G. L. van der Sluijs

Change ringing

Bachelor thesis, June 7, 2016
Supervisor: dr. M. J. Bright

Mathematisch Instituut, Universiteit Leiden
Contents

Introduction ........................................ 3

1 Preliminaries ....................................... 4
   1.1 Change ringing terminology .................. 4
   1.2 Words ........................................ 7

2 The existence of an extent ......................... 8
   2.1 Plain changes ................................ 8
   2.2 The Cayley graph ............................. 8
   2.3 Existence of an extent using only three changes .... 10

3 Grandsire Triples ................................ 12
   3.1 Description and basic properties .............. 12
   3.2 Thompson’s proof ............................. 14
   3.3 The largest possible touch ................... 16

4 Rankin’s campanological theorem ................. 18
   4.1 Rankin’s theorem ............................. 18
   4.2 Application to Grandsire Triples .............. 18
   4.3 Application to Double Norwich Court Bob Major .. 19

5 The existence and construction of extents .......... 21
   5.1 Extent existence theorems ................... 21
   5.2 Existence of Plain Bob Major extent with special bob leads ... 23
   5.3 Extent construction of Plain Bob Doubles ....... 23

References ........................................ 26
Introduction

This bachelor thesis will be concerned with the old English art of ringing church bells called change ringing. The development of change ringing in the early 17th century was mainly due to the invention of the full-circle wheel on which the bells were mounted. By pulling a rope, a bell would make a rotation of almost 360 degrees with a period of approximately two seconds. The time between two strikes of the same bell could be controlled rather accurately, which made it possible to ring a certain number of bells all after each other and keep repeating this in the same order.

A practised bell ringer could also adjust the rotation period of his bell slightly. Two neighboring bells might be swapped by increasing the rotation period of the first bell and decreasing the rotation period of the second. In this fashion one could change the order in which the bells were rung. In such a transition from one ordering of the bells to the order, each bell could not move more than one place, since the rotation period of a bell could only be adjusted a little bit. Bell ringers became interested in ringing the bells in every possible order after each other, whereby the bells were not allowed to be rung more than once in the same order; that is, they wanted to ring an extent. Thus the art of change ringing was born.

Along with the development of change ringing came the (mathematical) study of ringing patterns, which is sometimes called campanology (from the Latin word campana, which means ‘bell’). Change ringers tried to create methods to ring all possible orderings of the bells in a smart way. An important 17th century study on change ringing was done by Fabian Stedman in his book Campanalogia (1677, see [3]). Initially, most of the campanologists were change ringers, but in the 20th century also mathematicians became interested in this field of study. For an extensive mathematical description of change ringing, the reader may consult [7] and [11]. [2] elaborates on Fabian Stedman as someone using group theoretical tools before the actual development of group theory.

This thesis gives a mathematical analysis of change ringing. In Section 2 we will introduce the important notion of a Cayley graph and prove the existence of an extent for an arbitrary number of bells. A historically interesting question about a method called Grandsire Triples will be answered in Section 3 and Section 4 covers Rankin’s campanological theorem, which is closely related to Section 3. Section 5 will be concerned with the existence and construction of extents using specific methods.
1 Preliminaries

1.1 Change ringing terminology

Suppose we have a set of \( n \in \mathbb{Z}_{>0} \) bells. In the art of change ringing, an ordering of the bells is called a row. There are \( n! \) different rows, each of them corresponding to a specific order in which the bells are rung. More interesting than the rows are the transitions from one row to the other. For technical reasons, a transition is only allowed if each bell moves no more than one place; such a transition is called a change. More formally, rows and changes are defined as follows.

**Definition 1.1.** A row is a bijection \( r : \{1, \ldots, n\} \to \{1, \ldots, n\} \) and a change is a permutation \( \sigma \in S_n \setminus \{(1)\} \) which is the product of disjoint transpositions of the form \((k \ k+1)\).

We view a row as a map that assigns a unique bell to each bell position and a change as a permutation of the set of bell positions. Precomposition defines a right action of \( S_n \) on the set of rows; this action defines the application of a change to a row. Despite of the different interpretation of rows and changes, we sometimes want to treat them both as elements of the same group \( S_n \). From this perspective, applying a change to a row comes down to right multiplication: applying the change \( \sigma \) to the row \( r \) yields the new row \( r\sigma \).

The following notation visualizes the positions of the bells in a row.

**Notation 1.2.** A row \( r \) is denoted as \( r(1)r(2) \ldots r(n) \).

Examples of special rows are \( 12 \ldots n \) (rounds), which is the identity map on \( \{1, \ldots, n\} \), and \( n \ldots 21 \) (back rounds), which is the map \( x \mapsto n + 1 - x \).

Bells that do not move in a change are said to ‘make places’. Given the number of bells \( n \) that is used, a change is determined by its fixed points (the positions in which the bells make places). This observation gives rise to the so-called place notation for changes, which will often be used instead of the usual cycle notation.

**Notation 1.3.** The change with fixed points \( i_1, \ldots, i_k \) is denoted by \( a_{i_1 \ldots i_k} \) and the change without any fixed points is denoted by \( a_x \).

Obviously, not every choice of positions in which the bells make places defines a change. If \( n \) is odd, for example, the change \( a_x \) is ill-defined. In general, the parity of the number of fixed points of a change equals the parity of \( n \).

**Example 1.4.** Consider the following successive rows (for \( n = 6 \) bells), starting with rounds:

\[
\begin{align*}
123456 \\
214365 \\
241356 \\
142356
\end{align*}
\]

Note that the first two transitions \( \sigma_1 = (12)(34)(56) \) and \( \sigma_2 = (23)(56) \) are changes; in place notation we can write \( \sigma_1 = a_x \) and \( \sigma_2 = a_{14} \). The transition
The first row to the third row is described by the identity $r_3 = r_1\sigma_1\sigma_2$. The transition (13) from the third to the fourth row is not a change since the bells in position 1 and 3 move more than one place.

The following relation between the number $c_n$ of possible changes using $n$ bells and the $(n+1)$th Fibonacci number $F_{n+1}$ can be found in [11, Theorem 2.1].

**Theorem 1.5.** For $n \geq 1$ we have $c_n = F_{n+1} - 1$.

**Proof.** We proceed by induction on $n$. Note that $c_1 = 0 = F_2 - 1$ and $c_2 = 1 = F_3 - 1$. Assume that the result is true for all $k \leq n$ for some $n \geq 2$ and consider the number of possible changes $c_{n+1}$ for $n+1$ bells. There are $c_n$ changes fixing position $n+1$ and $c_{n-1}$ changes exchanging the positions $n$ and $n+1$ and at least two other positions. Finally, there is the transposition $(n \, n+1)$. This yields

$$c_{n+1} = c_n + c_{n-1} + 1 = (F_{n+1} - 1) + (F_n - 1) + 1 = F_{n+2} - 1.$$ 

□

The aim of change ringing is to ring all possible rows in a sequence without any repetition. Traditionally one starts and ends with rounds, thus getting a sequence of $n!+1$ rows of which only the first and the last are identical. Such a sequence is called an *extent*. In more mathematical terms it can be defined as follows.

**Definition 1.6.** An extent on $n$ bells is a $n!+1$-tuple $(r_1, \ldots, r_{n!+1}) \in (S_n)^{n!+1}$ satisfying

1. $r_1 = r_{n!+1} = (1)$;
2. $r_i \neq r_j$ for all $1 \leq i < j \leq n!$;
3. There are changes $\sigma_1, \ldots, \sigma_{n!} \in S_n$ such that $r_{i+1} = r_i\sigma_i$ for $i = 1, \ldots, n!$.

A shorter sequence of rows satisfying the above conditions is called a *touch* on $n$ bells.

Ringing an extent on $n$ bells requires remembering a sequence of $n!$ changes, which can be a difficult job, even for relatively small $n$ like 6 or 7 (we have $6! = 720$ and $7! = 5040$). In order to make this easier, change ringers often use a *method*, a sequence of changes, described by a specific algorithm that is relatively simple. A method usually consists of repeating one or more short blocks of changes; a sequence of rows produced by such a block is called a *lead*. The first row of a lead is called the *lead head* and the last row the *lead end*. Typically, the first bell, called the *treble*, is in first position in these rows.

