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Self-Interacting Random Walk

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Summary

In this thesis we study a class of self-interacting random walk. The specific behaviour, which can differ from a simple random walk, is presented in the form of simulations and depends on some chosen parameters. The model for the random walk is as follows: the walker chooses a starting position on the integers (1-dimensional walk) and an initial value/weight is given to every edge between two neighbouring sites called the local time profile. Our stochastic model represents the probability to jump one position, after one unit of time, to the right which also depends on the local time profile. When the walker doesn't jump right the walker jumps one position to the left. The local time of the corresponding edge is raised with an amount of one and the local time profile is updated after every step. The underlying idea in this model is the possibility to choose parameters in such a way the local time profile determines the qualitative asymptotic behaviour of the walk. We start with the just described model and prove for a range of parameters that the walk eventually gets stuck on a single edge. This will be done in three steps. First we prove the walk gets stuck with positive probability. Next we show the walk must have a finite range and we use Rubin's Theorem to complete the proof. Further we investigate extended models where the walker can jump one or two positions at a time and find out that for one choice of parameters the type of behaviour is not unique. In the remainder we discuss the main result from the article *Stuck walks* by Erschler, Tóth and Werner. The theorem stated there gives a range of parameters such that the walk gets stuck on two, three, ... edges.

1 Introduction to self-interacting random walk

In a recent paper Erscher, Tóth and Werner [2] study a class of self-interacting random walk on \mathbb{Z} with ‘next-to-neighbouring’ interactions. This means that jumping left or right depends on some function of the number of times the neighbouring and next-to-neighbouring edges are crossed. We start with a similar model which is called an Edge Reinforced Random Walk (ERRW) because the transition probabilities depend on some nearby edges. In different models the transition probabilities could depend on nearby sites instead of edges, which is called Vertex Reinforced Random Walk (VRRW). Random walks have been studied thoroughly since the beginning of the twentieth century. Ordinary random walk is defined by a starting position, $X_0 \in \mathbb{Z}$, and a probability to jump one position to the left or right, $(1 - p)$ and p respectively ($p \in (0, 1)$ to avoid trivial walks). The following property holds for such walks:

$$\mathbb{P}(X_{n+1} = i_{n+1} | X_n = i_n, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = \mathbb{P}(X_{n+1} = i_{n+1} | X_n = i_n)$$

meaning that the positions before time n are irrelevant in determining the next position. This is called the ‘Markov property’. When studying *self-interacting* random walk we find however that these walks are non-Markovian because their past trajectories do influence their current behaviour. Before showing some examples we must start by defining such random walks. Let $L_0: \mathbb{Z} + \frac{1}{2} \rightarrow \mathbb{R}$ be a function that assigns a value, $L_0(i + \frac{1}{2})$, to every edge $\{i, i + 1\}$, $i \in \mathbb{Z}$, and L_0 is called the *initial local time profile*. After crossing an edge it is raised by one. When choosing the initial local profile to be identically zero for all edges, the values of $L_n(\cdot)$ can be interpreted as weights or the local time of the edge. With this clarified we start off with the following model and explain it in more detail.

Definition 1. Let L_0 , the initial local profile, k a positive integer, $a_{\pm i} \in \mathbb{R}$ where $i = 1, \dots, k$ be given. $L_n(\cdot)$ is updated after every step to $L_{n+1}(\cdot)$, $L_{n+2}(\cdot)$ and so on by letting $L_{n+1}(\cdot) = L_n(\cdot) + 1_e$ where $e = \{X_n, X_{n+1}\} = \{X_{n+1}, X_n\}$ denotes the crossed edge in the $(n + 1)$ -th jump. Furthermore let $\ell_n(\cdot) = L_n(\cdot + X_n)$ be the centralized local profile. A self-interacting random walk on the set of integers is a sequence of random variables (X_n) such that $|X_{n+1} - X_n| = 1$, for all $n \in \{0, 1, 2, \dots\}$. The random variables are defined inductively as follows: Given an arbitrary starting point $X_0 = i_0 \in \mathbb{Z}$ the probability to jump right equals

$$\begin{aligned} \mathbb{P}(X_{n+1} = X_n + 1 = i_n + 1 | X_0 = i_0, \dots, X_n = i_n, L_n) &= 1 - \mathbb{P}(X_{n+1} = X_n - 1 = i_n - 1 | X_0 = i_0, \dots, X_n = i_n, L_n) \\ &= \frac{e^{\Delta_n}}{e^{\Delta_n} + e^{-\Delta_n}} \end{aligned}$$

where

$$\Delta_n = \sum_{i=1}^k a_{-i} \ell_n(-i + \frac{1}{2}) - a_i \ell_n(i - \frac{1}{2}).$$

Note that the random variables are in fact the choices of jumping left or right. In [2] the definition of a self-interacting random walk is different because the most general form is given in a way that any function satisfies, namely $R(\ell_n(\cdot))$. Just like [2] and [3] we choose to study a particular class of self-interacting random walk and look at properties for specific initial values and parameters.

Example 1. For all i , let $a_{\pm i} = 0$, $X_0 = 0$ and let $L_0(\cdot)$ be randomly chosen. Then this model described a 1-dimensional *ordinary random walk* with transition probabilities

$$\mathbb{P}(X_{n+1} = X_n \pm 1 = i_n \pm 1 | X_0 = i_0, \dots, X_n = i_n, L_n) = \frac{e^0}{e^0 + e^0} = \frac{1}{2}.$$

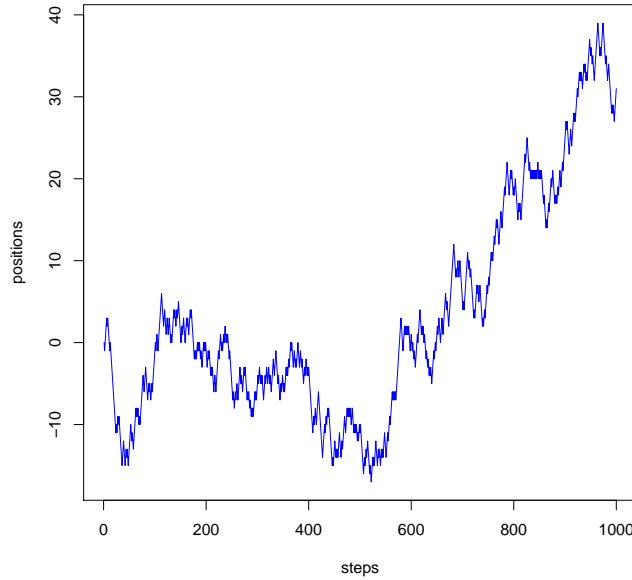
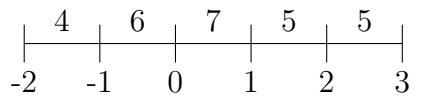
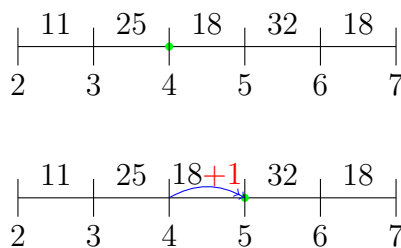


Figure 1: Trajectory of a 1-dimensional ordinary random walk where every step is independent of its previous visits.

