M. Straatman

The duality map

Bachelorscriptie

Scriptiebegeleider: Dr. O.W. van Gaans

Datum Bachelorexamen: 11 juni 2014

Mathematisch Instituut, Universiteit Leiden

Contents

Introduction

In this thesis we look at Calvert's theorem and elaborate its proof. Calvert's theorem connects the so-called duality map to contractive projections in Banach spaces. In the theorem some conditions are required. In this thesis we check if the theorem could be true when these conditions do not hold. There appear to be examples in which not all the conditions hold, while the result of the theorem actually remains intact. However, we also present examples in which not all of the conditions hold, and indeed the result of the theorem is no longer true.

In the previous paragraph the duality map was mentioned. The duality map on a Banach space X is given for any $x \in X$ with $x \neq 0$ by

$$J_x = \{\phi \in X^* : ||\phi|| = 1, \phi(x) = ||x||\}.$$

With the Hahn-Banach theorem we have that this set is non-empty. In this thesis we will see that if certain conditions hold on the Banach space X, J_x consists of exactly one element. This will be used in Calvert's theorem, which states that if X is strictly convex and reflexive, and its dual is also strictly convex, that a closed linear subspace is the range of a linear contractive projecton if and only if the duality map of the subspace is a linear subspace of the dual space of X, with a contractive projection being a projection where each element is projected upon an element with a norm smaller than or equal to the norm of its original.

Calvert's theorem was proved by Bruce Calvert in 1975, but this original proof is very short and many of the details are left out. Therefore we will rewrite the proof, including the details that were left out. The proof of Calvert's theorem is really based on all the conditions stated in the theorem, but we do not know for certain that all of these conditions are actually necessary. There might be a proof of an altered version of Calvert's theorem possible in which not all of the conditions are needed. Therefore we will investigate examples in which some of the conditions do no longer hold, to see if the result of Calvert's theorem still holds.

Some examples give the impression that certain conditions on X are necessary while other examples in which different conditions on X are left out suggest that some of the conditions are not always necessary. This is an interesting subject for further research. We have also considered the shape of the unit sphere of the dual norm of a norm in $\mathbb{R}^2$ when given the shape of the unit sphere of the norm itself. However this was not investigated fully enough to end up in the thesis, but might also be an interesting subject for further research.

In chapter one we will start with defining and explaining the Gateaux derivative, and consider some properties it has. This will be useful when considering the duality maps and their properties, which we will do in chapter two. Chapter three will be about the contractive projection, and also about the nearest-point projection and the connection between both, which will be useful when proving Calvert's theorem in chapter four. We will conclude with chapter five in which we investigate the examples stated above.

1 The Gateaux derivative

The Gateaux derivative is a generalization of the concept of a directional derivative which we know from the real analysis. It can be used to determine the derivative of a norm in an arbitrary Banach space. The Gateaux derivative has some interesting proporties, and because of those properties it will actually turn out to be a useful concept later on when considering the so-called duality map. For this chapter we consider a real normed space X with norm $||\cdot||$. First we will define the concepts of the unit sphere and unit ball, since we will need these concepts in various considerations.

Definition 1.1. *Let X be a normed vector space with norm $||\cdot||$. Then we define $S_X = \{x \in X : ||x|| = 1\}$ as the unit sphere.*

Definition 1.2. *Let X be a normed vector space with norm $||\cdot||$. Then we define $B_X = \{x \in X : ||x|| \leq 1\}$ as the unit ball.*

Before we can define the Gateaux derivative, we have to define the partial limits G_- and G_+. We will be able to define the Gateaux derivative only when these partial limits exist.

Definition 1.3. *Let $x_0, y_0 \in X$. Then we define*

$$G_-(x_0, y_0) = \lim_{t \uparrow 0} \frac{||x_0 + ty_0|| - ||x_0||}{t}$$

as the left-hand Gateaux derivative of the norm at x_0 in the direction of y_0, and

$$G_+(x_0, y_0) = \lim_{t \downarrow 0} \frac{||x_0 + ty_0|| - ||x_0||}{t}$$

as the right-hand Gateaux derivative of the norm at x_0 in the direction of y_0.

Lemma 1.4. *For any $x_0, y_0 \in X$ we have that $G_-(x_0, y_0)$ and $G_+(x_0, y_0)$ exist, and $G_-(x_0, y_0) \leq G_+(x_0, y_0)$.*

Proof. We will show that for any $x_0, y_0 \in X$ the function

$$f_{x_0,y_0} : \mathbb{R} \setminus \{0\} \to \mathbb{R},\ t \mapsto \frac{||x_0 + ty_0|| - ||x_0||}{t}$$

is non-decreasing. Let $x_0, y_0 \in X$ and let $0 < t_0 < t_1$. Then we have:

$$\begin{aligned}
\frac{||x_0 + t_0 y_0|| - ||x_0||}{t_0} &= \frac{||\left(1 - \frac{t_0}{t_1}\right) x_0 + \frac{t_0}{t_1}(x_0 + t_1 y_0)|| - ||x_0||}{t_0} \\
&\leq \frac{||\left(1 - \frac{t_0}{t_1}\right) x_0|| + ||\frac{t_0}{t_1}(x_0 + t_1 y_0)|| - ||x_0||}{t_0} \\
&= \frac{\left|1 - \frac{t_0}{t_1}\right| ||x_0|| + \left|\frac{t_0}{t_1}\right| ||(x_0 + t_1 y_0)|| - ||x_0||}{t_0} \\
&= \frac{\left(1 - \frac{t_0}{t_1}\right) ||x_0|| + \left(\frac{t_0}{t_1}\right) ||(x_0 + t_1 y_0)|| - ||x_0||}{t_0} \\
&= \frac{\frac{t_0}{t_1}(-||x_0|| + ||x_0 + t_1 y_0||)}{t_0} = \frac{||x_0 + t_1 y_0|| - ||x_0||}{t_1}.
\end{aligned}$$

So f_{x_0,y_0} is nondecreasing on the interval $(0,\infty)$. In the same way $f_{x_0,-y_0}$ is also nondecreasing on $(0,\infty)$. Now let $-t_1 < -t_0 < 0$. Then we get:

$$\frac{||x_0 - t_1 y_0|| - ||x_0||}{-t_1} = -\frac{||x_0 + t_1(-y_0)|| - ||x_0||}{t_1}$$
$$\leq -\frac{||x_0 + t_0(-y_0)|| - ||x_0||}{t_0} = \frac{||x_0 - t_0 y_0|| - ||x_0||}{-t_0}.$$

So f_{x_0,y_0} is also nondecreasing on the interval $(-\infty, 0)$. Finally, let $0 < t_0$. We have:

$$||x_0|| + ||x_0|| = ||2x_0|| = ||x_0 - t_0 y_0 + t_0 y_0 + x_0|| \leq ||x_0 - t_0 y_0|| + ||x_0 + t_0 y_0||.$$

So we have:

$$||x_0|| - ||x_0 - t_0 y_0|| \leq ||x_0 + t_0 y_0|| - ||x_0||.$$

Therefore we get:

$$\frac{||x_0 - t_0 y_0|| - ||x_0||}{-t_0} \leq \frac{||x_0 + t_0 y_0|| - ||x_0||}{t_0}.$$

Thus we can conclude that f_{x_0,y_0} is nondecreasing on $\mathbb{R} \setminus \{0\}$. From this we can immediately conclude that $G_-(x_0,y_0)$ and $G_+(x_0,y_0)$ exist and that $G_-(x_0,y_0) \leq G_+(x_0,y_0)$. □

Definition 1.5. *Let $x_0, y_0 \in X$. If $G_+(x_0,y_0) = G_-(x_0,y_0)$ then we call the norm Gateaux differentiable in x_0 in the direction of y_0 and we denote this by $G(x_0,y_0)$. If the Gateaux derivative exists in x_0 for all directions $y \in X$ we say that the norm is Gateaux differentiable in x_0. If the norm is Gateaux differentiable in all $x \in S_X$, we say that the norm is Gateaux differentiable.*

There is a connection between G_- and G_+, and they also have some interesting properties. As an example, it turns out that G_+ is sublinear. We will now consider these properties.

Lemma 1.6. *Let $x_0, y_0 \in X$. Then $G_-(x_0,y_0) = -G_+(x_0,-y_0)$.*

Proof. We have:

$$G_-(x_0,y_0) = \lim_{t\uparrow 0} \frac{||x_0 + t y_0|| - ||x_0||}{t} = \lim_{t\downarrow 0} \frac{||x_0 - t y_0|| - ||x_0||}{-t}$$
$$= -\lim_{t\downarrow 0} \frac{||x_0 - t y_0|| - ||x_0||}{t} = -\lim_{t\downarrow 0} \frac{||x_0 + t \cdot -y_0|| - ||x_0||}{t} = -G_+(x_0,-y_0).$$

□

Lemma 1.7. *Let $x_0, y_0, y_1 \in X$ and let $\lambda \in \mathbb{R}_{\geq 0}$. Then*

1. $G_+(x_0, \lambda y_0) = \lambda G_+(x_0,y_0)$
2. $G_+(x_0, y_0 + y_1) \leq G_+(x_0,y_0) + G_+(x_0,y_1)$
3. $G_-(x_0, \lambda y_0) = \lambda G_-(x_0,y_0)$

Proof. Let $x_0, y_0, y_1 \in X$ and let $\lambda \in \mathbb{R}_{\geq 0}$.

1. We have:

$$G_+(x_0, \lambda y_0) = \lim_{t\downarrow 0} \frac{||x_0 + t\lambda y_0|| - ||x_0||}{t}.$$

- Suppose $\lambda \neq 0$. Then we get:

$$G_+(x_0, \lambda y_0) = \lim_{t \downarrow 0} \frac{||x_0 + t\lambda y_0|| - ||x_0||}{t} = \lim_{\frac{t}{\lambda} \downarrow 0} \frac{||x_0 + \frac{t}{\lambda}\lambda y_0|| - ||x_0||}{\frac{t}{\lambda}}$$

$$= \lambda \lim_{\frac{t}{\lambda} \downarrow 0} \frac{||x_0 + ty_0|| - ||x_0||}{t} = \lambda \lim_{t \downarrow 0} \frac{||x_0 + ty_0|| - ||x_0||}{t} = \lambda G_+(x_0, y_0).$$

- Suppose $\lambda = 0$. Then we have:

$$G_+(x_0, 0 \cdot y_0) = \lim_{t \downarrow 0} \frac{||x_0 + t \cdot 0 \cdot y_0|| - ||x_0||}{t} = \lambda \lim_{t \downarrow 0} \frac{||x_0|| - ||x_0||}{t}$$

$$= \lambda \lim_{t \downarrow 0} 0 = 0 = 0 \cdot G_+(x_0, y_0) = \lambda G_+(x_0, y_0).$$

So $G_+(x_0, \lambda y_0) = \lambda G_+(x_0, y_0)$.

2. We have:

$$G_+(x_0, y_0 + y_1) = \lim_{t \downarrow 0} \frac{||x_0 + t(y_0 + y_1)|| - ||x_0||}{t} = \lim_{\frac{t}{2} \downarrow 0} \frac{||x_0 + \frac{t}{2}(y_0 + y_1)|| - ||x_0||}{\frac{t}{2}}$$

$$= \lim_{\frac{t}{2} \downarrow 0} 2 \frac{||x_0 + \frac{t}{2}(y_0 + y_1)|| - ||x_0||}{t} = \lim_{\frac{t}{2} \downarrow 0} \frac{||2(x_0 + \frac{t}{2}(y_0 + y_1))|| - 2||x_0||}{t}$$

$$= \lim_{t \downarrow 0} \frac{||x_0 + ty_0 + x_0 + ty_1|| - 2||x_0||}{t} \leq \lim_{t \downarrow 0} \frac{||x_0 + ty_0|| + ||x_0 + ty_1|| - 2||x_0||}{t}$$

$$= \lim_{t \downarrow 0} \left(\frac{||x_0 + ty_0|| - ||x_0||}{t} + \frac{||x_0 + ty_1|| - ||x_0||}{t} \right)$$

$$= \lim_{t \downarrow 0} \frac{||x_0 + ty_0|| - ||x_0||}{t} + \lim_{t \downarrow 0} \frac{||x_0 + ty_1|| - ||x_0||}{t}$$

$$= G_+(x_0, y_0) + G_+(x_0, y_1).$$

3. This proof is identical to the proof of 1.

□

Lemma 1.8. *Let $x_0, y_0, y_1 \in X$. Then we have*

$$G_-(x_0, y_0 + y_1) \geq G_-(x_0, y_0) + G_-(x_0, y_1).$$

Proof. Let $x_0, y_0, y_1 \in X$. With lemma 1.7 we have that

$$G_+(x_0, y_0 + y_1) \leq G_+(x_0, y_0) + G_+(x_0, y_1)$$

and thus that

$$G_+(x_0, -y_0 - y_1) \leq G_+(x_0, -y_0) + G_+(x_0, -y_1).$$

So this gives:

$$-G_+(x_0, -y_0 - y_1) \geq -G_+(x_0, -y_0) - G_+(x_0, -y_1).$$

We also know from lemma 1.6 that

$$G_-(x_0, y_0) = -G_+(x_0, -y_0)$$

and thus

$$G_-(x_0, y_0 + y_1) = -G_+(x_0, -y_0 - y_1).$$

If we combine this, we find:

$$G_-(x_0, y_0 + y_1) = -G_+(x_0, -y_0 - y_1) \geq -G_+(x_0, -y_0) - G_+(x_0, -y_1) = G_-(x_0, y_0) + G_-(x_0, y_1).$$

□

Because of the properties of G_+ and G_-, we can now determine some properties of G.

Proposition 1.9. *Let $x_0, y_0, y_1 \in X$ and let $\lambda \in \mathbb{R}$ and suppose that $||\cdot||$ is Gateaux differentiable in x_0 in the direction of y_0 and y_1. Then $||\cdot||$ is also Gateaux differentiable in x_0 in the direction of λy_0 and $y_0 + y_1$. Moreover we have that $G(x_0, \lambda y_0) = \lambda(x_0, y_0)$ and that $G(x_0, y_0 + y_1) = G(x_0, y_0) + G(x_0, y_1)$.*

Proof. Let $x_0, y_0, y_1 \in X$ and suppose that $||\cdot||$ is Gateaux differentiable in x_0 in the direction of y_0 and y_1.

1. Now we have:
$$G_+(x_0, y_0) = G_-(x_0, y_0) = -G_+(x_0, -y_0)$$
so
$$G_+(x_0, y_0) = -G_+(x_0, -y_0)$$
and thus
$$G_+(x_0, -y_0) = -G_+(x_0, --y_0) = -G_+(x_0, y_0).$$
Similarly we get
$$G_-(x_0, -y_0) = -G_-(x_0, y_0).$$
Because we have for all $\lambda \in \mathbb{R}_{\geq 0}$ that
$$G_+(x_0, \lambda y_0) = \lambda G_+(x_0, y_0)$$
we now get for all $\lambda \in \mathbb{R}$ that
$$G_+(x_0, \lambda y_0) = \lambda G_+(x_0, y_0).$$
Similarly we get that for all $\lambda \in \mathbb{R}$ we have
$$G_-(x_0, \lambda y_0) = \lambda G_-(x_0, y_0).$$
Now we get for all $\lambda \in \mathbb{R}$, because $G_+(x_0, y_0) = G_-(x_0, y_0)$:
$$G_+(x_0, \lambda y_0) = \lambda G_+(x_0, y_0) = \lambda G_-(x_0, y_0) = G_-(x_0, \lambda y_0)$$
and thus we have for all $\lambda \in \mathbb{R}$ that
$$G_+(x_0, \lambda y_0) = G_-(x_0, \lambda y_0).$$
So $||\cdot||$ is Gateaux differentiable in the direction of λy_0 and we have that $G(x_0, \lambda y_0) = \lambda(x_0, y_0)$.