The following lemma will be used several times.

**Lemma 1.7.** Let $n \geq 4$ and define the changes

$$\alpha = \begin{cases} a \times, & \text{if } n \text{ is even} \\ a_n, & \text{if } n \text{ is odd} \end{cases}, \quad \beta = \begin{cases} a_{1n}, & \text{if } n \text{ is even} \\ a_1, & \text{if } n \text{ is odd} \end{cases}$$

for $n$ bells. Then we have $\langle \alpha, \beta \rangle = D_n$. 

5
Proof. We consider the regular $n$-sided polygon with its vertices labelled as $2, 4, \ldots, n, n-1, n-3, \ldots, 1$ for even $n$ and $2, 4, \ldots, n-1, n, n-2, \ldots, 1$ for odd $n$. Then $\alpha$ and $\beta$ are reflections of this polygon. The reflection $\alpha$ and the rotation $\alpha \beta = \{ (12) \cdots (n-1 \ n \ n-2 \cdots 1), \text{ if } n \text{ is even} \}
\{ (12) \cdots (n-2 \ n-1) (23) \cdots (n-1 \ n), \text{ if } n \text{ is odd} \}$
of order $n$ together generate $D_n$. It follows that $\langle \alpha, \beta \rangle = D_n$. \qed

Example 1.8. Consider the method Plain Bob Minimus on 4 bells (see also [2] p. 774), which is shown in Figure 2. A line denotes the end of a lead so that each column corresponds to one lead. The rows of the first column, which are produced by an alternation of the changes $\alpha = a_\times = (12)(34)$ and $\beta = a_{14} = (23)$, make up the dihedral group $D_4 = \langle \alpha, \beta \rangle$ (Lemma 1.7).

The first lead is followed by the change $\gamma = a_{12} = (34)$ and after that the same sequence of alternating $\alpha$’s and $\beta$’s. This means that the second lead is the left coset $bD_4$ where $b = (\alpha \beta)^3 \alpha \gamma = (234)$. Likewise, the third lead is the left coset $b^2 D_4$. So the Plain Bob Minimus method can be seen as the decomposition of $S_4$ into the three left cosets $D_4, bD_4$ and $b^2 D_4$ of $D_4$. Since these cosets form a partition of $S_4$, we know immediately that this is indeed an extent.

Figure 2: Plain Bob Minimus.
1.2 Words

The notion of words will be helpful to describe the succession of changes.

Definition 1.9. Let \( G \) be a group and \( X \subset G \) a subset. A word in \( G \) is a formal expression \( w = g_1 g_2 \ldots g_k \) with letters \( g_1, g_2, \ldots, g_k \in G \). The length of \( w \) is \(|w| = k\) and a word \( u = g_i g_{i+1} \ldots g_j \) with \( 1 \leq i \leq j \leq k \) is called a subword of \( w \). By \( W(G, X) \) we denote the set of words in \( G \), made up of letters from \( X \).

If a word contains repetitions of subwords, we use exponential notation. For example, the word \( g_1 g_1 g_2 g_3 g_2 g_3 g_1 \) can be written as \( g_1^3 (g_2 g_3)^2 g_1 \).

Let \( G \) be a group and \( X \subset G \) a subset. A product operation on \( W(G, X) \) can be defined by concatenation:

\[
g_1 g_2 \ldots g_k \cdot h_1 h_2 \ldots h_l = g_1 g_2 \ldots g_k h_1 h_2 \ldots h_l.
\]

Also we define the evaluation map

\[
\varpi : W(G, X) \to G
\]

\[
g_1 g_2 \ldots g_k \mapsto g_1 g_2 \ldots g_k = g_1 g_2 \ldots g_k.
\]

It is surjective if and only if \( X \) is a generating set for \( G \).

A word \( w \in W(G, X) \) can be considered as the product of its letters. However, we will always separate its letters by dots to avoid confusion with its evaluation \( \varpi \in G \).

Definition 1.10. Let \( G \) be a group with neutral element \( e \). A word \( w \) in \( G \) is called an irreducible identity word if \( \varpi = e \) and \( \varpi \neq e \) for every proper subword \( u \) of \( w \).
The existence of an extent

2.1 Plain changes

An important question one might ask, is whether ringing an extent on \( n \) bells is possible for all \( n \geq 2 \). Notice that there is no extent on 1 bell since the neutral element \( (1) \in S_1 \) is not a change. Obviously, an extent on 2 bells does exist, and as early as 1677, Fabian Stedman demonstrated in [3] pp. 16-20 how to ring an extent on 3, 4 and 5 bells by using a method called plain changes, which is nowadays known as the (Steinhaus–)Johnson–Trotter algorithm (see [4] for S. M. Johnson’s original description of it). The name plain changes indicates that in every change only two bells are exchanged. The treble plain hunts, which means that it occupies the successive positions \( 1, 2, \ldots, n, n, n - 1, \ldots, 1, 1, 2, \ldots \) While the treble is moving, the other bells remain in the same relative order. When the treble makes places, the other bells behave as if they were doing plain changes with one fewer bell (i.e. the second bell plain hunts).

If \( r \) is a row of \( n - 1 \) bells, we define \( b_i = r(i) + 1 \) for \( i = 1, \ldots, n - 1 \). Now for \( j = 1, \ldots, n \) we construct rows

\[ r_j = b_1 b_2 \ldots b_{j-1} b_j b_{j+1} \ldots b_{n-1} \]

of \( n \) bells by ‘inserting’ bell 1 in every possible position. Suppose that an extent of plain changes with \( n - 1 \) bells is given. By replacing each even row \( r \) by the successive rows \( r_1, \ldots, r_n \) and each odd row \( r \) by the successive rows \( r_n, \ldots, r_1 \), we obtain an extent of plain changes for \( n \) bells (for the case \( n = 4 \) this construction is shown in Figure 3). This argument could be turned into a proof that an extent on \( n \) bells exists for all \( n \geq 2 \) by using induction on \( n \). However, in Section 2.3 we choose an alternative route, proving in addition the existence of an extent that uses only three different changes.

\[
\begin{array}{ccc}
234 & 342 & 423 \\
1234 & 1342 & 1423 \\
2134 & 3142 & 4123 \\
2314 & 3412 & 4213 \\
2341 & 3421 & 4231 \\
324 & 432 & 243 \\
3241 & 4321 & 2431 \\
3214 & 4312 & 2413 \\
3124 & 4132 & 2143 \\
312 & 413 & 2143 \\
1324 & 1432 & 1243 \\
234 & 1234 \\
\end{array}
\]

Figure 3: The construction of plain changes with 4 bells out of plain changes with 3 bells. The bell in bold is the inserted treble.

2.2 The Cayley graph

We now introduce the notion of a Cayley graph. It will appear to be closely related to words.
Definition 2.1. Let $G$ be a group and $X \subseteq G$ a subset. The Cayley graph $\Gamma(G, X)$ is the directed graph with vertex set $G$ and arrow set $\{(g, gx) : g \in G, x \in X\}$.

If $X$ only contains elements of order 2 (for example if its elements are changes), there is an arrow from vertex $v$ to vertex $w$ if and only if there is one from $w$ to $v$. In this case we can consider the Cayley graph as an undirected graph.

Remark 2.2. If $G$ is a group and $X \subseteq G$ a subset, the Cayley graph $\Gamma(G, X)$ is connected if and only if $X$ is a generating set of $G$.

Definition 2.3. Let $\Gamma = (V, E)$ be an undirected graph. A $k$-cycle in $\Gamma$ is a $k+1$-tuple $(v_1, v_2, \ldots, v_k)$ with $v_i, v_{i+1} \in V$ all distinct such that $\{v_i, v_{i+1}\} \in E$ for $i = 1, \ldots, k \mod k$. A $k$-cycle in a directed graph is defined analogously.

Definition 2.4. A $\#V$-cycle in a graph $\Gamma = (V, E)$ is called a Hamiltonian cycle. A graph is called Hamiltonian if it contains a Hamiltonian cycle.

The next theorem, which clarifies the relation between extents, words and Cayley graphs, is crucial (see for instance [1] Theorem 2.2] for the equivalence $(1) \iff (3)$).

