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A Categorical Approach to Varieties over k

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Contents

1 Introduction

Affine varieties form the building blocks of algebraic geometry. In this thesis we will define the category of affine varieties as a specific category that is anti-equivalent to the category of finitely generated k-algebras that are also an integral domain. Then we will compare the affine varieties obtained this way with the usual definition of affine varieties. We will also see that we can fix the topology on an affine variety using categorical notions. We will give a description of the construction of varieties out of affine varieties. Again we will see that we can view the topology on a variety in categorical terms.

2 The category of affine varieties

We will introduce the category of affine varieties as the essential image of some functor from the category of finitely generated k-algebras that are also an integral domain to the category of locally ringed spaces.

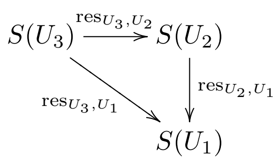
2.1 Sheaves

A locally ringed space is a topological space together with a sheaf of rings on this topological space satisfying some extra conditions. We will now look at what sheaves are.

We will work exclusively with categories in which all objects are sets (sometimes with a specific structure) and all morphisms are functions between these objects. (As opposed to for example the category in which the objects are vertices of some directed graph and the morphisms between two vertices v_1 and v_2 are the paths from v_1 to v_2.)

Definition 2.1 (Presheaf)**.** Let X be a topological space. Let C be a category. A presheaf S on X with values in C consists of the following data:

- for all open $U \subset X$ an object $S(U)$ of C
- and for all pairs of open $U_1, U_2 \subset X$ with $U_1 \subset U_2$ a (restriction) map $\mathrm{res}_{U_2,U_1} : S(U_2) \to S(U_1)$ that is a morphism of C such that
  - $\mathrm{res}_{U,U} = \mathrm{id}_{S(U)}$ for all open $U \subset X$
  - and such that for open $U_1, U_2, U_3 \subset X$ with $U_1 \subset U_2 \subset U_3$ the following diagram commutes



We can view presheaves as functors between categories, which are defined as follows.

Definition 2.2 (Co(ntra)variant functor)**.** A co(ntra)variant functor $F \colon C \to D$ with C and D categories is a mapping that

- associates to each object X of C an object $F(X)$ of D and
- associates to each morphism $m \colon X \to Y$ in C a morphism $F(m) \colon F(X) \to F(Y)$ in the covariant case and a morphism $F(m) \colon F(Y) \to F(X)$ in the contravariant case such that
  - $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all objects X of C
  - and for all morphisms $f \colon X \to Y$ and $g \colon Y \to Z$ we have $F(g \circ f) = F(g) \circ F(f)$ in the case of a covariant functor and $F(g \circ f) = F(f) \circ F(g)$ in the case of a contravariant functor

For a topological space X we can create a category OpX by taking the open subsets of X as the objects and the inclusions between these open sets as the morphisms. Define the category Sets as the category with as objects sets and as morphisms functions between sets. We can view a presheaf S on X as a contravariant functor from OpX to Sets by sending an open U to $S(U)$ and an inclusion $U \hookrightarrow V$ between opens U and V to the restriction map $\mathrm{res}_{V,U}$. However we will not be using this.

Definition 2.3 (Morphism of presheaves)**.** Let S_1 and S_2 be two presheaves on X with restriction maps denoted by respectively res and res$'$. A morphism $\phi\colon S_1 \to S_2$ between these two presheaves is a collection of maps $\{\phi(U)\colon S_1(U) \to S_2(U)\}_{U\subset X \text{ open}}$ such that for all pairs of open $U_1, U_2 \subset X$ with $U_1 \subset U_2$ the following diagram commutes

$$\begin{array}{ccc} S_1(U_2) & \xrightarrow{\phi(U_2)} & S_2(U_2) \\ {\scriptstyle \mathrm{res}_{U_2,U_1}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}'_{U_2,U_1}} \\ S_1(U_1) & \xrightarrow[\phi(U_1)]{} & S_2(U_1) \end{array}$$

Again we can view this as a categorical concept. If we view presheaves as functors we can view a map between presheaves as a natural transformation between functors.

Definition 2.4 (Natural transformation of co(ntra)variant functors)**.** If F and G are functors from the category C to D a natural transformation T from F to G associates to every object X of C a morphism $\alpha_X\colon F(X) \to G(X)$ between objects of D such that for every morphism $m\colon X \to Y$ in C in the case that F and G are covariant it holds that $\alpha_Y \circ F(m) = G(m) \circ \alpha_X$ and in the case that F and G are contravariant it holds that $\alpha_X \circ F(m) = G(m) \circ \alpha_Y$.

Definition 2.5 (Sheaf)**.** Let X be a topological space. Let C be a category. A sheaf on X is a presheaf S on X with values in C such that for every collection $\{U_i\}_{i\in I}$ of open sets in X with $U = \cup_{i\in I} U_i$ the following diagram is exact

$$S(U) \xrightarrow{\prod \mathrm{res}_{U,U_i}} \prod_{i\in I} S(U_i) \underset{\prod \mathrm{res}_{U_j,U_i\cap U_j}}{\overset{\prod \mathrm{res}_{U_i,U_i\cap U_j}}{\rightrightarrows}} \prod_{(i,j)\in I\times I} S(U_i \cap U_j)$$

by which we mean that

- the map $\prod \mathrm{res}_{U,U_i}$ is injective

- and $\mathrm{Im}(\prod \mathrm{res}_{U,U_i})$ is the set on which $\prod \mathrm{res}_{U_i,U_i\cap U_j}$ and $\prod \mathrm{res}_{U_j,U_i\cap U_j}$ agree

Suppose equalizers exist in C. Then by the second condition $(S(U), \prod \mathrm{res}_{U,U_i})$ is the equalizer of $\prod \mathrm{res}_{U_i,U_i\cap U_j}$ and $\prod \mathrm{res}_{U_j,U_i\cap U_j}$.

Definition 2.6 (Equalizer)**.** Let C be a category, A and B objects in C and $f, g\colon X \to Y$ morphisms in C. The equalizer of f and g consists of an object E and a morphism $q\colon E \to X$ satisfying $f \circ q = g \circ q$, and such that, given any object T and morphism $m\colon T \to X$, if $f \circ m = g \circ m$, then there exists a unique morphism $u\colon T \to E$ such that $q \circ u = m$.

We can interpret the definition of a sheaf as follows.
Let $\{U_i\}_{i\in I}$ be a collection of open sets in X with $U = \cup_{i\in I} U_i$.
Then the condition that $\prod \mathrm{res}_{U,U_i}$ is injective ensures that if for $\sigma_1, \sigma_2 \in S(U)$ it holds that for all $i \in I$ we have $\mathrm{res}_{U,U_i}(\sigma_1) = \mathrm{res}_{U,U_i}(\sigma_2)$ then $\sigma_1 = \sigma_2$. So every $\sigma \in S(U)$ is uniquely determined by local data.
The second condition ensures that if the coordinates of a tuple $(\sigma_i)_{i\in I} \in \prod_{i\in I} S(U_i)$ agree on all intersections i.e. for every $i, j \in I$ it holds that $\mathrm{res}_{U_i,U_i\cap U_j}(\sigma_i) = \mathrm{res}_{U_j,U_i\cap U_j}(\sigma_j)$ then there exists a $\sigma \in S(U)$ such that $\prod \mathrm{res}_{U,U_i}(\sigma) = (\sigma_i)_{i\in I}$. This is called "the gluing property".
The above can be found in [6] paragraph 1.4.

Example 2.7. Let X be a topological space and S a presheaf on X such that $S(U)$ is an abelian group and $\mathrm{res}_{V,U}$ is a group homomorphism for all open $U, V \subset X$ with $U \subset V$. Then by definition S is a sheaf if and only if for every collection $\{U_i\}_{i\in I}$ of open sets in X with $U = \cup_{i\in I} U_i$ the following diagram is exact

$$S(U) \xrightarrow{\prod \mathrm{res}_{U,U_i}} \prod_{i\in I} S(U_i) \underset{\prod \mathrm{res}_{U_j,U_i\cap U_j}}{\overset{\prod \mathrm{res}_{U_i,U_i\cap U_j}}{\rightrightarrows}} \prod_{(i,j)\in I\times I} S(U_i \cap U_j)$$

Define a map δ as follows

$$\delta\colon \prod_{i\in I} S(U_i) \longrightarrow \prod_{(i,j)\in I\times I} S(U_i\cap U_j)$$
$$(\sigma_i)_{i\in I} \longmapsto (\mathrm{res}_{U_i,U_i\cap U_j}(\sigma_i) - \mathrm{res}_{U_j,U_i\cap U_j}(\sigma_j)).$$

Then we can reformulate the above condition as follows: for every collection $\{U_i\}_{i\in I}$ of open sets in X with $U = \cup_{i\in I} U_i$ it must hold that $\prod \mathrm{res}_{U,U_i}$ is injective and that

$$\mathrm{Im}(\prod \mathrm{res}_{U,U_i}) = \ker(\delta).$$

When one has a sheaf S on a topological space X and there is a continuous map between topological spaces X and Y the following proposition gives a method to interpret S as a sheaf on Y.

Proposition 2.8. *Let X and Y be topological spaces, and $f\colon X \to Y$ a continuous map. Let S be a sheaf on X. Define f_*S as the presheaf with for all open $U \subset Y$ a set*

$$f_*S(U) := S(f^{-1}(U))$$

and for two open U, V with $U \subset V$ a map defined by the restriction map $\mathrm{res}_{f^{-1}(V),f^{-1}(U)}$ of S. This presheaf is a sheaf on Y.

Proof. For two opens $V \subset U$ in Y we have $f^{-1}(V) \subset f^{-1}(U)$. Hence we see that f_*S indeed forms a presheaf on Y.

Take an open $U \subset Y$. Take a collection $\{U_i\}_{i\in I}$ of open sets in Y with $U = \cup_{i\in I} U_i$. Because f is continuous, $f^{-1}(U)$ is open and $f^{-1}(U_i)$ is open for every $i \in I$. It holds that $f^{-1}(\cup_{i\in I} U_i) = \cup_{i\in I} f^{-1}(U_i)$, so $f^{-1}(U) = \cup_{i\in I} f^{-1}(U_i)$. Hence $\{f^{-1}(U_i)\}_{i\in I}$ is a collection of opens with $f^{-1}(U) = \cup_{i\in I} f^{-1}(U_i)$.

Because S is a sheaf we now know that $\prod \mathrm{res}_{f^{-1}(U),f^{-1}(U_i)}$ is injective and that the set on which $\prod \mathrm{res}_{f^{-1}(U_i),f^{-1}(U_i)\cap f^{-1}(U_j)}$ and $\prod \mathrm{res}_{f^{-1}(U_j),f^{-1}(U_i)\cap f^{-1}(U_j)}$ agree is $\mathrm{Im}(\prod \mathrm{res}_{f^{-1}(U),f^{-1}(U_i)})$.

We find that f_*S is a sheaf. □

Before we introduce the notions of stalks we first need some definitions.

Definition 2.9 (Directed set)**.** A directed set is a non-empty set I together with a reflexive and transitive relation $\leqslant$ with the additional property that for $i, j \in I$ there exists a $k \in I$ such that $i \leqslant k$ and $j \leqslant k$.

Definition 2.10 (Direct system over a directed set)**.** Let $\{A_i | i \in I\}$ be a family of sets indexed by a directed set I and for all $i, j \in I$, with $i \leqslant j$ let there be given a morphism $f_{ij}\colon A_i \to A_j$ such that

- $f_{ii} = \mathrm{id}_{A_i}$ for all $i \in I$ and

- $f_{ik} = f_{jk} \circ f_{ij}$ for all $i, j, k \in I$ with $i \leqslant j \leqslant k$.

Then the pair $(\{A_i\}_{i \in I}, \{f_{ij}\}_{i,j \in I, i \leq j})$ is called a direct system over I.

Definition 2.11 (Direct limit of a direct system)**.** Take a direct system $(\{A_i\}_{i \in I}, \{f_{ij}\}_{i,j \in I, i \leq j})$ over I. Define a relation $\sim$ on $\bigsqcup_i A_i$ as follows: for $(x_i, i), (x_j, j) \in \bigsqcup_i A_i$ we say that $(x_i, i) \sim (x_j, j)$ if there exists some $k \in I$ with $i \leq k$ nd $j \leq k$ such that $f_{ik}(x_i) = f_{jk}(x_j)$. The direct limit is defined as

$$\varinjlim A_i = \bigsqcup_i A_i / \sim .$$

Let X be a topological space and S a sheaf on this space. Let $p \in X$ be a point. The set of all opens that contain p form a directed set when we use the reflexive and transitive relation $\leqslant$ defined by $U \leqslant V$ if and only if $U \supset V$.

The conditions on the restriction maps of S required for S to be a presheaf coincide with the conditions on the morphisms in the definition of a direct system. We find that the collection of sets $S(U)$ and maps $\mathrm{res}_{U,V}$ for all open U and V containing p, form a direct system. We define a relation $\sim$ on $\bigsqcup_U S(U)$ as follows: $(x_U, U) \sim (y_V, V)$ if and only if there exists an open W such that $U \supset W$ and $V \supset W$ and $\mathrm{res}_{U,W}(x_U) = \mathrm{res}_{V,W}(y_V)$.

Definition 2.12 (Stalk at a point)**.** Let X be a topological space and S a sheaf on this space. Let $p \in X$ be a point. Then we define the stalk at p as the direct limit of the direct system of sets $S(U)$ and maps $\mathrm{res}_{U,V}$ for all open U and V that contain p.

Notation 2.13 (S_p)**.** Let X be a topological space and S a sheaf on this space and $p \in X$. Then we denote the stalk of S at p by S_p.

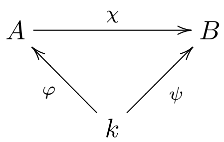
The stalk of a sheaf at a point tells us what the sheaf looks like near this point.

