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Gaussian Curvature and The Gauss-Bonnet Theorem

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1 Introduction

The simplest version of the Gauss-Bonnet theorem probably goes back to the time of Thales, stating that the sum of the interior angles of a triangle in the Euclidean plane equals π. This evolved to the nineteenth century version which applies to a compact surface M with Euler characteristic $\chi(M)$ given by:

$$\int_M K \; dA = 2\pi\chi(M)$$

where K is the Gaussian curvature of the surface M.
This thesis will focus on Gaussian curvature, being an intrinsic property of a surface, and how through the Gauss-Bonnet theorem it bridges the gap between differential geometry, vector field theory and topology, especially the Euler characteristic. For this, a short introduction to surfaces, differential forms and vector analysis is given.
Within the proof of the Gauss-Bonnet theorem, one of the fundamental theorems is applied: the theorem of Stokes. This theorem will be proved as well.
Finally, an application to physics of a corollary of the Gauss-Bonnet theorem is presented involving the behaviour of liquid crystals on a spherical shell.

2 Introduction to Surfaces

This section presents the basics of the differential geometry of surfaces through the first and second fundamental forms.

2.1 Surfaces

Definition 2.1 *A non-empty subset $M \subset \mathbb{R}^3$ is called a regular surface if for every point $p \in M$ there exists a neighbourhood $V \subset \mathbb{R}^3$, an open subset $U \subset \mathbb{R}^2$ and a differentiable map $\phi : U \to V \cap M \subset \mathbb{R}^3$ with the following properties:*

(i) $\phi : U \to \phi(U) \subset M$ is a homeomorphism.

(ii) For every $q \in U$, the differential $D\phi(q) : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

The map ϕ is called a local parametrization of the surface M and the pair (ϕ, U) is called a local chart for M.

Definition 2.2 *Given a regular surface M, a k-differentiable atlas $\mathcal{A}$ is a family of charts $\{(\phi_i, U_i), i \in I\}$ for some index set I, such that:*

1. *$\forall i \in I$, $\quad \phi_i(U_i) \subseteq \mathbb{R}^3$.*
2. *The U_i cover M, meaning*
$$M = \bigcup_{i \in I} U_i.$$
3. *For $i, j \in I$, the map*
$$(\phi_j \circ \phi_i^{-1})_{|\phi_i(V_i \cap V_j)} : \phi_i(V_i \cap V_j) \to \phi_j(V_i \cap V_j)$$
is k-differentiable.

M is now called a 2-dimensional manifold with differentiable atlas $\mathcal{A}$.

Definition 2.3 *A curve in M through a point $p \in M$ is a differentiable map $\gamma : (-\epsilon, \epsilon) \to M, \epsilon \in \mathbb{R}^+$ and $\gamma(0) = p$.*

It holds that
$$\dot{\gamma}(0) = \frac{d}{dt}\gamma(0) \in \mathbb{R}^3$$
is a tangent vector to M at the point $p = \gamma(0)$.

Definition 2.4 *The set*
$$T_pM = \{\dot{\gamma}(0) \mid \gamma \textit{ is a curve in } M \textit{ through } p \in M\} \subset \mathbb{R}^3$$
is called the tangent space to M in p.

Proposition 2.1 *Let (ϕ, U) be a local chart for M and let $q \in U$ with $\phi(q) = p \in M$. It holds that*
$$T_pM = (D\phi(q))(\mathbb{R}^2).$$

This implies that $(\frac{\partial \phi}{\partial u}(q), \frac{\partial \phi}{\partial v}(q))$ forms a basis for T_pM, with (u, v) coordinates in U.

By restricting the natural inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^3$ to each tangent plane T_pM, we get an inner product on T_pM. We call this inner product on T_pM the first fundamental form and denote it by I_p.

2.2 First Fundamental Form

The first fundamental form tells how the surface inherits the natural inner product of $\mathbb{R}^3$.

Definition 2.5 *Let M be a regular surface in $\mathbb{R}^3$ and let $p \in M$. The first fundamental form of M at p is the map:*

$$\begin{array}{rcl} I_p : T_pM \times T_pM & \to & \mathbb{R} \\ (X, Y) & \to & \langle X, Y \rangle. \end{array}$$

We will, for convenience, use the notation $f_x = \frac{\partial f}{\partial x}$ for the partial derivative of a function f with respect to a variable x.

We want to write the first fundamental form in terms of the basis associated with the local chart (ϕ, U). Remember that an element of T_pM is a tangent vector at a point $p = \gamma(0) \in M$ to a parametrized curve $\gamma(t) = \phi(u(t), v(t)), t \in (-\epsilon, \epsilon)$.
It holds that:

$$\begin{array}{rcl} I_p(\gamma', \gamma') & = & \langle \phi_u \cdot \frac{du}{dt} + \phi_v \cdot \frac{dv}{dt}, \phi_u \cdot \frac{du}{dt} + \phi_v \cdot \frac{dv}{dt} \rangle \\ & = & \langle \phi_u, \phi_u \rangle (\frac{du}{dt})^2 + 2\langle \phi_u, \phi_v \rangle \frac{du}{dt}\frac{dv}{dt} + \langle \phi_v, \phi_v \rangle (\frac{dv}{dt})^2 \\ & = & E(\frac{du}{dt})^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G(\frac{dv}{dt})^2, \end{array}$$

where we define

$$\begin{array}{rcl} E & = & \langle \phi_u, \phi_u \rangle, \\ F & = & \langle \phi_u, \phi_v \rangle, \\ G & = & \langle \phi_v, \phi_v \rangle. \end{array}$$

The first fundamental form is often written in the modern notation of the metric tensor:

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

2.3 Orientation and Gauss map

Orientability is a property of surfaces measuring whether is it possible to make a consistent choice for a normal vector at every point of the surface.

Definition 2.6 *A regular surface M is orientable if it is possible to cover it with an atlas $\mathcal{A}$, so that*

$$\forall i, j \in I, \forall p \in U_i \cap U_j \ : \ \det(\phi_j \circ \phi_i^{-1})(\phi_i(p)) > 0.$$

A choice of an atlas that satisfies this condition is called an orientation of M, and M is then called oriented. A local chart (ϕ, U) is then called a positive local chart. If no such atlas exists, M is called non-orientable.

The differential geometry of surfaces frequently involves moving frames of reference. Before going any further, we distinguish two frames used: the Frenet-Serret and the Darboux-Cartan frames of reference. The first one depends on a regular curve $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^3$, that is, $\|\dot{\gamma}(t)\| > 0$ for every $t \in (-\epsilon, \epsilon)$. Let γ be parametrized by arc length ds, then the frame of reference is constructed by vectors T, N, B where $T = \frac{d\gamma}{ds}$ is a unit speed vector, $N = \frac{\frac{dT}{ds}}{\|\frac{dT}{ds}\|}$ and $B = N \times T$.
The Darboux-Cartan frame of reference depends on a regular curve γ on a regular surface M and is constructed by orthonormal vectors $(e_1, e_2, e_3) = (T, n \times T, n)$ where $T = \dot{\gamma}$, n is a unit normal vector to the surface M at $p \in \gamma$.
An oriented surface in $\mathbb{R}^3$ yields a pair (M, n) where $M \subset \mathbb{R}^3$ is a regular surface and $n : M \to S^2 \subset \mathbb{R}^3$ a differentiable map such that $n(p)$ is a unit vector orthogonal to T_pM for each $p \in M$ as defined below.

Definition 2.7 *Let (M, n) be an oriented surface in $\mathbb{R}^3$ and (ϕ, U) a local chart for M with basis (ϕ_u, ϕ_v) for T_pM, with (u, v) coordinates in U. The Gauss map is the map:*

$$\begin{aligned} n : M &\to S^2 \subset \mathbb{R}^3 \\ p &\to n(p) = \frac{\phi_u \times \phi_v}{\|\phi_u \times \phi_v\|}(p). \end{aligned}$$

The Gauss map is a differentiable map. Its differential induces the Weingarten-map.