Theorem 2.5. Let $n$ be a positive integer and $X \subseteq S_n$ a set of changes. The following are equivalent:

1. There is an extent on $n$ bells using only changes from $X$.
2. There is an irreducible identity word $w \in W(S_n, X)$ of length $|w| = n!$.
3. The Cayley graph $\Gamma(S_n, X)$ is Hamiltonian.

Proof. Let $(r_1, \ldots, r_{nl+1})$ be an extent on $n$ bells using changes $g_i = r_i^{-1}r_{i+1} \in X$ for $i = 1, \ldots, nl$. The word $w = g_1g_2\ldots g_{nl} \in W(S_n, X)$ satisfies $w = (1)$ since we have $r_1 = r_{nl+1} = r_1w$. Furthermore, if a proper subword $u = g_{l_1}g_{l_1+1}\ldots g_{m}$ of $w$ satisfied $u = (1)$, then we would have $r_l = r_lw = r_{m+1}$, which is a contradiction. Hence $w$ is an irreducible identity word, which proves $(1) \Rightarrow (2)$. If $w = g_1g_2\ldots g_{nl} \in W(S_n, X)$ is an irreducible identity word, then the subwords $g_1g_2\ldots g_k$ of $w$ for $k = 1, \ldots, n!$ evaluate to $n!$ different elements of $S_n$. Hence $(g_1g_2g_3g_4, g_1g_2\ldots g_{n!})$ is a Hamiltonian cycle in $\Gamma(S_n, X)$, which proves $(2) \Rightarrow (3)$. Finally, the implication $(3) \Rightarrow (1)$ is clear. \hfill \Box

Figure 4: The Cayley graph $\Gamma(S_3, \{(12), (23)\})$. The $(12)$ changes are denoted by solid lines and the $(23)$ changes by dashed lines.
Example 2.6. There are two possible extents on 3 bells. They correspond to the two Hamiltonian cycles in the Cayley graph $\Gamma(S_3, \{(12), (23)\})$ (see Figure 4) and the two irreducible identity words $((12)(23))^3$ and $((23)(12))^3$ of length 3! in $W(S_3, \{(12), (23)\})$.

2.3 Existence of an extent using only three changes

We will now prove the existence of an extent on $n \geq 2$ bells using only three different changes (compare [11] pp. 206–209). Following the proof of [5], we need the lemma below.

Lemma 2.7. Let $\Gamma = (V, E)$ be a connected 3-regular undirected graph. Suppose that there exist a set $C$ of cycles and a set $Q$ of 4-cycles, each partitioning $V$, such that there is no $c \in C$ containing all vertices of some $q \in Q$. Then $\Gamma$ is Hamiltonian.

Proof. We define the following ‘gluing operation’. Suppose we have a 4-cycle $q = (v_1, v_2, v_3, v_4, v_1) \in Q$ and two different cycles $c_1, c_2 \in C$ such that the edge $\{v_1, v_2\}$ is contained in $c_1$ and the edge $\{v_3, v_4\}$ is contained in $c_2$. Then we glue $c_1$ and $c_2$ together by removing the edges $\{v_1, v_2\}$ and $\{v_3, v_4\}$ and adding the edges $\{v_2, v_3\}$ and $\{v_1, v_4\}$, thus making a new cycle $c$. By repeating this gluing operation until it cannot be done anymore, we get a set $C'$ of cycles, still partitioning $V$.

We claim that $C'$ contains only one cycle, which then has to be Hamiltonian. If not, $C'$ must contain two different cycles $c'_1$ and $c'_2$, joined by an edge $e$ that connects the vertices $x_1$ in $c'_1$ and $x_2$ in $c'_2$. By 3-regularity and the assumption that $c'_1$ and $c'_2$ cannot be glued together, it follows that $e$ is not contained in any 4-cycle of $Q$. But then the 4-cycle $q \in Q$ that contains $x_1$ must also contain the two edges $\{a, x_1\}$ and $\{b, x_1\}$ in $c'_1$ incident to $x_1$. Note that the vertices $a, b, x_1$ belong to the same cycle $c_1 \in C$ and to the same 4-cycle $q \in Q$. Therefore, the fourth vertex of $q$ cannot belong to $c_1$. But by 3-regularity it cannot be contained in another cycle $c \in C$ either, which is a contradiction.

![Figure 5: The gluing operation in the proof of Lemma 2.7](image)

Lemma 2.8. For $n \geq 2$ the permutations

$$a = (12), \quad \beta = (12)(34)(56) \cdots, \quad \gamma = (23)(45)(67) \cdots$$

generate $S_n$. 

10
Proof. Since the case $n = 2$ is clear, we assume that $n \geq 3$. Fix $a \in \{2, 3, \ldots, n\}$. It follows from

$$a \beta \gamma = \begin{cases} (2 \ 4 \ 6 \ \ldots \ n \ n - 1 \ n - 3 \ \ldots \ 3), & \text{if } n \text{ is even} \\ (2 \ 4 \ 6 \ \ldots \ n \ n - 1 \ n - 2 \ \ldots \ 3), & \text{if } n \text{ is odd} \end{cases}$$

that we can choose a positive integer $k$ such that $(a \beta \gamma)^k a (a \beta \gamma)^{-k} = (1a)$. Since any transposition $(ab)$ with $a, b \neq 1$ can be written as $(ab) = (1a)(1b)(1a)$ and $S_n$ is generated by its transpositions, this concludes the proof.

**Theorem 2.9.** For $n \geq 4$ the Cayley graph $\Gamma(S_n, \{\alpha, \beta, \gamma\})$ with

$$\alpha = (12), \ \beta = (12)(34)(56) \cdots, \ \gamma = (23)(45)(67) \cdots$$

is Hamiltonian.

Proof. The Cayley graph $\Gamma = \Gamma(S_n, \{\alpha, \beta, \gamma\})$ is 3-regular, and by Lemma 2.8 and Remark 2.2 it is connected. We will show that the other conditions of Lemma 2.7 are satisfied. For any vertex the irreducible identity word $(\alpha \beta)^2$ corresponds to a 4-cycle in $\Gamma$, and if two of such 4-cycles share a common vertex, they are identical. This yields a partition $Q$ of $S_n$ consisting of 4-cycles. Likewise, the irreducible identity word $(\alpha \gamma)^3$ (for $n = 4$) or $(\alpha \gamma)^6$ (for $n > 4$) corresponds to a partition $C$ of $S_n$, consisting of 6-cycles or 12-cycles, respectively. If all vertices of a 4-cycle in $Q$ were contained in some cycle in $C$, we could write $\beta$ as a product of alternating $\alpha$’s and $\gamma$’s. However, this is not possible. Hence we can apply Lemma 2.7 and find that $\Gamma$ is Hamiltonian.

**Corollary 2.10.** An extent on $n$ bells exists for all $n \geq 2$. Moreover, this can be accomplished by using only the changes

$$\alpha = (12), \ \beta = (12)(34)(56) \cdots, \ \gamma = (23)(45)(67) \cdots$$

Proof. The case $n = 2$ is trivial and the case $n = 3$ follows from Example 2.6. For $n \geq 4$, Theorems 2.5 and 2.9 together give the result.
3 Grandsire Triples

3.1 Description and basic properties

We now consider a method on 7 bells called Grandsire Triples (for a more extensive description of Grandsire Triples see [5, pp. 13–22]). Three different lead types occur in this method: plain leads, bob leads and single leads, each consisting of 14 rows. These three lead types are described by the words

\[ L_P = a_3.(a_1.a_7)^6.a_1, \quad L_B = a_3.(a_1.a_7)^5.a_1.a_3.a_1, \quad L_S = a_3.(a_1.a_7)^5.a_1.a_3.a_{123}, \]

respectively. The successive letters of the word that defines a certain lead type are the changes producing a lead of that type, where we read from left to right. Notice that a lead is determined by its lead head and lead type. We will write \( P = \overline{L_P}, B = \overline{L_B} \) and \( S = \overline{L_S} \) for the evaluated values in \( S_7 \) of each lead type.