2. We also have:
$$G_+(x_0, y_0 + y_1) \leq G_+(x_0, y_0) + G_+(x_0, y_1) = G_-(x_0, y_0) + G_-(x_0, y_1)$$
$$\leq G_-(x_0, y_0 + y_1) \leq G_+(x_0, y_0 + y_1)$$
so
$$G_+(x_0, y_0 + y_1) = G_+(x_0, y_0) + G_+(x_0, y_1) = G_-(x_0, y_0 + y_1)$$
and thus $||\cdot||$ is Gateaux differentiable in x_0 in the direction of $y_0 + y_1$ and we have that
$$G(x_0, y_0 + y_1) = G(x_0, y_0) + G(x_0, y_1).$$

□

Corollary 1.10. *Let $x_0 \in X$ and suppose that $||\cdot||$ is Gateaux differentiable in x_0. Then G is linear in y_0.*

In conclusion, we will show how a calculation of a Gateaux derivative is made. Therefore we consider $\mathbb{R}^n$ with the p-norm.

Example 1.11. Let $X = \mathbb{R}^n$ and consider for $1 < p < \infty$ the following norm:

$$|| \cdot ||_p : (x_1, ..., x_n) \longmapsto (|x_1|^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}}.$$

Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$ with $x \neq 0$. First we will calculate the Gateaux derivative in the direction $y = (1, 0, ..., 0)$.

- Suppose $x_1 \neq 0$. Then we have:

$$\lim_{t\to 0} \frac{||x + ty|| - ||x||}{t} = \lim_{t\to 0} \frac{(|x_1 + t|^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}} - (|x_1|^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}}}{t}.$$

  We assume that t is small enough such that the sign of $x_1 + t$ is equal to the sign of x_1.

  - Suppose $x_1 > 0$. Then we get:

$$\lim_{t\to 0} \frac{(|x_1 + t|^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}} - (|x_1|^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}}}{t}$$

$$= \lim_{t\to 0} \frac{((x_1 + t)^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}} - ((x_1)^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}}}{t}.$$

    With l'Hopital we get:

$$\lim_{t\to 0} \frac{((x_1 + t)^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}} - ((x_1)^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}}}{t}$$

$$= \lim_{t\to 0} \frac{((x_1 + t)^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}-1} \cdot \frac{1}{p} \cdot p \cdot (x_1 + t)^{p-1} - 0}{1}$$

$$= \lim_{t\to 0} |x_1 + t|^{p-1} \cdot (|x_1 + t|^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}-1} = |x_1|^{p-1} \cdot (|x_1|^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}-1}.$$

  - Suppose $x_1 < 0$. Then we get:

$$\lim_{t\to 0} \frac{(|x_1 + t|^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}} - (|x_1|^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}}}{t}$$

$$= \lim_{t\to 0} \frac{((-x_1 - t)^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}} - ((-x_1)^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}}}{t}.$$

    With l'Hopital we get:

$$\lim_{t\to 0} \frac{((-x_1 - t)^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}} - ((-x_1)^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}}}{t}$$

$$= \lim_{t\to 0} \frac{((-x_1 - t)^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}-1} \cdot \frac{1}{p} \cdot p \cdot (-x_1 - t)^{p-1} \cdot -1 - 0}{1}$$

$$= \lim_{t\to 0} -|x_1 + t|^{p-1} \cdot (|x_1 + t|^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}-1} = -|x_1|^{p-1} \cdot (|x_1|^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}-1}.$$

  So when $x_1 \neq 0$, we get

$$G(x, y) = \text{sgn}(x_1)|x_1|^{p-1} \cdot (|x_1|^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}-1}.$$

- Suppose $x_1 = 0$. Then we have:

$$\lim_{t\to 0}\frac{||x+ty||-||x||}{t} = \lim_{t\to 0}\frac{(|x_1+t|^p+|x_2|^p+...+|x_n|^p)^{\frac{1}{p}}-(|x_1|^p+|x_2|^p+...+|x_n|^p)^{\frac{1}{p}}}{t}$$
$$= \lim_{t\to 0}\frac{(|t|^p+|x_2|^p+...+|x_n|^p)^{\frac{1}{p}}-(|x_2|^p+...+|x_n|^p)^{\frac{1}{p}}}{t}.$$

Then:

$$\lim_{t\downarrow 0}\frac{(|t|^p+|x_2|^p+...+|x_n|^p)^{\frac{1}{p}}-(|x_2|^p+...+|x_n|^p)^{\frac{1}{p}}}{t}$$
$$= \lim_{t\downarrow 0}\frac{(t^p+|x_2|^p+...+|x_n|^p)^{\frac{1}{p}}-(|x_2|^p+...+|x_n|^p)^{\frac{1}{p}}}{t}.$$

With l'Hopital we get:

$$\lim_{t\downarrow 0}\frac{(t^p+|x_2|^p+...+|x_n|^p)^{\frac{1}{p}}-(|x_2|^p+...+|x_n|^p)^{\frac{1}{p}}}{t}$$
$$= \lim_{t\downarrow 0}\frac{(t^p+|x_2|^p+...+|x_n|^p)^{\frac{1}{p}-1}\cdot\frac{1}{p}\cdot p\cdot t^{p-1}-0}{1}$$
$$= \lim_{t\downarrow 0}|t|^{p-1}\cdot(|t|^p+|x_2|^p+...+|x_n|^p)^{\frac{1}{p}-1}$$
$$= 0^{p-1}\cdot(|x_2|^p+...+|x_n|^p)^{\frac{1}{p}-1}.$$

Because $p > 1$ we have that $0^{p-1} = 0$ and thus:

$$0^{p-1}\cdot(|x_2|^p+...+|x_n|^p)^{\frac{1}{p}-1} = 0\cdot(|x_2|^p+...+|x_n|^p)^{\frac{1}{p}-1} = 0.$$

We also have:

$$\lim_{t\uparrow 0}\frac{(|t|^p+|x_2|^p+...+|x_n|^p)^{\frac{1}{p}}-(|x_2|^p+...+|x_n|^p)^{\frac{1}{p}}}{t}$$
$$= \lim_{t\uparrow 0}\frac{((-t)^p+|x_2|^p+...+|x_n|^p)^{\frac{1}{p}}-(|x_2|^p+...+|x_n|^p)^{\frac{1}{p}}}{t}.$$

With l'Hopital we get:

$$\lim_{t\uparrow 0}\frac{((-t)^p+|x_2|^p+...+|x_n|^p)^{\frac{1}{p}}-(|x_2|^p+...+|x_n|^p)^{\frac{1}{p}}}{t}$$
$$= \lim_{t\uparrow 0}\frac{((-t)^p+|x_2|^p+...+|x_n|^p)^{\frac{1}{p}-1}\cdot\frac{1}{p}\cdot p\cdot(-t)^{p-1}\cdot -1-0}{1}$$
$$= \lim_{t\uparrow 0}-|t|^{p-1}\cdot(|t|^p+|x_2|^p+...+|x_n|^p)^{\frac{1}{p}-1}$$
$$= -0^{p-1}\cdot(|x_2|^p+...+|x_n|^p)^{\frac{1}{p}-1}.$$

Because $p > 1$ we have that $0^{p-1} = 0$ and thus:

$$-0^{p-1}\cdot(|x_2|^p+...+|x_n|^p)^{\frac{1}{p}-1} = -0\cdot(|x_2|^p+...+|x_n|^p)^{\frac{1}{p}-1} = 0.$$

So when $x_1 = 0$, we get

$$G(x,y) = 0.$$

So now we can write:

$$G(x,y) = \text{sgn}(x_1)|x_1|^{p-1} \cdot (|x_1|^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}-1}.$$

Because when $x_1 = 0$ we get that $|x_1|^{p-1} = 0$ and thus $G(x,y) = 0$.
So the Gateaux derivative in x in the direction of $y = (1, 0, ..., 0)$ is

$$G(x,y) = \text{sgn}(x_1)|x_1|^{p-1} \cdot (|x_1|^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}-1}.$$

We can calculate the Gateaux derivative in x in the direction of $y = (y_1, ..., y_n)$ with for every $i \in \{1, ..., n\}$ that $y_i \in \{0, 1\}$ and $\sum\limits_{i=1}^{n} y_i = 1$ in the same way as above, and this results in the following Gateaux derivative in the direction of y_0 for which $y_i = 1$:

$$G(x,y) = \text{sgn}(x_i)|x_i|^{p-1} \cdot (|x_1|^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}-1}.$$

With lemma 1.9 we have that $|| \cdot ||_p$ is Gateaux differentiable in any direction $y \in \mathbb{R}^n$ and thus the Gateaux derivative in $x \in \mathbb{R}^n$ in the direction of $y = (y_1, ..., y_n) \in \mathbb{R}^n$ is given by:

$$G(x,y) = \sum_{i=1}^{n} y_i \text{sgn}(x_i)|x_i|^{p-1} \cdot (|x_1|^p + |x_2|^p + ... + |x_n|^p)^{\frac{1}{p}-1}.$$

2 The duality map

In the previous chapter we defined the Gateaux derivative, and said that it would turn out to be useful when considering the duality map. So in this chapter we will define the duality map, and consider the connection with the Gateaux derivative. We will also define an alternate duality map, which is only slightly different from the normal duality map. The duality map has several properties that are of interest for us, and all of these properties also hold for the alternate duality map, although sometimes with a little adjustment. In this chapter we will also consider the smoothness, strict convexity and reflexivity of a space, and the connection between these notions and the uniqueness of the duality map.

The images of the duality map are in fact subsets of the dual space, and thus we will start with the definition of the dual space and dual norm of a normed vector space, after which we can define de duality map and its alternative form.

Definition 2.1. *Let X be a normed vector space over $\mathbb{F}$ with norm $||\cdot||$. Then*

$$X^* = \{f : X \longrightarrow \mathbb{F} \mid f \text{ is linear and bounded}\}.$$

X^* is a normed vector space with the norm given by:

$$||\cdot||_* : f \longmapsto \sup\{||f(x)||_{\mathbb{F}} : ||x|| \leq 1\}.$$

Definition 2.2. *Let E be a Banach space. Let $x \in E$ with $x \neq 0$. Then we call*

$$J_x = \{\varphi \in E^* : ||\varphi|| = 1, \varphi(x) = ||x||\}$$

the duality map.

Definition 2.3. *Let E be a Banach space. Let $x \in E$ with $x \neq 0$. Then*

$$\tilde{J}_x = \{\varphi \in E^* : ||\varphi|| = ||x||, \varphi(x) = ||x||^2\}$$

is an alternative definition of the duality map, and is also called the duality map.

Proposition 2.4. *Let X be a normed space, and let $x \in X$. Then we have that $\#J_x \geq 1$, and thus $\#\tilde{J}_x \geq 1$.*

Proof. See [4, corollary 5.22]. This is a corollary of the Hahn Banach theorem. □

Theorem 2.5. *Let E be a Banach space with norm $||\cdot||$ which is Gateaux differentiable in x for all $x \in E$ with $x \neq 0$. Let $x \in E$ with $x \neq 0$. Let $G(x, \cdot)$ denote the Gateaux derivative of the norm in x as a function of y. Then:*

$$G(x, \cdot) \in J_x.$$

Proof. Let $||\cdot||$ be a norm on E which is Gateaux differentiable for all $x \in E$ with $x \neq 0$. Let $x \in E$ with $||x|| = 1$. Let $G(x, \cdot)$ denote the Gateaux derivative of the norm in x as a function of y.

1. With corollary 1.10 we have that G linear is in y_0.
2. Let $y \in E$ such that $||y|| \leq 1$. Then we have:

$$G(x, y) = G_+(x, y) = \lim_{t \downarrow 0} \frac{||x + ty|| - ||x||}{t} \leq \lim_{t \downarrow 0} \frac{||x|| + ||ty|| - ||x||}{t}$$

$$= \lim_{t \downarrow 0} \frac{|t| \cdot ||y||}{t} = \lim_{t \downarrow 0} \frac{t||y||}{t} = \lim_{t \downarrow 0} ||y|| = ||y|| \leq 1.$$

So with [4, Lemma 4.1] we now know that $G(x, \cdot)$ is bounded and thus $G(x, \cdot) \in E^*$.

3. Let $y \in E$ such that $y = x$. Then we have:

$$G(x,y) = G(x,x) = G_+(x,x) = \lim_{t\downarrow 0} \frac{||x+tx|| - ||x||}{t} = \lim_{t\downarrow 0} \frac{||(1+t)x|| - ||x||}{t}$$

$$= \lim_{t\downarrow 0} \frac{|1+t| \cdot ||x|| - ||x||}{t} = \lim_{t\downarrow 0} \frac{(1+t)||x|| - ||x||}{t} = \lim_{t\downarrow 0} \frac{t||x||}{t} = \lim_{t\downarrow 0} ||x|| = ||x||.$$

   So $G(x,x) = ||x||$.

4. Let $y \in E$ such that $y = \frac{x}{||x||}$. From 2 it follows that

$$||G|| = \sup\{||G(x,y)|| : ||y|| \leq 1\} \leq 1.$$

   G is linear, so it follows by 3 that

$$G(x,y) = G(x, \frac{x}{||x||}) = \frac{1}{||x||} G(x,x) = \frac{1}{||x||} ||x|| = 1$$

   so it follows that

$$||G|| = \sup\{||G(x,y)|| : ||y|| \leq 1\} \geq 1$$

   so now we have that $||G|| = 1$.

So now it follows that $G \in J_x$. □

Corollary 2.6. *Let E be a Banach space with norm $||\cdot||$ which is Gateaux differentiable in x for all $x \in E$ with $x \neq 0$. Let $x \in E$ with $x \neq 0$. Let $G(x,\cdot)$ denote the Gateaux derivative of the norm in x as a function of y. Then:*

$$||x||G(x,\cdot) \in \tilde{J}_x.$$

If a normed space meets certain requirements, the duality map and also its alternative form will be unique for every element x. We will now consider the requirements needed for this result. When the duality map is unique, we can define it as a function from a normed space into its dual space.

Definition 2.7. *Let X be a normed space. X is called reflexive if*

$$\tau : X \to X^{**}, \ x \mapsto (y \mapsto y(x))$$

is an isomorphism.

Proposition 2.8. *Every finite dimensional normed vector space is reflexive.*

Proof. See [4, example 5.39]. □

Definition 2.9. *Let X be a normed vector space. Let $x^* \in X^*$ be a linear functional and let $A \subset X$ be a subset of X. Then x^* is a support functional for A if there exists an $x_0 \in A$ such that $x^*(x_0) = \sup\{x^*(x) : x \in A\}$. Then x_0 is a support point of A. The set $\{x \in X : x^*(x) = x^*(x_0)\}$ is a support hyperplane for A. The support hyperplane and x^* are both said to support A at x_0.*

Definition 2.10. *Let X be a normed space and let $x \in S_X$. Then x is a point of smoothness of B_X if there is only one hyperplane that supports B_X at x.*
X is called smooth if for every $x \in S_X$, x is a point of smoothness of B_X.

