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Belyi Pairs, Dessins d'Enfants & Hypermaps

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Introduction

Recently I have been introduced to some theory of compact Riemann surfaces, in which I found it remarkable that many structures allow for different perspectives in the form of equivalent categories. The equivalence of five of these categories, most notably of Belyi pairs, dessins d'enfants and hypermaps, has been selected as topic for this thesis.

We begin our introduction with summarizing the five selected categories and four functors between them, where in the body of the text the former are constructed in more detail and the latter are all shown to be equivalences of categories. Note that the summary is already rather technical, which hopefully has the advantage of giving the reader some guidance in the main text. For keeping the eventual goal in mind will motivate the subsequent steps in the fairly long build-up to our main theorem. We end this introduction with some remarks on concrete categories, the structure of this thesis and notation.

Summary

Call a meromorphic map on a compact Riemann surface $\mathcal{M}$ unramified outside $\{0, 1, \infty\}$ and non-constant on each connected component of $\mathcal{M}$ a *Belyi map*, and a pair $(\mathcal{M}, f)$ such that f is a Belyi map on $\mathcal{M}$ a *Belyi pair.* A morphism of Belyi pairs $\varphi : (\mathcal{M}, f) \to (\mathcal{M}', f')$ is a holomorphic map on the underlying Riemann surfaces such that $f' \circ \varphi = f$. Denote the resulting category by Bel and the subspace $\mathbb{C} - \{0, 1\}$ of $\mathbb{C}$ by $\mathbb{P}_\circ$. Then we have a finite covering $\mathcal{X}_\mathcal{B}$ over $\mathbb{P}_\circ$ associated to a given Belyi pair $\mathcal{B} = (\mathcal{M}, f)$, namely the subspace $X_\mathcal{B} := \mathcal{M} - f^{-1}\{0, 1, \infty\}$ together with the map $p_\mathcal{B} : X_\mathcal{B} \to \mathbb{P}_\circ$ induced by restriction of f. This construction will induce a functor from Bel to finite coverings over $\mathbb{P}_\circ$ (called the *puncture functor*), which in fact is an equivalence.

The fundamental group of $\mathbb{P}_\circ$ with base point $1/2$ (denoted by π) is a free group generated by the equivalence classes of counter-clockwise parametrizations of the circles $\partial B_{1/2}(0)$ resp. $\partial B_{1/2}(1)$, written as σ_B resp. σ_W. Now define a *finite π-set* $\mathcal{S}$ as a pair $(|\mathcal{S}|, \rho)$, where $|\mathcal{S}|$ is a finite set and ρ a left π-action on $|\mathcal{S}|$. It is known that the category of finite π-sets (denoted by $\pi\text{–}\mathsf{Set}_\mathsf{f}$) is equivalent with the category of finite coverings over $\mathbb{P}_\circ$ (denoted by $\mathsf{Cov}(\mathbb{P}_\circ)_\mathsf{f}$).

Let $\mathcal{C}$ be a *finite cyclic set*, i.e. a pair (X, R) with X a finite set and R a cyclic order on X. Then we have a unique *successor function on* $\mathcal{C}$, i.e. an injection $s : X \to X$ such that $(x, y, s(x)) \notin R$ for all $x, y \in X$ and with, for all $z \in X$, $s(z) = z$ if and only if $|X| = 1$. If (Y, S) is another finite cyclic set, then a function $\varphi : X \to Y$ is called *order preserving* if $\varphi(s(x)) = s(\varphi(x))$ for all $x \in X$.

For a finite bicolored graph $\mathcal{G}$ (with colors black and white), denote the set of vertices of $\mathcal{G}$ by $\mathrm{V}\mathcal{G}$, the set of edges joined at a vertex v by $\mathrm{E}v$, and the set of all edges of $\mathcal{G}$ by $\mathrm{E}\mathcal{G}$. Now let $\mathcal{G}, \mathcal{G}'$ be finite bicolored graphs and define a *graph morphism* from $\mathcal{G}$ to $\mathcal{G}'$ as a function $\varphi : \mathrm{E}\mathcal{G} \to \mathrm{E}\mathcal{G}'$ such that for each white resp. black vertex v of $\mathcal{G}$, there is a white resp. black vertex v' of $\mathcal{G}'$ with $\mathrm{E}v' = \varphi[\mathrm{E}v]$. A *dessin* $\mathcal{D}$ is defined as a pair $(|\mathcal{D}|, \mathcal{R}\mathcal{D})$, where $|\mathcal{D}|$ is a finite bicolored graph and $\mathcal{R}\mathcal{D}$ a *cyclic structure on* $|\mathcal{D}|$, i.e. a collection $\{\mathrm{C}v \mid v \in \mathrm{V}|\mathcal{D}|\}$ such that $\mathrm{C}v$ is a cyclic order on $\mathrm{E}v$ for each vertex $v \in \mathrm{V}|\mathcal{D}|$. A *dessin morphism* is defined as a graph morphism $\varphi : |\mathcal{D}| \to |\mathcal{D}'|$ such that for each $v \in \mathrm{V}|\mathcal{D}|$ the restriction $(\mathrm{E}v, \mathrm{C}v) \xrightarrow{\varphi} (\varphi[\mathrm{E}v], \mathrm{C}\varphi(v))$ is order preserving. Write Des for the category of dessins.

Now let $\mathcal{S}$ be a finite π-set. We construct a finite bicolored graph $\mathcal{G}_\mathcal{S}$ from $\mathcal{S}$ by taking the orbits of $s \in |\mathcal{S}|$ under σ_B resp. σ_W (made disjoint by construction) as black resp. white vertices, and $|\mathcal{S}|$ as the set of edges. The π-action of $\mathcal{S}$ induces a cyclic order on $\mathrm{E}v$ for each vertex v of $\mathcal{G}_\mathcal{S}$, which gives us a dessin $\mathcal{D}_\mathcal{S}$. Because an equivariant map $\mathcal{S} \to \mathcal{S}'$ becomes a dessin morphism $\mathcal{D}_\mathcal{S} \to \mathcal{D}_{\mathcal{S}'}$, this gives the *orbit functor* from the category of finite π-sets to Des, which again is an equivalence.

Finally, a *hypermap* is a triple $(\mathcal{G}, \Sigma, g)$ with $\mathcal{G}$ a finite bicolored graph, Σ a compact oriented surface, g an embedding of the associated polyhedron $\hat{\mathcal{G}}$ of $\mathcal{G}$ into Σ such that the complement of $g[\hat{\mathcal{G}}]$ in Σ is a finite union of open sets, each homeomorphic to an open disc, and with for each connected component Σ_i of Σ a unique connected component $\hat{\mathcal{G}}_i$ of $\hat{\mathcal{G}}$ such that $g^{-1}[\Sigma_i] = \hat{\mathcal{G}}_i$. A *hypermorphism* $(\mathcal{G}, \Sigma, g) \to (\mathcal{G}', \Sigma', g')$ is a pair $(\varphi, [f])$ such that φ is a graph morphism $\mathcal{G} \to \mathcal{G}'$ and $[f]$ the equivalence class under homotopy relative to $g[\hat{\mathcal{G}}]$ of an orientation preserving, open map $f : \Sigma \to \Sigma'$ *associated to* φ, i.e. with $g' \circ \hat{\varphi} = f \circ g$ for the continuous map $\hat{\varphi} : \hat{\mathcal{G}} \to \hat{\mathcal{G}}'$ induced by φ. The resulting category is denoted by HoHyp.

For a given hypermap $\mathcal{H} = (\mathcal{G}, \Sigma, g)$ we can use the orientation on Σ to make $\mathcal{G}$ into a dessin $\mathcal{D}_\mathcal{H}$ such that for each hypermorphism $(\varphi, [f]) : \mathcal{H} \to \mathcal{H}'$, the graph morphism φ becomes a dessin morphism $\mathcal{D}_\mathcal{H} \to \mathcal{D}_{\mathcal{H}'}$. The result will be functor from HoHyp to Des (called the *cut functor*); again an equivalence of categories.

Concrete Categories

The categories outlined in the summary will all be constructed as concrete categories by means of a recipe.[1] The idea of this approach is relatively straightforward, as seen in the following example.

Let Set be the category of sets. We can construct the category Top of topological spaces by taking all pairs $(X, \mathcal{T}_X)$ such that $\mathcal{T}_X$ is a topology on the set X as Top-objects and continuous functions as Top-morphisms. Then Top is concrete over Set by means of the forgetful functor $U : \mathsf{Top} \to \mathsf{Set}$, sending $(X, \mathcal{T}_X)$ to X and a continuous function φ to φ.

In general we proceed similarly. We begin with a given base category X, endow X-objects with structure and decide which X-morphisms respect these structures. Now if the collection of structure-respecting morphisms is closed under X-composition and contains the X-identity for each X-object endowed with a structure, then we have a concrete category A over X with:

- The collection of pairs (X, S) such that X is an X-object and S a structure associated to X as A-objects (if S is an n-tuple we consider (X, S) as an $n+1$-tuple);
- For two A-objects $(X, S), (X', S')$, the set of X-morphisms $X \to X'$ respecting the structures S, S' as A-morphisms $(X, S) \to (X', S')$, and with X-composition as A-composition.

For an A-object $\mathcal{Y} = (Y, T)$, we call Y resp. T the *underlying object of* $\mathcal{Y}$ (denoted by $|\mathcal{Y}|$) resp. *the structure associated to* $\mathcal{Y}$. Furthermore, we may refer to $|\mathcal{Y}|$ by means of $\mathcal{Y}$ itself if no confusion can arise.

More Preliminaries

The constructions of the concrete categories $\mathsf{Bel}, \mathsf{Cov}(\mathbb{P}_\circ)_\mathsf{f}, \pi\text{–}\mathsf{Set}_\mathsf{f}, \mathsf{Des}$ and HoHyp as outlined in the summary will be given in Section 1. For reference sake, we may call these categories the C*-categories*. Our goal will thus be to show the following:

Equivalence Theorem. *All* C*-categories are mutually equivalent.*

In the first section we will also look at 'empty objects' in each of the C-categories in order to deal with these at the start of our proof, together with more interesting examples to illustrate the C-categories. We furthermore show auxiliary results that are mostly aimed at giving some freedom while working in the C-categories, such as the fact that for an object $\mathcal{X}$ from a C-category we can replace the underlying object $|\mathcal{X}|$ of $\mathcal{X}$ with an isomorphic copy, where isomorphism is taken in the base category. In Section 2, the functors mentioned in the summary will first be constructed in more detail, and in part 2.2 these functors will be shown to be equivalences.

For me, the motivation for this thesis is the fact that various topics that I have encountered during my Bachelor's, not closely related on first sight, come together in a nice way. Furthermore, in the literature consulted I have only found a correspondence between the connected objects of $\mathsf{Des}, \mathsf{HoHyp}$ and Bel that respects isomorphisms. This thesis aims at adding some detail, for example by bringing morphisms of hypermaps into play.[2]

I would like to thank my supervisor and other teachers of the Mathematical Institute of Leiden University for all the knowledge and motivation they have given me, and my family and friends for their support.

Some Notation and Conventions

Let us agree on some notation that will be used throughout the text.

- For $n \in \mathbb{N}_{>0}$ let ϑ_n be the primitive nth root of unity $\exp(^{2\pi i}/_n)$ and $\mu_n := \{\vartheta_n^m \mid m \in \mathbb{Z}\}$.
- Denote the unit interval $[0, 1]$ by I.
- For a function $\varphi : X \to Y$ and $A \subset X, B \subset Y$ with $\varphi(A) \subset B$, denote the restriction of φ to $A \to B$ by $\varphi| : A \to B$. If $\psi : Z \to W$ is another function, by some abuse of notation we may write $\psi \circ \varphi$ for $(\psi| : Y \cap Z \to W) \circ (\varphi| : \varphi^{-1}[Y \cap Z] \to Y \cap Z)$ if no confusion can arise.
- We write a composition of morphisms $\varphi \circ \psi$ as $\varphi\psi$.
- Let $\mathbb{P}_\circ$ be the subspace $\mathbb{C} - \{0, 1\}$ of $\mathbb{C}$.
- Top-objects resp. Top-morphisms are called *spaces* resp. *maps*. We use standard topological notions from [10] without reference, and agree that the empty space is both compact and connected.
- We denote the open unit ball in $\mathbb{R}^n$ by D^n and may consider D^2 as subset of $\mathbb{C}$.

[1] We adopt the notion of concrete categories from [1], Def. 5.1 and use the conventions presented in ibid., Rem. 5.3. The recipe we use for the construction of concrete categories is spelled out in more detail in the appendix.

[2] See [13] and §4 of [4] for the correspondence between the connected objects of $\mathsf{Des}, \mathsf{HoHyp}$ and Bel.

1 Definitions and Examples

In this section we construct the C-categories as outlined in the summary and give some results and examples for illustrative purposes and to aid the proof of the equivalence theorem. We first define the concrete categories Surf resp. Riem over Top of topological resp. Riemann surfaces, together with three equivalent constructions of the Riemann sphere and a notion of ramification of Riem-morphisms, which will be enough to subsequently define the category Bel of Belyi pairs.[3] We will then consider the categories π–$\mathsf{Set_f}$ of finite π-sets and $\mathsf{Cov}(\mathbb{P}_\circ)_\mathsf{f}$ of finite coverings over $\mathbb{P}_\circ$ briefly and Des of dessins d'enfants in some detail, to end this section with hypermaps.

A large part of the following is taken up by definitions, which are used for constructing the five C-categories, and by verifying that these definitions are sound. This is perhaps not the most fun part of our work, although the examples hopefully make up for this. Still, the reader may wonder for the reasons behind the definitions or the relations between the constructions. In this case, we can of course refer to the proof of the equivalence theorem. However, the proof that $\mathsf{Bel}, \mathsf{Cov}(\mathbb{P}_\circ)_\mathsf{f}, \pi$–$\mathsf{Set_f}$ and Des are all mutually equivalent is still rather technical: to me it almost seems to be a little magical. Fortunately, we can give a picture of the idea behind the equivalence theorem, which is more intuitive, to serve as a leitmotif. We do this in a series of informal previews throughout this section, using terminology introduced in the summary. The category HoHyp of hypermaps will make this idea more precise, and the proof that HoHyp is equivalent to Des will show that it is correct.

1.1 Surfaces, Riemann Surfaces and Belyi Pairs

Let us agree on some terminology. With a *chart* on a given space X we will always mean a pair (U, z) such that U (the *coordinate neighborhood*) is an open subset of X and z (the *coordinate function*) a homeomorphism from U to an open subset of $\mathbb{R}^2$ or, equivalently, of $\mathbb{C}$. For two charts (U, z) and (V, w) on X the composition $z \circ w^{-1}$ from $w[U \cap V]$ to $z[U \cap V]$ is called a *transition map*, which is homeomorphic by construction. An *atlas* on X is then a collection of charts whose coordinate neighborhoods cover X.[4]

Construction 1.1.1. A **topological surface** $\mathcal{M}$ (or simply *surface*) is a pair (M, Ψ) such that M is a Hausdorff space and Ψ an atlas on M.[5] We construct the concrete category Surf over Top by taking surfaces as objects and maps as morphisms.

Notation. For a surface $\mathcal{M}$, we denote the atlas on $|\mathcal{M}|$ associated to $\mathcal{M}$ by $\Phi\mathcal{M}$ and the open cover $\{U \mid (U, z) \in \Phi\mathcal{M}\}$ of $\mathcal{M}$ by $\mathcal{U}\mathcal{M}$. For a second surface $\mathcal{N}$, a function $f : |\mathcal{M}| \to |\mathcal{N}|$ and charts (U, z) resp. (V, w) from $\Phi\mathcal{M}$ resp. $\Phi\mathcal{N}$, we define ${}_z f_w$ as the function $w \circ f \circ z^{-1}$ from $z[f^{-1}[V] \cap U]$ to $w[f[U] \cap V]$.

We call a surface $\mathcal{M}$ *orientable* if translating a oriented circle around a simple closed curve on $|\mathcal{M}|$ preserves the sense of this circle. From the classification theorem of compact surfaces, it follows that a compact surface $\mathcal{M}$ is orientable if and only if each connected component of the underlying space $|\mathcal{M}|$ is homeomorphic to a finite connect sum of tori or to a sphere.[6]

Preview 1.1.2. A given compact surface $\mathcal{M}$ is orientable if and only if $\mathcal{M}$ admits an *orientation*, which we will define in paragraph 1.4. This is essentially a choice of direction around each point on $|\mathcal{M}|$, i.e. for a given circle around such a point we can traverse this circle in two direction, where the orientation determines one of these as the positive one.

Now suppose $\mathcal{M}$ is a connected, orientable surface, and furthermore suppose we have endowed $\mathcal{M}$ with an orientation. Then if we draw a finite bicolored graph $\mathcal{G}$ on $\mathcal{M}$ such that the edges of $\mathcal{G}$ do not intersect and with the complement of $\mathcal{G}$ in $\mathcal{M}$ a finite disjoint union of open discs, then the result will be a hypermap. Because $\mathcal{M}$ is oriented, we have a sense of rotation around each vertex v of $\mathcal{G}$, and thus a notion of succession on the set $\mathrm{E}v$ of edges connected to v.

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[3] See [4], in particular Def. 1.1, 1.2, 1.16, resp. ibid., Def. 4.19, [13] for some theory of topological and Riemann surfaces resp. Belyi pairs.

[4] Note this can be generalized to different dimensions, as is done in [6] for example. However, because in this thesis only the real two-dimensional and complex one-dimensional case is considered, we will not be needing this.

[5] Unless otherwise stated, for an atlas Ψ on M we assume for each chart $(U, z) \in \Psi$ that $z[U]$ is contained in $\mathbb{C}$.

[6] We use the definition of orientability as given in [2], p. 154. See ibid., §7 for the classification theorem of surfaces. Observe that a space X is connected if and only if the only clopen sets of X are the empty set and X itself. By definition, a connected component of X is a connected subspace of X not properly contained in any other connected subspace of X. It follows that X is a disjoint union of its connected components, each closed in X, and that $\varnothing$ is a connected component of X if and only if $X = \varnothing$.

The latter will be an example of a cyclic order, and taking such an order on each Ev induced by the orientation on $\mathcal{M}$, with v ranging over the vertices of $\mathcal{G}$, will give a cyclic structure on $\mathcal{G}$. If we endow $\mathcal{G}$ with this cyclic structure, we have an example of a dessins d'enfant. The equivalence theorem will imply we can make this dessin into a Belyi pair. Defining the category of Belyi pairs is our first goal.

Riemann Surfaces with Holomorphisms

The first ingredient in the definition of Belyi pairs is the concept of Riemann surfaces.

Construction 1.1.3. A **Riemann surface** is a surface $\mathcal{M}$ such that $\Phi\mathcal{M}$ is a complex structure on $|\mathcal{M}|$, i.e. an atlas with each transition map holomorphic. A map $\varphi : \mathcal{M} \to \mathcal{N}$ of Riemann surfaces is called *holomorphic* if for each pair of charts (U, z) resp. (V, w) on $\mathcal{M}$ resp. $\mathcal{N}$ the map ${}_z\varphi_w$ is holomorphic. Because the identity function on a Riemann surface is holomorphic and a composition of holomorphic maps on Riemann surfaces is again holomorphic, we can construct the concrete category Riem over Top by taking Riemann surfaces as Riem-objects and maps that are holomorphic as Riem-morphisms (called **holomorphisms**).

Remark 1.1.4. Let $\mathcal{M}_\varnothing$ be the pair $(X_\varnothing, \varnothing)$ with $X_\varnothing$ the empty space. Then $\mathcal{M}_\varnothing$ is both a Riemann surface and a surface, called the *empty surface*.

For a (Riemann) surface $\mathcal{M}$ and $A \subset \mathcal{M}$ open, $\{(U \cap A, z|_A) \mid (U, z) \in \Phi\mathcal{M}\}$ is an atlas (complex structure) on A. We denote it by $\Phi\mathcal{M}|_A$ and call it the *restriction of* $\Phi\mathcal{M}$ *to* A. Notice, without loss of generality, we may assume each coordinate neighborhood $U \in \mathcal{U}\mathcal{M}$ is connected. It follows each connected component N of $|\mathcal{M}|$ is open, and $\mathcal{N}$ defined as $(N, \Phi\mathcal{M}|_N)$ is a (Riemann) surface, called a *component of* $\mathcal{M}$. It is thus clear that:

Lemma 1.1.5. *A compact (Riemann) surface $\mathcal{M}$ has a finite number of components.* □

Preview 1.1.6. As we will see in Paragraph 1.4, the complex structure on a given compact Riemann surface $\mathcal{M}$ induces an orientation on the underlying space $|\mathcal{M}|$ of $\mathcal{M}$ in a natural way. Thus, drawing a finite bicolored graph on each connected component of $|\mathcal{M}|$ as in Preview 1.1.2 will induce a dessin.

Let us familiarize ourselves with the concept of Riemann surfaces some more.

Example 1.1.7. For $q \in \mathbb{N}_{\geq 1}$ we construct the affine Fermat curve of degree q.[7] First define

$$F_q := \{(\zeta, \xi) \in \mathbb{C}^2 \mid \zeta^q + \xi^q = 1\},$$

considered as a subspace of $\mathbb{C}^2$, so that F_q is Hausdorff. We give a complex structure on this space, making it into a Riemann surface.

Pick $\varsigma \in \mathbb{C}\backslash\mu_q$. From the Implicit Function Theorem it follows there is some $\epsilon \in \mathbb{R}_{>0}$ such that the holomorphic map $f : \mathbb{C} \to \mathbb{C}$ given by $\zeta \mapsto 1 - \zeta^q$ is injective and non-zero on $B_\varsigma := B_\epsilon(\varsigma)$. With the existence of analytic branches of logarithms we have an analytic function $H : B_\varsigma \to \mathbb{C}$ such that $H(\zeta)^q = f(\zeta)$ for all $\zeta \in B_\varsigma$.[8]

Now for $1 \leq i \leq q$ define H_i as $\vartheta_q^i H$. Notice the equation $\varsigma^q + \xi^q = 1$ has q solutions in $\mathbb{C}$. From the fact that ϑ_q is primitive it follows $H_i(\varsigma)$ for $1 \leq i \leq q$ are q distinct solutions to this equation, and therefore all possible solutions. Now define $W_{\varsigma,i}$ as $(B_\varsigma \times H_i[B_\varsigma]) \cap F_q$, and observe H_i is injective because f is, so (with the Open Mapping Theorem) $W_{\varsigma,i}$ is open in F_q and the continuous projection $w_{\varsigma,i} : W_{\varsigma,i} \to B_\varsigma$, mapping (ζ, ξ) to ζ, has a two-sided continuous inverse given by $(\zeta, H_i(\zeta)) \leftarrow\!\shortmid \zeta$.[9]

For the points $(\vartheta_q^i, 0) \in F_q$ with $1 \leq i \leq q$ we use above construction but with the first and second coordinate interchanged. This gives open sets $V_i := (H_i[B_0] \times B_0) \cap F_q$ and homeomorphisms $v_i : V_i \to B_0$, mapping (ζ, ξ) to ξ, with inverses given by $(H_i(\xi), \xi) \leftarrow\!\shortmid \xi$. Notice $\{W_{\varsigma,i} \cup V_i \mid \varsigma \in \mathbb{C}\backslash\mu_q, 1 \leq i \leq q\}$ is an open cover of F_q. Moreover, for $\varsigma, \varsigma' \in \mathbb{C}\backslash\mu_q, 1 \leq i, j \leq q$, we have

$$w_{\varsigma,i} \circ v_j^{-1} = H_j; \qquad w_{\varsigma,i} \circ w_{\varsigma',j}^{-1} = \mathrm{id}; \qquad v_j \circ w_{\varsigma,i}^{-1} = H_i; \qquad v_j \circ v_i^{-1} = \mathrm{id}\,.$$

Therefore, the collection $\Upsilon_q := \{(W_{\varsigma,i}, w_{\varsigma,i}), (V_j, v_j) \mid \varsigma \in \mathbb{C}\backslash\mu_q, 1 \leq i, j \leq q\}$ is a complex structure on F_q. We write $\mathcal{F}_q$ for the Riemann surface (F_q, Υ_q) and call it the *affine Fermat curve of degree* q.

Remark 1.1.8. Observe for two Hausdorff spaces $M, \tilde{M}$ that a complex structure Ψ on M and a homeomorphism $\varphi : M \to \tilde{M}$ induce an complex structure $\tilde{\Psi}$ on $\tilde{M}$, making φ an isomorphism of Riemann surfaces. With $\tilde{\Psi} := \{(\varphi[U], z \circ \varphi^{-1}) \mid (U, z) \in \Psi\}$, the verification is straightforward.

[7] This example is taken from [4], Exm. 1.10 and [13], but our construction differs in some details.
[8] See [3], Thm. I.5.7 for the Implicit Function Theorem and ibid., Cor. II.2.9$_1$ for analytic branches of logarithms.
[9] See ibid, Thm. III.3.3.

Lemma 1.1.9. *Let $\mathcal{M}$ be a Riemann surface. Then the following statements hold:*
(i) If Ψ is a complex structure on $|\mathcal{M}|$ with $\Phi\mathcal{M} \subset \Psi$ then $\mathcal{M} \cong (|\mathcal{M}|, \Psi)$;
(ii) There is a unique maximal complex structure $\Phi\mathcal{M}_m$ on $|\mathcal{M}|$ containing $\Phi\mathcal{M}$.

