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The Katowice Problem

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Contents

Introduction

Given a cardinal κ, its power set $\mathcal{P}(\kappa)$ is a Boolean algebra, with union, intersection and complementation as its operations. Simply by looking at the number of atoms, it follows that for any two distinct cardinals these algebras are not isomorphic. If we remove the atoms, i.e., take the quotient algebra $\mathcal{P}(\kappa)/\mathit{fin}(\kappa)$ of the given algebra modulo the ideal consisting of the finite subsets of κ, is it possible that for two distinct infinite cardinals these quotient algebras are isomorphic? It turns out that this question is not so simple to answer. In [1978], Balcar and Frankiewicz showed, building on an earlier result by Frankiewicz in [1977], for all but one pair of distinct infinite cardinals that the corresponding quotient algebras are indeed not isomorphic. For the only pair left, consisting of the first and second smallest infinite cardinals ω and ω_1, it remains to this day, an open problem to determine if it is consistent that the quotient algebras $\mathcal{P}(\omega)/\mathit{fin}(\omega)$ and $\mathcal{P}(\omega_1)/\mathit{fin}(\omega_1)$ are isomorphic. This problem is the main subject of this thesis; since it originates and has been studied extensively at the University of Silesia in the Polish city Katowice, it is commonly called the Katowice problem.

Due to Stone's duality [1937], it is possible to give an equivalent topological variant of the problem above. Assume that all cardinals carry the discrete topology and let $\beta\kappa$ denote the Čech-Stone compactification of a cardinal κ. The subspace $\kappa^* = \beta\kappa \backslash \kappa$ of this compactification is called the Čech-Stone remainder of κ. By the duality, the quotient algebras $\mathcal{P}(\kappa)/\mathit{fin}(\kappa)$ and $\mathcal{P}(\lambda)/\mathit{fin}(\lambda)$ are isomorphic precisely when the Čech-Stone remainders of the cardinals κ and λ are homeomorphic. Consequently, the Katowice problem is equivalent to the problem of determining if it is consistent that the spaces ω^* and ω_1^* are homeomorphic. This problem is often called one of the most interesting problems or even the most interesting problem about the Čech-Stone compactification of ω. Note that this variant only makes sense if the axiom of choice holds; in the absence of this axiom, the spaces ω^* and ω_1^* need not exist.

The generally accepted belief is that the Katowice problem has a negative answer, or, in other words, that there does not exist a model in which the remainders ω^* and ω_1^* are homeomorphic. The strategy to actually show this, is to determine consequences of the assumption that the given remainders are homeomorphic and to try to derive from these a contradiction. One consequence, similar to what was used so successfully by Balcar and Frankiewicz to obtain their result, is the existence of an ω_1-scale. However, this consequence alone is not enough to obtain a contradiction since such a scale exists for example in every model in which the continuum hypothesis holds. Another important consequence is the existence of an uncountable strong Q-sequence. This consequence has been shown to be consistent by

Steprāns in [1985]. Furthermore, a model given by Chodounský in [2011] shows that it is consistent that these two consequences hold simultaneously. A fairly new and still unpublished consequence, due to the advisor of this thesis, is the existence of a nontrivial autohomeomorphism of ω^*. A result by Rudin given in [1956] shows that it is consistent that such a nontrivial autohomeomorphism exists. It is however an open question whether a model exists in which all three consequences hold simultaneously. If such a model does not exist, then the structure used to define this autohomeomorphism seems a good candidate to derive a contradiction from since the other two consequences follow directly from it. Although we have studied this structure quite a bit, we did not find a contradiction.

In this thesis we treat the Katowice problem and the corresponding results discussed above. It is assumed that the reader is familiar with the basic concepts of general topology and set theory.

Already used in this introduction, a number in square brackets denotes the year of a publication by a given author and refers to the bibliography at the end of this thesis.

Each chapter in this thesis ends with a section containing notes. The purpose of these notes is to give references to the different books and papers that were used during the writing of that chapter.

In the first chapter we explain the structures and concepts that are needed to understand the arguments that are used in the two chapters that follow. First, the Čech-Stone compactification of a topological space and the corresponding remainder are defined. To prove that the two variants of the Katowice problem given above are equal, and to give a better description of the Čech-Stone remainder of a discrete space, we define Boolean algebras and prove Stone's duality.

In Chapter 2 we define scales and give a proof, based on the proof given by Chodounský in [2011], of the aforementioned result by Balcar and Frankiewicz.

In Chapter 3 we discuss in each section a consequence that follows from a positive answer to the Katowice problem. These are, in the order in which they are given, a consequence concerning the weights of the Čech-Stone remainders of ω and ω_1, the existence of an uncountable strong Q-sequence, a consequence concerning the small cardinals $\mathfrak{t}$, $\mathfrak{b}$ and $\mathfrak{d}$, and, finally, the existence of a nontrivial autohomeomorphism of ω^*.

CHAPTER 1

Basic notions

The Čech-Stone remainders of discrete spaces play a central role in this thesis. This chapter is intended to introduce the reader to these remainders and to give all the relevant terminology and notations.

Some of the results of this chapter are not needed for the main purpose of this thesis but are included for their own sake.

1.1. The Čech-Stone compactification

A topological space Y is a *compactification* of a topological space X if Y is a compact Hausdorff space and X can be embedded in Y as a dense subspace. If e is the corresponding embedding then we may (and always will) identify the space X with $e[X]$ and consider X as a subspace of Y.

If X is a subspace of the topological space Y and f a function of X to a topological space Z, then f is said to be *continuously extendable* over Y if there is a continuous mapping g of Y to Z such that $g(x) = f(x)$ for every point $x \in X$; the mapping g is called a *continuous extension* of f over Y.

Let I denote the closed unit interval $[0, 1]$.

1.1. Definition. A compactification Y of a topological space X is called a *Čech-Stone compactification* of X if every continuous mapping of X to I is continuously extendable over Y.

1.2. Remark. If the condition given in the definition above is replaced by the requirement that for every compact Hausdorff space K and every continuous mapping f of X to K there is a continuous extension of f over Y, then we get an equivalent definition.

1.3. Example. Since the mapping $f : (0, 1] \to [-1, 1]$ defined by $f(x) = \sin(1/x)$ does not have a continuous extension to the compact Hausdorff space $[0, 1]$, it follows that $[0, 1]$, the one-point compactification of $(0, 1]$, is not a Čech-Stone compactification of $(0, 1]$.

1.4. Example. Consider the sets ω_1 and $\omega_1 + 1$ together with the order topology. Then ω_1 is a dense subspace of the compact Hausdorff space $\omega_1 + 1$. Any continuous function f of ω_1 to I must eventually be constant; if we define a function g to be equal to f on ω_1 and let $g(\omega_1)$ be this constant, then g is a continuous extension of the mapping f over $\omega_1 + 1$. This shows that $\omega_1 + 1$ is a Čech-Stone compactification of ω_1.

If a topological space X has two Čech-Stone compactifications then one can show that these compactifications are equivalent, i.e., there exists a homeomorphism between the two compactifications that maps every point of the space X, considered as a subspace of both compactifications, onto itself. For this reason, any compactification Y that satisfies the condition of Definition 1.1 is called *the* Čech-Stone compactification of X and is always denoted by the symbol βX.

Since X is a dense subspace of βX, it follows that the continuous extension of a function f of X to I over βX is unique; this extension is called the *Čech-Stone extension* of f and is usually denoted by βf.

Let $C(X, I)$ be the collection of all continuous functions from X to I.

1.5. Theorem. *Every Tychonoff space has a Čech-Stone compactification.*

Proof. Let X be a Tychonoff space. Take the embedding e of X into $I^{C(X,I)}$ defined by $e(x)(f) = f(x)$. Then βX is the closure of $e[X]$. If π_f is the projection on the f-th coordinate of $I^{C(X,I)}$, then the restriction of π_f to βX extends the mapping f; this shows the extension property. □

The following theorem gives another property that characterizes the space βX among all compactifications of a normal space X.

1.6. Theorem. *Let X be a normal space; a compactification Y of X is the Čech-Stone compactification of X if and only if for any two disjoint closed subsets F and G of X, the closures of F and G in Y are disjoint.*

Proof. Let us first prove that the condition of the theorem is necessary. If F and G are two disjoint closed subsets of X, then by Urysohn's lemma the sets F and G are completely separated, i.e., there exists a continuous function f of X to I such that $f(x) = 0$ for $x \in F$ and $f(x) = 1$ for $x \in G$. Let g be the continuous extension of f over Y. Since

$$\mathrm{cl}_Y\, f^{-1}(0) \cap \mathrm{cl}_Y\, f^{-1}(1) \subseteq g^{-1}(0) \cap g^{-1}(1) = \varnothing,$$

it follows that the closures of F and G in the space Y are disjoint.

A theorem by Taĭmanov [1952] shows that the condition of the theorem is sufficient. □

Let Y be a compactification of the space X; the subspace $Y \backslash X$ of Y is called the *remainder* or *growth* of the compactification Y. The remainder $\beta X \backslash X$ of the compactification βX is called the *Čech-Stone remainder* of X and is denoted by X^*.

The following proposition gives an important property of remainders.

1.7. Proposition. *If Y and Z are compactifications of a space X and $f : Y \to Z$ a continuous mapping such that f restricted to X is an autohomeomorphism of X, then $f[Y \backslash X] = Z \backslash X$.*

Proof. Since $f[Y]$ is a closed subset of Z that contains the dense subset X, it follows that f is onto.

Suppose that there exists a point $x \in Y \backslash X$ such that $f(x) \in X$. Take $y \in X$ such that $f(y) = f(x)$ and let $U, V \subset Y$ be disjoint neighborhoods of x and y respectively. Since $X \backslash V$ is a closed subset of X, there is a closed subset F of Z such that $F \cap X = f[X \backslash V]$. Then the set $f^{-1}[F] \supseteq X \backslash V$ is closed in Y and does not contain the point x. Since X is a dense subset of Y, this implies that x is contained in the closure of V in Y, a contradiction. □

1.2. Boolean algebras

An *algebra of sets*, or sometimes called a *field of sets*, is a family of subsets of a nonempty set X that is closed under the set-theoretic operations of union, intersection and complementation. Boolean algebras can be seen as a generalization of algebras of sets.

1.8. Definition. A *Boolean algebra* is a set B, with at least two distinct elements 0 and 1, binary operations $\vee$ and $\wedge$ and a unary operation $'$, such that for all elements a, b and c in B the following identities hold.

(i) $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$,
(ii) $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$,
(iii) $a \wedge (b \vee c) = a \wedge b \vee a \wedge c$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
(iv) $a \vee 0 = a$ and $a \wedge 1 = a$, and
(v) $a \vee a' = 1$ and $a \wedge a' = 0$.

The operations $\vee$, $\wedge$ and $'$ are called *join*, *meet* and *complement* respectively. The elements 0 and 1 are called *zero* and *one*.

If $A = \{a_1, \ldots, a_n\}$ is a finite subset of a Boolean algebra B, then we let the symbol $\bigwedge A$ denote the meet $a_1 \wedge \cdots \wedge a_n$ and call this the meet of A.

The *symmetric difference* of two elements a and b in a Boolean algebra is denoted by $a \triangle b$ and is equal to the element $(a \wedge b') \vee (a' \wedge b)$.

1.9. Example. Let $\mathcal{P}(X)$ be the family of all subsets of the nonempty set X. If we let the set-theoretic operations union, intersection and complementation be the join, meet and complement operations respectively and zero and one be equal to the elements $\varnothing$ and X, then $\mathcal{P}(X)$ is a Boolean algebra. Using the same operations and constants it follows that any algebra of sets is a Boolean algebra.

1.10. Example. Let X be a topological space; the family $\mathrm{CO}(X)$, consisting of all the closed-and-open subsets of X, is an algebra of sets and therefore a Boolean algebra.

If B is a Boolean algebra, then the relation $\leqslant$ defined by

$$a \leqslant b \text{ if and only if } a \wedge b = a,$$

is a partial order on B. A nonzero element b is called an *atom* if for every element a such that $a \leqslant b$, either a is zero or a is equal to b.