2.2 A topology on $\mathrm{Hom}(A, k)$

Definition 2.14 (k-algebra)**.** Let k be a field. Then a k-algebra (A, φ) is a commutative ring A (with $1 \in A$) together with a ring homomorphism $\varphi\colon k \to A$.

Note that for a a ring homomorphism $\varphi\colon k \to A$ we have $\varphi(1) = 1$. Hence if A is not the trivial ring the morphism φ is injective. (Since φ is a ring homomorphism its kernel is an ideal and a field only has the ideals (0) and the entire field.) We will usually omit the map φ.

Definition 2.15 (Morphism between k-algebras)**.** A k-algebra homomorphism $\chi\colon (A, \varphi) \to (B, \psi)$ is a ring homomorphism $\chi\colon A \to B$ such that the following diagram commutes



Definition 2.16 (Finitely generated)**.** A k-algebra A is finitely generated if there exist $y_1, \ldots, y_n \in A$ such that for all $y \in A$ there exist $\lambda_1, \ldots, \lambda_n \in k$ and $b_1, \ldots, b_n \in \mathbb{Z}_{\geq 0}$ with $y = \sum_{i=1}^{n} \lambda_i y_i^{b_i}$.

Let A be finitely generated by $y_1, \ldots, y_n \in A$. Let $k[x_1, \ldots, x_n]$ be the polynomial ring over k in n variables. Consider the map

$$\begin{aligned} m\colon k[x_1, \ldots, x_n] &\to A \\ x_i &\mapsto y_i. \end{aligned}$$

This map is well-defined, and because A is generated by the y_i this map is surjective. So $A \cong k[x_1, \ldots, x_n]/\ker(m)$.

Take an algebraically closed field k and take a finitely generated k-algebra A that is also an integral domain.

Notation 2.17 ($V(I)$)**.** For an ideal I of A we define

$$V(I) := \{\varphi \in \mathrm{Hom}(A, k) | I \subset \ker(\varphi)\}.$$

We will check that setting the $V(I)$'s as closed sets we obtain a topology on $\mathrm{Hom}(A, k)$.

We have that

$$\begin{aligned} V((0)) &= \{\varphi \in \mathrm{Hom}(A, k) | (0) \subset \ker(\varphi)\} \\ &= \mathrm{Hom}(A, k). \end{aligned}$$

Since a homomorphism maps 1 to 1 and in any field we have $1 \neq 0$ we also have that

$$\begin{aligned} V((1)) &= \{\varphi \in \mathrm{Hom}(A, k) | (1) \subset \ker(\varphi)\} \\ &= \emptyset. \end{aligned}$$

Take two ideals I and J of $\mathrm{Hom}(A, k)$. Suppose φ is such that $I \not\subset \ker(\varphi)$ and $J \not\subset \ker(\varphi)$. Because $\ker(\varphi)$ is maximal (and thus prime) we now know there exist $i \in I$ and $j \in J$ such that $ij \notin \ker(\varphi)$ hence $I \cdot J \not\subset \ker(\varphi)$. So

$$\begin{aligned} &\{\varphi \in \mathrm{Hom}(A, k) | I \cdot J \subset \ker(\varphi)\} \\ \subset\ &\{\varphi \in \mathrm{Hom}(A, k) | I \subset \ker(\varphi)\} \cup \{\varphi \in \mathrm{Hom}(A, k) | J \subset \ker(\varphi)\}. \end{aligned}$$

Suppose φ is such that $I \subset \ker(\varphi)$ or $J \subset \ker(\varphi)$ then $I \cdot J \subset \ker(\varphi)$. So

$$\begin{aligned} &\{\varphi \in \mathrm{Hom}(A, k) | I \cdot J \subset \ker(\varphi)\} \\ \supset\ &\{\varphi \in \mathrm{Hom}(A, k) | I \subset \ker(\varphi)\} \cup \{\varphi \in \mathrm{Hom}(A, k) | J \subset \ker(\varphi)\}. \end{aligned}$$

So we have

$$\begin{aligned} V(I \cdot J) &= \{\varphi \in \mathrm{Hom}(A, k) | I \cdot J \subset \ker(\varphi)\} \\ &= \{\varphi \in \mathrm{Hom}(A, k) | I \subset \ker(\varphi)\} \cup \{\varphi \in \mathrm{Hom}(A, k) | J \subset \ker(\varphi)\} \\ &= V(I) \cup V(J). \end{aligned}$$

Take a set of ideals I_j with $j \in J$ and J an index set. Suppose φ is such that $\sum_{j \in J} I_j \subset \ker(\varphi)$ then we must have that $I_j \subset \ker(\varphi)$ for all j because for any $i_j \in I_j$ we can take the element $i_j = 0 + \cdots + 0 + i_j + 0 + \cdots + 0 \in \sum_{j \in J} I_j$ and this element must be contained in $\ker(\varphi)$. So

$$\begin{aligned} &\{\varphi \in \mathrm{Hom}(A, k) | \sum_{j \in J} I_j \subset \ker(\varphi)\} \\ \subset\ &\cap_{j \in J} \{\varphi \in \mathrm{Hom}(A, k) | I_j \subset \ker(\varphi)\}. \end{aligned}$$

Suppose φ is such that $I_j \subset \ker(\varphi)$ for all $j \in J$ then $\sum_{j\in J} \varphi(x_j) = 0$ for $x_j \in I_j$ so $\varphi(\sum_{j\in J} x_j) = 0$ so $\sum_{j\in J} I_j \subset \ker(\varphi)$. So

$$\begin{aligned} &\{\varphi \in \mathrm{Hom}(A,k) | \sum_{j\in J} I_j \subset \ker(\varphi)\} \\ &\supset \cap_{j\in J} \{\varphi \in \mathrm{Hom}(A,k) | I_j \subset \ker(\varphi)\}. \end{aligned}$$

So we have

$$\begin{aligned} V\big(\sum_{j\in J} I_j\big) &= \{\varphi \in \mathrm{Hom}(A,k) | \sum_{j\in J} I_j \subset \ker(\varphi)\} \\ &= \cap_{j\in J} \{\varphi \in \mathrm{Hom}(A,k) | I_j \subset \ker(\varphi)\} \\ &= \cap_{j\in J} V(I_j). \end{aligned}$$

Hence indeed setting the $V(I)$'s as the closed sets we obtain a topology on $\mathrm{Hom}(A,k)$.

Proposition 2.18. *The* $\mathrm{Hom}(A,k) \setminus V((f))$ *with* $f \in A$ *form a basis for the topology on* $\mathrm{Hom}(A,k)$.

Proof. First we will prove that every $V(I)$ with I an ideal of $\mathrm{Hom}(A,k)$ can be written as the intersection of $V((f))$'s with $f \in I$. Note that if $f \in I$ then $(f) \subset I$, so

$$\begin{aligned} V(I) &= \{\varphi \in \mathrm{Hom}(A,k) | I \subset \ker(\varphi)\} \\ &\subset \{\varphi \in \mathrm{Hom}(A,k) | (f) \subset \ker(\varphi)\} \\ &= V((f)). \end{aligned}$$

Hence $V(I) \subset \cap_{f\in I} V((f))$.

Take a $\varphi \in \mathrm{Hom}(A,k)$. If for all $f \in I$ we have $(f) \subset \ker(\varphi)$ then for all $f \in I$ we have $f \in \ker(\varphi)$ so $I \subset \ker(\varphi)$. Therefore

$$\begin{aligned} \cap_{f\in I} V((f)) &= \cap_{f\in I} \{\varphi \in \mathrm{Hom}(A,k) | (f) \subset \ker(\varphi)\} \\ &\subset \{\varphi \in \mathrm{Hom}(A,k) | I \subset \ker(\varphi)\} \\ &= V(I). \end{aligned}$$

Hence $V(I) = \cap_{f\in I} V((f))$.

Take an open $U \subset \mathrm{Hom}(A,k)$. Then $\mathrm{Hom}(A,k) \setminus U$ is closed. Then there exists an ideal I such that $V(I) = \mathrm{Hom}(A,k) \setminus U$. Then $\cap_{f\in I} V((f)) = \mathrm{Hom}(A,k) \setminus U$. Note that $\cap_{f\in I} V((f)) = \mathrm{Hom}(A,k) \setminus \cup_{f\in I}(\mathrm{Hom}(A,k) \setminus V((f)))$. Hence we have

$$\mathrm{Hom}(A,k) \setminus U = \mathrm{Hom}(A,k) \setminus \cup_{f\in I}(\mathrm{Hom}(A,k) \setminus V((f))).$$

Taking complements we find

$$U = \cup_{f\in I}(\mathrm{Hom}(A,k) \setminus V((f))).$$

□

2.3 A contravariant functor from $\mathrm{Alg_k}$ to LoRiSp

Let k be an algebraically closed field.

Notation 2.19 ($\mathrm{Alg_k}$)**.** Let $\mathrm{Alg_k}$ denote the category of finitely generated k-algebras that are an integral domain.

Definition 2.20 (Ringed space)**.** A ringed space $(X, \mathcal{O}_X)$ is a topological space X together with a sheaf $\mathcal{O}_X$ on X with values in the category of rings.

Definition 2.21 (Locally ringed space)**.** A ringed space $(X, \mathcal{O}_X)$ is a locally ringed space if for each point $\varphi \in X$ the stalk $\mathcal{O}_{X,\varphi}$ is a local ring.

Definition 2.22 (Morphism of ringed spaces)**.** We define a morphism between two ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ as a pair (f, ϕ_f) where $f: X \to Y$ is a continuous map and $\phi_f: \mathcal{O}_Y \to f_*\mathcal{O}_X$ a morphism of sheaves, with $f_*\mathcal{O}_X$ as in proposition 2.8.

Definition 2.23 (Local homomorphism of local rings)**.** Let A and B be local rings with respectively maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$. Then a morphism $f: A \to B$ is a local homomorphism if $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

Definition 2.24 (Morphism of locally ringed spaces)**.** A morphism between two locally ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a morphism of ringed spaces (f, ϕ_f) such that for each point $\varphi \in X$ the induced map of local rings $\phi_{f,\varphi}: \mathcal{O}_{Y,f(\varphi)} \to \mathcal{O}_{X,\varphi}$ is a local homomorphism.

Notation 2.25 (LoRiSp)**.** Let LoRiSp denote the category of locally ringed spaces.

Notation 2.26 ($X(A)$)**.** For any object A of $\mathrm{Alg_k}$ define

$$X(A) := \mathrm{Hom}(A, k)$$

with as topology on $X(A)$ the topology described in section 2.2.

Definition 2.27 ($\mathcal{O}_{X(A)}(U)$)**.** For any object A of $\mathrm{Alg_k}$ and open U of $X(A)$ we define $\mathcal{O}_{X(A)}(U)$ as the intersection of all subsets $A_{\ker(\varphi)}$ of the field of fractions of A for which φ is contained in U i.e.

$$\mathcal{O}_{X(A)}(U) := \cap_{\varphi \in U} A_{\ker(\varphi)}.$$

We have $A_{\ker(\varphi)} = \{\frac{f}{g} | f, g \in A, g \notin \ker(\varphi)\} = \{\frac{f}{g} | f, g \in A, \varphi(g) \neq 0\}$.
Note that $\mathcal{O}_{X(A)}(U)$ is indeed a ring for all open U of $X(A)$.

Lemma 2.28. *If we take the inclusions inside the field of fractions as restriction maps $\mathcal{O}_{X(A)}$ is a sheaf.*

Proof. It is clear that $\mathcal{O}_{X(A)}$ is a presheaf.
Take an open $U \subset X(A)$ and an open cover $\{U_i\}_{i \in I}$ of U, i.e. $\cup_{i \in I} U_i = U$.
Let $(\frac{f_i}{g_i})_{i \in I} \in \mathrm{Im}(\prod \mathrm{res}_{U,U_i})$. Then $\frac{f_i}{g_i} = \frac{f_j}{g_j}$ for all $i, j \in I$. Suppose $\frac{f}{g}, \frac{f'}{g'} \in \mathcal{O}_{X(A)}(U)$ and $\prod_{i \in I} \mathrm{res}_{U,U_i}(\frac{f}{g}) = \prod_{i \in I} \mathrm{res}_{U,U_i}(\frac{f'}{g'})$. Then $\frac{f}{g} = \frac{f_i}{g_i} = \frac{f'}{g'}$ for all $i \in I$. Hence $\prod \mathrm{res}_{U,U_i}$ is injective.
Now consider $\prod \mathrm{res}_{U_i, U_i \cap U_j}$ and $\prod \mathrm{res}_{U_j, U_j \cap U_i}$. Let $(\frac{f_i}{g_i})_{i \in I} \in \prod_{i \in I} \mathcal{O}_{X(A)}(U_i)$. Recall that res_{U,U_i} is the inclusion for all $i \in I$. Hence

$$\mathrm{Im}(\prod \mathrm{res}_{U,U_i}) = \{\left(\frac{f_i}{g_i}\right)_{i \in I} | \frac{f_i}{g_i} \in \mathcal{O}_{X(A)}(U), \frac{f_i}{g_i} = \frac{f_j}{g_j} \text{ for all } i, j \in I\}.$$

(Note that for $(\frac{f_i}{g_i})_{i\in I} \in \prod_{i\in I} \mathcal{O}_{X(A)}(U_i)$ if $\frac{f_i}{g_i} \notin \mathcal{O}_{X(A)}(U)$ for some $i \in I$ then there exists a $j \in I$ with $\frac{f_j}{g_j} \neq \frac{f_i}{g_i}$ since $U = \cup_{i\in I} U_i$.) Suppose $(\frac{f_i}{g_i})_{i\in I} \notin \mathrm{Im}(\prod \mathrm{res}_{U,U_i})$ then there exist $k, l \in I$ with $\frac{f_k}{g_k} \neq \frac{f_l}{g_l}$. Then

$$\mathrm{res}_{U_k,U_k\cap U_l}\left(\frac{f_k}{g_k}\right) \neq \mathrm{res}_{U_l,U_k\cap U_l}\left(\frac{f_l}{g_l}\right)$$

so

$$\prod \mathrm{res}_{U_i,U_i\cap U_j}((\frac{f_i}{g_i})_{i\in I}) \neq \prod \mathrm{res}_{U_i,U_j\cap U_i}((\frac{f_i}{g_i})_{i\in I}).$$

Note that if $(\frac{f_i}{g_i})_{i\in I} \in \mathrm{Im}(\prod \mathrm{res}_{U,U_i})$ then $\prod \mathrm{res}_{U_i,U_i\cap U_j}((\frac{f_i}{g_i})_{i\in I}) = \prod \mathrm{res}_{U_j,U_j\cap U_i}((\frac{f_i}{g_i})_{i\in I})$. We find that $\prod \mathrm{res}_{U_i,U_i\cap U_j}$ and $\prod \mathrm{res}_{U_j,U_j\cap U_i}$ agree exactly on the image of $\prod \mathrm{res}_{U,U_i}$.