After choosing the parameters $a_{\pm i}$ and letting $L_0(\cdot) \equiv 0$, a fraction of the edges could look like this when the walker started at zero:



Example 2. We visualize the effect of jumping over an edge ones more without choosing the $a_{\pm i}$ explicitly which is not necessary to grasp the idea. Let the walk start at $X_0 = 0$ and suppose at the moment we are at $X_n = 4$. When $X_{n+1} = 5$ the edge between four and five is raised by one.



The probability that this happens can be computed by going through the definition. Lets compute this probability explicitly for some choice of parameters. Suppose $k = 2$, $a_{-2} = a_2 = 3$, $a_{-1} = 1$ and $a_1 = 2$. To be exact regarding notion we note that the weight of the edge between three and four is $L(-\frac{1}{2}) = \ell(\frac{7}{2}) = 25$. Now we see that

$$\begin{aligned} \mathbb{P}(X_{n+1} = 5 | X_n = 4, X_{n-1} = i_{n-1}, \dots, X_0 = 0, L_n) &= \frac{e^{11a_{-2}+25a_{-1}-18a_1-32a_2}}{e^{11a_{-2}+25a_{-1}-18a_1-32a_2} + e^{-(11a_{-2}+25a_{-1}-18a_1-32a_2)}} \\ &= \frac{e^{33+25-36-96}}{e^{33+25-36-96} + e^{-(33+25-36-96)}} \end{aligned}$$

which equals zero by approximation. So it is very unlikely to jump right in this particular case. What is rather surprising is that the walker ended up in this setting at all since the probability to jump right is close to zero.

We can simulate such walks on a computer. Because we don't have time to compute such probabilities by hand every time it is natural to use a tool as simulation. This is done in section 3 and we see that the choices of $a_{\pm i}$ determines the characteristic behaviour of the walk.

A random walk is *height-invariant* if the local time profile raised by a certain amount doesn't influence the walker. In the special case when $a_{-i} = a_i$ for all i (the parameters are symmetric) the random walk is height-invariant:

$$\begin{aligned} \Delta_n((m + \ell(\cdot))) &= a_{-k}(m + \ell_{-k+\frac{1}{2}}(\cdot)) + \dots + a_{-1}(m + \ell_{-\frac{1}{2}}(\cdot)) \\ &\quad - a_1(m + \ell_{\frac{1}{2}}(\cdot)) - \dots - a_k(m + \ell_{k+\frac{1}{2}}(\cdot)) \\ &= a_{-k}\ell_{-k+\frac{1}{2}}(\cdot) + \dots + a_{-1}\ell_{-\frac{1}{2}}(\cdot) - a_1\ell_{\frac{1}{2}}(\cdot) - \dots - a_k\ell_{k+\frac{1}{2}}(\cdot) \\ &= \Delta(\ell(\cdot)) \end{aligned}$$

for all $m \in \mathbb{R}$. So the local time profile raised by an amount of m doesn't change $\Delta_n(\ell(\cdot))$ and consequently doesn't change the transition probabilities.

The a_i play an important role. Whenever a $a_{\pm i}$ is negative, the walker is attracted to the corresponding edge $L(X_n + i - \frac{1}{2})$ or $L(X_n - i + \frac{1}{2})$. Whenever it is positive it is repelled by it. This effect increases when a_i is larger or when the local profile is larger. This repelling effect is also shown in Example 2 where the walker most repelled from the edge between five and six with value 32. Also, whenever Δ_n is positive or negative the walker tends to the right or left respectively.

An interesting property of random walk in general is whether or not the walk is recurrent. A state i is said to be transient if there is a nonzero probability to never *return* to state i . If a state i is not transient it is recurrent.

It is still unknown (open question) whether the random walk stated in Definition 1 in dimension $d > 1$ is recurrent or not. However, Merkl and Rolles (*Recurrence of edge-reinforced random walk on a two-dimensional graph*) proved for Linearly Edge Reinforced Random Walk, after dividing every edge in \mathbb{Z}^2 in r pieces that the walk on that graph with certain initial conditions (also on the edges) is recurrent when $r \geq 130$. Here linearly means that the transition probabilities proportional to the weights of (only) the

neighbouring edges. On the other hand if we consider Linear Vertex Reinforced Random Walk on \mathbb{Z} , Tarrès (*Vertex-reinforced random walk on \mathbb{Z} eventually gets stuck on five sites*) proved that such a walk (under certain conditions) eventually gets stuck on five sites almost surely. So there are results known about random walks being transient or recurrent but not a lot is known about them in higher dimensions.

2 Basic properties

In this section we prove some basic properties of interacting random walks. Fortunately we find that it is not necessary to use simulation from the start and now find properties without computing the probabilities explicitly. We say a random walk gets stuck on an edge $\{i, i + 1\}$ when only finitely many times a different edge is crossed by the walker. Analogous definitions hold for more than one edge.

Theorem 1. *Let $(X_n, n \geq 0)$ as in Definition 1 where $a_{-1}, a_1 < 0$, other a_i, L_0, X_0 are given. Then for every edge there is a positive probability to be stuck on that particular edge.*

To prove this we let the random walk jump back and forth on the edge $\{i, i + 1\}$, $i \in \mathbb{Z}$ arbitrary.