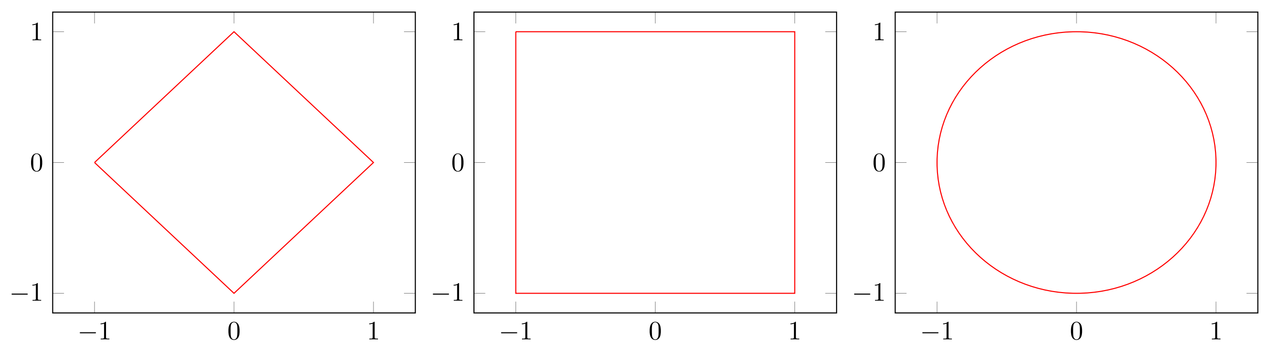
Example 2.11. Consider $\mathbb{R}^2$, with the following norms:

$$||\cdot||_1 : (x_1, x_2) \longmapsto |x_1| + |x_2|$$

$$||\cdot||_2 : (x_1, x_2) \longmapsto \sqrt{x_1^2 + x_2^2}$$

$$||\cdot||_\infty : (x_1, x_2) \longmapsto \max(|x_1|, |x_2|).$$

These norms have the following unit spheres:



(a) (b) (c)

Figure 1: The unit sphere of $||\cdot||_1$, $||\cdot||_2$ and $||\cdot||_\infty$.

As visible in the pictures, $\mathbb{R}^2$ is smooth with the $||\cdot||_2$ norm, but not smooth with the $||\cdot||_1$ and the $||\cdot||_\infty$ norm.

Definition 2.12. *Let X be a normed space with norm $||\cdot||$. $\{x \in X : ||x|| \leq 1\}$ is called strictly convex if for all $y, z \in \{x \in X : ||x|| \leq 1\}$ with $y \neq z$ and for all $t \in (0,1)$ we have that*

$$||yt + (1-t)z|| < 1.$$

X is called strictly convex if $\{x \in X : ||x|| \leq 1\}$ is strictly convex.

Proposition 2.13. *If X is a reflexive normed space, then the following are equivalent:*

1. *The norm is Gateaux differentiable*
2. *X is smooth*
3. *X^* is strictly convex*

Proof. See [3, corollary 5.4.18 and proposition 5.4.7]. □

Theorem 2.14. *For $x \in X$ with $x \neq 0$ we have that $\frac{x}{||x||}$ is a point of smoothness of B_X if and only if we have that $\#J_x = 1$.*
In particular it follows that if the conditions of proposition 2.13 hold, then for all $x \in X$ with $x \neq 0$ we have that $\#J_x = 1$.

Proof. Suppose $x \in X$ with $x \neq 0$ such that $\frac{x}{||x||}$ is a point of smoothness of B_X. By theorem 2.4 we have that $\#J_x \geq 1$. Let $f \in J_x$. Then we have:

$$f\left(\frac{x}{||x||}\right) = \frac{1}{||x||} f(x) = \frac{1}{||x||} ||x|| = 1 = \left|\left|\frac{x}{||x||}\right|\right|.$$

We also have $||f(x)|| = 1$. So we have
$$f \in J_{\frac{x}{||x||}}.$$
Then we obtain
$$f\left(\frac{x}{||x||}\right) = 1 = ||f|| = \sup\{f(y) : y \in B_X\}.$$
So f is a support functional for B_X at $\frac{x}{||x||}$. So $\left\{y \in X : f(y) = f\left(\frac{x}{||x||}\right)\right\}$ is a support hyperplane for B_X at $\frac{x}{||x||}$. Since X is smooth there is only one such support hyperplane. Suppose we also have $g \in J_x$ with $f \neq g$. Then with the same reasoning we have that $\left\{y \in X : g(y) = g\left(\frac{x}{||x||}\right)\right\}$ is a support hyperplane for B_X at $\frac{x}{||x||}$. Because $\frac{x}{||x||}$ is a point of smoothness of B_X, we get that:
$$\{y \in X : f(y) = f\left(\frac{x}{||x||}\right) = 1\} = \{y \in X : g(y) = g\left(\frac{x}{||x||}\right) = 1\}.$$
Let $y \in X$ with $f(y) \neq g(y)$. Suppose $f(y) = 0$. Then $g(y) \neq 0$. However, then we obtain:
$$g\left(\frac{1}{g(y)}y\right) = \frac{1}{g(y)}g(y) = 1.$$
Thus we get $\frac{1}{g(y)}y \in \{z \in X : g(z) = g\left(\frac{x}{||x||}\right) = 1\}$. However, we have $f\left(\frac{1}{g(y)}y\right) = 0$, so $\frac{1}{g(y)}y \notin \left\{z \in X : f(z) = f\left(\frac{x}{||x||}\right) = 1\right\}$. So that gives a contradiction. In the same way we get a contradiction if $f(y) \neq 0$ and $g(y) = 0$. Now suppose $f(y) \neq 0$ and $g(y) \neq 0$. Just like earlier, we have $\frac{1}{g(y)}y \in \left\{z \in X : g(z) = g\left(\frac{x}{||x||}\right) = 1\right\}$. However, we have:
$$f\left(\frac{1}{g(y)}y\right) = \frac{1}{g(y)}f(y) \neq 1$$
because $f(y) \neq g(y)$. So then $\frac{1}{g(y)}y \notin \left\{z \in X : f(z) = f\left(\frac{x}{||x||}\right) = 1\right\}$. So again, we get a contradiction. So we must have that $f = g$. So we obtain that $\#J_x = 1$.

Now let $x \in X$ with $x \neq 0$ and suppose that $\#J_x = 1$. Suppose that $\frac{x}{||x||}$ is not a point of smoothness of B_X. Then we can let f and g be two different support functionals for B_X at $\frac{x}{||x||}$ such that $\left\{y \in X : f(y) = f\left(\frac{x}{||x||}\right)\right\} \neq \left\{y \in X : g(y) = g\left(\frac{x}{||x||}\right)\right\}$. This is possible because $\frac{x}{||x||}$ is not a point of smoothness of B_X. Then we see that f is not a constant multiple of g. Then it follows that
$$f\left(\frac{x}{||x||}\right) = \sup\{f(x) : x \in B_X\} = ||f||$$
and
$$g\left(\frac{x}{||x||}\right) = \sup\{g(x) : x \in B_X\} = ||g||.$$
We define $\tilde{f} = \frac{f}{||f||}$ and $\tilde{g} = \frac{g}{||g||}$. Then we have:
$$\tilde{f}(x) = ||x||\tilde{f}\left(\frac{x}{||x||}\right) = ||x||\frac{1}{||f||}f\left(\frac{x}{||x||}\right) = ||x||\frac{1}{||f||}||f|| = ||x||$$
and
$$\tilde{g}(x) = ||x||\tilde{g}\left(\frac{x}{||x||}\right) = ||x||\frac{1}{||g||}g\left(\frac{x}{||x||}\right) = ||x||\frac{1}{||g||}||g|| = ||x||.$$
We also have:
$$||\tilde{f}|| = \left\|\frac{f}{||f||}\right\| = \frac{1}{||f||}||f|| = 1$$

and

$$||\tilde{g}|| = \left|\left|\frac{g}{||g||}\right|\right| = \frac{1}{||g||}||g|| = 1.$$

So we get that $\tilde{f}, \tilde{g} \in J_x$, but $\tilde{f} \neq \tilde{g}$ since f is not a constant multiple of g. So this gives a contradiction. So $\frac{x}{||x||}$ must be a point of smoothness of B_X.

So $\frac{x}{||x||}$ is a point of smoothness of B_X if and only if we have that $\#J_x = 1$.

Now if the conditions in proposition 2.13 hold, then for all $x \in X$ with $x \neq 0$ we have that $\frac{x}{||x||}$ is a point of smoothness of B_X and thus it follows that $\#J_x = 1$. □

Corollary 2.15. *For $x \in X$ with $x \neq 0$ we have that $\frac{x}{||x||}$ is a point of smoothness of B_X if and only if we have that $\#\tilde{J}_x = 1$.*

Corollary 2.16. *If the conditions of proposition 2.13 hold, then for $x \in X$ with $x \neq 0$ we have:*

$$J_x = \{G(x, \cdot)\}.$$

In that case we can define the following map, which we will also call the duality map:

$$J : X \to X^*, \ x \mapsto \begin{cases} 0, \ \textit{if } x = 0 \\ G(x, \cdot), \ \textit{if } x \neq 0 \end{cases} .$$

Corollary 2.17. *If the conditions of proposition 2.13 hold, then we get the map $\tilde{J}$, the alternate duality map:*

$$\tilde{J} : X \to X^*, \ x \mapsto \begin{cases} 0, \ \textit{if } x = 0 \\ ||x||G(x, \cdot), \ \textit{if } x \neq 0 \end{cases} .$$

3 Projections

In this chapter we will consider two types of projections, namely the contractive projection and the nearest point projection. The contractive projection has some properties that only hold if the projection is linear, and the nearest point projection does not always exist. We will show the properties of a linear nearest point projection, and also show under which conditions a nearest point projection is unique. Under the conditions that a nearest point projection is unique, we will see a connection between the range of a linear contractive projection and the nullspace of a linear nearest point projection. This connection will later on be useful when proving Calvert's theorem. To define a contractive projection, first we need to define a contractive map and a projection.

Definition 3.1. *Let (M, d) be a metric space. A contractive map is a map $f : M \to M$ such that for all $x, y \in M$ we have:*

$$d(f(x), f(y)) \leq d(x, y).$$

From now on we let E be a Banach space.

Definition 3.2. *A projection is a map $f : E \to E$ such that $f \circ f = f$.*

Combining the two definitions above we can now define a contractive projection.

Definition 3.3. *A contractive projection on a normed space E is a map $f : E \to E$ such that $f \circ f = f$, i.e. f is a projection, and such that for all $x, y \in E$ we have:*

$$||f(x) - f(y)|| \leq ||x - y||$$

i.e. f is a contractive map.

A contractive projection has, as stated above, some properties if it is linear. So we will now consider some of the properties a linear contractive projection has. These properties will be useful when stating and proving the connection between the range of a linear contractive projection and the nullspace of a linear nearest point projection.

Remark 3.4. *If $f : E \to E$ is linear, then an alternative (and equivalent) definition for f to be a contractive projection is that f is a projection and that for all $z \in E$ we must have:*

$$||f(z)|| \leq ||z||.$$

Lemma 3.5. *Suppose $f : E \to E$ is a linear contractive projection. Then f is either the zero map, or $||f|| = 1$.*

Proof. Suppose $f \neq 0$. Let $z \in E$ with $||z|| = 1$. Since f is a contractive map we have:

$$||f(z)|| = ||f(z) - f(0)|| \leq ||z - 0|| = ||z||.$$

So with [4, lemma 4.1] we now know that f is bounded and that $||f|| \leq 1$.
We have that f is not the zero map, so we let y in the image of f such that $||y|| = 1$. We write $y = f(x)$. Since f is a projection we get:

$$||f(y)|| = ||f(f(x))|| = ||f(x)|| = ||y||.$$

So we get that $||f|| \geq 1$. Combining this gives that $||f|| = 1$. □

A nearest point projection is a completely different type of projection, which does exactly as the name states: The nearest point projection of a point x on a subset K is the point in K that is closest to x with respect to the norm. It is clear that this point is not unique for every subset K, and thus a nearest point projection is not always unique. Before we define a nearest point projection we have to define a nearest point, and determine when a nearest point is unique.

Definition 3.6. *Let X be a normed space, and let $K \subset X$ and $x \in X$. If $z \in K$ satisfies*

$$||x - z|| = \inf\{||x - y|| : y \in K\}$$

then we call z a nearest point of x in K.

Lemma 3.7. *Let X be a normed space. Suppose X is strictly convex and reflexive and S is a non-empty, closed, convex subset of X. Then for all $x \in X$ there exists a unique $y \in S$ such that y is a nearest point of x.*

Proof. See [3, corollary 5.1.19]. □

Definition 3.8. *Let X be a normed space with norm $|| \cdot ||$. Let $K \subset X$. A nearest point projection on K is a map*

$$Q : X \to X$$

such that $Q \circ Q = Q$ and such that for all $x \in X$:

$$||x - Qx|| = \inf\{||x - y|| : y \in K\}.$$

Remark 3.9. *If X and $S \subset X$ satisfy the conditions of lemma 3.7, then there is a unique nearest point projection on S.*

Now we can finally state and prove the lemma that connects the contractive projection to the nearest point projection. As stated before, this lemma will be used in the proof of Calvert's theorem.

Lemma 3.10. *Let X be a normed space. A set $S \subset X$ is the range of a linear contractive projection if and only if S is the nullspace of a linear nearest point projection.*

Proof. Let X be a normed space with norm $|| \cdot ||$. Let $S \subset X$.

1. Suppose S is the range of a linear contractive projection. Let $(I - Q) : X \to X$ be that linear contractive projection. Then we have:
   (a) Q is a linear map, since I and $I - Q$ are and since $Q = -(I - Q) + I$.
   (b) Let $x \in X$. We have:

   $$(I - Q)((I - Q)(x)) = (I - Q)(I(x) - Q(x)) = (I - Q)(x - Q(x)) = (I - Q)(x) - (I - Q)(Q(x))$$
   $$= I(x) - Q(x) - I(Q(x)) + Q(Q(x)) = I(x) - Q(x) - Q(x) + Q(Q(x)) = x - 2Q(x) + Q(Q(x))$$

   and we have, because $I - Q$ is a projection:

   $$(I - Q)((I - Q)(x)) = (I - Q)(x) = I(x) - Q(x) = x - Q(x).$$

   So now we have

   $$x - 2Q(x) + Q(Q(x)) = x - Q(x)$$

   and thus

   $$-Q(x) + Q(Q(x)) = 0$$

   so

   $$Q(Q(x)) = Q(x).$$

   So Q is a projection.

(c) Define R as the range of Q. Let $x \in X$ and let $y \in R$. Then, because Q is a projection, we have that $y = Qy$. Then:

$$||x - Qx|| = ||(I - Q)(x)|| = ||(I - Q)(x) - y + Qy|| = ||(I - Q)(x - y)|| \leq ||x - y||$$

so

$$||x - Qx|| \leq \inf\{||x - y|| : y \in R\}.$$

Let $y = Qx$. Then we have that $y \in R$ and we have:

$$||x - y|| = ||x - Qx||$$

so

$$||x - Qx|| \geq \inf\{||x - y|| : y \in R\}.$$

So now we have

$$||x - Qx|| = \inf\{||x - y|| : y \in R\}.$$

So Q is a nearest point projection.

Thus Q is a linear nearest point projection. Let $x \in S$. Then we have that $(I - Q)(x) = x$, because S is the range of $I - Q$ and $I - Q$ is a projection and thus

$$(I - Q)(x) = Ix - Qx = x - Qx = x$$

and thus $Qx = 0$. Let $x \in X$ and suppose $Q(x) = 0$. Then we have $x - Q(x) = 0$ and thus $(I - Q)(x) = x$. So we have $x \in S$, since S is the range of $I - Q$. So S is the nullspace of Q, and thus S is the nullspace of a linear nearest point projection.