Notation. Call $\Phi\mathcal{M}_m$ *maximal with respect to* $\Phi\mathcal{M}$ and write $\mathcal{M}_m$ for $(|\mathcal{M}|, \Phi\mathcal{M}_m)$.

Proof. The first claim follows from considering the identity function id on $|\mathcal{M}|$. Then ${}_w\mathrm{id}_z$ is a transition map of Ψ for all coordinate functions w resp. z from Ψ resp. $\Phi\mathcal{M}$.

For (ii), take the union $\Phi\mathcal{M}_m$ of all complex structures Υ on $|\mathcal{M}|$ such that $\Upsilon \cup \Phi\mathcal{M}$ is a complex structure on $\mathcal{M}$. From the chain rule it follows $\Phi\mathcal{M}_m$ is a complex structure on $|\mathcal{M}|$. It is clear that $\Phi\mathcal{M}_m$ is maximal. For uniqueness, if $\Phi\mathcal{M}_{m'}$ is a maximal complex structure on $|\mathcal{M}|$ containing $\Phi\mathcal{M}$, then $\Phi\mathcal{M}_m \cup \Phi\mathcal{M}_{m'}$ is one as well, containing both $\Phi\mathcal{M}_m$ and $\Phi\mathcal{M}_{m'}$. The claim follows from the assumption that $\Phi\mathcal{M}_m$ and $\Phi\mathcal{M}_{m'}$ are both maximal. □

Remark 1.1.10. Let $\mathcal{M}$ be a Riemann surface. Then $\mathcal{M} \cong \mathcal{M}_m$, and for a chart $(U, z) \in \Phi\mathcal{M}$ and a biholomorphic function $f : W \to W'$ we have $(U \cap z^{-1}[W], f \circ z) \in \Phi\mathcal{M}_m$. Moreover, for each $x \in \mathcal{M}$ we have a chart $(V, w) \in \Phi\mathcal{M}_m$ around x such that $w(x) = 0$ and with $w[V]$ the open unit disc D^2.

The Riemann Sphere

We give three equivalent constructions of the Riemann sphere. Let in the following $\mathbb{P}^1$ be the complex projective line $\mathbb{P}^1(\mathbb{C})$ with the quotient topology and $\hat{\mathbb{C}}$ the one-point compactification of $\mathbb{C}$ with underlying set $\mathbb{C} \cup \{\infty\}$, which is Hausdorff. Although the constructions are well-known, because the Riemann sphere plays an important role in the category of Belyi pairs, we sketch some arguments for further reference.[10]

Proposition 1.1.11. *There are complex structures $\Psi_{\hat{\mathbb{C}}}, \Psi_{S^2}$ resp. $\Psi_{\mathbb{P}^1}$ on $\hat{\mathbb{C}}, S^2$ resp. $\mathbb{P}^1$ such that $(\hat{\mathbb{C}}, \Psi_{\hat{\mathbb{C}}}), (S^2, \Psi_{S^2})$ and $(\mathbb{P}^1, \Psi_{\mathbb{P}^1})$ are isomorphic Riemann surfaces.*

Proof. Let $N := (0,0,1), Z := (0,0,-1) \in S^2$, take $U_N := S^2 - \{N\}, U_Z := S^2 - \{Z\}$ and define $u_N : U_N \to \mathbb{C}$ resp. $u_Z : U_Z \to \mathbb{C}$ by sending (x,y,z) to $(x+iy)(1-z)^{-1}$ resp. $(x-iy)(1+z)^{-1}$. Then u_N, u_Z are stereographic projections, and thus homeomorphisms onto their respective images.

Moreover, for $\zeta \in \mathbb{C}\backslash\{0\}$, it turns out that $u_N \circ u_Z^{-1}(\zeta) = {}^1\!/\!_\zeta = u_Z \circ u_N^{-1}(\zeta)$. Therefore Ψ_{S^2}, defined as the set $\{(U_N, u_N), (U_Z, u_z)\}$, is a complex structure on S^2. Now if we define

$$\varphi : S^2 \to \hat{\mathbb{C}}; \quad (x,y,z) \mapsto \begin{cases} \frac{x+iy}{1-z} & : z \neq 1; \\ \infty & : z = 1; \end{cases} \quad \& \quad \psi : \hat{\mathbb{C}} \to \mathbb{P}^1; \quad \zeta \mapsto \begin{cases} (\zeta : 1) & : \zeta \neq \infty; \\ (1 : 0) & : \zeta = \infty, \end{cases}$$

then both φ and ψ are homeomorphisms. For φ this follows from the fact $\varphi| : S^2\backslash\{N\} \to \mathbb{C}$ equals u_N and the uniqueness up to homeomorphism of a one-point compactification.[11] For ψ, this can be shown by first noticing $\mathbb{P}^1$ is Hausdorff. Moreover, $\mathbb{P}^1$ equals $q(S^3)$ with $q : \mathbb{C}^2\backslash\{(0,0)\} \to \mathbb{P}^1$ the projection and $S^3 \subset \mathbb{C}^2$ the unit sphere, which shows that $\mathbb{P}^1$ is compact. So again by uniqueness of a one-point compactification, ψ is a homeomorphism.

Now we use Ψ_{S^2}, together with φ resp. $\psi\varphi$ to induce the required complex structures on $\hat{\mathbb{C}}$ resp. $\mathbb{P}^1$. For the former we get $\Psi_{\hat{\mathbb{C}}} = \{(V_N, v_N), (V_Z, v_Z)\}$, with coordinate neighborhoods $V_N = \mathbb{C}$ and $V_Z = \hat{\mathbb{C}} - \{0\}$, and coordinate functions $v_N = \mathrm{id}_{V_N}$ and $v_Z(\zeta) = {}^1\!/\!_\zeta$ for $\zeta \neq \infty$ and zero otherwise. For the latter we have $\Psi_{\mathbb{P}^1} = \{(W_N, w_n), (W_Z, w_Z)\}$, with $W_N = \{(\zeta : 1) \in \mathbb{P}^1 \mid \zeta \in \mathbb{C}\}$ and $W_Z = \{(1 : \xi) \in \mathbb{P}^1 \mid \xi \in \mathbb{C}\}$, and coordinate functions $w_N : W_N \to \mathbb{C}; (\zeta : 1) \mapsto \zeta$ and $w_Z : W_Z \to \mathbb{C}; (1 : \xi) \mapsto \xi$. □

Definition 1.1.12. Call the complex structures $\Psi_{\hat{\mathbb{C}}}, \Psi_{S^2}$ resp. $\Psi_{\mathbb{P}^1}$ on $\hat{\mathbb{C}}, S^2$ resp. $\mathbb{P}^1$ (with notation for their charts as in the proof above) *canonical* and define the *Riemann sphere* $\mathbb{P}$ as $(\hat{\mathbb{C}}, \Psi_{\hat{\mathbb{C}}})_m$.

Ramification Indices of Holomorphic Maps

The notion of ramification of holomorphisms introduced here will be a main ingredient in the definition of Belyi pairs and an important tool in considering how a holomorphism can be restricted to give a covering. In the following, let $\mathcal{M}, \mathcal{N}$ be Riemann surfaces and $\varphi : \mathcal{M} \to \mathcal{N}$ a holomorphism.

[10] See [4], Exmp. 1.19.
[11] See [10], Thm. 3.3.26 for the uniqueness up to homeomorphism of a one-point compactification.

Definition 1.1.13. For $x \in \mathcal{M}$ let (U, w) resp. (V, z) be charts around x resp. $\varphi(x) = y$. Because ${}_w\varphi_z$ is holomorphic, we can pick $\epsilon \in \mathbb{R}_{>0}$ and $\{a_n \in \mathbb{C} \mid n \in \mathbb{N}\}$ such that:

$$\forall \zeta \in B_\epsilon(w(x)) : \frac{d}{d\zeta}\, {}_w\varphi_z = \sum_{n=0}^{\infty} a_n(\zeta - w(x))^n.$$

Now define the *ramification index* of x over y as ∞ if each $a_n = 0$ and as $1 + \min\{n \in \mathbb{N} \mid a_n \neq 0\}$ otherwise, and denote it by $e_{x \mapsto y}$.[12]

Proposition 1.1.14. *For each $x \in \mathcal{M}$ and $\varphi(x) = y \in \mathcal{N}$, the ramification index of x over y is:*
(i) Well-defined for each choice of charts around x and y;
(ii) Independent of the choice of charts around x and y.

Proof. (i) follows from the facts that holomorphic functions have holomorphic derivatives of every order and that a holomorphic function on an open set $W \subset \mathbb{C}$ has a unique power series expansion around every $\xi \in W$. For the second point one uses that the transition maps are biholomorphic.[13] □

Remark 1.1.15. Notice for isomorphisms $\psi : \mathcal{M}' \to \mathcal{M}, \chi : \mathcal{N} \to \mathcal{N}'$ of Riemann surfaces, for all $x \in \mathcal{M}'$ we have $e_{x \mapsto \chi\varphi\psi(x)} = e_{\psi(x) \mapsto \varphi\psi(x)}$. Therefore, with Lemma 1.1.9, in calculating ramification indices we may assume without loss of generality that complex structures are maximal.

Corollary 1.1.16. *Let $x \in \mathcal{M}$ and $\varphi(x) = y \in \mathcal{N}$. Then $e_{x \mapsto y} = k \in \mathbb{N}$ if and only if there are charts (U, w) resp. (V, z) around x resp. y such that ${}_w\varphi_z(\zeta) = \zeta^k$.*

Proof. Pick charts (U, w) resp. (V, z) around x resp. y such that $w(x) = 0 = z(y)$. Then $e_{x \mapsto y} = k$ if and only if there is an $\epsilon \in \mathbb{R}_{>0}$ and a biholomorphic map $h : B_\epsilon(0) \to W$ for some open set W of $\mathbb{C}$ such that ${}_w\varphi_z(\zeta) = h(\zeta)^k$ for each $\zeta \in B_\epsilon(0)$.[14] The claim now follows with Remark 1.1.10. □

Definition 1.1.17. Let $\varphi : \mathcal{M} \to \mathcal{N}$ be holomorphic and $y \in \mathcal{N}$. If there is some $x \in \varphi^{-1}(y)$ with $e_{x \mapsto y} > 1$, then x resp. y is called a *ramification point* resp. *branch point* of φ and φ is called *ramified at* x resp. *branched at* y. For $A \subset \mathcal{N}$ such that $\mathcal{N} - A$ contains no branch points, φ is called *unramified outside* A.

Belyi Pairs with Belyi Morphisms

The category Riem, the Riemann sphere and the notion of ramification indices are sufficient for the construction of Belyi pairs.

Construction 1.1.18. For a Riemann surface $\mathcal{M}$, define a *meromorphic map on* $\mathcal{M}$ as a holomorphism $\mathcal{M} \to \mathbb{P}$ and a *Belyi map on* $\mathcal{M}$ as a meromorphic map on $\mathcal{M}$ unramified outside $\{0, 1, \infty\}$ and non-constant on each component of $\mathcal{M}$. Moreover:

- A **Belyi pair** $\mathcal{B}$ is a pair $(\mathcal{M}, f)$, where $\mathcal{M}$ is a compact Riemann surface endowed with a maximal complex structure and f a Belyi map on $\mathcal{M}$;
- For Belyi pairs $(\mathcal{M}, f), (\mathcal{M}', f')$, a holomorphism $\varphi : \mathcal{M} \to \mathcal{M}'$ such that $f'\varphi = f$ is called a **Belyi morphism**;

Notice the identity function on a Belyi pair is a Belyi morphism and that a composition of Belyi morphisms is again a Belyi morphism. We therefore have a concrete category Bel over Riem, with Belyi pairs as objects and Belyi morphisms as morphisms.

Preview 1.1.19. For a Belyi pair $\mathcal{B} = (\mathcal{M}, f)$ we can draw a bicolored graph $\mathcal{G}$ on $\mathcal{M}$ with the black resp. white vertices equal to $f^{-1}(0)$ resp. $f^{-1}(1)$ and with each edge sent to I under f. The result will be a hypermap $\mathcal{H}_\mathcal{B}$ and will thus induce a dessin $\mathcal{D}_\mathcal{B}$ associated to $\mathcal{B}$.

Conversely, for a given dessin $\mathcal{D}$, we can glue open discs to cycles in $\mathcal{D}$ in such a way that the result will be a compact topological surface that admits an orientation such that the notion of succession of edges connected to a given vertex induced by this orientation is the same as the one coming from the cyclic structure on $\mathcal{D}$. This again gives a hypermap $\mathcal{H}$. The compact oriented surface of $\mathcal{H}$ can even be endowed with a complex structure together with a Belyi map f, inducing a Belyi pair $\mathcal{B}$ associated to $\mathcal{D}$.

The proof of the equivalence theorem will show that these two construction are, up to isomorphism, inverse to each other.

[12] Note that if $w(x) = 0 = z(y)$ and ${}_w\varphi_z(\zeta) = \sum_{m=0}^{\infty} b_m(\zeta - w(x))^m$ for ζ around 0, then $e_{x \mapsto y}$ equals $\min\{m \in \mathbb{N} \mid b_m \neq 0\}$ if some $b_m \neq 0$ and ∞ otherwise.

[13] See [3], Thm. III.2.2 and [4], Def. 1.30, 1.31.

[14] See [3], Cor. II.2.9$_1$ and Thm. III.3.3.

Remark 1.1.20. The pair $(\mathcal{M}_\varnothing, \varnothing)$ with $\mathcal{M}_\varnothing$ the empty surface is a Belyi pair, called the *empty Belyi pair* and denoted by $\mathcal{B}_\varnothing$. Now let $\mathcal{N}, \mathcal{M}, \mathcal{O}$ be Riemann surfaces with $\mathcal{M}$ compact, $\psi : \mathcal{N} \to \mathcal{M}, \varphi : \mathcal{O} \to \mathbb{P}$ isomorphisms and $f : \mathcal{M} \to \mathcal{O}$ a holomorphism such that $g := \varphi f \psi$ is unramified outside $g^{-1}(\{0, 1, \infty\})$ and non-constant on each component of $\mathcal{N}$. Then $\psi : (\mathcal{N}_m, g) \to (\mathcal{M}_m, \varphi f)$ is an isomorphism of Belyi pairs such that $(\mathcal{N}_m, g)$ is the empty Belyi pair if and only if $(\mathcal{M}_m, \varphi f)$ is. This follows from Remark 1.1.15 and Lemma 1.1.9, together with the fact that isomorphisms of Riemann surfaces are homeomorphisms on the underlying spaces.

Example 1.1.21. Let $q \in \mathbb{N}_{\geq 1}$. We construct a Belyi pair with as underlying compact Riemann surface a projective version of the affine Fermat curve $\mathcal{F}_q = (F_q, \Upsilon_q)$ of Example 1.1.7. Let $\mathbb{P}^2$ be the projective plane $\mathbb{P}(\mathbb{C}^3)$ with the quotient topology and consider the following as subspace of $\mathbb{P}^2$:

$$\mathbb{P}F_q := \{(\xi : \eta : \zeta) \in \mathbb{P}^2 \mid \xi^q + \eta^q = \zeta^q\},$$

which is well-defined because $\xi^q + \eta^q = \zeta^q$ is homogeneous. Moreover, it can be shown that $\mathbb{P}F_q$ is Hausdorff and compact, using that $\mathbb{P}^2$ has these properties. The latter follows from similar arguments as given in the construction of the Riemann sphere, the former from the fact that $\mathbb{P}F_q$ is closed as subset of $\mathbb{P}^2$. Now let $\iota_q = \exp(\pi i/q)$, which is a qth root of -1, and consider the functions:

$$g : F_q \to \mathbb{P}F_q; \quad (\xi, \eta) \mapsto (\xi : \eta : 1); \qquad \& \qquad h : F_q \to \mathbb{P}F_q; \quad (\xi, \eta) \mapsto (\iota_q : \xi : \iota_q \eta).$$

With the universal property of the quotient space it follows that g and h are embeddings. Because $\mathbb{P}F_q = g[F_q] \cup h[F_q]$, we can thus use g and h to transport the complex structure Υ_q of $\mathcal{F}_q$ to construct the following complex structure on $\mathbb{P}F_q$ (with notation as in Example 1.1.7):

$$\begin{aligned}\Psi_q := &\{(g[W_{\varsigma,i}], w_{\varsigma,i} \circ g^{-1}), (h[W_{\varsigma,i}], w_{\varsigma,i} \circ h^{-1}) \mid \varsigma \in \mathbb{C} \backslash \mu_q, 1 \leq i \leq q\} \\ &\cup \{(g[V_j], v_j \circ g^{-1}), (h[V_j], v_j \circ h^{-1}) \mid 1 \leq j \leq q\}.\end{aligned}$$

It is clear that all transition maps are holomorphic. We denote the compact Riemann surface $(\mathbb{P}F_q, \Psi_q)_m$ by $\mathbb{P}\mathcal{F}_q$ and call it the *projective Fermat curve of degree* q. Now consider the map

$$f_q : \mathbb{P}F_q \to \mathbb{P}; \quad (\xi : \eta : \zeta) \mapsto \begin{cases} \left(\frac{\xi}{\zeta}\right)^q & \text{if } \zeta \neq 0; \\ \infty & \text{if } \zeta = 0. \end{cases}$$

We show that f_q is a Belyi map on $\mathbb{P}\mathcal{F}_q$. First notice f_q is non-constant on each component of $\mathbb{P}\mathcal{F}_q$ (of which there is only one). It is clear f_q is holomorphic, so what remains are the ramification indices. The fibers above $\{0, 1, \infty\}$ work out nicely as follows:

$$f_q^{-1}(0) = \mathbb{P}F_q - h[F_q]; \qquad f_q^{-1}(1) = \{(\xi : \eta : \zeta) \in \mathbb{P}F_q \mid \eta = 0\}; \qquad f_q^{-1}(\infty) = \mathbb{P}F_q - g[F_q].$$

Let $p := (\xi : \eta : \zeta) \in \mathbb{P}F_q$ and suppose $f_q(p) \notin \{0, 1, \infty\}$. Then $\xi, \eta, \zeta \neq 0$ so $g\,(\xi/\zeta, \eta/\zeta) = p$ and we can take $(W_{\varsigma,i}, w_{\varsigma,i}) \in \Upsilon_q$ such that $(\xi/\zeta, \eta/\zeta) \in W_{\varsigma,i}$. This gives some $\{a_n \in \mathbb{C} \mid n \in \mathbb{N}\}$ such that for $\nu \in B_\varsigma$ the following holds:

$$\frac{d}{d\nu} f_q \circ g \circ w_{\varsigma,i}^{-1}(\nu) = q\nu^{q-1} = \sum_{n=0}^{\infty} a_n \left(\nu - \xi/\zeta\right)^n.$$

Thus, $a_0 = q\,(\xi/\zeta)^{q-1}$, which is non-zero because $\xi \neq 0$, so $e_{p \mapsto f_q(p)} = 1$. Therefore, f_q is unramified outside $\{0, 1, \infty\}$ and the pair $\mathcal{B}_q := (\mathbb{P}\mathcal{F}_q, f_q)$ is a Belyi pair. As an illustration, we compute the ramification indices above $0, 1, \infty$.

If $f_q(p) = 0$, then $p = g\,(0, \eta/\zeta)$, so we have a chart $(g[W_{\varsigma,i}], w_{\varsigma,i} \circ g^{-1}) \in \Psi_q$ around p. Because $w_{\varsigma,i} \circ g^{-1}(p) = 0$ and $f_q \circ g \circ w_{\varsigma,i}^{-1}(\nu) = \nu^q$ for $\nu \in B_\varsigma$, we have $e_{p \mapsto 0} = q$. If $f_q(p) = 1$ we have $p = g\,(\xi/\zeta, 0)$, which gives us a chart $(g[V_j], v_j \circ g^{-1})$ around p, with $v_j \circ g^{-1}(p) = 0$. If we let $t : \mathbb{C} \to \mathbb{C}$ be the translation $\nu \mapsto \nu - 1$, then $t \circ f_q \circ g \circ v_j^{-1}(\nu) = -\nu^q$, so again $e_{p \mapsto 1} = q$.

For $f_q(p) = \infty$ we have $p = h(\xi, 0)$ and thus a chart $(h[V_i], v_i \circ h^{-1})$ around p and the chart (V_Z, v_Z) around $f_q(p)$ from the proof of Proposition 1.1.11. The composition $v_Z \circ f_q \circ h \circ v_i^{-1}$ gives the map $\nu \mapsto (H_i(\nu), \nu) \mapsto (\iota_q, H_i(\nu), \iota_q \nu) \mapsto \nu^{-q} \mapsto \nu^q$ for $\nu \neq 0$ and $0 \mapsto (\vartheta_q^i, 0) \mapsto (\iota_q, \vartheta_q^i, 0) \mapsto \infty \mapsto 0$, showing that $v_Z \circ f_q \circ h \circ v_i^{-1}(\nu) = \nu^q$ and thus $e_{p \mapsto \infty} = q$ as well.

1.2 Coverings over $\mathbb{P}_\circ$ and π-Sets

In the following paragraph we review some aspects of covering spaces and group actions.

Finite Coverings over $\mathbb{P}_\circ$ with Covering Morphisms

Construction 1.2.1. For a given space Y, denote the category of coverings over Y by $\mathsf{Cov}(Y)$. We call a covering $p : X \to Y$, written as $\mathcal{X} = (X, p)$, **finite** if p has finite fibers above each point $y \in Y$.

Let $\mathsf{Cov}(Y)_\mathsf{f}$ over Top be the full subcategory of finite coverings over Y, and call $\mathsf{Cov}(Y)_\mathsf{f}$-morphisms **covering morphisms**.

Preview 1.2.2. To show that the categories of Belyi pairs and of dessins are equivalent to each other, we will first show that Bel is equivalent to $\mathsf{Cov}(\mathbb{P}_\circ)_\mathsf{f}$. This has the advantage that a given Belyi pair $(\mathcal{M}, f)$ is already 'almost' a finite covering over $\mathbb{P}_\circ$, because we only need to remove the points above $\{0, 1, \infty\}$ and forget the complex structure on $\mathcal{M}$ to induce the desired covering. Moreover, we know that the category of finite coverings over $\mathbb{P}_\circ$ is equivalent to the category of finite sets endowed with group action of $\pi_1(\mathbb{P}_\circ, 1/2)$. It is this latter category that will be shown to be equivalent to the category of dessins, which will turn out to be relatively straightforward.

Remark 1.2.3. The pair $\mathcal{X}_\varnothing := (X_\varnothing, \varnothing)$ with $X_\varnothing$ the empty space is a finite covering over $\mathbb{P}_\circ$, called the *empty covering*. Now let X, Y, O be spaces with homeomorphisms $\varphi : Y \to X$ and $\psi : O \to \mathbb{P}_\circ$ and let $p : X \to O$ be a finite covering. Then $(Y, \psi p \varphi)$ and $(X, \psi p)$ are finite coverings over $\mathbb{P}_\circ$, making φ into an $\mathsf{Cov}(\mathbb{P}_\circ)_\mathsf{f}$-isomorphism. Of course, $(Y, \psi p \varphi) = \mathcal{X}_\varnothing$ if and only if $(X, \psi p) = \mathcal{X}_\varnothing$.

The following lemma will be convenient in the proof of the equivalence theorem.

Lemma 1.2.4. *Let $\mathcal{X} = (X, p)$ be a finite covering over $\mathbb{P}_\circ$ and for all $\eta \in \mathbb{P}_\circ$ write I_η for the subspace $p^{-1}(\eta)$ of X. Then the space I_η is discrete for all $\eta \in \mathbb{P}_\circ$, and moreover:*
(i) For all $\zeta, \xi \in \mathbb{P}_\circ$, I_ζ and I_ξ are homeomorphic;
(ii) If $\mathcal{X}$ is not the empty covering, p must be surjective.

Proof. The first claim follows immediately from the definition of a covering space. For (i), one uses the fact that for each $\zeta \in \mathbb{P}_\circ$ there is some open neighborhood $V_\zeta \subset \mathbb{P}_\circ$ of ζ such that $p| : p^{-1}[V_\zeta] \to V_\zeta$ is isomorphic as covering space over V_ζ to the projection $V_\zeta \times I_\zeta \to V_\zeta$ of the first coordinate. Then $I_\zeta \cong I_{\zeta'}$ as spaces for all $\zeta' \in V_\zeta$, and because $\{V_\xi \mid \xi \in \mathbb{P}_\circ\}$ is an open cover for the connected space $\mathbb{P}_\circ$, the claim follows.[15] Now (ii) follows from (i): if $\mathcal{X} \neq \mathcal{X}_\varnothing$, then there must be some $\zeta \in \mathbb{P}_\circ$ with $I_\zeta \neq \varnothing$, and therefore all fibers of p must be non-empty. □

Example 1.2.5. Let $q \in \mathbb{N}_{\geq 1}$ and take the notation as in the previous examples. Define X_q as the subspace $F_q - \{(\xi, \eta) \in F_q \mid \xi \in \mu_q \cup \{0\}\}$, which equals $\{(\xi, H_i(\xi)) \mid \xi \notin \mu_q \cup \{0\}, 1 \leq i \leq q\}$, where the H_i's are again the analytic qth roots around ξ of the map $f : \mathbb{C}\backslash\mu_q \to \mathbb{C}$ with $f(\zeta) = 1 - \zeta^q$. Consider the function $p_q : X_q \to \mathbb{P}_\circ$ given by $(\xi, \eta) \mapsto \xi^q$. We show that the pair $\mathcal{X}_q$ defined as (X_q, p_q) is a finite covering over $\mathbb{P}_\circ$.