1.11. Definition. Let B be a Boolean algebra. A *filter* in B is a nonempty subset F of B that satisfies the following conditions:

(F1) $0 \notin F$,
(F2) If $a, b \in F$, then $a \wedge b \in F$, and
(F3) If $a \in F$, $b \in B$ and $a \leqslant b$, then $b \in F$.

A filter F in a Boolean algebra B is called a *maximal filter* or *ultrafilter* if every filter in B that contains F is equal to F.

A subset A of a Boolean algebra B has the *finite intersection property* if it is nonempty and the meet of every finite subset of A is not equal to zero.

1.12. LEMMA. *Let A be a subset of a Boolean algebra B. If A has the finite intersection property then there exists an ultrafilter in B that contains A.*

PROOF. It is easily verified that that the set

$$F = \{b \in B : \text{there exists a finite set } C \subseteq A \text{ such that } \bigwedge C \leqslant b\}$$

is a filter in B containing A. By Zorn's lemma, every filter can be enlarged to an ultrafilter. □

1.13. LEMMA. *A nonempty subset F of a Boolean algebra B is an ultrafilter in B if and only if F has the following properties.*

(U1) $0 \notin F$,
(U2) *If $a, b \in F$, then $a \wedge b \in F$, and*
(U3) *If $a \in B$, then either $a \in F$ or $a' \in F$.*

PROOF. We will only prove that the condition of the theorem is necessary. A straightforward proof shows that the condition is sufficient.

If F is an ultrafilter, then properties (U1) and (U2) follow from the fact that F is a filter. Let $a \in B$. If a and a' are both contained in F, then from property (F2) it follows that $a \wedge a' \in F$, a contradiction with property (F1). Suppose that the set $\{a\} \cup F$ has the finite intersection property. Then it follows from Lemma 1.12 that this set is contained in an ultrafilter, hence $a \in F$. If the set does not have the finite intersection property, then there is an element $b \in F$ such that $a \wedge b = 0$. But then $b \leqslant a'$, hence $a' \in F$ by property (F3). This shows that property (U3) holds. □

The dual notion of a filter is an ideal.

1.14. DEFINITION. Let B be a Boolean algebra. An *ideal* in B is a nonempty subset G of B that satisfies the following conditions:

(I1) $1 \notin G$,
(I2) If $a, b \in G$, then $a \vee b \in G$, and
(I3) If $a \in G$, $b \in B$ and $b \leqslant a$, then $b \in G$.

Let G be an ideal in B. The relation $\sim$ defined by

$$a \sim b \text{ if and only if } a \triangle b \in G,$$

is a equivalence relation on B. Let B/G denote the quotient set $B/{\sim}$. If we define for all equivalence classes $[a]$ and $[b]$ in B/G the operations

(i) $[a] \wedge [b] = [a \wedge b]$,
(ii) $[a] \vee [b] = [a \vee b]$, and
(iii) $[a]' = [a']$,

and let $[0]$ and $[1]$ be the zero and one, then B/G is a Boolean algebra. A Boolean algebra of this form is called a *quotient algebra*.

1.15. DEFINITION. A function ϕ of a Boolean algebra A to a Boolean algebra B is called a *(Boolean) homomorphism* if for all elements a and b in A the following three properties hold.

(i) $\phi(a \vee b) = \phi(a) \vee \phi(b)$,
(ii) $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$, and
(iii) $\phi(a') = (\phi(a))'$.

A Boolean homomorphism that is a bijection, i.e., it is one-to-one and onto, is called a *(Boolean) isomorphism*.

If $\phi : A \to B$ is a homomorphism between Boolean algebras A and B then its kernel,

$$\ker(\phi) = \{b \in B : \phi(b) = 0\},$$

is an ideal in A.

1.16. LEMMA. *Let $\phi : A \to B$ be a homomorphism between the two Boolean algebras A and B that is onto. Then B is isomorphic to the quotient algebra $A/\ker(\phi)$.*

PROOF. The mapping $\psi : A/\ker(\phi) \to B$ defined by $\psi([a]) = \phi(a)$ is an isomorphism. $\square$

1.3. Stone duality

To every topological space we can associate a natural Boolean algebra, see Example 1.10. We shall now show that the reverse is also possible.

Consider a Boolean algebra B. Let $\mathrm{St}(B)$ denote the family of all ultrafilters in B and for every element b in B define the set

$$U_b = \{F \in \mathrm{St}(B) : b \in F\},$$

consisting of all ultrafilters in B that contain b.

1.17. LEMMA. *Let B be a Boolean algebra. The following properties hold for all elements a and b of B.*

(i) $U_a \cup U_b = U_{a\vee b}$,
(ii) $U_a \cap U_b = U_{a\wedge b}$, *and*
(iii) $\mathrm{St}(B)\backslash U_a = U_{a'}$.

PROOF. We will only prove the first property. Similar arguments show that the remaining properties hold.

If $F \in U_a \cup U_b$, then $a \in F$ or $b \in F$. Since $a, b \leqslant a \vee b$ and F is a filter in B, we find that $a \vee b \in F$ and thus $F \in U_{a\vee b}$.

Conversely, if $F \in U_{a\vee b}$, then $a \vee b \in F$. Suppose $F \notin U_a$, i.e., $a \notin F$. Since F is an ultrafilter, it follows from property (U3) that $a' \in F$. This implies, by property (F2), that $a' \wedge (a \vee b) = a' \wedge b \in F$. Since $a' \wedge b \leqslant b$, it follows that $b \in F$ and therefore that $F \in U_b$. $\square$

Lemma 1.17 shows that the family $\mathcal{B} = \{U_b : b \in B\}$ is closed under finite intersections and obviously the elements of this family cover $\mathrm{St}(B)$. Hence the family $\mathcal{B}$ is a base for a topology on $\mathrm{St}(B)$; this topology is called the

Stone topology on $\mathrm{St}(B)$ and the resulting space is called the *Stone space* of B. From now on we let $\mathrm{St}(B)$ denote the Stone space of B.

1.18. THEOREM. *Every Stone space is a compact Hausdorff space.*

PROOF. Let B be a Boolean algebra and F and G two distinct ultrafilters in B. Without loss of generality we can assume that there exists an element b in $F \backslash G$. Since G is an ultrafilter, by property (U3) we find that b' is contained in G. Then $F \in U_b$, $G \in U_{b'}$ and $U_b \cap U_{b'} = U_{b \wedge b'} = U_0 = \varnothing$ by Lemma 1.17. Thus $\mathrm{St}(B)$ is a Hausdorff space.

It remains to show that $\mathrm{St}(B)$ is a compact space. Let $\mathcal{A}$ be a family of closed subsets of $\mathrm{St}(B)$ with the finite intersection property. The complement of every set in $\mathcal{A}$ is open and can therefore be written as the union of a subfamily of $\mathcal{B}$. Since the family $\mathcal{B}$ is closed under complements it follows that every set in $\mathcal{A}$ is equal to the intersection of a subfamily of $\mathcal{B}$. Hence, without loss of generality, we can assume that $\mathcal{A}$ is a subfamily of $\mathcal{B}$. Let A be a subset of B such that $\mathcal{A} = \{U_a : a \in A\}$. Then for any finite subset C of A we have

$$U_{\bigwedge C} = \bigcap_{a \in C} U_a \neq \varnothing,$$

and therefore $\bigwedge C \neq 0$. This shows that A has the finite intersection property. By Lemma 1.12 there is an ultrafilter F in B such that $A \subseteq F$. But then

$$F \in \bigcap_{a \in A} U_a = \bigcap \mathcal{A}.$$

We conclude that $\mathrm{St}(B)$ is a compact space. □

1.19. DEFINITION. A topological space X is called *zero-dimensional* if it has a base consisting of closed-and-open sets.

From Lemma 1.17 it follows that the Stone space of a Boolean algebra B is zero-dimensional.

The remaining theorems in this section describe the so-called Stone duality between Boolean algebras and zero-dimensional compact Hausdorff spaces.

1.20. THEOREM. *Every Boolean algebra B is isomorphic to the Boolean algebra* $\mathrm{CO}(\mathrm{St}(B))$.

PROOF. Let $\phi : B \to \mathrm{CO}(\mathrm{St}(B))$ be the function defined by $\phi(b) = U_b$. From Lemma 1.17 it follows that U_b is a closed-and-open subset of $\mathrm{St}(B)$, thus ϕ is well-defined. The same lemma implies that ϕ is a homomorphism.

If a and b are distinct elements in B, then $a \triangle b > 0$ and

$$\varnothing \neq \phi(a \triangle b) = \phi(a) \triangle \phi(b).$$

Thus $\phi(a) \neq \phi(b)$ and this shows that the function ϕ is one-to-one.

It remains to show that ϕ is onto. Let A be a closed-and-open subset of $\mathrm{St}(B)$. Since A is open, it follows that there exists a subset C of B such that $A = \bigcup_{a \in C} U_a$. Since A is a closed subset of the compact space $\mathrm{St}(B)$, it is compact. Hence there is a finite subset D of C such that

$$A = \bigcup_{a \in D} U_a = U_{\bigvee D}.$$

This shows that ϕ is onto. We conclude that ϕ is an isomorphism. □

Notice that the proof of this theorem shows that the family of closed-and-open subsets of the Stone space of a Boolean algebra B is equal to the family $\{U_b : b \in B\}$.

1.21. Theorem. *Every zero-dimensional compact Hausdorff space X is homeomorphic to* $\mathrm{St}(\mathrm{CO}(X))$.

Proof. Define for every $x \in X$ the family

$$\mathcal{F}_x = \{A \in \mathrm{CO}(X) : x \in A\}$$

of closed-and-open subsets of X. For every $x \in X$ the family $\mathcal{F}_x$ is an ultrafilter in $\mathrm{CO}(X)$ since it satisfies the properties (U1)–(U3). Hence the function $f : X \to \mathrm{St}(\mathrm{CO}(X))$ that maps every point $x \in X$ to $\mathcal{F}_x$, is well-defined.

Let x and y be distinct points in X. Since X is a zero-dimensional Hausdorff space, there is a closed-and-open neighborhood A of x that does not contain the point y. But then $A \in \mathcal{F}_x$ and $X \backslash A \in \mathcal{F}_y$. Hence $f(x) \neq f(y)$ and this shows that the function f is one-to-one.

To show that f is onto, let $\mathcal{F}$ be a point in $\mathrm{St}(\mathrm{CO}(X))$. Since $\mathcal{F}$ is a filter, it follows that $\mathcal{F}$ has the finite intersection property. Because X is a compact space and $\mathcal{F}$ consists of closed subsets of X, the intersection $\bigcap \mathcal{F}$ is nonempty; let x be a point in this intersection. Then $\mathcal{F} \subseteq \mathcal{F}_x$ and since $\mathcal{F}$ is an ultrafilter, it follows that $\mathcal{F}$ and $\mathcal{F}_x$ are equal. Hence $f(x) = \mathcal{F}_x = \mathcal{F}$.

To prove that f is a homeomorphism, it remains to show that f is an open mapping. Let A be a nonempty closed-and-open subset of X. Then

$$f[A] = \{\mathcal{F}_x : x \in A\} \subseteq \{\mathcal{F} \in \mathrm{St}(\mathrm{CO}(X)) : A \in \mathcal{F}\} = U_A.$$

Conversely, let $\mathcal{F}$ be an ultrafilter in U_A. Since f is onto, there is an $x \in X$ that is mapped onto $\mathcal{F}$ by f. But this implies that $A \in \mathcal{F}_x$, thus $x \in A$, and therefore $\mathcal{F} \in f[A]$. We conclude that f is an open mapping. □

For every continuous function $f : X \to Y$ between two zero-dimensional compact Hausdorff spaces, define the function $\mathrm{CO}(f) : \mathrm{CO}(Y) \to \mathrm{CO}(X)$ by $\mathrm{CO}(f)(A) = f^{-1}[A]$. Clearly, each such a function is a homomorphism. Likewise, define for every homomorphism $\phi : A \to B$ between two Boolean algebras, the function $\mathrm{St}(\phi) : \mathrm{St}(B) \to \mathrm{St}(A)$ by $\mathrm{St}(\phi)(F) = \phi^{-1}[F]$. It is easily verified that each such a function is well-defined. If $\phi : A \to B$ is a Boolean homomorphism, then for every $a \in A$ we have

$$\begin{aligned}(\mathrm{St}(\phi))^{-1}[U_a] &= \{F \in \mathrm{St}(B) : \phi^{-1}[F] \in U_a\} = \{F \in \mathrm{St}(B) : a \in \phi^{-1}[F]\} \\ &= \{F \in \mathrm{St}(B) : \phi(a) \in F\} = U_{\phi(a)},\end{aligned}$$

and this shows that the function $\mathrm{St}(\phi)$ is continuous.