2.4 Second Fundamental Form

The Weingarten map is a linear map related to the Gauss map as follows:

Definition 2.8 *The Weingarten map is the map*

$$\begin{aligned} L_p : T_pM &\to T_pM \\ v &\mapsto -Dn(p)(v) = -\frac{d}{dt}(n \circ \gamma)(0) \end{aligned}$$

with $\gamma(0) = p \in M$ *and* $v = \dot{\gamma}(0) \in T_pM$*.*

Definition 2.9 *The second fundamental form of a regular oriented surface* M *at a point* $p \in M$ *is the map:*

$$\begin{aligned} II_p : T_pM \times T_pM &\to \mathbb{R} \\ (X, Y) &\to \langle L_p(X), Y \rangle . \end{aligned}$$

Theorem 2.2 L_p *is self-adjoint, meaning that* $\langle L_p(X), Y \rangle = \langle X, L_p(Y) \rangle$*.*

Proof It holds that L_p is linear thus we only have to check the above claim for the basis vectors ϕ_u, ϕ_v.
It holds that:

$$\left\langle n \circ \phi, \frac{\partial \phi}{\partial u} \right\rangle = 0.$$

From this it follows that:

$$0 = \frac{\partial}{\partial v} \langle n \circ \phi, \phi_u \rangle = \left\langle \frac{\partial}{\partial v}(n \circ \phi), \phi_u \right\rangle + \langle n \circ \phi, \phi_{uv} \rangle .$$

Using the chain rule we get:

$$L_p(\phi_v) = -Dn(\phi_v) = -\frac{\partial}{\partial v}(n \circ \phi).$$

Thus it holds that

$$\langle L_p(\phi_v), \phi_u \rangle = -\left\langle \frac{\partial}{\partial v}(n \circ \phi), \phi_u \right\rangle = \langle n \circ \phi, \phi_{uv} \rangle .$$

This is symmetric in u and v. $\square$

This also shows that the differential $Dn(p) : T_pM \to T_pM$ of the Gauss map is self-adjoint.

As with the first fundamental form, it is useful to write the second fundamental form in terms of the basis (ϕ_u, ϕ_v) associated with the local parametrization ϕ.
It holds that:

$$\begin{aligned}
II_p(\gamma') &= -\langle Dn(p)(\gamma'), \gamma' \rangle \\
&= -\langle Dn(p)(\phi_u)\frac{du}{dt} + Dn(p)(\phi_v)\frac{dv}{dt}, \phi_u\frac{du}{dt} + \phi_v\frac{dv}{dt} \rangle \\
&= -\langle n_u\frac{du}{dt} + n_v\frac{dv}{dt}, \phi_u\frac{du}{dt} + \phi_v\frac{dv}{dt} \rangle \\
&= e(\frac{du}{dt})^2 + 2f\frac{du}{dt}\frac{dv}{dt} + g(\frac{dv}{dt})^2,
\end{aligned}$$

where

$$n_u = Dn(p)(\phi_u) \qquad n_v = Dn(p)(\phi_v)$$

and

$$\begin{aligned}
e &= -\langle n_u, \phi_u \rangle = -\langle n, \phi_{uu} \rangle, \\
f &= -\langle n_v, \phi_u \rangle = -\langle n, \phi_{uv} \rangle = -\langle n_u, \phi_v \rangle, \\
g &= -\langle n_v, \phi_v \rangle = -\langle n, \phi_{vv} \rangle.
\end{aligned}$$

3 Gaussian Curvature

The fundamental idea behind the Gaussian curvature is the Gauss map, as defined in definition 2.7. The Gaussian curvature can be defined as follows:

Definition 3.1 *The Gaussian curvature of the regular surface M at a point $p \in M$ is*

$$K(p) = \det(Dn(p)),$$

where $Dn(p)$ is the differential of the Gauss map at p.

It holds that for a local parametrization $\phi(u, v)$ and a curve $\gamma(t) = \phi(u(t), v(t))$ with $\gamma(0) = p \in M$, the tangent vector to the curve γ at $t = 0$ equals

$$\dot\gamma(0) = \phi_u(u(0), v(0))\frac{du}{dt}(0) + \phi_v(u(0), v(0))\frac{dv}{dt}(0),$$

and thus

$$Dn(\dot\gamma(0)) = n_u\frac{du}{dt}(0) + n_v\frac{dv}{dt}(0).$$

Since n_u and n_v lie in the tangent plane T_pM, we can write them in terms of the basis (ϕ_u, ϕ_v) as

$$\begin{aligned} n_u &= \alpha_{11}\phi_u + \alpha_{21}\phi_v, \\ n_v &= \alpha_{12}\phi_u + \alpha_{22}\phi_v, \end{aligned}$$

for some $(\alpha_{ij})_{i,j=1,2} \in \mathbb{R}$, which gives the matrix of the linear map $Dn(p)$:

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}.$$

We can now link the coefficients of the first fundamental form to those of the second fundamental form.
Since $\langle n, \phi_u \rangle = \langle n, \phi_v \rangle = 0$ it holds that

$$\begin{aligned} -e = \langle n_u, \phi_u \rangle &= \langle \alpha_{11}\phi_u + \alpha_{21}\phi_v, \phi_u \rangle = \alpha_{11}E + \alpha_{21}F, \\ -f = \langle n_u, \phi_v \rangle &= \langle \alpha_{11}\phi_u + \alpha_{21}\phi_v, \phi_v \rangle = \alpha_{11}F + \alpha_{21}G, \\ -f = \langle n_v, \phi_u \rangle &= \langle \alpha_{12}\phi_u + \alpha_{22}\phi_v, \phi_u \rangle = \alpha_{12}E + \alpha_{22}F, \\ -g = \langle n_v, \phi_v \rangle &= \langle \alpha_{12}\phi_u + \alpha_{22}\phi_v, \phi_v \rangle = \alpha_{12}F + \alpha_{22}G, \end{aligned}$$

which, written in matrix form, gives

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

It holds that $EG - F^2 > 0$ since the inner product is positive definite and that $\det(A \cdot B) = \det(A) \cdot \det(B)$. Thus we get:

$$K = \det(\alpha_{ij})_{i,j=1,2} = \frac{eg - f^2}{EG - F^2}.$$

4 Theorema Egregium

The following theorem represents one of the most important theorems within the field of differential geometry and the study of surfaces. Gauss called it *Theorema Egregium*, meaning "remarkable theorem", since it tells us that the curvature of a surface can be measured without knowing how the surface is embedded in space.

Theorem 4.1 ***Theorema Egregium*** *The Gaussian curvature $K(p)$ of a surface M at a point $p \in M$ is an intrinsic value of the surface itself at this point, that is, it only depends on the metric tensor g on M.*

For the purpose of the proof of this theorem, we are going to denote ϕ_u, ϕ_v as ϕ_{u_1} and ϕ_{u_2}. In general it will be written as ϕ_{u_i} and ϕ_{u_j} with $i, j = 1, 2$.

Proof Let (ϕ, U) be a local chart for M with coordinates (u, v) in U, $q \in U$ and $p = \phi(q)$. It holds that (ϕ_u, ϕ_v, n) forms a basis for $\mathbb{R}^3$, where n is a unit normal vector to M at $p = \phi(q)$. This means that every $a \in \mathbb{R}^3$ can be written as:

$$a = \sum_{i=1}^{2} a_i \phi_{u_i} + \langle a, n \rangle n$$

for some $a_1, a_2 \in \mathbb{R}$.
Computing the second derivative of ϕ yields:

$$\frac{\partial^2 \phi}{\partial u_i \partial u_j}(q) = \sum_{k=1}^{2} \Gamma_{ij}^k \cdot \frac{\partial \phi}{\partial u_k}(q) + \left\langle \frac{\partial^2 \phi}{\partial u_i \partial u_j}(q), n \right\rangle n, \tag{$*$}$$

where Γ_{ij}^k are coefficients called the Christoffel symbols.
It holds by symmetry that $\frac{\partial^2 \phi}{\partial u_i \partial u_j}(q) = \frac{\partial^2 \phi}{\partial u_j \partial u_i}(q)$ and thus $\Gamma_{ij}^k = \Gamma_{ji}^k$.
We can now compute the derivative of the first fundamental form $(g_{ij})_{i,j=1,2}$:

$$\begin{aligned} \frac{\partial g_{ij}}{\partial u_k}(q) &= \frac{\partial}{\partial u_k} \left\langle \frac{\partial \phi(q)}{\partial u_i}, \frac{\partial \phi(q)}{\partial u_j} \right\rangle \\ &= \left\langle \frac{\partial^2 \phi(q)}{\partial u_i \partial u_k}, \frac{\partial \phi(q)}{\partial u_j} \right\rangle + \left\langle \frac{\partial \phi(q)}{\partial u_i}, \frac{\partial^2 \phi(q)}{\partial u_j \partial u_k}(q) \right\rangle. \end{aligned}$$