Pick $\zeta \in \mathbb{P}_\circ$. Then $p_q^{-1}(\zeta) = \{(\xi, H_i(\xi)) \mid \xi^q = \zeta, 1 \leq i \leq q\}$, so the fibers of p_q are finite. For $(\xi, H_i(\xi)) \in p_q^{-1}(\zeta)$ and $1 \leq i \leq q$, let $W_{\xi,i}$ be the open subset $(B_\xi, H_i[B_\xi]) \cap F_q$ of F_q. From the fact that f is injective on each $B_{\vartheta_q^i \xi}$ and because we may assume $B_{\vartheta_q^i \xi} = \vartheta_q^i B_\xi$ without loss of generality, it follows that $W_{\xi,i} \cap W_{\xi,j} = \varnothing$ for distinct $1 \leq i, j \leq q$. Now notice that the restriction of p_q to $W_{\xi,i} \to p_q[W_{\xi,i}]$ is the composition

$$W_{\xi,i} \to B_\xi \to f[B_\xi] \to p_q[W_{\xi,i}]; \quad (\zeta, H_i(\zeta)) \mapsto \zeta \mapsto 1 - \zeta^q \mapsto \zeta^q,$$

which are all homeomorphic by construction. Thus, $\mathcal{X}_q$ is indeed a finite covering over $\mathbb{P}_\circ$.

Finite π-Sets with Equivariant Maps

Construction 1.2.6. Let G be a group. Call a set endowed with left group action of G a **G-set**. Observe the identity function on a G-set and the composition of equivariant maps between G-sets are both equivariant.[16] We therefore have a concrete category G–Set over Set with G-sets as objects and equivariant maps as morphisms. Let G–Set_f be the full subcategory of G–Set such that all G–Set_f-objects have finite underlying sets. From hereon, all group actions are taken to be left group actions.

Definition 1.2.7. Let σ_B, σ_W be the equivalence classes under path-homotopy of counter-clockwise parametrizations of the circles $\partial B_{1/2}(0)$ and $\partial B_{1/2}(1)$ respectively, both starting at $1/2$.

Proposition 1.2.8. *The fundamental group $\pi_1(\mathbb{P}_\circ, 1/2)$ is a free group generated by σ_B, σ_W.*

Proof. Let $H_l := \{\zeta \in \mathbb{P}_\circ \mid \Re(\zeta) < 3/4\}$ and $H_r := \{\zeta \in \mathbb{P}_\circ \mid \Re(\zeta) > 1/4\}$. Then both H_l and H_r are homotopically equivalent to S^1 and $H_l \cap H_r$ to $\{1/2\}$. In this case, the van Kampen Theorem gives $\pi_1(\mathbb{P}_\circ, 1/2) \cong \langle \sigma_B \rangle / 0 * \langle \sigma_W \rangle / 0 \cong \mathbb{Z} * \mathbb{Z}$, with 0 the trivial group.[17] □

[15] See [12], Prop. 2.1.3 and Cor. 2.1.4.
[16] See [7], §5 and p. 55, where an equivariant map is called a G-map.
[17] See [5], Thm. 1.20.

Notation. Let π be the fundamental group $\pi_1(\mathbb{P}_\circ, 1/2)$, generated by σ_B, σ_W. We write a finite π-set $\mathcal{S}$ as a pair $(|\mathcal{S}|, \rho)$, where $|\mathcal{S}|$ is the underlying set and ρ the π-action on $|\mathcal{S}|$. For $\sigma_X \in \{\sigma_B, \sigma_W\}$ and $s \in |\mathcal{S}|$, we define the *orbit of s under σ_X* as $\{\sigma_X^n s \mid n \in \mathbb{Z}\}$ and denote it by $\langle\sigma_X\rangle s$. Notice $\langle\sigma_X\rangle s$ has a natural group action of $\mathbb{Z}$, given by $ns := \sigma_X^n s$ for all $n \in \mathbb{Z}$.

Remark 1.2.9. The empty set has unique group action of π. Denote the resulting finite π-set by $\mathcal{S}_\varnothing$. Now let S, T be finite sets, ρ a π-action on T and $\varphi : S \to T$ a bijection. Then the composition $\varphi^{-1} \circ \rho \circ (\mathrm{id}, \varphi) : \pi \times S \to S$ is a π-action on S, making φ a π–$\mathsf{Set_f}$-isomorphism. Observe $(T, \rho) = \mathcal{S}_\varnothing$ if and only if $(S, \varphi^{-1} \circ \rho \circ (\mathrm{id}, \varphi)) = \mathcal{S}_\varnothing$.

Example 1.2.10. For $q \in \mathbb{N}_{\geq 1}$, define S_q as the set $(\mathbb{Z}/q\mathbb{Z})^2$ and a group action ρ_q on S_q by setting $\sigma_B(a,b) = (a+1,b)$ and $\sigma_W(a,b) = (a,b+1)$. Denote the resulting finite π-set (S_q, ρ_q) by $\mathcal{S}_q$.

Next consider some $\varphi \in \mathrm{Aut}(\mathcal{S}_q)$. Because φ is equivariant, we have $\varphi(a,b) = \varphi(0,0) + (a,b)$ for all $(a,b) \in S_q$. Conversely, for each $(x,y) \in S_q$, the function $\varphi_{x,y} : S_q \to S_q$, defined by sending (a,b) to $(x+a, y+b)$, is equivariant and bijective. Therefore, $\mathrm{Aut}(\mathcal{S}_q) = \{\varphi_{x,y} : S_q \to S_q \mid (x,y) \in S_q\}$.

Now for $\varphi_{x,y} \in \mathrm{Aut}(\mathcal{S}_q)$, set $\sigma_B\varphi_{x,y} = \varphi_{x+1,y}$ and $\sigma_W\varphi_{x,y} = \varphi_{x,y+1}$. This induces a group action τ_q on $\mathrm{Aut}(\mathcal{S}_q)$ and thus a finite π-set $\mathcal{A}_q := (\mathrm{Aut}(\mathcal{S}_q), \tau_q)$. If we define $\psi : \mathcal{S}_q \to \mathcal{A}_q$ by $(x,y) \mapsto \varphi_{x,y}$, we see that $\mathcal{S}_q$ and $\mathcal{A}_q$ are even isomorphic as finite π-sets.

1.3 Dessins d'Enfants

The construction of the category of dessins is carried out in three stages. We first give the category of finite cyclic sets with functions that preserve cyclic orders. Then we construct the category of bicolored graphs and graph morphisms. Both constructions come together in the definition of dessins and their morphisms. In this way, the definitions hopefully remain insightful in each subsequent stage.

Finite Cyclic Sets with Order Preserving Functions

As mentioned in Preview 1.1.19, a Belyi pair $(\mathcal{M}, f)$ induces a graph $\mathcal{G}$ on $\mathcal{M}$ together with a sense of succession on the set of edges connected to a given vertex of $\mathcal{G}$. To get some idea how this sense of succession can be formalized, let us first consider how elements of μ_n succeed each other while traversing the unit circle counter-clockwise.

If we pick $m \in \mathbb{Z}$ and define the binary relation $\lessdot_m := \{(\vartheta_n^{m+k}, \vartheta_n^{m+l}) \mid 0 \leq k < l < n\}$ on μ_n, then $\lessdot_m$ is a linear order on μ_n with minimal element ϑ_n^m. This linear order depends however on the choice of m, which is not very nice. We can fix this with the introduction of a cyclic order, which is essentially 'forgetting the minimal element' by only considering how the entries of a given triple from μ_n^3 are mutually related under $\lessdot_m$. Alternatively, we can endow μ_n with an obvious transitive $\mathbb{Z}$-action. Both approaches will be shown to be equivalent.

Definition 1.3.1. For a ternary relation R on a set X, write $R(x,y,z)$ if $(x,y,z) \in R$, and call R:
- *Orbital* if $R(x,y,z)$ implies $R(y,z,x)$ for all $x,y,z \in X$;
- *Asymmetric* if $R(x,y,z)$ implies $\neg R(z,y,x)$ for all $x,y,z \in X$;
- *Transitive* if $R(x,y,z) \wedge R(x,z,w)$ implies $R(x,y,w)$ for all $x,y,z,w \in X$;
- *Total* if $|\{x,y,z\}| = 3$ implies $R(x,y,z) \vee R(z,y,x)$ for all $x,y,z \in X$.

Now R is called a *cyclic order* if it is orbital, asymmetric, transitive and total. Define a **finite cyclic set** $\mathcal{C}$ as a pair (X,R), with X a finite set and R a cyclic order on X.

Remark 1.3.2. Notice for a finite cyclic set (X,R), the relation $R^* := \{(x,y,z) \in X^3 \mid R(z,y,x)\}$ is a cyclic order on X such that $R^{**} = R$. Also notice $R(x,y,z)$ implies that x,y,z are distinct. Therefore, if $|X| \leq 2$, the only cyclic order on X is $R_\varnothing := \varnothing$.

Example 1.3.3. Let $n \in \mathbb{N}_{>0}$ and $R_n := \{(x, x+a, x+b) \in (\mathbb{Z}/n\mathbb{Z})^3 \mid x \in \mathbb{Z}, 1 \leq a < b < n\}$. Then $\mathcal{Z}_n$ defined as $(\mathbb{Z}/n\mathbb{Z}, R_n)$ is a finite cyclic set. As another example, define the ternary relation T_n on μ_n as $\{(\vartheta_n^k, \vartheta_n^l, \vartheta_n^m) \mid R_n\,(k+n\mathbb{Z}, l+n\mathbb{Z}, m+n\mathbb{Z})\}$. Then $\mathcal{C}_n := (\mu_n, T_n)$ is a finite cyclic set. Observe: for each $1 \leq j \leq n$, we have $T_n(x,y,z)$ if and only if either $x \lessdot_j y \lessdot_j z$, $y \lessdot_j z \lessdot_j x$ or $z \lessdot_j x \lessdot_j y$.

We will now define morphisms of finite cyclic sets. These will be used in the construction of the category Des of dessins d'enfants.

Definition 1.3.4. For a finite cyclic set (X,R), a *successor function of* (X,R) is an injection $s : X \to X$ with $\neg R(x,y,s(x))$ for all $x,y \in X$, and for all $z \in X$, $s(z) = z$ if and only if $|X| = 1$.

Example 1.3.5. For $n \in \mathbb{N}_{>0}$, $s : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ given by $x \mapsto x+1$ is a successor function for $\mathcal{Z}_n$.

Lemma 1.3.6. *For each finite cyclic set (X, R), there is a unique successor function s of (X, R).*

Proof. If $|X| < 3$ the statement is clear. So assume $|X| = n \geq 3$. Pick $x_0 \in X$ and define:

$$\prec := \{(y, z) \in X^2 \mid R(x_0, y, z)\} \cup \{(x_0, w) \in X^2 \mid w \neq x_0\}.$$

It is clear $\prec$ is a linear order on X with x_0 as minimal element. Thus we can index the elements of $X - \{x_0\}$ as $x_1, ..., x_{n-1}$ in such a way that $x_0 \prec x_1 \prec ... \prec x_{n-1}$. Now define the function $s : X \to X$ by sending x_i to x_{i+1} if $i < n-1$ and to x_0 otherwise. It is straightforward to show s is a successor function. For uniqueness, if s' is a successor function of (X, R) with $s \neq s'$, then take $m \in \mathbb{N}$ minimal such that $s(x_m) \neq s'(x_m)$. This implies $R(x_m, s(x_m), s'(x_m))$, contradicting the assumption on s'. □

Notation. For a finite cyclic set (X, R), a point $x \in X$ and $n \in \mathbb{N}$, the image of x after n times applying the unique successor function of (X, R) is denoted by $s^n x$. If $n = 1$ we simply write sx.

Definition 1.3.7. For two finite cyclic sets $(X, R), (Y, S)$, a function $\varphi : X \to Y$ with $\varphi(sx) = s\varphi(x)$ for all $x \in X$ is called **order preserving**.

Remark 1.3.8. Because the identity function on a finite cyclic set is order preserving and a composition of order preserving functions is again order preserving, we have a category $\mathsf{Cyc_f}$ with finite cyclic sets as objects and order preserving functions as morphisms.

Transitive $\mathbb{Z}$-Sets with Equivariant Maps

For a given finite cyclic set (X, R) with $|X| = n > 0$, we have n possible isomorphisms to $\mathcal{Z}_n$. Now let $m, k \in \mathbb{N}_{>0}$. If $k \nmid m$, then $\mathrm{Hom}(\mathcal{Z}_m, \mathcal{Z}_k) = \varnothing$. If k does divide m, then with $\varphi_l : \mathcal{Z}_m \to \mathcal{Z}_k$ defined by sending $i + m\mathbb{Z}$ to $i + l + k\mathbb{Z}$ for $l \in \mathbb{Z}$, we have $\mathrm{Hom}(\mathcal{Z}_m, \mathcal{Z}_k) = \{\varphi_l \mid l \in \mathbb{Z}\}$. Notice that although the objects and morphisms of $\mathsf{Cyc_f}$ are thus intuitive, the precise definitions are somewhat extensive. We therefore construct a category that is isomorphic to $\mathsf{Cyc_f}$, but more tractable.

Definition 1.3.9. Define a **finite transitive $\mathbb{Z}$-set** $\mathcal{K}$ as a pair (X, ρ) where X is a finite set and ρ is a transitive group action of $\mathbb{Z}$ on X. Denote the category with finite transitive $\mathbb{Z}$-sets as objects and equivariant maps as morphisms by $\mathsf{Trans_f}$.

Proposition 1.3.10. *The categories $\mathsf{Cyc_f}$ and $\mathsf{Trans_f}$ are isomorphic.*[18]

Proof. Let (X, R) be a finite cyclic set. Define a $\mathbb{Z}$-action ρ_R on X by setting $nx := s^n x$ for each $n \in \mathbb{Z}$. Then (X, ρ_R) is a finite transitive $\mathbb{Z}$-set. For another finite cyclic set (Y, S) and an order preserving function $\varphi : (X, R) \to (Y, S)$, we have $\varphi(mx) = \varphi(s^m x) = s^m \varphi(x) = m\varphi(x)$ for all $m \in \mathbb{Z}$. We therefore have a functor G from finite cyclic sets to finite transitive $\mathbb{Z}$-sets, sending (X, R) to (X, ρ_R) and with $G\varphi = \varphi$.

Conversely, for a finite transitive $\mathbb{Z}$-set (Y, τ) with $|Y| = k$, define the ternary relation R_τ on Y as $\{(x, px, qx) \in Y^3 \mid 0 < p < q < k\}$. Then (Y, R_τ) is a finite cyclic set. If (Z, σ) is another finite transitive $\mathbb{Z}$-set and $\psi : (Y, \tau) \to (Z, \sigma)$ is equivariant, then $\psi(sy) = \psi(1y) = 1\psi(y) = s\psi(y)$. We therefore have a functor H from finite transitive $\mathbb{Z}$-sets to finite cyclic sets, sending (Y, τ) to (Y, R_τ) and with $H\psi = \psi$ for morphisms ψ of $\mathsf{Trans_f}$.

It is clear that $H \circ G$ resp. $G \circ H$ are the identity functors on $\mathsf{Cyc_f}$ resp. $\mathsf{Trans_f}$, so the statement follows. □

Notation. Write $(X, \rho_R) := G(X, R)$ and $(Y, R_\tau) := H(Y, \tau)$.

Remark 1.3.11. Suppose $\mathcal{S}$ is a finite π-set, let $s, t \in |\mathcal{S}|$ and pick $\sigma_X \in \{\sigma_B, \sigma_W\}$. Furthermore, let ρ resp. τ be the natural group actions of $\mathbb{Z}$ on $\langle\sigma_X\rangle s$ resp. $\langle\sigma_X\rangle t$, which are transitive. Then if there is some $n \in \mathbb{Z}$ such that $s = \sigma_X^n t$, then $(\langle\sigma_X\rangle s, R_\rho) = (\langle\sigma_X\rangle t, R_\tau)$. Note that R_ρ induces a cyclic order R'_ρ on $\langle\sigma_X\rangle s \times \{*\}$ in an obvious way, where $*$ is a formal point outside $\langle\sigma_X\rangle s$. We call R_ρ resp. R'_ρ the *natural cyclic orders* on $\langle\sigma_X\rangle s$ resp. $\langle\sigma_X\rangle s \times \{*\}$.[19]

[18] See [1], Def. 3.24 for the definition of isomorphic categories.

[19] The formal point will be used in the construction of the orbit functor from $\pi\text{–}\mathsf{Set_f}$ to the category of dessins (to make the set of σ_B-orbits disjoint from the set of σ_W-orbits).

Bicolored Graphs with Graph Morphisms

For the second stage in constructing the category of dessins we need the category of bicolored graphs and graph morphisms. Note the former will be concrete over the latter, meaning that dessins will be bicolored graphs with additional structure.

Construction 1.3.12. A **bicolored graph** $\mathcal{G}$ is a quintuple (E, B, W, b, w), where E, B, W are disjoint sets and $b : E \to B, w : E \to W$ surjective functions. Elements of E are called the *edges of* $\mathcal{G}$, while B resp. W are the sets of (*black* resp. *white*) *vertices* of $\mathcal{G}$ and b resp. w the (*black* resp. *white*) *colorings* of $\mathcal{G}$.

For a given finite bicolored graph $\mathcal{G}$, we denote the set of edges of $\mathcal{G}$ by $\mathrm{E}\mathcal{G}$, the set of black resp. white vertices of $\mathcal{G}$ by $\mathrm{B}\mathcal{G}$ resp. $\mathrm{W}\mathcal{G}$ and the black resp. white colorings of $\mathcal{G}$ by $\mathrm{b}\mathcal{G}$ resp. $\mathrm{w}\mathcal{G}$. Furthermore, we write $\mathrm{V}\mathcal{G}$ for $\mathrm{B}\mathcal{G} \cup \mathrm{W}\mathcal{G}$, and for a black resp. white vertex $v \in \mathrm{V}\mathcal{G}$, define the set $\mathrm{E}v$ of *edges connected to* v as $\mathrm{b}\mathcal{G}^{-1}(v)$ resp. $\mathrm{w}\mathcal{G}^{-1}(v)$.

For two bicolored graphs $\mathcal{G}, \mathcal{G}'$, define a **graph morphism** as a function $\varphi : \mathrm{E}\mathcal{G} \to \mathrm{E}\mathcal{G}'$ such that for each white resp. black vertex v of $\mathcal{G}$, there is white resp. black vertex v' of $\mathcal{G}'$ with $\varphi[\mathrm{E}v] = \mathrm{E}v'$, written as $\varphi(v) = v'$.

The identity function and a composition of graph morphisms are both graph morphisms. We thus have a concrete category Bic over Set with bicolored graphs as objects, graph morphisms as Bic-morphisms and for a given bicolored graph $\mathcal{G}$, the set $\mathrm{E}\mathcal{G}$ as underlying object.[20]

Definition 1.3.13. Define the obvious notions of connectedness and bicolored subgraphs (or simply *subgraphs*) for Bic-objects. A *component* of a bicolored graph $\mathcal{G}$ is a connected subgraph $\mathcal{G}' \subset \mathcal{G}$ not properly contained in any other connected subgraph of $\mathcal{G}$.

Remark 1.3.14. The quintuple $\mathcal{G}_\varnothing := (\varnothing, \varnothing, \varnothing, \varnothing, \varnothing)$ is a finite bicolored graph, called the *empty graph*. Notice each finite bicolored graph $\mathcal{G}$ is a disjoint union of its components, and that $\mathcal{G}_\varnothing$ is a component of $\mathcal{G}$ if and only if $\mathcal{G} = \mathcal{G}_\varnothing$.

Dessins d'Enfants with Dessin Morphisms

Everything is in place for the construction of the category of dessins d'enfants.

Construction 1.3.15. A *cyclic structure* $\mathcal{R}$ on a bicolored graph $\mathcal{G}$ is a collection of cyclic orders $\mathrm{C}v$ on $\mathrm{E}v$, with v ranging over $\mathrm{V}\mathcal{G}$. Define a **dessin d'enfant** $\mathcal{D}$ (or simply *dessin*) as a pair $(|\mathcal{D}|, \mathcal{R}\mathcal{D})$, consisting of a finite bicolored graph $|\mathcal{D}|$ and a cyclic structure $\mathcal{R}\mathcal{D}$ on $|\mathcal{D}|$.

For dessins $\mathcal{D}, \mathcal{D}'$, a graph morphism $\varphi : |\mathcal{D}| \to |\mathcal{D}'|$ is called a **dessin morphism** if for each $v \in \mathrm{V}|\mathcal{D}|$ the restriction $\varphi| : (\mathrm{E}v, \mathrm{C}v) \to (\mathrm{E}\varphi(v), \mathrm{C}\varphi(v))$ is order preserving. This gives a concrete category Des over Bic with dessins d'enfants as objects and dessin morphisms as morphisms.

Remark 1.3.16. The pair $\mathcal{D}_\varnothing := (\mathcal{G}_\varnothing, \varnothing)$ is a dessin, called the *empty dessin*. Now let $\varphi : \mathcal{G} \to \mathcal{G}'$ be an isomorphism of finite bicolored graph. Then a cyclic structure $\mathcal{R}$ on $\mathcal{G}$ induce a cyclic structure $\mathcal{R}'$ on $\mathcal{G}'$, making φ an isomorphism of dessins. Note $(\mathcal{G}, \mathcal{R}) = \mathcal{D}_\varnothing$ if and only if $(\mathcal{G}', \mathcal{R}') = \mathcal{D}_\varnothing$

Example 1.3.17. Let $\mathcal{S}_d$ be the finite π-set (S_d, ρ_d) as constructed in Example 1.2.10. For color index $X \in \{B, W\}$, define the set $V_X := \{\langle\sigma_X\rangle(a, b) \mid (a, b) \in S_d\}$ and the function $c_X : S_d \to V_X$ sending (a, b) to $\langle\sigma_X\rangle(a, b)$. Then the quintuple $(S_d, V_B, V_W, c_B, c_W)$ is a finite bicolored graph $\mathcal{G}_d$ and the pair $(\mathcal{G}_d, \mathcal{R}_d)$ with $\mathcal{R}_d$ the cyclic structure on $\mathcal{G}_d$ given by the natural cyclic orders on the orbits under σ_B, σ_W a dessin.

1.4 Hypermaps

As mentioned previously, the equivalence between Bel and Des will be shown in stages, going from Bel to $\mathsf{Cov}(\mathbb{P}_\circ)_\mathsf{f}$ to $\pi\text{–}\mathsf{Set}_\mathsf{f}$ to Des. Although this approach is convenient because in the first resp. last step the objects of our categories 'look alike', i.e. topological spaces with additional structure resp. finite sets with additional structure, while in the second step we can use a well known result, one can argue that some of the intuition or insight is lost.

[20] Observe our definition of bicolored graphs with graph morphisms differs from the usual one, where a graph consists of a set V of vertices together with a set E of unordered pairs of vertices (the edges), and a morphism of graphs is defined as a function on vertices such that two adjacent vertices in the domain remain adjacent under this function. The definition used here is more suitable for our needs, because for example we need not define colorings of graphs nor morphisms φ of graphs that respect these colorings and induce surjections on $\mathrm{E}v \to \mathrm{E}\varphi(v)$ separately.

In the previews it was promised we would try and revive the intuition with the category of hypermaps, namely by showing (in the next section) how we can use the equivalence F from Bel to Des (induced by composition) to draw a finite bicolored graph $\mathcal{G}$ on the underlying Riemann surface $\mathcal{M}$ of a given Belyi pair $\mathcal{B}$ in such a way that the orientation of $\mathcal{M}$ induces a cyclic structure $\mathcal{R}$ on $\mathcal{G}$, giving a dessin $\mathcal{D} = (\mathcal{G}, \mathcal{R})$ isomorphic to $F\mathcal{B}$. Now the result of drawing $\mathcal{G}$ on $\mathcal{M}$ will be an example of a hypermap. Defining the category of hypermaps is the goal of this paragraph.

Compact Oriented Surfaces and Orientation Preserving Maps

Remark 1.4.1. Let $\mathcal{M}$ be a non-empty surface. First notice we may assume for each coordinate neighborhood U of $\mathcal{M}$ that the closure $\bar{U}$ is homeomorphic to a closed disc, which is done in this part from hereon. Furthermore, it is known that $H_2(\mathcal{M}, \mathcal{M} - \{x\}) \cong \mathbb{Z}$ for all $x \in \mathcal{M}$ (where a generator of this group is called an *orientation of $\mathcal{M}$ at x*).[21] Now for each coordinate neighborhood U of $\mathcal{M}$ and points $p, q \in U$, we give a canonical isomorphism $H_2(\mathcal{M}, \mathcal{M} - \{p\}) \cong H_2(\mathcal{M}, \mathcal{M} - \{q\})$ depending on U that will be used throughout the following.

