Jesse Hemerik

Abstract Cauchy problems and stochastic integral equations

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Thesis supervisor: dr. O. van Gaans

Mathematisch Instituut, Universiteit Leiden

Contents

1 Preface

An (inhomogeneous) abstract Cauchy problem is a type of differential equation of Banach space valued functions. The only derivate in a Cauchy problem is $\frac{du}{dt}$, where u is the sought-after solution. However, because we consider such a general setting, we could for example also formulate some systems of multiple higher order differential equations as a Cauchy problem.

In section 3 we consider certain stochastic integral equations. The connection with the (inhomogeneous) abstract Cauchy problem is, that the latter is (almost) a special case of the stochastic integral equations that we study. The idea is that the stochastic element adds the resulting effect of random 'white noise'. This can be useful in all sorts of applications, since it can model the effects of small changes one could not predict.

We first consider a stochastic integral equation in a separable Hilbert space, for which an existence and uniqueness proof is known (see [GA]). We would like to generalise this by allowing values in an arbitrary Banach space. This is done in section 3.3. We will also prove a boundedness result there. We finish in section 3.4 with some specific examples of stochastic integral equations. I would like to thank my thesis supervisor Onno van Gaans for the effort he has put into guiding me in the right direction. Thanks to his regular help I have always had the sense of really getting somewhere.

2 Inhomogeneous abstract Cauchy problems

2.1 A simple case of the Cauchy problem

Let X be a Banach space. By the (homogeneous) abstract Cauchy problem we mean finding solutions of the system

$$\begin{cases} \frac{du}{dt}(t) = Au(t), \\ u(0) = u_0. \end{cases}$$

Here $u_0 \in X$ is given, $A : X \to X$ is a closed linear operator with dense domain $D(A) \subset X$ and we look for solutions[1] $u : [0, \infty) \to X$. For $t \geq 0$ we define $\frac{du}{dt}(t) = \lim_{h \to 0} \frac{1}{h}(u(t+h) - u(t))$ if this limit exists.

In this section we consider a simplification of the Cauchy problem: the case that A is bounded and the domain of A is the whole set X. We will show that in this case $u(t) = (\sum_{n=0}^{\infty} (tA)^n / n!) u_0$ is a solution to the abstract Cauchy problem.

[1]We define $u : [0, \infty) \to X$ to be a solution if $\frac{du}{dt}(t)$ exists for all $t \geq 0$ and u satisfies the system for all $t \geq 0$.

To see that $\sum_{n=0}^{\infty}\frac{(tA)^n}{n!}$ converges in $B(X)$, first note that $\lim_{N\to\infty}\sum_{n=N}^{\infty}\|\frac{(tA)^n}{n!}\| \leq \lim_{N\to\infty}\sum_{n=N}^{\infty}\frac{\|tA\|^n}{n!} = 0$. Let $\epsilon > 0$. Choose $M \in \mathbb{N}$ such that for all $N \geq M$: $\sum_{n=N}^{\infty}\|\frac{(tA)^n}{n!}\| < \epsilon$. Then for all $N_1 \geq N_2 > M$ we have

$$\|\sum_{n=1}^{N_1}\frac{(tA)^n}{n!} - \sum_{n=1}^{N_2}\frac{(tA)^n}{n!}\| \leq \sum_{N_2-1}^{\infty}\|\frac{(tA)^n}{n!}\| < \epsilon.$$

So $(\sum_{n=0}^{N}\frac{(tA)^n}{n!})_{N=1}^{\infty}$ is Cauchy sequence in $B(X)$, so it converges.

Note that for $t \geq 0$ and $N \in \mathbb{N}$,

$$\lim_{h\to 0}\frac{1}{h}\left(\sum_{n=0}^{N}\frac{(t+h)^n A^n}{n!} - \sum_{n=0}^{N}\frac{t^n A^n}{n!}\right) = \sum_{n=1}^{N}\lim_{h\to 0}\frac{1}{h}\frac{(t^n + nht^{n-1} + ... - t^n)A^n}{n!} =$$

$$\sum_{n=1}^{N}\frac{nt^{n-1}A^n}{n!} = \sum_{n=1}^{N}\frac{t^{n-1}A^n}{(n-1)!} =$$

$$A\sum_{n=0}^{N-1}\frac{A^n t^n}{n!}.$$

So this limit exists and $\frac{d}{dt}\sum_{n=0}^{N}\frac{A^n t^n}{n!}u_0 = A\sum_{n=0}^{N-1}\frac{A^n t^n}{n!}u_0$.

This raises the question whether $A\sum_{n=0}^{\infty}\frac{(At)^n}{n!}u_0 = \frac{d}{dt}\sum_{n=0}^{\infty}\frac{(At)^n}{n!}u_0$. That this is true is shown in Theorem 2.2.

Lemma 2.1. *For s, $t > 0$,*

$$\sum_{n=0}^{\infty}\frac{(At)^n}{n!}\sum_{n=0}^{\infty}\frac{(As)^n}{n!} = \sum_{n=0}^{\infty}\frac{(A(t+s))^n}{n!}.$$

Proof. Because $\sum_{n=0}^{\infty}\frac{(At)^n}{n!}$ is bounded, for all $\phi \in B(X)'$,

$$\phi\left(\lim_{N\to\infty}\sum_{n=0}^{N}\frac{(At)^n}{n!}\lim_{K\to\infty}\sum_{k=0}^{K}\frac{(As)^k}{k!}\right) =$$

$$\phi\left(\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(At)^n}{n!}\frac{(As)^k}{k!}\right) =$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\phi(\frac{(At)^n}{n!}\frac{(As)^k}{k!}) =: (*).$$

Since we are allowed to change the order of summation (by absolute convergence (which follows from $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\|\frac{(At)^n}{n!}\frac{(As)^k}{k!}\| < \infty$ and boundedness of ϕ)),

$$(*) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\phi(\frac{A^n t^{n-k}s^k}{(n-k)!k!}) =$$

$$= \phi\left(\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{A^n t^{n-k}s^k}{(n-k)!k!}\right) = \phi\left(\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{A^n}{n!}\frac{n!}{k!(n-k)!}t^{n-k}s^k\right)$$

$$= \phi\left(\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{A^n}{n!}\binom{n}{k}t^{n-k}s^k\right) = \phi\left(\sum_{n=0}^{\infty}\frac{(A(t+s))^n}{n!}\right).$$

Since $\phi \in B(X)'$ was arbitrary, the result follows (see [RY, Cor. 5.22(c)]). □

Theorem 2.2. *For $t \geq 0$,*

$$A\sum_{n=0}^{\infty}\frac{(At)^n}{n!}u_0 = \frac{d}{dt}\sum_{n=0}^{\infty}\frac{(At)^n}{n!}u_0.$$

So $\sum_{n=0}^{\infty}\frac{(A(t))^n}{n!}u_0$ is a solution to the abstract Cauchy problem (assuming A is bounded and everywhere defined).

Proof. Note that

$$\lim_{h\to 0}\frac{1}{h}\left(\sum_{n=0}^{\infty}\frac{(A(t+h))^n}{n!}u_0 - \sum_{n=0}^{\infty}\frac{(At)^n}{n!}u_0\right) =$$

$$\lim_{h\to 0}\frac{1}{h}\sum_{n=0}^{\infty}\frac{(At)^n}{n!}\left(\sum_{n=0}^{\infty}\frac{(Ah)^n}{n!}u_0 - u_0\right) =$$

$$\lim_{h\to 0}\sum_{n=0}^{\infty}\frac{(At)^n}{n!}\frac{1}{h}(\sum_{n=1}^{\infty}\frac{(Ah)^n}{n!})u_0.$$

Since $\lim_{h\to 0}\frac{1}{h}\sum_{n=1}^{\infty}\frac{A^n}{n!}h^n = A$, this equals

$$\sum_{n=0}^{\infty}\frac{(At)^n}{n!}Au_0 = A\sum_{n=0}^{\infty}\frac{(At)^n}{n!}u_0.$$

□

2.2 Semi-group theory

Recall that by a *homogeneous Cauchy problem* we mean finding solutions of the system

$$\begin{cases}\frac{du}{dt}(t) = Au(t),\\ u(0) = u_0,\end{cases}$$

where $A : X \to X$ is a closed, densely defined operator.

We will in fact discuss the more general *inhomogeneous abstract Cauchy problem*

$$\begin{cases} \frac{du}{dt}(t) = Au(t) + f(t, u(t)), \\ u(0) = u_0, \end{cases}$$

where $f : [0, \infty) \times X \to X$ is assumed to be continuous.

The study of abstract Cauchy problems is tightly related to *semi-group* theory.

Definition 2.3. A *strongly continuous semi-group* is a family $\{T(t)\}$, $0 \leq t < \infty$ of bounded linear operators from X to X with the following properties:

(a) $T(0) = I$;

(b) $T(t)T(s) = T(t+s)$, s, $t \geq 0$;

(c) For each $x \in X$, $t \mapsto T(t)x$ is continuous.

If, in addition,

(d) $t \mapsto T(t)$ is continous,

we speak of a *uniformly continuous semi-group*.

It can be seen that if A is bounded, then $\{e^{tA}\}_{t\geq 0}$ is a uniformly continuous semi-group. (e^{tA} is defined according to the operator calculus, see Appendix A.) The following theorem says that every uniformly continuous semi-group is of this form.

Theorem 2.4. *Let $\{T(t)\}_{t\geq 0}$ be a uniformly continuous semi-group. Then there exists an operator $A \in B(X)$ such that $T(t) = e^{tA}$ for all $t \geq 0$. The operator A is given by the formula $A = \lim_{h\to 0} \frac{1}{h}(T(h) - I)$.*

Proof. See [DU, Thm VIII.1.2]. □

Definition 2.5. Let $\{T(t)\}$ be a strongly continuous semi-group. Let $D(A)$ be the set of all $x \in X$ for which $\lim_{h\to 0} \frac{T(h)x - x}{h}$ exists, and define the operator A with domain $D(A)$ by $Ax = \lim_{h\to 0} \frac{T(h)x-x}{h}$. A is called the *infinitesimal generator* of the semi-group $\{T(t)\}_{t\geq 0}$.

In the previous section we have seen that if A is bounded and everywhere defined, for $u_0 \in X$,

$$A \sum_{n=0}^{\infty} \frac{(At)^n}{n!} u_0 = \frac{d}{dt} \sum_{n=0}^{\infty} \frac{(At)^n}{n!} u_0.$$

It can be seen that $e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$ and this is a uniformly continuous semi-group with generator A (if A is bounded and everywhere defined).

