$, for which we write $i_-(\lambda)$ is, one and the Morse index of A_∞, $i_+(\lambda)$, is zero, for all λ in region II. Therefore, according to Theorem 3.3 in (Sandstede), all the values of λ in Region II (including its borders) are in the essential spectrum of L, which is part of the spectrum of L.

Note that in Sandstede the spectrum is defined as λ such that $(L - \lambda)$ is not invertible. In general this does not mean that λ is an eigenvalue, i.e. λ is such that $Lv = \lambda v$. Therefore, we will illustrate why values of λ between the two curves are eigenvalues. To do this, we use Theorem 2 from Levinson to prove that solutions of (9) approach solutions of the linear system induced by $A_{\pm\infty}$ for $\xi \to \pm\infty$. In order to apply the theorem we split the matrix $A(\xi, \lambda)$ from system (9) into a the constant matrix A_∞ and a matrix $R_\infty(\xi)$

$$A(\xi, \lambda) = \begin{pmatrix} 0 & 1 \\ 2\phi^*(\xi) - 1 + \lambda & -c \end{pmatrix} = A_\infty + R_\infty$$

$$\text{with } A_\infty = \begin{pmatrix} 0 & -1 \\ -1 + \lambda & -c \end{pmatrix} \text{ (as before) and } R_\infty(\xi) = \begin{pmatrix} 0 & 0 \\ 2\phi^*(\xi) & 0 \end{pmatrix}$$

We do the same for $\xi \to -\infty$. That is, we apply Theorem 2 from Levinson by reversing 'time' (ξ), to conclude that solutions form (9) approach solutions of $A_{-\infty}$

for $\xi \to -\infty$. We find:

$$A(\xi,\lambda) = \begin{pmatrix} 0 & 1 \\ 2\phi^*(\xi) - 1 + \lambda & -c \end{pmatrix} = A_{-\infty} + R_{-\infty}$$

$$\text{with } A_{-\infty} = \begin{pmatrix} 0 & 1 \\ 1+\lambda & -c \end{pmatrix} \text{ (as before) and } R_{-\infty}(\xi) = \begin{pmatrix} 0 & 0 \\ 2\phi^*(\xi) - 2 & 0 \end{pmatrix}$$

Theorem 2 from Levinson requires that all the elements $r_{ij}(\xi)$ of $R_\infty(\xi)$ satisfy

$$\int_0^\infty |r_{ij}(\xi)| d\xi < \infty.$$

Therefore, we must show $\int_0^\infty |2\phi(\xi)|\, d\xi < \infty$ and $\int_\infty^0 |2\phi(\xi) - 2|\, d\xi < \infty$. To do this, we refer to the corollary of the stable manifold theorem in paragraph 2.7, page 115 of Perko. Recall that ϕ^* is a heteroclinic orbit and by definition it lies on the unstable manifold of $(1,0)$ of (6) and on the stable manifold of $(0,0)$ (which is two-dimensional). Hence, the stable manifold theorem applies. The corollary states that any solution on a stable manifold of a hyperbolic fixed point s converges exponentially fast to s for $t \to \infty$, and that any solution on an unstable manifold of a hyperbolic fixed point u converges exponentially to u for $t \to -\infty$. Thus, there are $\epsilon_1, \epsilon_2 > 0$, $\lambda_1 < 0$, $\lambda_2 > 0$ such that:

$$\begin{aligned} \phi(\xi) &\le \epsilon_1 e^{\lambda_1 \xi} && \text{for} \quad \xi > 0 \quad \text{and} \\ 1 - \phi(\xi) = |1 - \phi(\xi)| &\le \epsilon_2 e^{\lambda_2 \xi} && \text{for} \quad \xi < 0 \end{aligned}$$

Therefore, we find:

$$\int_0^\infty |2\phi(\xi)| d\xi = \int_0^\infty 2\phi(\xi) d\xi \le 2\int_0^\infty \epsilon_1 e^{\lambda_1 \xi} d\xi = -\frac{2\epsilon_1}{\lambda_1} < \infty$$

Notice that $-\frac{2\epsilon}{\lambda_1}$ is positive, since λ_1 is negative. We also have:

$$\int_{-\infty}^0 |2\phi(\xi) - 2| d\xi = \int_{-\infty}^0 2 - 2\phi(\xi) d\xi = 2\int_{-\infty}^0 1 - \phi(\xi) d\xi \le 2\int_{-\infty}^0 \epsilon_2 e^{\lambda_2 \xi} d\xi = \frac{2\epsilon_2}{\lambda_2} < \infty$$

Thus, the conditions of Theorem 2 from Levinson are satisfied. Let v_1, v_2 be eigenvectors of $A_{-\infty}$ and let v_3, v_4 be eigenvectors of A_∞. Theorem 2 from Levinson implies that there are solutions x_1, x_2 of (9) such that:

$$x_1 \sim v_1 e^{\gamma_{-\infty}^- \xi} \text{ and } x_2 \sim v_2 e^{\gamma_{-\infty}^+ \xi} \text{ for } \xi \to -\infty,$$

and there are solutions x_3, x_4 of (9) such that:

$$x_3 \sim v_3 e^{\gamma_\infty^- \xi} \text{ and } x_4 \sim v_4 e^{\gamma_\infty^+ \xi} \text{ for } \xi \to \infty.$$

Finally, Theorem 2 for Levinson (1998) also states that v_1, v_2 are linearly independent, as are v_3, v_4.

Note that for λ in region II we have $\gamma_\infty^\pm < 0$. Therefore, x_3 and x_4 tend to zero for $\xi \to \infty$. Since v_3 and v_4 are linearly independent, x_3 and x_4 are linearly independent, and therefore span the space of all solutions. Thus, every solution of (9) tends to zero for $\xi \to \infty$. Therefore, x_1 and x_2 tend to zero for $\xi \to \infty$ as well, and, crucially, x_1 also tends to zero for $\xi \to -\infty$. Since x_1 is also continuous, it is bounded, and thus an eigenfunction associated to an eigenvalue in Region II. Note that region II has elements with positive real part. Therefore, ϕ^* is not stable against perturbations in $B_0(\mathbb{R})$.

4 Stability of traveling waves for a weighted norm

In the previous section we found that the traveling waves constructed in Section 2 are unstable against perturbations in $B_0(\mathbb{R})$. However, the traveling waves might be stable against perturbations from other spaces. Therefore, we analyze the stability using the weighted norm $\|\cdot\|_{B_\mu(\mathbb{R})}$. The new space of perturbations, which is defined by the weighted norm, is:

$$B_\mu(\mathbb{R}) = \{v \in B_0(\mathbb{R}) \ : \ \sup_{\xi\in\mathbb{R}} |v(\xi)e^{\mu\xi}| < \infty\} \text{ with } \mu > 0$$

Note that the stability of the wave will depend on the value of μ. We will determine the set of values of μ for which ϕ^* is stable. For $\mu > 0$ the norm $\|\cdot\|_{B_\mu(\mathbb{R})}$ adds the restriction that v must decay exponentially for $\xi \to \infty$. We will find that the traveling waves are stable under this norm for certain values of $\mu > 0$ and $c > 2$, implying that the part of the where $\phi^*(\xi)$ is closer to one, where $0 \ll \phi^*(\xi) < 1$, is "more stable" than the part where $0 < \phi^*(\xi) \ll 1$

The stability analysis entails that we analyze the spectrum of the differential operator L, which is defined as $Lv := v_{\xi\xi} - cv_\xi + (1 - 2\phi^*)v$.