Let (U, z) be a chart of $\mathcal{M}$ and x a point in U. Excision gives us an isomorphism f_x from $H_k(\mathcal{M}, \mathcal{M} - \{x\})$ to $H_k(\bar{U}, \bar{U} - \{x\})$, while the exact sequence for the triple $(\bar{U}, \bar{U} - \{x\}, \partial U)$, the dimension property and the fact that $\bar{U} - \{x\}$ and ∂U are homotopy equivalent gives us an isomorphism $g_x : H_k(\bar{U}, \bar{U} - \{x\}) \to H_k(\bar{U}, \partial U)$. Now for $p, q \in U$, our isomorphism from $H_2(\mathcal{M}, \mathcal{M} - \{p\})$ to $H_2(\mathcal{M}, \mathcal{M} - \{q\})$ depending on U will be the composition

$$H_2(\mathcal{M}, \mathcal{M} - \{p\}) \xrightarrow{f_p} H_2(\bar{U}, \bar{U} - \{p\}) \xrightarrow{g_p} H_2(\bar{U}, \partial U) \xrightarrow{g_q^{-1}} H_2(\bar{U}, \bar{U} - \{q\}) \xrightarrow{f_q^{-1}} H_2(\mathcal{M}, \mathcal{M} - \{q\}).$$

Consequently, the assertion $H_2(\mathcal{M}, \mathcal{M} - \{x\}) \cong \mathbb{Z}$ for all $x \in \mathcal{M}$ is equivalent to the claim $H_2(\bar{D}^2, S^1) \cong \mathbb{Z}$, which is perhaps better known.[22] For by assumption we have $(\bar{U}, \partial\bar{U}) \cong (\bar{D}^2, S^1)$ as pairs of spaces, with $\bar{D}^2$ the closed unit disc. In particular, $(\bar{U}, \partial\bar{U})$ and $(\bar{D}^2, S^1)$ are homotopy equivalent, so $H_2(\bar{U}, \partial U) \cong H_2(\bar{D}^2, S^1)$. Thus, for $x \in \mathcal{M}$, combining this with the isomorphisms f_x, g_x, we see that $H_2(\mathcal{M}, \mathcal{M} - \{x\}) \cong H_2(\bar{D}^2, S^1)$.

With the canonical isomorphisms $H_2(\mathcal{M}, \mathcal{M} - \{p\}) \to H_2(\mathcal{M}, \mathcal{M} - \{q\})$ depending on U we can now define the category of compact oriented surfaces, which will play an important role in the discussion of hypermaps.

Construction 1.4.2. For a non-empty surface $\mathcal{M}$, define an *orientation on $\mathcal{M}$* as a collection $\{\mu_p \mid p \in \mathcal{M}\}$ of orientations of $\mathcal{M}$ at p such that for each coordinate neighborhood U on $\mathcal{M}$ and points $q, q' \in U$ the isomorphism $H_2(\mathcal{M}, \mathcal{M} - \{q\}) \to H_2(\mathcal{M}, \mathcal{M} - \{q'\})$ depending on U maps μ_q to $\mu_{q'}$. By convention, an orientation on the empty surface $\mathcal{M}_\varnothing$ is a choice from the set $\{\mu_\varnothing, -\mu_\varnothing\}$. Now define a **compact oriented surface** Σ as a pair $(|\Sigma|, \mathcal{O}\Sigma)$ where $|\Sigma|$ is a compact surface and $\mathcal{O}\Sigma$ an orientation on $|\Sigma|$.

Let Σ, Σ' be compact oriented surfaces and $f : |\Sigma| \to |\Sigma'|$ a map. We call f **orientation preserving** if each $p \in |\Sigma|$ is an isolated point in $f^{-1}f(p)$ with positive local degree $\deg_p(f)$. Because the local degree is multiplicative and the identity mapping is orientation preserving, we have a concrete category coSurf over Surf with compact oriented surfaces as objects and orientation preserving, open maps as morphisms.

Remark 1.4.3. Call the compact oriented surface $\Sigma_\varnothing := (\mathcal{M}_\varnothing, \{\mu_\varnothing\})$ the *empty compact oriented surface*. Now suppose we are given a compact oriented surface Σ. Then $\Sigma^* := (|\Sigma|, \mathcal{O}^*)$, where $\mathcal{O}^*$ is defined as $\{-\mu_p \mid \mu_p \in \mathcal{O}\Sigma\}$, is a compact oriented surface as well, with $\Sigma^{**} = \Sigma$ and $\mathrm{id} : \Sigma \to \Sigma^*$ an isomorphism if and only if $|\Sigma| = \mathcal{M}_\varnothing$. Next suppose $\varphi : \mathcal{M} \to |\Sigma|$ is an isomorphism of surfaces. Then $H_2(\mathcal{M}, \mathcal{M} - \{p\}) \cong H_2(|\Sigma|, |\Sigma| - \{\varphi(p)\})$ for each $p \in \mathcal{M}$, inducing an orientation on $\mathcal{M}$ such that φ becomes an isomorphism of compact oriented surfaces.

Example 1.4.4. Let K be a simplicial complex (hereafter simply *complex*) such that the associated polyhedron $\hat{K} := \bigcup K$ is an orientable combinatorial surface.[23] Choose a collection $\mathcal{R}$ of cyclic orders R_σ on the vertices of σ for each 2-simplex $\sigma \in K$ in a compatible manner. We endow $\hat{K}$ with an atlas ΦK by taking, for each vertex $v \in K$, the interior of the union of all triangles in K of which v

[21] We use some algebraic topology from [9], specifically pp. 2 - 15 for homology theory and pp. 18 - 25 for orientations and local degrees of maps.

[22] See for example [5], Exmp. 2.17.

[23] See [2], Def. 6.1 and pp. 154 - 155. We take all complexes to be finite. Notice for compact surfaces, this example illustrates the relation between the notion of orientability (mentioned in Paragraph 1.1 and presented in [2]), and that of an orientation (the one we use, taken from [9]). This follows from the fact that any compact surface admits a triangulation (see [2], §7.2).

is a vertex as coordinate neighborhoods, together with obvious coordinate functions, and denote the resulting compact topological surface $(\hat{K}, \Phi K)$ by $\mathcal{K}$. We show that $\mathcal{R}$ induces an orientation on $\mathcal{K}$. So let $p \in \hat{K}$.

First suppose p is contained in the interior of a 2-simplex $\alpha \in K$ with vertices v_0, v_1, v_2 such that $R_\alpha(v_0, v_1, v_2)$. Then define μ_p as the class $[\langle v_0, v_1, v_2 \rangle]$ in $H_2(\hat{K}, \hat{K} - \{p\})$.

If p is contained in the interior of a 1-simplex $\tau \in K$, then τ must be the edge of exactly two triangles σ, σ'. Number the vertices of σ resp. σ' as w_0, w_1, w_2 resp. w_0', w_1', w_2' such that $R_\sigma(w_0, w_1, w_2), R_{\sigma'}(w_0', w_1', w_2')$ and $w_0, w_0' \notin \tau$. In this case, we define μ_p as the class $[\langle w_0, w_1, w_2 \rangle + \langle w_0', w_1', w_2' \rangle]$. The assumption on $\mathcal{R}$ implies $w_1 = w_2', w_2 = w_1'$, so μ_p spans $H_2(\hat{K}, \hat{K} - \{p\})$.

Finally, if p is a vertex of K, then there are at least three 2-simplexes $\sigma_1, \sigma_2, \sigma_3, ..., \sigma_n$ that form a cone with apex p. By possibly rearranging the indices of these triangles, we can number their vertices such that σ_i has vertices $p, u_1^{(i)}, u_2^{(i)}$ with $(p, u_1^{(i)}, u_2^{(i)}) \in R_{\sigma_i}$, and with $u_2^{(i)} = u_1^{(i+1)}$ for $1 \leq i < n$ and $u_2^{(n)} = u_1^{(0)}$. The group $H_2(\hat{K}, \hat{K} - \{p\})$ is thus spanned by the class $\sum_i [\langle p, u_1^{(i)}, u_2^{(i)} \rangle]$.

Now let U be a coordinate neighborhood on $\mathcal{K}$. By assumption, we have triangles $\sigma_1, \sigma_2, ..., \sigma_n$ in K such that $\bar{U}$ is the union of these triangles (which share exactly one vertex $v \in K$, namely the apex of the cone $\bar{U}$). We again number their vertices such that $(v, u_1^{(i)}, u_2^{(i)}) \in R_{\sigma_i}$. Now pick points $p, q \in U$ and observe that $H_2(\bar{U}, \partial \bar{U})$ is spanned by $[u] := \sum_i [\langle v, u_1^{(i)}, u_2^{(i)} \rangle]$. The construction of the orientations of $\mathcal{K}$ at p resp. q imply that $[u]$ actually equals μ_p resp. μ_q. Because the isomorphism $H_2(\mathcal{K}, \mathcal{K} - \{p\}) \to H_2(\mathcal{K}, \mathcal{K} - \{q\})$ depending on U only depends on the composition $H_2(\bar{U}, \bar{U} - \{p\}) \to H_2(\bar{U}, \partial U) \to H_2(\bar{U}, \bar{U} - \{q\})$, which is induced by inclusions and thus sends μ_p to $[u]$ to μ_q, the assertion follows. Denote the resulting compact oriented surface by Σ_K.

Remark 1.4.5. The concept of compact oriented surfaces and coSurf-morphisms can be seen as a generalization of compact Riemann surfaces as follows. Let $\mathcal{M}$ be a compact Riemann surface, considered as real two-dimensional smooth manifold.[24] For $p \in \mathcal{M}$, let μ_p be an orientation of $\mathcal{M}$ at p induced by a chart around p and the orientation of $\mathbb{R}^2$ given by its standard basis. It can be shown this generator is independent of the choice of a chart around p, using the fact that each transition map of $\mathcal{M}$, as a diffeomorphism with positive Jacobian everywhere, has local degree 1 at every point in its domain.[25] We thus define $\mathcal{OM}$ as the collection $\{\mu_q \mid q \in \mathcal{M}\}$, giving us a compact oriented surface $(\mathcal{M}, \mathcal{OM})$.

Now let $\varphi : \mathcal{M} \to \mathcal{M}'$ be a morphism of compact Riemann surfaces that is non-constant on each component of $\mathcal{M}$. From the discussion of ramification indices it follows, for each $p \in \mathcal{M}$, we can find a coordinate neighborhood U_p around p such that $\varphi(x) = \varphi(p)$ implies $x = p$ for all $x \in U_p$, and thus that p is isolated in $\varphi^{-1}\varphi(p)$. One shows that φ, considered as mapping $(\mathcal{M}, \mathcal{OM}) \to (\mathcal{M}', \mathcal{OM}')$, is orientation preserving, using the fact that ${}_w\varphi_z$ is orientation preserving for each pair of charts $(U, w) \in \Phi\mathcal{M}$ and $(V, z) \in \Phi\mathcal{M}'$. The map φ is furthermore open, as we shall see in the next section.

Hypermaps with Hypermorphisms

Using bicolored graphs and compact oriented surfaces, we can finally construct the last of our C-categories, namely the category HoHyp of hypermaps with hypermorphisms. Recall the idea of HoHyp is to formalize the notion of drawing a graph on an underlying Riemann surface $\mathcal{M}$ of a given Belyi pair $(\mathcal{M}, f)$, induced by the inverse image of $[0, 1]$ under f.

As it turns out, we only need to consider graphs drawn on compact oriented surfaces. In other words, in the category of hypermaps we have no need for any complex structures. This has the advantage that the resulting category will be a purely topological description of Belyi pairs. The drawback however is that morphisms on compact oriented surfaces (i.e. orientation preserving, open maps) have a lot more ‘freedom’ than holomorphisms, let alone Belyi morphisms. For if the equivalence theorem is correct, $\mathrm{Hom}(\mathcal{B}, \mathcal{B}')$ will be finite for any given pair of Belyi pairs $\mathcal{B}, \mathcal{B}'$. Compare this for example to the number of coSurf-endomorphisms on $(\mathbb{P}, \mathcal{OP})$.

We will remedy this as follows. First we construct the category Hyp of hypermaps with some natural definition of Hyp-morphisms. Then we give a nice equivalence relation $\simeq$ on Hyp-morphisms, using relative homotopy. This will give a second category HoHyp, which again has hypermaps as objects but with equivalence classes of Hyp-morphisms under $\simeq$ as HoHyp-morphisms. The resulting category will induce an obvious forgetful functor HoHyp $\to$ Bic, showing that, in any case, the number of HoHyp morphisms between two hypermaps must be finite.

[24] See [6], §1.
[25] See [9], p. 25.

Definition 1.4.6. Let $\mathcal{G}$ be a finite bicolored graph with edges $e_1, e_2, ..., e_n$. For an edge $e_j \in \mathrm{E}\mathcal{G}$ let I_j be $I \times \{e_j\}$. Define an equivalence relation $\approx$ on $\bigcup\{I_j \mid 1 \leq j \leq n\}$ by identifying (s, e_i) with (t, e_j) if either $s = t = 0$ and $\mathrm{b}\mathcal{G}(e_i) = \mathrm{b}\mathcal{G}(e_j)$ or $s = t = 1$ and $\mathrm{w}\mathcal{G}(e_i) = \mathrm{w}\mathcal{G}(e_j)$. Call the resulting quotient space $\bigcup\{I_j \mid 1 \leq j \leq n\}/\approx$ the *polyhedron of* $\mathcal{G}$ and denote it by $\hat{\mathcal{G}}$.

Remark 1.4.7. Note that intuitively, the polyhedron of $\mathcal{G}$ is obtained by drawing the edges of $\mathcal{G}$ as distinct lines, say horizontally on a piece of paper. Now every edge has two endpoints, a black one and a white one. Draw the black ones left, the white ones right. Of course, because certain edges in $\mathcal{G}$ can share common vertices, we have possibly drawn the same vertex of $\mathcal{G}$ multiple times on our paper. The equivalence relation $\approx$ now identifies all vertices drawn on the paper that are identical as vertices of $\mathcal{G}$. Thus if we cut out the drawn lines and glue the appropriate vertices, we get $\hat{\mathcal{G}}$.

Notation. Write $[s, j]$ for the equivalence class of (s, e_j) under $\approx$. For a graph morphism $\varphi : \mathcal{G} \to \mathcal{G}'$, written as $\varphi(e_j) = e_{\varphi(j)}$, we have a map $\hat{\varphi} : \hat{\mathcal{G}} \to \hat{\mathcal{G}}'$, sending $[s, j]$ to $[s, \varphi(j)]$.

Construction 1.4.8. A **hypermap** $\mathcal{H}$ is a triple $(\mathcal{G}, \Sigma, g)$, where $\mathcal{G}$ is a finite bicolored graph, Σ a coSurf-object and g an injective map from $\hat{\mathcal{G}}$ to Σ (called the *embedding of* $\mathcal{G}$ *into* Σ) such that:

(i) For each component Σ_i of Σ there is a unique component $\mathcal{G}_i$ of $\mathcal{G}$ with $g^{-1}[\Sigma_i] = \hat{\mathcal{G}}_i$;

(ii) $\Sigma - g[\hat{\mathcal{G}}]$ is a finite collection of disjoint open sets (the *faces of* $\mathcal{H}$), each homeomorphic to D^2.

Define the category Hyp with hypermaps as objects and as Hyp-morphisms $(\mathcal{G}, \Sigma, g) \to (\mathcal{G}', \Sigma', g')$ pairs (φ, f) with $\varphi : \mathcal{G} \to \mathcal{G}'$ a graph morphism and $f : \Sigma \to \Sigma'$ a coSurf-morphism *associated to* φ, i.e. such that the following diagram commutes:

$$\begin{array}{ccc} \hat{\mathcal{G}} & \xrightarrow{\hat{\varphi}} & \hat{\mathcal{G}}' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ |\Sigma| & \xrightarrow{f} & |\Sigma'| \end{array}$$

We compose Hyp-morphisms element-wise and take pairs $(\mathrm{id}, \mathrm{id})$ as identity morphisms in Hyp. From a simple diagram chase and using that Bic and coSurf are both categories, it follows that Hyp is a category as well.

Notation. For a given hypermap $\mathcal{H}$ we denote the underlying bicolored graph of $\mathcal{H}$ by $|\mathcal{H}|$ and the compact oriented surface resp. embedding associated to $\mathcal{H}$ by $\Sigma\mathcal{H}$ resp. $g\mathcal{H}$. Furthermore, we write $[\mathcal{H}]$ for the image of the polyhedron of $|\mathcal{H}|$ under $g\mathcal{H}$.[26]

Remark 1.4.9. The triple $\mathcal{H}_\varnothing := (\mathcal{G}_\varnothing, \mathcal{M}_\varnothing, \varnothing)$ is a Hyp-object, called the *empty hypermap*. Now let $\mathcal{H}$ be a hypermap. Then $\mathcal{H}^*$ defined as $(|\mathcal{H}|, \Sigma\mathcal{H}^*, g\mathcal{H})$ is a hypermap as well such that $\mathcal{H}^{**} = \mathcal{H}$ and with $\mathrm{id} : \mathcal{H} \to \mathcal{H}^*$ an isomorphism if and only if $\mathcal{H} = \mathcal{H}_\varnothing$ or $\mathcal{H} = \mathcal{H}^*_\varnothing$ if and only if $|\mathcal{H}| = \mathcal{G}_\varnothing$.

Example 1.4.10. Let K be a complex such that $\hat{K}$ is an orientable combinatorial surface. Then K induces a finite bicolored graph $\mathcal{G}_K$ on Σ_K by taking the set of vertices of K as $\mathrm{B}\mathcal{G}_K$ and by adding one white vertex on the interior of each 1-simplex of K (and with obvious edges). Because the interiors of 2-simplexes are homeomorphic to open discs, this gives us a hypermap $\mathcal{H}_K$.

Let $\mathcal{H}, \mathcal{H}'$ be hypermaps. Given a graph morphism $\varphi : |\mathcal{H}| \to |\mathcal{H}'|$, there are in general many choices of morphism $\Sigma\mathcal{H} \to \Sigma\mathcal{H}'$ associated to φ. Luckily, up to homotopy relative to $[\mathcal{H}]$, this choice is unique. To show this, we first give CW-structures on our hypermaps after we introduce some terminology.[27] The uniqueness up to relative homotopy of coSurf-morphisms associated to a given graph morphism will be the main ingredient in the construction of HoHyp (besides, of course, hypermaps).

Definition 1.4.11. Let $\mathcal{H} = (\mathcal{G}, \Sigma, g)$ be a hypermap with edges $e_1, e_2, ..., e_n$. Then for $1 \leq j \leq n$:

- The points $g([0, j])$ resp. $g([1, j])$ in $|\Sigma|$ are called the (*black* resp. *white*) *vertices on* Σ;
- The subsets $g[I_j^\circ] \subset |\Sigma|$ resp. $g[I_j] \subset |\Sigma|$ are called the *edges on* Σ resp. *the closed edges on* Σ.

Note in the latter we have abused some notation, for g is defined as a map from the polyhedron $\bigcup\{I_j \mid 1 \leq j \leq n\}/\approx$ to $|\Sigma|$. This is justified because $(s, j) \approx (s', j)$ if and only if $s = s'$, so $I_j/\approx$ is just I_j itself.

For a given vertex x on Σ, we may refer to the singleton $\{x\} \subset |\Sigma|$ by means of x itself if no confusion can arise. This will be convenient in discussing the CW-structure $\mathcal{H}_\bullet$ associated to $\mathcal{H}$, for the vertices on Σ will be the 0-cells of $\mathcal{H}_\bullet$.

[26] Note that although we use notation as if Hyp is concrete over Bic, this is evidently not the case. However, the category HoHyp constructed later on, i.e. 'the' category of hypermaps, will be concrete over Bic.

[27] See [9], Def. 5.3 for relative homotopy and ibid. Def. 4.6, the appendix of [5], specifically Prop. A.2, and [11], §5 for some theory relating to CW-structures.

Lemma 1.4.12. *For each hypermap $\mathcal{H} = (\mathcal{G}, \Sigma, g)$ we have a CW-structure $\mathcal{H}_\bullet$ on $|\Sigma|$ with the vertices on Σ as 0-cells, the edges on Σ as 1-cells and the faces of $\mathcal{H}$ as 2-cells.*

Proof. Let $e_1^0, e_2^0, ..., e_{n_0}^0$ resp. $e_1^1, e_2^1, ..., e_{n_1}^1$ be the vertices resp. the edges on Σ. For $1 \leq i \leq n_0$, denote the unique map $\bar{D}^0 \to \bar{e}_i^0$ by χ_i^0. For $1 \leq j \leq n_1$, let $\chi_j^1 : \bar{D}^1 \to \bar{e}_i^1$ be the map given by $s \mapsto g([{}^s\!/\!_2 + {}^1\!/\!_2, j])$. Note that for each $\chi_i^l : \bar{D}^l \to \bar{e}_i^l$ with $l = 0, 1$, the restriction $\chi_i^l| : D^l \to e_i^l$ is a homeomorphism, where all e_i^l's are mutually disjoint. Thus these mappings are suitable candidates for the characteristic maps of $\mathcal{H}_\bullet$.

Now let $e_1^2, e_2^2, ..., e_{n_2}^2$ be the faces of $\mathcal{H}$ and pick one of these, say e_k^2. We want to show $\bar{e}_k^2$ is a closed disc with a suitable equivalence relation on its boundary, so that the characteristic map χ_k^2 can be easily defined. For this, notice that ∂e_k^2 is a union of closed edges on Σ, forming a cycle c, where a closed edge is repeated in c if and only if it is not contained in the boundary of any other face distinct from e_k^2. If we let E be the disjoint union of the closed edges in c, with m copies of each closed edge that is repeated m times in c, then $\approx$ is an equivalence relation on E as well, and $\hat{E} := E/\approx$ is homeomorphic to S^1. Next we define a second equivalence relation $\sim$ on $\hat{E} \cong S^1$ by identifying points that were identical in $|\Sigma|$. We then get ∂e_k^2 back as the space $S^1/\sim$. Because $e_k^2 \cong D^2$, if we extend $\sim$ to $\bar{D}^2$, this indeed shows $\bar{e}_k^2 \cong \bar{D}^2/\sim$.

Technicalities aside, the above argument says that the closure $\bar{e}_k^2$ of a face e_k^2 is homeomorphic to the closed disc $\bar{D}^2$, with the latter under an obvious equivalence relation: if we consider e_k^2 as an open disc in $|\Sigma|$ and trace its boundary as a closed loop on $|\Sigma|$, then we may trace segments of this boundary more than once. We remove these redundancies by the equivalence relation $\sim$ and are left with the desired homeomorphism $\bar{e}_k^2 \cong \bar{D}^2/\sim$. Thus we can define the map $\chi_k^2 : \bar{D}^2 \to \bar{e}_k^2$ as the quotient map. Indeed, $\chi_k^2| : D^2 \to e_k^2$ is then a homeomorphism, and all faces of $\mathcal{H}$ are disjoint.

Because our (candidate) cells are finite in number, we only need to check that the boundary of each cell e_i^l is contained in a union of a finite number of cells of dimension less then e_i^l. This is certainly satisfied for our 0- resp. 1-cells, being the vertices resp. the edges on Σ. For the 2-cells (the faces) this follows because $\mathcal{G}$ is finite. We thus have the desired CW-structure $\mathcal{H}_\bullet$ on $|\Sigma|$, with the mappings χ_i^l as characteristic maps. □

In the construction of the category HoHyp the following proposition is convenient. From hereon, for a given hypermap $\mathcal{H}$, we tacitly endow it with the CW-structure $\mathcal{H}_\bullet$ from the above lemma.

Proposition 1.4.13. *Let $\mathcal{H} = (\mathcal{G}, \Sigma, g), \mathcal{H}' = (\mathcal{G}', \Sigma', g')$ be hypermaps, φ, φ' graph morphisms $\mathcal{G} \to \mathcal{G}'$ and f, f' coSurf-morphisms $\Sigma \to \Sigma'$. If f is associated to φ, then the following holds:*
(i) The map f' is associated to φ as well if and only if f is homotopic to f' relative to $[\mathcal{H}]$;
(ii) Conversely, f is associated to φ' as well if and only if $\varphi = \varphi'$.

Proof. Let us first capture the data of the proposition in the following diagram:

$$\begin{array}{ccccccc}
\hat{\mathcal{G}} & \xrightarrow{g} & |\Sigma| & \xleftarrow{g} & \hat{\mathcal{G}} & \xrightarrow{g} & |\Sigma| \\
\downarrow{\scriptstyle \hat{\varphi}'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \hat{\varphi}} & & \downarrow{\scriptstyle f'} \\
\hat{\mathcal{G}}' & \xrightarrow{g'} & |\Sigma'| & \xleftarrow{g'} & \hat{\mathcal{G}}' & \xrightarrow{g'} & |\Sigma'|
\end{array}$$

By assumption, the middle square commutes. We need to show (i) the right square commutes if and only if f is homotopic to f' relative to $[\mathcal{H}]$ and (ii) the left square commutes if and only if $\varphi = \varphi$'. We begin with the latter. The implication from right to left is trivial. Conversely, if f is associated to both φ and φ', then $\hat{\varphi}' = \hat{\varphi}' g^{-1} g = g'^{-1} g' \hat{\varphi} = \hat{\varphi}$ and thus $\varphi = \varphi'$. For (i), first suppose f is homotopic to f' relative to $[\mathcal{H}]$. Then $f|_{[\mathcal{H}]} = f'|_{[\mathcal{H}]}$, thus $g'\hat{\varphi} = fg = f'g$, and therefore f' is associated to φ, which shows the implication from right to left.