□

Here follows a very strong lemma from commutative algebra.

Lemma 2.29 (Weak version of Hilbert's Nullstellensatz)**.** *Let k be an algebraically closed field and A a finitely generated k-algebra. Let $\mathfrak{m}$ be a maximal ideal. Then $A/\mathfrak{m} \cong k$.*

See [1] corollary 7.10.

So we have a bijective correspondence

$$\begin{aligned} \mathrm{Hom}(A,k) &\leftrightarrow \text{ Maximal ideals of } A \\ (\varphi\colon A \to k) &\mapsto \ker(\varphi) \\ (A \to A/\mathfrak{m}) &\leftarrow\!\shortmid \mathfrak{m} \end{aligned}$$

We have that $\mathcal{O}_{X(A)}(X(A)) = \cap_{\varphi\in X(A)} A_{\ker(\varphi)}$. By the weak version of the Nullstellensatz we have that $\cap_{\varphi\in X(A)} A_{\ker(\varphi)} = \cap_{\mathfrak{m}\subset A, \mathfrak{m} \text{ maximal ideal}} A_{\mathfrak{m}}$. By lemma 2.9 of [9] we have that $\cap_{\mathfrak{m}\subset A, \mathfrak{m} \text{ maximal ideal}} A_{\mathfrak{m}} = A$. So

$$\mathcal{O}_{X(A)}(X(A)) = A.$$

Lemma 2.30. *Let $\psi \in X(A)$. Then $\mathcal{O}_{X(A),\psi} = A_{\ker(\psi)}$.*

Proof. Note that the restriction map of $\mathcal{O}_{X(A)}$ is the inclusion so for opens U and V and $\frac{f_1}{g_1} \in \cap_{\varphi\in U} A_{\ker(\varphi)}$ and $\frac{f_2}{g_2} \in \cap_{\varphi\in V} A_{\ker(\varphi)}$ with $\frac{f_1}{g_1} = \frac{f_2}{g_2}$ we have $\mathrm{res}_{U,U\cap V}(\frac{f_1}{g_1}) = \mathrm{res}_{V,U\cap V}(\frac{f_2}{g_2})$. We find

$$\begin{aligned} (\mathcal{O}_{X(A)})_\psi &:= \bigsqcup_{U\ni\psi} (\cap_{\varphi\in U} A_{\ker(\varphi)})/\sim \\ &= \cup_{U\ni\psi}(\cap_{\varphi\in U} A_{\ker(\varphi)}). \end{aligned}$$

Note that for all U over which we take the union we have $\psi \in U$ so that $\cap_{\varphi\in U} A_{\ker(\varphi)} \subset A_{\ker(\psi)}$ for these U. Hence $\cup_{U\ni\psi}(\cap_{\varphi\in U} A_{\ker(\varphi)}) \subset A_{\ker(\psi)}$.

Let $\frac{f}{g} \in A_{\ker(\psi)}$, then $g \notin \ker(\psi)$. Consider $\mathrm{Hom}(A,k) \setminus V((g))$. We have that $\psi \in \mathrm{Hom}(A,k) \setminus V((g))$ and that this set is open. For all $\varphi \in \mathrm{Hom}(A,k) \setminus V((g))$ we have $g \notin \ker(\varphi)$. Hence $\frac{f}{g} \in \cap_{\varphi\in \mathrm{Hom}(A,k)\setminus V((g))} A_{\ker(\varphi)}$. Therefore $A_{\ker(\psi)} \subset \cup_{U\ni\psi}(\cap_{\varphi\in U} A_{\ker(\varphi)})$.

We can conclude that the stalk of $\mathcal{O}_{X(A)}$ at $\psi \in X(A)$ is $A_{\ker(\psi)}$.

□

Lemma 2.31. *The ringed space $(X(A), \mathcal{O}_{X(A)})$ is a locally ringed space.*

Proof. Let $\psi \in X(A)$. By proposition 2.30 the stalk of $\mathcal{O}_{X(A)}$ at ψ is $A_{\ker(\psi)}$. Note that $\ker(\psi)$ is the only maximal ideal of $A_{\ker(\psi)}$. Hence $(X(A), \mathcal{O}_{X(A)})$ is a locally ringed space. $\square$

Definition 2.32 (f^*)**.** For any morphism $f\colon A \to B$ in $\mathrm{Alg_k}$ we define a morphism

$$\begin{aligned} f^*\colon \operatorname{Hom}(B,k) &\to \operatorname{Hom}(A,k) \\ g &\mapsto g \circ f. \end{aligned}$$

Note that for a morphism $f^*\colon \operatorname{Hom}(B,k) \to \operatorname{Hom}(A,k)$ we have

$$\psi \in f^{*-1}(U) \Leftrightarrow \psi \circ f =: f^*(\psi) \in U.$$

For an open $U \subset X(A)$ we have

$$\begin{aligned} (f^*)_*\mathcal{O}_{X(B)}(U) &= \mathcal{O}_{X(B)}(f^{*-1}(U)) \\ &= \cap_{\varphi \in f^{*-1}(U)} B_{\ker(\varphi)} \\ &= \cap_{f^*(\varphi) \in U} B_{\ker(\varphi)}. \end{aligned}$$

Further note

$$\begin{aligned} h \notin \ker(f^*(\psi)) &\Leftrightarrow (f^*(\psi))(h) \neq 0 \\ &\Leftrightarrow (\psi \circ f)(h) \neq 0 \\ &\Leftrightarrow \psi(f(h)) \neq 0 \\ &\Leftrightarrow f(h) \notin \ker(\psi). \end{aligned}$$

Definition 2.33 (ϕ_{f^*})**.** For any morphism $f\colon A \to B$ in $\mathrm{Alg_k}$ and associated morphism $f^*\colon \operatorname{Hom}(B,k) \to \operatorname{Hom}(A,k)$, we define a morphism between sheaves

$$\phi_{f^*}\colon \mathcal{O}_{X(A)} \to (f^*)_*\mathcal{O}_{X(B)}$$

by the collection of maps $\phi_{f^*}(U)\colon \mathcal{O}_{X(A)}(U) \to (f^*)_*\mathcal{O}_{X(B)}(U)$ induced by maps

$$\begin{aligned} A_{\ker(f^*(\varphi))} &\to B_{\ker(\varphi)} \\ \frac{g}{h} &\mapsto \frac{f(g)}{f(h)}. \end{aligned}$$

Lemma 2.34. *The map $(f^*, \phi_{f^*})\colon (X(B), \mathcal{O}_{X(B)}) \to (X(A), \mathcal{O}_{X(A)})$ is a morphism of locally ringed spaces.*

Proof. Take a $\psi \in X(B)$. The stalk of $\mathcal{O}_{X(A)}$ at $f^*(\psi)$ is $A_{\ker(f^*(\psi))}$ and the stalk of $\mathcal{O}_{X(B)}$ at ψ is $B_{\ker(\psi)}$. The morphism induced by ϕ_{f^*} between these stalks is

$$\begin{aligned} \phi_{f^*,\psi}\colon A_{\ker(f^*(\psi))} &\to B_{\ker(\psi)} \\ \frac{g}{h} &\mapsto \frac{f(g)}{f(h)}. \end{aligned}$$

We find $\phi_{f^*,\psi}^{-1}(\ker(\psi)) = A_{\ker(f^*(\psi))}$. $\square$

Define the contravariant functor

$$\begin{aligned} F\colon \mathrm{Alg_k} &\to \mathrm{LoRiSp} \\ A &\mapsto (X(A), \mathcal{O}_{X(A)}) \\ (f\colon A \to B) &\mapsto \big((f^*\colon \operatorname{Hom}(B,k) \to \operatorname{Hom}(A,k)), (\phi_{f^*}\colon \mathcal{O}_{X(A)} \to (f^*)_*\mathcal{O}_{X(B)})\big) \end{aligned}$$

with f^* as in 2.32 and ϕ_{f^*} as in 2.33.

2.4 The category $\mathrm{AffVar}_\mathrm{k}$

Definition 2.35 (Image of a functor)**.** Let C and D be categories and $F\colon C \to D$ a functor. The image of F is the smallest subcategory of D that contains all objects for which there exists some object in C that is mapped to it by F and all morphisms for which there exists some morphism in C that is mapped to it by F.

Definition 2.36 (Isomorphism)**.** Let C be a category and A and B objects in this category. A morphism $f\colon A \to B$ is called an isomorphism if there exists a morphism $g\colon B \to A$ with $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Definition 2.37 (Replete subcategory)**.** A subcategory R of C is replete if for any object A of R and any isomorphism $f\colon A \to B$ between objects of C we have that B is an object of R and f is a morphism of R.

Definition 2.38 (Essential image of a functor)**.** The essential image of a functor $F\colon C \to D$ is the smallest replete subcategory of D containing the image of F.

Lemma 2.39. *Let $F\colon C \to D$ be a functor from C to D. Let R be the essential image of F. For an object Y of R there exist an object X in the image of F and a morphism $i\colon X \to Y$ that is an isomorphism in R.*

Proof. For an object Y of R there exist an object X in the image of F and an isomorphism $i\colon X \to Y$ in D that is a morphism in R. Thus there exists an isomorphism $i^{-1}\colon Y \to X$ in D (with $i \circ i^{-1} = \mathrm{id}_Y$ and $i^{-1} \circ i = \mathrm{id}_X$). Because R is a replete subcategory of D and Y is an object of R and $i^{-1}\colon Y \to X$ an isomorphism in D we know that i^{-1} is a morphism in R. Hence i is an isomorphism in R. □

Definition 2.40 ($\mathrm{AffVar}_\mathrm{k}$)**.** We define $\mathrm{AffVar}_\mathrm{k}$ to be the essential image of the contravariant functor F that we introduced in section 2.3.

2.5 An anti-equivalence

We restrict the target of F to obtain the following contravariant functor

$$\begin{aligned} G\colon\ \mathrm{Alg}_\mathrm{k} &\to \mathrm{AffVar}_\mathrm{k} \\ A &\mapsto (X(A), \mathcal{O}_{X(A)}) \\ (f\colon A \to B) &\mapsto \big((f^*\colon \mathrm{Hom}(B,k) \to \mathrm{Hom}(A,k)), (\phi_{f^*}\colon \mathcal{O}_{X(A)} \to f_*\mathcal{O}_{X(B)})\big) \end{aligned}$$

We will show that this contravariant functor is an anti-equivalence. Let $g\colon (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ be a morphism in $\mathrm{AffVar}_\mathrm{k}$. Take an i_Y as in lemma 2.39 for $(Y, \mathcal{O}_Y)$ and an i_Z^{-1} for $(Z, \mathcal{O}_Z)$. Consider the map $i_Z^{-1} \circ g \circ i_Y\colon (X(A), \mathcal{O}_{X(A)}) \to (X(B), \mathcal{O}_{X(B)})$. This is a map between locally ringed spaces and thus is a pair $(g', \phi_{g'})$ with $\phi_{g'}$ a morphism from $\mathcal{O}_{X(B)}$ to $g'_*\mathcal{O}_{X(A)}$. Now we would like there to exist an $f\colon B \to A$ with $G(f) = (g', \phi_{g'})$. But not all morphisms in the essential image of F are images of morphisms in Alg_k so a priori we would not for every morphism m in the essential image have an appropriate morphism in Alg_k to map to m.

Lemma 2.41. *Let $((j\colon X(B) \to X(A)), (\phi_j\colon \mathcal{O}_{X(A)} \to j_*\mathcal{O}_{X(B)}))$ be an isomorphism in the essential image of F (so an isomorphism of locally ringed spaces) between objects in the image of F. Then (j, ϕ_j) is an isomorphism in the image of F.*

Proof. Let $(j, \phi_j)\colon (X(B), \mathcal{O}_{X(B)}) \to (X(A), \mathcal{O}_{X(A)})$ be an isomorphism of locally ringed spaces. Note that ϕ_j induces a ring homomorphism $\phi_j(X(A))\colon \mathcal{O}_{X(A)}(X(A)) \to j_*\mathcal{O}_{X(B)}(X(A))$. Recall that $\mathcal{O}_{X(A)}(X(A)) = A$ and $j_*\mathcal{O}_{X(B)}(X(A)) = \mathcal{O}_{X(B)}(X(B)) = B$. We will think of $\phi_j(X(A))$ as a $f\colon A \to B$. We want to prove that the morphism of locally ringed spaces $(f^*, \phi_{f^*})\colon (X(B), \mathcal{O}_{X(B)}) \to (X(A), \mathcal{O}_{X(A)})$ induced by f equals (j, ϕ_j).