<table>
<thead>
<tr>
<th>Plain lead</th>
<th>Bob lead</th>
<th>Single lead</th>
</tr>
</thead>
<tbody>
<tr>
<td>1234567</td>
<td>1234567</td>
<td>1234567</td>
</tr>
<tr>
<td>2135476</td>
<td>2135476</td>
<td>2135476</td>
</tr>
<tr>
<td>2314567</td>
<td>2314567</td>
<td>2314567</td>
</tr>
<tr>
<td>3241657</td>
<td>3241657</td>
<td>3241657</td>
</tr>
<tr>
<td>3426175</td>
<td>3426175</td>
<td>3426175</td>
</tr>
<tr>
<td>4362715</td>
<td>4362715</td>
<td>4362715</td>
</tr>
<tr>
<td>4637251</td>
<td>4637251</td>
<td>4637251</td>
</tr>
<tr>
<td>6473521</td>
<td>6473521</td>
<td>6473521</td>
</tr>
<tr>
<td>6745312</td>
<td>6745312</td>
<td>6745312</td>
</tr>
<tr>
<td>7654132</td>
<td>7654132</td>
<td>7654132</td>
</tr>
<tr>
<td>7561423</td>
<td>7561423</td>
<td>7561423</td>
</tr>
<tr>
<td>5716243</td>
<td>5716243</td>
<td>5716243</td>
</tr>
<tr>
<td>5172634</td>
<td>5172634</td>
<td>5172634</td>
</tr>
<tr>
<td>1527364</td>
<td>1576243</td>
<td>1576243</td>
</tr>
<tr>
<td>1253746</td>
<td>1752634</td>
<td>1572634</td>
</tr>
</tbody>
</table>

Figure 6: The plain, bob and single lead of Grandsire Triples starting with rounds.

Change ringers prefer to use plain leads and bob leads and will only use single leads if strictly necessary. Thus the question arises whether ringing an extent of Grandsire Triples is possible by using only plain and bob leads. As we will see in Section 3.2, the answer is negative. Before we come to this, there is some preliminary work to be done.

**Lemma 3.1.** The following identities hold:

\[ P = (35764), \quad B = (274)(356), \quad PB^{-1} = (23467). \]
Proof. We compute
\[
P = a_3(a_1a_7)^6a_1 = (12)(45)(67)(135764)^6(23)(45)(67) = (35764);
B = a_3(a_1a_7)^5a_1a_3a_1
= (274)(356);
PB^{-1} = (35764)((274)(356))^{-1} = (23467).
\]

Lemma 3.2. Let \( r \) be a row of 7 bells and define

\[
n_r = \begin{cases} r^{-1}(1), & \text{if } r \text{ and } r^{-1}(1) \text{ have reverse parity} \\ 15 - r^{-1}(1), & \text{if } r \text{ and } r^{-1}(1) \text{ have equal parity} \end{cases}
\]

Then in any touch of Grandsire Triples using only plain and bob leads in which \( r \) occurs, \( r \) is the \( n_r \)th row of its lead.

Proof. Suppose that we have a touch of Grandsire Triples using only plain and bob leads in which \( r \) occurs as the \( k \)th row of its lead. Since the treble plain hunts, we must have \( k = r^{-1}(1) \) or \( k = 15 - r^{-1}(1) \). Now notice that \( r \) and \( r^{-1}(1) \) have reverse parity if and only if \( k \in \{1, \ldots, 7\} \): every lead head is even with the treble in first position, and the parity of the rows reverses every change, whereas the parity of the treble positions reverses every change except the 7th and 14th one. Therefore we have \( k = n_r \).

Notation 3.3. For \( n \geq 3 \) we let \( G_n \) denote the subgroup of \( S_n \) consisting of all permutations that keep 1 fixed and \( H_n = G_n \cap A_n \) the subgroup of \( S_n \) consisting of all even permutations that keep 1 fixed.

Notice that we have \( G_n \cong S_{n-1} \) and \( H_n \cong A_{n-1} \), where \( A_{n-1} \) is the alternating group on \( n-1 \) elements.

Remark 3.4. In the notation of Lemma 3.2, we have \( H_7 = \{ r \in S_7 : n_r = 1 \} \). So the elements of \( H_7 \) occurring in a touch of Grandsire Triples using only plain and bob leads, are precisely the lead heads occurring in the touch.

Every lead head in a touch of Grandsire Triples using only plain and bob leads is a product of \( P \)'s and \( B \)'s. So if \( \{ P, B \} \) did not even generate \( H_7 \), this would immediately disprove the existence of an extent of Grandsire Triples using only plain and bob leads. However, as the following lemma shows, \( \{ P, B \} \) does generate \( H_7 \) (see [7] Lemma 4.14).

Lemma 3.5. We have \( \langle P, B \rangle = H_7 \).

Proof. The inclusion \( \langle P, B \rangle \subset H_7 \) is clear from Lemma 3.1. It is a well-known fact that \( A_n \) is generated by \{\( (12k) : k = 3, \ldots, n \)\} (see for example [7] Proposition 4.2). So for the inclusion \( H_7 \subset \langle P, B \rangle \) it suffices to show that
\[(23k) \in (P, B) \text{ for } k = 4, \ldots, 7. \] We indeed have
\[(234) = (35764)^{-2}(274)(356) = P^{-2}B;
(235) = (35764)(234)^{-1}(35764)^{-1} = P(234)^{-1}P^{-1};\]
\[(23764) = (235)^{-1}(35764)(235) = (235)^{-1}P(235);\]
\[(237) = (23764)(234)(23764)^{-1};\]
\[(236) = (35764)^{-2}(237)^{-1}(35764)^2 = P^{-2}(237)^{-1}P^2.\]

\[\square\]

3.2 Thompson’s proof

In 1886 W. H. Thompson succeeded in disproving the existence of an extent of Grandsire Triples using only plain and bob leads (see \[\textit{[8]}\]). As T. J. Fletcher remarks in \[\textit{[2]}\, p. 624\], Thompson seems not to be a mathematician, nor does his proof suggest that he was aware of using group theoretical tools. Therefore we will use \[\textit{[6]}\, pp. 624–625\] to present Thompson’s interesting but rather long-winded proof in a streamlined version. The terminology is Thompson’s except Definition \[\textit{3.8}\].

**Definition 3.6.** A round block of length \(k > 0\) is a \(k\)-tuple \((x_1, \ldots, x_k) \in (H_7)^k\) with \(x_i \neq x_j\) for \(1 \leq i < j \leq k\) such that for all \(i = 1, \ldots, k \mod k\) we have either \(x_{i+1} = x_iP\) or \(x_{i+1} = x_iB\). A \(P\)-block is a round block \((x, xP, xP^2, xP^3, xP^4)\) and a \(B\)-block is a round block \((x, xB, xB^2)\).

**Remark 3.7.** Since no product of one or two elements of \(\{P, B\}\) equals the neutral element, there are no round blocks of length 1 and 2. The smallest round block is a \(B\)-block, which has length 3.

**Definition 3.8.** An \(H_7\)-decomposition is a set of round blocks such that every element of \(H_7\) appears exactly once as coefficient in one of the round blocks.

One can think about an \(H_7\)-decomposition as a set of cycles in the Cayley graph \(\Gamma(H_7, \{P, B\})\) that forms a partition of \(H_7\).

**Remark 3.9.** It follows from Remark \[\textit{3.4}\] that the existence of an extent of Grandsire Triples by using only plain and bob leads would imply the existence of a round block of length \(#H_7 = 360\). This is equivalent to the existence of an \(H_7\)-decomposition containing only 1 element, which is again equivalent to the existence of a Hamiltonian cycle in the Cayley graph \(\Gamma(H_7, \{P, B\})\).

**Definition 3.10.** The \(Q\)-set of a row \(x \in H_7\) is the left coset \(x(PB^{-1})\). Given an \(H_7\)-decomposition, a plained \(Q\)-set is a \(Q\)-set of which each element is followed by a plain lead and a bobbed \(Q\)-set is a \(Q\)-set of which each element is followed by a bob lead. We say that a bobbed \(Q\)-set is plained if the bob leads following its elements are replaced by plain leads; bobbing a plained \(Q\)-set is defined analogously.

Notice that a \(Q\)-set contains exactly 5 elements since \(PB^{-1} = (23467)\) has order 5. In Figure \[\textit{2}\] a \(Q\)-set is shown as part of the Cayley graph \(\Gamma(H_7, \{P, B\})\).
Figure 7: In this part of the Cayley graph $\Gamma(H_7, \{P, B\})$ the black vertices form a bobbed Q-set. The solid arrows are part of the cycles that form the $H_7$-decomposition.