For the implication from left to right of (i), suppose f' is associated to φ as well. Without loss of generality we assume that both $|\Sigma|$ and $|\Sigma'|$ are connected. We first give some properties of f that apply equally well to f'. Because f is open, $|\Sigma|$ is compact and $|\Sigma'|$ is Hausdorff and connected, f is surjective. Now let F be a face of $\mathcal{H}$. If $f[F]$ is contained in $[\mathcal{H}'] = g'[\hat{\mathcal{G}}']$, then for each point $p \in F$ we can choose an open neighborhood U of p that is mapped to an open interval in $g'[\hat{\mathcal{G}}']$, showing that p is not isolated in $f^{-1}f(p)$. Thus there must be some face F' of $\mathcal{H}'$ that meets $f[F]$, i.e. with $F' \cap f[F] \neq \varnothing$.

Now let v be a vertex on Σ and let $\mathrm{E}v$ be the collection of the closed edges on Σ connected to v. Endow $\mathrm{E}v$ resp. $\mathrm{E}f(v)$ with cyclic orders $R_v, R_{f(v)}$ induced by the orientations of Σ resp. Σ' as follows. If we take the orientation μ of $|\Sigma|$ at v, then there is a representative σ of μ such that $\sigma| : \partial\Delta^2 \to |\Sigma|$ is a non-intersecting loop around v with orientation inherited from the standard orientation in $\mathbb{R}^3$.

We may assume that each element in Ev meets this loop around v exactly once, which we do. Thus we can set $(x, y, z) \in R_v$ for distinct $x, y, z \in \mathrm{E}v$ if we meet z after y after x while traversing the loop around v in the positive direction exactly once. With the same procedure we get $R_{f(v)}$ with respect to $f(v)$. Therefore, the induced mapping $\tilde{f} : \mathrm{E}v \to \mathrm{E}f(v)$, with $\tilde{f}(e) = f[e] \in \mathrm{E}f(v)$ for each $e \in \mathrm{E}v$, is order preserving with respect to $R_v, R_{f(v)}$, because f is orientation preserving.

It can be shown above arguments imply the boundary ∂F of each face F of $\mathcal{H}$ is mapped under f to a cycle of edges on Σ' that is the boundary of a unique face of $\mathcal{H}'$. For this, one uses the successor function on the edges connected to a given vertex on Σ in the boundary of F resp. on Σ' in $f[\partial F]$, and the fact that f is orientation preserving, inducing order preserving functions $(\mathrm{E}w, R_w) \to (\mathrm{E}f(w), R_{f(w)})$ for each vertex w on Σ in ∂F.

Notice the assumption implies $f|_{[\mathcal{H}]} = f'|_{[\mathcal{H}]}$. Therefore, for each face F of $\mathcal{H}$, we have a unique face F' of $\mathcal{H}'$ equal to both $f[F]$ and $f'[F]$. Now the assertion follows from the fact that maps on CW-complexes and thus homotopies between such maps can be constructed cell-wise. In other words, $H : |\Sigma| \times I \to |\Sigma'|$ is continuous if and only if for each cell e^l_k of $\mathcal{H}$, the composition $H \circ (\chi^l_k \times \mathrm{id}_I) : \bar{D}^l \times I \to |\Sigma| \times I \to |\Sigma'|$ is continuous.[28]

To give the desired homotopy between f and f' cell-wise, let e^2_i resp. e^2_j be faces of $\mathcal{H}$ resp. $\mathcal{H}'$ such that e^2_j equals both $f[e^2_i]$ and $f'[e^2_i]$. Because $f|_{\partial e^2_i} = f'|_{\partial e^2_i}$, we have an obvious homotopy between $f \circ \chi^2_i$ and $f' \circ \chi^2_i$ which is relative to ∂D^2. Thus, f and f' are homotopic on $\bar{e}^2_i \to \bar{e}^2_j$ relative to ∂e^2_i, and the claim follows. □

Remark 1.4.14. Note that in the above proof, for a given hypermap $\mathcal{H} = (\mathcal{G}, \Sigma, g)$, we have given a cyclic order on Ev for each vertex v on Σ, induced by the orientation of Σ. This will be used in the next section, namely in the construction of a dessin associated to $\mathcal{H}$.

We use the previous proposition for the definition of a suitable equivalence relation on Hyp-morphisms, which in turn will be used for the construction of HoHyp.

Definition 1.4.15. Call Hyp-morphisms $(\varphi, f), (\varphi', f') : \mathcal{H} \to \mathcal{H}'$ *relative homotopic* if f, f' are homotopic relative to $[\mathcal{H}]$ (notation: $(\varphi, f) \simeq (\varphi', f')$).

Corollary 1.4.16. *The rule $\simeq$ is a well-defined equivalence relation on* Hyp*-morphisms, respecting* Hyp*-composition. Moreover, if $(\varphi, f) \simeq (\varphi', f')$, then $\varphi = \varphi'$.*

Proof. Let $\mathcal{H}, \mathcal{H}'$ be hypermaps. Using Proposition 2.1.18 and the fact that being homotopic relative to $[\mathcal{H}]$ is an equivalence relation on the set of coSurf-morphisms $\Sigma\mathcal{H} \to \Sigma\mathcal{H}'$, we see that $\simeq$ is indeed a well-defined equivalence relation on $\mathrm{Hom}_{\mathsf{Hyp}}(\mathcal{H}, \mathcal{H}')$ such that $(\varphi, f) \simeq (\varphi', f')$ implies $\varphi = \varphi'$.

Now for a third hypermap $\mathcal{H}''$, suppose $(\psi, h), (\psi', h') : \mathcal{H}'' \to \mathcal{H}$ resp. $(\varphi, f), (\varphi', f') : \mathcal{H} \to \mathcal{H}'$ are Hyp-morphisms such that $(\psi, h) \simeq (\psi', h')$ and $(\varphi, f) \simeq (\varphi', f')$. Then $\varphi = \varphi'$ and $\psi = \psi'$, so $(\varphi \circ \psi, f \circ h) = (\varphi' \circ \psi', f \circ h) \simeq (\varphi' \circ \psi', f' \circ h')$, showing that $\simeq$ respects composition in Hyp. □

Definition 1.4.17. Denote the equivalence class of a Hyp-morphism $(\varphi, f) : \mathcal{H} \to \mathcal{H}'$ under $\simeq$ by $(\varphi, [f])$, and call it a *hypermorphism* form $\mathcal{H}$ to $\mathcal{H}'$.

Remark 1.4.18. The notation $(\varphi, [f]) : \mathcal{H} \to \mathcal{H}'$ is unambiguous, because if $(\varphi, [f]) = (\varphi', [f'])$, then $\varphi = \varphi'$ and $[f] = [f']$, with the latter the equivalence class of f under homotopy relative to $[\mathcal{H}]$. Notice that hypermaps $\mathcal{H}, \mathcal{H}'$ are isomorphic in Hyp if and only if they are isomorphic in HoHyp.

All is in place for the construction of our final C-category:

Construction 1.4.19. Let HoHyp be the category with hypermaps as objects and hypermorphisms as morphisms.

The following is a direct result of the preceding discussion, showing that HoHyp is a concrete category over Bic.

Lemma 1.4.20. *We have a forgetful functor U_H :* HoHyp $\to$ Bic*, sending a hypermap $\mathcal{H} = (\mathcal{G}, \Sigma, g)$ to the bicolored graph $\mathcal{G}$, and a hypermorphism $(\varphi, [f])$ to the graph morphism φ.* □

[28] See [11], Prop. 5.5, 5.6.

2 Equivalence Theorem

Note the correspondence between Bel and Des as outlined in Preview 1.1.19 will follow from the equivalence theorem, which implies the objects and morphisms of Bel are completely determined by their induced dessins and dessin morphisms.

In this section we first give the puncture functor from Bel to $\mathsf{Cov}(\mathbb{P}_\circ)_\mathsf{f}$. Although this functor is perhaps the most obvious, a large part of the proof in the second part of this section will be devoted to the proof it is an equivalence, by showing that it is fully faithful and essentially surjective.[29] After the puncture functor we use the fiber functor to show that $\mathsf{Cov}(\mathbb{P}_\circ)_\mathsf{f}$ and $\pi{-}\mathsf{Set}_\mathsf{f}$ are equivalent. We then define the orbit functor from $\pi{-}\mathsf{Set}_\mathsf{f}$ to Des and the cut functor from HoHyp to Des. The proof in the second part of this section that both the orbit functor and the cut functor are equivalences will thus show that hypermaps can be seen as 'purely topological' descriptions of Belyi pairs.

2.1 Formulation of the Equivalence Theorem

From Belyi Pairs to Finite Coverings over $\mathbb{P}_\circ$

Recall that a Belyi pair $\mathcal{B}$ is a pair $(\mathcal{M}, f)$ with $\mathcal{M}$ a compact Riemann surface and f a Belyi map on $\mathcal{M}$, i.e. a meromorphic function on $\mathcal{M}$ non-constant on each component of $\mathcal{M}$ and unramified outside $\{0, 1, \infty\}$. As manifoldly promised, we show that removing all possible ramification points and forgetting the complex structure on $\mathcal{M}$ induces a finite covering over $\mathbb{P}_\circ$.

Definition 2.1.1. For a Belyi pair $\mathcal{B} = (\mathcal{M}, f)$, define $\mathcal{X}_\mathcal{B}$ as the pair $(X_\mathcal{B}, p_\mathcal{B})$, where $X_\mathcal{B}$ is the subspace $\mathcal{M} - f^{-1}\{0, 1, \infty\}$ and $p_\mathcal{B}$ is the restriction of f to $X_\mathcal{B} \to \mathbb{P}_\circ$. For a Belyi morphism $\varphi : \mathcal{B} \to \mathcal{B}'$ to another Belyi pair $\mathcal{B}'$, define $\varphi_\circ$ as the restriction $\varphi| : X_\mathcal{B} \to X_{\mathcal{B}'}$.

Remark 2.1.2. Observe for Belyi morphisms $\varphi : \mathcal{B} \to \mathcal{B}', \psi : \mathcal{B}' \to \mathcal{B}''$, the induced functions $\varphi_\circ, \psi_\circ$ are well-defined maps such that $p_{\mathcal{B}'}\varphi_\circ = p_\mathcal{B}$, and that $(\psi\varphi)_\circ = \psi_\circ\varphi_\circ$. Now to show $\mathcal{X}_\mathcal{B}$ is a finite covering over $\mathbb{P}_\circ$ for each Belyi pair $\mathcal{B}$, we use the fact that a given holomorphism $\varphi : \mathcal{M} \to \mathcal{N}$ non-constant on each component $\mathcal{M}_i$ of $\mathcal{M}$ is open, which follows from the Open Mapping Theorem and the fact that coordinate functions of surfaces are homeomorphisms.[30] If moreover $\mathcal{M}$ is compact while $\mathcal{N}$ is connected, then each restriction $\varphi| : \mathcal{M}_i \to \mathcal{N}$ is surjective, because in this case only $\varnothing$ and $|\mathcal{N}|$ itself are clopen in $|\mathcal{N}|$.

Proposition 2.1.3. *We have a functor* $\mathrm{Pun} : \mathsf{Bel} \to \mathsf{Cov}(\mathbb{P}_\circ)_\mathsf{f}$, *associating* $\mathcal{X}_\mathcal{B}$ *to a Belyi pair* $\mathcal{B}$, *and* $\varphi_\circ : \mathcal{X}_\mathcal{B} \to \mathcal{X}_{\mathcal{B}'}$ *to a Belyi morphism* $\varphi : \mathcal{B} \to \mathcal{B}'$.

Proof. Let $\mathcal{B}$ be a Belyi pair. Then $\mathcal{X}_\mathcal{B}$ is indeed a finite covering over $\mathbb{P}_\circ$, which follows from the local structure of morphisms of Riemann surfaces.[31] In particular, one uses that $p_\mathcal{B}$, considered as holomorphism unramified everywhere, is a local homeomorphism with finite fibers. Thus for $\zeta \in \mathbb{P}_\circ$ with $p_\mathcal{B}^{-1}(\zeta) = (x_1, x_2, ..., x_m)$ we can find open neighborhoods V of ζ resp. U_i of x_i such that $p_\mathcal{B}| : U_i \to V$ is a homeomorphism, and by taking V small enough we can guarantee $U_i \cap U_j = \varnothing$ for all distinct $1 \leq i, j \leq n$. Now $(\mathrm{id}_\mathcal{B})_\circ = \mathrm{id}_{\mathcal{X}_\mathcal{B}}$, so with Remark 2.1.2 Pun is a functor from Bel to $\mathsf{Cov}(\mathbb{P}_\circ)_\mathsf{f}$. ☐

Definition 2.1.4. We call $\mathrm{Pun} : \mathsf{Bel} \to \mathsf{Cov}(\mathbb{P}_\circ)_\mathsf{f}$ the **puncture functor**.

Example 2.1.5. Let $q \in \mathbb{N}_{\geq 1}$, take the Belyi pair $\mathcal{B}_q = (\mathbb{P}\mathcal{F}_q, f_q)$ from Example 1.1.21 and the finite covering $\mathcal{X}_q$ from Example 1.2.5. Note that the underlying space of $\mathrm{Pun}\,\mathcal{B}_q$ consists of the points $(\xi : \eta : 1) \in \mathbb{P}\mathcal{F}_q$ such that $\xi, \eta \notin \mu_q$, while the restriction $f_q| : \mathrm{Pun}\,\mathcal{B}_q \to \mathbb{P}_\circ$ maps $(\xi : \eta : 1)$ to ξ^q. Thus we have a $\mathsf{Cov}(\mathbb{P}_\circ)_\mathsf{f}$-isomorphism $\varphi : \mathrm{Pun}\,\mathcal{B}_q \to \mathcal{X}_q$, sending $(\xi : \eta : 1)$ to (ξ, η).

The Fiber Functor

We continue with the second functor, namely Fib from $\mathsf{Cov}(\mathbb{P}_\circ)_\mathsf{f}$ to $\pi{-}\mathsf{Set}_\mathsf{f}$. Because in this case we use a well-known result, after reviewing this, we shall immediately prove that Fib is an equivalence.

[29] We use the definitions of 'fully faithful' resp. 'essentially surjective' as given in [12], Def. 1.4.8, where the latter is called 'isomorphism-dense' in [1], Def. 3.33. Notice that ibid., Prop. 3.36. implies that 'being equivalent' defines an equivalence relation on the collection of categories.

[30] See [3], Thm. III.3.3 and [4], Rem. 1.17.

[31] See ibid., Thm. 1.74.

Remark 2.1.6. Let (X, x) be a pointed space, i.e. a non-empty space X with a base point $x \in X$. Then we have a functor $\mathrm{Fib}_x : \mathsf{Cov}(X) \to \pi_1(X,x)\text{–}\mathsf{Set}$, sending a covering (Y, p) over X to $(p^{-1}(x), \rho)$ with ρ the monodromy action of $\pi_1(X, x)$ on $p^{-1}(x)$, and a covering morphism $\varphi : (Y, p) \to (Z, q)$ to the restriction $\varphi| : p^{-1}(x) \to q^{-1}(x)$, which is $\pi_1(X, x)$-equivariant.[32]

Definition 2.1.7. Call a space X *locally simply connected* if each point $x \in X$ has a neighborhood basis of simply connected open neighborhoods.

Lemma 2.1.8. *Let X be a connected and locally simply connected space with base point $x \in X$. Then $\mathrm{Fib}_x : \mathsf{Cov}(X) \to \pi_1(X,x)\text{–}\mathsf{Set}$ is an equivalence such that connected coverings over X correspond to $\pi_1(X, x)$-sets endowed with transitive $\pi_1(X, x)$-action.*

Proof. This is said well-known result.[33] □

Lemma 2.1.9. *A space X is locally simply connected if and only if for each $x \in X$ and open neighborhood U of x there is a simply connected open neighborhood of x contained in U.*

Proof. The proof is straightforward. □

Lemma 2.1.10. *The space $\mathbb{P}_\circ$ is connected and locally simply connected.*

Proof. Because $\mathbb{P}_\circ$ is path-connected, it is certainly connected. For the second claim, for each $\zeta \in \mathbb{P}_\circ$ and open neighborhood U of ζ we can find an $\epsilon \in \mathbb{R}_{>0}$ such that the open ball $B_\epsilon(\zeta) \subset \mathbb{C}$ is entirely contained in U, thus with $B_\epsilon(\zeta) \cap \{0, 1\} = \varnothing$, implying that $B_\epsilon(\zeta)$ is contractible as subset of $\mathbb{P}_\circ$. Thus, $\mathbb{P}_\circ$ is locally simply connected as well. □

Observe the above lemma implies $\mathrm{Fib}_{1/2} : \mathsf{Cov}(\mathbb{P}_\circ) \to \pi\text{–}\mathsf{Set}$ is an equivalence of categories.

Definition 2.1.11. Define the **fiber functor** $\mathrm{Fib} : \mathsf{Cov}(\mathbb{P}_\circ)_\mathsf{f} \to \pi\text{–}\mathsf{Set}_\mathsf{f}$ as the restriction of $\mathrm{Fib}_{1/2} : \mathsf{Cov}(\mathbb{P}_\circ) \to \pi\text{–}\mathsf{Set}$ to the finite case.

Lemma 1.2.4 and the fact that $\mathrm{Fib}_{1/2} : \mathsf{Cov}(\mathbb{P}_\circ) \to \pi\text{–}\mathsf{Set}$ is an equivalence imply that $|\mathrm{Fib}_{1/2}\,\mathcal{X}|$ is a finite set if and only if $\mathcal{X}$ is a finite covering over $\mathbb{P}_\circ$, so Fib is well-defined on objects and essentially surjective. Furthermore, because $\mathsf{Cov}(\mathbb{P}_\circ)_\mathsf{f}$ resp. $\pi\text{–}\mathsf{Set}_\mathsf{f}$ are precisely the full subcategories of $\mathsf{Cov}(\mathbb{P}_\circ)$ resp. $\pi\text{–}\mathsf{Set}$ whose objects have finite underlying sets resp. are finite coverings over $\mathbb{P}_\circ$, Fib is well-defined on morphisms and fully faithful. Therefore:

Proposition 2.1.12. *The fiber functor is an equivalence from $\mathsf{Cov}(\mathbb{P}_\circ)_\mathsf{f}$ to $\pi\text{–}\mathsf{Set}_\mathsf{f}$.* □

Example 2.1.13. Let again $q \in \mathbb{N}_{\geq 1}$, take the finite covering $\mathcal{X}_q = (X_q, p_q)$ over $\mathbb{P}_\circ$ from Example 1.2.5 and the finite π-set $\mathcal{S}_q$ from Example 1.2.10. Using the previous example, identify the space X_q with $\{(\xi : \eta : 1) \in \mathbb{P}^2 \mid \xi^q + \eta^q = 1, \xi, \eta \notin \mu_q\}$ and p_q with the mapping $(\xi : \eta : 1) \mapsto \xi^q$. For $x \in \mathbb{R}_{\geq 0}$, let $\sqrt[q]{x}$ be the q-th positive root of x.

Denote $\sqrt[q]{2^{-1}}(\vartheta_q^n : \vartheta_q^m : 1)$ by $x_{n,m}$, and for $s \in I$, define $\xi(s)$ resp. $\eta(s)$ as $\sqrt[q]{2^{-1}}\exp(2\pi i(n+s)/q)$ resp. $\vartheta_q^m \sqrt[q]{(1-\xi(s)^q)}$. Now the fiber of p_q above $1/2$ equals $\{x_{n,m} \mid n, m \in \mathbb{Z}\}$, and $\tilde{\gamma}_B : I \to X_q$, mapping s to $(\xi(s) : \eta(s) : 1)$, is a lifting of a representative of σ_B, starting at $x_{n,m}$ and with $\tilde{\gamma}_B(1) = x_{n+1,m}$. Thus $\sigma_B x_{n,m} = x_{n+1,m}$. Likewise, p_q sends $(\xi : \eta : 1)$ to $\xi^q = 1 - \eta^q$, so with similar arguments as for σ_B, we get $\sigma_W x_{n,m} = x_{n,m+1}$. Therefore, $\mathrm{Fib}\,\mathcal{X}_q$ and $\mathcal{S}_q$ are isomorphic as finite π-sets, under the identification of $x_{n,m} \in p_q^{-1}(1/2)$ with $(n, m) \in (\mathbb{Z}/q\mathbb{Z})^2$.

From Finite π-Sets to Dessins d'Enfants

Recall that a dessin is a bicolored graph $\mathcal{G}$ endowed with a cyclic structure, i.e. a cyclic order $\mathrm{C}v$ on $\mathrm{E}v$ for each vertex v of $\mathcal{G}$, and that a morphism of dessins $\varphi : \mathcal{D} \to \mathcal{D}'$ is a graph morphism on the underlying bicolored graphs such that $\varphi| : \mathrm{E}v \to \mathrm{E}\varphi(v)$ is order preserving with respect to $\mathrm{C}v, \mathrm{C}\varphi(v)$ for each $v \in \mathrm{V}|\mathcal{D}|$ (and surjective by definition of a graph morphism).

We moreover showed an isomorphism $\mathsf{Trans}_\mathsf{f} \cong \mathsf{Cyc}_\mathsf{f}$ and gave natural $\mathbb{Z}$-actions on the orbits $\langle\sigma_X\rangle s$ under $\sigma_X \in \{\sigma_B, \sigma_W\}$ of $s \in |\mathcal{S}|$ for a given finite π-set $\mathcal{S}$.

[32] See [12], Constr. 2.3.3.

[33] See [12], Thm. 2.3.4. Note that in the literature, the functor Fib_x is called the fiber functor as well. However, because in this thesis we mainly consider finite coverings over $\mathbb{P}_\circ$ resp. finite π-sets, no confusion should arise.

Definition 2.1.14. Let $\mathcal{S}$ be a finite π-set.

(i) For $s \in |\mathcal{S}|$, call $\Sigma_B(s) := \langle\sigma_B\rangle s \times \{\bullet\}$ resp. $\Sigma_W(s) := \langle\sigma_W\rangle s \times \{\circ\}$ the *disjoint orbits of s under σ_B resp. σ_W*. Define $\mathrm{B}_\mathcal{S} := \{\Sigma_B(s) \mid s \in S\}, \mathrm{W}_\mathcal{S} := \{\Sigma_W(s) \mid s \in S\}$, the functions:
$$\mathrm{b}_\mathcal{S} : |\mathcal{S}| \to \mathrm{B}_\mathcal{S};\quad s \mapsto \Sigma_B(s); \qquad \& \qquad \mathrm{w}_\mathcal{S} : |\mathcal{S}| \to \mathrm{W}_\mathcal{S};\quad s \mapsto \Sigma_W(s),$$
and write $\mathcal{G}_\mathcal{S}$ for the quintuple $(\mathrm{E}_\mathcal{S}, \mathrm{B}_\mathcal{S}, \mathrm{W}_\mathcal{S}, \mathrm{b}_\mathcal{S}, \mathrm{w}_\mathcal{S})$, where $\mathrm{E}_\mathcal{S} := |\mathcal{S}|$.[34]

(ii) Denote the natural cyclic orders on $\Sigma_B(s)$ resp. $\Sigma_W(s)$ by $\mathrm{C}\Sigma_B(s)$ resp. $\mathrm{C}\Sigma_W(s)$ for each $s \in |\mathcal{S}|$, the set $\{\mathrm{C}\Sigma_B(s), \mathrm{C}\Sigma_W(s) \mid s \in S\}$ by $\mathcal{R}_\mathcal{S}$ and the pair $(\mathcal{G}_\mathcal{S}, \mathcal{R}_\mathcal{S})$ by $\mathcal{D}_\mathcal{S}$.

Proposition 2.1.15. *We have a functor* $\mathrm{Orb} : \pi\text{–}\mathsf{Set}_\mathsf{f} \to \mathsf{Des}$*, sending a finite π-set $\mathcal{S}$ to $\mathcal{D}_\mathcal{S}$ and an equivariant map φ to φ.*

Proof. For a given finite π-set $\mathcal{S}$, it is clear that $\mathcal{G}_\mathcal{S}$ is a finite bicolored graph with cyclic structure $\mathcal{R}_\mathcal{S}$, so $\mathcal{D}_\mathcal{S}$ is a dessin. Now let $\varphi : \mathcal{S} \to \mathcal{S}'$ be a $\pi\text{–}\mathsf{Set}_\mathsf{f}$-morphism and $\Sigma_X(s)$ a vertex of $\mathcal{D}_\mathcal{S}$. Then because φ is equivariant we have $\varphi[\langle\sigma_X\rangle s] = \langle\sigma_X\rangle\varphi(s)$, thus φ is a graph morphism $\mathcal{G}_\mathcal{S} \to \mathcal{G}_{\mathcal{S}'}$.