The same computation for $\frac{\partial g_{ki}}{\partial u_j}(q)$ and $\frac{\partial g_{jk}}{\partial u_i}(q)$ gives:

$$\frac{1}{2}\left(\frac{\partial g_{ki}}{\partial u_j}(q) + \frac{\partial g_{jk}}{\partial u_i}(q) - \frac{\partial g_{ij}}{\partial u_k}(q) \right) = \left\langle \frac{\partial^2 \phi}{\partial u_i \partial u_j}, \frac{\partial \phi}{\partial u_k} \right\rangle. \tag{$**$}$$

The inner product of $(*)$ with $\frac{\partial \phi}{\partial u_l}$ gives via $(**)$:

$$\frac{1}{2}\left(\frac{\partial g_{ki}}{\partial u_j}(q) + \frac{\partial g_{jk}}{\partial u_i}(q) - \frac{\partial g_{ij}}{\partial u_k}(q) \right) = \sum_k \Gamma_{ij}^k g_{kl},$$

and thus

$$\Gamma_{ij}^k = \frac{1}{2} \sum_l \left(\frac{\partial g_{ki}}{\partial u_j}(q) + \frac{\partial g_{jk}}{\partial u_i}(q) - \frac{\partial g_{ij}}{\partial u_k}(q) \right) g^{kl},$$

where g^{kl} is the inverse of g_{ij}.

We can now compute the third derivative of ϕ:

$$\frac{\partial^3 \phi}{\partial u_i \partial u_j \partial u_l} = \sum_k \frac{\partial \Gamma_{ij}^k}{\partial u_l} \cdot \frac{\partial \phi}{\partial u_k} + \sum_k \Gamma_{ij}^k \cdot \frac{\partial^2 \phi}{\partial u_k \partial u_l} + \frac{\partial g_{ij}}{\partial u_l} \cdot n + g_{ij} \cdot \frac{\partial n}{\partial u_l}.$$

It is obvious that

$$(\phi_{uu})_v - (\phi_{uv})_u = 0.$$

Furthermore, combining this with $(*)$ we deduce:

$$\begin{aligned}&\Gamma_{11}^1 \phi_{uv} + \Gamma_{11}^2 \phi_{vv} + e\, n_v + (\Gamma_{11}^1)_v \phi_u + (\Gamma_{11}^2)_v \phi_v + e_v\, n \\ &\qquad = \Gamma_{12}^1 \phi_{uu} + \Gamma_{12}^2 \phi_{vu} + f\, n_u + (\Gamma_{12}^1)_u \phi_u + (\Gamma_{12}^2)_v \phi_v + f_v\, n,\end{aligned}$$

with e, f the coefficients we defined for the second fundamental form. If we substitute all second-order derivatives using $(*)$ again and equating the coefficients of ϕ_v we get

$$\Gamma_{11}^1 \Gamma_{12}^2 + \Gamma_{11}^2 \Gamma_{22}^2 + e\, \alpha_{22} + (\Gamma_{11}^2)_v = \Gamma_{12}^1 \Gamma_{11}^2 + \Gamma_{12}^2 \Gamma_{12}^2 + f\, \alpha_{12} + (\Gamma_{12}^2)_u,$$

with α_{21}, α_{22} the coefficients given in the matrix of $Dn(p)$ introduced earlier. Solving the $Dn(p)$ matrix for these coefficients gives:

$$\alpha_{21} = \frac{eF - fE}{EG - F^2} \qquad \alpha_{22} = \frac{fF - gE}{EG - F^2}.$$

Replacing these coefficients we obtain

$$\begin{aligned}(\Gamma_{12}^2)_u - (\Gamma_{11}^2)_v + \Gamma_{12}^1 \Gamma_{11}^2 + \Gamma_{12}^2 \Gamma_{12}^2 - \Gamma_{11}^1 \Gamma_{12}^2 - \Gamma_{11}^2 \Gamma_{22}^2 &= e\, \alpha_{22} - f\, \alpha_{21} \\ &= -E \frac{eg - f^2}{EG - F^2} \\ &= -EK,\end{aligned}$$

which gives the desired formula for the Gaussian curvature in terms of the first fundamental form and its first and second derivatives. $\square$

This shows that the Gaussian curvature is independent of the embedding of the surface in $\mathbb{R}^3$. Thus, it is an intrinsic value of the surface itself since it depends only on the metric tensor g.

5 Gauss-Bonnet Theorem

In this section, the Gauss-Bonnet theorem is proved, through the Gauss-Bonnet formula. To do this, some elementary tools from differential geometry are needed: differential forms.

5.1 Differential Forms

Definition 5.1 *A 1-form on* $\mathbb{R}^3$ *is a map*

$$\begin{aligned} \omega : \mathbb{R}^3 &\to \sqcup_{p\in\mathbb{R}^3}(\mathbb{R}^3_p)^* \\ p &\to \omega(p), \end{aligned}$$

where $(\mathbb{R}^3_p)^*$ *is the dual space of the tangent space* $\mathbb{R}^3_p = \{v|v = q - p, q \in \mathbb{R}^3\}$*, that is, the set of linear maps* $\varphi : \mathbb{R}^3_p \to \mathbb{R}$*. It holds that* ω *can be written as*

$$\omega(p) = \sum_{i=1}^{3} a_i(p)(dx^i)_p,$$

where a_i *are real functions in* $\mathbb{R}^3$ *and the set* $\{(dx^i)_p;\ i = 1, 2, 3\}$ *is the dual basis of* $\{(e_i)_p\}$*, that is*

$$(dx^i)_p(e_j)_p = \frac{\partial x_i}{\partial x_j} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

This map should be compatible with the projection map $P_1 : \sqcup_{p\in\mathbb{R}^3}(\mathbb{R}^3)^* \to \mathbb{R}^3$, that is, $P_1 \circ \omega = id_{\mathbb{R}^3}$. If the functions a_i are differentiable, ω is called a differential 1-form.

Let $\Lambda^2(\mathbb{R}^3_p)^*$ be the set of maps $\varphi : \mathbb{R}^3_p \times \mathbb{R}^3_p \to \mathbb{R}$ that are bilinear and alternating, meaning $\varphi(v_1, v_2) = -\varphi(v_2, v_1)$ for $(v_1, v_2) \in \mathbb{R}^3_p \times \mathbb{R}^3_p$. This is a vector space. The set $\{(dx^i \wedge dx^j)_p, i < j\}$ forms a basis for $\Lambda^2(\mathbb{R}^3_p)^*$. It holds that

$$(dx^i \wedge dx^j)_p = -(dx^j \wedge dx^i)_p, \qquad i \neq j$$

and

$$(dx^i \wedge dx^i)_p = 0.$$

Definition 5.2 *A 2-form on* $\mathbb{R}^3$ *is a map*

$$\omega : \mathbb{R}^3 \to \sqcup_{p\in\mathbb{R}^3}\Lambda^2(\mathbb{R}^3_p)^*.$$

It can be written as:

$$\mathbb{R}^3 \ni p \to \omega(p) = \sum_{i<j} a_{ij}(p)(dx^i \wedge dx^j)_p, \qquad i, j = 1, 2, 3,$$

where a_{ij} *are real functions in* $\mathbb{R}^3$*.*

Here again, the map should be compatible with the projection map $P_2 : \sqcup_{p\in\mathbb{R}^3}(\mathbb{R}^3)^* \to \mathbb{R}^3$, that is, $P_2 \circ \omega = id_{\mathbb{R}^3}$. If the functions a_{ij} are differentiable then ω is called a differential 2-form.

The above definitions are for 1- and 2-forms on $\mathbb{R}^3$. From here on we are going to work with 1- and 2-forms on the regular surface M.