Proof. First we note that with positive probability the walker successively jumps towards i until it arrives at i , after n steps. Note that when $X_0 = 0$ we could say: after $n = |i|$ number of steps. This is possible because the probability to jump left or right is never zero. At i we have the following probability to jump to the right for the m th time:

$$\mathbb{P}(X_{n+2m-1} = i + 1 | X_{n+2m-2} = i, L_{n+2m-2}) = \frac{e^{\Delta_{1,m}}}{e^{\Delta_{1,m}} + e^{-\Delta_{1,m}}}$$

where

$$\begin{aligned} \Delta_{1,m} = & \sum_{j=2}^k (a_{-j} L_{n+2(m-1)}(i - j + \frac{1}{2}) - a_j L_{n+2(m-1)}(i + j - \frac{1}{2})) \\ & + a_{-1} L_{n+2(m-1)}(i - \frac{1}{2}) - a_1 (L_{n+2(m-1)}(i + \frac{1}{2}) + 2(m - 1)). \end{aligned}$$

At $i + 1$ we have the following probability to jump to the left for the m th time:

$$\mathbb{P}(X_{n+2m} = i | X_{n+2m-1} = i + 1, L_{n+2m-1}) = \frac{e^{-\Delta_{2,m}}}{e^{\Delta_{2,m}} + e^{-\Delta_{2,m}}}$$

where

$$\begin{aligned} \Delta_{2,m} = & \sum_{j=2}^k (a_{-j} L_{n+2(m-1)+1}(i - j + \frac{3}{2}) - a_j L_{n+2(m-1)+1}(i + j + \frac{1}{2})) \\ & + a_1 L_{n+2(m-1)+1}(i + \frac{3}{2}) - a_{-1} (L_{n+2(m-1)+1}(i + \frac{1}{2}) + 2(m - 1) + 1). \end{aligned}$$

The information concerning the weights (due to the actual history of the walker) is inside L_n so we don't write the whole history of the walker in the conditional probability. Let τ be the set of all times up to which the walker follows the stated strategy, jumping right and left:

$$\tau = \{t \in \mathbb{N} : X_{n+2k} = X_n = i, X_{n+2k-1} = X_n + 1 = i + 1 \text{ for all } k = 1, \dots, t\}.$$

Multiplying all probabilities up to $2t$ and after simplifying the equation we notice that

$$\mathbb{P}(\max \tau > 2t) = \prod_{m=1}^t \frac{1}{1 + e^{-2\Delta_{1,m}}} \frac{1}{1 + e^{2\Delta_{2,m}}}.$$

Hence

$$\begin{aligned} \ln \mathbb{P}(\max \tau > 2t) &= \sum_{m=1}^t \ln((1 + e^{-2\Delta_{1,m}})(1 + e^{2\Delta_{2,m}})) \\ &= \sum_{m=1}^t \ln(1 + e^{-2\Delta_{1,m}}) + \sum_{m=1}^t \ln(1 + e^{2\Delta_{2,m}}). \\ &= \sum_{m=1}^t \ln(1 + e^{-2\Delta'_1 e^{4a_{-1}m}}) + \sum_{m=1}^t \ln(1 + e^{2\Delta'_2 e^{4a_1m}}). \end{aligned}$$

where Δ'_1 and Δ'_2 are constants, possibly large or small. These summations are bounded uniformly in t by the next observation. There exist $m_1, m_2 \in \mathbb{N}$ such that

$$e^{-2\Delta'_1 e^{4a_{-1}m_1}}, e^{2\Delta'_2 e^{4a_1m_2}} < 1.$$

Letting $t \rightarrow \infty$, we find

$$\sum_{m=1}^{\infty} \ln(1 + e^{-2\Delta'_1 e^{4a_{-1}m}}) = \sum_{0 < m < m_1} \ln(1 + e^{-2\Delta'_1 e^{4a_{-1}m}}) + \sum_{m \geq m_1} \ln(1 + e^{-2\Delta'_1 e^{4a_{-1}m}}).$$

Here the first expression is a finite sum and the second is dominated by a geometric series with common ratio $e^{4a_{-1}}$. This also holds for

$$\sum_{m=1}^{\infty} \ln(1 + e^{2\Delta'_2 e^{4a_1m}}) = \sum_{0 < m < m_2} \ln(1 + e^{2\Delta'_2 e^{4a_1m}}) + \sum_{m \geq m_2} \ln(1 + e^{2\Delta'_2 e^{4a_1m}})$$

where the common ratio equals e^{4a_1} . Note that

$$\limsup_{t \rightarrow \infty} \mathbb{P}(\tau > t) = \mathbb{P}(\limsup_{t \rightarrow \infty} \{\tau > t\}) = \mathbb{P}(\tau = \infty).$$

Hence $-\ln \mathbb{P}(\tau = \infty) < \infty$ and consequently $\mathbb{P}(\tau = \infty) > 0$. Because at the start we move to i with positive probability and stay on the edge $\{i, i + 1\}$ with positive probability, the walk gets stuck on $\{i, i + 1\}$ with positive probability. \square

Before improving the statement of Theorem 1 to almost sure behaviour, we point out the following interesting aspect of Definition 1. When we consider the initial values $L_0(\cdot)$ notice that the local time profile is not bounded just as in Theorem 1 and find the following remarkable fact.

Lemma 1. *Let $(X_n, n \geq 0)$ be a random walk as in Definition 1 with $a_{\pm i} = 0$ if $i = 2, 3, \dots, k$ and $a_{-1} = a_1 = a < 0$ and let $X_0 = 0$. Then there exist initial conditions L_0 such that the walk gets stuck on one edge with positive probability and the walk goes to infinity with positive probability.*

To prove that the walk gets stuck on one edge almost surely, we need the initial local time profile to be bounded otherwise this lemma would contradict.

Proof. The first part of the lemma follows directly from Theorem 1 since no special initial conditions are required to get stuck. For the second part we construct initial conditions on the edges such that the walk is ‘pulled to’ infinity with positive probability.

$$\begin{array}{cccccc} & L(-\frac{1}{2}) & L(\frac{1}{2}) & L(\frac{3}{2}) & L(\frac{5}{2}) & L(\frac{7}{2}) \\ | & | & | & | & | & | \\ -1 & 0 & 1 & 2 & 3 & 4 \end{array}$$

Now forcing the walk to the right, from the start at X_0 :

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P}(X_n \rightarrow \infty) &> \mathbb{P}(\forall n : X_{n+1} = X_n + 1) \\ &= \lim_{t \rightarrow \infty} \prod_{n=1}^t \frac{e^{a(L_{n-1}(n-\frac{3}{2})+1-L_{n-1}(n-\frac{1}{2}))}}{e^{a(L_{n-1}(n-\frac{3}{2})+1-L_{n-1}(n-\frac{1}{2}))} + e^{-a(L_{n-1}(n-\frac{3}{2})+1-L_{n-1}(n-\frac{1}{2}))}} \\ &= \lim_{t \rightarrow \infty} \prod_{n=1}^t \frac{1}{1 + e^{-2a(L_{n-1}(n-\frac{3}{2})+1-L_{n-1}(n-\frac{1}{2}))}}. \end{aligned}$$