Next notice the natural $\mathbb{Z}$-actions ρ_s on $\langle\sigma_X\rangle s$ and $\rho_{\varphi(s)}$ on $\langle\sigma_X\rangle\varphi(s)$ make the restriction $\varphi| : \langle\sigma_X\rangle s \to \langle\sigma_X\rangle\varphi(s)$ a $\mathsf{Trans}_\mathsf{f}$-morphism. Because under the isomorphism $\mathsf{Cyc}_\mathsf{f} \cong \mathsf{Trans}_\mathsf{f}$ the cyclic orders $\mathrm{C}\Sigma_X(s)$ resp. $\mathrm{C}\Sigma_X(\varphi(s))$ are induced by ρ_s resp. $\rho_{\varphi(s)}$, the restriction $\varphi|$ is order preserving and thus the function φ is a dessin morphism $\mathcal{D}_\mathcal{S} \to \mathcal{D}_{\mathcal{S}'}$.

The claim that Orb is a functor follows straightforwardly from a direct argument. We give a somewhat indirect approach, which is more illustrative. For this, first notice $\pi\text{–}\mathsf{Set}_\mathsf{f}$ is constructed as concrete over Set. Let $U_\pi : \pi\text{–}\mathsf{Set}_\mathsf{f} \to \mathsf{Set}$ be the associated forgetful functor, i.e. with $U_\pi\mathcal{S} = |\mathcal{S}|$ and $U_\pi\varphi = \varphi$ for finite π-sets $\mathcal{S}$ and $\pi\text{–}\mathsf{Set}_\mathsf{f}$-morphisms φ. Next, Des is constructed as concrete over Bic, which in turn is concrete over Set. Let $U_D : \mathsf{Des} \to \mathsf{Set}$ be its associated forgetful functor, i.e. with $U_D\mathcal{D} = \mathrm{E}\mathcal{D}$ and $U_D\psi = \psi$ for dessins $\mathcal{D}$ and dessin morphisms ψ. Then $U_D \circ \mathrm{Orb} = U_\pi$, from which the claim follows. □

Definition 2.1.16. We call $\mathrm{Orb} : \pi\text{–}\mathsf{Set}_\mathsf{f} \to \mathsf{Des}$ the **orbit functor**.

Example 2.1.17. For $q \in \mathbb{N}_{\geq 1}$, let $\mathcal{S}_q$ resp. $\mathcal{D}_q$ be as in Example 1.2.10 resp. 1.3.17. Then $\mathrm{Orb}\,\mathcal{S}_q \cong \mathcal{D}_q$.

From Hypermaps to Dessins d'Enfants

Finally, we remind the reader that a hypermap $\mathcal{H}$ is a triple $(|\mathcal{H}|, \Sigma\mathcal{H}, g\mathcal{H})$ with $|\mathcal{H}|$ a finite bicolored graph, $\Sigma\mathcal{H}$ a compact oriented surface and $g\mathcal{H}$ an embedding of $|\mathcal{H}|$ into $\Sigma\mathcal{H}$. A hypermorphism $(\varphi, [f]) : \mathcal{H} \to \mathcal{H}'$ consists of a graph morphism $\varphi : |\mathcal{H}| \to |\mathcal{H}'|$ and the equivalence class of a coSurf-morphism $f : \Sigma\mathcal{H} \to \Sigma\mathcal{H}'$ associated to φ under homotopy relative to $[\mathcal{H}]$.

Lemma 2.1.18. *For each hypermap $\mathcal{H}$, the orientation on $\Sigma\mathcal{H}$ induces a cyclic structure $\mathcal{R}\mathcal{H}$ on $|\mathcal{H}|$ such that $\mathcal{D}_\mathcal{H} := (|\mathcal{H}|, \mathcal{R}\mathcal{H})$ is a dessin and such that for each hypermorphism $(\varphi, [f]) : \mathcal{H} \to \mathcal{H}'$, the graph morphism φ becomes a dessin morphism $\mathcal{D}_\mathcal{H} \to \mathcal{D}_{\mathcal{H}'}$.*

Proof. Let $\mathcal{H}$ be a hypermap. For the first claim observe the orientation on $\Sigma\mathcal{H}$ induces a small, oriented circle around each vertex v on $\Sigma\mathcal{H}$ and thus a cyclic order R_v on the set $\mathrm{E}v$ of the closed edges on $\Sigma\mathcal{H}$ that are connected to v, as was shown in the proof of Proposition 1.4.13. We use these cyclic orders, together with the mapping $g\mathcal{H}$ (a homeomorphism onto its image), to induce a cyclic structure $\mathcal{R}\mathcal{H}$ on $|\mathcal{H}|$ in an obvious way. It is thus clear that $\mathcal{D}_\mathcal{H}$, defined as $(|\mathcal{H}|, \mathcal{R}\mathcal{H})$, is a dessin.

Now let $(\varphi, [f_0]) : \mathcal{H} \to \mathcal{H}'$ be a hypermorphism. The proof of Proposition 1.4.13 moreover showed each representative $f : \Sigma\mathcal{H} \to \Sigma\mathcal{H}'$ of $[f_0]$ induces order preserving functions $(\mathrm{E}u, R_u) \to (\mathrm{E}f(u), R_{f(u)})$ for each vertex u on $\Sigma\mathcal{H}$. Because $f|_{[\mathcal{H}]} = g\mathcal{H}' \circ \hat{\varphi} \circ g\mathcal{H}^{-1}$, it follows that φ is order preserving at $\mathrm{E}v \to \mathrm{E}\varphi(v)$ for each vertex v of $|\mathcal{H}|$ as well, which shows the second claim. □

Proposition 2.1.19. *Sending each hypermap $\mathcal{H}$ to its associated dessin $\mathcal{D}_\mathcal{H}$ and each hypermorphism $(\varphi, [f])$ to φ is a functor from* HoHyp *to* Des *(denoted by* Cut*).*

Proof. Notice for the forgetful functors $U_H : \mathsf{HoHyp} \to \mathsf{Bic}$ and $U_D : \mathsf{Des} \to \mathsf{Bic}$, we have $\mathrm{Cut} \circ U_D = U_H$.[35] Thus, with the previous lemma, the claim follows. □

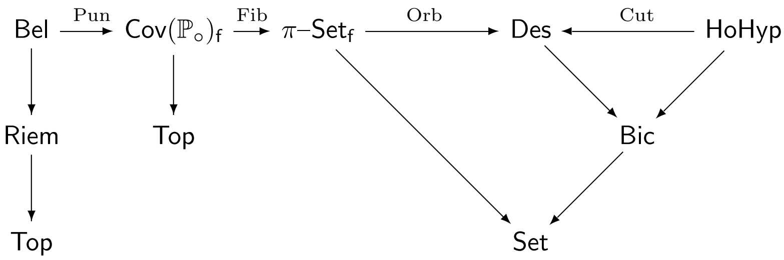
Definition 2.1.20. We call $\mathrm{Cut} : \mathsf{HoHyp} \to \mathsf{Des}$ the **cut functor**.

[34] Note that we add formal points $\bullet, \circ$ to the orbits $\langle\sigma_B\rangle s, \langle\sigma_W\rangle s$ to make them disjoint.

[35] See the proof of Proposition 2.1.15 resp. Lemma 1.4.20 for the forgetful functors U_D resp. U_H.

2.2 Proof

The main results up until now can be summarized in the following commutative diagram of functors:



with the forgetful functors as downward arrows and Fib already an equivalence. In this part we show that Pun, Orb and Cut are equivalences as well. First notice for each of these functors, say F from A to B, that F sends the empty object $A_\varnothing$ of A to the empty object of B.[36] Furthermore, for each A-object X, the hom-set restrictions $F^{A_\varnothing X}$ and $F^{X A_\varnothing}$ are both bijective.[37] Therefore, in the following we restrict ourselves to objects in the C-categories which are non-empty.

The Puncture Functor is an Equivalence

Proposition 2.2.1. *The functor* Pun *is essentially surjective.*

We first give the proof strategy and some definitions. The latter are used later on as well.

Approach and Notation. Let B_0 resp. B_1 be the open unit disc $D^2 \subset \mathbb{C} \subset \mathbb{P}$ resp. the open disc $B_1(1) \subset \mathbb{C} \subset \mathbb{P}$, and $B_\infty \subset \mathbb{P}$ the complement of $\bar{B}_{1/2}(1/2)$, which is open as well. For $i \in \{0, 1, \infty\}$, denote the punctured version $B_i - \{i\}$ of B_i by $\dot{B}_i$, choose a biholomorphic map $g_i : B_i \to D^2$ such that $g_i(i) = 0$ and let $\dot{g}_i$ be the restriction $g_i| : \dot{B}_i \to \dot{D}^2$, which is biholomorphic. Write $\dot{D}^2$ for $D^2 - \{0\}$ as well (this redundancy will aid the clearness of our proof).

Now let $\mathcal{X} = (X, p)$ be a finite covering over $\mathbb{P}_\circ$. Our approach is to associate a Belyi pair $\mathcal{B}_\mathcal{X}$ to $\mathcal{X}$ such that $\mathrm{Pun}\,\mathcal{B}_\mathcal{X} = \mathcal{X}$. For this we first show the following.

(i) For $i \in \{0, 1, \infty\}$, the open set $p^{-1}(\dot{B}_i)$ is a finite disjoint union of n_i open sets in X, denoted by $\dot{V}_{ij}$ with $1 \le j \le n_i$, such that each restriction $p| : \dot{V}_{ij} \to \dot{B}_i$, written as $p|_{ij}$, is a finite connected covering over $\dot{B}_i$;

(ii) For $x \in \dot{V}_{ij} \subset X$, we can take an open neighborhood U_x of x such that $p| : U_x \to p[U_x]$, denoted by $p|_x$, is a homeomorphism (and thus p is a local homeomorphism);

(iii) For $k \in \mathbb{N}_{>0}$, let $\dot{h}_k$ resp. h_k be the holomorphic map $\dot{D}^2 \to \dot{D}^2$ resp. $D^2 \to D^2$, both sending ζ to ζ^k. Then for each $i \in \{0, 1, \infty\}, 1 \le j \le n_i$, we have a homeomorphism $\dot{\varphi}_{ij} : \dot{V}_{ij} \to \dot{D}^2$ and some $k_{ij} \in \mathbb{N}_{>0}$ such that $\dot{h}_{k_{ij}} \circ \dot{\varphi}_{ij} = \dot{g}_i \circ p|_{ij}$. We may refer to this property by saying that p behaves like $\zeta \mapsto \zeta^{k_{ij}}$ on $\dot{D}^2$;

(iv) Now we can form a Hausdorff space $\tilde{X}$, containing X as subspace, by taking distinct copies D^2_{ij} of D^2 with $i \in \{0, 1, \infty\}$ and $1 \le j \le n_i$, the disjoint union of X with these copies, and by identifying points in the resulting space, using the mappings $\dot{\varphi}_{ij}$, i.e. by taking the quotient space

$$\tilde{X} := X \sqcup \left(\textstyle\bigsqcup_{ij} D^2_{ij} \right) \Big/ \sim \,,$$

with for $x \in X$ and $y \in D^2_{ij}$, $x \sim y$ if $\dot{\varphi}_{ij}(x) = y$. Define T as $\tilde{X} - X$;

(v) For each $\dot{V}_{ij}$ we have a unique open set $V_{ij} \subset \tilde{X}$ containing $\dot{V}_{ij}$ such that $V_{ij} - \dot{V}_{ij}$ contains exactly one point, which is denoted by y_{ij}. Now we can extend $\dot{\varphi}_{ij}$ to a homeomorphism $\varphi_{ij} : V_{ij} \to D^2$, i.e. with $\varphi_{ij}|_{\dot{V}_{ij}} = \dot{\varphi}_{ij}$ and $\varphi_{ij}(y_{ij}) = 0$.

The construction will give a complex structure Ψ on $\tilde{X}$, with charts (V_{ij}, φ_{ij}) for $i \in \{0, 1, \infty\}$ and $1 \le j \le n_i$, giving us a compact Riemann surface $\mathcal{M} := (\tilde{X}, \Psi)_m$, i.e. $\tilde{X}$ endowed with the unique maximal complex structure Ψ_m that contains Ψ. We can furthermore extend the map $p : X \to \mathbb{P}_\circ$ to a Belyi map $\tilde{p} : \mathcal{M} \to \mathbb{P}$. In conclusion, $\mathcal{B}_\mathcal{X} := (\mathcal{M}, \tilde{p})$ will be a Belyi pair that is sent to $\mathcal{X}$ under the puncture functor.

[36] See Rem. 1.1.20; 1.2.3; 1.2.9; 1.3.16; 1.4.9 for the definitions of the empty objects in each of the C-categories.

[37] This follows from the fact that for $A_\varnothing$ resp. X the empty object resp. a given object of the C-category A, the hom-sets $\mathrm{Hom}_\mathsf{A}(A_\varnothing, X)$ resp. $\mathrm{Hom}_\mathsf{A}(X, A_\varnothing)$ are equal to $\mathrm{Hom}_\mathsf{Set}(A_\varnothing, |X|)$ resp. $\mathrm{Hom}_\mathsf{Set}(|X|, A_\varnothing)$, with $|X|$ the underlying set of X.

We first show the points (i) - (v). Then, in a small intermezzo, recapitulate this proof and draw a plan for the remainder of the construction of $\mathcal{B}_\mathcal{X}$, to finish of course with carrying out this plan.

Proof of (i) - (v). Let $\mathcal{X} = (X, p)$ be a finite covering over $\mathbb{P}_\circ$ and pick $i \in \{0, 1, \infty\}$. Then the restriction $p|_i$ of p to $p^{-1}(\dot{B}_i) \to \dot{B}_i$ is a finite covering over $\dot{B}_i$. Thus, $p^{-1}(\dot{B}_i)$ is a finite disjoint union of its connected components, say $\dot{V}_{ij}$ with $1 \leq j \leq n_i$, making the restrictions $p|_{ij} : \dot{V}_{ij} \to \dot{B}_i$ of $p|_i$ finite connected coverings over $\dot{B}_i$. In particular, for each $x \in X$ we can find some $\dot{V}_{ij}$ containing x, and because $p|_{ij}$ is a covering, also an open neighborhood $U_x \subset \dot{V}_{ij}$ of x with $p| : U_x \to p[U_x]$ a homeomorphism. This shows (i) and (ii).

Now for (iii), we use the fact that each finite connected covering over $\dot{D}^2$ is isomorphic to $\dot{h}_k : \dot{D}^2 \to \dot{D}^2; \zeta \mapsto \zeta^k$ for some $k \in \mathbb{N}_{\geq 1}$.[38] In particular, for each $\dot{V}_{ij}$, the composition $\dot{g}_i \circ p|_{ij} : \dot{V}_{ij} \to \dot{D}^2$ is a finite connected covering over $\dot{D}^2$, which gives us the desired homeomorphism $\dot{\varphi}_{ij} : \dot{V}_{ij} \to \dot{D}^2$, i.e. such that the diagram:

$$\begin{array}{ccc} \dot{V}_{ij} & \xrightarrow{\dot{\varphi}_{ij}} & \dot{D}^2 \\ {\scriptstyle p|_{ij}}\downarrow & & \downarrow{\scriptstyle \dot{h}_{k_{ij}}} \\ \dot{B}_i & \xrightarrow{\dot{g}_i} & \dot{D}^2 \end{array}$$

commutes for some $k_{ij} \in \mathbb{N}_{\geq 1}$.

For (iv) we first need to check that setting $x \sim y$ if $x = y$ or if there is some $i \in \{0, 1, \infty\}$ and $1 \leq j \leq n_i$ with $\dot{\varphi}_{ij}(x) = y$ or $\dot{\varphi}_{ij}(y) = x$ is a well-defined equivalence relation on the space $X \sqcup \left(\bigsqcup_{ij} D^2_{ij} \right)$, which is straightforward. Notice this furthermore implies we can consider X as a subspace of $\tilde{X}$, which we do from hereon. To show that $\tilde{X}$ is Hausdorff, we first prove (v).

Notice we can embed each D^2_{ij} homeomorphically into $\tilde{X}$ by means of the projection π_{ij} of D^2_{ij} into $\tilde{X}$. Let V_{ij} be its image $\pi_{ij}[D^2_{ij}]$ in $\tilde{X}$. Then indeed $V_{ij} - \dot{V}_{ij}$ is the singleton $\{\pi(0)\}$. Denote this point by y_{ij}. Now extend $\dot{\varphi}_{ij}$ to $\varphi_{ij} : V_{ij} \to D^2$ by setting $\varphi_{ij}|_{\dot{V}_{ij}} = \dot{\varphi}_{ij}$ and $\varphi_{ij}(y_{ij}) = 0$. Then φ_{ij} is continuous, and has as two-sided continuous inverse $\pi_{ij}| : D^2_{ij} \to V_{ij}$. This shows (v).

Observe that T defined as $\tilde{X} - X$ equals $\{y_{ij} \mid i \in \{0, 1, \infty\}, 1 \leq j \leq n_i\}$. Now to finish our argument in (iv), note p is a local homeomorphism $X \to \mathbb{P}_\circ$ with finite fibers, so X is Hausdorff. Because T is discrete in $\tilde{X}$ and $\tilde{X} = X \cup T$, we only need to check that a given pair of point $x \in X$ resp. $y_{ij} \in T$ can be separated by open neighborhoods. It is clear we may assume $x \in V_{ij}$ (using p is a finite covering over $\mathbb{P}_\circ$), and with $V_{ij} \cong D^2$ the statement follows. □

Intermezzo. The proof of (i - v) showed we can think of X as a punctured version of a compact space (namely $\tilde{X}$), just like $\mathbb{P}_\circ$ is a punctured unit sphere. Now p behaves like $\zeta \mapsto \zeta^k$ on $\dot{D}^2$ for suitable $k \in \mathbb{N}_{\geq 1}$ around each puncture in X, i.e. missing point above $0, 1$ or ∞. Thus, 'patching up' $\mathcal{X}$, i.e. gluing open discs to X around each missing point, and extending p to $\tilde{X} \to \mathbb{P}$ should result in a ramified covering over $\mathbb{P}$ unramified outside $\{0, 1, \infty\}$. The claim is of course this can be made into a Belyi pair. Let us review what is still needed for this:

(a) The collection $\Psi := \{(V_{ij}, \varphi_{ij}) \mid i \in \{0, 1, \infty\}, 1 \leq j \leq n_i\}$ is a complex structure on $\tilde{X}$;
(b) The map $p : X \to \mathbb{P}_\circ$ can be extended to a holomorphism $\tilde{p} : (\tilde{X}, \Psi) \to (\mathbb{P}, \Psi_{\hat{\mathbb{C}}})$;[39]
(c) $\tilde{p}$ is unramified outside $\{0, 1, \infty\}$ and non-constant on each component $\tilde{X}_i$ of $\tilde{X}$;
(d) The pair $\mathcal{M} := (\tilde{X}, \Psi)_m$ is a compact Riemann surface;
(e) Therefore $\mathcal{B}_\mathcal{X}$, defined as $(\mathcal{M}, \tilde{p})$, is a Belyi pair such that $\operatorname{Pun} \mathcal{B}_\mathcal{X} = \mathcal{X}$.

Proof of Proposition 2.2.1. Retain the data and notation for the given finite covering $\mathcal{X} = (X, p)$ over $\mathbb{P}_\circ$ as in (i) - (v). First we show p can be extended to a map $\tilde{p} : \tilde{X} \to \mathbb{P}$. So set $\tilde{p}|_X = p$ and $\tilde{p}(y_{ij}) = i \in \{0, 1, \infty\}$ for each $y_{ij} \in T$. Then each restriction $\tilde{p}|_{ij} : V_{ij} \to B_i$ is continuous, so $\tilde{p}$ is.

For (a), suppose $(V_{ij}, \varphi_{ij}), (V_{st}, \varphi_{st}) \in \Psi$ are distinct charts on $\tilde{X}$ such that $V := V_{ij} \cap V_{st} \neq \varnothing$. Then the construction implies $i \neq s$, and it gives us the following commutative diagram:

$$\begin{array}{ccccc} \varphi_{st}[V] & \xleftarrow{\varphi_{st}} & V & \xrightarrow{\varphi_{ij}} & \varphi_{ij}[V] \\ {\scriptstyle h_{k_{st}}}\downarrow & & {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle h_{k_{ij}}} \\ g_s\tilde{p}[V] & \xleftarrow{g_s} & \tilde{p}[V] & \xrightarrow{g_i} & g_i\tilde{p}[V] \end{array}$$

[38] This follows from the universal covering $\{\zeta \in \mathbb{C} \mid \Re(\zeta) < 0\} \to \dot{D}^2$, sending ζ to $\exp(\zeta)$. See [12], Exmpl. 2.4.12.

[39] Recall that the complex structure $\Psi_{\hat{\mathbb{C}}}$ on $\hat{\mathbb{C}}$ is the set $\{(V_N, v_N), (V_Z, v_Z)\}$, with coordinate neighborhoods $V_N = \mathbb{C}$ resp. $V_Z = \hat{\mathbb{C}} - \{0\}$ and coordinate functions $v_N = \mathrm{id}_{V_N}$ resp. $v_Z(\zeta) = 1/\zeta$ for $\zeta \neq \infty$ and $v_Z(\infty) = 0$.

Now for $\zeta \in \varphi_{st}[V]$ we can find some open neighborhood $U_\zeta \subset \varphi_{st}[V]$ of ζ such that

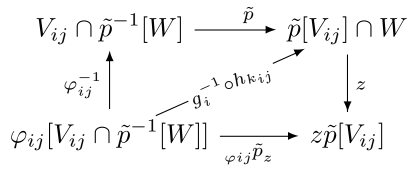
$$h_{k_{st}}| : U_\zeta \to h_{k_{st}}[U_\zeta] \qquad \& \qquad h_{k_{ij}}| : \left(h_{k_{ij}}^{-1} g_i g_s^{-1} h_{k_{st}}[U_\zeta]\right) \to \left(g_i g_s^{-1} h_{k_{st}}[U_\zeta]\right),$$

are both homeomorphisms (using $i \neq s$, thus $i, s \notin V$), i.e. such that

$$\left(\varphi_{ij} \circ \varphi_{st}^{-1} : U_\zeta \to \varphi_{ij}\varphi_{st}^{-1}[U_\zeta]\right) = \left(h_{k_{ij}}^{-1} \circ g_i \circ g_s^{-1} \circ h_{k_{st}} : U_\zeta \to \varphi_{ij}\varphi_{st}^{-1}[U_\zeta]\right).$$

Because the right-hand side of above equation is holomorphic, the left-hand side is as well. Therefore, $\varphi_{ij} \circ \varphi_{st}^{-1} : \varphi_{st}[V] \to \varphi_{ij}[V]$ is holomorphic and Ψ is indeed a complex structure on $\tilde{X}$.

Showing (b), i.e. that $\tilde{p}$ is holomorphic, is relatively straightforward: for each $(V_{ij}, \varphi_{ij}) \in \Psi$ and chart $(W, z) \in \Psi_{\hat{\mathbb{C}}}$ from Proposition 1.1.11, we again have a commutative diagram:



with ${}_{\varphi_{ij}}\tilde{p}_z = (z \circ \tilde{p} \circ \varphi_{ij}^{-1})$, which is holomorphic because $z \circ g_i^{-1} \circ h_{k_{ij}}$ is.

For (c), first suppose $x \in \tilde{X}$ is such that $\tilde{p}(x) \notin \{0, 1, \infty\}$. Then $x \in \dot{V}_{ij}$ for some $\dot{V}_{ij} \subset X$, giving us an open neighborhood U_x of x contained in $\dot{V}_{ij}$ with $\tilde{p}| : U_x \to \tilde{p}[U_x]$ a homeomorphism equal to $p|_x$. This implies $\varphi_{ij}|_{U_x} = \dot{\varphi}_{ij}|_{U_x}, g_i|_{\tilde{p}[U_x]} = \dot{g}_i|_{p[U_x]}$ and $h_{k_{ij}}|_{\varphi_{ij}[U_x]} = \dot{h}_{k_{ij}}|_{\dot{\varphi}_{ij}[U_x]}$. Moreover, in above commutative diagram, we can assume $z = \mathrm{id}$ because $\tilde{p}(x) \neq \infty$. Thus we have some $\epsilon \in \mathbb{R}_{>0}$ with $B_\epsilon(\dot{\varphi}_{ij}(x)) \subset \dot{\varphi}_{ij}[U_x]$ and $\{a_n \in \mathbb{C} \mid n \in \mathbb{N}\}$ such that for all $\zeta \in B_\epsilon(\dot{\varphi}_{ij}(x))$ it holds

$$\frac{d}{d\zeta}\,{}_{\varphi_{ij}}\tilde{p}_z(\zeta) = \sum_{n=0}^{\infty} a_n(\zeta - \dot{\varphi}_{ij}(x))^n = \frac{d}{d\zeta}(\dot{g}_i^{-1} \circ \dot{h}_{k_{ij}})(\zeta) = \frac{d}{d\zeta}\dot{g}_i(\zeta^{k_{ij}}) \cdot k_{ij}\zeta^{k_{ij}-1}.$$

Notice $\dot{g}_i$ is defined as a biholomorphic map into $\dot{D}^2$, so $\frac{d}{d\zeta}\dot{g}_i(\zeta^{k_{ij}}) \neq 0$ for each $\zeta \in B_\epsilon(\dot{\varphi}_{ij}(x))$. Moreover, k_{ij} by definition is not equal to zero, so if we take $\zeta = \dot{\varphi}_{ij}(x)$ in above equation, we get

$$a_0 = \frac{d}{d\zeta}\dot{g}_i(\dot{\varphi}_{ij}(x)^{k_{ij}}) \cdot k_{ij}\dot{\varphi}_{ij}(x)^{k_{ij}-1} \neq 0,$$

thus $e_{x \mapsto \tilde{p}(x)} = 1$. Therefore, $\tilde{p}$ is unramified outside $\{0, 1, \infty\}$.