One can show that CO and St are actually, in the language of category theory, contravariant functors between the category of Boolean algebras and Boolean homomorphisms and the category of zero-dimensional compact Hausdorff spaces and continuous functions. Furthermore, one can show that the isomorphism given in Theorem 1.20 and the homeomorphism given in Theorem 1.21 are in fact natural. Hence the given functors are pseudo inverses of each other and we obtain the following theorem.

1.22. THEOREM. *The category of Boolean algebras and Boolean homomorphisms and the opposite of the category of zero-dimensional compact Hausdorff spaces and continuous functions are equivalent.*

The following corollary follows from the fact that St is a functor.

1.23. COROLLARY. *If $\phi : A \to B$ is an isomorphism between Boolean algebras, then the mapping $\mathrm{St}(\phi) : \mathrm{St}(B) \to \mathrm{St}(A)$ is a homeomorphism.*

1.4. Čech-Stone remainders of discrete spaces

A topological space X is called a discrete space if every subset of X is open. It follows from Theorem 1.5 that every nonempty discrete space has a Čech-Stone compactification.

For every subset A of a discrete space X, let $\overline{A}$ denote the closure of A in βX and A^* the subset $\overline{A} \cap X^*$ of the Čech-Stone remainder X^*.

1.24. PROPOSITION. *Let X be a discrete space. The set $\overline{A}$ is a closed-and-open subset of βX for every subset A of X.*

PROOF. If A is a subset of X then it follows from Theorem 1.6 that $\overline{A} \cap \overline{X \backslash A} = \varnothing$. Since also $\overline{A} \cup \overline{X \backslash A} = \overline{X} = \beta X$, we find that $\overline{A}$ is a closed-and-open subset of βX. □

1.25. LEMMA. *Let X be a discrete space. The family $\{\overline{A} : A \subseteq X\}$ is a base for βX.*

PROOF. Let x be a point in βX and U a neighborhood of x. Since βX is a normal space, there exists an open set V in βX such that $x \in V \subseteq \mathrm{cl}_{\beta X} V \subseteq U$. Now $\mathrm{cl}_{\beta X} V = \mathrm{cl}_{\beta X} V \cap X = \overline{V \cap X}$. This shows that the given family is a base for βX. □

Notice that the lemma above also shows that if X is a discrete space, then the family $\{A^* : A \subseteq X\}$ is a base for the space X^*. Hence both the Čech-Stone compactification and the Čech-Stone remainder of a discrete space X are zero-dimensional spaces.

1.26. LEMMA. *If X is a discrete space, then the function ϕ from $\mathcal{P}(X)$ to $\mathrm{CO}(\beta X)$ defined by $\phi(A) = \overline{A}$ is an isomorphism.*

Proof. From Proposition 1.24, it follows that the function ϕ is well-defined.

If A and B are subsets of X such that $\phi(A) = \phi(B)$, then $A = \overline{A} \cap X = \overline{B} \cap X = B$. Thus ϕ is a one-to-one function.

Let us now show that ϕ is onto. If A is a closed-and-open subset of βX, then $A = \mathrm{cl}_{\beta X} A = \mathrm{cl}_{\beta X} A \cap X = \overline{A \cap X}$, hence $\phi(A \cap X) = A$.

It is seen quite easily that ϕ is a homomorphism. □

1.27. Theorem. *If X is a discrete space, then the space βX is homeomorphic to the Stone space of the Boolean algebra $\mathcal{P}(X)$.*

Proof. Since βX is a zero-dimensional compact Hausdorff space we can apply Theorem 1.21 and find that βX is homeomorphic to $\mathrm{St}(\mathrm{CO}(\beta X))$. Since X is a discrete space it follows from Lemma 1.26 that the Boolean algebra $\mathrm{CO}(\beta X)$ is isomorphic to the Boolean algebra $\mathcal{P}(X)$ and thus, by Corollary 1.23, the space βX is homeomorphic to $\mathrm{St}(\mathcal{P}(X))$. □

The theorem above shows that we can identify the points of the Čech-Stone compactification of a discrete space X with the ultrafilters in $\mathcal{P}(X)$. If $\mathcal{F}$ is an ultrafilter in the Boolean algebra $\mathcal{P}(X)$, then we call $\mathcal{F}$ an ultrafilter *on* the set X.

1.28. Definition. An ultrafilter $\mathcal{F}$ on a set X is called *fixed* if the intersection $\bigcap \mathcal{F}$ is nonempty and it is called *free* if it is not fixed.

Let $\mathcal{F}_x$ denote the ultrafilter $\{A \subseteq X : x \in A\}$ on a set X.

1.29. Proposition. *An ultrafilter $\mathcal{F}$ on a set X is fixed if and only if there is a unique element $x \in X$ such that $\mathcal{F}$ is equal to $\mathcal{F}_x$.*

Proof. If $\mathcal{F}$ is fixed, then there is an element $x \in X$ such that x is a point in the intersection $\bigcap \mathcal{F}$. Then $\mathcal{F} \subseteq \mathcal{F}_x$ and since $\mathcal{F}$ is an ultrafilter we find that $\mathcal{F}$ is equal to $\mathcal{F}_x$. If $\mathcal{F}$ is also equal to $\mathcal{F}_y$ then it follows that the intersection $\{x\} \cap \{y\}$ is nonempty, hence $y = x$.

It is obvious that $\mathcal{F}$ is a fixed ultrafilter if it is equal to $\mathcal{F}_x$ for a certain element $x \in X$. □

1.30. Proposition. *An ultrafilter $\mathcal{F}$ on a set X is free if and only if every set A in $\mathcal{F}$ is infinite.*

Proof. From Proposition 1.29 it follows that if $\mathcal{F}$ is not free, then there is an element $x \in X$ such that $\mathcal{F}$ is equal to $\mathcal{F}_x$. Then $\{x\}$ is a finite set in $\mathcal{F}$.

Conversely, let A be a finite subset of $\mathcal{F}$ and suppose that $\mathcal{F}$ is free. Then there is a finite subfamily $\mathcal{A}$ of $\mathcal{F}$ such that $\varnothing = A \cap \bigcap \mathcal{A} \in \mathcal{F}$, a contradiction. □

From now on we will identify the points of the Čech-Stone compactification of a discrete space X with the ultrafilters on X. This implies that every point $x \in X$ corresponds to the ultrafilter $\mathcal{F}_x$ on X. Hence, by Proposition 1.29, the fixed ultrafilters on X correspond to the points of X and the free ultrafilters on X correspond to the points of X^*. We now have the following proposition.

1.31. Proposition. *If A is a subset of a discrete space X then*

$$\overline{A} = U_A = \{\mathcal{F} \in \beta X : A \in \mathcal{F}\}.$$

Proof. If $x \in A$, then $A \in \mathcal{F}_x$ and thus $\mathcal{F}_x \in U_A$. This shows that A is a subset of U_A and since U_A is a closed-and-open subset of βX it follows that the closure $\overline{A}$ of A is a subset of U_A.

Conversely, if $\mathcal{F} \notin \overline{A}$ then there is a subset B of X such that $\mathcal{F} \in U_B$ and U_B and $\overline{A}$ are disjoint. Hence if $x \in A$ then $\mathcal{F}_x \notin U_B$, thus $B \notin \mathcal{F}_x$ and therefore $x \notin B$. This shows that A and B are disjoint and consequently that $\mathcal{F} \notin U_A$. □

1.32. Definition. An ultrafilter $\mathcal{F}$ on a set X is called *uniform* if the cardinality of every set A in $\mathcal{F}$ is equal to the cardinality of X and $\mathcal{F}$ is called *κ-uniform* if every set A in $\mathcal{F}$ has cardinality at least κ.

Let $\mathcal{U}(X)$ denote the family consisting of all uniform ultrafilters on the set X and $\mathcal{U}_\kappa(X)$ the family of all κ-uniform ultrafilters on X.

If κ is an infinite cardinal and X a set of cardinality at least κ, then the family

$$\mathcal{F}_\kappa = \{A \subseteq X : X \backslash A \text{ has cardinality less than } \kappa\}$$

is a filter on X and is called the *κ-Frechet* filter on X.

1.33. Lemma. *Let κ be an infinite cardinal and X a set of cardinality at least κ. An ultrafilter $\mathcal{F}$ on X is κ-uniform if and only if $\mathcal{F}_\kappa \subseteq \mathcal{F}$.*

Proof. If $\mathcal{F}$ is a κ-uniform ultrafilter and A a set in $\mathcal{F}_\kappa \backslash \mathcal{F}$, then the set $X \backslash A$ is contained in $\mathcal{F}$ and has cardinality less than κ, a contradiction.

Conversely, if a subset A of X has cardinality less than κ and is contained in $\mathcal{F}$, then the set $X \backslash A$ is contained in $\mathcal{F}_\kappa \backslash \mathcal{F}$. □

A family $\mathcal{F}$ of subsets of X is called *κ-centered* if for every finite subfamily $\mathcal{A}$ of $\mathcal{F}$ the intersection $\bigcap \mathcal{A}$ has cardinality at least κ.

1.34. Lemma. *If κ is an infinite cardinal and X is a set of cardinality at least κ, then every κ-centered family $\mathcal{F}$ of subsets of X extends to a κ-uniform ultrafilter.*

Proof. The family $\mathcal{A} = \mathcal{F} \cup \mathcal{F}_\kappa$ has the finite intersection property. By Lemma 1.12 there is an ultrafilter that contains $\mathcal{A}$. It follows from Lemma 1.33 that this ultrafilter is κ-uniform. □

For every discrete space X we will consider the family $\mathcal{U}_\kappa(X)$ as a topological subspace of the space βX.

1.35. LEMMA. *If κ is an infinite cardinal and X a discrete space of cardinality at least κ then $\mathcal{U}_\kappa(X)$ is a closed subspace of βX.*

PROOF. Let $\mathcal{F}$ be an ultrafilter on the set X that is not contained in $\mathcal{U}_\kappa(X)$. Then there is a set A in $\mathcal{F}$ with cardinality less than κ. Now $\mathcal{F}$ is contained in the open subset $\overline{A}$ of βX that is disjoint from $\mathcal{U}_\kappa(X)$. □

If κ is an infinite cardinal and X a set of cardinality at least κ, let

$$[X]^{<\kappa} = \{A \subseteq X : A \text{ has cardinality less than } \kappa\}.$$

We now have the following lemma.

1.36. LEMMA. *If κ is an infinite cardinal and X a discrete space of cardinality at least κ, then the function $\phi : \mathcal{P}(X) \to \mathrm{CO}(\mathcal{U}_\kappa(X))$ defined by $\phi(A) = \overline{A} \cap \mathcal{U}_\kappa(X)$ is a surjective homomorphism with $\ker(\phi) = [X]^{<\kappa}$.*

PROOF. It follows from Lemma 1.24 that the function ϕ is well-defined.

To show that ϕ is a homomorphism let A and B be subsets of X. It is clear the equality $\phi(A \cup B) = \phi(A) \cup \phi(B)$ holds. From the fact that $\overline{X\backslash A} = \beta X \backslash \overline{A}$ it follows that $\phi(X\backslash A) = \mathcal{U}_\kappa(X)\backslash\overline{A} = \phi(X)\backslash\phi(A)$. This proves that ϕ is a homomorphism.

Let U be a closed-and-open subset of $\mathcal{U}_\kappa(X)$. Since $\mathcal{U}_\kappa(X)$ is a compact subspace of βX it follows that there is a closed-and-open subset V of βX such that $U = V \cap \mathcal{U}_\kappa(X)$. If we let $A = V \cap X$, then $\overline{A} = \overline{V \cap X} = \overline{V} = V$ and thus $\phi(A) = U$. Hence ϕ is onto.