In the previous section we found instability. Therefore, there was no need to analyze the whole spectrum of L, as we knew ϕ^* was unstable, as soon as we found spectrum with positive real part. Since we expect to find stability for a weighted norm, we would like to find the whole spectrum. We find there are two subsets of $\sigma(L)$, the essential spectrum $\sigma_{ess}(L)$ and the point spectrum $\sigma_p(L)$, each of which has to be considered separately. (Sandstede). We will do this in the next two subsections.

4.1 The essential spectrum σ_{ess} of L

The essential spectrum is the part of the spectrum that is found using the methods of subsection 3.1. More specifically, we use Theorem 3.3 of (Sandstede) to reduce the problem to the asymptotic dynamics as $\xi \to -\infty$ and $\xi \to \infty$.

Take $\mu > 0$. We restrict the space of perturbations to bounded functions v that satisfy:

$$\sup_{\xi\in\mathbb{R}} |v(\xi)e^{\mu\xi}| < \infty$$

To analyze the implications of this restriction we compute the derivatives w_ξ and z_ξ of $w(\xi) = v(\xi)e^{\mu\xi}$ and $z(\xi) = q(\xi)e^{\mu\xi}$, where $q(\xi) = v_\xi(\xi)$:

$$w_\xi = \frac{\partial}{\partial\xi}\left(e^{\mu\xi}v\right) = \mu e^{\mu\xi}v + e^{\mu\xi}q = \mu w + z$$

$$\begin{aligned} z_\xi = \frac{\partial}{\partial\xi}\left(e^{\mu\xi}q\right) &= \mu e^{\mu\xi}q + e^{\mu\xi}((2\phi^*(\xi) - 1 + \lambda)v - cq) \\ &= \mu z + e^{\mu\xi}v(2\phi^*(\xi) - 1 + \lambda) + ce^{\mu\xi}q \\ &= \mu z + w(2\phi^*(\xi) - 1 + \lambda) + ce^{\mu\xi}q \\ &= (\mu - c)z + (2\phi^*(\xi) - 1 + \lambda)w \end{aligned}$$

Writing this in matrix form we have:

$$\begin{pmatrix} w_\xi \\ z_\xi \end{pmatrix} = A(\xi,\mu;\lambda)\begin{pmatrix} w \\ z \end{pmatrix} \quad \text{with} \quad A(\xi,\mu;\lambda) = \begin{pmatrix} 0 & 1 \\ 2\phi^*(\xi) - 1 + \lambda & -c \end{pmatrix} + \mu I \tag{12}$$

Thus, $v \in B_\mu(\mathbb{R})$ solves the eigenvalue problem $Lv = \lambda v$ if and only if $(w,z)^\mathsf{T} = (v, v_\xi)^\mathsf{T} e^{\mu\xi}$ solves (12) and $w, z \in B_0(\mathbb{R})$.

Similar to the stability analysis for the supremum norm, we will use (Sandstede), to reduce the problem of finding the essential spectrum to analyzing the matrix $A(\xi,\mu;\lambda)$ for $\xi \to -\infty$ and $\xi \to \infty$. Letting $A_{\pm\infty} = \lim_{\xi\pm\infty} A(\xi,\mu;\lambda)$, we find:

$$A_{\pm\infty} = \begin{pmatrix} 0 & 1 \\ \mp 1 + \lambda & -c \end{pmatrix} + \mu I$$

As with perturbations in $B_0(\mathbb{R})$, we determine where the γ-eigenvalues have a real part equal to zero by solving $|A_{\pm\infty} - \zeta i I| = 0$ for λ. We have:

$$|A_{\pm\infty} - \zeta i I| = \begin{vmatrix} \mu - i\zeta & 1 \\ \mp 1 + \lambda & \mu - c - i\zeta \end{vmatrix} = \mu^2 - \mu c - 2i\zeta\mu + i\zeta c - \zeta^2 \pm 1 - \lambda$$

Therefore, the real part of the eigenvalues of $A_{\pm\infty}$ is zero for:

$$\lambda = -\zeta^2 \pm 1 + \mu^2 - \mu c + i(\zeta c - 2\zeta\mu) \tag{13}$$

In Section 3.1 we found that the region II (see Figure 4) is part of the spectrum and are eigenvalues. More specifically, for any $\lambda \in \mathbb{C}$ in Region II there exists a bounded solution $(v(\xi), q(\xi))^\mathsf{T}$ of (9) corresponding to an eigenfunction $v \in B_0(\mathbb{R})$ solving $Lv = \lambda v$. Similarly, for any λ between the curves (13), there exists a bounded solution $(w(\xi), z(\xi))^\mathsf{T}$ to (12) corresponding to an eigenfunction $v = e^{\mu\xi} w \in B_\mu(\mathbb{R})$ solving $Lv = \lambda v$.

Since the traveling wave is unstable if there are λ-eigenvalues with real part greater than zero, a necessary, but not sufficient, condition for stability is that the curves (13) both lie left of the origin. We attempt to find a value of μ for which this occurs. The real part of the boundaries (13) of the essential spectrum for the weighted norm differ only by the term $\mu^2 - \mu c$. Recall that TW only exist for $c \geq 2$. The difference $\mu^2 - \mu c$ is maximized for $\mu = \frac{c}{2}$, and the curves shift to the left by $\frac{c^2}{4} = \left(\frac{c}{2}\right)^2$. Therefore, for $c > 2$ and $\mu = \frac{c}{2}$ the curves lie strictly left of the origin, since we have $-\zeta^2 \pm 1 + \mu^2 - \mu c < \pm 1 + \mu^2 - \mu c = \pm 1 + \left(\frac{c}{2}\right)^2 - c\frac{c}{2} = \pm 1 - \frac{c^2}{4} = \pm 1 - \left(\frac{c}{2}\right)^2 < 0$ for $\zeta \in \mathbb{R}$. Thus, for these values of c and μ all elements of the essential spectrum have real part strictly less than zero.

The region left and right of (13) are not part of the essential spectrum. This can be proven using Theorem 3.3 of Sandstede (2002).

Note that for $\mu = \frac{c}{2}$ the curves are $\lambda = -\zeta \pm 1 - \left(\frac{c}{2}\right)^2$, and the these curves lie on the real axis. On the curves the real part of the γ-eigenvalues is zero. Therefore, $A_\pm\infty$ is hyperbolic on the corresponding curve, and, according to Theorem 3.3 in Sandstede (2002), the curves are also part of $\sigma_{ess}(L)$. Thus, for $\mu = \frac{c}{2}$ we have the essential spectrum of L is the real interval $\left(-\infty, 1 - \left(\frac{c}{2}\right)^2\right)$.

4.2 The point spectrum σ_p of L

The essential spectrum is only part of the spectrum of a linear differential operator. To complete our analysis, we must also determine the complete spectrum. The part of the spectrum that is not part of the essential spectrum is called the point spectrum.