We will often use the following lemma to estimate the norm of a semi-group.

Theorem 2.6. *Let $\{T(t)\}$ be a strongly continuous semi-group. The limit $\omega_0 = \lim_{t\to\infty} \log \frac{\|T(t)\|}{t}$ exists. For each $\delta > \omega_0$ there is a constant M_δ such that $\|T(t)\| < M_\delta e^{\delta t}$ for all $t \geq 0$.*

Proof. See [DU, Cor. VIII.1.5]. □

Let $\{T\}$ be a strongly continuous semi-group and A the infinitesimal generator with domain $D(A)$. The following properties hold true. We interpret integrals of Banach space valued functions as Bochner integrals[2].

Lemma 2.7. *(a) $D(A)$ is a linear manifold and A is linear.*

(b) If $x \in D(A)$, then $T(t)x$ is in $D(A)$, and $\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax$.

(c) If $x \in D(A)$, then $[T(t) - T(s)]x = \int_s^t T(r)Axdr$, $0 \leq s < t < \infty$.

(d) If $t \geq 0$ and $g : [0, \infty) \to \mathbb{C}$ is Lebesgue integrable and continuous at t, then

$$\lim_{h\to\infty} \frac{1}{h} \int_t^{t+h} g(r)T(r)xdr = g(t)T(t)x.$$

(e) $D(A)$ is dense in X and A is closed on $D(A)$.

Proof. See [DU, Lemma's VIII.1.7, VIII.1.8]. □

Lemma 2.8. *A semi-group has a bounded infinitesimal generator if and only if it is uniformly continuous.*

Proof. See [DU, Lemma VIII.1.9]. □

Theorem 2.9. *Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous semi-group. Define $A_h = \frac{1}{h}(T(h) - I)$. Then, for $x \in X$,*

$$T(t)x = \lim_{h\to 0} e^{tA_h}x,$$

uniformly in t for any finite interval.

Proof. See [DU, Lemma VIII.1.10]. □

We will need the next result for Theorem 2.11.

Theorem 2.10. *Let X, Y be Banach spaces, and $\{T_n\}_{n=1}^\infty$ a sequence in $B(X, Y)$. Then $Tx = \lim_{n\to\infty} T_n x$ exists for every $x \in X$ if and only if*

(1) the limit exists for every x in a dense set, and

(2) for each $x \in X$, $\sup_n \|T_n x\| < \infty$.

[2] see Appendix B

In that case

$$\|T\| \leq \sup_n \|T_n\| < \infty.$$

Proof. See [DU, Thm II.3.6]. □

The following theorem gives a necessary and sufficient condition that a densely defined, closed operator A in X be the generator of a strongly continuous semi-group. By $R(\lambda; A)$ is meant $(\lambda I - A)^{-1}$.

Theorem 2.11. *(Hille-Yosida-Phillips) Let $A : X \to X$ be a closed linear operator with dense domain $D(A) \subset X$. Then A generates a strongly continuous semi-group if and only if there exist M, $\omega \in \mathbb{R}$ such that for every real $\lambda > \omega$, λ is in $\rho(A)$ and for all $n \in \mathbb{N}$,*

$$\|R(\lambda; A)^n\| \leq M(\lambda - \omega)^{-n}.$$

Proof. See [DU, Thm VIII.1.13]. We give a sketch of the proof of the sufficiency of the condition.

For $\lambda > \omega$, define $B_\lambda = -\lambda[I - \lambda R(\lambda; A)]$, $\lambda > \omega$. The idea is to show that for $x \in D(A)$, $Ax = \lim_{\lambda\to\infty} B_\lambda x$, and then extend the bounded operator family $\{e^{tB_\lambda}\}$ defined on $D(A)$ to the whole set X.

It is immediate that

$$e^{tB_\lambda} = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda^2 t)^n R(\lambda; A)^n}{n!}.$$

Using the assumptions it can then be derived that for all $\omega_1 > \omega$, for all $\lambda \in \mathbb{R}$ large enough,

$$\|e^{tB_\lambda}\| < M e^{t\omega_1}.$$

Note that $I - \lambda R(\lambda; A) = R(\lambda; A)A$. For $x \in D(A)$,

$$\|\lambda R(\lambda; A)x - x\| = \|R(\lambda; A)Ax\| \leq M\|Ax\|(\lambda - \omega)^{-1} \to 0$$

as $\lambda \to \infty$. Since $\|\lambda R(\lambda; A)\| \leq M\lambda(\lambda - \omega)^{-1} < 2M$ for large λ, we have $\sup_{\lambda\in\mathbb{R}} \|\lambda R(\lambda; A)\| < \infty$. It follows with Theorem 2.10 that $\lambda R(\lambda; A)x \to x$ as $\lambda \to \infty$ for any $x \in X$. So $B_\lambda x = \lambda R(\lambda; A)Ax \to Ax$ for $x \in D(A)$.

Using Lemma 2.7(b) it can be obtained that for $\omega_1 > \omega$, for large values of λ, $\mu \in \mathbb{R}$,

$$\|e^{tB_\lambda}x - e^{tB_\mu}x\| \leq M^2 t e^{t\omega_1} \|B_\lambda x - B_\mu x\|.$$

Remember that $\lim_{\lambda\to\infty} B_\lambda x$ exists for $x \in D(A)$. So $e^{tB_\lambda}x$ converges to a limit as $\lambda \to \infty$ uniformly on each finite (t-)interval. Remember that for large enough values of λ, $\|e^{tB_\lambda}\| < M e^{t\omega_1}$. With Theorem 2.10 it follows that for each $t \geq 0$ there exists an operator $T(t) \in B(X)$ with $T(t)x = \lim_{\lambda\to\infty} e^{tB_\lambda}x$, $x \in X$.

From the uniformity of the convergence follows the continuity in t of $T(t)x$. Since $\{e^{tB_\lambda}\}$ is a semi-group it easily follows that $\{T(t)\}$ is a semi-group.

The final step is to show that the generator of $\{T(t)\}$ is in fact A. □

The sufficient condition that an operator generate a strongly continuous semi-group in the Hille-Yosida-Phillips Theorem is complicated. There are corollaries with less complicated conditions, but they pose some additional assumptions. We now state one such a corollary.

Corollary 2.12. *Let A be a closed linear operator in X with dense domain and let $\omega \in \mathbb{R}$. Then A generates a strongly continuous semi-group $\{T(t)\}$ with $\|T(t)\| \leq e^{\omega t}$ if and only if for all $\lambda > \omega$,*

$$\|R(\lambda; A)\| \leq (\lambda - \omega)^{-1}.$$

Proof. See [DU, Cor. VIII.1.14]. □

2.3 The inhomogeneous Cauchy problem

Let $A : X \to X$ be a densely defined, closed linear operator. Moreover, assume that A satisfies the properties from the Hille-Yosida-Phillips Theorem 2.11, that is, there exist ω, $M \in \mathbb{R}$ such that for every real $\lambda > \omega$, λ is in $\rho(A)$ and for all $n \in \mathbb{N}$

$$\|R(\lambda; A)^n\| \leq M(\lambda - \omega)^{-n},$$

that is, A generates a strongly continuous semi-group (Theorem 2.11).

Let $f : [0, \infty) \times X \to X$ be continuous. We consider the inhomogeneous Cauchy problem

$$\begin{cases} \frac{d}{dt}u(t) = Au(t) + f(t, u(t)), \\ u(0) = u_0. \end{cases} \tag{1}$$

Denote by $\{S(t)\}$ the strongly continuous semi-group generated by A. We need the following lemma for theorems 2.14 and 2.17.

Lemma 2.13. *Suppose $g : [0, \infty) \to X$ is continuous. Then the map $s \mapsto S(t - s)g(s)$, $s \in [0, t]$, is integrable. Moreover, the map $v : [0, \infty) \to X$,*

$$t \mapsto S(t)u_0 + \int_0^t S(t - s)g(s)ds,$$

is continuous.

Proof. For the first part it clearly suffices to show that $s \mapsto S(t - s)g(s)$ is continuous. Since $\{\|S(t')\| : \ t' \in [0, t]\}$ is bounded by Lemma 2.6, let $c := \sup_{t' \in [0,t]} \|S(t')\|$. Fix $s' \in [0, t]$. Then for $s \in [0, T]$, $\|S(t - s')g(s') - S(t - s)g(s)\| = \|S(t - s')g(s') + S(t - s)(g(s') - g(s)) - S(t - s)g(s')\| \leq$

$c\|g(s') - g(s)\| + \|S(t-s')g(s') - S(t-s)g(s')\|$. So (since S is a strongly continuous semi-group) our map is continuous at s'.

To see that v is continuous, let $t' > 0$ and note that if $t < t'$, $\|v(t') - v(t)\| \leq \int_0^t \|(S(t'-s) - S(t-s))\| \cdot \|f(s)\| ds + \int_t^{t'} \|S(t'-s)f(s)\| ds$. In the first integral, note that the integrand is bounded by $2 \sup_{s' \in [0,t']} \|S(s')\| \cdot \|f(s)\|$. Moreover the integrand converges to 0 as $t \uparrow t'$, so the integral does as well by the Dominated Convergence Theorem. The second integral equals $\int_0^{t'} 1_{[t,t']} \|S(t'-s)f(s)\|$ which also converges to 0 as as $t \uparrow t'$ by the Dominated Convergence Theorem. In a similar way it can be shown that v is right continuous. So v is continuous.

□

Theorem 2.14. *If u is such that*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s, u(s))ds \tag{2}$$

and $u(t)$, $f(t, u(t)) \in D(A)$ for all $t \geq 0$, then

$$u(t) - u(0) = \int_0^t (Au(s) + f(s, u(s))ds.$$

Proof.

$$\int_0^t Au(s)ds = A \int_0^t u(s)ds =$$

$$A\left(\int_0^t S(s)u_0 ds\right) + A\left(\int_0^t \int_0^s S(s-r)f(r, u(r))drds\right).$$

By Fubini's Theorem this equals

$$A\left(\int_0^t S(s)u_0 ds\right) + A\left(\int_0^t \int_r^t S(s-r)f(r, u(r))dsdr\right) =$$

$$\int_0^t AS(s)u_0 ds + \int_0^t \int_0^{t-r} AS(s)f(r, u(r))dsdr.$$

By Lemma 2.7(b), (c) this equals

$$(S(t) - S(0))u_0 + \int_0^t (S(t-r) - S(0))f(r, u(r))dr =$$

$$S(t)u_0 - u_0 - \int_0^t f(r, u(r))dr + u(t) - S(t)u_0 =$$

$$-\int_0^t f(s, u(s))ds + u(t) - u(0).$$

□

Corollary 2.15. *If u satisfies the properties stated in the previous theorem, then it is a solution to the inhomogeneous Cauchy problem (1).*

Proof. By the previous theorem, $u(t) - u(0) = \int_0^t (Au(s) + f(s))ds$. From [DU, Thm III.12.8] it follows that (1) holds for almost all $t \geq 0$. □

We need the following theorem.