It remains to show that the kernel of ϕ is equal to $[X]^{<\kappa}$. If A is a subset of X of cardinality less than κ, then every ultrafilter in $\overline{A}$ contains the set A and is therefore not κ-uniform, i.e., $\phi(A) = \varnothing$. Conversely, if A is a subset of X of cardinality at least κ then the family $\{A\}$ is κ-centered and thus, by Lemma 1.34, there is an ultrafilter $\mathcal{F}$ contained in $\overline{A} \cap \mathcal{U}_\kappa(X)$. □

We now have the following theorem.

1.37. THEOREM. *If κ is an infinite cardinal and X a discrete space of cardinality at least κ then $\mathcal{U}_\kappa(X)$ is homeomorphic to the Stone space of the Boolean algebra $\mathcal{P}(X)/[X]^{<\kappa}$.*

PROOF. From Lemma 1.35 it follows that $\mathcal{U}_\kappa(X)$ is a compact space and since it also a zero-dimensional Hausdorff space, we can apply Theorem 1.21 and find that $\mathcal{U}_\kappa(X)$ is homeomorphic to $\mathrm{St}(\mathrm{CO}(\mathcal{U}_\kappa(X)))$. It follows from Lemma 1.36 that the Boolean algebra $\mathrm{CO}(\mathcal{U}_\kappa(X))$ is isomorphic to the Boolean algebra $\mathcal{P}(X)/[X]^{<\kappa}$ and thus, by Corollary 1.23, the space $\mathcal{U}_\kappa(X)$ is homeomorphic to $\mathrm{St}(\mathcal{P}(X)/[X]^{<\kappa})$. □

Let $\mathit{fin}(X)$ denote the family of finite subsets of X, i.e., $\mathit{fin}(X) = [X]^{<\omega}$. If X is an infinite discrete space then X^* consists of all the free ultrafilters on X. Hence X^* is equal to $\mathcal{U}_\omega(X)$ and this gives the following corollary.

1.38. COROLLARY. *If X is an infinite discrete space, then X^* is homeomorphic to the Stone space of the Boolean algebra $P(X)/\mathit{fin}(X)$.*

Since the Čech-Stone remainders of two discrete spaces of equal cardinality are homeomorphic, it follows that we can (and always will) consider each cardinal as a discrete space and work only with these spaces.

The general question that we are concerned with in this thesis can now be formulated as follows.

Does there exist a pair $\{\kappa, \lambda\}$ of distinct infinite cardinals such that it is consistent that the Čech-Stone remainders of κ and λ are homeomorphic?

In Chapter 2 we give a result by Balcar and Frankiewicz that shows for all but one pair of distinct infinite cardinals that the corresponding Čech-Stone remainders are not homeomorphic. Hence only the remaining pair, consisting of the cardinals ω and ω_1, is important and to answer this question we only need to answer the following question.

Is it consistent that the Čech-Stone remainders of ω and ω_1 are homeomorphic?

This question is known as the Katowice problem. It follows from Stone's duality that it is equivalent to the following question.

Is it consistent that the Boolean algebras $\mathcal{P}(\omega)/\mathit{fin}(\omega)$ and $\mathcal{P}(\omega_1)/\mathit{fin}(\omega_1)$ are isomorphic?

Our language in the two following chapters is topological. Let us, for this reason, take a better look at the Čech-Stone remainder X^* of a discrete space X.

It follows from Lemma 1.35 that X^* is a closed subspace of the compact space βX and therefore that X^* is a compact space.

If A and B are subsets of X, then by Lemma 1.36 it follows that the equality $(A \cap B)^* = A^* \cap B^*$ holds.

Lemma 1.26 shows that every closed-and-open subset of βX is of the form $\overline{A}$ where A is a subset of X. It follows from Lemma 1.36 that the function $\phi : \mathcal{P}(X) \to \mathrm{CO}(X^*)$ defined by $\phi(A) = A^*$ is a surjective homomorphism with $\ker(\phi) = \mathit{fin}(X)$. Hence for every nonempty closed-and-open subset U of X^* there is an infinite subset A of X such that U is equal to A^*. This function ϕ also shows for all subsets A and B of X that

(i) A^* is empty if and only if the set A is finite,
(ii) $A^* = B^*$ if and only if the symmetric difference $A \triangle B$ is finite,
(iii) $A^* \subseteq B^*$ if and only if the difference $A \backslash B$ is finite, and
(iv) $A^* \cap B^* = \varnothing$ if and only if the intersection $A \cap B$ is finite.

If the symmetric difference $A \triangle B$ is finite then A and B are said to be *almost equal*; we denote this by $A =^* B$. Similarly, if the difference $A \backslash B$ is finite then A is said to be *almost contained* in B; this is denoted by $A \subseteq^* B$. The sets A and B are said to be *almost disjoint* if the set $A \cap B$ is finite.

The following definition will be used to determine for every infinite discrete space X the cardinality of the spaces βX and X^*,

1.39. Definition. Let X be a set of cardinality κ. A family $\mathcal{F}$ of subsets of X is called an *independent family* over X if for every pair $\mathcal{A}$, $\mathcal{B}$ of disjoint finite subfamilies of $\mathcal{F}$, the intersection

$$\bigcap \mathcal{A} \cap (X \backslash \bigcup \mathcal{B})$$

has cardinality κ.

1.40. THEOREM. *If X is an infinite set of cardinality κ, then there exists an independent family $\mathcal{F}$ over X of cardinality 2^κ.*

PROOF. The family of finite subsets of X, $fin(X)$, has cardinality $1 + \kappa + \kappa^2 + \cdots = \aleph_0 \cdot \kappa = \kappa$. Hence also the set

$$Y = \{\langle x, y\rangle : x \in fin(X), y \subseteq \mathcal{P}(x)\}$$

has cardinality κ. For every subset A of X define

$$Y_A = \{\langle x, y\rangle \in Y : A \cap x \in y\}.$$

We will show that the family $\mathcal{A} = \{Y_A : A \subseteq X\}$ is an independent family over Y of cardinality 2^κ.

If A and B are distinct subsets of X and x is an element in the symmetric difference $A \triangle B$, then the element $\langle\{x\}, \{\{x\}\}\rangle \in Y$ is contained in the symmetric difference $Y_A \triangle Y_B$. This shows that the sets Y_A and Y_B are different and therefore that the given family has cardinality 2^κ.

Let $A_1, \ldots, A_n; B_1, \ldots, B_m$ be distinct subsets of X and choose an element x_{ij} in the symmetric difference $A_i \triangle B_j$ for $i = 1, \ldots, n; j = 1, \ldots, m$. For any finite subset x of X that contains these chosen elements we have that the sets $A_i \cap x$ and $B_j \cap x$ are different for $i = 1, \ldots n$; $j = 1, \ldots m$ and hence

$$\langle x, \{A_1 \cap x, \ldots, A_n \cap x\}\rangle \in Y_{A_1} \cap \cdots \cap Y_{A_n} \cap (Y \backslash Y_{B_1}) \cap \cdots \cap (Y \backslash Y_{B_j}).$$

This shows that the intersection above has cardinality κ and therefore that the family $\mathcal{A}$ is independent.

Since the sets X and Y have the same cardinality, it follows that there is an independent family over X of cardinality 2^κ. □

1.41. THEOREM. *There are 2^{2^κ} ultrafilters on an infinite set X of cardinality κ.*

PROOF. Theorem 1.40 states that there is an independent family $\mathcal{F}$ over X of cardinality 2^κ. Define for every function $f : \mathcal{F} \to \{0, 1\}$ the family

$$\mathcal{A}_f = \{A \in \mathcal{F} : f(A) = 1\} \cup \{X \backslash A : A \in \mathcal{F}, f(A) = 0\}$$

and note that each family $\mathcal{A}_f$ is κ-centered. By Lemma 1.34, for every function f there is a κ-uniform ultrafilter u_f that extends $\mathcal{A}_f$.

Let f and g be distinct functions from $\mathcal{F}$ to $\{0, 1\}$ and A a set in $\mathcal{F}$ such that $f(A) \neq g(A)$. Then $A \in u_f \triangle u_g$ and this shows that there are at least 2^{2^κ} distinct ultrafilters on X.

Because every ultrafilter on X is contained in $\mathcal{PP}(X)$ there are at most 2^{2^κ} ultrafilters on X. □

From the theorem above we immediately get the following corollary.

1.42. COROLLARY. *If X is an infinite discrete space of cardinality κ, then the spaces βX, $\mathcal{U}(X)$ and X^* have cardinality 2^{2^κ}.*

We end this section with a lemma that will be used in the following chapters to simplify a number of proofs.

1.43. LEMMA. *Let X be an infinite discrete space. For every bijective function $f : X \to X$, there exists an autohomeomorphism h of X^* such that for every subset A of X the equality $h[A^*] = (f[A])^*$ holds.*

PROOF. Since X is a subspace of βX, we can consider the function f as a mapping from X to βX. It follows from Remark 1.2 that there exists a continuous extension βf of f over βX. Similarly, let $\beta(f^{-1})$ be the continuous extension of f^{-1} over βX.

Since the composition $\beta f \circ \beta(f^{-1})$ is continuous and it is equal to the identity function when restricted to X, it follows that it is the identity function on βX. A similar argument shows that the composition $\beta(f^{-1}) \circ \beta f$ is equal to the identity function on βX. From this is follows that βf is an autohomeomorphism of βX and therefore the restriction of this mapping to the remainder X^*, let us call it h, is an autohomeomorphism of X^*.

Now if A is a subset of X, then

$$h[A^*] \subseteq \beta f[\overline{A}] = \beta f[\overline{A \cap X}] \subseteq \overline{f[A]},$$

and this shows that $h[A^*]$ is a subset of $(f[A])^*$. Analogously it follows that $h[(X \backslash A)^*]$ is a subset of $(f[X \backslash A])^*$. Since the sets $(f[A])^*$ and $(f[X \backslash A])^*$ are disjoint and the mapping h is onto, we find the required equality. □

Notes

Everything in this chapter is well-known and can be found for example in the book by Comfort and Negrepontis [1974] or the book by Frankiewicz and Zbierski [1994].

The construction of the Čech-Stone compactification given in the proof of Theorem 1.5 is due to Čech [1937]. In his paper Čech shows the existence of the compactification βX, gives, among other characterizations of βX, the characterization found in Theorem 1.6 and uses βX to derive properties of X. Čech used $\beta(X)$ to denote the Čech-Stone compactification of X and it is this β that is still used today.

In the same year Stone [1937] published a paper about the relation between algebra and topology. This paper gives a different construction of the space βX and plays an important role in the development of the Stone duality as given in Section 1.3.

Example 1.4 is based on an example used by Tong in [1949] to answer a question posed by Čech.

The proof of Theorem 1.40 is due to Hausdorff [1936].

CHAPTER 2

A partial answer

The general question, as given in Section 1.4, is the following: does there exist a pair of distinct infinite cardinals such that it is consistent that the corresponding Čech-Stone remainders are homeomorphic? The purpose of this chapter is to show for all but one pair of distinct infinite cardinals that the corresponding remainders are not homeomorphic.

The remaining pair gives rise to the Katowice problem: is it consistent that the remainders ω^* and ω_1^* are homeomorphic? This problem will be treated in the following chapter.

2.1. Scales

Let ω^ω denote the set of all functions from ω to ω and define, on this set, the quasi-order $\leqslant^*$ by

$$f \leqslant^* g \text{ if } f(n) \leqslant g(n) \text{ for all but finitely many } n \in \omega.$$

Two functions f and g in ω^ω are called *almost equal* if $f \leqslant^* g$ and $g \leqslant^* f$.

A subset of ω^ω is called *dominating* if it is cofinal in ω^ω with respect to the ordering $\leqslant^*$, i.e., for every function f in ω^ω there is a function g in this set such that $f \leqslant^* g$. If a subset of ω^ω is both dominating and well-ordered by $\leqslant^*$, then it is called a *scale* or, if κ is the cardinality of this subset, a *κ-scale*. If the set $\{f_\alpha : \alpha < \kappa\}$ is a scale, then we will always assume that $f_\alpha \leqslant^* f_\beta$ whenever $\alpha < \beta < \kappa$.