To use proposition 3.2, we must prove that zero is the eigenvalue of the operator L in the point spectrum with the largest real part. The spectrum of L with respect to $B_\mu(\mathbb{R})$ is equivalent to the spectrum of an operator L_μ, that we will define now, with respect to $B_0(\mathbb{R})$. The first coordinate of the bounded solutions of (12) are eigenfunctions of an operator L_μ with respect to $B_0(\mathbb{R})$. We will compute the expression for L_μ. From (12) we find:

$$w_\xi = \mu w + z$$

and

$$z_\xi = w_{\xi\xi} - \mu w_\xi(2\phi^* - 1 + \lambda)w + (\mu - c)z$$

Combining these two equations we find:

$$\begin{aligned} w_{\xi\xi} - \mu w_\xi &= (2\phi^* - 1 + \lambda)w + (\mu - c)(w_\xi - \mu w) \\ &= (2\phi^* - 1 + \lambda)w + (\mu - c)w_\xi - (\mu - c)\mu w, \end{aligned}$$

or equivalently:

$$w_{\xi\xi} - (2\mu - c)w_\xi + (1 - 2\phi^* + \mu(\mu - c))w = \lambda w$$

Thus $v = we^{\mu\xi}$ is an eigenfunction of L with respect to $B_\mu(\mathbb{R})$ if and if and only if w is eigenfunction of the operator L_μ with respect to $B_0(\mathbb{R})$ with:

$$L_\mu(w) := w_{\xi\xi} - (2\mu - c)w_\xi + (1 - 2\phi^* + \mu^2 - c\mu)w \tag{14}$$

Next, we show that zero is an eigenvalue of L_μ for $\mu = \frac{c}{2}$ with eigenfunction $\tilde{\phi}^* = \phi^*_\xi e^{\mu\xi}$. For $\mu = \frac{c}{2}$ we have:

$$L_\mu(v) = v_{\xi\xi} + (1 - 2\phi^* - \mu^2)v$$

Thus, it holds that:

$$\begin{aligned} L_\mu(\tilde{\phi}^*) &= \tilde{\phi}^*_{\xi\xi} + (1 - 2\phi^* - \mu^2)\tilde{\phi}^* \\ &= e^{\mu\xi}(\mu^2\phi^*_\xi + 2\mu\phi^*_{\xi\xi} + \phi^*_{\xi\xi\xi}) + e^{\mu\xi}(1 - 2\phi^* - \mu^2)\phi^*_\xi \\ &= e^{\mu\xi}(\phi^*_{\xi\xi\xi} + c\phi^*_{\xi\xi} + (1 - 2\phi^*)\phi^*_\xi) = e^{\mu\xi}((\phi^*_{\xi\xi} + c\phi^*_\xi + \phi^*(1 - \phi^*))_\xi) = e^{\mu\xi} \cdot 0 = 0 \end{aligned}$$

In the last step equation (8) was used.

Therefore, zero is an eigenvalue of L with respect to $B_\mu(\mathbb{R})$. Theorem 2.3.3 of (Kapitula and Promislow) implies that L has a finite number of simple eigenvalues λ_i for $i = 0, ..., N$ in the point spectrum, such that every λ_i has an eigenfunction with i simple zeros. Moreover, the theorem implies that the eigenvalues in the point spectrum are are real and ordered in a strictly descending order:

$$\lambda_0 > \lambda_1 ... > \lambda_N \quad \text{or equivalently:} \quad \lambda_N < ... < \lambda_1 < \lambda_0$$

Since the the traveling wave solution ϕ^* is strictly decreasing, we have $\phi^*_\xi(\xi) < 0$ for all $\xi \in \mathbb{R}$, and the eigenfunction $\phi^*_\xi(\xi)e^{\mu\xi}$ has no zeros. Therefore, theorem 2.3.3 implies that the eigenvalue zero is the largest eigenvalue in the point spectrum. Note that Theorem 2.3.3 is essentially Sturm-Liouville Theory.

4.3 Stability of ϕ^* against perturbations from $B_{\frac{c}{2}}(\mathbb{R})$

Now that the spectrum of L has been investigated, we can apply proposition 3.2. Since the point spectrum has not been determined exactly, we consider two cases.

The first case is $\sigma_p(L) = \{0\}$. In this case it holds $\sigma(L) \setminus \{0\} = \sigma_{ess}$. Recall that for $\mu = \frac{c}{2}$ the the essential spectrum is equal to the interval $\left(-\infty, 1 - \left(\frac{c}{2}\right)^2\right)$. Therefore, L trivially satisfies the conditions of Proposition 3.2 in this case for $c > 2$.

In the other case, $\sigma_p(L) \neq \{0\}$, we still know that all eigenvalues in the point spectrum lie left of the origin, as we argued in the previous section. Moreover, the point spectrum is real and finite, and is there is a second largest eigenvalue λ_1 in $\sigma_p(L)$. Therefore for $\mu = \frac{c}{2}$, all of $\sigma_p(L)$ lies in $\{\lambda \in \mathbb{R} : \lambda < \frac{\lambda_1}{2}\} \subset \{\lambda \in \mathbb{C} : (\mathrm{Im}(\lambda))^2 - \frac{\lambda_1}{2} + \mathrm{Re}(\lambda) < 0\}$. The essential spectrum is equal to the interval $\left(-\infty, 1 - \left(\frac{c}{2}\right)^2\right)$, therefore $\sigma(L) \setminus \{0\}$ is contained in:

$$\left\{\lambda \in \mathbb{C} : (\mathrm{Im}(\lambda))^2 - \max\left(\frac{\lambda_1}{2}, 1 - \left(\frac{c}{2}\right)^2\right) + \mathrm{Re}(\lambda) < 0\right\}$$

and the conditions of Proposition 3.2 are again satisfied.

Therefore, ϕ^* is stable against perturbations from $B_{\frac{c}{2}}(\mathbb{R})$ for $c > 2$.

5 Traveling wave solutions in the Fisher-KPP equation with Allee effect

In this section we will determine if the Fisher-KPP equation with Allee effect (FKPPA) has traveling wave solutions. The FKPPA equation is:

$$u_t = u_{xx} + u(1-u)(u-a) \tag{15}$$

The parameter a is chosen in $(0, \frac{1}{2})$. Systems with values between one half and one need not be studied separately. This is without loss of generality, due to symmetry in the system. We will explain this symmetry in detail. Let $L_a(u) = u_t - (u_{xx} + u(1-u)(u-a)) = u_t - u_{xx} - u(1-u)(u-a)$ be a differential operator. Note that L is unbounded. We have $L_a(u) = 0$ if and only if u is solution of (15). Let u be a solution of (15), then we have $L_{1-a}(1-u) = -u_t + u_{xx} - (1-u)u(1-u-(1-a)) = -u_t + u_{xx} - (1-u)u(-u+a) = -u_t + u_{xx} + u(1-u)(u-a) = -L_a(u) = 0$. Therefore, $1-u$ is a solution on (15) with a parameter value of $1-a$. Thus, if we know a solution of (15) for $a \in (0, \frac{1}{2})$, then we know a solution of of (15) for $1-a \in (\frac{1}{2}, 1)$, and we may restrict our analysis to $a \in (0, \frac{1}{2})$.