2. Suppose S is the nullspace of a linear nearest point projection. Let $Q : X \to R$ be that linear nearest point projection. Then we have:

(a) $I - Q$ is a linear map, since I and Q are.

(b) Let $x \in X$. Then we have

$$((I - Q) \circ (I - Q))(x) = (I - Q)((I - Q)(x)) = (I - Q)(I(x) - Q(x))$$
$$= (I - Q)(x - Q(x)) = (I - Q)(x) - (I - Q)(Q(x)) = I(x) - Q(x) - I(Q(x)) + Q(Q(x))$$
$$= x - Q(x) - Q(x) + Q(x) = x - Q(x) = (I - Q)(x)$$

so $I - Q$ is a projection.

(c) We know that $||x - Qx|| = \inf\{||x - y|| : y \in R\}$. R is the range of Q and thus a linear subspace, so $0 \in R$. So now we have:

$$||(I - Q)(x)|| = ||x - Qx|| \leq ||x - 0|| = ||x||.$$

So $I - Q$ is a contractive map.

Thus $I - Q$ is a linear contractive projection. Let $x \in S$. Then we have that $Qx = 0$ and thus

$$(I - Q)(x) = Ix - Qx = x - 0 = x$$

so $S \subset (I - Q)(X)$. Let $x \in (I - Q)(X)$. Let $y \in X$ such that $(I - Q)(y) = x$. Since $I - Q$ is a projection, we get

$$(I - Q)(x) = (I - Q)(I - Q)(y) = (I - Q)(y) = x.$$

Therefore we get $(I - Q)(x) = x$. Thus we have $x - Q(x) = x$. So we get $Q(x) = 0$. Therefore we have $(I - Q)(X) \subset S$. So $S = (I - Q)(X)$. So S is the range of a linear contractive projection.

□

4 Calvert's theorem

In the previous chapters we have seen several definitions such as the definition of the duality map in chapter 2 and the definitions of a contractive projection and a nearest point projection in chapter 3. We also saw some lemmas and propositions, such as lemma 3.10, which stated that a subset of X is the range of a linear contractive projection if and only if S is the nullspace of a linear nearest point projection. In this chapter several of these definitions and some of the lemma's come together in one big theorem, called Calvert's theorem. This theorem combines the range of a contractive projection with the range of the alternative duality map. Before we can actually formulate and prove this theorem, we need a couple more definitions, notations, and even lemmas.

Notation 4.1. *Let X be a normed space. If $\varphi \in X^*$ and $x \in X$ then $(\varphi, x) = \varphi(x)$.*

Definition 4.2. *Suppose X is a normed space and let $W \subset X^*$. Then we define $W^\perp$, the annihilator of W, to be:*

$$W^\perp = \{x \in X : (\varphi, x) = 0 \ \forall \varphi \in W\}.$$

If $B \subset X$, then we can also define the annihilator of B to be:

$$^\perp B = \{\varphi \in X^* : (\varphi, x) = 0 \ \forall x \in B\}.$$

Proposition 4.3. *Let X, Y be normed spaces and $P \in B(X, Y)$ a bounded linear map. Then there exists a unique bounded linear map*

$$P^* : Y^* \to X^*$$

which satisfies

$$P^*(f)(x) = f(Px)$$

for all $x \in X, f \in X^*$.

Proof. See [4, theorem 5.50]. □

Lemma 4.4. *Let X be a normed space. Suppose $S \subset X^*$ is a linear subspace. Then $S^\perp$ is a non-empty, closed, convex linear subspace of X.*

Proof. By [3, proposition 1.10.15] we have that $S^\perp$ is a closed linear subspace of X. Since it is a subspace, it follows immediately that it is also non-empty and convex. □

Lemma 4.5. *Let X be a normed space and $P : X \to X$ a linear bounded map. If P is a contractive projection, then so is P^*.*

Proof. By definition, we have that P^* is also a linear bounded map.
Let $f \in X^*$ and $x \in X$. Then we have, since P is a projection:

$$P^*(P^*(f))(x) = P^*(f(Px)) = f(P(P(x))) = f(P(x)) = P^*(f(x)).$$

So we have:

$$P^*(P^*(f)) = P^*(f).$$

So P^* is a projection.
Let $f \in X^*$. Suppose $||P^*|| = 1$. Let $x \in X$ with $||x|| = 1$. Then we have:

$$||P^*(f(x))|| = ||P(f(x))|| \leq ||f(x)||$$

since P is a linear contractive projection. So then we have:

$$||P^*(f)|| = \sup_{x \in X, ||x||=1} ||P^*(f(x))|| \leq \sup_{x \in X, ||x||=1} ||f(x)|| = ||f||.$$

By remark 3.4 we have that P^* is a contractive projection. □

Lemma 4.6. *Let X be a strictly convex and reflexive normed space with strictly convex dual space and let $\tilde{J}$ be duality map on X. Then $\tilde{J}$ is bijective and $\tilde{J}^{-1}$ is the alternative duality map on X^*.*

Proof. In this proof we identify X with X^{**}, which is possible since X is reflexive. An element x can therefore represent an element of X and an element of X^{**} at the same time. Since X^* is strictly convex, we have by corollary 2.17 that $\tilde{J}$ is a map from X to X^*.

Let $\phi \in X^{**}$. Since X is strictly convex, X^{**} is also strictly convex, so by corollary 2.17 we get that the alternative duality map on X^*, $\tilde{J}^*$, is a map from X^* to $X^{**} = X$. Let $x = \tilde{J}^*(\phi)$. Since $\tilde{J}^*$ is the alternative duality map on X^*, it follows that $||x|| = ||\phi||$. We also get:

$$\langle \phi, x\rangle = \langle x, \phi\rangle = ||\phi||^2 = ||x||^2$$

since $\tilde{J}^*$ is the alternative duality map on X^*. So now we immediately see that $\tilde{J}(x) = \phi$. So $\tilde{J} \circ \tilde{J}^* = id_{X^*}$. In the same way we find that $\tilde{J} \circ \tilde{J}^* = id_X$. So we see that $\tilde{J}$ is bijective and that $\tilde{J}^{-1} = \tilde{J}^*$. So $\tilde{J}^{-1}$ is the alternative duality map on X^*. □

Lemma 4.7. *Let X and $\tilde{J}$ be as in lemma 4.6. Then $\tilde{J}^{-1}$ is continuous from the strong to the weak topology.*

Proof. See [1, theorem 4.12]. □

Theorem 4.8. ***Calvert's theorem***
Let X be a strictly convex reflexive Banach space with strictly convex dual X^*. Consider the alternative duality map $\tilde{J} : X \to X^*$, thus $||\tilde{J}x|| = ||x||$ and $(\tilde{J}x, x) = ||x||^2$. Then a closed linear subspace $M \subset X$ is the range of a linear contractive projection if and only if $\tilde{J}(M)$ is a linear subspace of X^*.

Proof. Let M be a closed linear subspace.

1. Suppose $\tilde{J}(M)$ is a linear subspace. Let Q be the nearest point projection on $\tilde{J}(M)^\perp$, which exists because of lemma 3.7 and lemma 4.4.
   Let $x \in X$ and let $y \in \tilde{J}(M)^\perp$. Then we have $Qx \in \tilde{J}(M)^\perp$, since Q is a projection on $\tilde{J}(M)^\perp$.
   We will show that Q is linear. To do so, we will first prove two claims.

   <u>Claim 1</u> Let $x \in X$. Then for $z \in X$ we have $z = Qx$ if and only if $z \in \tilde{J}(M)^\perp$ and for all $y \in \tilde{J}(M)^\perp$ it holds that $(\tilde{J}(x - z), y) = 0$.
   Proof of Claim 1. Suppose $z = Qx$ and suppose $x = Qx$. Then we clearly have for all $y \in \tilde{J}(M)^\perp$ that $(\tilde{J}(x - Qx), y) = 0$. So suppose $x \neq Qx$. For $y \in \tilde{J}(M)^\perp$ we define:

$$f : \tilde{J}(M)^\perp \to \mathbb{R},\ y \mapsto \frac{1}{2}||x - y||^2.$$

   Since we know that the point with minimal distance from a point of $\tilde{J}(M)^\perp$ to x is Qx, we get that f is minimal for $y = Qx$. Now we let $y \in \tilde{J}(M)^\perp$ and define:

$$g : \mathbb{R} \to \mathbb{R},\ t \mapsto f(Qx - tz) = \frac{1}{2}||x - Qx + ty||^2.$$

   By lemma 4.4 we have that $\tilde{J}(M)^\perp$ is a linear subspace of X. So since for all $t \in \mathbb{R}$ we have $(Qx+ty) \in \tilde{J}(M)^\perp$ we get with the same reasoning as before that g is minimal in $t = 0$. Because $x \neq Qx$ we have that g is differentiable in $t = 0$. So we must have that $g'(0) = 0$. Now we will calculate $g'(0)$ in terms of the Gateaux derivative. Since the norm is differentiable at $x - Qx$, we can write for $h \in \mathbb{R}$ by definition of the Gateaux derivative:

$$||x - Qx + hy|| = ||x - Qx|| + hG(x - Qx, y) + \psi(h).$$

Where ψ is a function satisfying:

$$\lim_{h\to 0} \frac{\psi(h)}{h} = 0.$$

Then in particular:

$$\lim_{h\to 0} \psi(h) = 0.$$

Now we can write for $h \in \mathbb{R}$:

$$\frac{1}{2}||x - Qx + hy||^2 - \frac{1}{2}||x - Qx||^2$$

$$= \frac{1}{2}\left(h^2 G(x - Qx, y)^2 + 2hG(x - Qx, y)||x - Qx|| + 2||x - Qx||\psi(h) + 2\psi(h)hG(x - Qx, y) + \psi(h)^2\right).$$

Now we obtain:

$$g'(0) = \lim_{h\to 0} \frac{\frac{1}{2}||x - Qx - hy||^2 - \frac{1}{2}||x - Qx||^2}{h} =$$

$$\lim_{h\to 0} \frac{\frac{1}{2}h^2 G(x - Qx, y)^2 + hG(x - Qx, y)||x - Qx|| + ||x - Qx||\psi(h) + \psi(h)hG(x - Qx, y) + \frac{1}{2}\psi(h)^2}{h} =$$

$$\lim_{h\to 0} \left(\frac{1}{2}hG(x - Qx, y)^2 + G(x - Qx, y)||x - Qx|| + ||x - Qx||\frac{\psi(h)}{h} + \psi(h)G(x - Qx, y) + \frac{\psi(h)^2}{h}\right) =$$

$$G(x - Qx, y)||x - Qx|| = (\tilde{J}(x - Qx), y).$$

So we get:

$$0 = (\tilde{J}(x - Qx), y).$$

So if $z = Qx$, then $Qx \in \tilde{J}(M)^{\perp}$ and for all $y \in \tilde{J}(M)^{\perp}$ we have $(\tilde{J}(x - Qx), y)$.

Now suppose that $z \in \tilde{J}(M)^{\perp}$ and that for all $y \in \tilde{J}(M)^{\perp}$ we have $(\tilde{J}(x - z), y) = 0$.
Let $y \in \tilde{J}(M)^{\perp}$. We define the following function:

$$g : \mathbb{R} \to \mathbb{R}, t \mapsto \frac{1}{2}||x - z + t(z - y)||^2.$$

Similarly to the proof of claim 1, we have that the derivative of g in $t = 0$ is given by:

$$g'(0) = ||x - z||G(x - z, z - y) = \tilde{J}(x - z, z - y).$$

Since $z - y \in \tilde{J}(M)^{\perp}$, we have that $g'(0) = 0$. Since the span of the vector $z - y$ is a non-empty closed convex subset of X, lemma 3.7 tells us that we have a unique $w \in \text{span}(z - y)$ such that w is a nearest point of $x - z$ in $\text{span}(z - y)$. This also means that if we have $t \in \mathbb{R}$ such that $t(z - y) \neq w$ that then g is either strictly increasing or strictly decreasing at that point, so then $g'(t) \neq 0$. Since $g'(0) = 0$, we must have that g is minimal at $t = 0$. So in particular:

$$g(1) = ||x - z + z - y|| = ||x - y|| \geq ||x - z||.$$

So for all $y \in \tilde{J}(M)^{\perp}$ we have $||x - y|| \geq ||x - z||$. So then z is a nearest point of x in $\tilde{J}(M)^{\perp}$. By lemma 3.7 this means that $z = Qx$.

So $z = Qx$ if and only if $z \in \tilde{J}(M)^{\perp}$ and for all $y \in \tilde{J}(M)^{\perp}$ it holds that $(\tilde{J}(x - z), y) = 0$.

Now we will prove our second claim.
<u>Claim 2</u> Let $x \in X$. Then for $z \in X$ we have $z = Qx$ if and only if $z \in \tilde{J}(M)^{\perp}$ and $(x - z) \in M$
Proof of Claim 2. Suppose $z = Qx$. From claim 1 we know that for all $y \in \tilde{J}(M)^{\perp}$ we have $(\tilde{J}(x - Qx), y) = 0$. So we have:

$$\tilde{J}(x - Qx) \in {}^{\perp}(\tilde{J}(M)^{\perp}).$$

By lemma 4.7 we have that $\tilde{J}^{-1}$ is continuous from the strong to the weak topology. Since M is closed, this means that $\tilde{J}(M)$ is closed. By theorem 5.47 of (1), we have that ${}^{\perp}(\tilde{J}(M)^{\perp}) = \tilde{J}(M)$, because X is reflexive and $\tilde{J}(M)$ is closed. So we get:

$$\tilde{J}(x - Qx) \in \tilde{J}(M).$$

Since $\tilde{J}$ is injective, this means that $(x - Qx) \in M$.

Now suppose $z \in \tilde{J}(M)^{\perp}$ and $(x - z) \in M$. Then we also have $\tilde{J}(x - z) \in \tilde{J}(M)$. This means that for all $y \in \tilde{J}(M)^{\perp}$ we have $(\tilde{J}(x - z), y) = 0$. So then by claim 1 we have $z = Qx$.

So we find $z = Qx$ if and only if $z \in \tilde{J}(M)^{\perp}$ and $x - z \in M$.

Now let $x, y \in X$ and $\alpha \in \mathbb{R}$. Then from claim 2 we have $(x - Qx) \in M$ and $(y - Qy) \in M$. Since M is a linear subspace we also have that

$$((x - Qx) + (y - Qy)) \in M$$

and thus

$$((x + y) - (Qx + Qy)) \in M.$$

Since by lemma 4.4 $\tilde{J}(M)^{\perp}$ is a linear subspace, we have that

$$(Qx + Qy) \in \tilde{J}(M)^{\perp}$$

and thus by claim 2 we now have that

$$Qx + Qy = Q(x + y).$$

Because M is a linear subspace we have that

$$\alpha(x - Qx) \in M$$

and thus that

$$(\alpha x - \alpha Qx) \in M.$$

Since by lemma 4.4 $\tilde{J}(M)^{\perp}$ is a linear subspace, we have that

$$\alpha Qx \in \tilde{J}(M)^{\perp}$$

and thus by claim 2 we now have that

$$\alpha Qx = Q(\alpha x).$$

So Q is a linear map.

Let $x \in M$. Then we have that $Qx \in \tilde{J}(M)^{\perp}$ and thus we have, because we showed above that $(x - Qx) \in M$ and because M is a linear subspace, that $Qx \in M$. So now we have that $\tilde{J}(Qx) \in \tilde{J}(M)$. Thus $(\tilde{J}(Qx), Qx) = 0$. But we also know that $(\tilde{J}(Qx), Qx) = ||Qx||^2$, so now we have that $||Qx||^2 = 0$ and thus $Qx = 0$.
Let $x \in X$ and suppose $Qx = 0$. We have showed above that $(x - Qx) \in M$, and because $x - Qx = x - 0 = x$, we now get that $x \in M$.
So now we have that $Qx = 0$ if and only if $x \in M$.
So M is the nullspace of a linear nearest point projection, and thus by lemma 3.10 we get that M is the range of a linear contractive projection.