2.1. Lemma. *There is at most one regular cardinal κ such that a κ-scale exists in ω^ω.*

Proof. Let κ and λ be regular cardinals such that the set $\{f_\alpha : \alpha < \kappa\}$ is a κ-scale and the set $\{g_\alpha : \alpha < \lambda\}$ a λ-scale.

Now suppose that $\kappa < \lambda$. For every $\alpha \in \kappa$, let $h_\alpha : \omega \to \omega$ be the function defined by $h_\alpha(n) = f_\alpha(n) + 1$ and choose an element $\beta(\alpha) \in \lambda$ such that $h_\alpha \leqslant^* g_{\beta(\alpha)}$; this implies that $f_\alpha \leqslant^* g_{\alpha(\beta)}$ and that f_α and $g_{\beta(\alpha)}$ are not almost equal. By the regularity of λ, there is an element $\mu \in \lambda$ such that $\beta(\alpha) < \mu$ for every $\alpha \in \kappa$. Let $\alpha \in \kappa$ such that $g_\mu \leqslant^* f_\alpha$. Since also $f_\alpha \leqslant^* g_{\beta(\alpha)} \leqslant^* g_\mu$, it follows that f_α and $g_{\beta(\alpha)}$ are almost equal, a contradiction. A similar argument shows that λ is not less than κ. □

The following theorem is the first of a number of consequences that are given in this thesis that follow from the assumption that the Čech-Stone remainders of two specific distinct infinite cardinals are homeomorphic.

2.2. Theorem. *If κ is an uncountable regular cardinal such that κ^* is homeomorphic to ω^*, then there exists a κ-scale in ω^ω.*

Proof. Consider the discrete spaces $\omega \times \kappa$ and $\omega \times \omega$ instead of κ and ω respectively and let $h : (\omega \times \kappa)^* \to (\omega \times \omega)^*$ be a homeomorphism. For each $n \in \omega$ define V_n to be the subset $\{n\} \times \kappa$ of $\omega \times \kappa$ and choose an infinite subset v_n of $\omega \times \omega$ such that $h[V_n^*] = v_n^*$. Since the family $\{V_n : n \in \omega\}$ is pairwise disjoint, it follows that the family $\{v_n : n \in \omega\}$ is almost pairwise disjoint. We can rearrange the sets v_n in such a way that the family $\{v_n : n \in \omega\}$ forms a partition of $\omega \times \omega$. Now Lemma 1.43 shows, using a one-to-one mapping of $\omega \times \omega$ onto itself such that for every $n \in \omega$ the set v_n is mapped onto the set $\{n\} \times \omega$, that we can even assume that $v_n = \{n\} \times \omega$ for every $n \in \omega$.

For every $\alpha \in \kappa$ define E_α to be the subset $\omega \times [\alpha, \kappa)$ of $\omega \times \kappa$, choose a subset e_α of $\omega \times \omega$ such that $h[E_\alpha^*] = e_\alpha^*$ and let $f_\alpha : \omega \to \omega$ be the function defined by

$$f_\alpha(n) = \min\{k \in \omega : \langle n, k \rangle \in e_\alpha\}.$$

Since the intersection of e_α with v_n is nonempty for every $n \in \omega$, it follows that the function above is well-defined. We claim that the subset $\{f_\alpha : \alpha \in \kappa\}$ of ω^ω is a κ-scale.

If $\alpha < \beta < \kappa$, then E_β is a subset of E_α, hence the set $e_\beta \backslash e_\alpha$ is finite and this implies that $f_\alpha \leqslant^* f_\beta$. Thus the set $\{f_\alpha : \alpha \in \kappa\}$ is well-ordered by $\leqslant^*$.

It remains to show that the given subset is dominating. Let f be a function in ω^ω and define l_f (lower f) by

$$l_f = \{\langle n, m \rangle : n \in \omega, m \leqslant f(n)\}.$$

Choose an infinite subset L_f of $\omega \times \kappa$ such that $h[L_f^*] = l_f^*$. Since for every $n \in \omega$ the set $l_f \cap v_n$ is finite, it follows that the set $L_f \cap V_n$ is finite. Since κ is an uncountable regular cardinal, there exists an $\alpha \in \kappa$ such that E_α is disjoint from L_f. This implies that $e_\alpha \cap l_f$ is a finite subset of $\omega \times \omega$ and therefore $f \leqslant^* f_\alpha$. □

2.2. An answer for all but one pair

2.3. Lemma. *If κ, λ and μ are infinite cardinals such that $\kappa \leqslant \lambda \leqslant \mu$ and κ^* is homeomorphic to μ^*, then κ^* and λ^* are homeomorphic.*

Proof. Suppose that $\langle \kappa, \lambda, \mu \rangle$ is, using the lexicographical order, the smallest triple for which the lemma fails. Let $h : \mu^* \to \kappa^*$ be a homeomorphism and choose an infinite subset A of κ such that $h[\lambda^*] = A^*$. Since κ^* and λ^* are not homeomorphic, it follows that the cardinality of A, say ν, is less than κ. But then the lemma also fails for the triple $\langle \nu, \kappa, \lambda \rangle$, a contradiction with our previous choice of smallest triple. □

Let κ^+ denote the successor cardinal of a cardinal number κ.

2.4. Lemma. *If κ is an infinite cardinal such that κ^* and $(\kappa^+)^*$ are homeomorphic, then κ^* is homeomorphic to ω^*.*

PROOF. Let λ be the smallest cardinal number such that κ^* and λ^* are homeomorphic. From Lemma 2.3 it follows that λ^* and $(\lambda^+)^*$ are homeomorphic; let $h : \lambda^* \to (\lambda^+)^*$ be a homeomorphism. For every $\alpha \in \lambda$ choose a subset A_α of λ^+ such that $h[\alpha^*] = A_\alpha^*$. By the minimality of λ it follows that the cardinality of A_α is less than λ. Hence the set

$$A = \lambda^+ \backslash \bigcup_{\alpha\in\lambda} A_\alpha$$

has cardinality λ^+. Choose a subset B of λ such that $h[B^*] = A^*$. For every $\alpha \in \lambda$ the intersection of A_α with A is empty and thus, translating this back to λ^*, the intersection of α^* with B^* is empty; this shows that the set $\alpha \cap B$ is finite for every $\alpha \in \lambda$. Hence the set

$$B = \bigcup_{\alpha\in\lambda} B \cap \alpha$$

is the union of an increasing sequence of finite sets and therefore countable. This shows that ω^* is equal to $(\lambda^+)^*$ and, as a consequence, that ω^* and κ^* are homeomorphic. □

2.5. THEOREM. *The remainders ω_1^* and ω_2^* are not homeomorphic.*

PROOF. Suppose that ω_1^* is homeomorphic to ω_2^*. It then follows from Lemma 2.4 that ω^* is homeomorphic to ω_1^*. Hence by Theorem 2.2 there exists both an ω_1-scale and ω_2-scale in ω^ω. But this is a contradiction with Lemma 2.1. □

The following theorem shows for all but one pair of distinct infinite cardinals, that the corresponding Čech-Stone remainders are not homeomorphic.

2.6. THEOREM. *If κ and λ are infinite cardinals, $\kappa < \lambda$ and κ^* is homeomorphic to λ^*, then κ is equal to ω and λ is equal to ω_1.*

PROOF. If κ^* and λ^* are homeomorphic, then, by Lemma 2.3, it follows that κ^* and $(\kappa^+)^*$ are homeomorphic and therefore, as a result of Lemma 2.4, the remainders κ^* and ω^* are homeomorphic. Let us first show that κ is equal to ω. If not, then Lemma 2.3 implies that ω_1^* is homeomorphic to ω_2^*. This however, is a contradiction with Theorem 2.5. Now suppose that λ is not equal to ω_1. Then again, by Lemma 2.3, it follows that ω_1^* and ω_2^* are homeomorphic and this is a contradiction. □

We shall end this section by a result that we do not actually need but that is worth mentioning.

Let $\mathrm{cf}(\kappa)$ denote the cofinality of a cardinal κ.

Recall that $\mathcal{U}(\kappa)$ denotes the subspace of $\beta\kappa$ consisting of the uniform ultrafilters on the infinite cardinal κ. For every subset A of κ define $A^\wedge$ to be the set $\overline{A} \cap \mathcal{U}(\kappa)$. From Lemma 1.36 it follows that if A is a subset of κ, then $A^\wedge$ is empty precisely when the cardinality of A is less than κ. Furthermore this lemma shows that for all subsets A and B of κ the equality $(A \cap B)^\wedge = A^\wedge \cap B^\wedge$ holds.

We now have the following theorem.

2.7. THEOREM. *If κ and λ are infinite cardinals and* $\mathrm{cf}(\kappa) \neq \mathrm{cf}(\lambda)$, *then the spaces $\mathcal{U}(\kappa)$ and $\mathcal{U}(\lambda)$ are not homeomorphic.*

PROOF. Suppose for example that $\mathrm{cf}(\kappa) < \mathrm{cf}(\lambda)$. Let $\mathcal{A}$ be a partition of λ into $\mathrm{cf}(\kappa)$ many sets of cardinality λ and define $\mathcal{B} = \{A\hat{} : A \in \mathcal{A}\}$. If B is a subset of λ with cardinality λ then, by our assumption, there is a set $A \in \mathcal{A}$ that meets B in a set of cardinality λ, so that

$$B\hat{} \cap \bigcup \mathcal{B} \supseteq B\hat{} \cap A\hat{} = (B \cap A)\hat{} \neq \varnothing.$$

This proves that the family $\mathcal{B}$ has a dense union in $\mathcal{U}(\lambda)$. Hence, to prove the theorem, it is enough to show that every family of $\mathrm{cf}(\kappa)$ many pairwise disjoint closed-and-open subsets of $\mathcal{U}(\kappa)$ does not have a dense union in $\mathcal{U}(\kappa)$. If $\mathcal{B}$ is such a family, then there is a family $\mathcal{A}$ of subsets of κ such that $\mathcal{B} = \{A\hat{} : A \in \mathcal{A}\}$. By replacing the elements in $\mathcal{A}$ if necessary, it follows that we can assume that $\mathcal{A}$ is a pairwise disjoint family. Then there is a subset B of κ with cardinality κ such that B meets each element of $\mathcal{A}$ in a set of cardinality less than κ, so that

$$B\hat{} \cap \bigcup \mathcal{B} = \varnothing.$$

This shows that the family $\mathcal{B}$ does not have a dense union in $\mathcal{U}(\kappa)$. □

A result of the theorem above is that the spaces $\mathcal{U}(\omega)$ and $\mathcal{U}(\omega_1)$ are not homeomorphic. It is an open question whether it is consistent that there are distinct infinite cardinals κ and λ such that $\mathcal{U}(\kappa)$ and $\mathcal{U}(\lambda)$ are homeomorphic.

Notes

Lemma 2.1 and Theorem 2.2, 2.5 and 2.6 are given in [1978] by Balcar and Frankiewicz. A proof of Theorem 2.5 and 2.6 is not given in this paper but is, according to the authors, essentially contained in the proof of the theorem given in [1977] by Frankiewicz. In this paper by Frankiewicz it is proven that if the Čech-Stone remainders of two distinct infinite cardinals are homeomorphic, then ω^* is homeomorphic to ω_1^*. The proof of this theorem can indeed, using induction, be used to prove the two aforementioned theorems. Depending on whether the smallest of the two distinct infinite cardinals is regular of singular, the proof is split in two distinct cases; this however, is not necessary as is shown by a proof given in [1977] by Comfort. The proof that we have given above is based on the proof given by Chodounský in [2011] which in turn is based on the proof given in the paper by Comfort.

Theorem 2.7 can be found for example in the paper by Comfort [1977].

CHAPTER 3

The remaining pair

In the previous chapter we have shown for all but one pair of distinct infinite cardinals that the corresponding Čech-Stone remainders are not homeomorphic. For the remaining pair, consisting of ω and ω_1, it is still an open problem to determine if it is consistent that the corresponding remainders ω^* and ω_1^* are homeomorphic. The purpose of this chapter is to discuss this problem.