We added one to $L(-\frac{1}{2})$ so the notation is somewhat better. Take the logarithm on both sides and notice that

$$-\lim_{t \rightarrow \infty} \sum_{n=1}^t \log(1 + e^{-2a(L_{n-1}(n-\frac{3}{2})+1-L_{n-1}(n-\frac{1}{2}))}) \leq -\lim_{t \rightarrow \infty} \sum_{n=1}^t e^{-2a(L_{n-1}(n-\frac{3}{2})+1-L_{n-1}(n-\frac{1}{2}))}$$

if $|e^{-2a(L_{n-1}(n-\frac{3}{2})+1-L_{n-1}(n-\frac{1}{2}))}| < 1$. Here we use the identity $\ln(1+x) \leq x$ if $|x| < 1$. Since $a < 0$ we need $L_{n-1}(n-\frac{3}{2})+1-L_{n-1}(n-\frac{1}{2}) < 0$ so $L(n-\frac{1}{2}) > L(n-\frac{3}{2})+1$. Take $L(-\frac{1}{2}) = 1$ (due to notation earlier), $L(\frac{1}{2}) = 3$ and $L(n-\frac{1}{2}) = 3L(n-\frac{3}{2})$ for $n \geq 2$ and the condition is satisfied for all n so the sum is finite and thus $\lim_{n \rightarrow \infty} \mathbb{P}(X_n \rightarrow \infty) > 0$.

Here the final picture of the initial local time profile is:

$$\begin{array}{cccccc} & 1 & 3 & 9 & 27 & 81 \\ | & | & | & | & | & | \\ -1 & 0 & 1 & 2 & 3 & 4 \end{array}$$

□

3 Simulations

We now give a short overview of some possible types of behaviour that can occur, by using simulations. Fortunately, self-interacting random walks are simulated just as easy as ordinary random walks which gives us the opportunity to see what happens for different values of $a_{\pm 1}$ and $a_{\pm 2}$. A ‘phase diagram’ in terms of $a = a_{\pm 1}$ and $b = a_{\pm 2}$ is given in [2]. For convenience, we also let $a_i = 0$ for all $i = 3, 4, \dots, k$ and $a_1 = a_{-1}$ and $a_2 = a_{-2}$. We have already seen the ordinary random walk in section 1. We start with a different type of behaviour, the so-called ‘ballistic’ random walk.

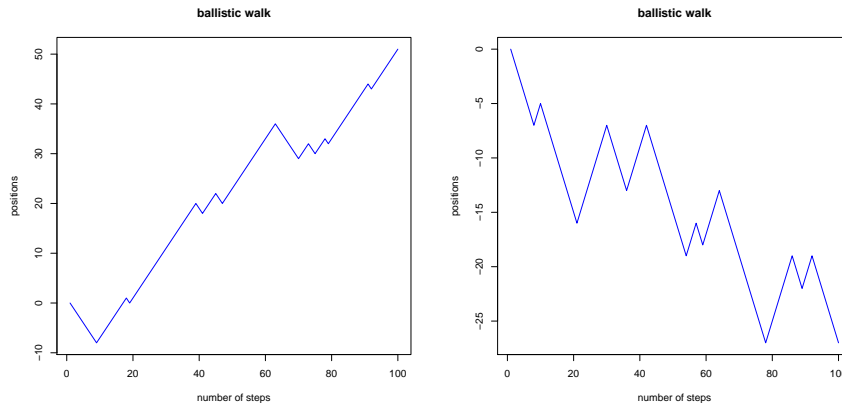


Figure 2: Two different walks for $a_1 = 0$ and $a_2 = 1$.

These walks are very similar to simple random walk where $p < \frac{1}{2}$ or $p > \frac{1}{2}$ but not quite since it looks very structured and less jumpy. When it turns around, it tends to walk in the new direction. Now we look at some cases where the walk gets stuck on a number of edges.

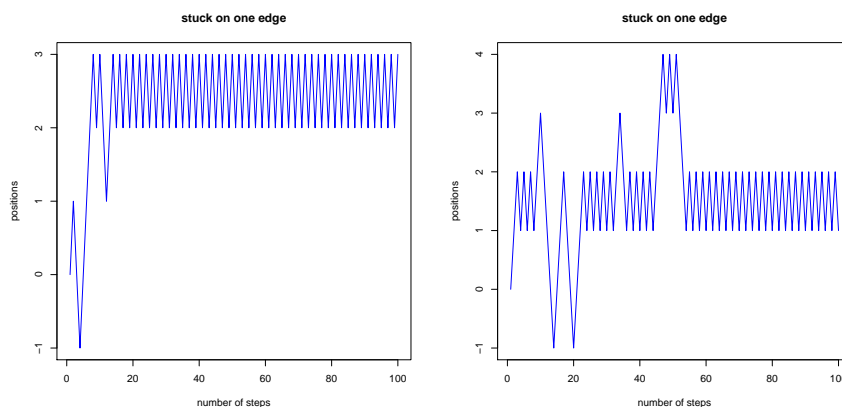


Figure 3: Two cases where the walker gets stuck on a single edge. Here $a_1 = 0$, $a_2 = -.1$ respectively $-.06$. In the latter case the walker slowly builds up enough local time to get stuck on the edge $\{1, 2\}$.

An example of a random walk where it takes very long before getting stuck.

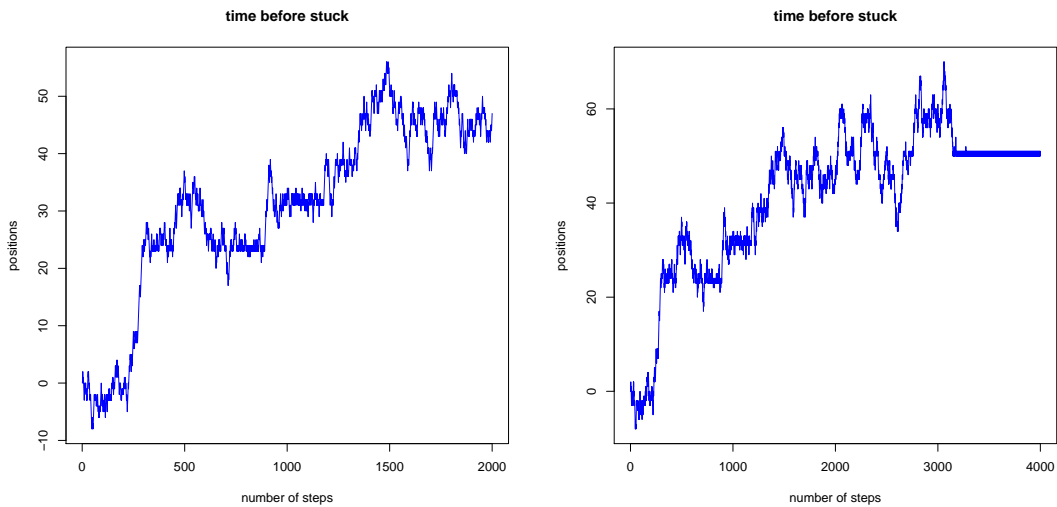


Figure 4: Similar case where the walker gets stuck on a single edge. These walks are the same, the second one is longer and we see it gets stuck. Here $a_1 = 0$ and $a_2 = -.01$.