2. Suppose that M is the range of a linear contractive projection P, thus $M = R(P)$. By lemma 3.5 we have that either $||P|| = 1$ or $||P|| = 0$. Then by [4, lemma 5.52] we have that $||P^*|| = 1$ or $||P^*|| = 0$. If $P^* = 0$, we have that $M = \{0\}$, and then we have that $\tilde{J}(M) = \{0\}$ and that $R(P^*) = \{0\}$, and thus $\tilde{J}(M) = R(P^*)$. So then $\tilde{J}(M)$ is a linear subspace of X^*.

Suppose $||P^*|| = 1$. Let $m \in M$.
Suppose $m = 0$. Then $P^*\tilde{J}m = P^*0 = 0 = \tilde{J}m$.
Suppose $m \neq 0$. Then $\tilde{J}m \neq 0$ and thus

$$||P^*\tilde{J}m|| = \left\| ||\tilde{J}||P^*\left(\frac{\tilde{J}m}{||\tilde{J}m||}\right)\right\| = ||\tilde{J}m|| \cdot \left\|P^*\left(\frac{\tilde{J}m}{||\tilde{J}m||}\right)\right\| \leq ||\tilde{J}m|| \cdot 1 = ||\tilde{J}m||$$

since P^* is linear and $\left\|\frac{\tilde{J}m}{||\tilde{J}m||}\right\| = 1$. Since $||\tilde{J}m|| = ||m||$, we get

$$||P^*\tilde{J}m|| \leq ||\tilde{J}m|| = ||m||.$$

We also have

$$(P^*\tilde{J}m, m) = (\tilde{J}m, Pm) = (\tilde{J}m, m) = ||m||^2$$

since $M = R(P)$. So

$$\left(P^*\tilde{J}m, \frac{m}{||m||}\right) = \frac{1}{||m||}(P^*\tilde{J}m, m) = \frac{1}{||m||}||m||^2 = ||m||.$$

Because $\left\|\frac{m}{||m||}\right\| = 1$, we get

$$||m|| = \left(P^*\tilde{J}m, \frac{m}{||m||}\right) \geq ||P^*\tilde{J}m||.$$

So we must have

$$||P^*\tilde{J}m|| = ||\tilde{J}m|| = ||m||.$$

So $P^*\tilde{J}m$ satisfies $||P^*\tilde{J}m|| = ||m||$ and $(P^*\tilde{J}m, m) = ||m||^2$, so by theorem 2.14 we have that $P^*\tilde{J}m = \tilde{J}m$. So now we have that $\tilde{J}m \in R(P^*)$, and thus $\tilde{J}(M) \subset R(P^*)$.

Now we let $f \in R(P^*)$. By lemma 4.5 we have that P^* is again a linear contractive projection, by lemma 4.6 we have that $\tilde{J}^{-1}$ is the duality map from X^* to X, and we have that $P^{**} = P$, so now we get similarly to the previous part that $P\tilde{J}^{-1}f = \tilde{J}^{-1}f$. Applying $\tilde{J}$ this gives $R(P^*) \subset \tilde{J}(M)$.

So we get that $R(P^*) = \tilde{J}(M)$. So then $\tilde{J}(M)$ is a linear subspace, since $R(P^*)$ is.

□

5 Calvert's theorem without all of the conditions

Calvert's theorem shows that if we have a normed vector space that meets certain conditions, and of which its dual also meets certain conditions, then there is a connection between a subspace being the range of a linear contractive projection and the range of that subspace under the alternative duality map being a subspace of the dual space.

Now it seems interesting to consider spaces that do not satisfy all of the conditions that Calvert's theorem require, to see if the result of Calvert's theorem remains intact. Stated differently, are all of the conditions that Calvert's theorem requires actually necessary, or do there exist examples where the assertion holds and in which not all the requirements are met.

It is clear that the proof of Calvert's theorem is completely built upon the conditions the theorem lists, but maybe there is an alternative proof possible, that does not use all the conditions and is therefore a generalisation of Calvert's theorem.

To investigate the possibility of generalizing Calvert's theorem, we will consider several examples, to see if the result of Calvert's theorem remains intact, when ignoring one or more conditions of the theorem. To start with, we will consider $\mathbb{R}^2$ with the 1-norm and with the ∞-norm.

Example 5.1. Consider $\mathbb{R}^2$ with the 1-norm. With lemma 2.8 we have that this vector space is reflexive. Example 2.11 shows us that the 1-norm is not strictly convex. With proposition 2.13 we have, because the 1-norm is not smooth, that the dual space is also not strictly convex. Now we consider the standard basis e_1, e_2 on $\mathbb{R}^2$ and its dual basis f_1, f_2. With [3, theorem C.13] we know that, in terms of this dual basis, the dual norm is the ∞-norm on the dual space. By [4, corollary 2.20] we have that every subspace of $\mathbb{R}^2$ is closed in the 1-norm.

1. Consider $M = \{(0, b) : b \in \mathbb{R}\}$. It is clear that M is a subspace, and thus that M is closed.
   Now consider
   $$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
   $$(x_1, x_2) \longmapsto (0, x_2).$$
   This is the orthogonal projection of $\mathbb{R}^2$ on M. It is a well-known result that this map is a linear projection and that M is the range of this projection. Now let $(x, y) \in \mathbb{R}^2$. Then we have that
   $$||f((x, y))|| = ||(0, y)|| = |y|$$
   and we have that
   $$||(x, y)|| = |x| + |y| \geq 0 + |y| = |y|$$
   and thus
   $$||f((x, y))|| \leq ||(x, y)||.$$
   So with remark 3.4 we have that f is a contractive projection.
   So M is the range of a linear contractive projection.

   Now we will determine $\tilde{J}(M)$. As always, we have $\tilde{J}((0, 0)) = \{(0, 0)\}$.
   Let $b \in \mathbb{R}_{>0}$. Let $a \in \mathbb{R}$ with $-b \leq a \leq b$. Then we have, since the dual space has the ∞-norm:
   $$||(a, b)||_\infty = b = 0 + |b| = ||(0, b)||_1$$
   and
   $$\langle (a, b), (0, b) \rangle = b^2 = (0 + |b|)^2 = ||(0, b)||_1^2.$$
   Therefore we see that $(a, b) \in \tilde{J}((0, b))$. So we obtain:
   $$\tilde{J}((0, b)) \supset \left\{(a, b) \in (\mathbb{R}^2)^* : -b \leq a \leq b\right\}.$$

Now let $(u,v) \in \tilde{J}((0,b))$. Then we have:

$$\|(u,v)\|_\infty = b$$

and

$$b^2 = \langle (u,v),(0,b)\rangle = vb.$$

So we obtain $b^2 - bv = 0$, i.e. $b = 0$ or $b = v$. Since $b > 0$, we must have that $b = v$. Since we have:

$$||(u,v)||_\infty = b$$

we must have that $|u| \leq b$, so we get $-b \leq u \leq b$. So we get:

$$\tilde{J}((0,b)) \subset \left\{(a,b) \in (\mathbb{R}^2)^* : -b \leq a \leq b\right\}.$$

Therefore we get:

$$\tilde{J}((0,b)) = \left\{(a,b) \in (\mathbb{R}^2)^* : -b \leq a \leq b\right\}.$$

In the same way we find for $b < 0$:

$$\tilde{J}((0,b)) = \left\{(a,b) \in (\mathbb{R}^2)^* : b \leq a \leq -b\right\}.$$

Now we can determine $\tilde{J}(M)$ as follows:

$$\tilde{J}(M) = \cup_{x\in M}\tilde{J}(x)$$

$$= \left\{(a,b) \in (\mathbb{R}^2)^* : b > 0, -b \leq a \leq b\right\} \cup \left\{(a,b) \in (\mathbb{R}^2)^* : b < 0, b \leq a \leq -b\right\} \cup \{(0,0)\}.$$

So we see that $(1,1),(1,-1) \in \tilde{J}(M)$. However, we see that $(1,1)+(1,-1) = (2,0) \notin \tilde{J}(M)$. So $\tilde{J}(M)$ is not a linear subspace of $(\mathbb{R}^2)^*$.
So even though M is the range of a linear contractive projection, $\tilde{J}(M)$ is not a linear subspace of $(\mathbb{R}^2)^*$.

2. Consider $M = \{(a,0) : a \in \mathbb{R}\}$. It is clear that M is a subspace, and thus that M is closed.
Now consider

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x_1,x_2) \longmapsto (x_1,0).$$

This is the orthogonal projection of $\mathbb{R}^2$ on M and just like before we have have that this is a linear contractive projection with range M.
So M is the range of a linear contractive projection.

In the same way as before, we get:

$$\tilde{J}(M) = \cup_{x\in M}\tilde{J}(x)$$

$$= \left(\left\{(a,b) \in (\mathbb{R}^2)^* : a > 0, -a \leq b \leq a\right\}\right) \cup \left(\left\{(a,b) \in (\mathbb{R}^2)^* : a < 0, a \leq b \leq -a\right\}\right) \cup \{(0,0)\}.$$

So we see that $(1,1),(-1,1) \in \tilde{J}(M)$. However, we see that $(1,1)+(-1,1) = (0,2) \notin \tilde{J}(M)$. So $\tilde{J}(M)$ is not a linear subspace of $(\mathbb{R}^2)^*$.
So again, even though M is the range of a linear contractive projection, $\tilde{J}(M)$ is not a linear subspace of $(\mathbb{R}^2)^*$.

3. Let $a \in \mathbb{R}_{>0}$ and consider $M = \{(x,ax) : x \in \mathbb{R}\}$. It is clear that M is a subspace, and thus that M is closed.
Now consider

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x_1,x_2) \longmapsto \left(\frac{x_1+x_2}{a+1}, \frac{ax_1+ax_2}{a+1}\right).$$

Let $(x, y) \in \mathbb{R}$. Then we have:

$$(f \circ f)((x,y)) = f(f((x,y))) = f\left(\left(\frac{x+y}{a+1}, \frac{ax+ay}{a+1}\right)\right) = \left(\frac{\frac{x+y}{a+1} + \frac{ax+ay}{a+1}}{a+1}, \frac{a\frac{x+y}{a+1} + a\frac{ax+ay}{a+1}}{a+1}\right)$$

$$= \left(\frac{\frac{x+y+ax+ay}{a+1}}{a+1}, \frac{a\frac{x+y+ax+ay}{a+1}}{a+1}\right) = \left(\frac{\frac{(1+a)(x+y)}{a+1}}{a+1}, \frac{a\frac{(1+a)(x+y)}{a+1}}{a+1}\right) = \left(\frac{(x+y)}{a+1}, \frac{a(x+y)}{a+1}\right) = f((x,y))$$

and thus f is a projection.

Let $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ and let $\lambda \in \mathbb{R}$.

- Now we have:

$$f((x_1,x_2)) + f((y_1,y_2)) = \left(\frac{x_1+x_2}{a+1}, \frac{ax_1+ax_2}{a+1}\right) + \left(\frac{y_1+y_2}{a+1}, \frac{ay_1+ay_2}{a+1}\right)$$

$$= \left(\frac{x_1+x_2}{a+1} + \frac{y_1+y_2}{a+1}, \frac{ax_1+ax_2}{a+1} + \frac{ay_1+ay_2}{a+1}\right) = \left(\frac{x_1+x_2+y_1+y_2}{a+1}, \frac{ax_1+ax_2+ay_1+ay_2}{a+1}\right)$$

$$= \left(\frac{(x_1+y_1)+(x_2+y_2)}{a+1}, \frac{a(x_1+y_1)+a(x_2+y_2)}{a+1}\right)$$

  and

$$f((x_1,x_2)+(y_1,y_2)) = f((x_1+y_1, x_2+y_2)) = \left(\frac{(x_1+y_1)+(x_2+y_2)}{a+1}, \frac{a(x_1+y_1)+a(x_2+y_2)}{a+1}\right)$$

  and thus

$$f((x_1,x_2)) + f((y_1,y_2)) = f((x_1,x_2)+(y_1,y_2)).$$

- We also have:

$$\lambda f((x_1,x_2)) = \lambda\left(\frac{x_1+x_2}{a+1}, \frac{ax_1+ax_2}{a+1}\right) = \left(\lambda\frac{x_1+x_2}{a+1}, \lambda\frac{ax_1+ax_2}{a+1}\right)$$

  and

$$f(\lambda(x_1,x_2)) = f((\lambda x_1, \lambda x_2)) = \left(\frac{\lambda x_1+\lambda x_2}{a+1}, \frac{a\lambda x_1+a\lambda x_2}{a+1}\right)$$

$$= \left(\frac{\lambda(x_1+x_2)}{a+1}, \frac{\lambda(ax_1+ax_2)}{a+1}\right) = \left(\lambda\frac{x_1+x_2}{a+1}, \lambda\frac{ax_1+ax_2}{a+1}\right)$$

  and thus

$$\lambda f((x_1,x_2)) = f(\lambda(x_1,x_2)).$$

So f is linear.

Now let $(x_1, x_2) \in \mathbb{R}$. Then we have:

$$||f((x_1,x_2))||_1 = \left\|\left(\frac{x_1+x_2}{a+1}, \frac{ax_1+ax_2}{a+1}\right)\right\|_1 = \left|\frac{x_1+x_2}{a+1}\right| + \left|\frac{ax_1+ax_2}{a+1}\right| = \left|\frac{1}{a+1}\right| \cdot |x_1+x_2| + \left|\frac{a}{a+1}\right| \cdot |x_1+x_2|$$

$$= \frac{1}{a+1}|x_1+x_2| + \frac{a}{a+1}|x_1+x_2| = \frac{a+1}{a+1}|x_1+x_2| = |x_1+x_2| \leq |x_1| + |x_2| = \|(x_1,x_2)\|_1$$

because $a > 0$, so by remark 3.4 we have that f is a linear contractive projection.

Let $x \in \mathbb{R}$. Then $(x, ax) \in M$. Let $y \in \mathbb{R}$. Then we have:

$$f((y, (a+1)x - y)) = \left(\frac{y + ((a+1)x - y)}{a+1}, \frac{ay + a((a+1)x - y)}{a+1}\right)$$

$$= \left(\frac{y+(a+1)x-y}{a+1}, \frac{ay+a(a+1)x-ay}{a+1}\right) = \left(\frac{(a+1)x}{a+1}, \frac{a(a+1)x}{a+1}\right) = (x, ax).$$

So $(x, ax) \in \text{Im}(f)$ and thus $M \subset \text{Im}(f)$.
Let $x_1, x_2 \in \mathbb{R}$, then

$$\left(\frac{x_1+x_2}{a+1}, \frac{ax_1+ax_2}{a+1}\right) \in \text{Im}(f).$$

Then:

$$\left(\frac{x_1+x_2}{a+1}, \frac{ax_1+ax_2}{a+1}\right) = \left(\frac{x_1+x_2}{a+1}, a\frac{x_1+x_2}{a+1}\right) = \frac{x_1+x_2}{a+1}(1,a) \in M$$

because M is a linear subspace. So

$$\left(\frac{x_1+x_2}{a+1}, \frac{ax_1+ax_2}{a+1}\right) \in M$$

and thus $\text{Im}(f) \subset M$ and thus $\text{Im}(f) = M$.
So M is the range of a linear contractive projection.