The generally accepted belief is that there does not exist a model in which the remainders ω^* and ω_1^* are homeomorphic. The strategy to actually show this, is to determine consequences of the assumption that the given remainders are homeomorphic and to try to derive from these a contradiction. In each section of this chapter we give such a consequence. The consequences in the first three sections are well-known and a result by Chodounský given in [2011] shows that there exists a model in which all three consequences hold simultaneously. A fairly new consequence, that is known to be consistent, is given in Section 3.4. It is an open question whether a model exists in which not only this consequence but also the other three mentioned consequences hold simultaneously.

3.1. Weights

Let us start with one of the most obvious consequences. From the Boolean algebraic variant of the Katowice problem it easily follows that unless the equality $2^\omega = 2^{\omega_1}$ holds, the remainders ω^* and ω_1^* are not homeomorphic. We will now prove this fact using the weights of these two remainders.

3.1. Definition. The least cardinality of a base for a topological space X is called the *weight* of X.

3.2. Theorem. *The remainder κ^* of an infinite cardinal κ has weight 2^κ.*

Proof. From Lemma 1.25 it follows that the family $\{A^* : A \subseteq \kappa\}$ is a base for κ^*. This implies that the weight of κ^* is at most 2^κ.

Now suppose that $\mathcal{B}$ is a base for κ^*. Theorem 1.40 shows that there exists an independent family $\{X_A : A \subseteq \kappa\}$ over κ of cardinality 2^κ. For every subset A of κ the set X_A^* is a closed-and-open subset of κ^* and therefore of the form $X_A^* = \bigcup \mathcal{B}_A$ where $\mathcal{B}_A$ is a finite subset of $\mathcal{B}$. If A and B are distinct subsets of κ then $X_A^* \neq X_B^*$ since $X_A \backslash X_B = X_A \cap (X \backslash X_B)$ is an infinite subset of κ and thus $\mathcal{B}_A \neq \mathcal{B}_B$. Hence $\mathcal{B}$ has at least 2^κ distinct finite subsets and this implies that $\mathcal{B}$ has cardinality at least 2^κ. We conclude that the remainder κ^* has weight 2^κ. $\square$

Two homeomorphic spaces must have equal weights. This implies the following theorem.

3.3. Theorem. *If the remainders ω^* and ω_1^* are homeomorphic, then the equality $2^\omega = 2^{\omega_1}$ holds.*

One consequence of the theorem above is that in any model in which the continuum hypothesis (CH) holds, the remainders ω^* and ω_1^* are not homeomorphic. On the other hand, Cohen has shown that there exists a model in which the equality $2^\omega = 2^{\omega_1}$ holds and in this model it is still possible that the remainders ω^* and ω_1^* are homeomorphic.

3.2. Strong Q-sequences

If we choose for every set in a disjoint family of subsets of ω a subset of this set, then it is trivial to find a single subset S of ω such that the intersection of S with an element of the family is the chosen subset of this element. Now consider an uncountable family of subsets of ω; this family is of course not disjoint. Choose for every set in this family a subset of this set. Does there exist a subset S of ω such that the intersection of S with an element of this family is almost equal to the chosen subset of this element? A homeomorphism between the remainders ω^* and ω_1^* associates to every uncountable family of disjoint infinite subsets of ω_1 a family of almost disjoint subsets of ω that has the above property. Such a family is called a strong Q-sequence.

3.4. Definition. Let $\mathcal{F}$ be a family of subsets of ω and $\mathcal{F}^*$ the set consisting of all functions $f : \mathcal{F} \to \mathcal{P}(\omega)$ such that for every set $A \in \mathcal{F}$ the set $f(A)$ is a subset of A; the family $\mathcal{F}$ is called a *strong Q-sequence* if for every function f in $\mathcal{F}^*$ there exists a subset S of ω such that for every element $A \in \mathcal{F}$ the intersection of S with A is almost equal to $f(A)$.

3.5. Theorem. *If the remainders ω^* and ω_1^* are homeomorphic, then there exists a strong Q-sequence of cardinality ω_1.*

Proof. Consider the discrete spaces of cardinality ω and ω_1 in the guises of $S_0 = \mathbb{Z} \times \omega$ and $S_1 = \mathbb{Z} \times \omega_1$ respectively and let $\gamma : S_1^* \to S_0^*$ be a homeomorphism between the corresponding remainders. For each $\alpha \in \omega_1$ define H_α to be the subset $\mathbb{Z} \times \{\alpha\}$ of S_1 and choose an infinite subset h_α of S_0 such that $\gamma[H_\alpha^*] = h_\alpha^*$. We will show that the family $\{h_\alpha : \alpha \in \omega_1\}$ is a strong Q-sequence.

For every $\alpha \in \omega_1$ choose a subset a_α of h_α and pick a subset A_α of H_α such that $\gamma[A_\alpha^*] = a_\alpha^*$. Let

$$A = \bigcup_{\alpha \in \omega_1} A_\alpha$$

and choose a subset a of S_0 such that $\gamma[A^*] = a^*$. Since for every $\alpha \in \omega_1$ we have that

$$A^* \cap H_\alpha^* = (A \cap H_\alpha)^* = A_\alpha^*,$$

it follows that

$$a^* \cap h_\alpha^* = (a \cap h_\alpha)^* = a_\alpha^*$$

and this shows that the intersection of a with h_α is almost equal to a_α. □

The following lemma shows that in a model in which there exists a strong Q-sequence of cardinality ω_1, the equality $2^\omega = 2^{\omega_1}$ holds.

3.6. LEMMA. *If there exists a strong Q-sequence of cardinality ω_1, then the equality $2^\omega = 2^{\omega_1}$ holds.*

PROOF. Let $\mathcal{A}$ be a strong Q-sequence of cardinality ω_1 and $\mathcal{B}$ the family consisting of all the infinite members of $\mathcal{A}$. Since there are at most countably many finite subsets of ω, it follows that $\mathcal{B}$ has cardinality ω_1. Let $\mathcal{B}^*$ be the set consisting of all functions $f : \mathcal{B} \to \mathcal{P}(\omega)$ such that for every set $A \in \mathcal{B}$ either $f(A) = A$ or $f(A) = \varnothing$. Now $\mathcal{B}^*$ has cardinality 2^{ω_1} and from the fact that $\mathcal{A}$ is a strong Q-sequence it follows that for every function f in $\mathcal{B}^*$ there exists a subset S_f of ω such that for every set $A \in \mathcal{B}$ the set $f(A)$ is almost equal to the intersection of A with S_f.

We will now show that the function that maps every element f in $\mathcal{B}^*$ to the subset S_f of ω is one-to-one. Let f and g be two distinct functions in $\mathcal{B}^*$ and suppose that $S_f = S_g$. Let A be a set in $\mathcal{B}$ such that $f(A) \neq g(A)$. Then

$$f(A) =^* S_f \cap A = S_g \cap A =^* g(A).$$

Thus $f(A)$ is almost equal to $g(A)$. However, the symmetric difference of $f(A)$ and $g(A)$ is equal to A and therefore infinite, a contradiction.

We conclude that $2^{\omega_1} \leqslant 2^\omega$. □

Steprāns showed in [1985] that there exists a model in which a strong Q-sequence of cardinality ω_1 exists and the lemma above shows that in this model the equality $2^\omega = 2^{\omega_1}$ holds. Hence in this model it is still possible that the remainders ω^* and ω_1^* are homeomorphic.

3.3. The small cardinals $\mathfrak{b}$, $\mathfrak{t}$ and $\mathfrak{d}$

A cardinal number is called *small* if it is defined as the cardinality of a set that is in some way associated with the natural numbers. A simple example is the cardinality of the set of real numbers: $\mathfrak{c}$. In this section we take a look at the small cardinals $\mathfrak{b}$, $\mathfrak{t}$ and $\mathfrak{d}$.

A subset A of ω^ω is called a *bounded family* if there exists a function g in ω^ω such that for every function f in A we have that $f \leqslant^* g$; we call the function g a bound for A and denote this by $A \leqslant^* g$. A subset of ω^ω that is not a bounded family is called an *unbounded family*.

3.7. DEFINITION. The least cardinality of an unbounded family is called the *bounding number* and is denoted by $\mathfrak{b}$.

3.8. PROPOSITION. *Every countable subset A of ω^ω is bounded.*

PROOF. Let $\{f_n : n \in \omega\}$ be an enumeration of A. Define, with recursion on $n \in \omega$, the function $g : \omega \to \omega$ by

$$g(n) = \max\{f_1(n), \dots, f_n(n)\}.$$

Then g is a bound for the set A. □

A set A is called a *pseudo-intersection* of a family $\mathcal{F}$ if $A \subseteq^* F$ for each element $F \in \mathcal{F}$. A family $\mathcal{F}$ of infinite subsets of ω is called a *tower* if $\mathcal{F}$ is well-ordered by $\supseteq^*$ and has no infinite pseudo-intersection.

3.9. DEFINITION. The least cardinality of a tower is denoted by $\mathfrak{t}$.

3.10. LEMMA. $\mathfrak{t} \leqslant \mathfrak{b}$.

PROOF. Let κ be a cardinal such that $\kappa < \mathfrak{t}$ and $B = \{b_\alpha : \alpha \in \kappa + 1\}$ a subset of ω^ω. We will show that B is a bounded family. Pick, with recursion on $\alpha \in \kappa + 1$, strictly increasing functions $f_\alpha \in \omega^\omega$ such that

(i) $\mathrm{ran}(f_\alpha) \subseteq^* \mathrm{ran}(f_\beta)$ for all $\beta \in \alpha$, and
(ii) $f_\alpha(n) \geqslant \max\{b_\alpha(k) : k \leqslant 2n\}$ for all $n \in \omega$.

At stage α first pick an infinite pseudo-intersection A of $\{\mathrm{ran}(f_\beta) : \beta \in \alpha\}$ and then put for every $n \in \omega$

$$f_\alpha(n) = \min\{a \in A \backslash \{f(k) : k < n\} : \forall\, k \leqslant 2n \ [b_\alpha(k) \leqslant a]\}.$$

Now consider any $\alpha \in \kappa+1$. Since $\mathrm{ran}(f_\kappa)$ is almost contained in $\mathrm{ran}(f_\alpha)$ there is an $m \in \omega$ such that $f_\kappa(m+n) \in \mathrm{ran}(f_\alpha)$ for all $n \in \omega$. Since both f_κ and f_α are strictly increasing it follows that $f_\kappa(m+n) \geqslant f_\alpha(n)$ for all $n \in \omega$. Because of (ii) we now have for all $n \in \omega$: if $n \geqslant 2m$, hence if $2(n-m) \geqslant n$, then

$$f_\kappa(n) = f_\kappa(n - m + m) \geqslant f_\alpha(n - m) \geqslant b_\alpha(n).$$

This shows that f_κ is a bound for the family B. □

3.11. LEMMA. $\omega_1 \leqslant \mathfrak{t}$.

PROOF. Let $\mathcal{F} = \{A_n : n \in \omega\}$ be a countable family of infinite subsets of ω such that $A_n \supseteq^* A_m$ whenever $n < m < \omega$. Define, with recursion on $n \in \omega$, the function $f : \omega \to \omega$ by

$$f(n) = \min\Big(\bigcap_{k \leqslant n} A_k \setminus \{a_k : k \in n\}\Big).$$

Then $\mathrm{ran}(f)$ is an infinite pseudo-intersection of $\mathcal{F}$. □

Let us recall that a subset of ω^ω is called dominating if it is cofinal in ω^ω with respect to the ordering $\leqslant^*$.

3.12. DEFINITION. The least cardinality of a dominating family is called the *dominating number* and is denoted by $\mathfrak{d}$.

It is clear that every dominating family is an unbounded family and therefore $\mathfrak{b} \leqslant \mathfrak{d}$. Of course the set ω^ω is dominating and therefore $\mathfrak{d} \leqslant \mathfrak{c}$. The lemmas above show that $\omega_1 \leqslant \mathfrak{t} \leqslant \mathfrak{b} \leqslant \mathfrak{d} \leqslant \mathfrak{c}$.

3.13. Lemma. *There exists an ω_1-scale in ω^ω if and only if $\mathfrak{d} = \omega_1$.*

Proof. Since every scale is a dominating family, it follows that the existence of an ω_1-scale implies that $\mathfrak{d} = \omega_1$.