Take any $\varphi \in X(B)$. We then have an induced local homomorphism $\phi_{j,\varphi}\colon \mathcal{O}_{X(A),j(\varphi)} \to \mathcal{O}_{X(B),\varphi}$ on stalks. By proposition 2.30 this is a homomorphism from $A_{\ker(j(\varphi))}$ to $B_{\ker(\varphi)}$. As $\phi_{j,\varphi}$ is local we have $(\phi_{j,\varphi}^{-1})(\ker(\varphi)) = \ker(j(\varphi))$. Recall that the map $f^*\colon X(B) \to X(A)$ is given by $\varphi \mapsto \varphi \circ f$. This map equals j if and only if $j(\varphi) = \varphi \circ f$ for all $\varphi \in X(B)$. As a surjective homomorphism $A \to k$ is determined by its kernel this is equivalent with $\ker(j(\varphi)) = \ker(\varphi \circ f)$ for all $\varphi \in X(B)$. Hence j and f^* are equal.

Note that the morphism

$$\begin{aligned} \phi_{f^*}(X(A))\colon A &\to B \\ a &\mapsto f(a) \end{aligned}$$

is equal to f and $f = \phi_j(X(A))$.

Recall that the restriction maps of the sheaves $\mathcal{O}_{X(A)}$ and $j_*\mathcal{O}_{X(B)}$ are simply inclusions. By the definition of a map between sheaves the following diagram must commute for all open $U \subset X(A)$

$$\begin{array}{ccc} A & \xrightarrow{\phi_j(X(A))} & B \\ {\scriptstyle \mathrm{res}_{X(A),U}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{X(A),U}} \\ \mathcal{O}_{X(A)}(U) & \xrightarrow[\phi_j(U)]{} & j_*\mathcal{O}_{X(B)}(U) \end{array}$$

Until now we have not used the assumption that (j, ϕ_j) is an isomorphism. Using [1] proposition 3.1 multiple times we find there is a unique morphism from the quotient field of A to the quotient field of B such that the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\phi_j(X(A))} & B \\ \downarrow & & \downarrow \\ Q(A) & \xrightarrow[\phi_j(U)]{} & Q(B) \end{array}$$

For an open $U \subset X(A)$ we can view $\mathcal{O}_{X(A)}(U)$ and $j_*\mathcal{O}_{X(B)}(U)$ as inclusions of the quotient field of respectively A and B. Hence for every open $U \subset X(A)$ the map $\phi_j(X(A))$ determines the map $\phi_j(U)$.

Hence $\phi_j = \phi_{f^*}$. We can conclude $(j, \phi_j) = (f^*, \phi_{f^*})$. □

Fix for every object $(Y, \mathcal{O}_Y)$ of $\mathrm{AffVar_k}$ an isomorphism i_Y as in 2.39. Note that we use the axiom of choice here.

Lemma 2.42. *For every morphism $m\colon (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ in the essential image of F there exists a morphism $f^*\colon (X(A), \mathcal{O}_{X(A)}) \to (X(B), \mathcal{O}_{X(B)})$ in the image of F*

such that the following diagram commutes

$$\begin{array}{ccc} (Y, \mathcal{O}_Y) & \xrightarrow{m} & (Z, \mathcal{O}_Z) \\ \uparrow{\scriptstyle i_Y} & & \downarrow{\scriptstyle i_Z^{-1}} \\ (X(A), \mathcal{O}_{X(A)}) & \xrightarrow[f^*]{} & (X(B), \mathcal{O}_{X(B)}) \end{array}$$

Proof. Every morphism $m\colon (X(A), \mathcal{O}_{X(A)}) \to (X(B), \mathcal{O}_{X(B)})$ in the image of F is of this form. For we can take $i^{-1}_{X(B)} \circ m \circ i_{X(A)}$ as f^*. Note that $i^{-1}_{X(B)}$ and $i_{X(A)}$ are in the image of F by lemma 2.41. Further note that the image of F is a category, so the composition $i^{-1}_{X(B)} \circ m \circ i_{X(A)}$ is contained in the image of F.

Every isomorphism $m\colon (X(A), \mathcal{O}_{X(A)}) \to (Y, \mathcal{O}_Y)$ in the essential image of F is of this form. Note that m and i_Y^{-1} are both isomorphisms and that $i_Y^{-1} \circ m$ is an isomorphism between objects that are in the image of F, so by lemma 2.41 the morphism $i_Y^{-1} \circ m$ is in the image of F. Take $i_Y^{-1} \circ m \circ i_{X(A)}$ as f^*.

Every identity $\mathrm{id}_{(Y,\mathcal{O}_Y)}\colon (Y, \mathcal{O}_Y) \to (Y, \mathcal{O}_Y)$ is of this form. Suppose $i_Y^{-1}((Y, \mathcal{O}_Y)) = (X(A), \mathcal{O}_{X(A)})$. Then we can take the identity $\mathrm{id}_{(X(A),\mathcal{O}_{X(A)})}$ as f^*.

Take two morphisms $m\colon (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ and $n\colon (Z, \mathcal{O}_Z) \to (W, \mathcal{O}_W)$ in the essential image of F for which there are morphisms

$$f^*\colon (X(A), \mathcal{O}_{X(A)}) \to (X(B), \mathcal{O}_{X(B)}) \text{ and } g^*\colon (X(C), \mathcal{O}_{X(C)}) \to (X(D), \mathcal{O}_{X(D)})$$

such that the following two diagrams commute

$$\begin{array}{ccc} (Y, \mathcal{O}_Y) & \xrightarrow{m} & (Z, \mathcal{O}_Z) \\ \uparrow{\scriptstyle i_Y} & & \downarrow{\scriptstyle i_Z^{-1}} \\ (X(A), \mathcal{O}_{X(A)}) & \xrightarrow[f^*]{} & (X(B), \mathcal{O}_{X(B)}) \end{array} \qquad \begin{array}{ccc} (Z, \mathcal{O}_Z) & \xrightarrow[n]{} & (W, \mathcal{O}_W) \\ {\scriptstyle i_Z}\uparrow & & {\scriptstyle i_W^{-1}}\downarrow \\ (X(B), \mathcal{O}_{X(B)}) & \xrightarrow[g^*]{} & (X(C), \mathcal{O}_{X(C)}) \end{array}$$

Consider the diagram

$$\begin{array}{ccc} (Y, \mathcal{O}_Y) & \xrightarrow{n \circ m} & (W, \mathcal{O}_W) \\ {\scriptstyle i_Y}\downarrow & & \downarrow{\scriptstyle i_W^{-1}} \\ (X(A), \mathcal{O}_{X(A)}) & \xrightarrow[g^* \circ f^*]{} & (X(C), \mathcal{O}_{X(C)}) \end{array}$$

The composition $g^* \circ f^*$ is again in the image of F, so $n \circ m$ is again of the desired form.

Therefore the smallest category that contains all $m\colon (X(A), \mathcal{O}_{X(A)}) \to (X(B), \mathcal{O}_{X(B)})$ that are in the image of F and all isomorphisms $i\colon (X(A), \mathcal{O}_{X(A)}) \to (Y, \mathcal{O}_Y)$ has morphisms of the form described in the lemma we are proving. □

Notation 2.43 ($\mathcal{O}_{i^{-1}(Y)}$)**.** For an object $(Y, \mathcal{O}_Y)$ in AffVar_k and an isomorphism i_Y, consider $i_Y^{-1}((Y, \mathcal{O}_Y)) = (X(A), \mathcal{O}_{X(A)})$ with $(X(A), \mathcal{O}_{X(A)})$ in the image of F. We define

$$\mathcal{O}_{i^{-1}(Y)} := \mathcal{O}_{X(A)}(X(A)).$$

Now fix for every morphism $(g\colon (Y,\mathcal{O}_Y) \to (Z,\mathcal{O}_Z))$ in $\mathrm{AffVar_k}$ an $f_g^*\colon (X(A),\mathcal{O}_{X(A)}) \to (X(B),\mathcal{O}_{X(B)})$ in the image of F such that $g = i_Z^{-1} \circ f_g^* \circ i_Y$. We form a functor from $\mathrm{AffVar_k}$ to $\mathrm{Alg_k}$ that sends a morphism g to the f_g with $g = i_Z^{-1} \circ f_g^* \circ i_Y$.

Define a contravariant functor

$$\begin{aligned} H\colon \mathrm{AffVar_k} &\to \mathrm{Alg_k} \\ (Y,\mathcal{O}_Y) &\mapsto \mathcal{O}_{i^{-1}(Y)} \\ (g\colon (Y,\mathcal{O}_Y) \to (Z,\mathcal{O}_Z)) &\mapsto (f_g\colon \mathcal{O}_{i^{-1}(Z)} \to \mathcal{O}_{i^{-1}(Y)}). \end{aligned}$$

We have $H \circ G = \mathrm{id}_{\mathrm{Alg_k}}$.

Definition 2.44 (Natural isomorphism)**.** Let F and G be functors from the category C to D. A natural transformation T from F to G is a natural isomorphism if for every object X of C the morphism $\alpha_X\colon F(X) \to G(X)$ is an isomorphism.

Definition 2.45 ((Anti)-equivalence of categories)**.** Two categories C and D are (anti)-equivalent if there exist co(ntra)variant functors $G\colon C \to D$ and $H\colon D \to C$ such that there exist natural isomorphisms $T_1\colon H \circ G \to \mathrm{id}_C$ and $T_2\colon G \circ H \to \mathrm{id}_D$.

Theorem 2.46. *The categories* $\mathrm{Alg_k}$ *and* $\mathrm{AffVar_k}$ *are anti-equivalent.*

Proof. Note that for an object $(Y,\mathcal{O}_Y)$ of $\mathrm{AffVar_k}$ we have

$$(G \circ H)((Y,\mathcal{O}_Y)) = (X(A),\mathcal{O}_{X(A)})$$

for some $(X(A),\mathcal{O}_{X(A)})$ in the image of G and that

$$\mathrm{id}((Y,\mathcal{O}_Y)) = (Y,\mathcal{O}_Y).$$

For an object $(Y,\mathcal{O}_Y)$ of $\mathrm{AffVar_k}$ define

$$(\alpha_{(Y,\mathcal{O}_Y)}\colon (G \circ H)((Y,\mathcal{O}_Y)) \to \mathrm{id}((Y,\mathcal{O}_Y))) := (i_Y\colon (X(A),\mathcal{O}_{X(A)}) \to (Y,\mathcal{O}_Y)).$$

Take a morphism $g\colon (Y,\mathcal{O}_Y) \to (Z,\mathcal{O}_Z)$ in $\mathrm{AffVar_k}$. Consider

$$(G \circ H)(g)\colon (X(A),\mathcal{O}_{X(A)}) \to (X(B),\mathcal{O}_{X(B)}).$$

We have that $\mathrm{id}(g) \circ \alpha_{(Y,\mathcal{O}_Y)} = \alpha_{(Z,\mathcal{O}_Z)} \circ (G \circ H)(g)$. Hence the $\alpha_{(Y,\mathcal{O}_Y)}$ form a natural isomorphism from $G \circ H$ to id_{AffVar_k}. □

In fact there is a less tedious approach which we will explore in chapter 2.

2.6 A categorical characterization of the topology on $X(A)$ using the anti-equivalence

We will see that the topology on $X(A)$ is exactly the topology that is obtained by setting the finite unions of images of extremal monomorphisms as the closed sets.

Definition 2.47 (Monomorphism)**.** A monomorphism $m\colon B \to C$ is a morphism such that if $f, g\colon A \to B$ are morphisms then $m \circ f = m \circ g$ implies $f = g$.

Proposition 2.48. *If a morphism is injective, it is a monomorphism.*

Proof. Take an injective morphism $f\colon B \to C$. Suppose $g, h\colon A \to B$ are morphisms with $f \circ g = f \circ h$. Now assume that there exists an $a \in A$ such that $g(a) \neq h(a)$. Since f is injective we then have $f(g(a)) \neq f(h(a))$. But this is in contradiction with $f \circ g = f \circ h$. Hence f is a monomorphism. □

Definition 2.49 (Embedding)**.** A morphism $f\colon Y \to X$ is said to be an embedding if it is an injection and the map induced by restricting the codomain to the image of f is an isomorphism.

Lemma 2.50. *An embedding is a monomorphism.*

Proof. An embedding is injective so by the proposition above, we have that an embedding is a monomorphism. □

The dual notion of a monomorphism is that of an epimorphism.

Definition 2.51 (Epimorphism)**.** An epimorphism $e\colon A \to B$ is a morphism such that if $g, h\colon B \to C$ are morphisms then $g \circ e = h \circ e$ implies $g = h$.

Lemma 2.52. *If a morphism is surjective, it is an epimorphism.*

Proof. Let $f\colon A \to B$ be a surjective morphism. Suppose $g, h\colon B \to C$ are morphisms with $g \circ f = h \circ f$. Now assume that there exists a $b \in B$ with $g(b) \neq h(b)$. But f is surjective so there exists an $a \in A$ with $f(a) = b$. Then $g(f(a)) \neq h(f(a))$. But this is in contradiction with $g \circ f = h \circ f$. Hence f is an epimorphism. □

Definition 2.53 (Extremal monomorphism)**.** An extremal monomorphism is a monomorphism $m\colon A \to C$ such that if $g\colon B \to C$ is a morphism and $e\colon A \to B$ is an epimorphism such that $m = g \circ e$ then e is an isomorphism.

The dual notion of an extremal monomorphism is that of an extremal epimorphism.

Definition 2.54 (Extremal epimorphism)**.** An extremal epimorphism is an epimorphism $e\colon A \to C$ such that if $g\colon A \to B$ is a morphism and $m\colon B \to C$ is a monomorphism such that $e = m \circ g$ then m is an isomorphism.

Lemma 2.55. *In* Alg_k *if a morphism* $f\colon A \to B$ *is a monomorphism then it is an injection.*

Proof. Let $f\colon A \to B$ be a morphism. Let I be the kernel of f. Suppose f is not injective so there exists an $a \in I$ that is not equal to zero. Consider the morphism $g_1\colon k[x] \to A$ defined by $g_1(x) = 0$ and the morphism $g_2\colon k[x] \to A$ defined by $g_2(x) = a$. Then $f \circ g_1 = f \circ g_2$ but $g_1 \neq g_2$. Hence f is not a monomorphism. □

See [7].