As always we have $\tilde{J}((0,0)) = \{(0,0)\}$.
Let $x \in \mathbb{R}$ with $x \neq 0$. Then we have, because $a > 0$:

$$||((1+a)x, (1+a)x||_\infty = (1+a)|x| = |x| + a|x| = |x| + |ax| = ||(x, ax)||_1.$$

We also have:

$$\langle((1+a)x, (1+a)x), (x, ax)\rangle = (1+a)x^2 + (1+a)x^2a = (1+2a+a^2)x^2 = ((1+a)|x|)^2 = ||(x, ax)||_1^2.$$

So we have $((1+a)x, (1+a)x) \in \tilde{J}((x, ax))$.
Let $(u, v) \in \tilde{J}((x, ax))$. Then we have, because $a > 0$:

$$\max\{|u|, |v|\} = ||(u, v)||_\infty = ||(x, ax)||_1 = |x| + |ax| = (1+a)|x|.$$

So $|u| \leq (1+a)|x|$ and $|v| \leq (1+a)|x|$ and thus $u \leq (1+a)x$ and $v \leq (1+a)x$ for $x > 0$ and $u \geq (1+a)x$ and $v \geq (1+a)x$ for $x < 0$. We also have:

$$ux + vax = \langle(u, v), (x, ax)\rangle = ||(x, ax)||_1^2 = (1+2a+a^2)x^2$$

Since $x \neq 0$, this gives $u + va = (1+2a+a^2)x$. Now we get if $x > 0$:

$$(1+2a+a^2)x = u + va \leq (1+a)x + a(1+a)x = (1+2a+a^2)x$$

or if $x < 0$:

$$(1+2a+a^2)x = u + va \geq (1+a)x + a(1+a)x = (1+2a+a^2)x.$$

So in both cases we must have equality and thus we obtain that $u = (1+a)x, v = (1+a)x$. Thus $(u, v) = ((1+a)x, (1+a)x)$. Thus $\tilde{J}((x, ax)) = \{((1+a)x, (1+a)x)\}$. Therefore we obtain:

$$\tilde{J}(M) = \cup_{x \in M} \tilde{J}(x)$$

$$= (\{((1+a)x, (1+a)x), x \in \mathbb{R}, x \neq 0\}) \cup \{(0,0)\} = \{(x, x), x \in \mathbb{R}\}$$

This is clearly a linear subspace of $(\mathbb{R}^2)^*$. So in this case we have that both M is the range of a linear contractive projection and that $\tilde{J}(M)$ is a linear subspace.

4. Let $a \in \mathbb{R}_{<0}$ and consider $M = \{(x, ax) : x \in \mathbb{R}\}$. It is clear that M is a subspace, and thus that M is closed.
Now consider

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x_1,x_2) \longmapsto \left(\frac{-x_1+x_2}{a-1}, \frac{-ax_1+ax_2}{a-1}\right).$$

Let $(x,y)\in\mathbb{R}$. Then we have:

$$(f\circ f)((x,y)) = f(f((x,y))) = f\left(\left(\frac{-x+y}{a-1}, \frac{-ax+ay}{a-1}\right)\right)$$

$$= \left(\frac{-\frac{-x+y}{a-1}+\frac{-ax+ay}{a-1}}{a-1}, \frac{-a\frac{-x+y}{a-1}+a\frac{-ax+ay}{a-1}}{a-1}\right) = \left(\frac{\frac{-(-x+y)+(-ax+ay)}{a-1}}{a-1}, \frac{a\frac{-(-x+y)+(-ax+ay)}{a-1}}{a-1}\right)$$

$$= \left(\frac{\frac{x-y-ax+ay}{a-1}}{a-1}, \frac{a\frac{x-y-ax+ay}{a-1}}{a-1}\right) = \left(\frac{\frac{(1-a)x+(a-1)y}{a-1}}{a-1}, \frac{a\frac{(1-a)x+(a-1)y}{a-1}}{a-1}\right)$$

$$= \left(\frac{\frac{(a-1)(-x+y)}{a-1}}{a-1}, \frac{a\frac{(a-1)(-x+y)}{a-1}}{a-1}\right) = \left(\frac{(-x+y)}{a-1}, \frac{a(-x+y)}{a-1}\right) = f((x,y))$$

and thus f is a projection.

Let $(x_1,x_2),(y_1,y_2)\in\mathbb{R}^2$ and let $\lambda\in\mathbb{R}$.

- Now we have:

$$f((x_1,x_2))+f((y_1,y_2)) = \left(\frac{-x_1+x_2}{a-1}, \frac{-ax_1+ax_2}{a-1}\right)+\left(\frac{-y_1+y_2}{a-1}, \frac{-ay_1+ay_2}{a-1}\right)$$

$$= \left(\frac{-x_1+x_2}{a-1}+\frac{-y_1+y_2}{a-1}, \frac{-ax_1+ax_2}{a-1}+\frac{-ay_1+ay_2}{a-1}\right)$$

$$= \left(\frac{-x_1+x_2-y_1+y_2}{a-1}, \frac{-ax_1+ax_2-ay_1+ay_2}{a-1}\right)$$

$$= \left(\frac{-(x_1+y_1)+(x_2+y_2)}{a-1}, \frac{-a(x_1+y_1)+a(x_2+y_2)}{a-1}\right)$$

and

$$f((x_1,x_2)+(y_1,y_2)) = f((x_1+y_1,x_2+y_2)) = \left(\frac{-(x_1+y_1)+(x_2+y_2)}{a-1}, \frac{-a(x_1+y_1)+a(x_2+y_2)}{a-1}\right)$$

and thus

$$f((x_1,x_2))+f((y_1,y_2)) = f((x_1,x_2)+(y_1,y_2)).$$

- We also have:

$$\lambda f((x_1,x_2)) = \lambda\left(\frac{-x_1+x_2}{a-1}, \frac{-ax_1+ax_2}{a-1}\right) = \left(\lambda\frac{-x_1+x_2}{a-1}, \lambda\frac{-ax_1+ax_2}{a-1}\right)$$

and

$$f(\lambda(x_1,x_2)) = f((\lambda x_1,\lambda x_2)) = \left(\frac{-\lambda x_1+\lambda x_2}{a-1}, \frac{-a\lambda x_1+a\lambda x_2}{a-1}\right)$$

$$= \left(\frac{\lambda(-x_1+x_2)}{a-1}, \frac{\lambda(-ax_1+ax_2)}{a-1}\right) = \left(\lambda\frac{-x_1+x_2}{a-1}, \lambda\frac{-ax_1+ax_2}{a-1}\right)$$

and thus

$$\lambda f((x_1,x_2)) = f(\lambda(x_1,x_2)).$$

So f is linear.

Now let $(x_1, x_2) \in \mathbb{R}$. Then we have:

$$||f((x_1, x_2))||_1 = \left\| \left(\frac{-x_1 + x_2}{a - 1}, \frac{-ax_1 + ax_2}{a - 1} \right) \right\|_1$$

$$= \left| \frac{-x_1 + x_2}{a - 1} \right| + \left| \frac{-ax_1 + ax_2}{a - 1} \right| = \left| \frac{1}{a - 1} \right| \cdot | - x_1 + x_2| + \left| \frac{a}{a - 1} \right| \cdot | - x_1 + x_2|$$

$$= \frac{1}{-a + 1} | - x_1 + x_2| + \frac{-a}{-a + 1} | - x_1 + x_2| = \frac{1 - a}{-a + 1} | - x_1 + x_2|$$

$$= | - x_1 + x_2| \leq | - x_1| + |x_2| = |x_1| + |x_2| = ||(x_1, x_2)||_1$$

because $a < 0$, so by remark 3.4 we have that f is a linear contractive projection.

Let $x \in \mathbb{R}$. Then $(x, ax) \in M$. Let $y \in \mathbb{R}$. Then we have:

$$f((y, (a - 1)x + y)) = \left(\frac{-y + ((a - 1)x + y)}{a - 1}, \frac{-ay + a((a - 1)x + y)}{a - 1} \right)$$

$$= \left(\frac{-y + (a - 1)x + y}{a - 1}, \frac{-ay + a(a - 1)x + ay}{a - 1} \right) = \left(\frac{(a - 1)x}{a - 1}, \frac{a(a - 1)x}{a - 1} \right) = (x, ax).$$

So $(x, ax) \in \text{Im}(f)$ and thus $M \subset \text{Im}(f)$.
Let $x_1, x_2 \in \mathbb{R}$, then

$$\left(\frac{-x_1 + x_2}{a - 1}, \frac{-ax_1 + ax_2}{a - 1} \right) \in \text{Im}(f).$$

Then:

$$\left(\frac{-x_1 + x_2}{a - 1}, \frac{-ax_1 + ax_2}{a - 1} \right) = \left(\frac{-x_1 + x_2}{a - 1}, a\frac{-x_1 + x_2}{a - 1} \right) = \frac{-x_1 + x_2}{a - 1}(1, a) \in M$$

because M is a linear subspace. So

$$\left(\frac{-x_1 + x_2}{a - 1}, \frac{-ax_1 + ax_2}{a - 1} \right) \in M$$

and thus $\text{Im}(f) \subset M$ and thus $\text{Im}(f) = M$.
So M is the range of a linear contractive projection.

As always we have $\tilde{J}((0, 0)) = \{(0, 0)\}$.
Let $x \in \mathbb{R}$ with $x \neq 0$. Then we have, because $a < 0$:

$$||((1 - a)x, (-1 + a)x||_\infty = |(-1 + a)||x| = (1 - a)|x| = |x| - a|x| = |x| + |ax| = ||(x, ax)||_1.$$

We also have:

$$\langle((1-a)x, (-1+a)x), (x, ax)\rangle = (1-a)x^2 + (-1+a)x^2a = (1-2a+a^2)x^2 = ((1-a)|x|)^2 = ||(x, ax)||_1^2.$$

So we have $((1 - a)x, (-1 + a)x) \in \tilde{J}((x, ax))$.
Let $(u, v) \in \tilde{J}((x, ax))$. Then we have, because $a < 0$:

$$\max\{|u|, |v|\} = ||(u, v)||_\infty = ||(x, ax)||_1 = |x| + |ax| = (1 - a)|x|.$$

So $|u| \leq (1 - a)|x|$ and $|v| \leq (1 - a)|x|$ and thus $u \leq (1 - a)x$ and $v \geq (1 - a)(-x) = (a - 1)x$ for $x > 0$ and $u \geq (1 - a)x$ and $v \leq (1 - a)(-x) = (a - 1)x$ for $x < 0$. We also have:

$$ux + vax = \langle(u, v), (x, ax)\rangle = ||(x, ax)||_1^2 = (1 - 2a + a^2)x^2.$$

Since $x \neq 0$, this gives $u + va = (1 - 2a + a^2)x$. Now we get if $x > 0$, because $a < 0$:

$$(1 - 2a + a^2)x = u + va \leq (1 - a)x + a(a - 1)x = (1 - 2a + a^2)x$$

or if $x < 0$:

$$(1 + 2a + a^2)x = u + va \geq (1 - a)x + a(a - 1)x = (1 - 2a + a^2)x.$$

So in both cases we must have equality and thus we obtain that $u = (1 - a)x, v = (a - 1)x$. Thus $(u, v) = ((1 - a)x, (a - 1)x)$. Thus $\tilde{J}((x, ax)) = \{((1 - a)x, (a - 1)x)\}$. Therefore we obtain:

$$\tilde{J}(M) = \cup_{x \in M} \tilde{J}(x)$$

$$= (\{((1 - a)x, (a - 1)x), x \in \mathbb{R}, x \neq 0\}) \cup \{(0, 0)\} = \{(x, -x), x \in \mathbb{R}\}.$$

This is clearly a linear subspace of $(\mathbb{R}^2)^*$. So in this case we have that both M is the range of a linear contractive projection and that $\tilde{J}(M)$ is a linear subspace.

5. Consider $M = \mathbb{R}^2$. Then clearly M is a closed linear subspace. Let f be the identity map on $\mathbb{R}^2$. Then $f \circ f = f$ and f is linear, so f is a linear projection. Clearly M is the range of f. Let $x \in \mathbb{R}^2$. Then we have $||f(x)||_1 = ||x||_1$, so f is a contractive map. So f is a linear contractive projection. So M is the range of a linear contractive projection. Let M_1 be the x-axis and M_2 be the y-axis. Then we have, using what we already found:

$$\tilde{J}(M) \supset \tilde{J}(M_1) \cup \tilde{J}(M_2)$$

$$= \left(\left\{(a, b) \in (\mathbb{R}^2)^* : a > 0, -a \leq b \leq a\right\}\right) \cup \left(\left\{(a, b) \in (\mathbb{R}^2)^* : a < 0, a \leq b \leq -a\right\}\right)$$

$$\cup \left(\left\{(a, b) \in (\mathbb{R}^2)^* : b > 0, -b \leq a \leq b\right\}\right) \cup \left(\left\{(a, b) \in (\mathbb{R}^2)^* : b < 0, b \leq a \leq -b\right\}\right) \cup \{(0, 0)\} = \mathbb{R}^2.$$

So we get that $\tilde{J}(M) = (\mathbb{R}^2)^*$, which is a linear subspace.
So in this case we have that M is the range of a linear contractive projection and that $\tilde{J}(M)$ is a linear subspace.

6. Consider $M = \{(0, 0)\}$. It is clear that M is a closed linear subspace.
Let f be the zero map. Then f is linear and for $x \in \mathbb{R}^2$ we have $f(f(x)) = (0, 0) = f(x)$. So f is a projection. Clearly, M is the range of this projection. For $x \in \mathbb{R}^2$ we have $||f(x)||_1 = ||(0, 0)||_1 = 0 \leq ||x||_1$. So f is a linear contractive projection.
So M is the range of a linear contractive projection.
We also have:

$$\tilde{J}(M) = \tilde{J}((0, 0)) = \{(0, 0)\}.$$

So $\tilde{J}(M)$ is a linear subspace.
So in this case we have that M is the range of a linear contractive projection and that $\tilde{J}(M)$ is a linear subspace.

Example 5.2. Consider $\mathbb{R}^2$ with the ∞-norm. With lemma 2.8 we have that this normed vector space is reflexive. Example 2.11 shows us that the ∞-norm is not strictly convex. With proposition 2.13 we have, because the ∞-norm is not smooth, that the dual space is also not strictly convex. Now we consider the standard basis e_1, e_2 on $\mathbb{R}^2$ and its dual basis f_1, f_2. With [3, theorem C.13] we know that, in terms of this dual basis, the dual norm is the 1-norm on the dual space. Again by [4, corollary 2.20] we have that every subspace of $\mathbb{R}^2$ is closed in the ∞-norm.

1. Consider $M = \{(x, x) : x \in \mathbb{R}\}$. It is clear that M is a subspace, and thus that M is closed.
Now consider

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x_1, x_2) \longmapsto (\frac{1}{2}(x_1 + x_2), \frac{1}{2}(x_1 + x_2)).$$

This is the orthogonal projection of $\mathbb{R}^2$ on M. It is a well-known result that this map is a linear projection and that M is the range of this projection. Now let $(x, y) \in \mathbb{R}^2$. Then we have that

$$||f((x,y))|| = ||(\frac{1}{2}(x+y), \frac{1}{2}(x+y))|| = \max\{|\frac{1}{2}(x+y)|, |\frac{1}{2}(x+y)|\} = \frac{1}{2}|x+y|$$

and we have that

$$||(x,y)|| = \max\{|x|,|y|\} \geq \frac{1}{2}(|x|+|y|) \geq \frac{1}{2}|x+y|$$

and thus

$$||f((x,y))|| \leq ||(x,y)||.$$

So with remark 3.4 we have that f is a contractive projection.
So M is the range of a linear contractive projection.