Now suppose that $\mathfrak{d} = \omega_1$ and let $D = \{f_\alpha : \alpha \in \omega_1\}$ be a dominating subset of ω^ω. Pick, with recursion on $\alpha \in \omega_1$, increasing functions g_α in D such that $f_\alpha \leqslant^* g_\alpha$. At stage α, pick a function g_α in D such that it is a bound for the set $\{g_\beta : \beta < \alpha\} \cup \{f_\alpha\}$. Now the family $\{g_\alpha : \alpha \in \omega_1\}$ is dominating and well-ordered by $\leqslant^*$. $\square$

3.14. Theorem. *If the remainders ω^* and ω_1^* are homeomorphic, then the small cardinals $\mathfrak{t}$, $\mathfrak{b}$ and $\mathfrak{d}$ are equal to ω_1.*

Proof. From Theorem 2.2 it follows that if the remainders ω^* and ω_1^* are homeomorphic, then there exists an ω_1-scale in ω^ω. By Lemma 3.13 this implies that $\mathfrak{d} = \omega_1$. $\square$

It is known that Martin's Axiom (MA) implies that $\mathfrak{d} = \mathfrak{c}$ and thus, by the theorem above, it follows that $\mathsf{MA} + \neg\mathsf{CH}$ implies that the remainders ω^* and ω_1^* are not homeomorphic. On the other hand, in any model in which CH holds also the equality $\mathfrak{d} = \omega_1$ holds. But if the remainders ω^* and ω_1^* are homeomorphic, then not only the equality $\mathfrak{d} = \omega_1$ must hold but there must also exist, by Theorem 3.5, a strong Q-sequence of cardinality ω_1. Chodounský showed in [2011] that it is consistent to have both simultaneously. Hence we cannot exclude that in the model constructed by Chodounský the remainders ω^* and ω_1^* are homeomorphic.

We shall end this section with a consequence that was given by Nyikos in [2007]. The following lemma shows that this consequence is closely related to the small cardinal $\mathfrak{t}$.

3.15. Lemma. *The small cardinal $\mathfrak{t}$ is equal to the least cardinality of a family of proper closed-and-open subsets of ω^* that is well-ordered by $\subseteq$ and such that the union of this family is a dense subset of ω^*.*

Proof. Let $\mathcal{F} = \{F_\alpha^* : \alpha \in \kappa\}$ be a family with the properties as given in the lemma and suppose that κ is the least cardinality of such a family. We will show that the family $\{\omega \backslash F_\alpha : \alpha \in \kappa\}$, which is well-ordered by $\supseteq^*$, is a tower. Suppose that A is an infinite pseudo-intersection of this family. Then A is almost disjoint from F_α for every $\alpha \in \kappa$ and this implies that the union of the family $\mathcal{F}$ is not dense in ω^*, a contradiction. Hence the small cardinal $\mathfrak{t}$ is at most κ. A similar argument shows that κ can not be larger than the small cardinal $\mathfrak{t}$. $\square$

3.16. Definition. A point x in a topological space is called a *P-point* if the intersection of a countable family of its neighborhoods is a neighborhood of x.

Let us recall that $\mathcal{U}(\kappa)$ is the subspace of κ^* consisting of all uniform ultrafilters on κ.

3.17. THEOREM. *The space $\mathcal{U}(\omega_1)$ does not contain any P-points.*

PROOF. Let $\mathcal{F}$ be a uniform ultrafilter on ω_1 and suppose that it is a P-point. Take a one-to-one function f from ω_1 to the subset $\mathbf{R}\backslash\mathbf{Q}$ of the real numbers. For every rational number $q \in \mathbf{Q}$ let

$$A_q = \{\alpha \in \omega_1 : f(\alpha) < q\}, \text{ and}$$
$$B_q = \{\alpha \in \omega_1 : f(\alpha) > q\},$$

then either A_q or B_q is a member of $\mathcal{F}$; let C_q be equal to this set. Since the sets C_q^* ($q \in \mathbf{Q}$) form a countable family of neighborhoods of $\mathcal{F}$, it follows using our assumption that there exists a subset U of ω_1 such that

$$\mathcal{F} \in U^* \subseteq \bigcap_{q \in \mathbf{Q}} C_q^*.$$

This implies that the set U has cardinality ω_1 and is almost contained in C_q for every $q \in \mathbf{Q}$. We can leave out a countable number of elements from the set U in such a way that U is actually contained in C_q for every $q \in \mathbf{Q}$. Take an element α in U and let $r = f(\alpha)$. Now note that if $q \in \mathbf{Q}$ such that $q > r$ then $A_q \in \mathcal{F}$ and since U is a subset of $C_q = A_q$ we find that for every $\alpha \in U$ the inequality $f(\alpha) < q$ holds. Thus for every $\alpha \in U$ we have that $f(\alpha) \leqslant r$. A similar argument shows that $f(\alpha) \geqslant r$ holds for every $\alpha \in U$. Hence the function f is constant on the set U, a contradiction. □

Let $\mathcal{S}(\omega_1)$ denote the subspace of ω_1^* consisting of all free ultrafilters on ω_1 that are not uniform. This subspace is known as *the space of subuniform ultrafilters.*

3.18. LEMMA. *The space $\mathcal{S}(\omega_1)$ is dense in ω_1^*.*

PROOF. Let A be an infinite subset of ω_1 and B a countable subset of A. Then B^* is nonempty and every ultrafilter in B^* is subuniform and an element of A^*. □

3.19. LEMMA. *The space $\mathcal{S}(\omega_1)$ is equal to the union of a family of proper closed-and-open subsets of ω_1^* that is well-ordered by $\subseteq$ and of cardinality ω_1.*

PROOF. Consider the discrete space of cardinality ω_1 in the guise of $S_1 = \mathbb{Z} \times \omega_1$. For each $\alpha \in \omega_1$ define B_α to be the subset $\mathbb{Z} \times \alpha$ of S_1. We will show that the union of the family $\{B_\alpha^* : \alpha \in \omega_1\}$ is equal to $\mathcal{S}(\omega_1)$.

Let $\mathcal{F}$ be an ultrafilter contained in the union above, say $\mathcal{F} \in B_\alpha^*$. Then the set B_α is contained in $\mathcal{F}$ and thus $\mathcal{F}$ is a member of $\mathcal{S}(\omega_1)$.

Conversely, if $\mathcal{F}$ is contained in $\mathcal{S}(\omega_1)$ then there exists a set $A \in \mathcal{F}$ such that A is countable. Let α be a countable limit ordinal such that A is a subset of B_α. Then $\mathcal{F}$ is contained in B_α^*. □

3.20. THEOREM. *If the remainders ω^* and ω_1^* are homeomorphic, then there is a family of proper closed-and-open subsets of ω^* that is well-ordered by $\subseteq$ and of cardinality ω_1 such that the union of this family is a dense subset of ω^* and the complement of this union does not contain any P-points.*

Proof. Theorem 3.17 together with Lemma 3.18 and Lemma 3.19 show that such a family exists in ω_1^* and a transfer of this family to ω^* using a homeomorphism between the remainders ω^* and ω_1^* gives the required family. □

Since there are no P-points in the model constructed by Chodounský in [2011], it follows that the consequence given in the theorem above also holds in this model.

3.4. A nontrivial autohomeomorphism

In this section we consider again the discrete spaces of cardinality ω and ω_1 in the guises of $S_0 = \mathbb{Z} \times \omega$ and $S_1 = \mathbb{Z} \times \omega_1$ respectively. Assume that the remainders ω^* and ω_1^* are homeomorphic and let $\gamma : S_1^* \to S_0^*$ be a homeomorphism between the Čech-Stone remainders of S_0 and S_1.

3.21. Definition. Let κ be an infinite cardinal. An autohomeomorphism $\rho : \kappa^* \to \kappa^*$ is called *trivial* if there exists a one-to-one function f of A onto B, where the subsets A and B of κ are almost equal to κ, such that $\rho[S^*] = (f[S \cap A])^*$ for all subsets S of κ. The function f is said to *induce* ρ.

Now consider the right shift σ on S_1 defined by $\langle k, \alpha\rangle \mapsto \langle k+1, \alpha\rangle$. Since this function is a bijection, by Lemma 1.43 the restriction σ^* of its Čech-Stone extension $\beta\sigma$ to S_1^* is an autohomeomorphism and therefore $\rho = \gamma \circ \sigma^* \circ \gamma^{-1}$ is an autohomeomorphism of S_0^*. We will prove in this section that ρ is a nontrivial autohomeomorphism.

Let us first list a number of properties of the mapping σ^*. Divide the set S_1 into vertical lines $V_k = \{k\} \times \omega_1$, horizontal lines $H_\alpha = \mathbb{Z} \times \{\alpha\}$ and sets $E_\alpha = \mathbb{Z} \times [\alpha, \omega_1)$.

3.22. Lemma. *The following properties hold.*

(i) *For every $k \in \mathbb{Z}$ we have that $\sigma^*[V_k^*] = V_{k+1}^*$.*
(ii) *The equality $\sigma^*[H_\alpha^*] = H_\alpha^*$ holds for every $\alpha \in \omega_1$.*
(iii) *The set $\{H_\alpha^* : \alpha \in \omega_1\}$ is a maximal disjoint family of σ^*-invariant closed-and-open subsets of S_1^*.*
(iv) *If S is a subset of S_1 such that $V_k^* \subseteq S^*$ for every $k \in \mathbb{Z}$, then there exists a $\beta \in \omega_1$ such that $H_\alpha \subseteq S$ for all $\alpha > \beta$.*
(v) *If S is a subset of S_1 such that $H_\alpha^* \subseteq S^*$ for uncountably many $\alpha \in \omega_1$, then there is a subset A of V_0 such that $\sigma^k[A] \subseteq S$ for all but finitely many $k \in \mathbb{Z}$ and $A \cap E_\alpha$ is infinite for all $\alpha \in \omega_1$.*

Proof. For every $k \in \mathbb{Z}$ we have that

$$\sigma^*[V_k^*] \subseteq \beta\sigma[\overline{V_k}] \subseteq \overline{\sigma[V_k]} = \overline{V_{k+1}}$$

and because σ^* is onto, property (i) follows.

A similar argument shows that property (ii) holds.

To show property (iii), let S be an infinite subset of S_1 such that the intersection $S \cap H_\alpha$ is finite for every $\alpha \in \omega_1$ and let A be the set consisting of all $\alpha \in \omega_1$ such that $S \cap H_\alpha$ is nonempty. Note that A is infinite and for every $\alpha \in A$ let $k_\alpha = \max\{k \in \mathbb{Z} : \langle k, \alpha\rangle \in S\}$. Then $\{\langle k_\alpha + 1, \alpha\rangle : \alpha \in A\}$ is an infinite subset of $\sigma[S]$ disjoint from S. Hence $\sigma^*[S^*] = (\sigma[S])^* \neq S^*$.

The condition of property (iv) implies that the set $V_k \backslash S$ is finite for every $k \in \mathbb{Z}$. Thus for every $k \in \mathbb{Z}$ there exists an $\alpha_k \in \omega_1$ such that $V_k \backslash S \subseteq \alpha_k$. Let $\beta \in \omega_1$ such that $\beta > \alpha_k$ for every $k \in \mathbb{Z}$. Then for every $\alpha \geqslant \beta$ we have that $H_\alpha \subseteq S$.

For every $\alpha \in \omega_1$ such that $H^*_\alpha \subseteq S^*$, let F_α be the finite set $\{k \in \mathbb{Z} : \langle k, \alpha\rangle \notin S\}$. Since there are only countably many finite subsets of $\mathbb{Z}$, the condition of property (v) implies that there are a finite subset F of $\mathbb{Z}$ and an uncountable subset J of ω_1 such that $F_\alpha = F$ for every $\alpha \in J$. Let $A = \{0\} \times J$, then $A \subseteq V_0$ and for every $k \notin F$ we have that $\sigma^k[A] \subseteq S$. Since A is uncountable it follows that $A \cap E_\alpha$ is infinite for all $\alpha \in \omega_1$. Thus property (v) holds. □

We will translate these properties to the mapping ρ and the remainder S_0^*. Choose infinite subsets v_k, h_α and e_α of S_0 such that

$$\gamma[V_k^*] = v_k^*$$
$$\gamma[H_\alpha^*] = h_\alpha^*$$
$$\gamma[E_\alpha^*] = e_\alpha^*$$

for every $k \in \mathbb{Z}$ and $\alpha \in \omega_1$. Since the family $\{V_k : k \in \mathbb{Z}\}$ is pairwise disjoint it follows that the family $\{v_k : k \in \mathbb{Z}\}$ is pairwise almost disjoint. We can rearrange the sets v_k in such a way that the family $\{v_k : k \in \mathbb{Z}\}$ forms a partition of S_0 and, by Lemma 1.43, even assume that $v_k = \{k\} \times \omega$ for every $k \in \mathbb{Z}$. In the following lemma we formulate all the properties from the previous lemma in terms of the mapping ρ and the sets v_k, h_α and e_α.