**Remark 3.11.** By plaining a bobbed Q-set in an $H_7$-decomposition $R_b$, one obtains a different $H_7$-decomposition $R_p$.

The following observation is essential.

**Lemma 3.12.** Given an $H_7$-decomposition, a Q-set $x(PB^{-1})$ is either plained or bobbed.

*Proof.* Suppose that $x(PB^{-1})^i$ is followed by a plain lead while $x(PB^{-1})^{i+1}$ is followed by a bob lead for some $i = 1, \ldots, 5 \mod 5$. Then the row $x(PB^{-1})^i P = x(PB^{-1})^{i+1} B$ appears more than once in the $H_7$-decomposition, which is a contradiction. \hfill $\Box$

The key ingredient for the proof of the main result of this section is the following lemma.

**Lemma 3.13.** Plaining a bobbed Q-set in an $H_7$-decomposition does not change the parity of the number of round blocks.

Thompson proves Lemma 3.13 by listing the 28 different ways in which the elements of a Q-set can be distributed over round blocks (see [8] pp. 15–17) and even Fletcher says in [9] p. 625] that “it is very difficult to see any means by which this could have been avoided.” However, D. J. Dickinson provided a rather short proof in [10]. We will give a slightly different proof, suggested by H. W. Lenstra, which is perhaps even more elegant. For this we use the following property of the sign map (see [11] p. 25).

**Lemma 3.14.** Let $\sigma$ be a permutation of a finite set $X$, having cycle type $(k_1, \ldots, k_r)$. Then $\text{sgn} (\sigma) = (-1)^{|X|-r}$. 

15
Proof. It follows from \( \sum_{i=1}^{r} k_i = \#X \) that
\[
\text{sgn}(\sigma) = \prod_{i=1}^{r} (-1)^{k_i-1} = (-1)^{\sum_{i=1}^{r} (k_i-1)} = (-1)^{\#X-r}.
\]

\[\square\]

Proof of Lemma 3.13. Let \( R_b = \{(x_{1,1}, \ldots, x_{1,k_1}), \ldots, (x_{r,1}, \ldots, x_{r,k_r})\} \) be an \( H_7 \)-decomposition and \( \sigma_b = (x_{1,1}, \ldots, x_{1,k_1}) \cdots (x_{r,1}, \ldots, x_{r,k_r}) \) the corresponding permutation of \( H_7 \). Suppose that \( X = x(PB^{-1}) \) is a bobbed \( Q \)-set in \( R_b \). Let \( R_p \) be the \( H_7 \)-decomposition obtained by plaining \( X \) and define \( \sigma_p \) analogously to \( \sigma_b \). Writing \( x_i = x(PB^{-1})^i \) for \( i = 1, \ldots, 5 \), we have
\[
\sigma_p(x_i) = x(PB^{-1})^i P = x(PB^{-1})^i B = \sigma_b(x_{i+1})
\]
for \( i = 1, \ldots, 5 \mod 5 \) and \( \sigma_p(x) = \sigma_b(x) \) for \( x \in H_7 \setminus X \); hence we find \( \sigma_p = \sigma_b \circ (x_1 \ldots x_5) \). It follows by Lemma 3.14 that
\[
(-1)^{\#B} \cdot \text{sgn}(\sigma_p) = \text{sgn}(\sigma_b) \cdot \text{sgn}(x_1 \ldots x_5) = (-1)^{\#B} \cdot 1 = (-1)^{\#B}
\]
and thus \( \#B \equiv \#R_p \mod 2 \). \[\square\]

Theorem 3.15. There is no extent of Grandsire Triples using only plain leads and bob leads.

Proof. Suppose there were an extent of Grandsire Triples using only plain leads and bob leads and thus a round block of length 360 (Remark 3.6). Let \( R \) denote the corresponding \( H_7 \)-decomposition and \( T \) the \( H_7 \)-decomposition consisting of only \( B \)-blocks, so that \( \#R = 1 \) and \( \#T = \frac{360}{5} = 120 \). It follows from Lemma 3.12 that \( R \) can be obtained from \( T \) by repeatedly plaining bobbed \( Q \)-sets. However, since the parity of the number of round blocks is invariant under plaining bobbed \( Q \)-sets by Lemma 3.13, this is a contradiction. \[\square\]

3.3 The largest possible touch

Since any \( H_7 \)-decomposition contains at least two round blocks of which the smallest has length at least 3 (Remark 3.7), the length of a round block in an \( H_7 \)-decomposition cannot exceed 357. As early as 1751, more than a century before Thompson’s proof was published, John Holt had attained a touch of this length (see [10]). In his extent of Grandsire Triples, which can be found in [12] part III, p. 73], only the 357th and 360th lead are single leads. Notice that the number of single leads cannot be reduced to one: the lead heads following the first single lead are contained in \( G_7 \setminus H_7 \) instead of \( H_7 \), so that at least one other single lead is needed to get back to rounds.

Is there a round block of length 358 or 359 (and thus a larger touch than Holt’s one)? Such a round block would not fit in an \( H_7 \)-decomposition, but it is not a priori clear that it does not exist. However, as will be shown below, there is no round block of length greater than 357.
Definition 3.16. The extended Q-set of a row $x \in H_7$ is $x\langle PB^{-1} \rangle \cup x\langle PB^{-1}B \rangle$.

Figure 7 might clarify this definition: the extended Q-set of a row $x \in H_7$ consists of 5 elements from $x\langle PB^{-1} \rangle$, forming the (ordinary) Q-set of $x$ (the black vertices), and 5 elements from $x\langle PB^{-1}B \rangle = x\langle PB^{-1}P \rangle$ (the white vertices).

Suppose we have a round block with corresponding cycle $c$ in $\Gamma(H_7, \{P, B\})$. Then $c$ contains as many arrows going into the extended Q-set as arrows going out of it. As a consequence we have the following remark.

Remark 3.17. Let $x \in H_7$ be a row. A round block contains an equal number of elements from $x\langle PB^{-1} \rangle$ and $xB^{-1}$.

Remark 3.18. A row $x \in H_7$ is contained in exactly two extended Q-sets, namely the extended Q-sets of $x$ and $xB^{-1}$.

Theorem 3.19. There is no round block of length greater than 357.

Proof. Let $R$ be a round block. By Theorem 3.15 there is a row $x \in H_7$ that is not contained in $R$. Let

$$
X = x\langle PB^{-1} \rangle \cup x\langle PB^{-1}B \rangle; \\
X' = xB^{-1}\langle PB^{-1} \rangle \cup xB^{-1}\langle PB^{-1}B \rangle = x\left(\langle PB^{-1} \rangle B^{-1} \cup x\langle B^{-1}P \rangle \right);
$$

be the two extended Q-sets in which $x$ is contained. It follows from

$$
\langle PB^{-1} \rangle \cap B^{-1}\langle PB^{-1} \rangle = \emptyset; \\
\langle PB^{-1} \rangle \cap \langle B^{-1}P \rangle = ((23467)) \cap \langle 24675 \rangle = \{1\}; \\
\langle PB^{-1} \rangle B \cap \left(\langle PB^{-1} \rangle B^{-1}\right) = ((23467)) \cap (\langle 24675 \rangle \cap (\langle 23467 \rangle (274)(356) \cap ((\langle 23467 \rangle (274)(356))^{-1} \\
\quad = \{B, (235), (235)(2467), (26)(26)(35764), (235), (35)(2467), (26)(3547)\}^{-1} \\
\quad = \emptyset; \\
\langle PB^{-1} \rangle B \cap B^{-1}\langle PB^{-1}B \rangle = \emptyset;
$$

that $X \cap X' = \{x\}$. By Remark 3.17 there are rows $y \in X \setminus \{x\}$ and $z \in X' \setminus \{x\}$ that are not contained in $R$. It follows from $X \cap X' = \{x\}$ that $y \neq z$. So we have found three distinct rows $x, y, z \in H_7$ that are not contained in $R$. Hence $R$ has length at most 357.