Theorem 2.16. *(*Banach fixed point theorem*, or* Banach contraction theorem*) Let (X, d) be a non-empty, complete metric space. Let $R : X \to X$ be a contraction mapping, that is, there is an $0 \leq \alpha < 1$ such that for all x, $y \in X$, $d(R(x), R(y)) \leq \alpha d(x, y)$. Then R has exactly one fixed point.*

Proof. See [BU, Thm 7.14]. □

We will use Banach's fixed point theorem to obtain the following existence and uniqueness result.

Theorem 2.17. *If f is Lipschitz continuous in the second argument[3], then equation (2) has exactly one continuous solution $u \in C([0, \infty), X)$.*

Proof. Let $T > 0$. Let L be a Lipschitz constant for f. Choose M, $\omega > 0$ such that $\|T(t)\| < Me^{\omega t}$ for all $t \geq 0$. Choose $\gamma > \omega$ and $0 < c < 1$ such that

$$ML \frac{1 - e^{-(\gamma-\omega)T}}{\gamma - \omega} < c.$$

Define the norm $|||\cdot|||$ on $C([0, T], X)$ by

$$|||u||| := \sup_{t\in[0,T]} e^{-\gamma t} \|u(t)\|.$$

Observe that

$$e^{-\gamma T} \sup_{t\in[0,T]} \|u(t)\| \leq |||u||| \leq \sup_{t\in[0,T]} \|u(t)\|$$

so $|||\cdot|||$ is equivalent with the supremum norm. Since $C([0, T], X)$ with the supremum norm is complete, so is $(C([0, T], X), |||\cdot|||)$.

We will show that the map $R : (C([0, T], X), |||\cdot|||) \to (C([0, T], X), |||\cdot|||)$ defined by

$$(Ru)(t) := S(t)u_0 + \int_0^t S(t - s)f(s, u(s))ds$$

is a contraction mapping. Let u, $v \in C([0, T], X)$. So

$$|||Ru - Rv||| = \sup_{t\in[0,T]} e^{-\gamma t} \| \int_0^t S(t - s)(F(s, u(s)) - F(s, v(s))ds \|$$

[3]That is, there is an $L \geq 0$ such that for all $t \geq 0$, $f(t, \cdot)$ is Lipschitz continuous with Lipschitz constant L.

$$
\begin{aligned}
&\leq \sup_{t\in[0,T]} \int_0^t e^{-\gamma t}\|S(t-s)\| \cdot \|(F(s,u(s)) - F(s,v(s))\| ds \\
&\leq \sup_{t\in[0,T]} \int_0^t e^{-\gamma t} M e^{\omega(t-s)} Lip(f)\|u(s) - v(s)\| ds \\
&\leq \sup_{t\in[0,T]} \int_0^t e^{-\gamma t} M e^{\omega(t-s)} Lip(f) e^{\gamma s} |||u-v||| ds \\
&= \sup_{t\in[0,T]} \int_0^t M e^{(\omega-\gamma)(t-s)} Lip(f)|||u-v||| ds \\
&= \sup_{t\in[0,T]} M e^{(\omega-\gamma)(t)} Lip(f)|||u-v||| \int_0^t e^{(\gamma-\omega)s} ds \\
&= \sup_{t\in[0,T]} M e^{(\omega-\gamma)t} Lip(f)|||u-v||| \frac{e^{(\gamma-\omega)t} - 1}{\gamma-\omega} \\
&= \sup_{t\in[0,T]} M Lip(f)|||u-v||| \frac{1 - e^{(\omega-\gamma)t}}{\gamma-\omega} \\
&= M Lip(f)|||u-v||| \frac{1 - e^{(\omega-\gamma)T}}{\gamma-\omega} \\
&< c \cdot |||u-v|||.
\end{aligned}
$$

So R is a contraction, so it has exactly one fixed point by Theorem 2.16. So equation (2) has exactly one continuous solution on $[0, T]$. Since T was arbitrary, there is exactly one continuous solution on $[0, \infty)$. □

We have seen that

$$
u(t) = S(t)u_0 + \int_0^t S(t-s) f(s, u(s)) ds
$$

has exactly one *continous* solution if f is Lipschitz. In fact, there are no other solutions, since any solution is continuous by Lemma 2.13. The uniqueness also follows from Theorem 2.19. We need the following version of Gronwall's Lemma.

Lemma 2.18. *(Gronwall) If $g : [0, T] \to \mathbb{R}$ is continuous, $C \in \mathbb{R}$, $D \geq 0$ and for all $t \in [0, T]$*

$$
g(t) \leq C + D \int_0^t g(s) ds,
$$

then for all $t \in [0, T]$,

$$
g(t) \leq C e^{Dt}.
$$

Proof. See [ME, Lemma 3.13]. □

Theorem 2.19. *The solution to*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s,u(s))ds,$$

where f is Lipschitz in the second argument, depends continuously on u_0 (and is unique as we have already seen).

Proof. Choose constants M, ω such that $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. Choose a Lipschitz constant L for f. Let u and v be two solutions, with initial values $u(0) = u_0$, $v(0) = v_0$. So for $t \in [0,T]$,

$$\begin{aligned}\|u(t) - v(t)\| &\leq \|S(t)\| \cdot \|u_0 - v_0\| + \int_0^t \|S(t-s)\| \cdot \|f(s,u(s)) - f(s,v(s))\|ds \\ &\leq Me^{\omega t} \cdot \|u_0 - v_0\| + \int_0^t Me^{\omega(t-s)} L\|u(s) - v(s)\|ds \\ &\leq Me^{\omega t} \cdot \|u_0 - v_0\| + Me^{\omega T} L \int_0^t \|u(s) - v(s)\|ds.\end{aligned}$$

From Lemma 2.18 follows that for all $t \in [0,T]$,

$$\|u(t) - v(t)\| \leq Me^{\omega T} \cdot \|u_0 - v_0\| e^{Me^{\omega T}Lt}.$$

Since $T > 0$ was arbitary, it follows that

$$\|u(T) - v(T)\| \leq Me^{\omega T} \cdot \|u_0 - v_0\| e^{Me^{\omega T}LT}$$

for all $T > 0$. The claims follow from this. □

We now show that the solution is bounded if $\|S(t)\|$ converges exponentially fast to zero and f does not depend on t.

Lemma 2.20. *Consider the equation*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(u(s))ds, \tag{3}$$

$u(0) = u_0$, with $f : X \to X$ Lipschitz. This is the same setting as before, except that f does not depend on t. Assume that there are M, $\alpha > 0$ such that $\|S(t)\| \leq Me^{-\alpha t}$ for all $t \geq 0$. Let $L > 0$ be a Lipschitz constant for f. Then the solution is bounded if $L > 0$ can be chosen small enough. (More precisely, if α, M and L can be chosen such that $\alpha > ML$.)

Proof. Let u be the continuous unique solution of (3) with $u(0) = u_0$. Let $\tau \in [0,1]$.

For all $t \geq 0$ we have

$$u(\tau + t) = S(\tau + t)u_0 + \int_0^{\tau+t} S(\tau + t - s)f(u(s))ds =$$

$$S(\tau+t)u_0 + \int_0^\tau S(\tau+t-s)f(u(s))ds + \int_\tau^{\tau+t} S(\tau+t-s)f(u(s))ds =$$

$$S(\tau+t)u_0 + \int_0^\tau S(\tau+t-s)f(u(s))ds + \int_0^t S(t-s)f(u(s+\tau))ds.$$

Denote $\int_0^\tau S(\tau+t-s)f(u(s))ds = I$. Note that for all $\tau \in [0,1]$ and $t \geq 0$ we have $I \leq e^{-\alpha t} \int_0^1 Me^{\alpha s}\|f(u(s))\|ds$. Define $K = \int_0^1 Me^{\alpha s}\|f(u(s))\|ds$. So

$$I \leq Ke^{-\alpha t} \quad \text{for all } \tau \in [0,1],\, t \geq 0.$$

Note that $\|u(t+\tau) - u(t)\| =$

$$\|(S(\tau+t) - S(t))u_0 + I + \int_0^t S(t-s)(f(u(s+\tau)) - f(u(s)))ds\| \leq$$

$$2Me^{-\alpha t}\|u_0\| + Ke^{-\alpha t} + \int_0^t Me^{-\alpha(t-s)}L\|u(s+\tau) - u(s)\|ds. \quad \text{So}$$

$$\|(u(t+\tau) - u(t))e^{\alpha t}\| \leq 2M\|u_0\| + K + ML\int_0^t e^{\alpha s}\|u(s+\tau) - u(s)\|ds.$$

So by Gronwall's Lemma 2.18, denoting $C = 2M\|u_0\| + K$, we have

$$\|(u(t+\tau) - u(t))e^{\alpha t}\| \leq Ce^{MLt}. \quad \text{So}$$

$$\|u(t+\tau) - u(t)\| \leq Ce^{(ML-\alpha)t} \quad \text{for all } \tau \in [0,1],\, t \geq 0. \tag{4}$$

So for $n \in \mathbb{N}$,

$$\|u(n\tau) - u_0\| \leq \sum_{k=1}^n \|u(k\tau) - u((k-1)\tau)\| \leq$$

$$\sum_{k=1}^n Ce^{(ML-\alpha)(k-1)\tau} = C\sum_{k=1}^n \left(e^{(ML-\alpha)\tau}\right)^{k-1}.$$

If $ML - \alpha < 0$ (so if L is small enough), then this is a converging geometric series. So then

$$\sup_{n\in\mathbb{N}} \|u(n\tau) - u_0\| < \infty \qquad \text{for all } \tau \in [0,1],\, t \geq 0.$$

So in particular

$$\sup_{n\in\mathbb{N}} \|u(n) - u_0\| < \infty. \quad \text{So} \tag{5}$$

$$\sup_{t\geq 0} \|u(t) - u_0\| = \sup_{n\in\mathbb{N},\tau\in[0,1]} \|u(n+\tau) - u_0\| \leq$$

$$\sup_{n\in\mathbb{N},\tau\in[0,1]} (\|u(n+\tau) - u(n)\| + \|u(n) - u_0\|).$$

By equation (4), $\|u(n+\tau) - u(n)\| \leq C$. Combined with (5) this yields that the above supremum is finite. So the solution u is bounded.