Let (ϕ, U) be a local chart for M with coordinates (u, v) and let

$$\vec{x} = (x, y, z) = (\phi_1(u, v), \phi_2(u, v), \phi_3(u, v))$$

be a moving point through M. Its differential yields

$$\begin{aligned} d\vec{x}(p) &= (d\phi_1(u, v), d\phi_2(u, v), d\phi_3(u, v)) \\ &= \left(\frac{\partial\phi_1}{\partial u}(p)du + \frac{\partial\phi_1}{\partial v}(p)dv, \frac{\partial\phi_2}{\partial u}(p)du + \frac{\partial\phi_3}{\partial v}(p)dv, \frac{\partial\phi_3}{\partial u}(p)du + \frac{\partial\phi_3}{\partial v}(p)dv\right) \\ &= \left(\frac{\partial\phi}{\partial u}(p)du + \frac{\partial\phi}{\partial v}(p)dv\right). \end{aligned}$$

It holds that $(\frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v})$ forms a basis for the tangent space T_pM. Thus, $d\vec{x}(p) \in T_pM^*$.
Now let (e_1, e_2, e_3) be a Darboux-Cartan frame for the regular surface M with e_1, e_2 tangential and $e_3 = e_1 \times e_2$ normal to M. It holds that we can write $d\vec{x}$ as

$$d\vec{x}(p) = \omega_1(p) \cdot e_1 + \omega_2(p) \cdot e_2$$

for some $\omega_1, \omega_2 \in T_pM^*$ since (e_1, e_2) forms a basis for T_pM as well.
At each point $p \in M$, the basis $\{(\omega_i)_p\}$ is the dual of the basis $\{(e_i)_p\}$. The set of differential 1-forms $\{\omega_i\}$ is called the coframe associated to $\{e_i\}$.
It also holds that each vector field e_i is a differentiable map $e_i : \mathbb{R}^3 \to \mathbb{R}^3$. Its differential at $p \in U, (de_i)_p : \mathbb{R}^3 \to \mathbb{R}^3$, is a linear map. Thus, for each p and each $v \in \mathbb{R}^3$ we can write

$$(de_i)_p(v) = \sum_j (\omega_{ij})_p(v) \cdot e_j.$$

It holds that $(\omega_{ij})_p(v)$ depends linearly on v. Thus $(\omega_{ij})_p$ is a linear form on $\mathbb{R}^3$ and since e_i is a differentiable vector field, w_{ij} is a differential 1-form. For convenience, we abbreviate the above expression to

$$de_i = \sum_j \omega_{ij} \cdot e_j.$$

We can now work with these expressions. Differentiating $\langle e_i, e_j \rangle = \delta_{ij}$ gives

$$0 = \langle de_i, e_j \rangle + \langle e_i, de_j \rangle = \omega_{ij} + \omega_{ji}$$

and thus $\omega_{ij} = -\omega_{ji}$ and in particular $\omega_{ii} = 0$.
The three forms $(\omega_{ij})_{i<j}$ are called the connection forms of $\mathbb{R}^3$ in the moving frame $\{e_i\}$.
Cartan's structural equations give an expression for the exterior differential of these 1-forms.

Theorem 5.1 *The structural equations are given by:*

- $d\omega_1 = \omega_{12} \wedge \omega_2$ *and* $d\omega_2 = \omega_1 \wedge \omega_{12}$,
- $d\omega_{ij} = \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj}$.

Proof To prove the first statement, we use the fact that $d(d\vec{x}) = 0$:

$$\begin{aligned} 0 = d(d\vec{x}) &= d(\omega_1 \cdot e_1 + \omega_2 \cdot e_2) \\ &= d\omega_1 \cdot e_1 - \omega_1 \wedge de_1 + d\omega_2 \cdot e_2 - \omega_2 \wedge de_2. \end{aligned}$$

Using the fact that $de_i = \sum_j \omega_{ij} \cdot e_j$, this yields:

$$\begin{aligned} 0 = d(d\vec{x}) &= d\omega_1 \cdot e_1 - \omega_1 \wedge (\omega_{12} \cdot e_2 + \omega_{13} \cdot e_3) + d\omega_2 \cdot e_2 - \omega_2 \wedge (\omega_{21} \cdot e_1 + \omega_{23} \cdot e_3) \\ &= (d\omega_1 - \omega_2 \wedge \omega_{21}) \cdot e_1 + (d\omega_2 - \omega_1 \wedge \omega_{12}) \cdot e_2 - (\omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23}) \cdot e_3. \end{aligned}$$

To prove the second statement we use the fact that $d(de_i) = 0$:

$$\begin{aligned} 0 = d(de_i) &= d(\sum_{j=1}^{3} \omega_{ij} \cdot e_j) \\ &= \sum_{j=1}^{3} d(\omega_{ij} \cdot e_j) \\ &= \sum_{j=1}^{3} (d\omega_{ij} \cdot e_j - \omega_{ij} \wedge de_j) \\ &= \sum_{j=1}^{3} \left(d\omega_{ij} \cdot e_j - \omega_{ij} \wedge \sum_{k=1}^{3} \omega_{jk} \cdot e_k \right). \end{aligned}$$

From here we see that in front of a basis vector e_l we have the coefficient:

$$d\omega_{il} - \sum_{j=1}^{3} \omega_{ij} \wedge \omega_{jl},$$

which gives the desired result. □

It holds that the differential of the normal vector $n = e_3 = e_1 \times e_2$ is contained in the plane spanned by (e_1, e_2) and given by:

$$dn = de_3 = \omega_{31} \cdot e_1 + \omega_{32} \cdot e_2.$$

Since ω_1 and ω_2 form a basis for 1-forms, we can write

$$\begin{aligned} \omega_{13} &= h_{11}\omega_1 + h_{12}\omega_2, \\ \omega_{23} &= h_{21}\omega_1 + h_{22}\omega_2. \end{aligned}$$

Thus the matrix of the Weingarten map with respect to the orthonormal basis (e_1, e_2) is given by:

$$L = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}.$$

Indeed we have

$$\begin{aligned} dn(e_1) &= -(\omega_{13} \cdot e_1 + \omega_{23} \cdot e_2)(e_1) \\ &= -(\omega_{13}(e_1) \cdot e_1 + \omega_{23}(e_1) \cdot e_2) \\ &= -(h_{11}e_1 + h_{12}e_2) \end{aligned}$$

and

$$\begin{aligned} dn(e_2) &= -(\omega_{13} \cdot e_1 + \omega_{23} \cdot e_2)(e_2) \\ &= -(\omega_{13}(e_2) \cdot e_1 + \omega_{23}(e_2) \cdot e_2) \\ &= -(h_{21}e_1 + h_{22}e_2) \end{aligned}$$

and since $L = -dn$, we get the wanted entries for the matrix form.
From the proof of the first statement of the previous theorem we know that $0 = \omega_1 \wedge \omega_{12} + \omega_2 \wedge \omega_{23}$. This means that $h_{12} = h_{21}$ and shows again the symmetry of the Weingarten map.
This also gives the Gaussian curvature, that is:

$$K = \det L = h_{11}h_{22} - h_{12}h_{21} = h_{11}h_{22} - h_{12}^2.$$

We now use Cartan's structural equations. It holds that:

$$d\omega_{12} = \omega_{13} \wedge \omega_{32} = -\omega_{13} \wedge \omega_{23}.$$

We have:

$$\omega_{13} \wedge \omega_{23} = (h_{11}\omega_1 + h_{12}\omega_2) \wedge (h_{21}\omega_1 + h_{22}\omega_2) = (h_{11}h_{22} - h_{12}^2)\omega_1 \wedge \omega_2.$$

This can be written as:

$$d\omega_{12} = -K \;\; dA.$$

Remark It can be easily shown that $dA = \omega_1 \wedge \omega_2$, even if the notation dA is a bad notation for the surface element. It holds that $\omega_1 \wedge \omega_2$ can never be an exact form, that is, the differential of a 1-form. But the notation above is tolerated and widely used.
For the orthonormal basis (e_1, e_2), it holds that

$$\begin{aligned}(\omega_1 \wedge \omega_2)(e_1, e_2) &= \omega_1(e_1)\omega_2(e_2) - \omega_1(e_2)\omega_2(e_1) \\ &= 1 \cdot 1 - 0 \cdot 0 = 1.\end{aligned}$$

Hence, $\omega_1 \wedge \omega_2$ is the volume element.

5.2 Gauss-Bonnet Formula

The Gauss-Bonnet formula (also called local Gauss-Bonnet theorem) relates the Gaussian curvature of a surface to the geodesic curvature of a curve and leads to the Gauss-Bonnet theorem. The geodesic curvature measures how far a curve on a surface is away from being a geodesic, that is, a curve on the surface for which its acceleration is either zero or parallel to its unit normal vector for each point on that curve.