We conclude that Holt’s touch is indeed the largest possible touch of Grand-sire Triples using only plain and bob leads.
4 Rankin’s campanological theorem

4.1 Rankin’s theorem

Thompson’s result concerning Grandsire Triples, which we discussed in Section 3.2, was in 1948 generalized by R. A. Rankin (see [13]). He proved a theorem that provides a tool to show in a more general setting that certain leads can never form an extent. We will follow the proof of [13].

Definition 4.1. Let $X$ be a finite set and $S$ a set of permutations of $X$. An $S$-cycle in $X$ is a nonempty tuple $(x_1, \ldots, x_k)$ with $x_i \in X$ and $x_i \neq x_j$ for $1 \leq i < j \leq k$ such that there exist $\sigma \in S$ with $x_{i+1} = \sigma(x_i)$ for all $i = 1, \ldots, k$ mod $k$.

Lemma 4.2. Let $X$ be a finite set and $S = \{\sigma_1, \sigma_2\}$ a set of permutations of $X$, where $\sigma_1$ and $\sigma_2$ have $a_1$ and $a_2$ orbits, respectively. Suppose that $\rho = \sigma_2^{-1} \sigma_1$ has odd order and that $X$ can be partitioned into $r$ distinct $S$-cycles. Then $r \equiv a_1 \equiv a_2 \mod 2$.

Proof. If $(x_{1,1}, \ldots, x_{1,k_1}), \ldots, (x_{r,1}, \ldots, x_{r,k_r})$ are the $r$ disjoint $S$-cycles partitioning $X$, we define the permutation $\pi = (x_{1,1} \ldots x_{1,k_1}) \cdots (x_{r,1} \ldots x_{r,k_r})$ of $X$. Also define the sets $A_i = \{x \in X : \pi(x) = \sigma_i(x)\}$ for $i = 1, 2$ and $\tau = \sigma_2^{-1} \pi$. Then we have $\tau|A_i = \rho|A_i$ and $\tau|A_2 = \text{id}|A_2$. It follows from $A_1 \cup A_2 = X$ that $A_1$ is stable under $\tau$ and therefore also under $\rho$. So $\tau|A_1 = \rho|A_1$ has odd order since $\rho$ has. Hence $\tau$, being the identity on $X \setminus A_1 \subset A_2$, has odd order as well and we find $\text{sgn}(\sigma_2^{-1})\text{sgn}(\pi) = \text{sgn}(\tau) = 1$. So we get by Lemma 4.1

$(-1)^{|X| - r} = \text{sgn}(\pi) = \text{sgn}(\sigma_2^{-1}) = \text{sgn}(\sigma_2) = (-1)^{|X| - a_2}$

and therefore $r \equiv a_2 \mod 2$. If $\sigma_2^{-1} \sigma_1$ has odd order, then $\sigma_1^{-1} \sigma_2 = (\sigma_2^{-1} \sigma_1)^{-1}$ has odd order as well. So by interchanging the roles of $\sigma_1$ and $\sigma_2$ we find $r \equiv a_1 \mod 2$. \hfill \Box

In Definition 4.1 we can choose $X$ to be the finite group $G$. If $S = \{s_1, s_2\} \subset G$ is a subset, then for $i = 1, 2$ the element $s_i$ induces the permutation $\sigma_i : g \mapsto gs_i$ of $G$. The $\{\sigma_1, \sigma_2\}$-cycles we just call $S$-cycles. For $i = 1, 2$ the orbits of $\sigma_i$ are the left cosets of $\langle s_i \rangle$, so that $\sigma_i$ has $[G : \langle s_i \rangle]$ orbits. Furthermore, $s_1 s_2^{-1}$ and $s_2^{-1} s_1 : g \mapsto gs_1 s_2^{-1}$ have the same order. Thus we find the following result as a consequence of Lemma 4.2

Theorem 4.3 (Rankin, 1948). Let $G$ be a finite group and $S = \{s_1, s_2\} \subset G$ a subset such that $s_1 s_2^{-1}$ has odd order. Suppose that $G$ can be partitioned into $r$ distinct $S$-cycles. Then $r \equiv [G : \langle s_1 \rangle] \equiv [G : \langle s_2 \rangle] \mod 2$. \hfill \Box

4.2 Application to Grandsire Triples

Rankin’s theorem can now be used to easily prove that there is no extent of Grandsire Triples using only plain and bob leads (compare [13, p. 24]).
Alternative proof of Theorem 3.15. Suppose there were an extent of Grandsire Triples using only plain and bob leads. Then we know by Remark 3.4 that the group \( H_7 \) is a \( \{ P, B \} \)-cycle; or in terms of Theorem 4.3, \( r = 1 \). Furthermore, by Lemma 3.1, \( PB^{-1} = (23467) \) has odd order. Therefore

\[
[H_7 : \langle B \rangle] = \frac{\#H_7}{\#(B)} = \frac{\#A_6}{\#((274)(356))} = \frac{360}{3} = 120 \not\equiv 1 \mod 2
\]

is a contradiction by Theorem 4.3.

Let us compare Thompson’s original proof of Theorem 3.15 with the proof using Rankin’s theorem. The notion of a round block used in Thompson’s proof appears in generalized form as a \( \{ P, B \} \)-cycle in Rankin’s proof, and an \( H_7 \)-decomposition is just a partition of \( H_7 \) into \( \{ P, B \} \)-cycles. Thompson’s key observation, Lemma 3.13, yields that the parity of the number of round blocks in the \( H_7 \)-decomposition of B-blocks equals that of an arbitrary \( H_7 \)-decomposition. This also follows from Theorem 4.3, which says that the parity of the number \( [H_7 : \langle B \rangle] \) of \( \{ B \} \)-cycles equals the parity of the number of \( \{ P, B \} \)-cycles in an arbitrary partition of \( H_7 \) into \( \{ P, B \} \)-cycles.

4.3 Application to Double Norwich Court Bob Major

In [13] pp. 24–25] Rankin applies his theorem to several other methods, showing that there is no extent using plain and bob leads only. As an example, we consider the method on 8 bells called Double Norwich Court Bob Major (for a more extensive description see [6] pp. 41–48]. Its plain and bob leads consist of 16 rows and are produced by the words

\[
L_P = a \cdot a_{14} \cdot a_{36} \cdot a_{58} \cdot a_{18} \cdot a_{36} \cdot a_{58} \cdot a_{14} \cdot a_{18};
\]

\[
L_B = a \cdot a_{14} \cdot a_{36} \cdot a_{58} \cdot a_{18} \cdot a_{36} \cdot a_{58} \cdot a_{14} \cdot a_{16}.
\]

It can be checked that

\[
P = \overline{L_P} = (2836547), \ B = \overline{L_B} = (2854736), \ PB^{-1} = (253),
\]

and we have the following analogue of Lemma 3.2.

Lemma 4.4. Let \( r \) be a row of 8 bells and define

\[
n_r = \begin{cases} 
  r^{-1}(1), & \text{if } r^{-1}(1) \equiv 1, 2 \mod 4 \text{ and } r \text{ is even} \\
  r^{-1}(1), & \text{if } r^{-1}(1) \equiv 0, 3 \mod 4 \text{ and } r \text{ is odd} \\
  17 - r^{-1}(1), & \text{else}
\end{cases}
\]

Then in any touch of Double Norwich Court Bob Major using only plain and bob leads in which \( r \) occurs, \( r \) is the \( n_r \)th row of its lead.

Proof. Suppose that we have a touch of Double Norwich Court Bob Major using only plain and bob leads in which \( r \) occurs as the \( k \)th row of its lead. Since the treble plain hunts, we must have \( k = r^{-1}(1) \) or \( k = 17 - r^{-1}(1) \). If \( r^{-1}(1) \equiv 1, 2 \mod 4 \) then \( r \) is even if and only if \( k \in \{1, \ldots, 8\} \), and if \( r^{-1}(1) \equiv 0, 3 \mod 4 \) then \( r \) is odd if and only if \( k \in \{1, \ldots, 8\} \). Therefore we have \( k = n_r \). □
Like in Remark 3.4 we have $H_8 = \{ r \in S_8 : n_r = 1 \}$, and analogously to the case of Grandsire Triples it follows that the existence of an extent using only plain and bob leads would imply that $H_8$ is an $\{P,B\}$-cycle. Therefore, noticing that

$$[H_8 : \langle B \rangle] = \frac{\#H_8}{\#(B)} = \frac{\#A_7}{\#(2854736)} = \frac{2520}{7} = 360 \not\equiv 1 \mod 2$$

and that $PB^{-1} = (253)$ has odd order gives a contradiction. Thus we find the following result.