□

3 Stochastic integral equations in a Banach space

3.1 Construction of the stochastic integral in a Hilbert space

Let E be a Banach space over $\mathbb{R}$. We would like to study differential equations of the form

$$\frac{d}{dt}X(t) = AX(t) + F(t, X(t)) + G(t, X(t)) \cdot \text{"}noise\text{"}, \tag{6}$$

where $A : [0, \infty) \times E \to E$ is densely defined and closed and F and $G : [0, \infty) \times E \to E$ are Lipschitz continuous. The noise term would be a stochastic process and therefore the other terms as well. Equation (6) would unfortunately only allow differentiable solutions; therefore we should instead consider an integral equation. Our goal is to make a definition of such an equation and to study the solutions. In this chapter we consider the case that all integrands take values in a separable Hilbert space. In the next chapter we will consider the more general case that some of the functions take values in a Banach space.

To model the effect of noise, we will use (sequences of) *normalised Brownian motions* (or *Wiener processes*) in $\mathbb{R}$.

Definition 3.1. If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, a *random variable* (in $\mathbb{R}$) on $(\Omega, \mathcal{F}, \mathbb{P})$ is a map $X : \Omega \to \mathbb{R}$ that is measurable with respect to the Borel σ-algebra of $\mathbb{R}$ and the σ-algebra $\mathcal{F}$.

Definition 3.2. A *normalised Brownian motion* (or *Wiener process*) in $\mathbb{R}$ is a family of random variables $\{W(t)\}_{t \geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for which the following hold:

(a) $W_0(\omega) = 0$ for all $\omega \in \Omega$,

(b) for every s, $t \geq 0$, $W(s + t) - W(s)$ is a Gaussian random variable with mean 0 and variance t,

(c) for every $0 = t_0 < t_1 < ... < t_n$, the random variables $W_{t_k} - W_{t_{k-1}}$, $k = 1, ..., n$, are independent,

(d) $t \mapsto W(t)(\omega)$ from $[0, \infty) \to \mathbb{R}$ is continuous for every $\omega \in \Omega$.

$W(s + t) - W(s)$ is a Gaussian random variable, so $W(t)$ seems a good way to model the effect of one-dimensional noise after t units of time. We will assume from now on that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and that there exists a Brownian motion $\{W(t)\}_{t \geq 0}$ on it.

We will state some definitions needed for the construction of the stochastic integral.

Definition 3.3. A *filtration* in $\mathcal{F}$ is an increasing family $(\mathcal{F}_t)_{t \geq 0}$ of sub-algebras of $\mathcal{F}$. A Wiener process $\{W(t)\}_{t \geq 0}$ on Ω is called a *Wiener process* *with respect to* $(\mathcal{F}_t)_{t\geq 0}$ if $W(t)$ is $\mathcal{F}_t$ measurable and $W(t+s)-W(t)$ is independent of $\mathcal{F}_t$, i.e. for every Borel set $B\subset\mathbb{R}$ and $C\in\mathcal{F}_t$, $\mathbb{P}(A\cap C)=\mathbb{P}(A)\mathbb{P}(C)$ holds, where $A=\{\omega\in\Omega:(W(t+s)-W(t))(\omega)\in B\}$, for all $s,t\geq 0$.

For example, one can take $(\mathcal{F}_t)_{t\geq 0}$ to be the filtration induced by W, i.e. $\mathcal{F}_t$ equals the σ-algebra induced by $\{W(s):0\leq s\leq t\}$.

For an index set $I\subset[0,\infty)$, a family $(\Phi_t)_{t\in I}$ of random variables on Ω (or, more precicely, $\mathcal{F}$) that take values on a separable Hilbert space H is called *adapted* to $(\mathcal{F}_t)_{t\geq 0}$ if Φ_t is measurable with respect to $\mathcal{F}_t$ and the Borel σ-algebra of H for every $t\in I$.

We will call a function $f:[0,T]\to L^2(\Omega;H)$ *piecewise uniformly continuous* if there are $0=a_0<a_1<...<a_n=T$ such that f is uniformly continuous on (a_{k-1},a_k), $k=1,...,n$.

Lemma 3.4. *Let H be a separable real Hilbert space and $(\Omega,\mathcal{F},\mathbb{P})$ a probability space. Let W_1, W_2 be independent normalised scalar Wiener processes on Ω with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ in $\mathcal{F}$. Let $T>0$ and let Φ_1, Φ_2 be adapted piecewise uniformly continuous functions from $[0,T]$ to $L^2(\Omega;H)$. Then the following hold:*

(i) The integral $\int_0^T\Phi_1(s)dW_1(s)$ is well-defined as a limit in $L^2(\Omega;H)$ of Riemann sums of the form

$$\sum_{k=1}^{n}\Phi_1(s_{k-1})(W_1(s_k)-W_1(s_{k-1})),$$

where $0=s_0<s_1<...<s_n=T$. That is, there is an element $\int_0^T\Phi_1(s)dW_1(s)$ in $L^2(\Omega;H)$ such that for every $\epsilon>0$ there is a $\delta>0$ with $\mathbb{E}\|\int_0^T\Phi_1(s)dW_1(s)-\sum_{k=1}^n\Phi_1(s_{k-1})(W_1(s_k)-W_1(s_{k-1}))\|<\epsilon$ for any partition $0=s_0<s_1<...<s_n=T$ with $\max_k\{s_k-s_{k-1}\}<\delta$.

(ii) The Itô isometry *holds true:*

$$\mathbb{E}\left\|\int_0^T\Phi_1(s)dW_1(s)\right\|^2=\int_0^T\mathbb{E}\|\Phi_1(s)\|^2ds$$

(iii)

$$\mathbb{E}\left\langle\int_0^T\Phi_1(s)dW_1(s),\int_0^T\Phi_2(s)dW_2(s)\right\rangle=0.$$

Proof. See [GA, Lemma 2.1]. □

Lemma 3.5. *(Construction of the stochastic integral). Let H be a separable real Hilbert space over $\mathbb{R}$ and $(\Omega,\mathcal{F},\mathbb{P})$ a probability space. Let $(W_i)_{i=0}^{\infty}$ be a sequence of independent normalised Wiener processes on Ω with respect to a* *filtration $(\mathcal{F}_t)_{t\geq 0}$ in $\mathcal{F}$. Let $T > 0$ and let $\Phi = (\Phi_i)_{i=0}^{\infty}$ be a sequence of adapted piecewise uniformly continuous functions from $[0, T]$ to $L^2(\Omega; H)$ such that*

$$\int_0^T \sum_{i=0}^{\infty} \mathbb{E}\|\Phi_i(s)\|^2 ds < \infty.$$

Then define

$$\int_0^T \Phi(s)dW(s) := \sum_{i=0}^{\infty} \int_0^T \Phi_i(s)dW_i(s), \tag{7}$$

which exists in $L^2(\Omega; H)$. Moreover,

$$\mathbb{E}\|\int_0^T \Phi(s)dW(s)\|^2 ds = \int_0^T \sum_{i=0}^{\infty} \mathbb{E}\|\Phi_i(s)\|^2 ds.$$

Proof. See [GA, Lemma 2.1] □

Remark 3.6. The assumption in Theorem 3.5 that the W_i are normalised is not a restriction, since each Φ_i can be scaled.

Lemma 3.7. *Let H be a separable real Hilbert space. Let $(W_i)_{i=0}^{\infty}$ be as in Theorem 3.5. Let $\Phi = (\Phi_i)_{i=0}^{\infty}$ be a sequence of adapted piecewise uniformly continuous functions from $[0, \infty)$ to $L^2(\Omega; H)$ such that $\int_0^t \sum_{i=0}^{\infty} \mathbb{E}\|\Phi_i(s)\|^2 ds < \infty$ for every $t \in [0, \infty)$. Then:*

(i) $t \mapsto \int_0^t \Phi(s)dW(s) : [0, \infty) \to L^2(\Omega; H)$ (see (7)) is continuous and adapted.

(ii) If $T : H \to H$ is a bounded linear operator, then for each $t \geq 0$

$$\int_0^t T\Phi(s)dW(s) = T \int_0^t \Phi(s)dW(s),$$

i.e.

$$\sum_{i=0}^{\infty} \int_0^t T \circ (\Phi_i(s))dW_i(s) = T \circ \left(\sum_{i=0}^{\infty} \int_0^t \Phi_i(s)dW_i(s) \right).$$

Proof. See [GA, Prop. 2.5]. □

3.2 A stochastic integral equation in a Hilbert space

Let H be a separable real Hilbert space. Let A be closed operator in H that generates a strongly continuous semi-group $(S(t))_{t\geq 0}$. Let $F : [0, \infty) \times H \to H$ be continuous and let $G = (G_i)_{i=0}^{\infty}$ be a sequence of continuous functions from $[0, \infty) \times H \to H$, such that there exist constants L_F, $L_G \geq 0$ with for all $t \geq 0$ and x, $y \in H$:

$$\|F(t, x) - F(t, y)\| \leq L_F\|x - y\|,$$

$$\sum_{i=0}^{\infty}\|G_i(t,x)-G_i(t,y)\|^2 \le L_B^2\|x-y\|^2,$$

$$\int_0^t \sum_{i=0}^{\infty}\|G_i(s,0)\|^2 ds < \infty.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $W = (W_i)_{i=0}^{\infty}$ a sequence of independent normalised scalar Wiener processes on Ω with respect to a filtration $(\mathcal{F}_t)_{t\ge 0}$ in $\mathcal{F}$. Let $X_0 \in L^2(\Omega; H)$ be $\mathcal{F}_0$-measurable. We consider the initial value problem

$$\begin{cases} X(t) - X(0) = \int_0^t AX(s) + F(s, X(s))ds + \int_0^t G(s, X(s))dW(s), \\ X(0) = X_0. \end{cases} \tag{8}$$

We now show that the integrals $\int_0^t F(s, X(s))ds$ and $\int_0^t G(s, X(s))dW(s) = \sum_{i=0}^{\infty}\int_0^t G_i(s)dW_i(s)$ are well-defined if X is a continuous and adapted function from [0,t] to $L^2(\Omega; H)$. First note that $s \mapsto F(s, X(s))$ is continuous (and therefore bounded) and adapted from $[0, t]$ to $L^2(\Omega; H)$, and thus Bochner integrable by Corollary B.8. $s \mapsto G_i(s)$ from $[0, t]$ to $L^2(\Omega; H)$ is also adapted and continuous. Furthermore,

$$\int_0^T \sum_{i=0}^{\infty}\mathbb{E}\|G_i(s, X(s))\|^2 ds =$$

$$\int_0^T \mathbb{E}\left(\sum_{i=0}^{\infty}\|G_i(s,0) + G_i(s,X(s)) - G_i(s,0)\|^2\right) ds \le$$

$$\int_0^T \mathbb{E}\left(\sum_{i=0}^{\infty}2\|G_i(s,0)\|^2 + 2\|G_i(s,X(s)) - G_i(s,0)\|^2\right) ds \le$$

$$2L_G^2\int_0^T \mathbb{E}\|X(s)\|^2 ds + \int_0^T 2\mathbb{E}\sum_{i=0}^{\infty}\|G_i(s,0)\|^2 ds < \infty.$$

So the integrals are well-defined. In a similar way it can be shown that the integrals in (9) are well-defined (see [GA, section 4]).