Definition 5.3 *Let γ be a regular curve on the regular surface M. The geodesic curvature of γ at a given point $p \in \gamma$ is defined as*

$$\kappa_\gamma(p) := \langle \ddot{\gamma}, N \times \dot{\gamma} \rangle .$$

Proposition 5.2 *Let M be a regular surface. For each $p \in M$, it holds that the Gaussian curvature $K(p)$, by choosing a moving orthonormal frame $(e_1, e_2, n = e_1 \times e_2)$ around p, satisfies*

$$d\omega_{12}(p) = -K(p) \ dA(p) = -K(p)(\omega_1 \wedge \omega_2)(p).$$

Since $d\omega_{12}$ and dA do not depend on $e_3 = n = e_1 \times e_2$, we see that K is an intrinsic value, as proven in *Theorema Egregium* earlier.

We can now state the Gauss-Bonnet formula.

Lemma 5.3 ***Gauss-Bonnet Formula**: Let (M, n) be an oriented surface with $M \subset \mathbb{R}^3$ and let (ϕ, U) be a coordinate patch with $\phi : U \to \mathbb{R}^3, \phi(U) \subset M$. Let γ be a piecewise regular curve on M enclosing a region $R \subset M$. Let $\{\gamma_i\}_{i=1}^n$ be the regular curves that form γ and denote by $\{\alpha_i\}_{i=1}^n$ the jump angles at the junction points (exterior angles).*
It holds that:

$$\int_R K \ dA + \int_\gamma \kappa_\gamma \ ds = 2\pi - \sum_{i=1}^n \alpha_i,$$

where K is the Gaussian curvature of M and dA is its Riemannian volume element.

Proof Let $(e_1, e_2, n = e_1 \times e_2)$ be an oriented orthonormal moving frame on a regular curve γ_i on M. Let (ω_1, ω_2) be the associated coframe with connection forms $\omega_{jk}, j, k = 1, 2$. Choose another oriented orthonormal frame $(\tilde{e_1} = \dot{\gamma_i}, \tilde{e_2})$ on M with associated coframe $(\tilde{\omega}_1, \tilde{w}_2)$ and connection forms $\tilde{\omega}_{jk}, j, k = 1, 2$. Changing from the first frame to the second is done by a rotation of angle θ. It holds that $(\tilde{e}_1, \tilde{e}_2)$ is written in the (e_1, e_2) basis as:

$$\begin{aligned}
\tilde{e}_1 &= \cos\theta \cdot e_1 + \sin\theta \cdot e_2, \\
\tilde{e}_2 &= -\sin\theta \cdot e_1 + \cos\theta \cdot e_2.
\end{aligned}$$

Furthermore

$$\begin{aligned}
d\tilde{e}_1 &= \tilde{\omega}_{12} \cdot \tilde{e_2} + \tilde{\omega}_{13} \cdot \tilde{e}_3, \\
\tilde{\omega}_{12} &= \langle d\tilde{e}_1, \tilde{e}_2 \rangle \\
&= \langle d\tilde{e}_1, \, -\sin\theta \cdot e_1 + \cos\theta \cdot e_2 \rangle \\
&= \langle -\sin\theta d\theta \cdot e_1 + \cos\theta \cdot de_1 + \cos\theta d\theta \cdot e_2 + \sin\theta \cdot de_2, -\sin\theta \cdot e_1 + \cos\theta \cdot e_2 \rangle \\
&= \omega_{12} + d\theta.
\end{aligned}$$

It holds that

$$\begin{aligned}
\kappa_{\gamma_i} \, ds &= \langle \ddot{\gamma}_i, n \times \dot{\gamma}_i \rangle \, ds \\
&= \langle dT, n \times T \rangle ds, \\
dT &= d\tilde{e}_1 = \tilde{\omega}_{12} \cdot \tilde{e}_2 + \tilde{\omega}_{13} \cdot \tilde{e}_3 \\
&= \tilde{\omega}_{12} \cdot n \times T + \tilde{w}_{13} \cdot \tilde{e}_3.
\end{aligned}$$

Thus taking the inner product of dT with $n \times T$ to get κ_{γ_i} gives:

$$\tilde{\omega}_{12} = \kappa_{\gamma_i} \, ds.$$

Going back to proposition 5.2, we get:

$$\begin{aligned}
\int_R K \, dA &= -\int_R d\omega_{12} \\
&\overset{(*)}{=} -\int_\gamma \omega_{12} \\
&= -\sum_i \int_{\gamma_i} (\tilde{\omega}_{12} - d\theta) \\
&= -\int_\gamma (\tilde{\omega}_{12} - d\theta) \\
&= -\int_\gamma \kappa_\gamma \, ds + \int_\gamma d\theta.
\end{aligned}$$

In $(*)$ we used the theorem of Stokes, which we will prove later. It holds that $d\theta$ is the change of angle. Integrated over a closed curve this gives the total rotational angle of the tangent vector starting in a point $p \in M$ (and ending there again). If the curve is simple, according to the Turning Tangent Theorem, this is equal to 2π. If there are singularities, thus kinks, in the curve γ, we can decompose γ into $\{\gamma_i\}_{i=1,\dots,n}$ simple curves. The exterior angle α_i can be defined as the jump angle between the tangent vector at the curve γ_i and the curve γ_{i+1}. It holds that the total rotational angle is (according to the Turning Tangent Theorem) 2π minus the sum over the 'added angles', namely the exterior angles. This gives $\int_\gamma d\theta = 2\pi - \sum_i \alpha_i$ and thus

$$\int_R K\, dA + \int_\gamma \kappa_g\, ds \quad = \quad 2\pi - \sum_i \alpha_i.$$

□

The Gauss-Bonnet formula leads to the Gauss-Bonnet Theorem with the aid of triangulations. This is a construction borrowed from algebraic topology.

Definition 5.4 *A triangulation of a compact regular surface M consists of a finite family of closed subsets $\{T_1, \dots, T_n\}$ and homeomorphisms $\{\phi_i : \mathcal{T}' \to T_i \in \mathbb{R}^2\}$ where $\mathcal{T}$ is a triangle in $\mathbb{R}^2$ such that:*

1. $\bigcup_i T_i = M$.
2. *For $i \neq j, T_i \cap T_j \neq \emptyset$ implies $T_i \cap T_j$ is either a single vertex or a single edge.*

Every compact regular surface possesses a triangulation. In fact, it was proved by Tibor Radó in 1925 that every compact topological 2-manifold possesses a triangulation.

Theorem 5.4 *(**Radó**) Every compact surface M admits a triangulation. Equivalently, M can be constructed, up to homeomorphism, by taking finitely many copies of the standard 2-simplex Δ, that is, triangles, and gluing them together appropriately along edges.*

5.3 Euler Characteristic

The Euler characteristic is a topological invariant of a 2-manifold which describes its shape regardless of the way it is bent. It moreover gives a simple way to determine topologically equivalent manifolds. It is defined as follows:

Definition 5.5 *If M is a triangulated 2-manifold, the Euler characteristic of M with respect to the given triangulation is defined as*

$$\chi(M) = V - E + F,$$

with V, E and F the number of vertices, edges and faces of the given triangulation.

It holds that the Euler characteristic is independent of the chosen triangulation.
We can examine what happens to the Euler characteristic if we add a hole to the surface, by attaching a handle. This can be done by removing two 'triangles' of the triangulation and gluing the handle. Then the original triangulation will be reduced by two faces and the new triangulation gains the triangulation of the handle, which can be triangulated into 6 triangles, thus having 6 new edges, and 6 faces, with the vertices coinciding with the vertices of the removed triangles.
Computing the Euler characteristic we get

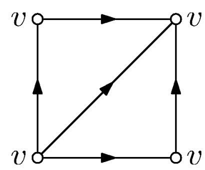
$$\chi(M) = V' - E' + F' = V - (E - 6 + 12) + (F - 2 + 6) = V - E + F - 2.$$

We see that $\chi(M)$ has been lowered by 2.