We see by simulations that indeed for all $a_{\pm 1} < 0$ the random walk gets stuck on a single edge. On the other hand the time until it finally gets stuck can vary due to the choice of parameters or initial local times. Finally, we find that the walker can get stuck on more than a single edge.

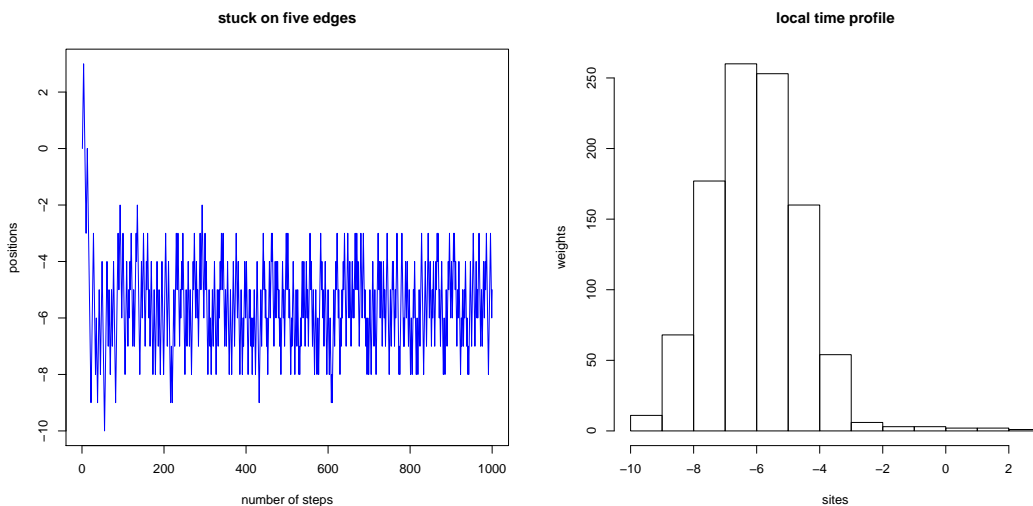


Figure 5: It appears that the walker gets stuck on five edges, note that the histogram plots six sites! $a_1 = -.45$, $a_2 = 1$ and make 1000 steps.

Here we see that the histogram reveals the local time profile on the edges. It is possible to get stuck on one, two, three, ... edges in the long run but impossible to show them all separately. A range of parameters is given in [3] which tells us on how many edges

the walker will get stuck on. This is shown in section 6.

These plots are made in R¹ and the program used can be found in the appendix as Model 1.

¹Search for R in your favorite internet search engine and you can download and use R for free.

4 Stuck on a single edge

Theorem 1 states the walk will get stuck with positive probability as soon as both $a_{\pm 1} < 0$. The aim of this section is to prove that the random walk gets stuck almost surely. In order to do so we need another result: the *finite range* of the walk. Here the range of the walker equals all sites ever visited, $\mathcal{R} = \{X_0, X_1, \dots\}$.

4.1 Finite range

Lemma 2. *The random walk from Theorem 1 with $a_{\pm i} = 0$ for $i = 2, 3, \dots, k$ and $\sup_e |L_0(e)| < \infty$ has finite range, $|\mathcal{R}| < \infty$.*

$$\frac{\left| \frac{L(2n - \frac{1}{2})}{2n} \right| \left| \frac{L(2n + \frac{1}{2})}{2n + 1} \right| \left| \frac{L(2n + \frac{3}{2})}{2n + 1} \right|}{}$$

Proof.

We show that there exist a $\delta > 0$ such that every time we encounter a new $2n$ th site, $n \in \mathbb{N}$, the probability that the walk gets immediately stuck on the next edge is more than δ . This δ can be viewed at as the worst case the walker could find itself in and so the probability of getting stuck immediately is least likely. Let $L(2n - \frac{1}{2})$, $L(2n + \frac{1}{2})$ and $L(2n + \frac{3}{2})$ be the weights on the edges as in the figure above, around $2n$. Let τ_x be the time of first arrival at x , $\tau_x = \inf\{m \in \mathbb{N} : X_m = x\}$. The first time we arrive at site $2n$ the probability to jump right equals

$$\mathbb{P}(X_{m+1} = 2n + 1 | \tau_{2n} = m) = \frac{e^{a_{-1}(L(2n - \frac{1}{2}) + 1) - a_1 L(2n + \frac{1}{2})}}{e^{a_{-1}(L(2n - \frac{1}{2}) + 1) - a_1 L(2n + \frac{1}{2})} + e^{-(a_{-1}(L(2n - \frac{1}{2}) + 1) - a_1 L(2n + \frac{1}{2}))}}$$

and jump over the same edge again (back to $2n$) with probability

$$\begin{aligned} \mathbb{P}(X_{m+2} = 2n | \tau_{2n} = m, X_{m+1} = 2n + 1) \\ = \frac{e^{-(a_{-1}(L(2n + \frac{1}{2}) + 1) - a_1 L(2n + \frac{3}{2}))}}{e^{-a_{-1}(L(2n + \frac{1}{2}) + 1) - a_1 L(2n + \frac{3}{2})} + e^{a_{-1}(L(2n + \frac{1}{2}) + 1) - a_1 L(2n + \frac{3}{2})}}. \end{aligned}$$