Lemma 3.8. *Suppose X is a continuous and adapted function from $[0, t]$ to $L^2(\Omega; H)$ that satisfies*

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(s, X(s))ds + \int_0^t S(t-s)G(s, X(s))dW(s), \tag{9}$$

where the second integral is defined as in (7). Assume that $X(t)(\omega)$, $F(t, X(t))(\omega)$ and $G_i(t, X(t))(\omega)$ are in $D(A)$ for almost all $\omega \in \Omega$ for all $t \ge 0$. Then X satisfies (8).

Proof. For $t \geq 0$,

$$\int_0^t AX(s)ds = A\int_0^t X(s)ds =$$

$$A\left(\int_0^t S(s)X_0 ds\right) + A\left(\int_0^t \int_0^s S(s-r)F(r,X(r))drds\right)$$
$$+A\left(\int_0^t \int_0^s S(s-r)G(r,X(r))dW(r)ds\right) =$$

$$A\left(\int_0^t S(s)X_0 ds\right) + A\left(\int_0^t \int_r^t S(s-r)F(r,X(r))dsdr\right)$$
$$+A\left(\int_0^t \sum_{i=0}^{\infty} \int_0^s S(s-r)G_i(r,X(r))dW_i(r)ds\right) =$$

$$\int_0^t AS(s)X_0 ds + \int_0^t \int_0^{t-r} AS(s)F(r,X(r))dsdr$$
$$+A\left(\sum_{i=0}^{\infty} \int_0^t \int_r^t S(s-r)G_i(r,X(r))dsdW_i(r)\right).$$

By Lemma 2.7 this equals

$$(S(t)-S(0))X(0) + \int_0^t (S(t-r)-S(0))F(r,X(r))dr+$$
$$+\sum_{i=0}^{\infty} \int_0^t (S(t-r)-S(0))G_i(r,X(r))dW_i(r) =$$

$$S(t)X_0 - X_0 - \int_0^t F(r,X(r))dr + \int_0^t S(t-r)F(r,X(r))dr$$
$$-\int_0^t G(r,X(r))dW(r) + \int_0^t S(t-r)G(r,X(r))dW(r) =$$

$$S(t)X_0 - X_0 - \int_0^t F(r,X(r))dr - \int_0^t G(r,X(r))dW(r) + X(t) - S(t)X_0 =$$
$$X(t) - X_0 - \int_0^t F(s,X(s))ds - \int_0^t G(s,X(s))dW(s).$$

□

If X is as in Lemma 3.8, it is called a *mild solution* of (9).

Theorem 3.9. *(9) has one and only one mild solution.*

Proof. See [GA, Thm 4.1]. The Banach Fixed Point Theorem is used. □

3.3 A stochastic integral equation in a Banach space

In the inhomogeneous Cauchy problem, we considered a differential equation of Banach space valued functions. In the previous section we considered a stochastic integral equation of functions with values in a separable Hilbert space. We would like to consider the same equation, however with Banach space valued functions. However, we defined the integral that modeled the effect of the noise as an infinite sum of integrals. This made sense since a separable Hilbert space has an orthonormal basis. A Banach space however generally does not. Moreover, the proof of Theorem 3.9 uses the Itô isometry. So we will consider a stochastic integral equation with Banach space valued functions except for the integrand of the integral that models the noise, which we assume to take values in a separable Hilbert space. This will be a generalisation of (9) (if we assume there that F is Lipschitz). We will prove existence and uniqueness of a solution, and a result on boundedness of the expected value of the norm of the solution.

Let E be a real Banach space and $(S(t))_{t\geq 0}$ a strongly continuous semi-group in E generated by some closed linear operator $A : E \to E$. Let $H \subset E$ be a separable real Hilbert space that is a subset of E. Denote the norms of E and H by $\|\cdot\|$ and $\|\cdot\|_H$ respectively and assume that for some $c > 0$, say, for all $x \in E$, $\|x\| \leq c\|x\|_H$. Further assume that $(S(t))_{t\geq 0}$ leaves H invariant, and that it is also a strongly continuous semi-group on H, that is, $S(t)x \in H$ for all $x \in H$ and there are constants $M_H > 0$, α_H such that $\|S(t)x\|_H \leq M_H e^{\alpha_H t}\|x\|_H$ for all $x \in H$ and $t \geq 0$.

Let $F : [0,\infty) \times E \to E$ be a continuous function and $G = (G_i)_{i=0}^{\infty}$ a sequence of continuous functions $G_i : [0,\infty) \times E \to H$, $i \in \mathbb{N}$ such that there exist constants L_F, $L_G \geq 0$ with for all $t \geq 0$ and x, $y \in E$:

$$\|F(t,x) - F(t,y)\| \leq L_F\|x - y\|,$$

$$\sum_{i=0}^{\infty}\|G_i(t,x) - G_i(t,y)\|_H^2 \leq L_B^2\|x - y\|^2.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $W = (W_i)_{i=0}^{\infty}$ a sequence of independent normalised scalar Wiener processes on Ω with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ in $\mathcal{F}$. Let $X_0 \in L^2(\Omega; E)$ be $\mathcal{F}_0$-measurable. We will study the equation

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(s,X(s))ds + \int_0^t S(t-s)G(s,X(s))dW(s). \quad (10)$$

Note that the analogue of Lemma 3.8 holds just as well for this setting. The following theorem is a generalisation of Theorem 3.9.

Theorem 3.10. *There exists one and only one function $X : [0, \infty) \to L^2(\Omega, E)$ that is continuous, adapted to $(\mathcal{F}_t)_{t\geq 0}$, and that satisfies (10).*

Proof. Let $T > 0$. It suffices to prove existence and uniqueness on $[0, T]$. Let C_{ad} denote the subspace of $C([0, T]; L^2(\Omega; H))$ consisting of the functions that are adapted to $(\mathcal{F}_t)_{t\geq 0}$. Let $M_T = \sup_{s\in[0,T]} \|S(t)\|$. For $X \in C_{ad}$ define the function

$$R(X)(t) := S(t)X_0 + \int_0^t S(t-s)F(s, X(s))ds + \int_0^t S(t-s)G(s, X(s))dW(s),$$

$t \in [0, T]$. Note that $R(X)$ is continuous (use the Itô isometry) from $[0, T]$ to $L^2(\Omega; H)$, and $R(X)$ is adapted, so R maps C_{ad} into itself.

Let X, $Y \in C_{ad}$. For $t \in [0, T]$ we have

$$\mathbb{E}\|R(X)(t) - R(Y)(t)\|^2$$

$$\leq \mathbb{E}2\|\int_0^t S(t-s)(F(s, X(s))ds\|^2 + \mathbb{E}2\|\sum_{i=0}^{\infty}\int_0^t S(t-s)(G_i(s, X(s)) - G_i(s, Y(s)))dW_i(s)\|^2$$

$$\leq 2\mathbb{E}(\int_0^t M_T L_F \|X(s) - Y(s)\|ds)^2 + 2\mathbb{E}c^2\|\sum_{i=0}^{\infty}\int_0^t S(t-s)(G_i(s, X(s)) - G_i(s, Y(s)))dW_i(s)\|_H^2.$$

By Tonelli's Theorem and the final statement of Theorem 3.5 this equals

$$2(\int_0^t M_T L_F \mathbb{E}\|X(s) - Y(s)\|ds)^2 + 2c^2\sum_{i=0}^{\infty}\int_0^t \mathbb{E}\|S(t-s)(G_i(s, X(s)) - G_i(s, Y(s)))\|_H^2 ds$$

$$\leq 2M_T L_F t\int_0^t \mathbb{E}\|X(s) - Y(s)\|^2 ds + 2c^2 M_T^2\int_0^t \mathbb{E}L_G^2\|X(s) - Y(s)\|^2 ds$$

$$\leq 2(L_F^2 T + 2c^2 L_G^2)M_T^2\int_0^t \mathbb{E}\|X(s) - Y(s)\|^2 ds.$$

The rest of the proof is analogous to the end of the proof of Theorem 3.9 (found in [GA, Thm 4.1]). That is, we note that for any $a > 0$ and $t \in [0, T]$,

$$e^{-\alpha t}\mathbb{E}\|R(X)(t) - R(Y)(t)\|^2$$

$$\leq 2(L_F^2 T + 2c^2 L_G^2)M_T^2\int_0^t e^{-a(t-s)}e^{as}\mathbb{E}\|X(s) - Y(s)\|^2 ds$$

$$\leq 2(L_F^2 T + 2c^2 L_G^2)M_T^2\int_0^t e^{-a(t-s)}ds \max_{0\leq s\leq t} e^{-as}\mathbb{E}\|X(s) - Y(s)\|^2 ds$$

$$\leq 2a^{-1}(L_F^2 T + 2c^2 L_G^2)M_T^2 \max_{0\leq s\leq t} e^{-as}\mathbb{E}\|X(s) - Y(s)\|^2 ds. \quad \text{So}$$

$$\max_{0\leq t\leq T} e^{-\alpha t}\mathbb{E}\|R(X)(t) - R(Y)(t)\|^2$$

$$\leq 2a^{-1}(L_F^2 T + 2c^2 L_G^2) M_T^2 \max_{0\leq s\leq T} e^{-as}\mathbb{E}\|X(s) - Y(s)\|^2 ds.$$