For the second part of (c), let $\tilde{X}_k$ be a component of $\tilde{X}$. Then $X_k := \tilde{X}_k - T$ is a connected component of X and thus $p| : X_k \to \mathbb{P}_\circ$ is a finite, non-empty covering over $\mathbb{P}_\circ$. In particular, it is surjective, and thus $\tilde{p}| : \tilde{X}_k \to \mathbb{P}$ is non-constant.

To show that $\tilde{X}$ is compact, let $\mathcal{U}$ be an open cover of $\tilde{X}$. Notice for $V \subset \mathbb{P}$ such that $\tilde{p}^{-1}[V] \cap T = \varnothing$, by taking V small enough, we can guarantee that $\tilde{p}^{-1}[V]$ is a finite disjoint union of open sets V_i in $X \subset \tilde{X}$ such that for each V_i we have $\tilde{p}| : V_i \to V$ a homeomorphism and V_i contained in some $U \in \mathcal{U}$. Thus, by possibly applying a refinement of $\mathcal{U}$, we assume for each $U \in \mathcal{U}$ with $U \cap T = \varnothing$ that $\tilde{p}| : U \to \tilde{p}[U]$ is a homeomorphism such that $\tilde{p}^{-1}\tilde{p}[U]$ is a finite disjoint union of open sets in X, each an element of $\mathcal{U}$.

Now observe T is finite, so it has a finite subcover $\mathcal{U}_T \subset \mathcal{U}$. We are done if we show $\tilde{X} - \bigcup \mathcal{U}_T$ has one as well. For this, notice that $\tilde{p}[\mathcal{U}] := \{\tilde{p}[U] \mid U \in \mathcal{U}\}$ is an open cover for $\mathbb{P}$, because $\tilde{p}$ is an open, surjective map. Thus we have a finite subcover $\mathcal{V} \subset \tilde{p}[\mathcal{U}]$. By assumption, for each $V \in \mathcal{V}$ such that $\tilde{p}^{-1}[V] \cap T = \varnothing$ we have $\tilde{p}^{-1}[V] = \bigsqcup V_i$, with the V_i's a finite collection of elements from $\mathcal{U}$. Thus taking all inverse images of every $V \in \mathcal{V}$ such that $\tilde{p}^{-1}[V] \cap T = \varnothing$ induces a finite subcover of $\tilde{X} - \bigcup \mathcal{U}_T$. Therefore, $\tilde{X}$ is compact, which implies (d).[40]

Notice $\tilde{p}$ is a Belyi map on the compact Riemann surface $\mathcal{M}$, defined as $(\tilde{X}, \Psi)_m$. So $\mathcal{B}_\mathcal{X} := (\mathcal{M}, \tilde{p})$ is indeed a Belyi pair. Furthermore, it is clear that $\tilde{X} - \tilde{p}^{-1}\{0, 1, \infty\} = \tilde{X} - T = X$ and $\tilde{p}|_X = p$. So $\mathrm{Pun}\,\mathcal{B}_\mathcal{X} = (X, p) = \mathcal{X}$, which shows (e) and concludes our proof. □

The second part of the proof that Pun is an equivalence is of course the following proposition.

Proposition 2.2.2. *The functor* Pun *is fully faithful.*

For the proof of this proposition we first give a little lemma.

Lemma 2.2.3. *Belyi pairs $\mathcal{B}, \mathcal{B}'$ such that* $\mathrm{Pun}\,\mathcal{B} \cong \mathrm{Pun}\,\mathcal{B}'$ *as coverings over $\mathbb{P}_\circ$ are isomorphic.*

[40] We sketch a second argument. One may first use $\tilde{p}$ to show that $\tilde{X}$ is a second countable space, using that $\mathbb{P}$ has this property. Thus, if $\tilde{X}$ is sequentially compact, it is compact (see [8], Lem. 4.44). To see that $\tilde{X}$ is sequentially compact, one uses that $\tilde{p}$ has finite fibers, that $\{V_{ij}\}_{ij}$ is a finite open cover of $\tilde{X}$, and the fact that $\tilde{p} : |V_{ij} \to \tilde{p}[V_{ij}]$ behaves like $\zeta \mapsto \zeta^k$ on D^2 for suitable $k \in \mathbb{N}$, together of course with sequential compactness of $\mathbb{P}$.

Proof. First suppose $\mathcal{X}, \mathcal{X}'$ are isomorphic finite coverings over $\mathbb{P}_\circ$. Then from the construction in the proof of Proposition 2.2.1 it follows $\mathcal{B}_\mathcal{X} \cong \mathcal{B}_{\mathcal{X}'}$. Moreover, it is clear that $\mathcal{B}_{\mathrm{Pun}\,\mathcal{E}} \cong \mathcal{E}$ as Belyi pairs for each Bel-object $\mathcal{E}$. Thus, if $\mathcal{B}, \mathcal{B}'$ is a pair of Belyi pairs such that $\mathrm{Pun}\,\mathcal{B} \cong \mathrm{Pun}\,\mathcal{B}'$ as finite covering over $\mathbb{P}_\circ$, then indeed $\mathcal{B} \cong \mathcal{B}_{\mathrm{Pun}\,\mathcal{B}} \cong \mathcal{B}_{\mathrm{Pun}\,\mathcal{B}'} \cong \mathcal{B}'$ as Belyi pairs. □

Proof of Proposition 2.2.2. Let $\mathcal{B}, \mathcal{B}'$ be Belyi pairs and denote $\mathrm{Pun}\,\mathcal{B}$ resp. $\mathrm{Pun}\,\mathcal{B}'$ by $\mathcal{X} = (X, p)$ resp. $\mathcal{X}' = (X', p')$. Observe with the previous lemma, to show the hom-set restriction $\mathrm{Pun}^{\mathcal{B}\mathcal{B}'}$ from $\mathrm{Hom}\,(\mathcal{B}, \mathcal{B}')$ to $\mathrm{Hom}\,(\mathcal{X}, \mathcal{X}')$, sending a Belyi morphism $\psi : \mathcal{B} \to \mathcal{B}'$ to $\psi_\circ : \mathcal{X} \to \mathcal{X}'$, is bijective, it is enough to show that $\mathrm{Pun}^{\mathcal{B}_\mathcal{X}\mathcal{B}_{\mathcal{X}'}}$ is bijective. It is clear this hom-set restriction is injective, so only surjectivity remains.

Suppose $\varphi : \mathcal{X} \to \mathcal{X}'$ is a covering morphism. Then $\tilde{p}^{-1}[B_i] = \bigsqcup_{j=1}^{n_i} V_{ij}$ and $\tilde{p}'^{-1}[B_i] = \bigsqcup_{t=1}^{m_i} V_{it}$ for $i \in \{0, 1, \infty\}$, with notation as in the proof of Proposition 2.2.1. Because $p'\varphi = p$, for each $\dot{V}_{ij} \subset X$ we have a unique $\dot{V}_{it} \subset X'$ such that $\varphi[\dot{V}_{ij}] = \dot{V}_{it}$. In particular, we can extend φ to a map $\tilde{\varphi} : \tilde{X} \to \tilde{X}'$, sending $y_{ij} \in V_{ij} - \dot{V}_{ij}$ to the unique $y_{it} \in V_{it} - \dot{V}_{it}$ such that $\varphi[\dot{V}_{ij}] = \dot{V}_{it}$. With a diagram chase, similar to the one carried out in Proposition 2.2.1, we see that $\tilde{\varphi} : \tilde{X} \to \tilde{X}'$ is holomorphic with $\tilde{p}'\tilde{\varphi} = \tilde{p}$. So $\tilde{\varphi}$ is a Belyi morphism $\mathcal{B}_\mathcal{X} \to \mathcal{B}_\mathcal{X}$. Moreover, it is straightforward that $\mathrm{Pun}\,\tilde{\varphi} = \varphi$, so the puncture functor is indeed fully faithful. □

The Orbit Functor is an Equivalence

We show that the functor $\mathrm{Orb} : \pi\mathrm{-Set_f} \to \mathsf{Des}$, sending a finite π-set $\mathcal{S}$ to the associated dessin $\mathcal{D}_\mathcal{S}$, is an equivalence. Recall that the latter equals $(\mathcal{G}_\mathcal{S}, \mathcal{R}_\mathcal{S})$, with $\mathcal{G}_\mathcal{S}$ a finite bicolored graph with $|\mathcal{S}|$ as edges, the disjoint orbits of elements in $|\mathcal{S}|$ under σ_B resp. σ_W as black resp. white vertices and with $\mathcal{R}_\mathcal{S}$ the cyclic structure on $\mathcal{G}_\mathcal{S}$ given by the natural cyclic orders on these disjoint orbits. Now:

Proposition 2.2.4. *The functor* Orb *is essentially surjective.*

Proof. Let $\mathcal{D} = (\mathcal{G}, \mathcal{R})$ be a dessin. We give a π-action on $\mathrm{E}\mathcal{G}$ such that the resulting finite π-set is sent under Orb to a dessin isomorphic to $\mathcal{D}$. For this, first let v be a given vertex of $\mathcal{G}$. Then $\mathrm{E}v$ has a $\mathbb{Z}$-action induced by its cyclic order $\mathrm{C}v \in \mathcal{R}$, using $\mathsf{Trans_f} \cong \mathsf{Cyc_f}$. Denote this by $\rho_v : \mathbb{Z} \times \mathrm{E}v \to \mathrm{E}v$. Now let $e \in \mathrm{E}\mathcal{G}$ be an edge and b resp. w the black resp. white vertices of $\mathcal{G}$ that are the endpoints of e. If we set $\sigma_B e = \rho_b(1, e)$ and $\sigma_W e = \rho_w(1, e)$, then this rule indeed induces a π-action on $\mathrm{E}\mathcal{G}$. Denote the resulting finite π-set by $\mathcal{S}_\mathcal{D}$.

Now consider the identity function $\varphi : \mathrm{E}\mathcal{G} \to \mathrm{E}\mathcal{G}$. We claim φ is an isomorphism $\mathcal{D} \to \mathrm{Orb}\,\mathcal{S}_\mathcal{D}$. For this, let $v \in \mathrm{V}\mathcal{G}$ be a vertex. If $e, e' \in \mathrm{E}v$ are edges of $\mathcal{G}$, then $\langle \sigma_X \rangle e = \langle \sigma_X \rangle e' = \mathrm{E}v$ (for suitable $\sigma_X \in \{\sigma_B, \sigma_W\}$, depending on the color of v), and thus we have a vertex $\Sigma_X(e_v)$ of $\mathrm{Orb}\,\mathcal{S}_\mathcal{D}$ with $\varphi(v) = \Sigma_X(e_v)$, namely the disjoint orbit of e_v under σ_X for any $e_v \in \mathrm{E}v$. Therefore, φ is a bijective graph morphism $\mathcal{G} \to |\,\mathrm{Orb}\,\mathcal{S}_\mathcal{D}|$ and thus an isomorphism of bicolored graphs.

To see that φ is an isomorphism of dessins, let again $v \in \mathrm{V}\mathcal{G}$ be a vertex. Then $\varphi| : (\mathrm{E}v, \mathrm{C}v) \to (\mathrm{E}\Sigma_X(e_v), \mathrm{C}\varphi(v))$ is order preserving, which follows from the isomorphism $\mathsf{Cyc_f} \cong \mathsf{Trans_f}$. For the cyclic order $\mathrm{C}\varphi(v)$ on $\mathrm{E}\varphi(v) = \mathrm{E}\Sigma_X(e_v)$ is induced by the natural transitive $\mathbb{Z}$-action on $\mathrm{E}\Sigma_X(e_v) = \mathrm{E}v$, which in turn is induced by $\mathrm{C}v$ itself. Thus, $\mathrm{C}v = \mathrm{C}\varphi(v)$, and the claim follows. □

Because $\mathrm{Orb}\,\varphi = \varphi$ for $\pi\mathrm{-Set_f}$-morphisms φ, the second part of the proof that Orb is an equivalence is relatively straightforward. So let us continue.

Proposition 2.2.5. *The functor* Orb *is fully faithful.*

Proof. Let $\mathcal{S}, \mathcal{S}'$ be two finite π-sets. Notice the hom-set restriction $\mathrm{Orb}^{\mathcal{S}\mathcal{S}'}$ from $\mathrm{Hom}\,(\mathcal{S}, \mathcal{S}')$ to $\mathrm{Hom}\,(\mathcal{D}_\mathcal{S}, \mathcal{D}_{\mathcal{S}'})$ of Orb, sending φ to φ, is injective by construction.

To show $\mathrm{Orb}^{\mathcal{S}\mathcal{S}'}$ is surjective, suppose we are given a dessin-morphism $\varphi : \mathcal{D}_\mathcal{S} \to \mathcal{D}_{\mathcal{S}'}$, i.e. a function $\mathrm{E}\mathcal{D}_\mathcal{S} \to \mathrm{E}\mathcal{D}_{\mathcal{S}'}$ which is a graph morphism $|\mathcal{D}_\mathcal{S}| \to |\mathcal{D}_{\mathcal{S}'}|$ respecting the cyclic structures on $\mathcal{D}_\mathcal{S}, \mathcal{D}_{\mathcal{S}'}$. Then notice

$$\mathrm{E}\mathcal{D}_\mathcal{S} = \bigcup_{b \in \mathrm{B}\mathcal{D}_\mathcal{S}} \mathrm{E}b = \bigcup_{w \in \mathrm{W}\mathcal{D}_\mathcal{S}} \mathrm{E}w; \qquad \& \qquad \mathrm{E}\mathcal{D}_{\mathcal{S}'} = \bigcup_{b' \in \mathrm{B}\mathcal{D}_{\mathcal{S}'}} \mathrm{E}b' = \bigcup_{w' \in \mathrm{W}\mathcal{D}_{\mathcal{S}'}} \mathrm{E}w',$$

and thus $\varphi = \bigcup_{b \in \mathrm{B}\mathcal{D}_\mathcal{S}} \varphi|_b = \bigcup_{w \in \mathrm{W}\mathcal{D}_\mathcal{S}} \varphi|_w$, with $\varphi|_v$ the restriction $\varphi| : \mathrm{E}v \to \mathrm{E}\varphi(v)$ for each vertex $v \in \mathrm{V}\mathcal{D}_\mathcal{S}$. Now let $e \in \mathrm{E}\mathcal{D}_\mathcal{S}$ be an edge of $\mathcal{D}_\mathcal{S}$ and suppose $v \in \mathrm{V}\mathcal{D}_\mathcal{S}$ is a vertex connected to e, say a black one. Let s be the successor function on $(\mathrm{E}v, \mathrm{C}v)$. Then we have $\varphi|_v(\sigma_B x) = \varphi|_v(sx) = s\varphi|_v(x) = \sigma_B \varphi|_v(x)$ for all $x \in \mathrm{E}v$, again using $\mathsf{Trans_f} \cong \mathsf{Cyc_f}$.

Thus, $\varphi|_b$ resp. $\varphi|_w$ is equivariant for all black resp. white vertices b resp. w of $\mathcal{D}_\mathcal{S}$ with respect to $\langle \sigma_B \rangle$ resp. $\langle \sigma_W \rangle$, implying that φ is equivariant from $\mathcal{S}$ to $\mathcal{S}'$. Because $\mathrm{Orb}\,\varphi = \varphi$, we are done. □

The Cut Functor is an Equivalence

Only one equivalence remains for the proof of our main theorem. We first claim:

Proposition 2.2.6. *The functor* Cut *is essentially surjective.*

For the proof of this proposition we give the following lemma, using $\mathrm{Orb} \circ \mathrm{Fib} \circ \mathrm{Pun}$ is an equivalence from Bel to Des. Note the lemma actually does most of the work.

Lemma 2.2.7. *For each Belyi pair $\mathcal{B}$, we have a hypermap $\mathcal{H}$ associated to $\mathcal{B}$ such that* $\mathrm{Cut}\,\mathcal{H}$ *and* $\mathrm{Orb}\,\mathrm{Fib}\,\mathrm{Pun}\,\mathcal{B}$ *are isomorphic as dessins.*

Proof. Let $\mathcal{B} = (\mathcal{M}, f)$ be a Belyi pair and retain the notation as in the proof of Proposition 2.2.1. Observe, with Lemma 2.2.3, we can replace $\mathcal{B}$ by its isomorphic copy $\mathcal{B}_{\mathrm{Pun}\,\mathcal{B}}$. Now denote $\mathrm{Fib}\,\mathrm{Pun}\,\mathcal{B}$ by $\mathcal{S}$, and $\mathrm{Orb}\,\mathcal{S}$ by $\mathcal{D} = (\mathcal{G}, \mathcal{R})$, which equals $\mathrm{Orb}\,\mathrm{Fib}\,\mathrm{Pun}\,\mathcal{B}$. Then we claim:

(i) We have an injective map $g : \hat{\mathcal{G}} \to \mathcal{M}$ such that the union of edges on $\mathcal{M}$ equals $f^{-1}[I^\circ]$, while the black resp. white vertices on $\mathcal{M}$ are the fibers $f^{-1}(0)$ resp. $f^{-1}(1)$;

(ii) If we let Σ be $\mathcal{M}$ endowed with the orientation $\mathcal{O}$ as given in Remark 1.4.5 and define $\mathcal{H}$ as $(\mathcal{G}, \Sigma, g)$, then $\mathcal{H}$ is a hypermap such that $\mathrm{Cut}\,\mathcal{H} = \mathcal{D}$.

For (i), note that $I^\circ = (0,1) \subset \dot{B}_0 \cap \dot{B}_1$. Thus, $f^{-1}[I^\circ] \subset \bigsqcup_j \dot{V}_{ij}$ for both $i = 0$ and $i = 1$. From the fact that $h_{k_{ij}} \varphi_{ij} = g_i f|_{ij}$, it follows each $\dot{V}_{ij}$ contains k_{ij} disjoint homeomorphic copies of I°. We denote these by Γ_{ijt} with $1 \leq t \leq k_{ij}$ for $i = 0, 1$ and $1 \leq j \leq n_i$, and observe:

- The inverse image $f^{-1}(I^\circ)$ is equal to $\bigsqcup_{j,t} \Gamma_{ijt}$ for both $i = 0$ and $i = 1$;
- For each $\bar{\Gamma}_{ijt}$ we have unique j_0, j_1, t_0, t_1 with $\bar{\Gamma}_{ijt} = \Gamma_{0 j_0 t_0} \cup \{y_{0j_0}, y_{1j_1}\} = \Gamma_{1 j_1 t_1} \cup \{y_{0j_0}, y_{1j_1}\}$;
- Each restriction $f| : \bar{\Gamma}_{ijt} \to I$ is a homeomorphism.

Let $\gamma_{ijt} : I \to \bar{\Gamma}_{ijt}$ be the two-sided continuous inverse of $f| : \bar{\Gamma}_{ijt} \to I$, and write $\{\bar{\Gamma}_{ijt}\}_{ijt}$ for the collection $\{\bar{\Gamma}_{ijt} \mid i \in \{0,1\}, 1 \leq j \leq n_i, 1 \leq t \leq k_{ij}\}$.[41]

Now consider the dessin $\mathcal{D} = (\mathcal{G}, \mathcal{R})$. The graph $\mathcal{G}$ has as edges $\mathrm{E}\mathcal{G} = f^{-1}(1/2)$ and as black resp. white vertices $\mathrm{B}\mathcal{G}$ resp. $\mathrm{W}\mathcal{G}$ the disjoint orbits of elements $x \in f^{-1}(1/2)$ under σ_B resp. σ_W, with monodromy action as π-action on $f^{-1}(1/2)$. The colorings $\mathrm{b}\mathcal{G} : f^{-1}(1/2) \to \mathrm{B}\mathcal{G}$ resp. $\mathrm{w}\mathcal{G} : f^{-1}(1/2) \to \mathrm{W}\mathcal{G}$ send $x \in f^{-1}(1/2)$ to its orbit under σ_B resp. σ_W. Furthermore, we can write

$$f^{-1}(1/2) = \{x_{0jt} \mid 1 \leq j \leq n_0, 1 \leq t \leq k_{0j}\} = \{x_{1jt} \mid 1 \leq j \leq n_1, 1 \leq t \leq k_{1j}\}$$

such that $x_{ijt} \in \Gamma_{ijt}$ for each $i = 0, 1$. We use this to show that, with $[s, ijt]$ the equivalence class of $I \times \{x_{ijt}\}$ under $\approx$, we have a well-defined, injective map

$$g : \hat{\mathcal{G}} \to \mathcal{M}; \quad [s, ijt] \mapsto \gamma_{ijt}(s),$$

from the polyhedron $\hat{\mathcal{G}}$ into $\mathcal{M}$. It is clear that $x_{0jt} = x_{1kv}$ implies $\gamma_{0jt}|_{I^\circ} = \gamma_{1kv}|_{I^\circ}$.

Now suppose $(s, ijt) \approx (u, klv)$. If $s = 0 = u$ and $\mathrm{b}\mathcal{G}(x_{ijt}) = \mathrm{b}\mathcal{G}(x_{klv})$, then we have a unique V_{0j} containing both x_{ijt} and x_{klv}. We may assume $i = k = 0$, giving us $\gamma_{0jt}(0) = y_{0j} = \gamma_{0lv}(0)$. The case $s = 1 = u$ is similar: assuming $i = k = 1$ gives us some $V_{1j'}$ with $\gamma_{1jt}(1) = y_{1j'} = \gamma_{1lv}(1)$. Thus g is well-defined. Using the universal property of quotient spaces, it follows g is continuous as well.

To show that g is injective, suppose $\gamma_{ijt}(s) = \gamma_{klv}(u)$ for some $[s, ijt], [u, klv] \in \hat{\mathcal{G}}$. If $ijt = klv$, then $s = u$ because γ_{ijt} is an injective map, in this case identical to γ_{klv}. If $ijt \neq klv$ then either $s = 0 = u$ and $\mathrm{b}\mathcal{G}(x_{ijt}) = \mathrm{b}\mathcal{G}(x_{klv})$ or $s = 1 = u$ and $\mathrm{w}\mathcal{G}(x_{ijt}) = \mathrm{w}\mathcal{G}(x_{klv})$. Therefore, $g[s, ijt] = g[u, klv]$ implies $[s, ijt] = [u, klv]$, and g is indeed an injective map. This concludes (i).

For (ii), let Σ be the compact oriented surface $(\mathcal{M}, \mathcal{O})$ associated to $\mathcal{M}$ as given in Remark 1.4.5. Then the following remains to show that $\mathcal{H} := (\mathcal{G}, \Sigma, g)$ is a hypermap:

(a) For each component Σ_i of Σ there is a unique component $\mathcal{G}_i$ of $\mathcal{G}$ with $g^{-1}[\Sigma_i] = \hat{\mathcal{G}}_i$;

(b) $\Sigma - g[\hat{\mathcal{G}}]$ is a finite collection of disjoint open sets, each homeomorphic to D^2.

For (a), we have a bijection $\{\bar{\Gamma}_{ijt}\}_{ijt} \leftrightarrow f^{-1}(1/2)$ given by $\bar{\Gamma}_{ijt} \leftrightarrow x_{ijt}$, and thus an induced π-action on $\{\bar{\Gamma}_{ijt}\}_{ijt}$. Furthermore, the fiber functor identifies connected coverings over $\mathbb{P}_\circ$ with transitive π-sets. Thus π acts transitively on a subset X of $f^{-1}(1/2)$ if and only if X is the maximal element (with respect to inclusion) of all subsets of $f^{-1}(1/2)$ contained in the connected component of $\mathcal{M} = |\Sigma|$ containing X. So if Σ_i is a component of Σ, we have a transitive π-action on the edges on Σ_i and thus a unique component $\mathcal{G}_i$ of $\mathcal{G}$, satisfying the requirement $g^{-1}[\Sigma_i] = \hat{\mathcal{G}}_i$.

[41] Note each element of $\{\bar{\Gamma}_{ijt}\}_{ijt}$ is indexed exactly twice, i.e. for $\bar{\Gamma}_{ijt} \in \{\bar{\Gamma}_{ijt}\}_{ijt}$ we have a unique $\bar{\Gamma}_{i'j't'} \in \{\bar{\Gamma}_{ijt}\}_{ijt}$ such that $i \neq i'$ but with $\bar{\Gamma}_{ijt} = \bar{\Gamma}_{i'j't'}$.