We want that the probability that we get stuck on that edge, $\{2n, 2n + 1\}$, immediately to be as small as possible and therefore we need both $a_{-1}(L(2n - \frac{1}{2}) + 1) - a_1 L(2n + \frac{1}{2})$ and $-a_{-1}(L(2n + \frac{1}{2}) + 1) - a_1 L(2n + \frac{3}{2})$ to be as small as possible because

$$\frac{e^x}{e^x + e^{-x}} > \frac{e^y}{e^y + e^{-y}}$$

if $x > y$. This can be done by maximizing $L(2n - \frac{1}{2})$, $L(2n + \frac{3}{2})$ and minimizing $L(2n + \frac{1}{2})$ and we see that for this worst case we need $L(2n - \frac{1}{2})$ and $L(2n + \frac{3}{2})$ to be the largest values allowed, say M , by our boundaries and $L(2n + \frac{1}{2})$ to be as small as possible. δ_{a_{-1}, a_1} depending on both a_{-1} and a_1 can be found explicitly and to calculate this we refer to the proof of Theorem 1 and note again it is greater than zero. Now every time we encounter a new $2n$ th site, the probability the walker gets stuck immediately is always more than δ . By symmetry the same holds for every new $-2n$ th site we encounter, $n \in \mathbb{N}$. It follows that $\mathbb{P}(|\mathcal{R}| > 2t) < (1 - \delta)^t$ and consequently $\mathbb{P}(|\mathcal{R}| = \infty) = \lim_{t \rightarrow \infty} \mathbb{P}(|\mathcal{R}| > 2t) = 0$. \square

4.2 Rubin's Theorem

In order to formulate Rubin's theorem we consider infinite sequences of the set $\{l, r\}$. Davis [1] shows an analogy with the generalized Polya urn model by drawing red (r) and white (w) balls from an urn. We interpret drawing a red/white ball as going left/right and we return a multiple of the same colored balls back into the urn.

We define two sequences $l = (l_0, l_1, \dots)$ and $r = (r_0, r_1, \dots)$ of nonnegative numbers and need $r_0, w_0 > 0$. Let $L_k = \sum_{i=0}^k l_i$ and $R_k = \sum_{i=0}^k r_i$. The infinite sequence we generate starts with an r with probability $\frac{R_0}{R_0 + L_0}$ and with l with probability $\frac{L_0}{R_0 + L_0}$. After n steps (entries) consisting of x r 's and $y = n - x$ l 's the probability that the $(n + 1)$ th entry is an r equals $\frac{R_x}{R_x + L_y}$ and it is an l with probability $\frac{L_y}{R_x + L_y}$. This looks very much like the model we start out with in Definition 1. Finally, let

$$p_r = \mathbb{P}(\text{all but finitely many elements of the sequence are 'r'})$$

and

$$p_l = \mathbb{P}(\text{all but finitely many elements of the sequence are 'l'})$$

$$\text{and } \phi(r) = \sum_{i=0}^{\infty} R_i^{-1} \text{ and } \phi(l) = \sum_{i=0}^{\infty} L_i^{-1}.$$

Theorem 2 (Rubin). *If $\phi(r) < \infty$ and $\phi(l) < \infty$ then $p_r, p_l > 0$ and $p_r + p_l = 1$.*

This proof is also stated in [1].

Proof. Let Y_0, Y_1, \dots be independent exponential random variables with $\mathbb{E}(Y_i) = R_i^{-1}$. Let Z_0, Z_1, \dots be independent exponential random variables with $\mathbb{E}(Z_i) = L_i^{-1}$ and note that all Y_i and Z_j are independent for $i, j \geq 0$. Let $A = \{\sum_{i=0}^k Y_i, k \geq 0\}$, and

$B = \{\sum_{i=0}^k L_i, k \geq 0\}$ and define $G = A \cup B$. Let ξ_i be the i th smallest number in

G . We define a random sequence of r 's and l 's by letting the i th number be an r if $\xi_i \in A$ and l if $\xi_i \in B$. The most important property of this sequence is that it has the same distribution as the generalized Polya sequence. We give two examples such that this is clarified. The probability that the first entry is an r equals $\mathbb{P}(\xi_1 \in A) = \mathbb{P}(Y_0 < Z_0)$. This can be computed easily because the joint probability density function is the product of the two exponential distributions with expectations R_0^{-1} and L_0^{-1} .

Integrating over the area $0 < y < z < \infty$ we get $\mathbb{P}(Y_0 < Z_0) = \frac{R_0}{R_0 + L_0}$ which is what is should be stated above. A different (representative) case is where the history $H = \{\text{the first four components are } rrlr\} = \{\xi_1 \in A, \xi_2 \in A, \xi_3 \in B, \xi_4 \in A\}$ is given. Given H , the distance α from ξ_4 to the smallest number in A greater than ξ_4 is Y_3 . So α has the distribution of Y_3 . Now let β be the smallest distance from ξ_4 to the smallest number of B greater than ξ_4 . Looking closely at the possibilities we find this distance to be $Z_1 + Z_0 - (Y_0 + Y_1 + Y_2)$. Due to the lack of memory property of exponential distributions

we note that β has the distribution of Z_1 . This all relies on the assumption that we condition on the given H . And because α and β are independent (given H) we conclude $\mathbb{P}(\xi_4 \in A|H) = \mathbb{P}(\alpha < \beta) = \frac{Y_3}{Y_3 + Z_1}$ which also correspondes to the probabilities of the

generalized Polya sequence. Finally, we observe that because $\sum_{i=0}^{\infty} R_i^{-1} < \infty$ it holds that

$\mathbb{P}(\sum_{i=0}^{\infty} Y_i < \infty) = 1$. The same result holds if we replace the $<$ sign by the $=$ sign in

both expressions. Further note that in the finite case $\sum_{i=0}^{\infty} Y_i$ has a positive density on $(0, \infty)$. All these results are also true if we replace R_i by L_i and the corresponding Y_i by Z_i . Furthermore we see that $p_r = \mathbb{P}(\sum_{i=0}^{\infty} Y_i < \sum_{i=0}^{\infty} Z_i)$ and $p_l = \mathbb{P}(\sum_{i=0}^{\infty} Z_i < \sum_{i=0}^{\infty} Y_i)$. These ‘piles’ must be compared in the long run and notice that one must be larger than the other with probability one, so $p_r + p_l = 1$ and both p_r and p_l are positive because both sums are supported on the whole interval $(0, \infty)$. \square

4.3 The argument completed

Theorem 3. *A random walk satisfying the conditions from Lemma 2 gets stuck on one edge almost surely.*