As in Section 2, we substitute the wave profile function ϕ applied to $\xi = x - ct$ for u, where $c \geq 0$. We find the ordinary differential equation:

$$-c\phi_\xi = \phi_{\xi\xi} + \phi(\phi-1)(\phi-a)$$

To study the behavior of the solutions, we write this equation as a two-dimensional system of first order equations, as we did in Section 2. We can do this by setting $\psi = \phi'$, as in Paragraph 3.1 of (Braun and Golubitsky):

$$\begin{aligned} \phi' &= \psi \\ \psi' &= -c\psi - \phi(1-\phi)(\phi-a) \end{aligned} \tag{16}$$

The fixed points of this system are $(0,0)$, $(a,0)$, and $(1,0)$ and the Jacobian is:

$$J(\phi,\psi) = \begin{pmatrix} 0 & 1 \\ 3\phi^2 - 2(a+1)\phi + a & -c \end{pmatrix},$$

we have:

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ a & -c \end{pmatrix}, \quad J(a,0) = \begin{pmatrix} 0 & 1 \\ a(a-1) & -c \end{pmatrix} \quad \text{and} \quad J(1,0) = \begin{pmatrix} 0 & 1 \\ 1-a & -c \end{pmatrix}$$

Let $\lambda^{\pm}(\phi,\psi)$ be the eigenvalues of $J(\phi,\psi)$. We have:

$$\begin{aligned}
\lambda^{\pm}(0,0) &= -\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 + a} \\
\lambda^{\pm}(a,0) &= -\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 + a(a-1)} \\
\lambda^{\pm}(1,0) &= -\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 + 1 - a}
\end{aligned}$$

Since we assume $a > 0$, (0,0) is a saddle point for all $c \in \mathbb{R}$. Likewise, since we assume $a < 1$, (1,0) is a saddle point as well for all $c \in \mathbb{R}$. The character of the fixed point $(a,0)$ depends on c. Specifically, if $\left(\frac{c}{2}\right)^2 + a(a-1) > 0$, or equivalently $c > 2\sqrt{a(1-a)}$ holds, then $(a,0)$ is a stable node, while it is a stable spiral, if $0 < c < 2\sqrt{a(1-a)}$ holds.

To find a solution of system (16), we make the ansatz $\psi = b\phi(1-\phi) = b(\phi - \phi^2)$, i.e. we assume that there is a heteroclinic orbit in the shape of a parabola. We calculate the derivative of ψ' using this equation, and then proceed with substitutions and algebraic manipulations, until we either derive a contradiction, or we find values for b and c:

$$\begin{aligned}
\psi' &= b(1-2\phi)\phi' \\
&= b(1-2\phi)\psi \\
&= b(1-2\phi)b\phi(1-\phi) \\
&= (b - 2b\phi)(b\phi - b\phi^2) \\
&= 2b^2\phi^3 - 3b^2\phi^2 + b^2\phi
\end{aligned} \tag{17}$$

From (16) we also have $\psi' = -c\psi - \phi(1-\phi)(\phi - a)$, that is,

$$\begin{aligned}
\psi' &= -c\psi - \phi(1-\phi)(\phi - a) \\
&= -c(b\phi - b\phi^2) - \phi(1-\phi)(\phi - a) \\
&= -bc\phi + bc\phi^2 - (\phi - \phi^2)(\phi - a) \\
&= -bc\phi + bc\phi^2 - (\phi^2 - a\phi - \phi^3 + a\phi^2) \\
&= -bc\phi + bc\phi^2 - \phi^2 + a\phi + \phi^3 - a\phi^2
\end{aligned} \tag{18}$$

Equating (17) and (18) gives:

$$-bc\phi + bc\phi^2 - \phi^2 + a\phi + \phi^3 - a\phi^2 = 2b^2\phi^3 - 3b^2\phi^2 + b^2\phi,$$

or equivalently:

$$-bc\phi + bc\phi^2 - \phi^2 + a\phi + \phi^3 - a\phi^2 - 2b\phi^3 + 3b^2\phi^2 - b^2\phi = 0.$$

Thus, we arrive at:

$$(1 - 2b^2)\phi^3 + (bc - 1 - a + 3b^2)\phi^2 + (-bc + a - b^2)\phi = 0 \tag{19}$$

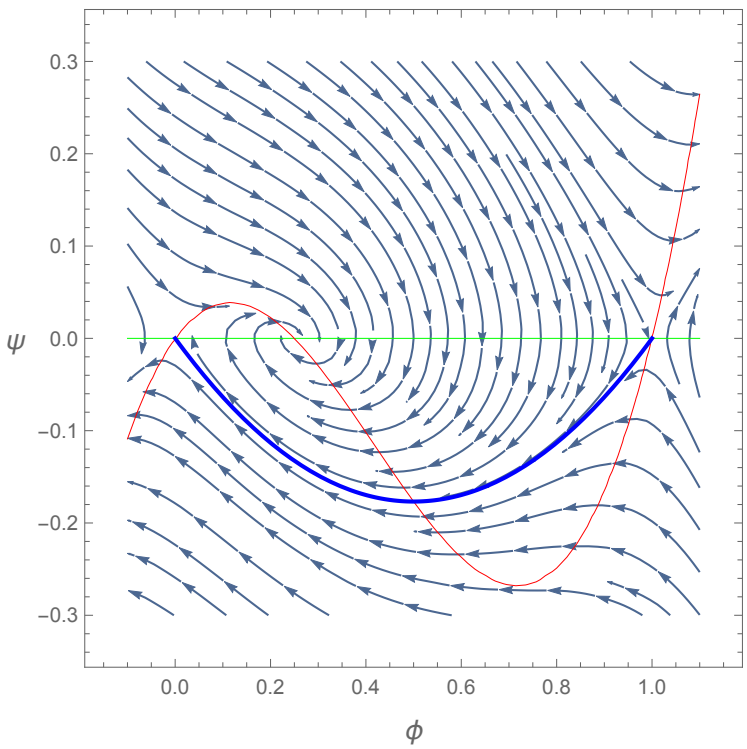

Figure 5: A phase plot showing the parabolic heteroclinic orbit corresponding to the traveling wave for $a = \frac{1}{4}$ and $c = \frac{1}{4}\sqrt{2}$

Since this last equation is valid for all ϕ we have in particular $1 - 2b^2 = 0$, and therefore $b = \pm\frac{1}{2}\sqrt{2}$. Using the same argument we find $-bc + a - b^2 = 0$, $bc = a - b^2$, hence:

$$c = \frac{a - b^2}{b} = \frac{a - \frac{1}{2}}{b}.$$

Note that the coefficient of ϕ^2 in (19) is a linear combination of the other coefficients, thus it neither contradicts previous conclusions, nor adds any new restriction.

Since we choose $a \in (0, \frac{1}{2})$, we have $\frac{1}{2} - a \leq 0$. Thus, if we want $c > 0$, we should choose $b < 0$. Choosing b as such, we find $b = -\frac{1}{2}\sqrt{2}$ and $c = \sqrt{2}(\frac{1}{2} - a)$.

Note that for $a = \frac{1}{2}$ we find $c = 0$, and thus a stationary solution of the FKPP equation with Allee effect. Also, for a fixed a we find only one value of c for which there is a TW solution, in contrast to the FKPP equation, where we find a continuum of TW solutions for $c \geq 2$.

6 Linear stability analysis of TW solutions of the FKPP equation with Allee effect

We will analyze the stability of traveling waves of the FKPP equation with Allee effect (FKPPA), using the same methods as in Section 3.

The first step is to transform the equation $u_t = u_{xx} + u(1-u)(u-a)$ to a moving reference frame. As before, we let $\xi = x - ct$ and $\phi(\xi, t) = u(x - ct, t)$. For convenience, we let $P(\phi) = \phi(1-\phi)(\phi - a)$.

$$\phi_t = \phi_{\xi\xi} + c\phi_x + \phi(1-\phi)(\phi - a) = \phi_{\xi\xi} + c\phi_x + P(\phi) \tag{20}$$

Like before, we denote the traveling wave solution of the transformed system by ϕ^*. Note that ϕ^* is a stationary solution of (**??**). The next step is to define a perturbation $v \in B_0(\mathbb{R})$ of ϕ^* and analyze the set of solutions of the form $\overline{v} = \phi^* + ve^{\lambda t}$. By substituting this into (20), applying $(\phi^*)_t = 0$ and linearizing with respect to v we find the linear operator L on $B_0(\mathbb{R})$ defined by:

$$Lv := v_{\xi\xi} + cv_\xi + vP'(\phi^*) = \lambda v \tag{21}$$

The corresponding 2-dimensional first order ODE is:

$$\begin{pmatrix} v \\ v_\xi \end{pmatrix}_\xi = A(\xi, \lambda) \begin{pmatrix} v \\ v_\xi \end{pmatrix} \quad \text{with} \quad A(\xi, \lambda) = \begin{pmatrix} 0 & 1 \\ \lambda - P'(\phi^*) & -c \end{pmatrix} \tag{22}$$

As in Section 3, we will determine the essential spectrum by analyzing $A_{\pm\infty} = \lim_{\xi \to \pm\infty} A(\xi, \lambda)$ and the point spectrum of L using Sturm-Liouville Theory.