Now choose $a > 2(L_F^2 T + 2c^2 L_G^2) M_T^2$. Then R is a strict contraction on C_{ad} with respect to the norm $|||\cdot|||$ given by

$$|||X||| = \max_{0\leq t\leq T} \left(e^{-at}\mathbb{E}\|X(t)\|^2\right)^{1/2}.$$

It can be shown that $(C_{ad}, |||\cdot|||)$ is a closed subspace, so a Banach space. So by Banach's fixed point theorem R has a unique fixed point in C_{ad}. So (10) has a unique solution on $[0, T]$. □

Theorem 3.11. *Suppose that there are M, $\alpha > 0$ such that $\|S(t)\| \leq Me^{-\alpha t}$ for all $t \geq 0$. Let F and $(G_i)_{i=0}^{\infty}$ be as usual, assuming however that they do not depend on t (i.e. that they are functions from $E \to E$ and $E \to H$ respectively). Then the unique solution $X \in C_{ad}$ of (10) satisfies $\sup_{t\geq 0}\mathbb{E}\|X(t)\| < \infty$ if L_F and L_G are small enough.*

Proof. Let X be the unique solution in C_{ad}. Choose constants c_1, $c_2 > 0$ such that for all $t \geq 0$, $\|\int_0^1 Me^{-\alpha(t-s)}F(X(s))ds\| \leq c_1 e^{-\alpha t}$ and $\|\int_0^1 Me^{-\alpha(t-s)}G(X(s))dW(s)\| \leq c_2 e^{-\alpha t}$. Let $\tau \in [0, 1]$. For $t \geq 0$,

$$\mathbb{E}\|X(t+\tau) - X(t)\|^2$$

$$\leq \mathbb{E}\|(S(t+\tau)-S(t))X_0 + \int_0^{\tau} S(t+\tau-s)F(X(s))ds + \int_{\tau}^{t+\tau} S(t+\tau-s)F(X(s))ds$$

$$+ \int_0^{\tau} S(t+\tau-s)G(X(s))dW(s) + \int_{\tau}^{t+\tau} S(t+\tau-s)G(X(s))dW(s)$$

$$- \int_0^t S(t-s)F(X(s))ds - \int_0^t S(t-s)G(X(s))dW(s)\|^2$$

$$\leq \mathbb{E}\|(S(t+\tau)-S(t))X_0 + \int_0^{\tau} S(t+\tau-s)F(X(s))ds + \int_0^t S(t-s)F(X(s+\tau))ds$$

$$+ \int_0^{\tau} S(t+\tau-s)G(X(s))dW(s) + \int_0^t S(t-s)G(X(s+\tau))dW(s)$$

$$- \int_0^t S(t-s)F(X(s))ds - \int_0^t S(t-s)G(X(s))dW(s)\|^2$$

$$\leq \mathbb{E}2\|(S(t+\tau)-S(t))X_0 + \int_0^{\tau} S(t+\tau-s)F(X(s))ds + \int_0^{\tau} S(t+\tau-s)G(X(s))dW(s)\|^2$$

$$+\mathbb{E}2\|\int_0^t S(t-s)(F(X(s+\tau))-F(s))ds + \int_0^t S(t-s)(G(X(s+\tau))-G(X(s))dW(s)\|^2$$

$$\leq 2\mathbb{E}\left(2Me^{-\alpha t}\|X_0\| + c_1 e^{-\alpha t} + c_2 e^{-\alpha t}\right)^2$$

$$+\mathbb{E}4\|\int_0^t S(t-s)(F(X(s+\tau))-F(s))ds\|^2 + \mathbb{E}4\|\int_0^t S(t-s)(G(X(s+\tau))-G(X(s))dW(s)\|^2$$

$$\leq 2\mathbb{E}\left(2M\|X_0\| + c_1 + c_2\right)^2 e^{-2\alpha t}$$

$$+\mathbb{E}4\left(\int_0^t Me^{-\alpha(t-s)}L_F\|X(s+\tau) - X(s)\|ds\right)^2 +$$

$$\mathbb{E}4c^2\|\int_0^t S(t-s)(G(X(s+\tau)) - G(X(s))dW(s)\|_H^2$$

$$\leq 2\mathbb{E}\left(2M\|X_0\| + c_1 + c_2\right)^2 e^{-2\alpha t}$$

$$+\mathbb{E}4\left(M^2e^{-2\alpha t}\int_0^t e^{\alpha s/2}e^{\alpha s/2}L_F\|X(s+\tau) - X(s)\|ds\right)^2 +$$

$$4c^2\sum_{i=0}^{\infty}\int_0^t \mathbb{E}\|S(t-s)(G_i(X(s+\tau)) - G_i(X(s)))\|_H^2 ds$$

$$\leq \mathbb{E}2\left(2M\|X_0\| + c_1 + c_2\right)^2 e^{-2\alpha t}$$

$$+\mathbb{E}4M^2L_F^2e^{-2\alpha t}\int_0^t (e^{\alpha s/2})^2 ds\int_0^t (e^{\alpha s/2})^2\|X(s+\tau) - X(s)\|^2 ds$$

$$+4c^2\int_0^t M^2e^{-2\alpha(t-s)}\mathbb{E}\sum_{i=0}^{\infty}\|(G_i(X(s+\tau)) - G_i(X(s)))\|_H^2 ds$$

$$\leq \mathbb{E}2\left(2M\|X_0\| + c_1 + c_2\right)^2 e^{-2\alpha t}$$

$$+\mathbb{E}4M^2L_F^2e^{-2\alpha t}\frac{e^{\alpha t}-1}{\alpha}\int_0^t e^{\alpha s}\|X(s+\tau) - X(s)\|^2 ds$$

$$+4c^2\int_0^t M^2e^{-\alpha(t-s)}\mathbb{E}L_G^2\|X(s+\tau) - X(s)\|^2 ds$$

$$\leq \mathbb{E}2\left(2M\|X_0\| + c_1 + c_2\right)^2 e^{-2\alpha t}$$

$$+4M^2L_F^2\alpha^{-1}e^{-\alpha t}\int_0^t e^{\alpha s}\mathbb{E}\|X(s+\tau) - X(s)\|^2 ds$$

$$+4c^2M^2L_G^2e^{-\alpha t}\int_0^t e^{\alpha s}\mathbb{E}\|X(s+\tau) - X(s)\|^2 ds.$$

Multiplying by $e^{\alpha t}$ on both sides gives

$$e^{\alpha t}\mathbb{E}\|X(t+\tau) - X(t)\|^2$$

$$\leq \mathbb{E}2\left(2M\|X_0\| + c_1 + c_2\right)^2$$

$$+4M^2(L_F^2\alpha^{-1} + c^2L_G^2)\int_0^t e^{\alpha s}\mathbb{E}\|X(s+\tau) - X(s)\|^2 ds$$

So by Gronwall's Lemma, for all $\tau \in [0,1]$ and $t \geq 0$,

$$e^{\alpha t}\mathbb{E}\|X(t+\tau) - X(t)\|^2$$
$$\leq \mathbb{E}2\left(2M\|X_0\| + c_1 + c_2\right)^2 \cdot e^{4M^2(L_F^2 a^{-1} + c^2 L_G^2)t}.$$

So for all $t \in [0,1]$ and $t \geq 0$,

$$\mathbb{E}\|X(t+\tau) - X(t)\|^2$$
$$\leq \mathbb{E}2\left(2M\|X_0\| + c_1 + c_2\right)^2 \cdot e^{(4M^2(L_F^2 a^{-1} + c^2 L_G^2) - \alpha)t}.$$

So if L_F and L_G are small enough, or more precicely, if $4M^2(L_F^2 a^{-1} + c^2 L_G^2) < \alpha$, then (for all $t \geq 0$, $0 \leq \tau \leq 1$)

$$\mathbb{E}\|X(t+\tau) - X(t)\|^2 < Ke^{-\beta t} \tag{11}$$

for some K, $\beta > 0$, say.

So in particular we see for $n \in \mathbb{Z}_{>0}$ that

$$\mathbb{E}\|X(n) - X_0\| \leq \sum_{k=1}^{n} \left(\mathbb{E}\|X(k) - X(k-1)\|^2\right)^{\frac{1}{2}} \leq \sum_{k=1}^{n} \sqrt{K}e^{-\frac{1}{2}\beta(k-1)} \leq \sum_{k=1}^{\infty} \sqrt{K}e^{-\frac{1}{2}\beta(k-1)},$$

which is a converging geometric series. So $\sup_{n\in\mathbb{N}} \mathbb{E}\|X(n) - X_0\| < \infty$.

By again using (11) it follows that

$$\sup_{n\in\mathbb{N},\tau\in[0,1]} \mathbb{E}\|X(n+\tau) - X_0\|$$
$$\leq \sup_{n\in\mathbb{N},\tau\in[0,1]} \left(\mathbb{E}\|X(n+\tau) - X(n)\| + \mathbb{E}\|X(n) - X(0)\|\right)$$
$$\leq K + \sum_{k=1}^{\infty} \sqrt{K}e^{-\frac{1}{2}\beta(k-1)} < \infty.$$

So the solution of (10) satisfies $\sup_{t\geq 0} \mathbb{E}\|X(t)\| < \infty$. □

3.4 Examples

We will give some simple examples of stochastic integral equations and apply our results. Let E be the Banach space $L^p[0,1]$, $1 \leq p \leq 2$. Take H to be the Hilbert space $L^2[0,1]$. Denote the (standard) norms by $\|\cdot\|$, $\|\cdot\|_H$ respectively. Then $H \subset E$ and $\|f\| \leq 1 \cdot \|f\|_H$ for all $f \in H$. Indeed, assuming $p \neq 2$ and taking $r = \frac{2}{p}$ and $s \geq 2$ such that $\frac{1}{r} + \frac{1}{s} = 1$, we have

$$\|f\| = \left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}}$$

$$\leq \left((\int_0^1 (|f(x)|^p)^r dx)^{1/r} \cdot (\int_0^1 1^s dx)^{1/s} \right)^{1/p}$$

$$= \left(\int_0^1 |f(x)|^2 dx \right)^{1/2} = \|f\|_H.$$

Define the family $\{S(t)\}_{t\geq 0}$, $S(t) : L^p[0,1] \to L^p[0,1]$, by

$$(S(t)f)(x) = \begin{cases} f(x+t) & \text{if} \quad 0 \leq x \leq 1-t, \\ 0 & \text{otherwise.} \end{cases}$$

Note that for each $t \geq 0$, $S(t) \in B(L^p[0,1])$. Moreover, $S(0) = I$ and $S(t+s) = S(t)S(s)$ for all t, $s \geq 0$.