Lemma 2.56. *In* Alg_k *a morphism* $f\colon A \to B$ *is an extremal epimorphism if and only if it is a surjection.*

Proof. Let $f\colon A \to B$ be a morphism in the category Alg_k.

$$\Longrightarrow$$

Suppose that $f\colon A \to B$ is not surjective. Define the map $f'\colon A \to f(A)$ as $f'(a) = f(a)$ for all $a \in A$. Define the map $i\colon f(A) \hookrightarrow B$ as $i(b) = b$ for all $b \in f(A)$. These are morphisms. Then f factorizes as follows $A \xrightarrow{f'} f(A) \overset{i}{\hookrightarrow} B$. An embedding is a monomorphism by lemma 2.50. But i is clearly not surjective (since f is not surjective), so it is not an isomorphism. Therefore we have $f = i \circ f'$ with i a monomorphism but not an isomorphism. Hence f is not an extremal epimorphism.

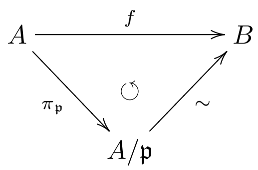
$$\Longleftarrow$$

Suppose f is surjective. Then by the proposition above we have that f is an epimorphism. Let $g\colon A \to C$ be a morphism and $h\colon C \to B$ a monomorphism such that $f = h \circ g$. Now h is injective by lemma 2.55. Because f is surjective and because $f = h \circ g$ we have that h is surjective. Hence h is an isomorphism. Therefore f is an extremal epimorphism. □

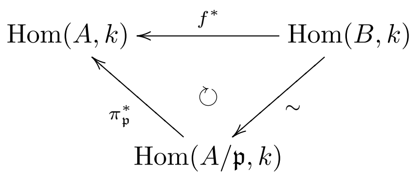
See [7].

Theorem 2.57. *Take an object A of Alg_k. Then the topology on $X(A)$ is the topology induced by setting finite unions of the images of extremal monomorphisms $f^*\colon \mathrm{Hom}(B,k) \to \mathrm{Hom}(A,k)$ to be the closed sets.*

Proof. Let $f\colon A \to B$ be an extremal epimorphism in the category Alg_k. Then f is surjective. Define $\mathfrak{p} := \ker(f)$. Then the map $\overline{f}\colon A/\mathfrak{p} \to B$, defined by $x + \mathfrak{p} \mapsto f(x)$ is an isomorphism. The following diagram commutes



Hence by the anti-equivalence the following diagram also commutes



We find $f^*(\mathrm{Hom}(B,k)) = \pi_{\mathfrak{p}}^*(\mathrm{Hom}(A/\mathfrak{p},k))$.
Every $\pi_{\mathfrak{p}}$ induced by a surjective $f\colon A \to B$ as in the upper diagram is a surjective morphism in Alg_k. Recall that every $\pi_{\mathfrak{p}}^*\colon \mathrm{Hom}(A/\mathfrak{p},k) \to \mathrm{Hom}(A,k)$ is defined by $g \mapsto g \circ \pi_{\mathfrak{p}}$. Hence

$$\{f^*(\mathrm{Hom}(B,k)) | f \text{ is surjective}\} \supset \{\pi_{\mathfrak{p}}^*(\mathrm{Hom}(A/\mathfrak{p},k)) | \mathfrak{p} \text{ is a prime ideal of } A\}.$$

On the other hand for every $f^*\colon \mathrm{Hom}(B,k) \to \mathrm{Hom}(A,k)$ induced by a surjective f, the above diagrams show there exists a $\pi_{\mathfrak{p}}^*\colon \mathrm{Hom}(A/\mathfrak{p},k) \to \mathrm{Hom}(A,k)$ with $f^*(\mathrm{Hom}(B,k)) = \pi_{\mathfrak{p}}^*(\mathrm{Hom}(A/\mathfrak{p},k))$. Therefore

$$\{f^*(\mathrm{Hom}(B,k)) | f \text{ is surjective}\} \subset \{\pi_{\mathfrak{p}}^*(\mathrm{Hom}(A/\mathfrak{p},k)) | \mathfrak{p} \text{ is a prime ideal of } A\}.$$

Hence we can conclude that

$$\{f^*(\mathrm{Hom}(B,k)) | f \text{ is surjective}\} = \{\pi_{\mathfrak{p}}^*(\mathrm{Hom}(A/\mathfrak{p},k)) | \mathfrak{p} \text{ is a prime ideal of } A\}.$$

Hence

$$\begin{aligned}
&\{\cup_{i=1}^n f_i^*(\mathrm{Hom}(B_i,k)) : f_i \text{ is surjective for all } i \in \{1,\cdots,n\}\} \\
= &\{\cup_{i=1}^n \pi_{I_i}^*(\mathrm{Hom}(A/I_i,k)) : I_i \text{ is a prime ideal of } A \text{ for all } i \in \{1,\cdots,n\}\}.
\end{aligned}$$

We have that $\pi_{\mathfrak{p}}$ sends elements of $\mathfrak{p}$ to $0 + \mathfrak{p}$. Hence for all $g \in \mathrm{Hom}(A/\mathfrak{p}, k)$ the map $g \circ \pi_{\mathfrak{p}}$ sends all elements of $\mathfrak{p}$ to zero. By definition we have $\pi_{\mathfrak{p}}^*(\mathrm{Hom}(A/\mathfrak{p}, k)) = \{g \circ \pi_{\mathfrak{p}} | g \in \mathrm{Hom}(A/\mathfrak{p}, k)\}$. Hence $\pi_{\mathfrak{p}}^*(\mathrm{Hom}(A/\mathfrak{p}, k)) \subset \{f \in \mathrm{Hom}(A, k) | \mathfrak{p} \subset \ker(f)\}$.

If $f \in \mathrm{Hom}(A, k)$ and $\mathfrak{p} \subset \ker(f)$ then $f = \overline{f} \circ \pi_{\mathfrak{p}}$ with $\overline{f} \colon A/\mathfrak{p} \to k$, defined by $x + \mathfrak{p} \to f(x)$. Hence $\pi_{\mathfrak{p}}^*(\mathrm{Hom}(A/\mathfrak{p}, k)) \supset \{f \in \mathrm{Hom}(A, k) | \mathfrak{p} \subset \ker(f)\}$. Therefore we have

$$\pi_{\mathfrak{p}}^*(\mathrm{Hom}(A/\mathfrak{p}, k)) = \{f \in \mathrm{Hom}(A, k) | \mathfrak{p} \subset \ker(f)\}.$$

Hence $\pi_{\mathfrak{p}}^*(\mathrm{Hom}(A/\mathfrak{p}, k)) = V(\mathfrak{p})$. Recall that the topology on $X(A)$ is the topology induced by setting the $V(I)$ with I an ideal of A as closed sets.

Every finitely generated k-algebra has a primary decomposition. Combine [1] corollary 7.7 with theorem 7.13. Hence every ideal I can be written as a finite intersection of primary ideals $\cap_{i=1}^n q_i$. In section 2.2 we checked that $V(I \cap J) = V(I) \cup V(J)$ for all ideals I and J of A. The radical of a primary ideal is a prime ideal. See [1] proposition 4.1. Let I be an ideal of A. Consider $V(I) = \{\varphi \in \mathrm{Hom}(A, k) | I \subset \ker(\varphi)\}$. Note that $\ker(\varphi)$ is a prime ideal, so $V(I) = V(\sqrt{I})$. Hence for every ideal I of A we have

$$\begin{aligned} V(I) &= V(\cap_{i=1}^n q_i) \\ &= \cup_{i=1}^n V(q_i) \\ &= \cup_{i=1}^n V(\sqrt{q_i})) \end{aligned}$$

for some primary ideals q_i. Hence every ideal I of A can be written as the intersection of finitely many prime ideals.

Therefore the topology on $X(A)$ is equal to the topology induced by setting finite unions of images of morphisms $f^* \colon \mathrm{Hom}(B, k) \to \mathrm{Hom}(A, k)$, defined by $g \mapsto g \circ f$ with $f \colon A \to B$ a surjective map, to be the closed sets. We have seen that for a morphism $f \colon A \to B$ in $\mathrm{Alg_k}$ the property of being surjective is equivalent to being an extremal epimorphism. Because the contravariant functor H is an anti-equivalence a morphism $f^* \colon \mathrm{Hom}(B, k) \to \mathrm{Hom}(A, k)$ is an extremal monomorphism if and only if $f \colon A \to B$ is an extremal epimorphism. Hence the topology on $X(A)$ is equal to the topology induced by setting finite unions of images of extremal monomorphisms as closed sets. □

The idea of this proof comes from [7].

3 The usual definition of an affine variety

3.1 A second definition of an affine variety

Most of this section can be found in [2] section 1.1.
Let k be an algebraically closed field.

Notation 3.1 ($\mathbb{A}^n$)**.** $\mathbb{A}^n = \{(a_1, \ldots, a_n) | a_1, \ldots, a_n \in k\}$

Notation 3.2 (A)**.** $A = k[x_1, \ldots, x_n]$

Definition 3.3 (Zero set of T)**.** For a subset T of A we define the zero set of T as

$$Z(T) := \{p \in \mathbb{A}^n | f(p) = 0 \text{ for all } f \in T\}.$$

Definition 3.4 (Algebraic set)**.** A subset Y of $\mathbb{A}^n$ is an algebraic set if there exists a subset T of A with $Y = Z(T)$.

Note that $Z(T)$ is equal to $Z(\langle T \rangle)$ with $\langle T \rangle$ the ideal generated by T. The inclusion $Z(T) \supset Z(\langle T \rangle)$ is obvious. For the other inclusion we can remark that if for a $p \in \mathbb{A}^n$ we have $f(p) = 0$ for all $f \in T$ then for any finite sum of elements of T filling in p gives 0.

Proposition 3.5. *Setting the algebraic sets as closed sets induces a topology on $\mathbb{A}^n$.*

Proof. Note that $Z((1)) = \emptyset$ and that $Z((0)) = \mathbb{A}^n$.
Let $Z(T_1)$ and $Z(T_2)$ be two algebraic sets.
Take a $p \in Z(T_1) \cup Z(T_2)$. It must hold that $p \in Z(T_1)$ or $p \in Z(T_2)$. So for $f \in T_1$ and $g \in T_2$ we have $f(p) = 0$ or $g(p) = 0$ so $f \cdot g(p) = 0$. So

$$Z(T_1) \cup Z(T_2) \subset Z(T_1 \cdot T_2).$$

Take a $p \in Z(T_1 \cdot T_2)$ Suppose $p \notin Z(T_1)$ then there exists an $f \in T_1$ with $f(p) \neq 0$. But for every $g \in T_2$ it must hold that $f \cdot g(p) = 0$. So $p \in Z(T_2)$, and thus $p \in Z(T_1) \cup Z(T_2)$. So

$$Z(T_1) \cup Z(T_2) \supset Z(T_1 \cdot T_2).$$

Take a family $\{Z(T_i)\}_{i \in I}$ of algebraic sets. Consider $\cap_{i \in I} Z(T_i)$.
Take a $p \in \cap_{i \in I} Z(T_i)$. Then for all i we have that p is a zero of all $f \in T_i$. So $p \in Z(\cup_{i \in I} T_i)$. So

$$\cap_{i \in I} Z(T_i) \subset Z(\cup_{i \in I} T_i).$$

Take a $p \in Z(\cup_{i \in I} T_i)$. Then for all i we have that p is a zero of all $f \in T_i$. So $p \in \cap_{i \in I} Z(T_i)$. So

$$\cap_{i \in I} Z(T_i) \supset Z(\cup_{i \in I} T_i).$$

□

Definition 3.6 (Zariski topology)**.** We call the topology defined by setting the algebraic sets as the closed sets the Zariski topology.

Definition 3.7 (Irreducible)**.** A non-empty topological space Y is called irreducible if it cannot be written as the union of two proper subsets (not equal to the whole set and not equal to the empty set) Y_1 and Y_2 that are closed in Y.

Proposition 3.8. *Let Y be an irreducible algebraic set and let $f \colon Y \to X$ be a continuous map, then $f(Y)$ is irreducible.*

Proof. Suppose $f(Y) = W_1 \cup W_2$ with $W_1, W_2 \subset X$ closed and non-empty subsets of X. Then

$$f^{-1}(f(Y)) = f^{-1}(W_1 \cup W_2) \text{ and } f^{-1}(W_1 \cup W_2) = f^{-1}(W_1) \cup f^{-1}(W_2).$$

We also have $Y = f^{-1}(f(Y))$ since $f^{-1}(f(Y))$ must be contained in the domain of f and we have $Y \subset f^{-1}(f(Y))$. So $Y = f^{-1}(W_1) \cup f^{-1}(W_2)$. Since f is continuous $f^{-1}(W_1)$ and $f^{-1}(W_2)$ are closed in Y. Since Y is irreducible this means that

$$Y = f^{-1}(W_1) \text{ or } Y = f^{-1}(W_2).$$

So

$$f(Y) = f(f^{-1}(W_1)) \subset W_1 \text{ or } f(Y) = f(f^{-1}(W_2)) \subset W_2.$$

But since $f(Y) = W_1 \cup W_2$ we have $f(Y) = W_1$ or $f(Y) = W_2$. So $f(Y)$ is irreducible. □

Definition 3.9 (Ideal of a subset of $\mathbb{A}^n$ in A)**.** The ideal of a subset Y of $\mathbb{A}^n$ in A is defined by

$$I(Y) := \{f \in A | f(p) = 0 \text{ for all } p \in Y\}.$$

Proposition 3.10.