6.1 The essential spectrum σ_{ess} of L

To find the essential spectrum of L, we determine the eigenvalues $\gamma^{\pm}_{\pm\infty}$ of $A_{\pm\infty}$, and the curves where the real parts of these eigenvalues are zero. The eigenvalues of $A(\lambda)_{\pm\infty}$ are:

$$\gamma^{\pm}_{-\infty} = -\frac{c}{2} \pm \sqrt{\frac{c}{2} + a + \lambda} \quad \text{and} \quad \gamma^{\pm}_{\infty} = -\frac{c}{2} \pm \sqrt{\frac{c}{2} + 1 - a + \lambda}$$

The curves in the complex plane for which the real parts of the eigenvalues of $A_{\pm\infty}$ are zero is determined by solving $|A(\lambda)_{\pm\infty} - \zeta i| = 0$ for λ. We have:

$$|A(\lambda)_{-\infty} - \zeta i| = \begin{vmatrix} -\zeta i & 1 \\ a + \lambda & -c - \zeta i \end{vmatrix} = ic\zeta - \zeta^2 - a - \lambda$$

and:

$$|A(\lambda)_{\infty} - \zeta i| = \begin{vmatrix} -\zeta i & 1 \\ 1 - a + \lambda & -c - \zeta i \end{vmatrix} = ic\zeta - \zeta^2 + a - 1 - \lambda$$

And thus, the real parts of the eigenvalues of $A_{-\infty}$ are zero for

$$\lambda = -a - \zeta^2 + ic\zeta \tag{23}$$

and the real parts of the eigenvalues of A_{∞} are zero for

$$\lambda = a - 1 - \zeta^2 + ic\zeta. \tag{24}$$

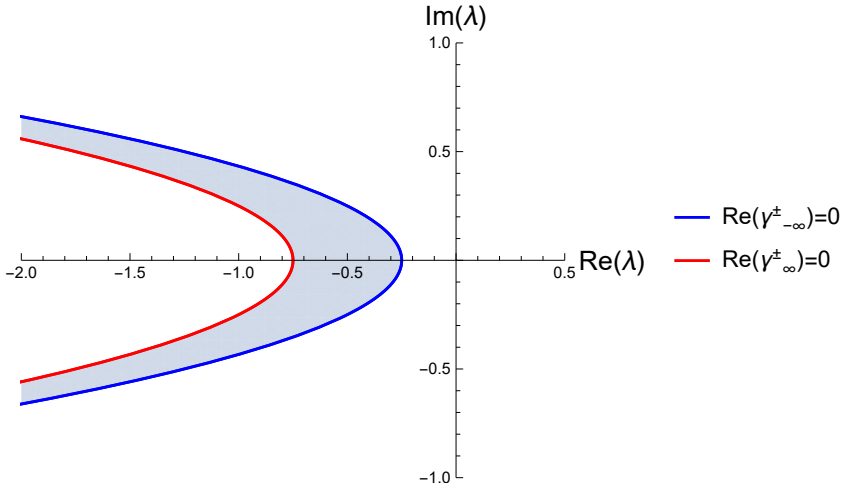

Figure 6: A plot of $\sigma_{ess}(L)$ for $a = \frac{1}{4}$ and $c = \sqrt{2}(\frac{1}{2} - a) = \frac{1}{4}\sqrt{2}$, with the essential spectrum shaded in blue.

Note that curves (23) and (24) lie strictly left of the origin, since we assume $a > 0$ which implies $-a < 0$, which places (23) left of the origin, and since we assume $a < \frac{1}{2}$ we have $a - 1 < \frac{1}{2}$, which places (24) left of the origin. Theorem 3.3 of Sandstede implies that the essential spectrum lies on and between the curves (23) and (24) . Therefore, the essential spectrum of L lies strictly left of the origin. See also Figure 6 for a plot of $\sigma_{ess}(L)$ for $a = \frac{1}{4}$ and $c = \sqrt{2}\frac{1}{2} - a = \frac{1}{4}\sqrt{2}$.

6.2 The point spectrum σ_p of L

As we saw in Subsection 4.2, the point spectrum of L is determined by applying theorem 2.3.3 of (Kapitula and Promislow), and, find that zero is an eigenvalue again with eigenfunction $(\phi^*)_\xi$. As before, the traveling wave solution is stationary, and therefore we have $\phi^*_{\xi\xi} + c\phi^*_\xi + P(\phi^*) = 0$. Thus it holds that:

$$0 = (\phi^*_{\xi\xi} + c\phi^*_\xi + P(\phi^*))_\xi = \phi^*_{\xi\xi\xi} + c\phi^*_{\xi\xi} + P'(\phi^*)\phi^*_\xi = L((\phi^*)_\xi)$$

The eigenfunction ϕ^* is again strictly decreasing. Therefore, the eigenfunction ϕ^*_ξ has no zeroes. Applying Theorem 2.3.3, we find that the point spectrum is real, that zero is the largest eigenvalue in the point spectrum, and that zero is an isolated, simple eigenvalue.

6.3 Stability of ϕ^* against perturbations from $B_0(\mathbb{R})$

To show that L satisfies the conditions of Proposition 3.2, we prove the following:

Proposition 6.1. The essential spectrum of L is contained in

$$\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) + a + \mathrm{Im}(\lambda)^2 < 0\}.$$

Proof. The essential spectrum lies left of and on the curve $\lambda = -a - \zeta^2 + ic\zeta$ (23), which implies that for any $\lambda \in \sigma_{ess}(L)$ there is a $\zeta \in \mathbb{R}$ such that $\mathrm{Re}(\lambda) \leq -a - \zeta^2$, or

equivalently $\mathrm{Re}(\lambda)+a+\zeta^2 \leq 0$, and $|\mathrm{Im}(\lambda)| \leq c\zeta$. Thus, we find $\mathrm{Im}(\lambda)^2 \leq c^2\zeta^2$. We assume $a > 0$ and in section 5 we showed $c = \sqrt{2}(\frac{1}{2} - a)$. Therefore, $c < \frac{1}{2}\sqrt{2} < 1$ holds, from which we conclude $\mathrm{Im}(\lambda)^2 \leq c^2\zeta^2 < \zeta^2$, and $\mathrm{Re}(\lambda) + a + \mathrm{Im}(\lambda)^2 < \mathrm{Re}(\lambda) + a + \zeta^2 \leq 0$ for all $\lambda \in \sigma_{ess}(L)$. □

We have shown in the previous subsection that the first condition of Proposition 3.2, which requires that zero is an eigenvalue of L, holds. As in Subsection 4.3, we discern two cases for the second condition. If $\sigma_p = \{0\}$, then the second condition of Proposition 3.2 is satisfied by Proposition 6.1. If $\sigma_p \neq \{0\}$, then let λ_1 be second largest eigenvalue. Then the second condition of Proposition 3.2 is satisfied by $\sigma(L) \setminus \{0\} \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) + \min\{-\frac{\lambda_1}{2}, a\} + \mathrm{Im}(\lambda)^2 < 0\ \}$.