Lemma 3.12. *For all $f \in L^p[0,1]$, the map $t \mapsto S(t)f : [0,\infty) \to L^p[0,1]$ is continuous.*

Proof. Choose $\epsilon > 0$. $C[0,1]$ is dense in $L^p[0,1]$, since the simple functions are dense in $L^p[0,1]$ and for every simple function on $[0,1]$ there is a sequence of continuous functions that converges to it. So choose $g \in C[0,1]$ with $\|g-f\| < \frac{1}{3}\epsilon$. Fix $t \geq 0$. As $t' \to t$, $S(t')g \to S(t)g$ in $L^p[0,1]$, since g is uniformly continuous. So choose $\delta > 0$ such that for all t' with $\|t'-t\| \leq \delta$, $\|S(t')g - S(t)g\| < \frac{\epsilon}{3}$. Then for such t' we have $\|S(t')f - S(t)f\| \leq \|S(t')f - S(t')g\| + \|S(t')g - S(t)g\| + \|S(t)g - S(t)f\| \leq 2\|f-g\| + \|S(t')g - S(t)g\| \leq 2 \cdot \frac{1}{3}\epsilon + \frac{1}{3}\epsilon = \epsilon$. So $t \mapsto S(t)f$ is continuous at t. Since $t \geq 0$ was arbitrary, $t \mapsto S(t)f$ is continuous on $[0,\infty)$. □

So $(S(t))$ is a strongly continuous semi-group on $L^p[0,1]$. Note that it leaves H invariant and is also a strongly continuous semi-group on H. The domain of the generator of $(S(t))$ is dense in $E = L^p[0,1]$, by Theorem 2.7(e). In case $p = 1$ we have the following result from [EN]:

Lemma 3.13. *In case $p = 1$, the generator A of $(S(t))$ is the map $f \mapsto f'$ with (dense) domain $D(A) = \{f \in L^1[0,1] : \quad f$ is absolutely continuous, $f' \in L^1[0,1]$ and $f(1) = 0\}$.*

Proof. See [EN, II.2.11]. □

Let $F : L^p[0,1] \to L^p[0,1]$ be Lipschitz continuous with Lipschitz constant L_F. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $W = (W_i)_{i=0}^{\infty}$ a sequence of independent normalised scalar Wiener processes on Ω with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ in $\mathcal{F}$. Let $X_0 \in L^2(\Omega; L^2[0,1])$ be $\mathcal{F}_0$-measurable.

In the following examples we will define a sequence $G = (G_i)_{i=0}^{\infty}$, $G_i : L^p[0,1] \to L^2[0,1]$. In each example we will show that $(G_i)_{i=0}^{\infty}$ is such that each G_i is continuous and there exists an $L_G > 0$ such that for all $f,\, g \in E = L^p[0,1]$,

$$\sum_{k=0}^{\infty} \|G_k(f) - G_k(g)\|_H^2 \leq L_G^2 \|f - g\|^2. \tag{12}$$

It then follows from Theorem 3.10 that the equation

$$X(t) = S(t)X_0 + \int_0^\infty S(t-s)F(X(s))ds + \int_0^\infty S(t-s)G(X(s))dW(s) \tag{13}$$

has exactly one solution $X : [0,\infty) \to L^2(\Omega; L^p[0,1])$ that is continuous and adapted. Theorem 3.11 says that if $L_F,\, L_G > 0$ are small enough, then $\mathbb{E}\|X(t)\|$ is bounded on $[0,\infty)$.

Example. For each $i \in \mathbb{N}$ define G_i by $G_i f = g_i f$, where the g_i are elements of $L^2[0,1]$ such that $\sum_{i=0}^{\infty} \|g_i\|_H^2 < \infty$. Then each G_i is a continuous map $L^p[0,1] \to L^2[0,1]$ that is Lipschitz with Lipschitz constant 0. The series in (12) is equal to zero for all $f,\, g \in E$. So (13) has a unique continuous, adapted solution. Moreover, if $L_f > 0$ is small enough, then $\mathbb{E}\|X(t)\|$ is bounded on $[0,\infty)$. This is in particular the case if F is identically zero.

Example. Define G_0 by $G_0(f)(x) = \int_0^x f(y)dy$, $x \in [0,1]$. It follows easily that G_0 is continuous. Note that $G_0(f)$ is continuous, so it is contained in $L^2[0,1]$. For $i \geq 1$, let G_i be identically 0. Note that for $f,\, g \in L^p[0,1]$,

$$\begin{aligned}\sum_{k=0}^{\infty} \|G_k(f) - G_k(g)\|_H^2 &= \|G_0(f) - G_0(g)\|_H^2 \\ &= \int_0^1 |G_0(f)(x) - G_0(g)(x)|^2 dx = \\ &= \int_0^1 |\int_0^x f(y) - g(y)dy|^2 dx := (*).\end{aligned}$$

Letting $q \geq 2$ such that $q^{-1} = 1 - p^{-1}$,

$$\begin{aligned}(*) &\leq \int_0^1 \left((\int_0^1 |f(y) - g(y)|^p dy)^{1/p} \cdot (\int_0^1 1^q dx)^{1/q} \right)^2 dx \\ &\leq \int_0^1 \left(\int_0^1 |f(y) - g(y)|^p dy \right)^{2/p} dx \\ &= 1 \cdot \left(\int_0^1 |f(y) - g(y)|^p dy \right)^{2/p} \\ &= \|f - g\|^2.\end{aligned}$$

In case $p = 1$, we have

$$\int_0^1 |\int_0^x f(y) - g(y)dy|^2 dx$$

$$\leq \int_0^1 \|f-g\|^2 dx = \|f-g\|^2.$$

So, taking $L_G = 1$, (12) holds for all $f, g \in E$. So like in the previous example we see that (13) has a unique continuous, adapted solution. Theorem 3.11 says that if L_F, $L_G > 0$ can be chosen small enough, then the expected value of $\|X(t)\|$ is bounded on $[0, \infty)$.

Example. For each $k \in \mathbb{N}$, define G_k by

$$G_k(f)(x) = \left(\int_0^1 f(y) sin(ky) dy \right) g_k(x),$$

where the $g_k \in L^2[0,1]$ are such that $\sum_{k=0}^{\infty} \|g_k\|_H^2 = K < \infty$ for some constant K. It follows easily that each G_k is continuous. Note that each G_k maps $L^p[0,1]$ to $L^2[0,1]$. For $f, g \in E = L^p[0,1]$ we have

$$\sum_{k=0}^{\infty} \|G_k(f) - G_k(g)\|_H^2$$

$$\leq \sum_{k=0}^{\infty} \int_0^1 \left| \left(\int_0^1 (f(y) - g(y)) sin(ky) dy \right) g_k(x) \right|^2 dx$$

$$\leq \sum_{k=0}^{\infty} \left(\int_0^1 |f(y) - g(y)| dy \right)^2 \cdot \|g_k\|_H^2$$

$$\leq \left(\int_0^1 |f(y) - g(y)|^p dy \right)^{2/p} \cdot \sum_{k=0}^{\infty} \|g_k\|_H^2 \qquad \text{(by Hölder)}$$

$$\leq \|f-g\|^2 \cdot K.$$

So, taking $L_G = K$, (12) holds for all $f, g \in E$. So again we see that (13) has a unique continuous, adapted solution and $\sup_{t \geq 0} \mathbb{E}\|X(t)\| < \infty$ if L_F, $L_G > 0$ are small enough.

A Appendix: The operator calculus

A.1 The operator calculus in a finite dimensional Banach space

In this thesis the operator calculus is used several times, for example in Theorem 2.11. For proofs in this subsection see [DU, VII.1]. Let X be a finite dimensional complex Banach space, and $T \in B(X)$. We will write $T^0 = I$, and if $P(\lambda) = \sum_{i=0}^{n} \alpha_i \lambda^i$ is a polynomial with complex coefficients, we will interpret $P(T)$ as $\sum_{i=0}^{n} \alpha_i T^i$.

Definition A.1. The *spectrum* $\sigma(T)$ of an operator T in a finite dimensional Banach space is the set of complex numbers λ such that $\lambda I - T$ is not injective. The *index* $v(\lambda)$ of a a complex number λ is the smallest non-negative integer v such that $(\lambda I - T)^v x = 0$ for every $x \in X$ for which $(\lambda I - T)^{v+1} x = 0$. Such a v always exists, as can be seen from the theory of generalised eigenvectors.

Theorem A.2. *If P and Q are complex polynomials, then $P(T) = Q(T)$ if and only if $P - Q$ has a zero of order $v(\lambda)$ for all $\lambda \in \sigma(T)$.*

Theorem A.3. *The spectrum of an operator in a finite dimensional Banachspace is a non-void finite set.*

Let $\mathcal{F}(T)$ be the set of all functions $\mathbb{C} \to \mathbb{C}$ which are analytic in an open neighbourhood of $\sigma(T)$. If $f \in \mathcal{F}$, let P be a complex polynomial such that $f^{(m)}(\lambda) = P^{(m)}(\lambda)$ for all $m \leq v(\lambda) - 1$, for each $\lambda \in \sigma(T)$. ($f^{(m)}(\lambda)$ is short for $\frac{d^m}{d\lambda^m} f(\lambda)$.) We define $f(T) = P(T)$. From A.2 can be seen that this definition is unambiguous.

The following theorem follows easily from the definition.