It holds that the Euler characteristic is related to the genus of the surface. The genus of a connected, orientable regular surface M is the maximum number of cuttings along non-intersecting closed simple curves without rendering the resultant manifold disconnected. It is equal to the number of holes in it. Alternatively, it can be defined for closed surfaces in terms of the Euler characteristic χ, through the relationship

$$\chi(M) = 2 - 2g,$$

where g is the genus.
This is consistent with the fact that $\chi(M)$ has been lowered by 2 by adding a hole.
To further illustrate the Euler characteristic, choose the following triangulation for the torus:



This triangulation is composed of 2 triangles with 1 vertex, 3 edges and 2 faces which gives an Euler characteristic of $\chi(\text{torus}) = 1 - 3 + 2 = 0$, which correponds to a genus $g = 1$.

5.4 Gauss-Bonnet Theorem

Theorem 5.5 ***Gauss-Bonnet Theorem****: Let $M \in \mathbb{R}^3$ be an oriented compact regular surface, K its Gaussian curvature and χ its Euler characteristic. Then*

$$\int_M K \ dA = 2\pi\chi(M).$$

Proof Let $\{R_i, i = 1, ..., F\}$ denote the triangles of the triangulation of M, and for each i let $\{\gamma_{ij} : j = 1, 2, 3\}$ be the edges of R_i.
Let $\{\theta_{ij} : j = 1, 2, 3\}$ denote its interior angles. Since each exterior angle is π minus the corresponding interior angle, applying the Gauss-Bonnet formula to each triangle and summing over the amount of triangles F gives:

$$\sum_{i=1}^{F} \int_{R_i} K \ dA + \sum_{i=1}^{F} \sum_{j=1}^{3} \int_{\gamma_{ij}} \kappa_\gamma \ ds + \sum_{i=1}^{F} \sum_{j=1}^{3} (\pi - \theta_{ij}) = \sum_{i=1}^{F} 2\pi. \tag{1}$$

It holds that each edge appears twice in the above sum, with opposite directions, therefore the integrals of κ_γ cancel out. Thus (1) becomes;

$$\int_M K \ dA + 3\pi F - \sum_{i=1}^{F} \sum_{j=1}^{3} \theta_{ij} = 2\pi F. \tag{2}$$

At each vertex, the sum of the interior angles of the surrounding triangles is equal to 2π and thus

$$\int_M K \ dA = 2\pi V - \pi F. \tag{3}$$

Since each edge appears in exactly two triangles, and each triangle has exactly three edges, the total number of edges counted with multiplicity is $2E = 3F$ where we count each edge once for each triangle in which it appears. This means that $F = 2E - 2F$, and thus (3) becomes:

$$\int_M K \ dA = 2\pi V - 2\pi E + 2\pi F = 2\pi\chi(M).$$

$\square$

6 Gaussian Curvature and the Index of a Vector Field

6.1 Change of frames

Equip the tangent space of the regular surface M with the standard inner product of $\mathbb{R}^3$ and let X be a vector field on M with isolated singularity $p \in M$, that is $X(p) = 0$, and let U_p be a neighbourhood of p containing no other singularities than p. Define $e_1 = X/|X|$ on $U_p \setminus \{p\}$ and complete e_1 to an oriented orthonormal moving frame (e_1, e_2) on U_p. Let (ω_1, ω_2) be the associated coframe with connection forms $\omega_{ij}, i, j = 1, 2$. Choose another oriented orthonormal frame $(\tilde{e_1}, \tilde{e_2})$ on U_p. This arbitrary frame has to be different from (e_1, e_2) since e_1 is not defined at p. Let's investigate what happens to the connection form ω_{12} if we change the local orthonormal frame. For each $q \in U_p$, denote by A the change of basis matrix from (e_1, e_2) to $(\tilde{e}_1, \tilde{e}_2)$. Since the transformation is a rotation of angle θ (between e_1 and $\tilde{e}_1$), it is clear that the matrix A is orthogonal and equals:

$$A = \begin{pmatrix} f & g \\ -g & f \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Thus the angle of rotation between the two orthonormal frames θ equals $\arctan \frac{g}{f}$. This is not well-defined since it is multi-valued. But it turns out that its 1-form

$$d\theta = d\left(\arctan \frac{g}{f}\right) = \frac{f dg - g df}{f^2 + g^2} = f dg - g df := \tau$$

is well-defined.
Now let $(\tilde{\omega}_1, \tilde{\omega}_2)$ be the coframe associated with $(\tilde{e}_1, \tilde{e}_2)$ and let $\tilde{\omega}_{12}, \tilde{\omega}_{21}$ be its corresponding connection forms.
It holds that:

$$\tilde{\omega}_{12} - \omega_{12} = \tau,$$

where $\tau = f dg - g df$ is called the $(e_1, e_2) \to (\tilde{e}_1, \tilde{e}_2)$ change of frame. Thus, if $(\tilde{e}_1, \tilde{e}_2)$ is another orthonormal frame we have

$$\begin{aligned} \tilde{e}_1 &= \cos\theta\, e_1 + \sin\theta\, e_2, \\ \tilde{e}_2 &= -\sin\theta\, e_1 + \cos\theta\, e_2. \end{aligned}$$

Therefore $\tilde{\omega}_1 \wedge \tilde{\omega}_2 = \omega_1 \wedge \omega_2$ and $\tilde{\omega}_{12} = \omega_{12} + d\theta$.

6.2 The Index of a Vector Field

Let C be a simple closed curve that bounds a region D in U_p, where D is homeomorphic to a unit disk in $\mathbb{R}^2$ and contains p in its interior and think of $(\tilde{e}_1, \tilde{e}_2)$ as the reference frame, then $\tau = \omega_{12} - \tilde{\omega}_{12}$ is the $(\tilde{e}_1, \tilde{e}_2) \to (e_1, e_2)$ change of frame (which is the opposite change of frame of what we defined earlier). The index of a vector field at an isolated singularity p is defined as follows:

Definition 6.1 *Let M be a regular surface and X a smooth vector field on M with an isolated singularity $p \in M$, that is $X(p) = 0$, and let C be a simple closed curve in M. The number*

$$Ind_X(p) = \frac{1}{2\pi} \int_C \tau$$

is called the index of X at p.

The index of X at the singularity p is an integer representing how many full turns the vector field makes around p, when moving along the curve C. We can illustrate that with the following example:

Example Consider the smooth vector field

$$X(x, y) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

The only singularity X has is the origin $(0, 0)$. Let $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ be the reference frame and let C be the unit circle. We obtain

$$\tau = d\left(\arctan\frac{-y}{x}\right) = \frac{-xdy + ydx}{x^2 + y^2},$$

so

$$Ind_X((0, 0)) = \frac{1}{2\pi} \int_C -xdy + ydx = \frac{1}{2\pi} \int_0^{2\pi} (-\cos^2 t - \sin^2 t)dt = -1.$$

Lemma 6.1 *The index of a smooth vector field X at a point p is independent of the choice of the curve C.*

Proof Let C_1, C_2 be two simple closed curves. First, assume they do not intersect. Let B be the region bounded by C_1 and C_2.

Thus $\partial B = C_1 \cup (-C_2)$. Then:

$$\begin{aligned}
\int_{C_1} \tau - \int_{C_2} \tau &= \int_{C_1 \cup (-C_2)} \tau \\
&= \int_{\partial B} \tau \\
&\overset{(*)}{=} \int_B d\tau \\
&= 0.
\end{aligned}$$

In $(*)$ we used the theorem of Stokes. If C_1 and C_2 intersect, take a simple closed curve C_0 that does not intersect C_1 and C_2 and apply the previous argument to show that $\int_{C_i} \tau = \int_{C_0} \tau$ for $i = 1, 2$. $\square$

Lemma 6.2 *The index of a smooth vector field X at a point p does not depend on the Riemannnian metric.*

Proof Let r_0, r_1 be two Riemannian metrics on M. Define

$$r_t = (1-t)r_0 + t \cdot r_1, \qquad 0 \leq t \leq 1.$$

Then r_t is a family of Riemannian metrics connecting r_0 with r_1. Denote the index of X at p with respect to r_t by $Ind[t]$. From the definition of the index it holds that $Ind[t]$ is a continuous function of t. Since it has integer values, it follows that $Ind[t]$ is constant, so $Ind[0] = Ind[1]$. $\square$