Proof. By Lemma 2 we know that the random walk has finite range and we proceed indirectly by assuming that at least two (neighbouring) edges are both crossed infinitely often. We refer to these as the left and right edge from the infinitely often visited site j . Now we define a sequence consisting of r ’s and l ’s as follows: let $T_1 = \inf\{k \geq 0 : X_k = j\}$ and $T_i = \inf\{k > T_{i-1} : X_k = j\}, i > 1$ and let the i th entry be r if $X_{T_i+1} = j+1$ and l if $X_{T_i+1} = j-1$. By the assumption, the random walker will produce a sequence consisting of infinitely many r ’s and infinitely many l ’s. Now we note that in general

$$\begin{aligned} \mathbb{P}(X_{n+1} = X_n + 1 | X_n, L_n) &= \frac{e^{a_{-1}l - a_1l_r}}{e^{a_{-1}l - a_1l_r} + e^{a_1l_r - a_{-1}l}} \\ &= \frac{e^{-a_1l_r}}{e^{-a_1l_r} + e^{a_1l_r - 2a_{-1}l}} \\ &= \frac{e^{-2a_1l_r}}{e^{-2a_1l_r} + e^{-2a_{-1}l}} = \frac{R_n}{R_n + L_n}. \end{aligned}$$

Here $R_n = e^{-2a_1l_r(n)}$ and $L_n = e^{-2a_{-1}l(n)}$. To finally apply Rubin’s Theorem, we need both $\phi(r) < \infty$ and $\phi(l) < \infty$. This holds because both $\phi(r)$ and $\phi(l)$ are geometric series with common ratio $e^{2a_1}, e^{2a_{-1}} < 1$, because in Lemma 2 $a_{-1}, a_1 < 0$. So by Rubin’s Theorem we now know that $p_r + p_l = 1$; implying the probability that we find both an infinite number of r ’s and l ’s equals zero. So the assumption is wrong and we conclude that only one edge is crossed infinitely often. \square

Sellke, *Reinforced Random Walk on the d -dimensional Integer Lattice (1994)*, proved the same result. Starting at the origin, an edge has weight w_k if it is crossed k times. He defines a walk on \mathbb{Z}^d and the transition probabilities are proportional to the weights on the edges connecting the sites around its position at distance one. He also proves

whenever $\sum_{k=0}^{\infty} w_{2k}^{-1} = \infty$ and $\sum_{k=0}^{\infty} w_{2k+1}^{-1} = \infty$ the range of the walker is finite and if both are finite the walker gets stuck on a single edge.

5 Extensions of the model

We consider extensions of the model where the walker is now able to jump to its neighbouring sites as well as to its next-to-neighbouring sites. We investigate whether or not it is possible to find parameters such that the qualitative asymptotic behaviour is not unique. We found such parameters in the original model but failed when considering bounded L_0 . We use simulations to find them for the following models.

The first extended model we consider is equal to the one we started with in Definition 1 but after choosing to jump left or right, we ‘flip a coin’ and with equal probability the walker jumps one or two positions in that direction. The edges that are crossed are raised by one.

We expect that whenever the original walker gets stuck on one edge this walker gets stuck on three. We find the following interesting behaviour: when the parameters $a_{\pm 2} = 1, a_{\pm 1} = -3$ the walk can get stuck on three or five edges with positive probability.

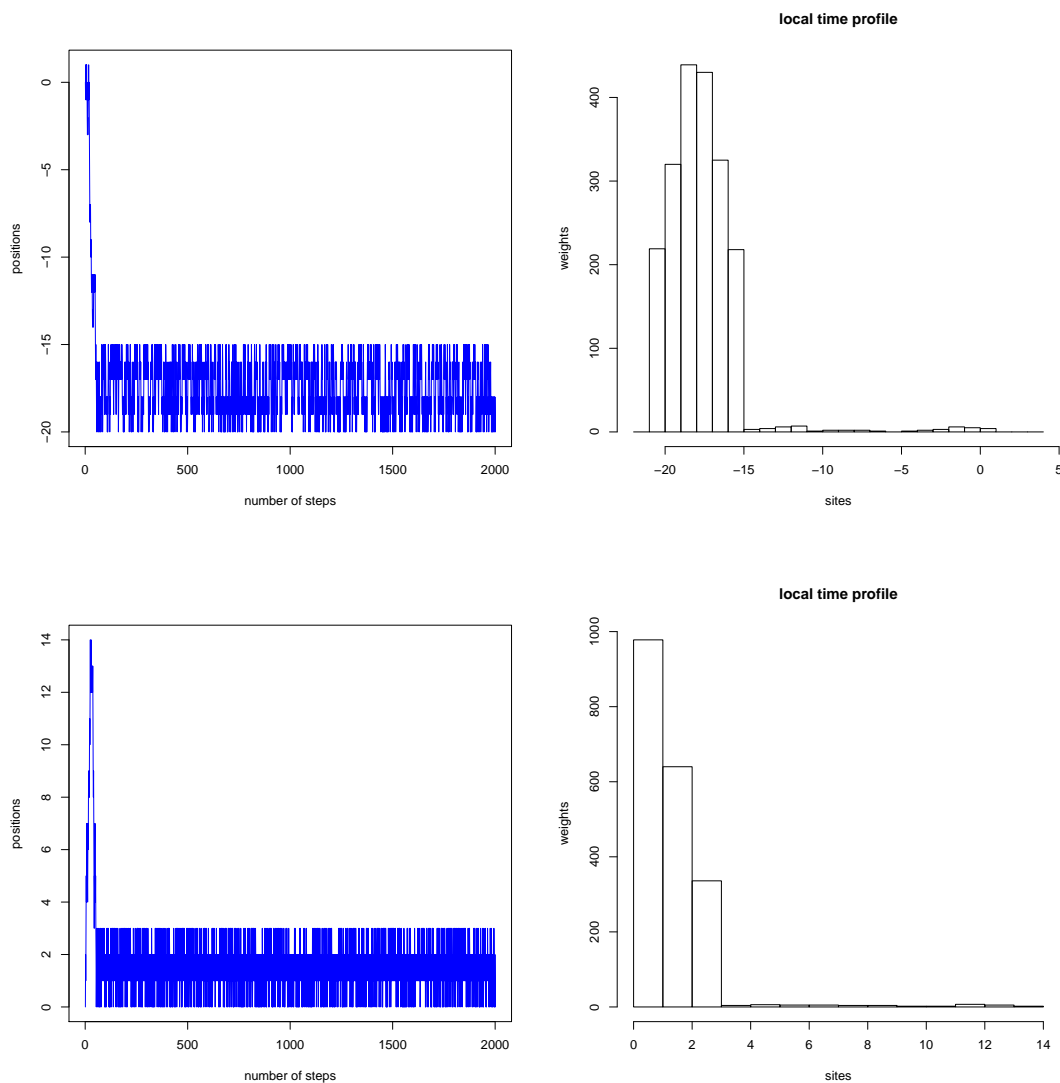


Figure 6: $a_{\pm 1} = -3, a_{\pm 2} = 1$, 2000 steps

After doing simulations we find that for $a_{\pm 1} < 0$ the walker gets stuck on five or three edges due to the possibility of jumping twice as far as normal.