Therefore, the entire family of TW solutions we found in Section 5 are stable against perturbations from $B_0(\mathbb{R})$. This is in contrast to the TW solution we found for the FKPP equation, which is stable against perturbations for $B_\mu(\mathbb{R})$ for $\mu = \frac{c}{2}$, but not for $\mu = 0$. Therefore one could consider solutions of the FKPPA equation to be "more stable" than solution of the equation.

7 Simulation of Traveling waves by numerical methods

In this section we will discuss two methods of simulating traveling waves by numerically. The first method is to use the PDEPE function (Mathworks inc.). The PDEPE function solves parabolic-eliptic initial-boundary value problems. To use the PDEPE function, one must specify a mesh of points on the spatial domain an a mesh of points on the time domain.

The PDEPE function will choose the appropriate algorithm and the appropriate time step to achieve stable and consistent time integration. It may therefore choose a smaller time step than the one implied by the user in the mesh of points in the time domain.

Another method we will discuss is simulating traveling waves on a 2D spatial domain by choosing a specific discretization scheme.

7.1 Simulation in one spatial dimension using pdepe

In order for a problem with a differential equation to be well-posed, one must choose an initial condition and boundary conditions. A necessary condition for well-posedness is the existence of a unique solution. In Section 2 we have derived that the quadratic Fisher-KPP equation has a traveling wave solution for every $c \geq 2$. Thus it is clear that the Fisher-KPP equation by itself is not well-posed.

An initial condition must be chosen for a PDE, much like an initial value for an ODE, but in the case of a PDE the initial condition is a function over the spatial domain rather than a value.

When running simulation boundary conditions are not just a matter of well-posedness, but also a practical matter. Since a simulation can only cover a finite amount of space, we choose a spatial interval $I \subset \mathbb{R}$ on which we run the simulation. The derivatives on the boundary of I cannot be computed in the same way as in the interior of the interval. Therefore, we set boundary conditions, which specify the derivatives. There are various ways to do this.

One way is to specify the value of $u(x,t)$ for $x \in \partial I$ and $t \in \mathbb{R}$. This type of boundary condition are called a Dirichlet boundary condition.

Another type of boundary condition, where the partial derivative $\frac{\partial u}{\partial x}$ is specified on ∂I, is called a Neumann boundary condition. This type of boundary condition is useful for simulating a population on a bounded habitat, where individuals cannot escape or enter the habitat. Such a "no flux" condition can be specified by setting $\frac{\partial u}{\partial x} = 0$.

Suppose we want to investigate the dynamics of the system on an infinite interval. In this case we choose an interval I that is big enough to see the development of a traveling wave. We will stop the simulation when the traveling wave reaches the boundary of the interval. This prevents the simulation being influenced by effects of the boundary conditions.

Therefore, the type of boundary conditions we choose does not matter as long the traveling waves do not reach the boundaries of our interval. Thus, for simplicity, will we choose $u(x,t) = 0$ for $x \in \partial I$.

Figure 7 shows the results of a simulation. In this figure we plotted the relative population density u for at various times. The initial condition is zero everywhere except near $x = 50$, where the initial condition is $(x - 50\frac{1}{2})(49\frac{1}{2} - x)$, i.e. a parabola with vertex at $(u,x) = (50, \frac{1}{4})$. We see two traveling waves, both starting near $x = 50$, with one going left, and one going right. One would see such a pattern when introducing a species into a new habitat. Note that the wave speed of both traveling waves is $c = 2$, as indicated in Figure 7, which is consistent with the finding of Kolmogorov et al. that any solution with an initial condition with finite support, consists of two traveling waves.

7.2 Simulation in two spatial dimensions

The matlab function PDEPE only simulates PDEs in one spatial dimension. (Mathworks inc.) Therefore, it is necessary to discretize the Fischer-KPP equation manually if we want to simulate it over a two-dimensional spatial domain.

The approach we will use is to approximate the Laplacian with a difference quotient M. Since we are simulating in 2D, u is now a function of three real variables: the spatial coordinates x and y and the time variable t. In the difference formula we will use a spatial step-size Δx. If we were calculating an exact derivative we would take a limit Δx approaches zero. In our case we must choose Δx sufficiently small to achieve our desired accuracy.

$$u_{xx}(t) + u_{yy}(t) \approx M = \frac{u(x-\Delta x, y, t) - 2u(x,y,t) + u(x+\Delta x, y, t)}{(\Delta x)^2}$$

$$+ \frac{u(x, y-\Delta x, t) - 2u(x,y,t) + u(x, y+\Delta x, t)}{(\Delta x)^2}$$

$$= \frac{u(x+\Delta x, y, t) + u(x-\Delta x, y, t) + u(x, y-\Delta x, t) + u(x, y+\Delta x, t) - 4u(x,y,t)}{(\Delta x)^2}$$

Now that we have an approximation of the Laplacian, we can compute an approximation of $u(x, y, t+\Delta t)$ for some time step Δt. We will use the approximation