**Theorem 4.5.** There is no extent of Double Norwich Court Bob Major using only plain leads and bob leads.
5 The existence and construction of extents

5.1 Extent existence theorems

In the previous sections we have mainly focused on how to show that certain lead types cannot produce an extent. Now we will study some cases in which it is possible to prove the existence of an extent or even to construct one. By Corollary 2.10 we know that an extent on \( n \) bells exists for all \( n \geq 2 \), but the extent that can be derived from the proof is far too unstructured to be used for real change ringing. It would be better to construct extents divided into leads. Sometimes this can be done by brute force, but hopefully the use of mathematical tools can make things easier. This section, which is mainly based on [2] pp. 735–743, describes some of these tools.

Definition 5.1. Let \( G \) be a group and \( H \) a subgroup. A right transversal for \( H \) in \( G \) is a subset \( S \subset G \) containing exactly one element of every right coset of \( H \). A left transversal for \( H \) in \( G \) is defined analogously.

The following theorem gives for two special lead types a necessary and sufficient condition for the existence of an extent using only leads of these two types.

Theorem 5.2. Let \( L_y = x_1x_2\ldots x_ky \) and \( L_z = x_1x_2\ldots x_kz \) be two lead types for \( n \) bells and suppose that the leads of type \( L_y \) and \( L_z \) are right transversals for \( H = \langle \{L_y, L_z\} \rangle \) in \( S_n \). Then there is an extent on \( n \) bells using only lead types \( L_y \) and \( L_z \) if and only if \( \Gamma(H, \{L_y, L_z\}) \) is Hamiltonian.

Proof. Define \( w_i = x_1x_2\ldots x_{i-1} \) for \( i = 2, \ldots, k+1 \) and \( w_1 = (1) \). Then the right cosets \( Hw_1, \ldots, Hw_{k+1} \) of \( H \) form a partition of \( S_n \) and a row \( r \in Hw_i \) occurring in a touch using only lead types \( L_y \) and \( L_z \) must be the \( i \)th row of its lead. Let \( rw_i \) and \( sw_i \) be rows occurring in leads of type \( L_y \) or \( L_z \) with different lead heads \( r, s \in H \), respectively. In the case \( i = j \) we obviously have \( rw_i \neq sw_j \). Otherwise, \( Hw_i \) and \( Hw_j \) are disjoint and it follows from \( rw_i \in Hw_i \) and \( sw_j \in Hw_j \) that \( rw_i \neq sw_j \). Hence leads with different lead heads are disjoint.

Suppose there is an extent on \( n \) bells using only lead types \( L_y \) and \( L_z \). Since the lead heads are precisely the elements of \( H = Hw_1 \), this implies the existence of a Hamiltonian cycle in \( \Gamma(H, \{L_y, L_z\}) \). Conversely, assume that there is a Hamiltonian cycle in \( \Gamma(H, \{L_y, L_z\}) \). Since leads with different lead heads are disjoint, this yields a touch on \( n \) bells using only lead types \( L_y \) and \( L_z \). The touch visits \( \#H \) different elements of every right coset of \( H \) and is therefore an extent.

The proof of the following lemma is similar to the proof of Lemma 2.7.

Lemma 5.3. Let \( G \) be a finite group containing elements \( g, h \in G \) such that \( gh^{-1} = hg^{-1} \) and \( \langle g, h \rangle = G \). Then \( \Gamma(G, \{g, h\}) \) is Hamiltonian.

Proof. Suppose that \( g \) is of order \( k \). The irreducible identity word \( g^k \) yields a partition \( C \) of \( G \) consisting of \( k \)-cycles in \( \Gamma = \Gamma(G, \{g, h\}) \). It follows from
$(g,h) = G$ that $\Gamma$ is connected. Therefore, every two distinct cycles $c_1, c_2 \in C$ must be connected by an arrow $(v,vh)$ from $c_1$ to $c_2$. It follows from $vgh^{-1} = vgh^{-1}$ that the cycles $c_1$ and $c_2$ can be glued together by removing the arrows $(v,vg)$ and $(vgh^{-1},vh)$ and adding the arrows $(v,vh)$ and $(vgh^{-1},vg)$, thus making a new cycle $c$. By repeating this gluing operation until it cannot be done any more, we get a set $C'$ of cycles, still partitioning $G$.

We claim that $C'$ contains only one cycle, which then has to be Hamiltonian. If not, there must be two distinct cycles $c'_1, c'_2 \in C'$ and an arrow $(v',v'h)$ from $c'_1$ to $c'_2$. Notice that the arrows $(v',v'h)$ and $(v'hg^{-1},v'g)$ are not contained in $c'_1$ and $c'_2$ whereas $(v',v'g)$ is contained in $c'_1$ and $(v'hg^{-1},v'h) in c'_2$. Therefore, $c'_1$ and $c'_2$ can be glued together as before, which is a contradiction.

![Figure 8: The situation in the proof of Lemma 5.3](image)

**Theorem 5.4.** Let $L_y = x_1x_2\ldots x_ky$ and $L_z = x_1x_2\ldots x_kz$ be two lead types for $n$ bells and suppose that the leads of type $L_y$ and $L_z$ are right transversals for $H = \langle L_y, L_z \rangle$ in $S_n$. If $y$ and $z$ satisfy $yz = zy$, then there is an extent on $n$ bells using only lead types $L_y$ and $L_z$.

**Proof.** It follows from $yz = zy$ and the fact that every change is of order 2 that

$\langle L_y, L_z \rangle^{-1} = (x_1x_2\ldots x_ky)(zx_k\ldots x_2x_1) = (x_1x_2\ldots x_kz)(yx_k\ldots x_2x_1) = L_z L_y^{-1}$.

By Lemma 5.3 we find that $\Gamma(H, \langle L_y, L_z \rangle)$ is Hamiltonian, so that Theorem 5.2 gives the result.

We find the following corollary (compare [7, Theorem 4.11]).

**Corollary 5.5.** Let $n > 0$ be an integer such that $n \equiv 0 \mod 4$ or $n \equiv 3 \mod 4$ and let $a, b, \gamma, \delta$ be changes of $n$ bells such that $\langle a, b \rangle = D_n$ and $\gamma \delta = \delta \gamma$. Suppose that the two lead types $L_\gamma = (a,b)^n-1.\gamma$ and $L_\delta = (a,b)^n-1.\delta$ satisfy $\langle L_\gamma, L_\delta \rangle = H_n$. Then there is an extent on $n$ bells using only lead types $L_\gamma$ and $L_\delta$.

**Proof.** By Theorem 5.4 it suffices to prove that leads of type $L_\gamma$ and $L_\delta$ are right transversals for $H_n$ in $S_n$. Notice that a lead of type $L_\gamma$ or $L_\delta$ with lead head $x \in H_n$ equals the left coset $xD_n = x\{1, a, a\beta, \ldots, (a\beta)^{n-1}a\}$ of $D_n$. Suppose that two rows $xr$ and $xs$ of this lead with $r, s \in D_n$ are both contained in the same right coset of $H_n$. It follows from $H_n r = H_n s$ that $rs^{-1} \in H_n \cap D_n$. Since we have $n \equiv 0 \mod 4$ or $n \equiv 3 \mod 4$, the neutral element $(1)$ is the only even element of $D_n$ that fixes 1. Therefore, we find $rs^{-1} = (1)$ and $xr = xs$. 

22
Hence different rows of the same lead are contained in different right cosets of $H_n$. Since a lead has $2n = #S_n/#A_{n-1} = [S_n : H_n]$ rows, this concludes the proof.