The second extended model we consider is equal to the one we started with in Definition 1 but after choosing to jump left or right we jump one or two positions depending on the weights on the edges. Once the walker decides to go left/right, it jumps two positions with probability

$$\frac{\ell(\pm 3/2)}{\ell(\pm 3/2) + \ell(\pm 1/2)}$$

and one position with probability

$$\frac{\ell(\pm 1/2)}{\ell(\pm 3/2) + \ell(\pm 1/2)}.$$

Remember that going left means only minus signs in these terms, and going right represents only plus signs. The edges that are crossed are raised by one and the local time profile $L_0 \geq 1$. We don't want to divide by zero at any time.

The problem we encounter here is that the simulations could be misleading. It may look like the walker is stuck on three edges for example when it actually needs four edges. That can happen when you look at the random walk (Model 3) in the appendix. You can use parameters $a_1 = .01, a_2 = -.1$ and `set.seed(1)` in R and first make only, say, 10000 steps and also one plot with at least 150000 steps (don't forget to use `set.seed(1)`, this is essential).

The authors of [2] ask whether not almost sure behaviour is possible. Our simulations suggest it is for model 2 and probably not for model 3. We note that we have not varied the initial conditions at any time so perhaps that could make a difference when the initial conditions are not bounded. For model 1 with suitable initial conditions we know it is possible (see Lemma 2) for different behaviours to occur with positive probability. However, we have proven that it is not possible in model 1 with bounded initial conditions and specified parameters in Theorem 3.

6 Discussion

Finally, we discuss some other results presented in [2] and [3] and state some open questions to be answered.

In [2] a symmetric case of the parameters is investigated and true self-repelling motion is shown as well as the stuck case and the nonsymmetric case for certain parameters. An overview of possible behaviour types is given in a ‘phase diagram’ in terms of $a(=a_{-2} = a_2)$ and $b(=a_{-1} = a_1)$. Furthermore, we state the main theorem of *Stuck walks* which gives sufficient conditions for the walker to get stuck on two, three, \dots edges. The Theorem states: Let $A_k := 1 + 2 \cos(\frac{2\pi}{k+2})$.

- If $b/|a| \in (A_k, A_{k+1})$ for some $k \geq 1$ then with positive probability the walk remains stuck on a set of $k + 2$ sites, and visits all infinitely often.
- If $b/|a| > A_k$ then, almost surely, the walk does not get stuck on a set with less than $k + 2$ sites.

The proof is given in [3], in two steps, consisting of a probabilistic part and a combinatorial one.

We wonder if it is possible to improve Theorem 3, so it hold for a larger range of parameters and if its possible for more types of asymptotic behaviour for one choice of parameters to occur when the initial conditions are bounded. Also, in higher dimensions there is not much known about self-interacting random walk. It is a challenge to even figure out what happens in \mathbb{Z}^2 .

References

- [1] Burgess Davis, *Reinforced random walk*, Probab. Theory Related Fields **84** (1990), no. 2, 203–229. MR 1030727 (91a:60179)
- [2] Anna Erschler, Bálint Tóth, and Wendelin Werner, *Some locally self-interacting walks on the integers*, (2010).
- [3] ———, *Stuck walks*, Probability Theory and Related Fields (2011), 1–15, 10.1007/s00440-011-0365-4.

7 Appendix

To run one of the following programs, first insert the program in R. You can run the program by entering *walk1(1,2,200)* for example. The first two entries (a_2 and a_1 respectively, where $a_2 = a_{-2}$ and $a_1 = a_{-1}$) are the parameters and the third entry is the number of steps. The programs speak for themselves.

```
#Model 1
walk1=function(a,b,c)
{
h=0*(1:c)
l=0*(1:(2*c+10))
n=c

for(k in 1:c)
{
h[k]=n
d=a*l[n-2]+b*l[n-1]-b*l[n]-a*l[n+1]
p=1/(1+exp(-2*d))
if(runif(1,0,1)<=p)
{
l[n]=l[n]+1
n=n+1
}
else
{
l[n-1]=l[n-1]+1
n=n-1
}
}
h=h-c
plot(h,xlab="steps", ylab="positions",type="l")
}
```

```

#Model 2
walk2=function(a,b,c)
{
h=0*(1:c)
l=0*(1:(2*c+10))
n=c

for(k in 1:c)
{
h[k]=n
d=a*l[n-2]+b*l[n-1]-b*l[n]-a*l[n+1]
p=1/(1+1*exp(-2*d))
q=1-p
v=runif(1,0,1)
if(v<=p/2)
{
l[n+1]=l[n+1]+1
l[n]=l[n]+1
n=n+2
} else
if(v<=p)
{
l[n]=l[n]+1
n=n+1
} else
if(v<=(p+(1-p)/2))
{
l[n-1]=l[n-1]+1
n=n-1
}
else
{
l[n-2]=l[n-2]+1
l[n-1]=l[n-1]+1
n=n-2
}
}
h=h-c
hist(h,main="local time profile", xlab="sites",ylab="weights")
#plot(h,type="l", xlab="number of steps", ylab="positions")
}

```

```

#Model 3
walk3=function(a,b,c)
{
h=0*(1:c)
l=0*(1:(2*c+10))
l=l+1
n=c

for(k in 1:c)
{
h[k]=n
d=a*l[n-2]+b*l[n-1]-b*l[n]-a*l[n+1]
p=1/(1+exp(-2*d))
v=runif(1,0,1)
w=runif(1,0,1)
if(v<=p)
{
if(w<=(l[n]/(l[n]+l[n+1])))
{
l[n]=l[n]+1
n=n+1}
else{
l[n+1]=l[n+1]+1
l[n]=l[n]+1
n=n+2}
}
else
{
if(w<=(l[n-1]/(l[n-1]+l[n-2])))
{
l[n-1]=l[n-1]+1
n=n-1}

else{
l[n-2]=l[n-2]+1
l[n-1]=l[n-1]+1
n=n-2}
}
}
h=h-c
plot(h,type="l",xlab="number of steps",ylab="positions")
}

```