- *For subsets $T_1, T_2 \subset A$ if $T_1 \subset T_2$, then $Z(T_1) \supset Z(T_2)$.*
- *For subsets $Y_1, Y_2 \subset \mathbb{A}^n$ if $Y_1 \subset Y_2$, then $I(Y_1) \supset I(Y_2)$.*
- *For subsets $Y_1, Y_2 \subset \mathbb{A}^n$ we have $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.*

Proof.

- If for a $p \in \mathbb{A}^n$ we have $f(p) = 0$ for all $f \in T_2$, then in particular we have $g(p) = 0$ for all $g \in T_1$.
- If for an $f \in A$ we have $f(p) = 0$ for all $p \in Y_2$, then in particular we have $f(q) = 0$ for all $q \in Y_1$.
- The set $I(Y_1 \cup Y_2)$ consists of all $f \in A$ with $f(p) = 0$ for all elements p of Y_1 and all elements p of Y_2. So does $I(Y_1) \cap I(Y_2)$.

□

Lemma 3.11 (Hilbert's Nullstellensatz)**.** *Let k be an algebraically closed field. Let $\mathfrak{a}$ be an ideal in $A = k[x_1, \ldots, x_n]$. Let $f \in A$ be a polynomial that vanishes at all points of $Z(\mathfrak{a})$. Then $f^n \in \mathfrak{a}$ for some $n \in \mathbb{Z}_{>0}$.*

See [2] theorem 1.3.
Hence for any ideal $\mathfrak{a} \subset A$, $I(Z(\mathfrak{a}))$ is equal to the radical of $\mathfrak{a}$.

Proposition 3.12. *For a subset $Y \subset \mathbb{A}^n$ we have $Z(I(Y)) = \overline{Y}$.*

Proof.

$$\supset$$

Note that $Y \subset Z(I(Y))$ and that $Z(I(Y))$ is an algebraic set. So $Z(I(Y)) \supset \overline{Y}$.

$$\subset$$

Let W be a closed set with $Y \subset W$. Then $W = Z(\mathfrak{a})$ for some ideal $\mathfrak{a}$. Then we have $Y \subset Z(\mathfrak{a})$. By 3.10 we now know that $I(Y) \supset I(Z(\mathfrak{a}))$. Hilbert's Nullstellensatz tells us in particular that $I(Z(\mathfrak{a})) \supset \mathfrak{a}$. So $I(Y) \supset \mathfrak{a}$. By 3.10 we know that $Z(I(Y)) \subset Z(\mathfrak{a})$. Because $\overline{Y}$ is a closed set with $Y \subset \overline{Y}$ we can conclude that $Z(I(Y)) \subset \overline{Y}$. □

Proposition 3.13. *An algebraic set is irreducible if and only if its ideal is prime.*

Proof.

$$\Longrightarrow$$

Let Y be an irreducible algebraic set. Suppose $f \cdot g \in I(Y)$, then $Y \subset Z(f \cdot g) = Z(f) \cup Z(g)$. Then $Y = Y \cap Z(f)$ or $Y = Y \cap Z(g)$, so $Y \subset Z(f)$ or $Y \subset Z(g)$. So $f \in I(Y)$ or $g \in I(Y)$.

$$\Longleftarrow$$

Let $\mathfrak{p}$ be a prime ideal. A prime ideal is a radical ideal. Suppose that Y_1 and Y_2 are closed sets and that $Z(\mathfrak{p}) = Y_1 \cup Y_2$. Then by Hilbert's Nullstellensatz we have $\mathfrak{p} = I(Z(\mathfrak{p}))$. By 3.10 we have $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$. So $\mathfrak{p} = I(Y_1) \cap I(Y_2)$.
Suppose that $\mathfrak{p} \subsetneq I(Y_1)$ and that $\mathfrak{p} \subsetneq I(Y_2)$. Then there exists an $i \in I(Y_1)$ and a $j \in I(Y_2)$ with $i, j \notin \mathfrak{p}$ so $i \cdot j \notin \mathfrak{p}$ but $i \cdot j \in I(Y_1) \cap I(Y_2)$. This is a contradiction.
So $\mathfrak{p} = I(Y_1)$ or $\mathfrak{p} = I(Y_2)$. So $Z(\mathfrak{p}) = Z(I(Y_1))$ or $Z(\mathfrak{p}) = Z(I(Y_2))$. By Hilbert's Nullstellensatz we have $Z(I(Y_1)) = Y_1$ and $Z(I(Y_2)) = Y_2$. So $Z(\mathfrak{p}) = Y_1$ or $Z(\mathfrak{p}) = Y_2$. So $Z(\mathfrak{p})$ is irreducible. □

Let $X \subset \mathbb{A}^n$ be an irreducible algebraic set and let $I(X)$ be the ideal of X. Then all elements of $I(X)$ are zero on X. So for $f, g \in k[x_1, \ldots, x_n]$ such that $g - f \in I(X)$ we have that they are as functions the same on X. Hence we can view elements of $k[x_1, \ldots, x_n]/I(X)$ as functions on X. Let $A := k[x_1, \ldots, x_n]/I(X)$ and consider A as a set of polynomial functions on X. Since X is irreducible we have that $I(X)$ is a prime ideal and thus A is an integral domain. So we can consider its field of fractions $Q(A)$.
We define a sheaf on X as follows.

Definition 3.14 (Sheaf on an irreducible algebraic set)**.** Let X be an irreducible algebraic set. For an $x \in X$ define $\mathcal{O}_x = \{\frac{f}{g} | f, g \in A, g(x) \neq 0\} \subset Q(A)$. For U open (with respect to the Zariski topology) in X we define $\mathcal{O}_X(U) = \cap_{x \in U} \mathcal{O}_x$. For opens $V \subset U$ of X we define the restriction map $\mathrm{res}_{U,V} \colon \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ simply as the restriction of functions, that is $\mathrm{res}_{U,V}(\frac{f}{g}) = \frac{f}{g}|_V$.

Definition 3.15 (Affine variety)**.** In this section we define an affine variety as a locally ringed space that is isomorphic to an irreducible algebraic set together with its structure sheaf.

In some literature an affine variety need not be irreducible in which case slightly different results are obtained.

Definition 3.16 (Morphism of affine varieties)**.** We define a morphism between affine varieties $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ as a map $m \colon X \to Y$ such that

- m is continuous and
- for all $U \subset Y$ open and for all $f \in \mathcal{O}_Y(U)$ we have $f \circ m \colon m^{-1}(U) \to k$ is in $\mathcal{O}_X(m^{-1}(U))$.

Lemma 3.17. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be affine varieties and let $Y \subset \mathbb{A}^n$. A map $\psi \colon X \to Y$ is a morphism if and only if $x_1 \circ \psi, \ldots, x_n \circ \psi \in \mathcal{O}_X(X)$ where $x_1, \ldots, x_n$ are the coordinate functions on $\mathbb{A}^n$.*

See [2] lemma 3.6.

Definition 3.18 (Category of affine varieties)**.** Let $\mathrm{affvar_k}$ denote the category of locally ringed spaces that are isomorphic to an affine variety that can be embedded into $\mathbb{A}^n$ for some $n \in \mathbb{Z}_{\geq 0}$.

Example 3.19. We know that $\mathbb{A}^n = Z((0))$ holds. Because (0) is a radical ideal the Nullstellensatz tells us that $I(\mathbb{A}^n) = I(Z((0))) = 0$. Note that (0) is a prime ideal. With proposition 3.13 we can conclude that $\mathbb{A}^n$ is irreducible. We can prove the Cayley-Hamilton theorem using the fact that $\mathbb{A}^n$ is irreducible.

> [Cayley-Hamilton] Let k be a field. Let $M \in \mathrm{Mat}_n(k)$ be an $n \times n$ matrix with coefficients in k. Let $P_M = \det(X \cdot I_n - M) = \det \begin{pmatrix} X - m_{11} & \dots & -m_{1n} \\ \vdots & \ddots & \vdots \\ -m_{n1} & \dots & X - m_{nn} \end{pmatrix}$ be the characteristic polynomial of M. Then $P_M(M) = 0$.

Proof. If the coefficients of M are elements of the field k, then they are also elements of $\overline{k}$. So we can assume that k is algebraically closed. We identify $\mathrm{Mat}_n(k)$ with $\mathbb{A}^{n^2}$ by sending a matrix M to its vector of coefficients (m_{ij}), where we order the m_{ij} lexicographically.

Consider $P_M(X) = \det(X \cdot I_n - M)$; this is a polynomial in X of degree n. So $P_M(M)$ is a polynomial in M of degree n, $P_M(M) = \sum_{i=0}^n b_i M^i$ where the b_k are polynomials in the m_{ij}. The sum $\sum_{i=0}^n b_i M^i$ itself is an $n \times n$ matrix, say M'.
Consider the set $Y = \{M \in \mathrm{Mat}_n(k) | P_M(M) = 0\}$. We have that

$$Y = \{M \in \mathrm{Mat}_n(k) | m'_{ij}(M) = 0 \, \forall i, j \in \{0, \dots, m\}\}.$$

If we identify the matrices M with the elements of $\mathbb{A}^n$ then Y is Zariski closed.

Suppose $\prod_{i<j}(\lambda_i - \lambda_j)^2 =: \mathrm{disc}(P_M) \neq 0$ (where the set of λ_i is the set of roots of P_M), then P_M has exactly n different zeros λ_i, so it is of the form $P_M = \prod_{i=1}^n (X - \lambda_i)$. Because P_M has n different zeros we know that M has exactly n different eigenvalues so M is diagonalizable and can be written as $Q^{-1}DQ$ for some invertible Q and $D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}$ We have that

$$\begin{aligned} P_M(M) &= \prod_{i=1}^n (M - \lambda_i I) \\ &= \prod_{i=1}^n (Q^{-1}DQ - \lambda_i I) \\ &= Q^{-1}(\prod_{i=1}^n (D - \lambda_i I))Q \\ &= Q^{-1} 0 Q \\ &= 0. \end{aligned}$$

So $M \in Y$. So $\{M \in \mathrm{Mat}_n(k) | \mathrm{disc}(P_M) \neq 0\} \subset Y$.

Consider $\text{disc}(P_M) = \prod_{i<j}(\alpha_i - \alpha_j)^2$ this is a polynomial in the coefficients of P_M, the coefficients m'_{ij} of $P_M = M'$ are polynomials in the m_{ij}. So $\text{disc}(P_M)$ is a polynomial in the entries of M.
So $Z = \{M \in \text{Mat}_n(k) | \, \text{disc}(P_M) = 0\}$ is Zariski closed.

We have that $\mathbb{A}^n = Y \cup Z$. Because $\mathbb{A}^n$ is irreducible and Y and Z are closed it must hold that $\mathbb{A}^n = Y$ or $\mathbb{A}^n = Z$. Now take n different $\lambda_1, \ldots, \lambda_n \in k$. The field k is algebraically closed and thus infinite so for every $n \in \mathbb{N}$ we can find n different $\lambda_1, \ldots, \lambda_n \in k$.

Then $\begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{pmatrix}$ is not an element of Z.

So $\mathbb{A}^n = Y$. □

See [5] 1.27.

3.2 How these two definitions of an affine variety coincide

3.2.1 Anti-equivalence

Just like AffVar_k the category affvar_k is anti-equivalent with Alg_k. Before we investigate the anti-equivalence between affvar_k and Alg_k we will introduce some definitions.

Definition 3.20 (Fully faithful)**.** A co(ntra)variant functor $F\colon C \to D$ is fully faithful if for all objects X and Y in C in the covariant case the map of sets $F_{X,Y}\colon \text{Hom}(X,Y) \to \text{Hom}(F(X),F(Y))$ induced by F is bijective and in the contravariant case the map of sets $F_{X,Y}\colon \text{Hom}(X,Y) \to \text{Hom}(F(Y),F(X))$ induced by F is bijective.

Definition 3.21 (Essentially surjective)**.** A functor $F\colon C \to D$ is essentially surjective if for all objects A in D there exists an object X in C such that $F(X)$ is isomorphic to A in D.

Lemma 3.22. *Two categories are (anti)-equivalent if and only if there is a co(ntra)variant functor between them that is fully faithful and essentially surjective.*

Proof. Let C and D be categories.

$$\Longleftarrow$$

Assume that $F\colon C \to D$ is fully faithful and essentially surjective. Because F is essentially surjective for every object A of D there exists an isomorphism $i_A\colon F(X) \to A$ with X an object in C. Define for every $F(X)$ in D the isomorphism $i_{F(X)} = \text{id}_{F(X)}$. Fix for every A in D for which there is no X in C such that $F(X) = A$ an isomorphism i_A. Note that this involves the axiom of choice.
Define a functor $G\colon D \to C$ by sending each object A in D to the object X in C with $i_A(F(X)) = A$ and by sending each morphism $m\colon A \to B$ with $i_A^{-1}(A) = F(X)$ and $i_B^{-1}(B) = F(Y)$ to $F_{X,Y}^{-1}(i_B^{-1} \circ m \circ i_A)$.
For every object A of D define morphisms $\alpha_A\colon F \circ G \to I_D$ as the isomorphisms i_A. Then $\alpha_B \circ i_B^{-1} \circ m \circ i_A = m \circ \alpha_A$, so $\alpha_B \circ F \circ F_{X,Y}^{-1}(i_B^{-1} \circ m \circ i_A) = m \circ \alpha_A$, so $\alpha_B \circ F \circ G(m) = I_D(m) \circ \alpha_A$. Hence there exists a natural isomorphism between $F \circ G$ and I_D.