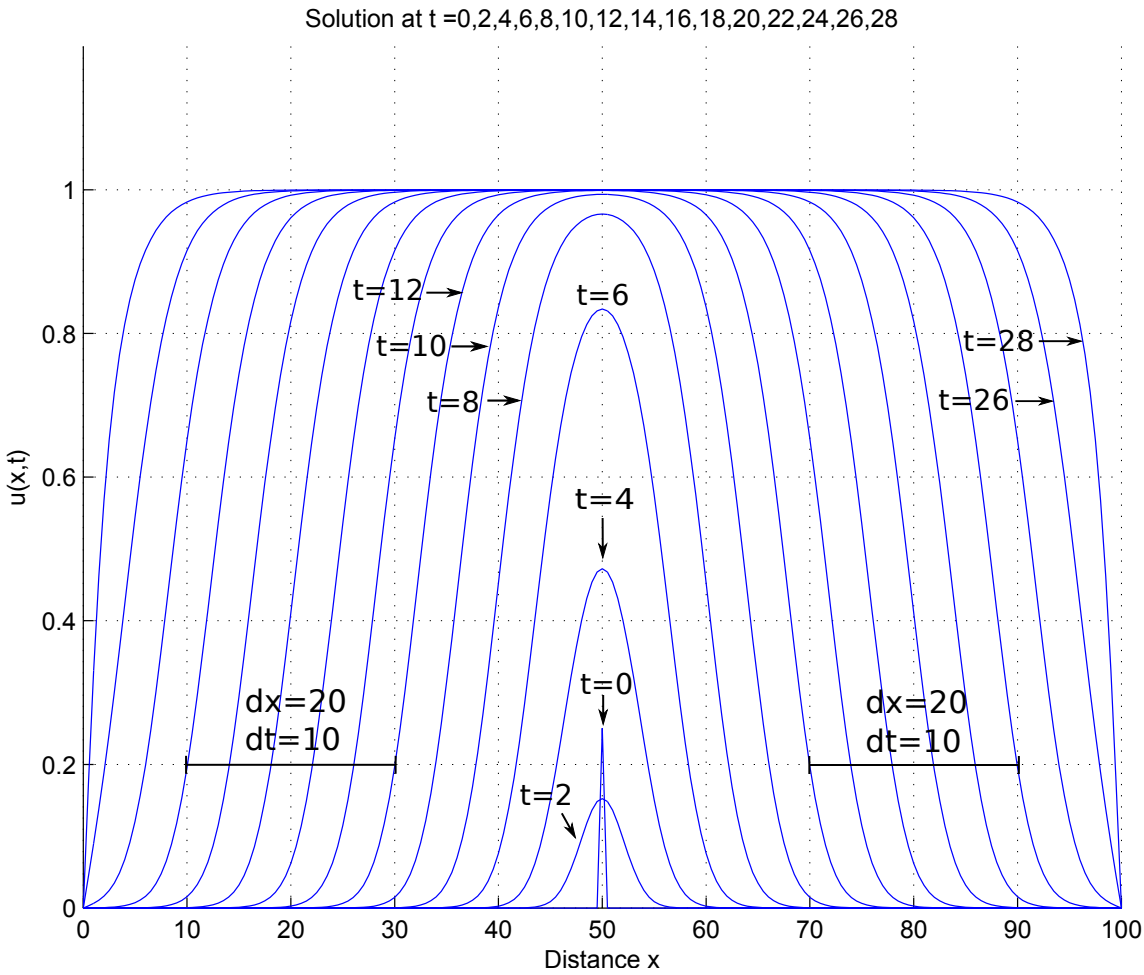

Figure 7: Plot of a Fischer-KPP system at various times.

of the Laplacian to approximate $u(x, y, t + \Delta t)$ using the straightforward method known as the Euler forward method:

$$u(x, y, t + \Delta t) = (M + u(x, y, t)(1 - u(x, y, t))) \, \Delta t$$

Background information on the the approximation of the second derivative, and Euler Forward method can be found in Chapters 3 and 6 of (Vuik et al. (2015)). Using such scheme a 2D Fischer-KPP system was simulated using software package "processing" (processing foundation). See Figure 8 for a plot of some results.

8 Conclusion

In this thesis we have shown that TW solutions exist for $c \geq 2$ in the FKPP system and TW solutions exist for the FKPPA system for $c = \sqrt{2}(\frac{1}{2} - a)$. We should mention that Kolmogorov et al. proved that for any initial condition with finite support, traveling waves develop with speed $c = 2$ in the FKPP system. The wave speed $c = 2$ was also seen in the numerical simulation. Therefore, the wave

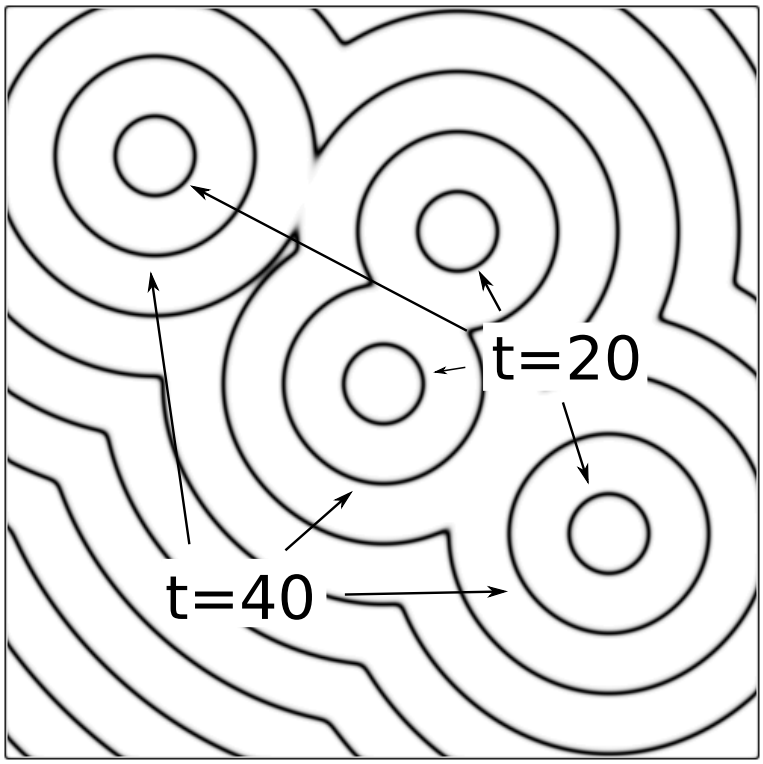

Figure 8: Plot of contours where $u = 1/2$ at various times at equal intervals

speed $c = 2$ is the wave speed one would often see in practice, for instance in when a non-indigenous species is introduced in a bounded region of a habitat.

The results concerning stability are as follows. The TW solutions of the FKPP equation are not stable against perturbations bounded by the supremum norm, while they are stable under perturbations that are bounded under the norm $\|\cdot\|_{B_{\frac{c}{2}}(\mathbb{R})}$ with $\|v\|_{B_{\frac{c}{2}}(\mathbb{R})} = \sup_{\xi\in\mathbb{R}} ve^{\frac{c}{2}\xi}$ for $c > 2$. Unfortunately, the stability of TW solutions of the FKPP equation for $c = 2$ has not been determined. The interpretation of what we do know about stability is as follows. The TW solutions are stable where the wave is near one, and unstable where the wave is near zero. This is easily understood, if one considers that away from the front of the wave the diffusion term u_{xx} in (2) is close to zero, and the system is resembles the logistic growth model in which zero is an unstable fixed point and one a stable fixed point.

In the same manner one can interpret the stability result for the FKPPA Equation. The fixed points zero and one of the ODE $u_t = u(1-u)(u-a)$ are both stable. Therefore, the TW solution, comprised of a front between amplitudes zero and one is stable on both sides. More specifically, in Section 6 we found that these TW solutions are stable against perturbations bounded by the supremum norm.

A traveling wave in a FKPP system is not very stable where the wave is near zero, since any increase in the population in such a place will be exponentially magnified by the logistic growth. Such growth can easily destabilize a traveling wave. However, in the FKPPA system, a population with a small population density is not viable, and therefore, a comparable perturbation will barely affect the traveling wave.

Unfortunately, further exploration of the relationship between the stability fixed points of a reaction equation and the locality of the stability of the traveling waves of the corresponding reaction diffusion equation was beyond the scope of this thesis.

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A Rigorous analysis of the Fischer-KPP system for $0 < c < 2$

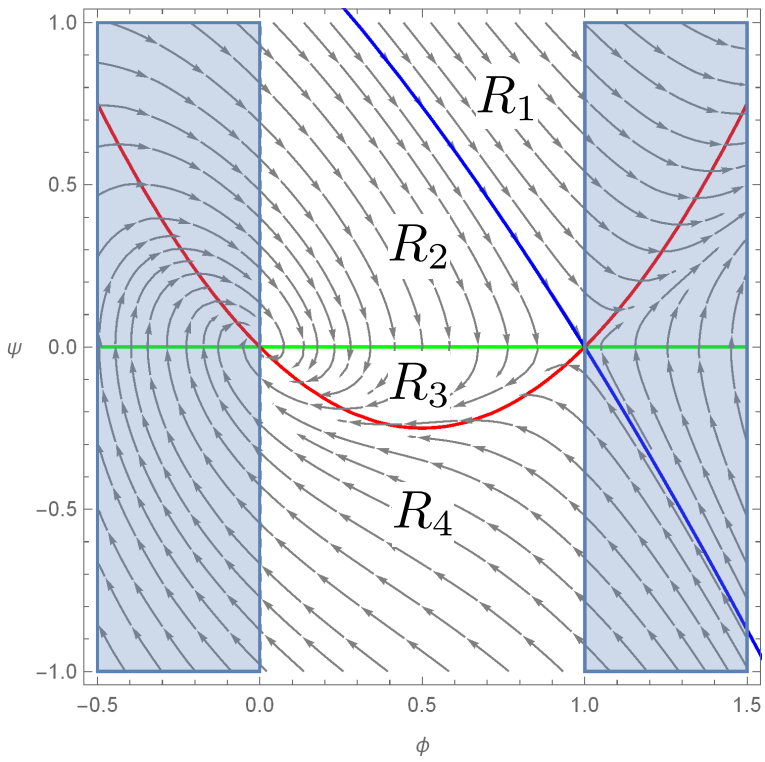

Figure 9: Phase plot of system (3) for $c = 1$

In Section 2.1, we showed that there are no orbits of the system (5) ($\phi' = \psi$, $\psi' = -c\psi + \phi(\phi - 1)$) that are relevant to applications in population dynamics. The arguments in 2.1 were made using observations in the phase plane. Since graphs can be misleading, we will give a more rigorous proof here.