Now we will determine $\tilde{J}(M)$. As always, we have $\tilde{J}((0,0)) = \{(0,0)\}$.
Let $x \in \mathbb{R}_{>0}$ and let $z \in \mathbb{R}$ with $0 \leq z \leq x$. Then we have:

$$||(z, x-z)||_1 = |z| + |x-z| = z + (x-z) = x$$

and

$$||(x,x)||_\infty = \max\{|x|,|x|\} = |x| = x.$$

We also have:

$$\langle (z, x-z), (x,x)\rangle = zx + (x-z)x = zx + x^2 - zx = x^2$$

and

$$||(x,x)||_\infty^2 = x^2.$$

So $(z, x-z) \in \tilde{J}((x,x))$ and thus $\{(z, x-z) : 0 \leq z \leq x\} \subset \tilde{J}((x,x))$.
Now let $(u,v) \in \tilde{J}((x,x))$. Then we have that:

$$ux + vx = \langle (u,v),(x,x)\rangle = x^2$$

and thus that $u + v = x$ because $x \neq 0$, and thus $v = x - u$. We also have that

$$|u| + |v| = ||(u,v)||_1 = x.$$

So

$$|u| = x - |v| = x - |x-u|.$$

So we get $-u \leq x - |x-u| \leq u$. From $-u \leq x - |x-u|$ we get $|x-u| \leq x+u$ and thus $x - u \leq x + u$ or $u \geq 0$. We also have

$$|v| = x - |u| = x - |x-v|$$

and in the same way we find $v \geq 0$, or $u \leq x$. So $(u,v) \in \{(z, x-z) : 0 \leq z \leq x\}$.
So $\tilde{J}((x,x)) \subset \{(z,x-z) : 0 \leq z \leq x\}$ and thus $\tilde{J}((x,x)) = \{(z, x-z) : 0 \leq z \leq x\}$.

In the same way we find for $x \in \mathbb{R}_{<0}$ that $\tilde{J}((x,x)) = \{(z, x-z) : x \leq z \leq 0\}$. So now we can determine $\tilde{J}(M)$:

$$\tilde{J}(M) = \cup_{x\in M}\tilde{J}(x)$$

$$= (\{(z, x-z) : x < 0, x \leq z \leq 0\}) \cup (\{(z, x-z) : x > 0, 0 \leq z \leq x\}) \cup \{(0,0)\}.$$

We see that $(0,1), (-1,0) \in \tilde{J}(M)$, but $(0,1) + (-1,0) = (-1,1) \notin \tilde{J}(M)$. So $\tilde{J}(M)$ is not a linear subspace of $(\mathbb{R}^2)^*$.

2. Consider $M = \{(x, -x) : x \in \mathbb{R}\}$. It is clear that M is a subspace, and thus that M is closed.
Now consider

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x_1, x_2) \longmapsto (\frac{1}{2}(x_1 - x_2), \frac{1}{2}(-x_1 + x_2)).$$

This is the orthogonal projection of $\mathbb{R}^2$ on M and just like before we have have that this is a linear contractive projection with range M.
So M is the range of a linear contractive projection.

In the same way as before, we get:

$$\tilde{J}(M) = \cup_{x \in M} \tilde{J}(x)$$
$$= (\{(z, z - x) : x < 0, x \leq z \leq 0\}) \cup (\{(z, z - x) : x > 0, 0 \leq z \leq x\}) \cup \{(0, 0)\}.$$

So we see that $(1, 0), (0, 1) \in \tilde{J}(M)$. However, we see that $(1, 0) + (0, 1) = (1, 1) \notin \tilde{J}(M)$. So $\tilde{J}(M)$ is not a linear subspace of $(\mathbb{R}^2)^*$.
So again, even though M is the range of a linear contractive projection, $\tilde{J}(M)$ is not a linear subspace of $(\mathbb{R}^2)^*$.

3. Let $a \in \mathbb{R}$ with $a < -1$ or $a > 1$. Consider $M = \{(x, ax) : x \in \mathbb{R}\}$. It is clear that M is a subspace, and thus that M is closed.
Now consider

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x_1, x_2) \longmapsto \left(\frac{1}{a}x_2, x_2\right).$$

Let $(x, y) \in \mathbb{R}$. Then we have:

$$(f \circ f)((x, y)) = f(f((x, y))) = f\left(\left(\frac{1}{a}y, y\right)\right) = \left(\frac{1}{a}y, y\right) = f((x, y))$$

and thus f is a projection.

Let $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ and let $\lambda \in \mathbb{R}$.

- We have:

$$f((x_1, x_2)) + f((y_1, y_2)) = \left(\frac{1}{a}x_2, x_2\right) + \left(\frac{1}{a}y_2, y_2\right) = \left(\frac{1}{a}(x_2 + y_2), x_2 + y_2\right)$$

and we have

$$f((x_1, y_1) + (x_2, y_2)) = f((x_1 + y_1, x_2 + y_2)) = \left(\frac{1}{a}(x_2 + y_2), x_2 + y_2\right)$$

and thus

$$f((x_1, x_2)) + f((y_1, y_2)) = f((x_1, y_1) + (x_2, y_2)).$$

- We also have:

$$\lambda f((x_1, x_2)) = \lambda \left(\frac{1}{a}x_2, x_2\right) = \left(\frac{1}{a}\lambda x_2, \lambda x_2\right)$$

and we have

$$\lambda f((x_1, x_2)) = f((\lambda x_1, \lambda x_2)) = \left(\frac{1}{a}\lambda x_2, \lambda x_2\right)$$

and thus

$$\lambda f((x_1, x_2)) = \lambda f((x_1, x_2)).$$

So f is linear.

Now let $(x_1, x_2) \in \mathbb{R}$. Then we have:

$$||f((x_1,x_2))||_\infty = \left\|\left(\frac{1}{a}x_2, x_2\right)\right\|_\infty = \max\left\{\left|\frac{1}{a}x_2\right|, |x_2|\right\} = |x_2| \leq \max\{|x_1|,|x_2|\} = ||(x_1,x_2)||_\infty$$

because $|a| > 1$, so by remark 3.4 we have that f is a linear contractive projection.

Let $x \in \mathbb{R}$. Then $(x, ax) \in M$, and we have:

$$f((0,ax)) = \left(\frac{1}{a}ax, ax\right) = (x, ax),$$

so $(x, ax) \in \text{Im}(f)$ and thus $M \subset \text{Im}(f)$.
Let $x \in \mathbb{R}$, then

$$\left(\frac{1}{a}x, x\right) \in \text{Im}(f)$$

and

$$\left(\frac{1}{a}x, x\right) = \left(\frac{1}{a}x, \frac{1}{a}ax\right) = \frac{1}{a}(x, ax) \in M$$

because M is a linear subspace. So

$$\left(\frac{-x_1 + x_2}{a-1}, \frac{-ax_1 + ax_2}{a-1}\right) \in M$$

and thus $\text{Im}(f) \subset M$ and thus $\text{Im}(f) = M$.
So M is the range of a linear contractive projection.

As always we have $\tilde{J}((0,0)) = \{(0,0)\}$.
Let $x \in \mathbb{R}$ with $x \neq 0$. Then we have, because $|a| > 1$:

$$||(0,ax)||_1 = 0 + |ax| = |ax| = ||(x,ax)||_\infty.$$

We also have:

$$\langle(0,ax),(x,ax)\rangle = (ax)^2 = |ax|^2 = ||(x,ax)||_\infty^2.$$

So we have $(0, ax) \in \tilde{J}((x, ax))$.
Let $(u, v) \in \tilde{J}((x, ax))$. Then we have:

$$|u| + |v| = ||(u,v)||_1 = ||(x,ax)||_\infty = |ax|$$

or $|v| = |ax| - |u|$. We also have:

$$ux + vax = \langle(u,v),(x,ax)\rangle = ||(x,ax)||_\infty^2 = |ax|^2 = a^2x^2.$$

Since $x \neq 0$, this gives $u + va = a^2x$. Now we get if $x > 0$ that $a^2x > 0$ and because $|a| > 1$ we obtain:

$$a^2x = |a^2x| = |u + va| \leq |u| + |v||a| = |u| + (|ax| - |u|)|a| = (1 - |a|)|u| + |a^2x| \leq |a^2x| = a^2x$$

or if $x < 0$:

$$a^2x = -|a^2x| = -|u+va| \geq -|u| - |v||a| = -|u| - (|ax| - |u|)|a| = -(1-|a|)|u| - |a^2x| \geq -|a^2x| = a^2x.$$

So in both cases we must have equality and thus we obtain that $(1 - |a|)u = 0$ and therefore $u = 0$. From $u + va = a^2x$, we then get $v = ax$. Thus $(u, v) = (0, ax)$. Thus $\tilde{J}((x, ax)) = \{(0, ax)\}$. Therefore we obtain:

$$\tilde{J}(M) = \cup_{x \in M}\tilde{J}(x)$$
$$= (\{(0,ax), x \in \mathbb{R}, x \neq 0\}) \cup \{(0,0)\} = \{(0,y), y \in \mathbb{R}\}.$$

This is clearly a linear subspace of $(\mathbb{R}^2)^*$. So in this case we have that both M is the range of a linear contractive projection and that $\tilde{J}(M)$ is a linear subspace.

4. Let $a \in \mathbb{R}$ with $-1 < a < 1$. Consider $M = \{(x, ax) : x \in \mathbb{R}\}$. It is clear that M is a subspace, and thus that M is closed.
Now consider

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x_1, x_2) \longmapsto (x_1, ax_1).$$

Let $(x, y) \in \mathbb{R}$. Then we have:

$$(f \circ f)((x, y)) = f(f((x, y))) = f((x, ax)) = (x, ax) = f((x, y))$$

and thus f is a projection.

Let $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ and let $\lambda \in \mathbb{R}$.

- We have:

$$f((x_1, x_2)) + f((y_1, y_2)) = (x_1, ax_1) + (y_1, ay_1) = (x_1 + y_1, a(x_1 + y_1))$$

and we have

$$f((x_1, y_1) + (x_2, y_2)) = f((x_1 + y_1, x_2 + y_2)) = (x_1 + y_1, a(x_1 + y_1))$$

and thus

$$f((x_1, x_2)) + f((y_1, y_2)) = f((x_1, y_1) + (x_2, y_2)).$$

- We also have:

$$\lambda f((x_1, x_2)) = \lambda(x_1, ax_1) = (\lambda x_1, a\lambda x_1)$$

and we have

$$\lambda f((x_1, x_2)) = f((\lambda x_1, \lambda x_2)) = (\lambda x_1, a\lambda x_1)$$

and thus

$$\lambda f((x_1, x_2)) = \lambda f((x_1, x_2)).$$

So f is linear.

Now let $(x_1, x_2) \in \mathbb{R}$. Then we have:

$$||f((x_1, x_2))||_\infty = ||(x_1, ax_1)||_\infty = \max\{|x_1|, |ax_1|\} = |x_1| \leq \max\{|x_1|, |x_2|\} = ||(x_1, x_2)||_\infty$$

because $|a| < 1$, so by remark 3.4 we have that f is a linear contractive projection.

Let $x \in \mathbb{R}$. Then $(x, ax) \in M$, and we have: $f((x, 0)) = (x, ax)$, so $(x, ax) \in \text{Im}(f)$ and thus $M \subset \text{Im}(f)$.
Let $x \in \mathbb{R}$, then $(x, ax) \in \text{Im}(f)$, but also $(x, ax) \in M$ and thus $\text{Im}(f) \subset M$ and thus $\text{Im}(f) = M$.
So M is the range of a linear contractive projection.

As always we have $\tilde{J}((0, 0)) = \{(0, 0)\}$.
Let $x \in \mathbb{R}$ with $x \neq 0$. Then we have, because $|a| < 1$:

$$||(x, 0)||_1 = |x| + |0| = |x| = ||(x, ax)||_\infty.$$

We also have:

$$\langle (x, 0), (x, ax) \rangle = (x)^2 = |x|^2 = ||(x, ax)||_\infty^2.$$

So we have $(x, 0) \in \tilde{J}((x, ax))$.
Let $(u, v) \in \tilde{J}((x, ax))$. Then we have:

$$|u| + |v| = ||(u, v)||_1 = ||(x, ax)||_\infty = |x|$$

or $|u| = |x| - |v|$. We also have:

$$ux + vax = \langle (u,v),(x,ax)\rangle = ||(x,ax)||_\infty^2 = |x|^2 = x^2.$$

Since $x \neq 0$, this gives $u + va = x$. Now we get if $x > 0$, because $|a| < 1$ that:

$$x = |x| = |u + va| \leq |u| + |v||a| = |x| - |v| + |v||a| = (|a| - 1)|v| + |x| \leq |x| = x$$

or if $x < 0$:

$$x = -|x| = -|u + va| \geq -|u| - |v||a| = -|x| + |v| - |v||a| = -(|a| - 1)|v| - |x| \geq -|x| = x.$$

So in both cases we must have equality and thus we obtain that $(|a| - 1)|v| = 0$ and therefore $v = 0$. From $u + va = a^2x$, we then get $u = x$. Thus $(u,v) = (x,0)$. Thus $\tilde{J}((x,ax)) = \{(x,0)\}$. Therefore we obtain:

$$\tilde{J}(M) = \cup_{x\in M}\tilde{J}(x)$$

$$= (\{(x,0), x \in \mathbb{R}, x \neq 0\}) \cup \{(0,0)\} = \{(x,0), x \in \mathbb{R}\}.$$

This is clearly a linear subspace of $(\mathbb{R}^2)^*$. So in this case we have that both M is the range of a linear contractive projection and that $\tilde{J}(M)$ is a linear subspace.

5. Consider $M = \{(0,y) : y \in \mathbb{R}\}$. Then clearly M is a closed linear subspace. Consider

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x_1, x_2) \longmapsto (0, x_2)$$

This is the orthogonal projection of $\mathbb{R}^2$ on M and just like before we have have that this is a linear contractive projection with range M.
So M is the range of a linear contractive projection.

As always we have $\tilde{J}((0,0)) = \{(0,0)\}$.
Let $y \in \mathbb{R}$ with $y \neq 0$. Then we have:

$$||(0,y)||_1 = |0| + |y| = |y| = \max\{|0|,|y|\} = ||(0,y)||_\infty$$

and

$$\langle (0,y),(0,y)\rangle = y^2 = |y|^2 = ||(0,y)||_\infty^2$$

so $(0,y) \in \tilde{J}((0,y))$.
Let $(u,v) \in \tilde{J}((0,y))$. Then we have:

$$|u| + |v| = ||(u,v)||_1 = |y|$$

and

$$vy = \langle (u,v),(0,y)\rangle = |y|^2 = y^2$$

and thus $v = y$, and thus $|u| = 0$, so $(u,v) = (0,y)$.
So $\tilde{J}((0,y)) = \{(0,y)\}$.