For every object X of C define $\beta_X\colon G\circ F(X)\to I_C(X)$ as the identity on X. Let $m\colon X\to Y$ be a morphism in C. Then $\beta_Y\circ m = m\circ\alpha_X$, so $\beta_Y\circ F^{-1}_{X,Y}(F(m)) = m\circ\beta_X$. Note that $F(m) = i^{-1}_{F(Y)}\circ F(m)\circ i_{F(X)}$ and that $G(F(m)) = F^{-1}_{X,Y}(i^{-1}_{F(Y)}\circ F(m)\circ i_{F(X)})$. Hence $\beta_Y\circ G(F(m)) = m\circ\beta_X$, so $\beta_Y\circ G(F(m)) = I_C(m)$. Hence there exists a natural isomorphism between $G\circ F$ and I_C.

$$\Longrightarrow$$

Let $T_1\colon F\circ G\to \mathrm{id}_D$ and $T_2\colon G\circ F\to\mathrm{id}_C$ be natural isomorphisms. Let X and Y be objects in C. Let $\alpha_X\colon G(F(X))\to I_C(X)$ and $\alpha_Y\colon G(F(Y))\to I_C(Y)$ be isomorphisms such that for every morphism $f\colon X\to Y$ in C it holds that $\alpha_Y\circ G(F(f)) = I_C(f)\circ\alpha_X$. Consider the maps

$$\begin{aligned} F_{X,Y}\colon\ &\mathrm{Hom}(X,Y)\to\mathrm{Hom}(F(X),F(Y))\\ &(f\colon X\to Y)\mapsto (F(f)\colon F(X)\to F(Y)) \end{aligned}$$

and

$$\begin{aligned} G_{F(X),F(Y)}\colon\ &\mathrm{Hom}(F(X),F(Y))\to\mathrm{Hom}(G(F(X)),G(F(Y)))\\ &(g\colon F(X)\to F(Y))\mapsto (G(g)\colon G(F(X))\to G(F(Y))) \end{aligned}$$

and

$$\begin{aligned} T_{G(F(X)),G(F(Y))}\colon\ &\mathrm{Hom}(G(F(X)),G(F(Y)))\to\mathrm{Hom}(X,Y)\\ &(h\colon G(F(X))\to G(F(Y)))\mapsto \alpha_Y\circ h\circ\alpha_X^{-1}. \end{aligned}$$

The composition $T_{G(F(X)),G(F(Y))}\circ G_{F(X),F(Y)}\circ F_{X,Y}$ is the identity. Hence $F_{X,Y}$ is a bijection, so F is fully faithful.

Let A be an object in D. Because there is a natural isomorphism $T_2\colon G\circ F\to\mathrm{id}_C$ there exists an isomorphism $\alpha_A\colon F(G(A))\to I_D(A)$. Hence F is essentially surjective.

The proof for the contravariant case is analogous to this proof. □

See [10] lemma 1.4.9.

Define the functor

$$\begin{aligned} F\colon\ \mathrm{affvar_k}&\to\mathrm{Alg_k}\\ (X,\mathcal{O}_X)&\mapsto\mathcal{O}_X(X)\\ \varphi&\mapsto\varphi^*. \end{aligned}$$

Note that $\mathcal{O}_X(X) = \cap_{x\in X}\mathcal{O}_x = A$ and A is a finitely generated k-algebra that is also an integral domain.

This functor is fully faithful. See [2] proposition I.3.5.

Let A be a finitely generated k-algebra that is an integral domain. So it can be generated by say $y_1,\dots,y_n\in A$. Then the following ring homomorphism is surjective

$$\begin{aligned} \psi\colon k[x_1,\dots,x_n]&\to A\\ x_i&\mapsto y_i. \end{aligned}$$

So $A\cong k[x_1,\dots,x_n]/\ker(\psi)$. So $\ker(\psi)$ is a prime ideal. Consider $X := V(\ker(\psi))\subset\mathbb{A}^n$. Then $\mathcal{O}_X(X)\cong A$. So F is essentially surjective.

Note that we can compose the functors

$$\begin{aligned} F\colon \text{affvar}_\text{k} &\to \text{Alg}_\text{k} \\ (X, \mathcal{O}_X) &\mapsto \mathcal{O}_X(X) \\ \varphi &\mapsto \varphi^* \end{aligned}$$

and

$$\begin{aligned} G\colon \text{Alg}_\text{k} &\to \text{AffVar}_\text{k} \\ A &\mapsto (X(A), \mathcal{O}_{X(A)}) \\ f &\mapsto f^* \end{aligned}$$

to obtain a covariant functor $G \circ F\colon \text{affvar}_\text{k} \to \text{AffVar}_\text{k}$.

3.2.2 Similarity

Take an irreducible algebraic set $X \subset \mathbb{A}^n$. Define $A = k[x_1, \dots, x_n]/I(X)$. Let $X(A) = \text{Hom}(A, k)$.

The Nullstellensatz (together with other lemmas that we have seen in section 3.1) asserts that there is an inclusion reversing bijection between prime ideals of A and closed subsets of $X \subset \mathbb{A}^n$. Hence maximal ideals correspond to points in $\mathbb{A}^n$. The weak version of the Nullstellensatz asserts there is a one-to-one correspondence between homomorphisms in $\text{Hom}(A, k)$ and maximal ideals of A. So we have that elements of $X(A)$ correspond to points in X via maximal ideals.

We also know which homomorphisms correspond to which points. Elements of $\text{Hom}(A, k)$ are fixed by their images of $x_1, \dots, x_n$. Take a homomorphism $\varphi\colon A \to k$. Then its kernel is $(x_1 - \varphi(x_1), \dots, x_n - \varphi(x_n))$ so $A/(x_1 - \varphi(x_1), \dots, x_n - \varphi(x_n)) \cong k$. The maximal ideal $(x_1 - \varphi(x_1), \dots, x_n - \varphi(x_n))$ corresponds to the point $(\varphi(x_1), \dots, \varphi(x_n))$. We find the following bijection

$$\begin{aligned} \text{Hom}(A, k) &\leftrightarrow X \\ \varphi &\mapsto (\varphi(x_1), \dots, \varphi(x_n)) \\ \begin{pmatrix} A & \to & k \\ x_i & \mapsto & p_i \end{pmatrix} &\leftarrow\!\shortmid (p_1, \dots p_n). \end{aligned}$$

3.2.3 Trade off

The second definition of an affine variety uses the embedding of the irreducible algebraic set into an $\mathbb{A}^n$ for some n. However this embedding is not very intrinsic since there exist isomorphisms between irreducible algebraic sets that are embedded in respectively $\mathbb{A}^n$ and $\mathbb{A}^m$ with n and m different.

Example 3.23. Let $X = \mathbb{A}^n$. Take a nonconstant $f \in k[x_1, \dots, x_n] = \mathcal{O}_X(X)$. Let $Y = X \setminus Z(f)$. Consider the following equalities

$$\begin{aligned} \mathcal{O}_X(Y) &\cong k[x_1, \dots, x_n]_f \\ &= k[x_1, \dots, x_n, \frac{1}{f}] \\ &\cong k[x_1, \dots, x_n, x_{n+1}]/(x_{n+1} \cdot f - 1) \\ &\cong k[x_1, \dots, x_n, x_{n+1}]/I(Z(x_{n+1} \cdot f - 1)) \end{aligned}$$

with $Z(x_{n+1} \cdot f - 1) \subset \mathbb{A}^{n+1}$.

This gives an explicit isomorphism

$$\begin{aligned} Z(x_{n+1} \cdot f - 1) &\to Y \\ (x_1, \ldots, x_n, x_{n+1}) &\mapsto (x_1, \ldots, x_n) \\ (x_1, \ldots, x_n, \frac{1}{f(x_1, \ldots, x_n)}) &\leftarrow\!\shortmid (x_1, \ldots, x_n). \end{aligned}$$

Note that $(Z(x_{n+1} \cdot f - 1), \mathcal{O}_{Z(x_{n+1} \cdot f - 1)})$ is an affine variety. Hence $(Y, \mathcal{O}_Y)$ is an affine variety as well and we have found an isomorphism between two affine varieties that have embeddings in respectively $\mathbb{A}^{n+1}$ and $\mathbb{A}^n$.

Hence we might conclude that the first definition of an affine variety is prettier.

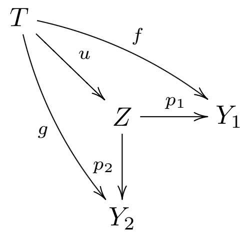
On the other hand some verifications are less confusing when one uses the second (and more widely used) definition. So it might be more convenient to start with the second definition and then look at the similarity between the two definitions when one wants to use the second definition.

4 The category of varieties

Varieties are a special kind of prevarieties. To formulate a version of the extra condition that is put on varieties we need to define the notion of a product.

4.1 Products

Definition 4.1 (Product)**.** Let C be a category and let Y_1 and Y_2 be objects in this category. A product of Y_1 and Y_2 is a triple $(Z, p_1 \colon Z \to Y_1, p_2 \colon Z \to Y_2)$ with the universal property that for all objects T and morphisms $f \colon T \to Y_1$ and $g \colon T \to Y_2$ there exists a unique morphism $u \colon T \to Z$ such that the following diagram commutes.



A product, once it exists, is unique up to a unique isomorphism. We denote it by $Y_1 \times Y_2$.

Proposition 4.2. *Let $(X, \mathcal{O}_X)$ with $X \subset \mathbb{A}^n$ and $(Y, \mathcal{O}_Y)$ with $Y \subset \mathbb{A}^m$ be affine varieties. Then $(X \times Y, \mathcal{O}_{X \times Y})$ with $X \times Y$ the set theoretic product of X and Y is again an affine variety and the projections $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$ make it into a product.*

Proof. Identify $\mathbb{A}^n \times \mathbb{A}^m$ with $\mathbb{A}^{n+m}$.

We will first check that $X \times Y$ is a closed subset and is irreducible for the Zariski topology on $\mathbb{A}^{n+m}$. Let $X = Z(f_1, \dots, f_r)$ for some polynomials $f_i \in k[x_1, \dots, x_n]$ and $Y = Z(g_1, \dots, g_s)$ for some polynomials $g_i \in k[x_{n+1}, \dots, x_{n+m}]$. Then $X \times Y$ is the zero set of the ideal $(f_1, \dots, f_r, g_1, \dots, g_s) \subset k[x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}]$. So $X \times Y$ is a closed subset of $\mathbb{A}^{n+m}$.

Now we will verify that $X \times Y$ is irreducible. Suppose that $X \times Y$ is not irreducible. Then there exist closed subsets $Z_1, Z_2 \subsetneq X \times Y$ whose union is $X \times Y$. Let $X_1 = \{p \in X | \{p\} \times Y \subset Z_1\}$ and $X_2 = \{p \in X | \{p\} \times Y \subset Z_2\}$. Suppose there is a $q \in X \setminus (X_1 \cup X_2)$. Then we have $\{q\} \times Y = [(\{q\} \times Y) \cap Z_1] \cup [(\{q\} \times Y) \cap Z_2]$. Note that $(\{q\} \times Y) \cap Z_1$ and $(\{q\} \times Y) \cap Z_2$ are both proper closed subsets. So $\{q\} \times Y$ is not irreducible which contradicts the irreducibility of Y. Hence $X = X_1 \cup X_2$. Next we observe that X_1 and X_2 are closed in X. Indeed if $p \notin X_1$ then this means there is a point $q \in Y$ with $(p, q) \notin Z$. The set $\{r \in X | (r, q) \notin Z_1\}$ is then open in X, disjoint from X_1 and containing p. Hence $X \setminus X_1$ is open. Therefore X_1 is closed. For the same reason X_2 is closed. But since Z_1 and Z_2 are proper subsets of $X \times Y$ we have that X_1 and X_2 are proper subsets of X which contradicts the irreducibility of X. Hence $X \times Y$ is irreducible.

Now we will verify that $(X \times Y, \mathcal{O}_{X \times Y})$ with projections $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$ indeed is the product of $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$. Note that the projections $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$ are indeed morphisms. Let $(T, \mathcal{O}_T)$ be an affine variety with maps $f \colon T \to X$ and $g \colon T \to X$. Consider the map

$$\begin{aligned} u \colon T &\to X \times Y \\ p &\mapsto (f(p), g(p)). \end{aligned}$$

Clearly this is the only map such that $f = p_1 \circ u$ and $g = p_2 \circ u$.

Now we will verify that u is a morphism. We know that f and g are morphisms. So by proposition 3.17 we know that $x_1 \circ f, \dots, x_n \circ f, x_{n+1} \circ g, \dots, x_{n+m} \circ g \in \mathcal{O}_X(X)$. Note that $x_1 \circ f = x_1 \circ u, \dots, x_n \circ f = x_n \circ u, x_{n+1} \circ g = x_{n+1} \circ u, \dots, x_{n+m} \circ g = x_{n+m} \circ u$. Hence by the same proposition we know that u is a morphism. □

See [5] proposition 2.26.

Since we have an anti-equivalence between affvar_k and Alg_k we could also have proved lemma 4.2 in terms of k-algebras. The coproduct, which is the dual notion of a product, of two k-algebras is the tensorproduct of these k-algebras (see proposition 6.1 of [4]). To assert lemma 4.2 is true we could have verified that the tensorproduct of two finitely generated k-algebras that are an integral domain is again a finitely generated k-algebra that also is an integral domain. Even though we already have another proof we can still use this fact to deduce the following.

Lemma 4.3. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be affine varieties and $(X \times Y, \mathcal{O}_{X\times Y})$ with the projections $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$ their product. Then $\mathcal{O}_{X\times Y}(X \times Y) = \mathcal{O}_X(X) \otimes_k \mathcal{O}_Y(Y)$.*

Proof. The anti-equivalence between affvar_k and Alg_k sends an affine variety $(X, \mathcal{O}_X)$ to $\mathcal{O}_X(X)$. The coproduct of two k-algebras is the tensorproduct of these k-algebras. □

4.2 Prevarieties

Definition 4.4 (Prevariety)**.** A locally ringed space $(X, \mathcal{O}_X)$, with $\mathcal{O}_X$ a sheaf of k-valued functions, is a prevariety if

- X is connected (that is cannot be represented as the union of two or more disjoint nonempty open subsets), and
- there is a finite open covering $\{U_i\}_{i\in I}$ of X such that for all $i \in I$ we have that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine variety.