3.23. Lemma. *The following properties hold.*

(i) *For every $k \in \mathbb{Z}$ we have that $\rho[v_k^*] = v_{k+1}^*$.*
(ii) *The equality $\rho[h_\alpha^*] = h_\alpha^*$ holds for every $\alpha \in \omega_1$.*
(iii) *The set $\{h_\alpha^* : \alpha \in \omega_1\}$ is a maximal disjoint family of ρ-invariant closed-and-open subsets of S_0^*.*
(iv) *If S is a subset of S_0 such that $v_k^* \subseteq S^*$ for every $k \in \mathbb{Z}$, then there exists a $\beta \in \omega_1$ such that $h_\alpha \subseteq^* S$ for all $\alpha > \beta$.*
(v) *If S is a subset of S_0 such that $h_\alpha^* \subseteq S^*$ for uncountably many $\alpha \in \omega_1$, then there is an infinite subset A of v_0 such that $\rho^k[A^*] \subseteq S^*$ for all but finitely many $k \in \mathbb{Z}$ and $A \cap e_\alpha$ is infinite for all $\alpha \in \omega_1$.*

Proof. All these properties follow from the corresponding properties from Lemma 3.22. Since we have that

$$\rho[v_k^*] = \gamma\sigma^*\gamma^{-1}[v_k^*] = \gamma\sigma^*[V_k^*] = \gamma[V_{k+1}^*] = v_{k+1}^*,$$

property (i) holds. A similar argument shows that property (ii) holds.

To show property (iii), let S be an infinite subset of S_0 such that the intersection $S^* \cap h_\alpha^*$ is empty for all $\alpha \in \omega_1$. It then follows that this also holds for the intersection $\gamma^{-1}[S^*] \cap H_\alpha^*$ for all $\alpha \in \omega_1$ and thus $\sigma^*[\gamma^{-1}[S^*]] \neq \gamma^{-1}[S^*]$. If we apply γ to both sides of the inequality we find that property (iii) holds.

If S is a subset of S_0 that satisfies the condition of property (iv), then $V_k^* \subseteq \gamma^{-1}[S^*]$ for every $k \in \mathbb{Z}$. Hence there exists a $\beta \in \omega_1$ such that $H_\alpha^* \subseteq \gamma^{-1}[S^*]$ for all $\alpha > \beta$. If we transfer this back to S_0^* using γ we find that $h_\alpha \subseteq^* S$ for all $\alpha > \beta$.

If S is a subset of S_0 that satisfies the condition of property (v), then $H_\alpha^* \subseteq \gamma^{-1}[S^*]$ for uncountably many $\alpha \in \omega_1$. Hence there exists a subset B of V_0 such that $(\sigma^*)^k[B^*] = (\sigma^k[B])^* \subseteq \gamma^{-1}[S^*]$ for all but finitely many $k \in \mathbb{Z}$. Let A be a subset of v_0 such that $\gamma[B^*] = A^*$. Then

$$S^* \supseteq \gamma[(\sigma^*)^k[B^*]] = \gamma[(\sigma^*)^k[\gamma^{-1}[A^*]] = \rho^k[A^*]$$

for all but finitely many $k \in \mathbb{Z}$ and this proves property (v). □

Now suppose that the autohomeomorphism ρ is trivial. Let $g : A \to B$ be a bijective function between cofinite subsets A and B of S_0 that induces ρ. By defining finitely many additional values we can assume that one of the sets A and B is equal to S_0. If B is equal to S_0 and A is not, then we can replace σ by its inverse to make sure that A is equal to S_0. Hence we will assume that A is equal to S_0. Let g^* be the restriction of βg to the Čech-Stone remainder S_0^*.

First we reformulate the properties that are given in Lemma 3.23 in terms of the function g. Note that if S is an infinite subset of S_0, then we have that $\rho[S^*] = (g[S])^* = g^*[S^*]$ since g is onto.

3.24. LEMMA. *The following properties hold.*

(i) *For every $k \in \mathbb{Z}$ we have that $g[v_k] =^* v_{k+1}$.*
(ii) *The equality $g[h_\alpha] =^* h_\alpha$ holds for every $\alpha \in \omega_1$.*
(iii) *The set $\{h_\alpha^* : \alpha \in \omega_1\}$ is a maximal disjoint family of g^*-invariant closed-and-open subsets of S_0^*.*
(iv) *If S is a subset of S_0 such that $v_k \subseteq^* S$ for every $k \in \mathbb{Z}$, then there exists a $\beta \in \omega_1$ such that $h_\alpha \subseteq^* S$ for all $\alpha > \beta$.*
(v) *If S is a subset of S_0 such that $h_\alpha \subseteq^* S$ for uncountably many $\alpha \in \omega_1$, then there is an infinite subset A of v_0 such that $g^k[A] \subseteq^* S$ for all but finitely many $k \in \mathbb{Z}$ and $A \cap e_\alpha$ is infinite for all $\alpha \in \omega_1$.*

We will now establish a contradiction. This then shows that the autohomeomorphism ρ is nontrivial. Let $\{a_n : n \in \omega\}$ be an enumeration of the set S_0. For $n \in \omega$, let $I_n = \{k \in \mathbb{Z} : g^k(a_n) \text{ is defined}\}$ and let $O_n = \{g^k(a_n) : k \in I_n\}$ be the orbit of a_n under g.

3.25. LEMMA. *Each set h_α splits only finitely many g-orbits.*

PROOF. If the set h_α splits the g-orbit O_n, then there is an integer $k \in I_n$ such that the point $g^k(a_n)$ is contained in h_α but at least one of the two points $g^{k-1}(a_n)$ and $g^{k+1}(a_n)$ is not. This implies that $g^k(a_n) \in h_\alpha \backslash g[h_\alpha]$ or $g^{k+1}(a_n) \in g[h_\alpha] \backslash h_\alpha$.

We conclude that the intersection of each split g-orbit with the symmetric difference of h_α and $g[h_\alpha]$, is nonempty. By Lemma 3.24, this symmetric difference is a finite set and since the different g-orbits are disjoint, it follows that only finitely many g-orbits intersect it. □

3.26. Lemma. *Each infinite g-orbit is almost covered by two h_α's.*

Proof. If $a_n \in S_0\backslash B$, then $I_n = \mathbb{Z}_{\geqslant 0}$ and the g-orbit O_n is infinite. Hence the set O_n^* is a nonempty closed-and-open subset of S_0^* that is g^*-invariant and thus, by Lemma 3.24, there is an $\alpha \in \omega_1$ such that the intersection of h_α with O_n is infinite. Let

$$J = \{k \in I_n : g^k(a_n) \in h_\alpha \text{ and } g^{k+1}(a_n) \notin h_\alpha\},$$

then $\{g^{k+1}(a_n) : k \in J\}$ is a subset of $g[h_\alpha]\backslash h_\alpha$ and since this latter set is finite it follows that J is finite. This implies that $O_n \subseteq^* h_\alpha$.

If $a_n \notin \bigcup\{O_k : a_k \in S_0\backslash B\}$ and the orbit O_n is infinite, then $I_n = \mathbb{Z}$ and we can split the orbit in an infinite forward and backward orbit. These orbits determine nonempty g^*-invariant closed-and-open subsets of S_0^* and using the same argument as above it follows that each of these two orbits is almost contained in an h_α for some $\alpha \in \omega_1$. Hence each infinite orbit is almost covered by two h_α's. □

Let F be the union of all finite g-orbits and G the union of all infinite g-orbits.

3.27. Lemma. *For all but countably many $\alpha \in \omega_1$, $h_\alpha \subseteq^* F$.*

Proof. Let D be the set consisting of all $\alpha \in \omega_1$ such that h_α meets an infinite orbit in an infinite set. Since there are only countably many orbits and by Lemma 3.26 each infinite orbit meets at most two h_α's in an infinite set, it follows that the set D is countable. If $\alpha \notin D$, then h_α meets every infinite orbit in a finite set. By Lemma 3.25 only finitely many g-orbits are split by h_α, thus only finitely many infinite g-orbits intersect h_α. Hence all but finitely many points of h_α are contained in F. □

We now have the following theorem.

3.28. Theorem. *If the remainders ω^* and ω_1^* are homeomorphic, then there exists a nontrivial autohomeomorphism of ω^*.*

Proof. It follows from Lemma 3.24, using Lemma 3.27, that there is an infinite subset A of v_0 such that $g^k[A]$ is almost contained in the set F for all but finitely many integers $k \in \mathbb{Z}$. In fact, since F is g-invariant, it follows that this actually holds for all integers. Now define $S = v_0 \cap F$ and note that this set almost contains the set A and is therefore infinite.

Let $E = \bigcup\{O_n : a_n \in S\}$. Since S is a subset of E, the set E is infinite and because it is a union of finite orbits, it is g-invariant and thus the corresponding set E^* is g^*-invariant. From Lemma 3.24 it follows that there is an $\alpha \in \omega_1$ such that the intersection $h_\alpha \cap E$ is infinite. By Lemma 3.25 this h_α only splits finitely many orbits. Hence h_α contains an infinite number of different orbits that all pass through S. This implies that $h_\alpha \cap v_0$ is infinite, a contradiction with the fact that h_α and v_0 are almost disjoint. Hence the autohomeomorphism ρ is nontrivial. □

Is it consistent that a nontrivial autohomeomorphism of ω^* exists? A result by Rudin in [1956] shows that if the continuum hypothesis is true, then ω^* has precisely $2^{\mathfrak{c}}$ autohomeomorphisms. Since there are only $\mathfrak{c}$ many trivial autohomeomorphisms of ω^* this implies that most of the autohomeomorphisms of ω^* are nontrivial.

It is an open question whether a model exists in which there are simultaneously a nontrivial autohomeomorphism of ω^*, a strong Q-sequence of cardinality ω_1 and an ω_1-scale.

Consider once again the discrete spaces $S_0 = \mathbb{Z} \times \omega$ and $S_1 = \mathbb{Z} \times \omega_1$. From the vertical lines $V_k = \{k\} \times \omega_1$, horizontal lines $H_\alpha = \mathbb{Z} \times \{\alpha\}$ and the sets $E_\alpha = \mathbb{Z} \times [\alpha, \omega_1)$ in S_1 we defined, using a homeomorphism between the two remainders S_0^* and S_1^*, the subsets v_k, h_α and e_α of S_0. We have used this structure to obtain all the consequences in this thesis: using the sets e_α we proved that there exists an ω_1-scale in ω^ω, the sets h_α formed an uncountable strong Q-sequence of cardinality ω_1 and in this section, using the sets v_k, h_α and e_α, we obtained a nontrivial autohomeomorphism of ω^*. Hence it is possible that the remainders ω^* and ω_1^* are homeomorphic, only if there exists a model in which the given three consequences hold and are linked together using the same structure. If such a model does not exist, then the given structure seems a good candidate to derive a contradiction from. Although we have studied this structure quite a bit, we did not find a contradiction.

Notes

An introduction to the Čech-Stone compactification $\beta\omega$ and the corresponding remainder is given in [1984] by Van Mill. A list of open problems concerning these two spaces is given in [1990] by Hart and Van Mill. The Katowice problem is given as Question 43 in this paper.

The most important small cardinals, including the three that are discussed in Section 3.3, are introduced by Van Douwen in [1984]. The proofs of Lemma 3.10 and Lemma 3.11 are from this paper.

Theorem 3.20 is given in [2007] by Nyikos. In this paper the Katowice problem is introduced and a number of known results and open questions are discussed.

The consequence given in Section 3.4 is a recent, still unpublished result of the advisor of this thesis.

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