5.2 Existence of Plain Bob Major extent with special bob leads

Let us try to find changes $\alpha, \beta, \gamma, \delta$ for 8 bells such that the conditions of Corollary 5.5 are fulfilled. We follow the reasoning of [7, pp. 736–737]. If we take $\alpha = a_x$ and $\beta = a_{18}$, then we know by Lemma 1.7 that $\langle \alpha, \beta \rangle = D_8$. Inspired by Plain Bob Minimus (see Example 1.8), we take $\gamma = a_{12}$. Finally, $\delta = (78)$ might be a good guess since we have $\gamma \delta = (34)(56) = \delta \gamma$. Thus we get lead types $L_{\gamma} = (a.\beta)^2.\alpha.\gamma$ and $L_{\delta} = (a.\beta)^2.\alpha.\delta$. The word $L_{\gamma}$ produces the plain leads of Plain Bob Major. However, for obtaining the usual bob leads of Plain Bob Major, $\delta = (78)$ in $L_{\delta}$ should be replaced by $a_{14}$. So the method under examination is a variation of Plain Bob Major. Define the evaluations $X = L_{\gamma}$ and $Y = L_{\delta}$ of $L_{\gamma}$ and $L_{\delta}$.

Lemma 5.6. We have $X = (3578642)$, $Y = (23)(45)(678)$ and $\langle X, Y \rangle = H_8$.

Proof. We have

\[
X = (a\beta)^2\alpha\gamma = (a\beta)^8\beta\gamma = \beta\gamma = (23)(45)(67)(34)(56)(78) = (3578642); \\
Y = (a\beta)^2\alpha\delta = (a\beta)^8\beta\delta = \beta\delta = (23)(45)(67)(78) = (23)(45)(678);
\]

and therefore $\langle X, Y \rangle \subset H_8$. Since $A_8$ is generated by $\{ (12k) : k = 3, \ldots, n \}$ (see for example [7, Proposition 4.2]), $H_8$ is generated by $\{ (68k) : k = 2, 3, 4, 5, 7 \}$. These elements can be obtained by conjugating $Y^2 = (687)$ by powers of $XY^2 = (23574)$, which shows that $H_8 \subset \langle X, Y \rangle$.

Theorem 5.7. Define the changes $\alpha = a_x$, $\beta = a_{18}$, $\gamma = a_{12}$ and $\delta = (78)$ for 8 bells. There is an extent on 8 bells using only lead types $L_{\gamma} = (a.\beta)^n.\alpha.\gamma$ and $L_{\delta} = (a.\beta)^n.\alpha.\delta$.

Proof. By Lemmas 5.6 and 1.7, and the observations $8 \equiv 0 \mod 4$ and $\gamma\delta = \delta\gamma$ the conditions of Corollary 5.5 are satisfied.

5.3 Extent construction of Plain Bob Doubles

Example 5.8. Reconsider the method Plain Bob Minimus, which we have discussed in Example 1.8. By taking $\delta = \gamma$ we have only one lead type $L_{\gamma} = L_{\delta} = (a.\beta)^3.\alpha.\gamma$, which satisfies $\langle L_{\gamma} \rangle = \langle (234) \rangle = H_4$. Also we have $\langle \alpha, \beta \rangle = D_4$ and $\gamma\delta = \gamma^2 = \delta\gamma$. Therefore, Corollary 5.5 guarantees that the word $L_{\gamma}^2 = ((a.\beta)^3.\alpha.\gamma)^2$ corresponds to an extent. Put the rows of Figure 2 (except for the last one) in a matrix $R = (r_{ij})_{1 \leq i \leq 8, 1 \leq j \leq 3}$. The columns of $R$ are the left cosets of $D_4$ and the rows of $R$ are the right cosets of $H_4$. By considering the 6 half leads in Figure 2 we can find a third coset decomposition of $S_4$. Each half lead is a right transversal for $G_4 \cong S_3$ in $S_4$. Notice that the lead heads and lead ends belong to $G_4$ and that the subword
\(w = (a, b)^2.a\) of \(L_r\) is palindromic. Therefore, the 4 right cosets of \(G_4\) are the sets \(\{r_{kj} : j = 1, 2, 3\text{ and } k = i, 9 - i\}\) for \(i = 1, 2, 3, 4\).

Unfortunately, there are many cases in which Theorem 5.4 cannot be applied. However, sometimes the third coset decomposition described in Example 5.8 helps us to find an extent. We illustrate this by explicitly constructing an extent of Plain Bob Doubles (see [2] pp. 730–732). The plain and bob leads of this method on 5 bells are produced by the words \(L_p = (as, a_1)^4.a_5.a_{125}\) and \(L_b = (as, a_1)^4.a_4.a_{145}\), respectively. We have \(P = L_p = (2354)\) and \(B = L_b = (45)\). Each lead consists of 10 rows and is a left coset of \(D_5\) (Lemma 1.7); see also Examples 4.8 and 5.8. Since \(P, B, a_{125}, a_{145}\) are all odd permutations fixing 1, both the lead heads and lead ends are contained in \(G_5\), but not necessarily in \(H_5\).

**Remark 5.9.** Let \(r \in S_{12}\) be a row occurring in a touch of Plain Bob Doubles using only plain and bob leads. Since the treble plain hunts, \(r\) is either the \(r^{-1}(1)\)th or the \((11 - r^{-1}(1))\)th row of its lead.

**Theorem 5.10.** Let \((h_1, h_2, \ldots, h_{12}, h_1)\) be a cycle in \(\Gamma(G_5, \{P, B\})\) such that \(h_1 = (1)\) and \(h_i \neq h_j(P^2B)^2\) for all \(i, j \in \{1, \ldots, 12\}\). Then the succession of plain and bob leads with lead heads \(h_1, \ldots, h_{12}\) forms an extent of Plain Bob Doubles.

**Proof.** Denote the \(i\)th row of the \(j\)th lead by \(r_{ij}\) for \(i = 1, \ldots, 10\) and \(j = 1, \ldots, 12\). It suffices to show that \(r_{ij} \neq r_{k,l}\) if \((i, j) \neq (k, l)\). Assume to the contrary that \(r_{ij} = r_{k,l}\) for some \((i, j) \neq (k, l)\). Since the elements of a lead are all distinct, we have \(j \neq l\). If \(i = k\) then the lead heads \(h_j = r_{1,j}\) and \(h_l = r_{1,j}\) would be identical, which is a contradiction. So we have \(i \neq k\) and by Remark 5.9 it follows that \(k = 11 - i\). Since the subword \(w = (as, a_1)^4.a_5\) of \(L_p\) and \(L_b\) is palindromic, it follows that the lead head \(h_l = r_{1,j}\) of the lead containing \(r_{1,j}\) equals the lead end \(r_{10,j}\) of the lead containing \(r_{k,l}\). However, this is a contradiction:

\[h_j = r_{10,j} = r_{1,j} \overline{w} = h_j(23)(45) = h_j(P^2B)^2.\]

We find that \(r_{ij} \neq r_{k,l}\) if \((i, j) \neq (k, l)\), which concludes the proof.

An extent of Plain Bob Doubles can now be constructed as follows. Write \(G_5 = \{x_1, x_2, \ldots, x_{12}; y_1, y_2, \ldots, y_{12}\}\) such that \(x_1 = (1)\) and \(x_i = y_i(P^2B)^2\) for \(i = 1, \ldots, 12\). Since \((P^2B)^2 = (23)(45)\) is of order 2, we also have \(y_i = x_i(P^2B)^2\) for \(i = 1, \ldots, 12\). By Theorem 5.10 it is now sufficient to find a cycle \((h_1, h_2, \ldots, h_{12}, h_1)\) in \(\Gamma(G_5, \{P, B\})\) with the property that for all \(i \in \{1, \ldots, 12\}\), there is exactly one \(j \in \{1, \ldots, 12\}\) such that \(h_j \in \{x_i, y_i\}\). If we look at Figure 9 (see also [2] Figure 4.1), this is an easy job. Examples are the cycles given by the words \(w_1 = (P^3B)^3\) and \(w_2 = B(P^3B)^2.P^3\), beginning at \(x_1 = (1)\). The words \(w_1, w_2 \in W(G_5, \{P, B\})\) are irreducible identity words of length 12, not containing a subword \(u\) with \(u = (P^2B)^2\), which is another way of formulating the condition in Theorem 5.10 on the 12-cycle in \(\Gamma(G_5, \{P, B\})\).
Figure 9: The Cayley graph $\Gamma(G_5, \{P, B\})$. The arrows correspond to the plain leads and the undirected edges to the bob leads (since $B = (45)$ has order 2).
References