Our argument starts by observing that solutions for which ϕ is not confined to $[0, 1]$ for all t are not relevant in the context of population dynamics, since negative populations are not meaningful and $\phi = 1$ corresponds to the maximum population that can be supported by the ecosystem.

Let M be the stable manifold of the fixed point $(1, 0)$.

To make our argument we divide the region $R = \{(\phi, \psi) : \phi \in (0, 1)\}$ into four parts, R_1, through R_4. Region R_1 is the part of R that lies strictly above the stable manifold of the fixed point $(0, 1)$, R_2 is the part of R that lies beneath the stable manifold M and above the ϕ axis. Region R_3 is the region that is enclosed by the nullclines $\psi = 0$ and $\psi = \frac{1}{c}(\phi^2 - \phi)$, and R_4 is the part of R that lies below the nullcline $\psi = \frac{1}{c}(\phi^2 - \phi)$. See also Figure 9.

The solutions that intersect region R_1 move down and towards the left, since we have $\phi' > 0$ and $\psi' < 0$. These solutions cannot intersect the stable manifold, since the part of the stable manifold M that borders R_1 is a solution, and solutions do not intersect in a two-dimensional autonomous system. Since we $\phi' > 0$ and $\psi' < 0$ there are no fixed points or periodic solutions. Therefore, the solutions that intersect R_1

must leave R_1 through the line $\phi = 1$. Hence the ϕ coordinates of these solutions are not confined to the interval $[0, 1]$. Thus, these solutions are not relevant in our context.

The solutions that intersect region R_2 move down and towards the left, since we have $\phi' > 0$ and $\psi' < 0$ in region R_2 as well. Here as well, the solutions cannot move through the stable manifold M and there are no fixed points and periodic orbits. Since they are moving down and to the left, they cannot leave through the line $\phi = 0$. Therefore, they must leave through the line $\psi = 0$, moving into region R_3.

The solutions in region R_3 move down and towards the right, since we have $\phi = \psi' < 0$ and $\psi' > 0$. These solutions can only leave through the nullcline $\psi = \frac{1}{c}(\phi^2 - \phi)$. Therefore, these solutions move towards region R_4.

The solutions in region R_4 move up and towards the right, since we have $\phi = \psi' < 0$ and $\psi' > 0$. These solutions can either move into R_3, after which they must move back into R_4, or they must move out of region R_4 through the line $\phi = 0$. The solutions cannot converge to the fixed point $(0, 0)$, without moving leaving R_4, since it is a focus.

Therefore, none of the solutions in the region R stay in R for all t. Thus, there are no solutions that are relevant in the context of population dynamics.

B Simulation Scripts

Matlab Script to simulate a Fischer-KPP system using pdepe

```
function pdex1

m = 0;

x_step = 0.5;
x_end = 100;
t_step = 0.05;
t_end = 30;

x = 0:x_step:x_end;
t = 0:t_step:t_end;

sol = pdepe(m,@pdex1pde,@pdex1ic,@pdex1bc,x,t);
% Extract the first solution component as u.
u = sol(:,:,1);

first_graph=1;
last_graph=round(t_end/t_step);
skip = 40;
titlestr='Solution at t = ';

figure('position',[100,100,800,600])
hold on;
grid on;
for i=first_graph:skip:last_graph
    plot(x,u(i,:));
    titlestr= strcat(titlestr, num2str((i-1)*t_step))
    if (i-1)*t_step~=28
        titlestr= strcat(titlestr,', ')
```

```
    end

end
axis([0 x_end 0 1.2]);
xlabel('Distance x');
ylabel('u(x,t)');
title(titlestr);

set(gcf,'PaperSize',[19 15])
set(gcf,'PaperPositionMode', 'manual')
set(gcf,'PaperPosition',[0 0 19 15])
set(gcf,'PaperUnits','inches')
print('bs_plot_two_fronts.pdf', '-dpdf');

%-----------------------------------------------------------------
function [c,f,s] = pdex1pde(x,t,u,DuDx)
c = 1;
f = DuDx;
s = u*(1-u);
%-----------------------------------------------------------------
function u0 = pdex1ic(x)
if x > 49.5 && x < 50.5
u0 = (x-50.5)*(49.5-x);
else
    u0=0;
end
%-----------------------------------------------------------------
function [pl,ql,pr,qr] = pdex1bc(xl,ul,xr,ur,t)
pl = ul;
ql = 0;
pr = ur;
qr = 0;
```

Processing script to simulate a 2d Fischer-KPP system.

```
static final double dx = 1;
static final double dt  = 0.025;

double[][] u;
double[][] uu;
static final float epsilon = 0.4;
static final int skip = 10;

void setup() {
  size(1501,1501); // sets the size of the screen
                   // and width and height
  background(255);
  noSmooth();
  u = new double[width][height];
  uu = new double[width][height];
  int xmid = (width-1)/2+1;
  int ymid = (height-1)/2+1;
  u[xmid][ymid] = 0.01;
  u[100][100] = 0.01;
  u[300][150] = 0.01;
  u[400][350] = 0.01;

  for (int i=0;i<55*15/skip;i++) {
    for (int j=0;j<skip/dt;j++) {
```

```
      iterate();
    }
    draw_countour();
  }
  String fmtstr = "graph dx=%.2f dt=%.2f eps=%.2f skip=%d time=%s.png";
  String filename = String.format(fmtstr,dx,dt,epsilon,skip,timestamp()
    );
  save(filename);
}

void iterate() {
  for (int x=1;x<width-1;x++) {
    for (int y=1;y<height-1;y++) {
      double uuu = u[x][y];
      double uxx = (u[x-1][y]+u[x][y-1]-4*uuu+u[x+1][y]+u[x][y+1])/dx/
    dx;
      double ut = uxx + uuu*(1-uuu);
      uu[x][y] = uuu + ut*dt;
    }
  }
  double[][] temp = u;
  u = uu;
  uu = temp;
}

void draw_countour() {
  for (int x=0;x< width;x++) {
    for (int y=0;y< height;y++) {
      double uuu = u[x][y];
      if ( 0.5 - epsilon < uuu && uuu < 0.5 + epsilon) {
        float delta = 2*abs((float)uuu-0.5f)/epsilon;
        float val = exp(-delta*delta);
        stroke(255-(int)255*val);
        point(x,y);
      }
    }
  }
}

String timestamp() {
  return ""+month()+"-"+day()+"-"+hour()+".."+minute()+"."+second()+
    millis();
}
```