Theorem A.4. *If f, $g \in \mathcal{F}(T)$ and α, $\beta \in \mathbb{C}$, the following properties hold:*

- *$\alpha f + \beta g \in \mathcal{F}(T)$ and $(\alpha f + \beta g)(T) = \alpha f(T) + \beta g(T)$;*
- *$f \cdot g \in \mathcal{F}(T)$ and $(f \cdot g)(T) = f(T) \cdot g(T)$;*
- *if $f(\lambda) = \sum \alpha_n \lambda^n$ then $f(T) = \sum \alpha_n T^n$;*
- *$f(T) = 0$ if and only if for all $\lambda \in \sigma(T)$ and $0 \leq m \leq v(\lambda) - 1$, $f^{(m)}(\lambda) = 0$.*

If $\lambda_0 \in \mathbb{C}$, let e_{λ_0} be identically equal to one in a neighbourhood of λ_0, and identically equal to zero in a neighbourhood of each point of $\sigma(T) \cap \{\lambda_0\}'$. Put $E(\lambda_0) = e_{\lambda_0}(T)$. Then $e_{\lambda_0} \in \mathcal{F}(T)$. The next theorem follows immediately from the last one.

Theorem A.5. *(a) $E(\lambda_0) \neq 0$ if and only if λ_0 is in $\sigma(T)$.*

(b) $E(\lambda_0)^2 = E(\lambda_0)$ and $E(\lambda_0)E(\lambda_1) = 0$ when $\lambda_0 \neq \lambda_1$.

(c)$I = \sum_{\lambda \in \sigma(T)} E(\lambda)$.

Let $\{\lambda_1, ..., \lambda_k\}$ be an enumeration of $\sigma(T)$. From theorem A.5 we see that

$$X = E(\lambda_1)X \oplus E(\lambda_2)X \oplus ... \oplus E(\lambda_k)X.$$

Also, since $TE(\lambda_i) = E(\lambda_i)T$, it follows that $TE(\lambda_i) \subset X_i$, $i = 1, ..., k$.

Theorem A.6. *If f is in $\mathcal{F}(T)$, then*

$$f(T) = \sum_{\lambda \in \sigma(T)} \sum_{i=0}^{v(\lambda)-1} \frac{(T - \lambda I)^i}{i!} f^{(i)}(\lambda) E(\lambda).$$

Proof. This is a consequence of A.4, since f and the function $g \in \mathcal{F}(T)$ defined by

$$g(\mu) = \sum_{\lambda \in \sigma(T)} \sum_{i=0}^{v(\lambda)-1} \frac{(\mu - \lambda)^i}{i!} f^{(i)}(\lambda) e_\lambda(\mu)$$

satisfy the relations $f^{(m)}(\lambda) = g^{(m)}(\lambda)$, $m \leq v(\lambda) - 1$, $\lambda \in \sigma(T)$. □

Theorem A.7. *Let $f \in \mathcal{F}(T)$ be analytic in a domain containing the closure of an open set U containing $\sigma(T)$, and suppose that the boundary B of U consists of a finite number of closed rectifiable Jordan curves, oriented in the positive sense customary in the theory of complex variables. Then $f(T)$ is equal to a Riemann contour integral over B:*

$$f(T) = \frac{1}{2\pi i} \int_B f(\lambda)(\lambda I - T)^{-1} d\lambda.$$

Proof. Let $\lambda \notin \sigma(T) = \{\lambda_1, ..., \lambda_k\}$, and let $r(\xi) = (\lambda - \xi)^{-1}$. Then, by theorems A.4 and A.6,

$$(\lambda I - T)^{-1} = r(T) = \sum_{j=1}^{k} \sum_{v=0}^{v(\lambda_j)-1} \frac{(T - \lambda_j I)^v}{(\lambda - \lambda_j)^{v+1}} E(\lambda_j).$$

So, if $f \in \mathcal{F}(T)$,

$$\frac{1}{2\pi i} \int_B f(\lambda)(\lambda I - T)^{-1} d\lambda = \sum_{j=1}^{k} \sum_{v=0}^{v(\lambda_j)-1} (T - \lambda_j I)^v \frac{f^{(v)}(\lambda_j)}{v!} E(\lambda_j) = f(T).$$

□

A.2 The operator calculus

Let $X \neq \{0\}$ be a complex Banach space, and $T \in B(X)$. In this section we generalise many results from the previous section to the case that X is infinite dimensional. Proof can be found in [DU, VII.3].

Definition A.8. The *resolvent set* $\rho(T)$ of T is the set of all $\lambda \in \mathbb{C}$ for which $(\lambda I - T)$ has a bounded inverse $\in B(X)$ (this inverse is then unique). Define the *spectrum* $\sigma(T)$ of T as $\mathbb{C} \setminus \rho(T)$. The function $R(\lambda; T) = (\lambda I - T)^{-1}$, defined on $\rho(T)$, is called the *resolvent function* of T, or simply the *resolvent* of T.

Lemma A.9. *The resolvent set $\rho(T)$ is open. The function $R(\lambda; T)$ is analytic in $\rho(T)$.*

Lemma A.10. *The closed set $\sigma(T)$ is bounded and not empty. Moreover $sup|\sigma(T)| \leq |T|$. For $|\lambda| > \sup|\sigma(T)|$ the series $R(\lambda;T) = \sum_{n=0}^{\infty} T^n/\lambda^{n+1}$ converges in the uniform operator topology.*

Denote by $\mathcal{F}(T)$ the family of functions $f : \mathbb{C} \to \mathbb{C}$ which are analytic on some neighbourhood of $\sigma(T)$.

Definition A.11. Let $f \in \mathcal{F}(T)$, and let U be an open set whose boundary B consists of a finite number of rectifiable Jordan curves, oriented in the positive sense customary in the theory of complex variables. Suppose that $U \subseteq \sigma(T)$, and that $U \cup B$ is contained in the domain of analyticity of f. Then the operator $f(T)$ is defined as

$$f(T) = \frac{1}{2\pi i}\int_B f(\lambda)R(\lambda;T)d\lambda.$$

It follows from lemma A.9 and from the Cauchy integral theorem (for vector valued functions of a complex variable), that $f(T)$ depends only on the function f, and not on the domain U.

Theorem A.12. *If f, g are in $\mathcal{F}(T)$, and α, β are complex numbers, then*

(a) $\alpha f + \beta g$ is in $\mathcal{F}(T)$ and $\alpha f(T) + \beta g(T) = (\alpha f + \beta g)(T)$;

(b) $f \cdot g$ is in $\mathcal{F}(T)$ and $f(T) \cdot g(T) = (f \cdot g)(T)$;

(c) if f has the power series expansion $f(\lambda) = \sum_{k=0}^{\infty} \alpha_k \lambda^k$, valid in a neighbourhood of $\sigma(T)$, then $f(T) = \sum_{k=0}^{\infty} \alpha_k T^k$.

Theorem A.13. *(Spectral mapping theorem.) If $f \in \mathcal{F}(T)$, then $f(\sigma(T)) = \sigma(f(T))$.*

Theorem A.14. *Let $f \in \mathcal{F}(T)$, $g \in \mathcal{F}(T)$, and $F(\xi) = g(f(\xi))$. Then F is in $\mathcal{F}(T)$, and $F(T) = g(f(T))$.*

B Appendix: Definition of the Bochner integral

We often consider integrals of certain Banach space valued functions. These integrals are all *Bochner integrals*, which we define here. Since any subset of $\mathbb{R}$ is separable, we can often use Corollary B.8. For proofs of the theorems in this section, see [DI]. Let X be a Banach space and (Ω, Σ, μ) a σ-finite measure space.

Definition B.1. A function $f : \Omega \to X$ is called Σ-*simple* (or just *simple*) if there exist E_1,...,$E_n \in \Sigma$ with $\mu(E_k) < \infty$ for all $1 \leq k \leq n$ and x_1,...,x_n such that

$$f(\omega) = \sum_{k=1}^{n} 1_{E_k}(\omega)x_k$$

for all $\omega \in \Omega$.

For such an f define

$$\int f d\mu := \sum_{k=1}^{n} \mu(E_k)x_k$$

which does not depend on the choice of the x_k and E_k.

Definition B.2. A function $f : \Omega \to X$ is called Σ-*strongly measurable* or simply *strongly measurable* if there exists a sequence of simple fucntions (f_n) such that $f_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$.

Recall that a set $E \subseteq X$ is called μ-*measurable* if there exists a set $A \in \Sigma$ such that the difference $(E \setminus A) \cup (A \setminus E)$ is contained in a set of Σ with μ-measure zero. We say that a function $f : \Omega \to X$ is μ-*strongly measurable* if it is μ-almost everywhere equal to a strongly measurable function.

If the measure space is complete, then the μ-strongly measurable fucntions are just the strongly measurable functions.

Lemma B.3. *If $f : \omega \to X$ is strongly measurable, then f is measurable (i.e. $f^{-1}(E) \in \Sigma$ for every Borel set E in X).*

If $f : \Omega \to X$ is strongly measurable, then the map $\|f(\cdot)\| : \omega \mapsto \|f(\omega)\| : \Omega \to \mathbb{R}$ is too. So $\|f(\cdot)\|$ is then measurable. So if f is just μ-strongly measurable, it follows that $\|f(\cdot)\|$ is μ-almost everywhere equal to a measurable function.

Definition B.4. A μ-strongly measurable function $f : \Omega \to X$ is called *Bochner integrable* if there exists a sequence (f_n) of simple functions such that

$$\int \|f - f_n\| d\mu \to 0$$

as $n \to \infty$.

In that case $(\int f_n d\mu)_n$ is a Cauchy sequence in X. Define the *Bochner integral* of f by

$$\int f d\mu := \lim_{n\to\infty} \int f_n d\mu.$$

This definition is independent of the choice for (f_n) (which is not hard to show).

Theorem B.5. *A μ-strongly measurable function $f : \Omega \to X$ is Bochner integrable if and only if $\int \|f\| d\mu < \infty$.*

A function $f : \Omega \to X$ is called *separably valued* if X has a closed separable subspace Y such that $f(\omega) \in Y$ for all $\Omega \in \Omega$.

A function $f : \omega \to X$ is called *weakly measurable* if $\omega \mapsto \phi(f(\omega))$ is measurable for all $\phi \in X'$.

Theorem B.6. *(Pettis measurability theorem.) For a function $f : \Omega \to X$ the following are equivalent:*

- *f is strongly measurable;*
- *f is separably valued and weakly measurable.*

The proof of the following corollary is now easy.

Corollary B.7. *For a separably valued function $f : \Omega \to X$, strong measurability, measurability and weak measurability are equivalent.*

Corollary B.8. *If X is separable, a function $f : \Omega \to X$ is Bochner integrable if and only if it is measurable and $\int \|f\| d\mu < \infty$.*

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