Definition 4.5 (Morphism of prevarieties)**.** We define a morphism between prevarieties $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ as a map $m \colon X \to Y$ such that

- m is continuous and
- for all $U \subset Y$ open and for all $f \in \mathcal{O}_Y(U)$ we have $f \circ m \colon m^{-1}(U) \to k$ is in $\mathcal{O}_X(m^{-1}(U))$.

Note that the product of two affine varieties in the category of prevarieties coincides with the product of two affine varieties in the category of affine varieties.

Proposition 4.6. *The product of two prevarieties $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is again a prevariety.*

Proof. Let $X = \cup_{i=1}^n U_i$ and let $Y = \cup_{i=1}^m V_i$ with $(U_i, \mathcal{O}_{U_i})$ and $(V_i, \mathcal{O}_{V_i})$ affine varieties. Note that $X \times Y$ is the union of the sets $U_i \times V_j$ and that all $(U_i \times V_j, \mathcal{O}_{U_i\times V_j})$ are affine varieties. Defining a subset $Z \subset X \times Y$ to be open if and only if $Z \cap (U_i \times V_j)$ is open in $U_i \times V_j$ for all $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$ we obtain a topology on $X \times Y$. Note that because X and Y are connected, $X \times Y$ is connected.

Further for an open subset $Z \subset X \times Y$ we define a function $f \colon Z \to k$ to be an element of $\mathcal{O}_{X\times Y}(Z)$ if and only if $f|_{Z\cap(U_i\times V_j)} \in \mathcal{O}_{U_i\times V_j}(Z \cap (U_i \times V_j))$ for all

$i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. Note that $(U_i \times V_j) \cap (U_k \times V_l) = (U_i \cap U_k) \times (V_j \cap V_l)$. Further note that $(U_i \cap U_k) \times (V_j \cap V_l) \subset U_i \times V_j$ and $(U_i \cap U_k) \times (V_j \cap V_l) \subset U_k \times V_l$ and that the topologies of $(U_i \cap U_k) \times (V_j \cap V_l)$ respectively induced by $U_i \times V_j$ and $U_k \times V_l$ coincide. Hence we obtain a prevariety $(X \times Y, \mathcal{O}_{X \times Y})$.

Now we will verify that this prevariety together with the projections is the product of $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$. Note that the projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are morphisms. Let $(T, \mathcal{O}_T)$ be an affine variety with maps $f : T \to X$ and $g : T \to Y$. Consider the map

$$\begin{aligned} u \colon T &\to X \times Y \\ p &\mapsto (f(p), g(p)). \end{aligned}$$

Clearly this is the only map such that $f = p_1 \circ u$ and $g = p_2 \circ u$. Recall that all $(U_i \times V_j, \mathcal{O}_{U_i \times V_j})$ are products. Note that if f and g respectively had codomains U_i and V_j the unique morphisms to let the diagram in the definition of a product commute would be $u|_{U_i \times V_j} : u^{-1}(X \times Y) \to X \times Y$. Hence $u|_{U_i \times V_j}$ is a morphism for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. Hence u is a morphism. □

See [5] proposition 5.7.

4.3 Varieties

Definition 4.7 (Variety)**.** Let $(X, \mathcal{O}_X)$ be a prevariety. Then $(X, \mathcal{O}_X)$ is a variety if for all prevarieties $(Y, \mathcal{O}_Y)$ and for all morphisms $f : Y \to X$ and $g : Y \to X$ the set $\{y \in Y | f(y) = g(y)\}$ is a closed subset of Y.

Consider the projections

$$\begin{aligned} p_1 \colon X \times X &\to X \\ (x, y) &\mapsto x \end{aligned}$$

and

$$\begin{aligned} p_2 \colon X \times X &\to X \\ (x, y) &\mapsto y. \end{aligned}$$

Consider the morphism

$$\begin{aligned} \triangle \colon X &\to X \times X \\ x &\mapsto (x, x). \end{aligned}$$

Then $\triangle\, (X) = \{z \in X \times X | p_1(z) = p_2(z)\}$.

Proposition 4.8. *Let X be a prevariety. Then X is a variety if and only if $\triangle\, (X)$ is closed in $X \times X$.*

Proof. Let X be a prevariety. Assume that X is a variety. Then we can take $f = p_1$ and $g = p_2$ in the definition of a variety. We find that $\triangle\, (X)$ is closed in $X \times X$.

Assume that $\triangle\, (X)$ is closed in $X \times X$. Let $f : Y \to X$ and $g : Y \to X$ be given. Consider the morphism

$$\begin{aligned} (f, g) \colon Y &\to X \times X \\ y &\mapsto (f(y), g(y)). \end{aligned}$$

This morphism is continuous. So since $\triangle\, (X)$ is closed in $X \times X$ we know that $(f, g)^{-1}[\triangle\, (X)]$ is closed in Y. Note that $\{y \in Y | f(y) = g(y)\} = (f, g)^{-1}[\triangle\, (X)]$. Hence $\{y \in Y | f(y) = g(y)\}$ is closed in Y. □

Definition 4.9 (Morphism of varieties)**.** A morphism of varieties is a morphism of prevarieties.

Notation 4.10 (Var_k)**.** We denote the category of varieties over k by Var_k.

4.4 Taking a closer look at the topology on a variety

From now on when we denote a variety we wil omit the sheaf.

Definition 4.11 (Section)**.** Let $g\colon Y \to X$ be a morphism then $f\colon X \to Y$ is called a section of g if $g \circ f = \mathrm{id}_X$.

A retraction is the dual notion of a section.

Lemma 4.12. *For a morphism $f\colon X \to Y$ the following are equivalent*

(a) f is an isomorphism

(b) f is an epimorphism and section

(c) f is an epimorphism and extremal monomorphism

(d) f is a monomorphism and retraction

(e) f is a monomorphism and extremal epimorphism

Proof.

$$(a) \Rightarrow (b)$$

Let $f\colon X \to Y$ be an isomorphism. Then there exists an $f^{-1}\colon Y \to X$ such that $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$. Hence f is a section.

Let $g, h\colon Y \to Z$ be morphisms such that $g \circ f = h \circ f$. Then $g \circ f \circ f^{-1} = h \circ f \circ f^{-1}$, so $g = h$. Hence f is an epimorphism.

$$(b) \Rightarrow (c)$$

Let $f\colon X \to Y$ be a section of h. Then f is a monomorphism since $f \circ g = f \circ g'$ implies $h \circ f \circ g = h \circ f \circ g'$ and $\mathrm{id}_X \circ g = \mathrm{id}_X \circ g'$ implies $g = g'$.

Suppose $f = g \circ e$ with $e\colon X \to Z$ an epimorphism. Then $f \circ \mathrm{id}_X = f \circ h \circ f = f \circ h \circ g \circ e$. Because f is a monomorphism this implies that $\mathrm{id}_X = h \circ g \circ e$. Hence e is a section of $h \circ g$. We also have $\mathrm{id}_Z \circ e = e = e \circ \mathrm{id}_X = e \circ h \circ g \circ e$. Because e is an epimorphism this implies $\mathrm{id}_Z = e \circ h \circ g$. Hence e is a retraction of $e \circ h$. Hence $e \circ (e \circ h) = \mathrm{id}_Z$ and $(e \circ h) \circ e = \mathrm{id}_X$, so e is an epimorphism. We can conclude that f is an extremal monomorphism.

$$(c) \Rightarrow (a)$$

Let $f\colon X \to Y$ be an epimorphism and an extremal monomorphism. We have $f = \mathrm{id}_X \circ f$. Because f is an epimorphism and an extremal monomorphism this implies that f is an isomorphism.

Note that (d) and (e) are the dual notions of respectively (b) and (c). □

See [3].

Definition 4.13 (Closed embedding)**.** A morphism $f\colon Y \to X$ is called a closed embedding if it is an embedding and we have $f(Y) = \overline{f(Y)}$.

So if a morphism f is a closed embedding then its image is closed in the codomain and isomorphic to its domain.

Definition 4.14 (Dense)**.** We call a morphism $f\colon Y \to X$ dense if we have $\overline{f(Y)} = X$.

Lemma 4.15. *In the category of varieties if a morphism $f\colon Z \to Y$ is dense then it is an epimorphism.*

Proof. Let $f\colon Z \to Y$ be dense. Let $g\colon Y \to X$ and $h\colon Y \to X$ be morphisms between varieties with $g \circ f = h \circ f$. Then g and h must agree on $f(Z)$. Because Y and X are varieties and g and h are morphisms between them, g and h must agree on a closed set (by the Hausdorff-axiom). So g and h must agree on $\overline{f(Z)}$. So $g = h$. We chose arbitrary g and h with $g \circ f = h \circ f$, so we may conclude that f is an epimorphism. We chose an arbitrary dense f so we can conclude that the lemma is proven. □

The subsets of a prevariety that are the intersection of a closed subset with an open subset are called subprevarieties. Remark I on page 53 of [6] says that any subprevariety of a variety is again a variety. Let X be a variety. Then any closed subset of X is the intersection of a closed set with the whole set, so every closed subset of X is a variety.

Theorem 4.16. *In the category of varieties, for a variety X the images of extremal monomorphisms that have X as co-domain are precisely the closed sets of the topology on X.*

Proof. Take a variety X with its topology.
Let $f\colon Y \to X$ be an extremal monomorphism. Note that $\overline{f(Y)}$ is a variety. Consider the morphism $f'\colon Y \to \overline{f(Y)}$ defined by $f'(y) = f(y)$ for all $y \in Y$. This morphism is dense and thus an epimorphism by lemma 4.15. Let $i\colon \overline{f(Y)} \hookrightarrow X$ be the injection defined by $i(x) = x$ for all $x \in \overline{f(Y)}$. Then we have $f = i \circ f'$. Because f is an extremal monomorphism we now know that f' is an isomorphism. So the image of f is equal to $\overline{f(Y)}$. In particular it is closed. This proof is analogous to a proof in [8].

Suppose $Y \subset X$ is a closed set. Then Y is a variety. Let $i\colon Y \hookrightarrow X$ be the inclusion map, defined by $i(y) = y$ for all $y \in Y$. The map $i'\colon Y \to Y$ defined by $i'(y) = i(y)$ for all $y \in Y$ is the identity on Y. So i is an embedding, so i is a monomorphism by lemma 2.50. Suppose there exists a morphism $f\colon T \to X$ and an epimorphism $e\colon Y \to U$ with $i = f \circ e$. Define $f'\colon e(Y) \to X$ by $f'(u) = f(u)$ for all $u \in e(Y)$. Then we have $i' = f' \circ e$. But i' was the identity on Y so e is a section. Now we have that e is an epimorphism and a section, so it is an isomorphism by lemma 4.12. We can conclude that i is an extremal monomorphism. Note that $i(Y) = Y$. So there exists an extremal monomorphism such that Y is the image of this morphism. □

Lemma 4.17. *Let X be a topological space and $Y \subset X$. Then Y is closed if and only if for any open covering $\cup_{i \in I} U_i$ of X one has $Y \cap U_i$ is closed in U_i for all i.*

Proof. Let X be a topological space and $Y \subset X$.

$$\Longrightarrow$$

Take an open covering $\cup_{i \in I} U_i$ of X. (Note that we can always take the union of all opens of X as the open covering.) Suppose Y is closed in X then $X \setminus Y$ is open.

Take an $i \in I$. Then $(X \setminus Y) \cap U_i$ is open in U_i for the induced topology on U_i. Note that $(X \setminus Y) \cap U_i = (X \cap U_i) \setminus (Y \cap U_i) = U_i \setminus (Y \cap U_i)$. Hence $U_i \setminus (Y \cap U_i)$ is open in U_i, so $Y \cap U_i$ is closed in U_i.

$\Longleftarrow$

Suppose there is an open covering $\cup_{i \in I} U_i$ of X such that $Y \cap U_i$ is closed in U_i for all i. Take a $j \in I$. Then $U_j \setminus (Y \cap U_j)$ is open in U_j. Hence there exists an open V_j in X such that $V_j \cap U_j = U_j \setminus (Y \cap U_j)$. Note that $U_j \setminus (Y \cap U_j) = (X \setminus Y) \cap U_j$. We find

$$\begin{aligned} \cup_{i \in I}(V_i \cap U_i) &= \cup_{i \in I}((X \setminus Y) \cap U_i) \\ &= ((X \setminus Y) \cap X) \\ &= X \setminus Y \end{aligned}$$

is open in X. Hence Y is closed in X. □

Theorem 4.18. *Let $(X, \mathcal{O}_X)$ be a variety and let $\cup_{i=1}^n U_i$ be a finite open cover of X such that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine variety for all i. Then the closed sets of X are the finite unions of extremal monomorphisms with affine varieties as domains and images that are for every i either fully contained in U_i or disjoint with U_i .*

Proof. Let $(X, \mathcal{O}_X)$ be a variety. Let $\cup_{i=1}^n U_i$ be a finite open cover of X such that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine variety for all i.

$\Longrightarrow$

Let Y be a closed subset of X. Then by lemma 4.17 for all i we have that $Y \cap U_i$ is closed in U_i. Hence for all i the set $Y \cap U_i$ is a finite union of images of extremal monomorphisms with affine varieties as domains. Note that $\cup_{i \in I}^n Y \cap U_i = Y$. Hence Y can be written as the finite union of finite unions of images of extremal monomorphisms with affine varieties as domains and images that are for every i either fully contained in U_i or disjoint with U_i .

$\Longleftarrow$

Let $Y \subset X$ be a finite union of images of extremal monomorphisms with affine varieties as domains and images that are for every i either fully contained in U_i or disjoint with U_i. Then $Y \cap U_i$ is closed in U_i for all i. □

Let X be a variety with finite open cover $\cup_{i=1}^n U_i$. We now know that the images of extremal monomorphisms in the category of varieties with codomain X are equal to the finite unions of extremal monomorphisms with affine varieties as domains and images that are for every i either fully contained in U_i or disjoint with U_i.

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