6.3 Relationship Between Curvature and Index

Theorem 6.3 *For every compact orientable regular surface M and every smooth vector field X on M with only isolated zeros $p_1, \ldots, p_n$, we have*

$$\int_M K \, dA = 2\pi \sum_{i=1}^{n} Ind_X(p_i).$$

Proof Let $U = M \setminus \{p_1, \ldots, p_n\}$, then X is non-vanishing on U. Thus we can define $e_1 = X/|X|$ and complete it to an oriented orthonormal frame of reference on U by constructing e_2 as a unit vector field orthogonal to e_1. Let (ω_1, ω_2) be the associated coframe with connection forms $(\omega_{ij})_{i,j=1,2}$.
For each $1 \leq i \leq n$, choose a small open disk B_i, with respect to the Riemannian distance on the regular surface M, centered at p_i, in a way that B_i contains no other singularities of X other than p_i. Let $V = M \setminus (\sum_{i=1}^{n} B_i)$. It is clear that V is a 2-manifold with boundary

the union of $\partial B_i, 1 \leq i \leq n$. The orientation of each ∂B_i as a boundary element of V is always the opposite of the orientation of ∂B_i as the boundary of B_i.
We now compute the integral over the Gaussian curvature. This gives:

$$\begin{aligned}
\int_V K\, dA &= -\int_V d\omega_{12} \\
&\overset{(*)}{=} -\int_{\partial V} \omega_{12} \\
&= \sum_{i=1}^{n} \int_{\partial B_i} \omega_{12}.
\end{aligned}$$

In $(*)$ we used the theorem of Stokes again.

To compute this we need the following lemma:

Lemma 6.4 *If B_r is the disk of radius r centered at $p \in M$ then:*

$$\lim_{r \to 0} \frac{1}{2\pi} \int_{\partial B_r} \omega_{12} = Ind_X(p).$$

Proof By the Cauchy criterion it holds that the above limit exists if and only if

$$\lim_{r,s \to 0} \left| \int_{\partial B_r} \omega_{12} - \int_{\partial B_s} \omega_{12} \right| = 0.$$

If $s < r$, denote the annulus bounded by ∂B_r and ∂B_s by A_{rs}. Then $\partial A_{rs} = \partial B_r - \partial B_s$. We get

$$\begin{aligned}
\int_{\partial B_r} \omega_{12} - \int_{\partial B_s} \omega_{12} &= \int_{\partial B_r - \partial B_s} \omega_{12} \\
&= \int_{\partial A_{rs}} \omega_{12} \\
&\overset{(*)}{=} \int_{A_{rs}} d\omega_{12} \\
&= 0
\end{aligned}$$

as $r, s \to 0$ since the area of A_{rs} goes to zero. In $(*)$, Stokes is applied. This shows that the limit exists. We now have to show that this limit is the index of the vector field on M. It holds that $d\omega_{12} = d\tilde{\omega}_{12} = -K\, dA$ and that $\omega_{12} = \tilde{\omega}_{12} + \tau$. This gives:

$$\int_{\partial B_r} \omega_{12} - \int_{\partial B_s} \omega_{12} = \int_{A_{rs}} d\omega_{12}.$$

Letting $s \to 0$, we see that $A_{rs} \to B_r$ and thus we get:

$$\begin{aligned}\int_{\partial B_r} \omega_{12} - 2\pi I &= \int_{B_r} d\omega_{12} \\ &= -\int_{B_r} K\ dA.\end{aligned}$$

This can be done since $d\omega_{12} = -K\ dA$ is globally defined on M.
It holds that $\int_{\partial B_r} \omega_{12} = -\int_{B_r} K\ dA$. This gives:

$$\begin{aligned}\int_{\partial B_r} \omega_{12} &= \int_{\partial B_r} (\tilde{\omega}_{12} + \tau) \\ &= -\int_{B_r} K\ dA + \int_{\partial B_r} \tau \\ &= -\int_{B_r} K\ dA + 2\pi \mathrm{Ind}_X(p_i).\end{aligned}$$

This shows that the limit indeed equals the index of the vector field on M.
Thus, letting the radii of B_i go to zero and using the above lemma we get:

$$\int_M K\ dA = 2\pi \sum_{i=1}^{n} Ind_X(p_i).$$

□

From this we can see how vector field theory relates to topology through differential geometry.

Corollary 6.5 *(Poincaré-Hopf) Let M be a regular orientable surface and X a smooth vector field on M with only isolated singularities $p_1, \ldots, p_n$. Then:*

$$\sum_{i=1}^{n} Ind_X(p_i) = \chi(M).$$

7 Stokes' Theorem

Another example to see how differential forms relate to the mathematics of vector fields is the theorem of Stokes.

Theorem 7.1 *Let ω denote a differential $(n-1)$-form on a compact oriented n-manifold M. Suppose that M has a smooth and compact boundary ∂M with the induced orientation. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

We are going to prove Stokes in the general case, for an n-manifold. Since we are not working on a surface now, we denote a manifold by X.
First we need some definitions we have not encountered before.

Definition 7.1 *Let X be a smooth oriented manifold. The support of a function $f : X \to \mathbb{R}$ is the closure of the set of $p \in X$ such that $f(p) \neq 0$, that is*

$$supp(f) := \overline{\{p \in X | f(p) \neq 0\}}.$$

Definition 7.2 *Let X be a manifold with open cover $\mathcal{U} = \bigcup_i (U_i)_{i\in I}$. A partition of unity subordinate to $\mathcal{U}$ is a family $(\rho_j)_{j\in J}$ of smooth functions $\rho_j : X \to [0,1]$ with compact supports such that*

1. *for each $j \in J$, there exists an $i_j \in I$ such that $supp(\rho_j) \subset U_{i_j}$,*
2. *for every compact subset $K \in X$, we have $K \cap supp(\rho_j) \neq \emptyset$ for only finitely many $j \in J$,*
3. *$\sum_{j\in J} \rho_j = 1$ on X.*

It holds that if $\mathcal{U}$ is an open cover of a manifold X then such a partition of unity subordinate to $\mathcal{U}$ exists.

Proof of the theorem of Stokes

Let $\mathcal{A}$ be an oriented atlas of charts $(\phi_i : U_i \to V_i)_{i\in I}$, with the corresponding orientation on X. Let $(\rho_j)_{j\in J}$ be a partition of unity subordinate to the open cover $\mathcal{U}_i$ of X. Then

$$\omega = \sum_{j\in J} \rho_j \, \omega.$$

It holds that:

$$\sum_j \int_{U_{i_j}} d(\rho_j \, \omega) \quad = \quad \sum_j \left(\int_{U_{i_j}} d\rho_j \wedge \omega + \int_{U_{i_j}} \rho_j (d\omega) \right).$$

Since $\sum_j \rho_j = 1$ then $\sum_j d\rho_j = 0$ and thus (since it is a finite sum) it holds that $\int_{U_{i_j}} \sum_j d\rho_j = 0$.
The fact that we can write ω as

$$\omega = \sum_{j \in J} \rho_j \; \omega$$

and due to the linearity of the integral, we can resume the proof to the case of the manifold X with boundary ∂X being the half-space

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 \leq 0\}$$

with boundary $\partial \mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 = 0\}$. This boundary $\partial \mathbb{H}^n$ can be viewed as $\mathbb{R}^{n-1}$. This means that we only need to prove Stokes' theorem for forms with support in a coordinate neighbourhood, that is, for differential forms with a compact support in $\mathbb{H}^n$.
Let ω be a differentiable $(n-1)$-form with compact support in $\mathbb{H}^n$.
For $n = 1$,we have 0-forms, that is, functions. The result follows directly from the Fundamental Theorem of Calculus. In this case ω is a smooth function of compact support on $(-\infty, 0]$ and thus

$$\int_{\mathbb{H}^1} d\omega = \int_{-\infty}^{0} \omega'(x) dx = \omega(0) = \int_{\partial \mathbb{H}^1} \omega.$$

For $n \geq 2$, we can write ω as

$$\omega = \sum_{i=1}^{n} f_i \;\; dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$$

with smooth functions $f_1, \ldots, f_n$, where the hat marks a missing term. Thus we get

$$d\omega = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x_i} \;\; dx_1 \wedge \ldots \wedge dx_n.$$