Therefore we obtain:

$$\tilde{J}(M) = \cup_{x\in M}\tilde{J}(x)$$

$$= (\{(0,y), y \in \mathbb{R}, y \neq 0\}) \cup \{(0,0)\} = \{(0,y), y \in \mathbb{R}\}.$$

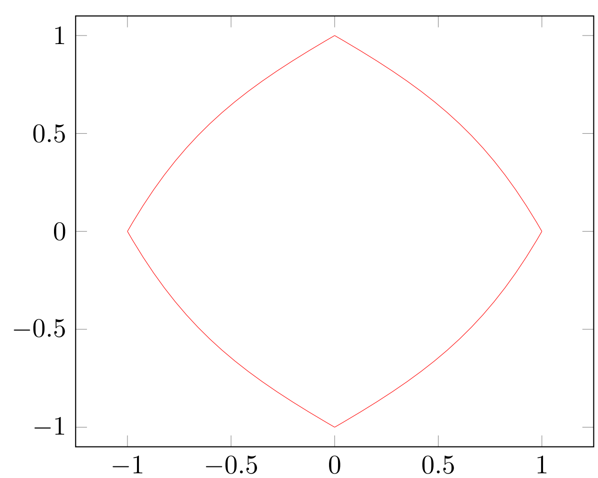
This is clearly a linear subspace of $(\mathbb{R}^2)^*$. So in this case we have that both M is the range of a linear contractive projection and that $\tilde{J}(M)$ is a linear subspace.

6. Consider $M = \mathbb{R}^2$. Then clearly M is a closed linear subspace. Let f be the identity map on $\mathbb{R}^2$. Then $f \circ f = f$ and f is linear, so f is a linear projection. Clearly M is the range of f. Let $x \in \mathbb{R}^2$. Then we have $||f(x)||_\infty = ||x||_\infty$, so f is a contractive map. So f is a linear contractive projection. So M is the range of a linear contractive projection. Let $M_1 = \{(x, x) : x \in \mathbb{R}\}$ and $M_2 = \{(x, -x) : x \in \mathbb{R}\}$. Then we have, using what we already found:
$$\tilde{J}(M) \supset \tilde{J}(M_1) \cup \tilde{J}(M_2)$$
$$= (\{(z, x - z) : x < 0, x \le z \le 0\}) \cup (\{(z, x - z) : x > 0, 0 \le z \le x\}) \cup$$
$$\{(0, 0)\} \cup (\{(z, z - x) : x < 0, x \le z \le 0\}) \cup (\{(z, z - x) : x > 0, 0 \le z \le x\}) = \mathbb{R}^2.$$
So we get that $\tilde{J}(M) = (\mathbb{R}^2)^*$, which is a linear subspace.
So in this case we have that M is the range of a linear contractive projection and that $\tilde{J}(M)$ is a linear subspace.

7. Consider $M = \{(0, 0)\}$. It is clear that M is a closed linear subspace.
Let f be the zero map. Then f is linear and for $x \in \mathbb{R}^2$ we have $f(f(x)) = (0, 0) = f(x)$. So f is a projection. Clearly, M is the range of this projection. For $x \in \mathbb{R}^2$ we have $||f(x)||_\infty = ||(0, 0)||_\infty = 0 \le ||x||_\infty$. So f is a linear contractive projection.
So M is the range of a linear contractive projection.
We also have:
$$\tilde{J}(M) = \tilde{J}((0, 0)) = \{(0, 0)\}.$$
So $\tilde{J}(M)$ is a linear subspace.
So in this case we have that M is the range of a linear contractive projection and that $\tilde{J}(M)$ is a linear subspace.

We have seen above that if we have a normed vector space that is not strictly convex and of which its dual space is also not strictly convex, then the result of Calvert's theorem does not remain intact for every subspace M. These are examples in which two conditions of the theorem are not met. Now we will consider examples in which only one of the conditions of the theorem is not met. To do this, we again consider $\mathbb{R}^2$, but this time we construct a norm in a way that only one requirement is left out.

Example 5.3. Consider the norm $|| \cdot ||$ in $\mathbb{R}^2$ with the following unit sphere:



As visible in the picture, this norm is strict convex, but not smooth. With proposition 2.13 it holds that the dual norm is not strict convex. With lemma 2.8 we have that this vector space is reflexive. Now we

consider the standard basis e_1, e_2 on $\mathbb{R}^2$ and its dual basis f_1, f_2. Again by [4, corollary 2.20] we have that every subspace of $\mathbb{R}^2$ is closed in this norm.
Now consider $M = \{(0,b) : b \in \mathbb{R}\}$.
Let $c \in \mathbb{R}$ with $c \neq 0$.

We see that $(0,c)$ is not a point of smoothness, so with corollary 2.15 we let $(a,b),(\tilde{a},\tilde{b}) \in \tilde{J}((0,c))$ with $(a,b) \neq (\tilde{a},\tilde{b})$. Suppose that $a = \tilde{a} = 0$. Then we must have that $b \neq \tilde{b}$, but then we have: $||(a,b)||_* = b \cdot ||(0,1)||_* \neq \tilde{(b)}||(0,1)|| = ||(\tilde{a},\tilde{b})||_*$. However we have that $(a,b),(\tilde{a},\tilde{b}) \in \tilde{J}((0,c))$ and thus that $||(a,b)||_* = ||(0,c)||$ and $||(\tilde{a},\tilde{b})||_* = ||(0,c)||$, so $||(a,b)||_* = ||(\tilde{a},\tilde{b})||_*$. So this gives a contradiction, and thus we must have that $a \neq 0$ or $\tilde{a} \neq 0$.

So let $(a,b) \in \tilde{J}((0,c))$ with $a \neq 0$.
Then we have:

$$||(a,-b)||^* = \sup_{(x,y)\in\mathbb{R}^2, ||(x,y)||=1} |\langle (a,-b),(x,y)\rangle| = \sup_{(x,-y)\in\mathbb{R}^2, ||(x,-y)||=1} |\langle (a,-b),(x,-y)\rangle| =$$

$$\sup_{(x,-y)\in\mathbb{R}^2, ||(x,-y)||=1} |\langle (a,b),(x,y)\rangle| = \sup_{(x,y)\in\mathbb{R}^2, ||(x,y)||=1} |\langle (a,b),(x,y)\rangle| = ||(a,b)||^* = ||(0,c)|| = ||(0,-c)||.$$

Here we used that the norm on $\mathbb{R}^2$ is symmetric in the x- and the y-axis and that $(a,b) \in \tilde{J}((0,c))$. We also have:

$$\begin{aligned}(a,-b)(0,-c) &= \langle (a,-b),(0,-c)\rangle = \langle (a,b)(0,c) = ||(0,c)||^2 \\ &= ||(-1)(0,-c)||^2 = |-1|^2||(0,-c)||^2 = ||(0,-c)||^2.\end{aligned}$$

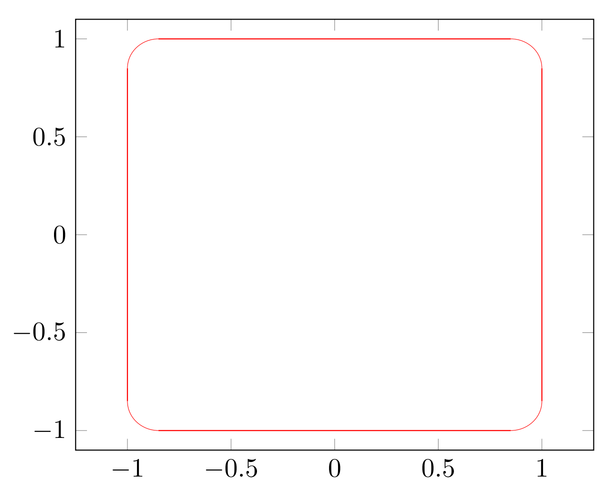
Here we used that $(a,b) \in \tilde{J}((0,c))$. So we get that $(a,-b) \in \tilde{J}((0,-c))$. So $(a,b) \in \tilde{J}(M)$ and $(a,-b) \in \tilde{J}(M)$.

Let $d \in \mathbb{R}$ with $d \neq 0$. Then we have:

$$(2a,0)(0,d) = \langle (2a,0),(0,d)\rangle = 0 \neq ||(0,d)||^2.$$

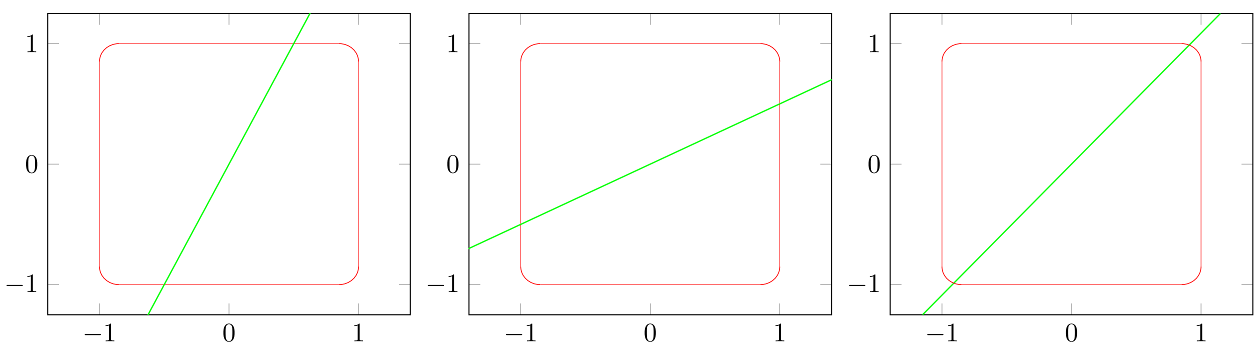
So we find that $(2a,0) \notin \tilde{J}((0,d))$. We also have $(2a,0) \notin \tilde{J}((0,0))$, for $\tilde{J}((0,0)) = \{(0,0)\}$ and $a \neq 0$. Therefore we have that $(2a,0) \notin \tilde{J}(M)$. So $\tilde{J}(M)$ is not a linear subspace of $(\mathbb{R}^2)^*$.

Example 5.4. Consider the norm $||\cdot||$ in $\mathbb{R}^2$ with the following unit sphere:



As visible in the picture, this norm is not strict convex, but it is smooth. With proposition 2.13 it holds that the dual norm is strict convex. With lemma 2.8 we have that this vector space is reflexive. Now we

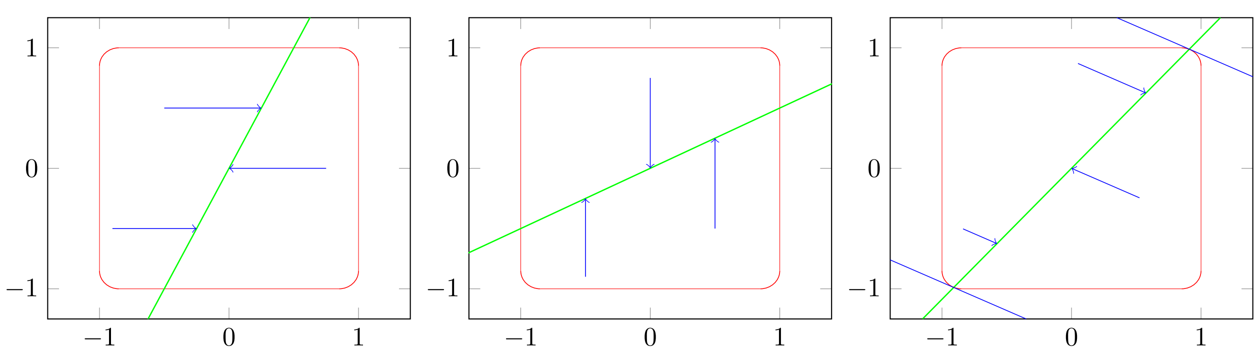
consider the standard basis e_1, e_2 on $\mathbb{R}^2$ and its dual basis f_1, f_2. Again by [4, corollary 2.20] we have that every subspace of $\mathbb{R}^2$ is closed in this norm.
Since the definition of the Gateaux derivative in a point $x \in \mathbb{R}^2$ in the direction y if $||y||_2 = 1$ coincides with the definition of the directional derivative in x in the direction of y, we see that the gradient in x is equal to the Gateaux derivative in x.
It is known from real analysis that the gradient of a function in a point x is always perpendicular to the level curve of f that contains x.
Now consider the following subspaces of $\mathbb{R}^2$:



(a) (b) (c)

Figure 2: with (a) we mean a line M_1 that goes through the upper and lower straight line segments of the unit sphere, with (b) we mean a line M_2 that goes through the right and the left straight line segments of the unit sphere, and with (c) we mean a line M_3 that goes throught the curved edges.

We can construct a linear contractive projection on each subspace in the following way:

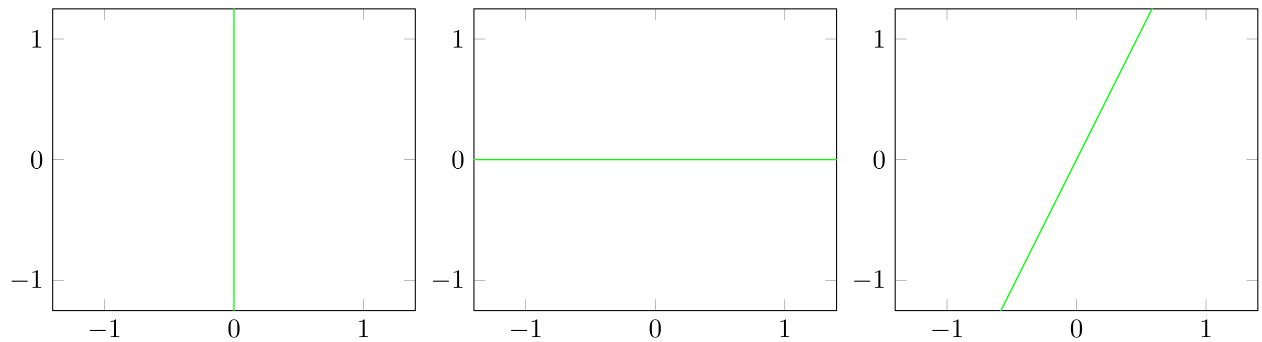


(a) (b) (c)

Figure 3: In (a) we project over lines parallel to the upper and lower straight line segments of the unit sphere, in (b) we project over lines parallel to the right and the left straight line segments of the unit sphere, and in (c) we project over lines parallel to the tangent lines in the points in which M_3 crosses the unit sphere.

In the figure we can see that the maps shown are linear contractive projections on respectively M_1, M_2 and M_3, as every point of the unit ball is clearly mapped to a point in the unit ball. We can now determine

$\tilde{J}(M_1)$, $\tilde{J}(M_2)$ and $\tilde{J}(M_3)$ by using the interpretation of the Gateaux derivative as the gradient as follows:



(a) (b) (c)

As seen in the figure, $\tilde{J}(M_1)$, $\tilde{J}(M_2)$ and $\tilde{J}(M_3)$ are linear subspaces of the dual space, and thus we see that the result of Calvert's theorem appears to hold.

We have seen several examples in which not all the conditions of Calvert's theorem were met. In three of those examples, the result of Calvert's theorem did not remain intact for every subspace M. In the last example however, the result of Calvert's theorem appears to remain intact, even though not all the requirements are met.

In example 5.3 we saw that the condition that X^* has to be strictly convex, i.e. that X is smooth, is a necessary condition for Calvert's theorem. However, in example 5.4 we saw that the condition that X has to be strictly convex might not be a necessary condtion. Moreover we have not found any examples where the only condition that was not met was this condition, but where the result of Calvert's theorem turned out to be false. Of course this does not prove that the condition might not be necessary and it might be interesting to investigate this further in the future.

References

[1] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic Publishers, Dordrecht, 1990.

[2] O. van Gaans, B. Lemmens and B. Randrianantoanina, Second derivatives of norms and contractive complementation in vector-valued spaces, *Studia Math.* **179** (2007), no. 2, 149-166.

[3] R.E. Megginson, *An introduction to Banach Space Theory*, Springer, New York, 1998.

[4] B.P. Rynne and M.A. Youngson, *Linear Functional Analysis*, Springer, SUMS, 2008.