For $i \geq 2$, since f_i has compact support in $\mathbb{H}^n$, and for every $c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n$ in $\mathbb{R}$ with $c_1 \leq 0$ it holds that

$$\int_{\mathbb{R}} \frac{\partial f_i}{\partial x_i}(c_1, \ldots, c_{i-1}, t, c_{i+1}, \ldots, c_n) dt = 0$$

by the Fundamental Theorem of Calculus. Thus it holds that

$$\int_{\mathbb{H}^n} \frac{\partial f_i}{dx_i} dx_1 \ldots dx_n = 0.$$

The pullback of ω, defined by the inclusion $i : \partial\mathbb{H}^n \hookrightarrow \mathbb{H}^n$, is $i^*\omega = f_1 dx_2 \wedge \ldots \wedge dx_n$ and thus

$$\begin{aligned}
\int_{\mathbb{H}^n} d\omega &= \int_{\mathbb{H}^n} \frac{\partial f_i}{\partial x_i} dx_1 \ldots dx_n \\
&= \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{0} \frac{\partial f_1}{\partial x_1} dx_1 \right) dx_2 \ldots dx_n \\
&= \int_{\mathbb{R}^{n-1}} f_1(0, x_2, \ldots, x_n) dx_2 \ldots dx_n \\
&= \int_{\partial\mathbb{H}^n} \omega.
\end{aligned}$$

using the Fundamental Theorem of Calculus again. This completes the proof of the theorem. □

In vector calculus, operations involving the gradient of a function and curl or divergence of a vector field are commonly used. The exterior differential is similar to these actions when working on differential forms. We restrict ourselves to $\mathbb{R}^3$ to establish the following correspondence:
Differentiable scalar functions correspond to 0-forms in $\mathbb{R}^3$.
Let $f(x, y, z)$ be a differentiable scalar function in $\mathbb{R}^3$. The gradient of f corresponds to the 1-form df:

$$\nabla f(x, y, z) \leftrightarrow df = \omega^1_{\nabla f} = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

Let F be a vector field in $\mathbb{R}^3$. It corresponds to the 1-form:

$$F = (F_x, F_y, F_z) \leftrightarrow \omega^1_F = F_x\ dx + F_y\ dy + F_z\ dz.$$

The curl out of F corresponds to the exterior differential of ω^1_F, which gives a 2-form:

$$\nabla \times F \leftrightarrow \omega^2_F = F_x\ dy \wedge dz - F_y\ dx \wedge dz + F_z\ dx \wedge dy.$$

The divergence of F corresponds to the 3-form:

$$\nabla \cdot F \leftrightarrow d\omega^2_F = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dx \wedge dy \wedge dz.$$

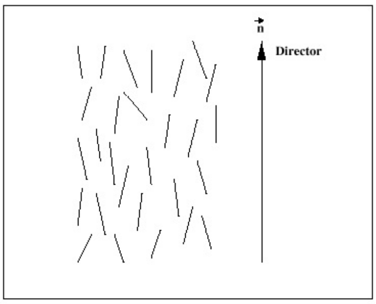
This correspondence allows us to formulate the classical version of the theorem of Stokes in $\mathbb{R}^3$:

Theorem 7.2 *Let M be an oriented regular surface with smooth boundary C, F be a differentiable vector field on M and n be the outward-pointing normal vector to M. Then:*

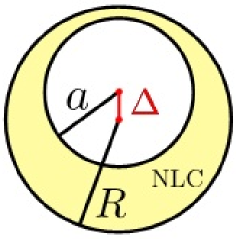
$$\iint_M (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\ dA = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r}.$$

8 Application to Physics

The Poincaré-Hopf theorem can be visualised through the following experiment including nematic liquid crystals [6, 7]. A nematic liquid crystal is a distinct phase observed between the crystalline (solid) and isotropic (liquid) states of matter. It is characterized by molecules having no positional order but tending to self-align in the same direction to have a long-range directional order with their long axes roughly parallel. We can then talk about a nematic direction vector, a director. Most nematic liquid crystals (referred to as nematics) are uni-axial: they have one axis which is longer and preferred. In the following diagram, we see a distribution of nematics:



Light is an electromagnetic wave and thus exhibits polarization: its waves can oscillate in more than one direction. When unpolarized light, that is, light with no defined polarization, passes through a polarizer, only waves of a certain polarization are transmitted (the waves that are in the plane of polarization) and waves of different polarizations are blocked. Two polarizers used together transmit light differently depending on their relative orientation. For example, when the polarizers are arranged so that their planes of polarization are perpendicular to each other, the light is blocked. When the second filter (called the analyzer) is parallel to the first, all of the light passed by the first filter is also transmitted by the second.
The behaviour of a nematic liquid crystal, which, due to the alignment of the molecules, can be seen as a vector field, is studied through the following experimental system, a nematic double emulsion droplet: a nematic liquid crystal droplet of radius R (yellow) encapsulates a smaller water droplet of radius a (white) as seen in the figure below:

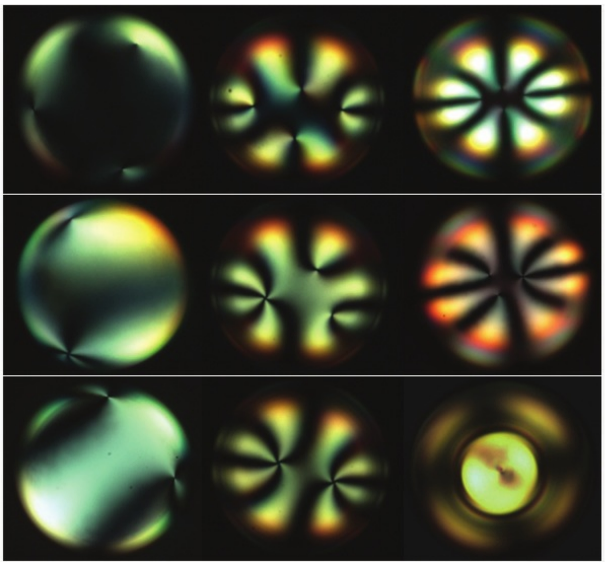


Schematic of a nematic double emulsion droplet of radius R. The inner water droplet of radius a is displaced by an amount Δ along the vertical direction, which makes the top of the shell thinner.

Since a nematic liquid crystal has one preferred long axis, it holds that this molecule, when rotated, returns to its preferred position after a rotation of only π. This means that we can have a singularity with index $\frac{1}{2}$.
Experimental model systems of spherical nematics have shown that the thickness of the shell plays a crucial role in the distribution and index of the singularities of a field of nematics. As an example, for a field of nematics with two singularities, there are two possible configurations depending on the homogeneity of the thickness of the shell. If the shell is rather thick and homogeneous over the whole sphere, then the singularities tend to align in such a way that the distance between them is maximized, that is, diametrically. But, when increasing the thickness inhomogeneity towards a critical value, one defect migrates towards the other one such that the singularities are all confined on the thinnest part of the shell.
The link to the mathematics of surfaces comes from the theorem of Poincaré-Hopf which tells us that the sum of the indices, that is, the singularities of a vector field on a sphere is always equal to two. This means that the following configurations can theoretically occur:

- $1\times$ index 2,
- $2\times$ index 1,
- $1\times$ index $1 + 2\times$ index $\frac{1}{2}$,
- $1\times$ index $\frac{3}{2} + 1\times$ index $\frac{1}{2}$.
- $4\times$ index $\frac{1}{2}$

The cases involving an index of 2 or $\frac{3}{2}$ are not likely to occur, due to energetical reasons. The possible configurations are shown in the following figure.



The first row shows different configurations of a nematic shell with four defects of charge $\frac{1}{2}$. *The second row shows different configurations of a nematic shell with three defects (one with charge* 1 *and two with* $\frac{1}{2}$. *The last row shows different configurations of a nematic shell with two defects of charge* 1.

In the experiment above, unpolarized light is sent through a polarizer and an analyzer positioned with their axes of polarization perpendicular to each other. The nematic droplet is used as an intermediate medium between the two polarizers. The nematic director of the nematic is uniform and makes an angle θ with the axis of the first polarizer.
If the angle θ is 0, the nematic does not change the polarization of the light and thus the light is blocked. If the angle θ is $\frac{\pi}{2}$ the polarized light is blocked by the nematic and no light passes. Finally, if θ is some other angle different from $0, \frac{\pi}{2} \pmod{\pi}$, the nematic changes the polarization of the light and light passes through the second polarizer and can be detected.
Using this last setup, simply counting the 'number' of 'brushes' (enlightened parts) at each defect and dividing it by 4 gives its index. (We get two brushes at a defect of index $\frac{1}{2}$ and four brushes at a defect of index 1).

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