For (b), notice that $\mathbb{C} - I \cong \dot{B}_\infty \cong \dot{D}^2$ as spaces, so $f^{-1}[\mathbb{C} - I]$ is a finite disjoint union $\bigsqcup_{j=1}^{n_\infty} \dot{W}_j$ of open sets such that each $\dot{W}_j$ contains $\dot{V}_{\infty j}$ and is homeomorphic to $\dot{D}^2$. It follows that $W_j := \dot{W}_j \cup \{y_{\infty j}\}$ is homeomorphic to D^2. So indeed:

$$\Sigma - g[\hat{\mathcal{G}}] = \Sigma - \bigcup_{i,j,t} \bar{\Gamma}_{ijt} = \mathcal{M} - f^{-1}[I] = \bigsqcup_{1 \le j \le n_\infty} W_j \cong \bigsqcup_{1 \le j \le n_\infty} D_j^2.$$

Note that (a) and (b) imply $\mathcal{H} = (\mathcal{G}, \Sigma, g)$ is indeed a hypermap, so what remains to show for (ii) is $\operatorname{Cut}\mathcal{H} = \mathcal{D}$. For this, recall that $\operatorname{Cut}\mathcal{H} = (\mathcal{G}, \mathcal{RH})$ with $\mathcal{RH}$ the cyclic structure on $\mathcal{G}$ induced by the orientation $\mathcal{O}$ on $\mathcal{M}$. We claim $\mathcal{R} = \mathcal{RH}$.

Let v be a vertex of $\mathcal{G}$, say a black one. Then v corresponds with a vertex on $|\Sigma|$, i.e. a point $y_{0j} \in V_{0j} \subset \mathcal{M}$ with $f(y_{0j}) = 0$ such that $\mathrm{E}v = \{x_{0j1}, x_{0j2}, ..., x_{0jk_{0j}}\} = V_{0j} \cap f^{-1}(1/2)$. Notice $\mathrm{C}v \in \mathcal{R}$ is just the cyclic order induced by the $\langle \sigma_B \rangle$-action on $\mathrm{E}v$. Index $\mathrm{E}v$ such that $\sigma_B x_{0jt} = x_{0j(t+1)}$ for $1 \le t < k$ and with $\sigma_B x_{0jk} = x_{0j1}$.

Now let $\mathrm{C}'v \in \mathcal{RH}$ be the cyclic order on $\mathrm{E}v$ coming from $\operatorname{Cut}\mathcal{H}$. Then if we take the orientation $\mu \in \mathcal{O}$ of $\mathcal{M}$ at y_{0j}, then μ has a representative $\sigma : \Delta^2 \to \mathcal{M}$ such that $\sigma| : \partial\Delta^2 \to \sigma[\partial\Delta^2]$ is a homeomorphism with $\sigma[\Delta^2]$ contained in V_{0j} and with $\{x_{0jt}\} \cap \sigma[\partial\Delta^2] = \{x_{0jt}\}$ for each $x_{0jt} \in \mathrm{E}v$. From the construction it follows for all $x, y, z \in \mathrm{E}v$ we have $\mathrm{C}'v(x, y, z)$ if and only if we meet $\sigma^{-1}(z)$ after $\sigma^{-1}(y)$ after $\sigma^{1}(x)$ while traversing $\partial\Delta^2$ in the counter-clockwise direction with respect to the standard orientation of $\mathbb{R}^3$. Furthermore, because f is non-constant on each component of $\mathcal{M}$, it is orientation preserving. We use this to show the successor functions on $(\mathrm{E}v, \mathrm{C}v)$ resp. $(\mathrm{E}v, \mathrm{C}'v)$ are identical.

If we identify $\partial\Delta^2$ with S^1 with $I/\{0,1\}$, by means of orientation preserving homeomorphisms, then $f \circ \sigma|_{\partial\Delta^2}$ is a representative of $\sigma_B^{k_{0j}} \in \pi$ (with some abuse of notation). Or, in other words, $f \circ \sigma|_{\partial\Delta^2}$ is homotopic to the loop $\psi : I/\{0,1\} \to \mathbb{P}_\circ$, sending s to $\gamma(k_{0j}s)$ with γ the representative $s \mapsto 1/2 \exp(2s\pi i)$ of σ_B. Thus ψ equals the k_{0j}-times concatenation $\gamma \odot \gamma \odot ... \odot \gamma$ of γ.

Denote the successor function on $(\mathrm{E}v, \mathrm{C}v)$ resp. $(\mathrm{E}v, \mathrm{C}'v)$ by s resp. s'. Then for each $x_{0jt} \in \mathrm{E}v$ we have $sx_{0jt} = \sigma_B x_{0jt}$. Furthermore, we can take liftings $\tilde{\gamma}_t : I \to \mathcal{M}$ of γ with $f \circ \tilde{\gamma}_t = \gamma$ and $\tilde{\gamma}_t(0) = x_{0jt}$ such that the closed loop $\tilde{\gamma}_1 \odot \tilde{\gamma}_2 \odot ... \odot \tilde{\gamma}_{k_{0j}}$, considered as map $\partial\Delta^2 \to \mathcal{M}$, is equal to $\sigma|_{\partial\Delta^2}$. Therefore, $s'x = \sigma_B x$ for each $x \in \mathrm{E}v$ as well, implying $s = s'$. Thus the cyclic orders $\mathrm{C}v$ and $\mathrm{C}'v$ are identical, and because v was arbitrary (the case $v \in \mathrm{W}\mathcal{G}$ is similar), $\mathcal{R}$ equals $\mathcal{RH}$, showing $\mathcal{D} = \operatorname{Cut}\mathcal{H}$. □

Proof of Proposition 2.2.6. Let $\mathcal{D}$ be a dessin. Then there is a Belyi pair $\mathcal{B}$ such that $\mathcal{D}$ and $\operatorname{Orb}\operatorname{Fib}\operatorname{Pun}\mathcal{B}$ are isomorphic. If we let $\mathcal{H}$ be the hypermap associated to $\mathcal{B}$ as given in Lemma 2.2.7, then indeed $\operatorname{Cut}\mathcal{H} = \operatorname{Orb}\operatorname{Fib}\operatorname{Pun}\mathcal{B} \cong \mathcal{D}$, so Cut is essentially surjective. □

To show Cut is an equivalence and thus concluding the proof of our main theorem, only full faithfulness of Cut remains. We first give one final lemma.

Lemma 2.2.8. *Hypermaps $\mathcal{H}, \mathcal{H}'$ such that $\operatorname{Cut}\mathcal{H} \cong \operatorname{Cut}\mathcal{H}'$ are isomorphic as hypermaps.*

Proof. Let $\mathcal{H} = (\mathcal{G}, \Sigma, g)$ and $\mathcal{H}' = (\mathcal{G}', \Sigma', g')$ be hypermaps such that $\operatorname{Cut}\mathcal{H} \cong \operatorname{Cut}\mathcal{H}'$, say with isomorphism $\varphi : \operatorname{Cut}\mathcal{H} \to \operatorname{Cut}\mathcal{H}'$, and denote $\operatorname{Cut}\mathcal{H}$ resp. $\operatorname{Cut}\mathcal{H}'$ by $\mathcal{D} = (\mathcal{G}, \mathcal{R})$ resp. $\mathcal{D}' = (\mathcal{G}', \mathcal{R}')$. Endow $\mathrm{E}\mathcal{G}$ and $\mathrm{E}\mathcal{G}'$ with π-actions ρ, ρ' coming from the equivalence $\operatorname{Orb} \circ \operatorname{Fib} \circ \operatorname{Pun}$. Then $\varphi : (\mathrm{E}\mathcal{G}, \rho) \to (\mathrm{E}\mathcal{G}', \rho')$ is equivariant and thus an isomorphism of finite π-sets.

Furthermore, let $\mathrm{E}\Sigma$ resp. $\mathrm{E}\Sigma'$ be the sets of closed edges on Σ resp. Σ'. Note we have an obvious one-to-one correspondence between $\mathrm{E}\mathcal{G}$ resp. $\mathrm{E}\mathcal{G}'$ on the one hand and $\mathrm{E}\Sigma$ resp. $\mathrm{E}\Sigma'$ on the other (given by $e_j \leftrightarrow g[I_j]$ resp. $e_{j'} \leftrightarrow g'[I_{j'}]$ for $e_j \in \mathrm{E}\mathcal{G}$ resp. $e_{j'} \in \mathrm{E}\mathcal{G}'$), thus giving us π-actions $\tilde{\rho}$ resp. $\tilde{\rho}'$ on $\mathrm{E}\Sigma$ resp. $\mathrm{E}\Sigma'$ induced by ρ resp. ρ'. Moreover, φ induces an obvious isomorphism $\tilde{\varphi} : (\mathrm{E}\Sigma, \tilde{\rho}) \to (\mathrm{E}\Sigma', \tilde{\rho}')$, sending $g[I_i] \in \mathrm{E}\Sigma$ to $g'\hat{\varphi}[I_j] \in \mathrm{E}\Sigma'$.

Now for a closed edge e on Σ resp. Σ' and distinct $\sigma_X, \sigma_Y \in \{\sigma_B, \sigma_W\}$, define the *$\sigma_X$-cycle* of e as $c_X(e) = \{e, \sigma_X e, \sigma_Y \sigma_X e, \sigma_X \sigma_Y \sigma_X e, ...\}$. It is clear for each subset A of $\mathrm{E}\Sigma$ resp. $\mathrm{E}\Sigma'$ that A is a σ_X-cycle for some $\sigma_X \in \{\sigma_B, \sigma_W\}$ if and only if $\cup A$ is the boundary of some face of $\mathcal{H}$ resp. $\mathcal{H}'$. Furthermore, $\tilde{\varphi}$ gives a one-to-one correspondence between σ_X-cycles in $\mathcal{H}$ and σ_X-cycles in $\mathcal{H}'$.

Define $\tilde{f}$ as the map $g' \circ \hat{\varphi} \circ g^{-1} : [\mathcal{H}] \to [\mathcal{H}']$, which is a homeomorphism such that for each vertex v' resp. closed edge e' on $\mathcal{H}'$, there is a unique vertex v resp. closed edge e on $\mathcal{H}$ equal to the inverse image of v' resp. e' under $\tilde{f}$, such that the restriction of $\tilde{f}$ to $v \to v'$ resp. to $e \to e'$ is a homeomorphism. The bijection between σ_X-cycles on Σ and σ_X-cycles on Σ' induced by $\tilde{\varphi}$ shows that for each face F' of $\mathcal{H}'$, we have a unique face F of $\mathcal{H}$ with ∂F equal to the inverse image of $\partial F'$ under $\tilde{f}$ such that $\tilde{f}| : \partial F \to \partial F'$ is a homeomorphism as well. Thus, using the associated

CW-structures to $\mathcal{H}$ and $\mathcal{H}'$, we extend $\tilde{f}$ to a homeomorphism $f : |\Sigma| \to |\Sigma'|$ cell-wise. It is clear that $g'\hat{\varphi} = fg$. So we are (almost) done if f is orientation preserving.

As before, the orientations on Σ resp. Σ' induce small, oriented circles around each point in Σ resp. Σ'. Now for each vertex v on Σ resp. Σ', the cyclic order on the set of closed edges on Σ resp. Σ' connected to v induced by $\mathcal{R}$ resp. $\mathcal{R}'$ is the same as the cyclic order induced by the small, oriented circle around v coming from the orientation on Σ resp. Σ'. Because φ preserves these orders, the local degree of f is 1 at the vertices on Σ. Therefore, each component of Σ contains at least one point such that f has local degree 1 at this point. From this the claim follows, using that Σ and Σ' are oriented and the fact that for each pair of points $x, y \in \Sigma$ contained in the same component, we have a finite number of coordinate neighborhoods $U_1, U_2, ..., U_k$ on Σ such that each $\bar{U}_i$ is homeomorphic to $\bar{D}^2$, with $x \in U_1, y \in U_k$ and $\bigcap_i \bar{U}_i$ homeomorphic to $\bar{D}^2$. It is clear that f^{-1} is orientation preserving as well and that $(\varphi^{-1}, [f^{-1}]) = (\varphi, [f])^{-1}$. Therefore, $\mathcal{H}$ and $\mathcal{H}'$ are isomorphic as hypermaps. □

Remark 2.2.9. Let $\mathcal{H}$ be a hypermap. Then we can associate the following data to $\mathcal{H}$, unique up to isomorphism. We have a Belyi pair $\mathcal{B} = (\mathcal{M}, f)$, a dessin $\mathcal{D} = (\mathcal{G}, \mathcal{R})$, an orientation $\mathcal{O}$ on $\mathcal{M}$ induced by the complex structure $\Phi\mathcal{M}$ such that $\Sigma := (\mathcal{M}, \mathcal{O})$ is a compact oriented surface, and an injective map $g : \hat{\mathcal{G}} \to \Sigma$ such that $\mathcal{H}$ is $(\mathcal{G}, \Sigma, g)$. Furthermore, the black resp. white vertices on Σ equal $f^{-1}(0)$ resp. $f^{-1}(1)$, while the edges on Σ equals $f^{-1}(I^\circ)$. The set of edges $\mathrm{E}\mathcal{G}$ of $\mathcal{G}$ itself equals the fiber of f above $1/2$, and the black resp. white vertices of $\mathcal{G}$ are the disjoint orbits of points in $f^{-1}(1/2)$ under σ_B resp. σ_W, with π-action on $\mathrm{E}\mathcal{G}$ induced by $\mathcal{R}$.

The homeomorphism $g : \hat{\mathcal{G}} \to [\mathcal{H}]$ identifies edges of $\mathcal{G}$ with edges on Σ and black resp. white vertices of $\mathcal{G}$ with black resp. white vertices on Σ bijectively, inducing a π-action on the edges on Σ, using the π-action on $\mathrm{E}\mathcal{G}$ induced by $\mathcal{R}$. The restriction $f| : \mathcal{M}\backslash f^{-1}\{0, 1, \infty\} \to \mathbb{P}_\circ$ is a finite covering over $\mathbb{P}_\circ$, and the monodromy action of π on the fiber $f^{-1}(1/2)$ induces a π-action on the edges on Σ as well, identical to the one coming from $\mathcal{R}$ and g. Thus, conversely, the cyclic structure $\mathcal{R}$ on $\mathcal{G}$ is identical to the cyclic structure induced on $\mathcal{G}$, using g, coming from this π-action (induced by monodromy) on the edges on Σ, and moreover identical to the cyclic structure induced by the orientation $\mathcal{O}$.

Notice the ramification index $e_{b\mapsto 0}$ resp. $e_{w\mapsto 1}$ of f equals the number of closed edges on Σ connected to the black vertex b resp. white vertex w on Σ. Moreover, for each face F of $\mathcal{H}$ we have a unique point $x_F \in F$ such that $f(x_F) = \infty$. With $k \in \mathbb{N}$ such that $e_{x_F\mapsto\infty} = k$, the finite connected covering $f| : F\backslash\{x_F\} \to \mathbb{C}\backslash I \cong \dot{D}^2$ is isomorphic to $\zeta \mapsto \zeta^k$ on $\dot{D}^2$.

Proposition 2.2.10. *The functor* Cut *is fully faithful.*

Proof. Let $\mathcal{H}, \mathcal{H}'$ be hypermaps. The hom-set restriction $\mathrm{Cut}^{\mathcal{H}\mathcal{H}'} : \mathrm{Hom}\,(\mathcal{H}, \mathcal{H}') \to \mathrm{Hom}\,(\mathcal{D}_{\mathcal{H}}, \mathcal{D}_{\mathcal{H}'})$, sending $(\varphi, [f])$ to φ, is injective by construction. To show it is surjective, let $\psi : \mathcal{D}_{\mathcal{H}} \to \mathcal{D}_{\mathcal{H}'}$ be a morphism of dessins.

Let $\mathcal{B}, \mathcal{B}'$ be Belyi pairs associated to $\mathcal{H}, \mathcal{H}'$. Then ψ induces a Belyi morphism $f : \mathcal{B} \to \mathcal{B}'$ such that $\mathrm{Orb}\,\mathrm{Fib}\,\mathrm{Pun}\, f = \psi$. Note that f is a coSurf-morphism with respect to the orientations induced by the complex structures on the underlying Riemann surfaces of $\mathcal{B}$ resp. $\mathcal{B}'$. Using above remark about the data unique up to isomorphism associated to $\mathcal{H}, \mathcal{H}'$, it is clear f is associated to ψ, so $(\psi, [f])$ is a hypermorphism from $\mathcal{H}$ to $\mathcal{H}'$ such that $\mathrm{Cut}(\psi, [f]) = \psi$, concluding our proof. □

Review 2.2.11. In conjunction, the propositions in this paragraph imply all C-categories are mutually equivalent, proving the equivalence theorem. Now let $\mathcal{M}$ be a compact, orientable surface with each component of $\mathcal{M}$ equal to a finite connected sum of tori or to a sphere. Then the following operations on $\mathcal{M}$ are all the same:

(i) Endowing $|\mathcal{M}|$ with a complex structure and the resulting Riemann surface with a Belyi map;
(ii) Removing a finite subset of $|\mathcal{M}|$ and giving a finite covering map on the remaining space to $\mathbb{P}_\circ$;
(iii) Doing (ii), and taking out the fiber above $1/2$, to endow it with monodromy action from π;
(iv) Drawing a connected, finite bicolored graph on each component of $\mathcal{M}$ such that the edges do not intersect and with the complement of the union of these graphs a finite union of open discs, and endowing $\mathcal{M}$ with an orientation;
(v) Doing the same as (iv), then using the orientation on $\mathcal{M}$ to endow the bicolored graph on $\mathcal{M}$ with a cyclic structure, and cutting out this bicolored graph with cyclic structure,

i.e. we can do these operations in such a way that the result is (i) a Belyi pair $\mathcal{B}$, (ii) a $\mathsf{Cov}(\mathbb{P}_\circ)_{\mathsf{f}}$-object $\mathcal{X}$, (iii) a finite π-set $\mathcal{S}$, (iv) a hypermap $\mathcal{H}$ and (v) a dessin $\mathcal{D}$ such that $\mathrm{Pun}\,\mathcal{B} = \mathcal{X}, \mathrm{Fib}\,\mathcal{X} = \mathcal{S}$ and $\mathrm{Orb}\,\mathcal{S} = \mathcal{D} = \mathrm{Cut}\,\mathcal{H}$. Furthermore, each object in any of the C-categories is (up to isomorphism) attained in this way, and the morphisms on each of these resulting objects are completely determined by the morphisms on any of the other associated objects, which is a nice result.

Appendix: Concrete Categories

Recall, as mentioned in the introduction, our definition of a concrete category is taken from [1], Def. 5.1. Thus, formally, a concrete category over a given base category X is a pair (A, U) such that A is a category and U a forgetful functor $\mathsf{A} \to \mathsf{X}$. We moreover agreed on the convention given in ibid., Rem. 5.3, so we tacitly assumed the forgetful functors in our concrete categories and considered, for a given concrete category (A, U) over X, the hom-sets of A as hom-sets of X (by means of U). In this appendix we explain in more detail how a concrete category over a given base category can be constructed and how the result is, in a certain sense, 'unique up to equivalence' (explained below).

Personally, I think this recipe is nice because it gives some convenience in constructing categories. Note for example that Des is constructed concrete over Bic, which in turn is concrete over Set. The advantage of this has been that the properties resp. well-definedness of dessin morphisms could be given resp. checked in stages. Moreover, we have introduced some notational convenience for all C-categories at once, using these are all concrete. Furthermore, in carrying out our recipe it seems to be obvious one has lots of choices in exactly how to add structure on objects in such a way that the resulting category is essentially the same. For example, it is to be expected that writing a chart on a space X as either a pair (U, z) or a triple (U, z, W), with U resp. W open in X resp. $\mathbb{R}^2$ (or $\mathbb{C}$) and z a homeomorphism $U \to W$, does not really matter. The uniqueness up to equivalence of constructed concrete categories makes this intuition more precise. Let us now be more concrete about our recipe, so that first this uniqueness can be explained in more detail, after which we show the promised results.

Let the base category X be given and say we want to construct A concrete over X. The idea is to select a class of X-objects, add structure on each of these selected objects, and to determine which X-morphisms respect these structures. This is formalized by taking a second category S for the structures associated to X-objects (note that although we only use S-objects, taking a whole category is convenient for our recipe). Now we endow an X-object with structure by associating an S-object to it. This is done by means of a property $\mathcal{P}$ of $\mathsf{X} \times \mathsf{S}$-objects, i.e. a subclass of the class of pairs (X, S) with X resp. S an X-object resp. an S-object. Likewise, selecting the X-morphisms that respect the associated structures is done by means of a second property $\mathcal{Q}$.

Now suppose (A, U) is a constructed concrete category over X with structure taken from S. Furthermore suppose (A', U') is a second constructed concrete category, this time over X' with structure taken from S', such that we have an equivalence $F : \mathsf{X} \to \mathsf{X}'$ of categories, a surjective class-function G from S-objects to S'-objects, and with S a structure on X if and only if $G(S)$ a structure on FX resp. φ a structure-respecting morphism if and only if $F\varphi$ is one as well, for each X-object X with structure S from S resp. each X-morphism φ that respects the structure from S. Then we have an equivalence of categories $H : \mathsf{A} \to \mathsf{A}'$ such that $U' \circ H = F \circ U$. This is the claim of uniqueness of constructed concrete categories. We will give H after the prove of our recipe, but first let us agree on some notation (only used in this appendix).

Notation. Let X, S be categories and say we want to construct A concrete over X with structure taken from S.

- For a property $\mathcal{P}$ of $\mathsf{X} \times \mathsf{S}$-objects, denote the class of $\mathsf{X} \times \mathsf{S}$-objects with property $\mathcal{P}$ by $\mathcal{P}(\mathsf{X} \times \mathsf{S})$, which will be the class of A-objects.
- Set $U(X, S) := X$ resp. $\Sigma(X, S) := S$ for all $\mathsf{X} \times \mathsf{S}$-objects (X, S). This will be the underlying object X resp. associated structure S of a given A-object (X, S).
- Denote the class of triples $(\mathcal{X}, f, \mathcal{Y})$ with $\mathcal{X}, \mathcal{Y} \in \mathcal{P}(\mathsf{X} \times \mathsf{S})$ and $f \in \mathrm{Hom}_{\mathsf{X}}(U\mathcal{X}, U\mathcal{Y})$ by $\mathcal{P}\,\mathrm{Mor}(\mathsf{X} \times \mathsf{S})$. The A-morphisms will be taken from the class $\mathcal{P}\,\mathrm{Mor}(\mathsf{X} \times \mathsf{S})$. We take triples to guarantee the disjointness of our hom-sets in A.
- Set $D(\mathcal{X}, f, \mathcal{Y}) := \mathcal{X}$, $U(\mathcal{X}, f, \mathcal{Y}) := f$ resp. $C(\mathcal{X}, f, \mathcal{Y}) := \mathcal{Y}$ for $(\mathcal{X}, f, \mathcal{Y}) \in \mathcal{P}\,\mathrm{Mor}(\mathsf{X} \times \mathsf{S})$, which will be the domain $\mathcal{X}$, underlying morphism f resp. codomain $\mathcal{Y}$ of an A-morphism $(\mathcal{X}, f, \mathcal{Y})$.
- For a property $\mathcal{Q}$ of elements of $\mathcal{P}\,\mathrm{Mor}(\mathsf{X} \times \mathsf{S})$, let $\mathcal{QP}\,\mathrm{Mor}(\mathsf{X} \times \mathsf{S})$ be the class of elements $\varphi \in \mathcal{P}\,\mathrm{Mor}(\mathsf{X} \times \mathsf{S})$ with property $\mathcal{Q}$, and for $\mathcal{X}, \mathcal{Y} \in \mathcal{P}(\mathsf{X} \times \mathsf{S})$ define $\mathcal{QP}\mathrm{Hom}(\mathcal{X}, \mathcal{Y})$ as the set $\{\varphi \in \mathcal{QP}\,\mathrm{Mor}(\mathsf{X} \times \mathsf{S}) \mid D(\varphi) = \mathcal{X}, C(\varphi) = \mathcal{Y}\}$. The class $\mathcal{QP}\,\mathrm{Mor}(\mathsf{X} \times \mathsf{S})$ will be the structure-respecting morphisms, i.e. the class of all A-morphisms, while the set $\mathcal{QP}\mathrm{Hom}(\mathcal{X}, \mathcal{Y})$ will be the A-morphisms $\mathcal{X} \to \mathcal{Y}$ for given A-objects $\mathcal{X}, \mathcal{Y}$.
- Now for $\varphi, \psi \in \mathcal{QP}\,\mathrm{Mor}(\mathsf{X} \times \mathsf{S})$ with $D\psi = C\varphi$, let $\psi \diamond \varphi$ be the triple $(D\varphi, U\psi \circ U\varphi, C\psi)$ and set $\overline{\mathrm{id}}_{\mathcal{Z}} := (\mathcal{Z}, \mathrm{id}_{U\mathcal{Z}}, \mathcal{Z})$ for all $\mathcal{Z} \in \mathcal{P}(\mathsf{X} \times \mathsf{S})$. Composition of A-morphisms will be given by $\diamond$ with identity $\overline{\mathrm{id}}_{\mathcal{Z}}$ on a given A-object $\mathcal{Z}$.

The recipe for the construction of a concrete category is formalized in the following lemma.

Lemma (Construction of Concrete Categories). *Let* X, S *be categories,* $\mathcal{P}$ *a property of* $\mathsf{X} \times \mathsf{S}$*-objects and* $\mathcal{Q}$ *a property of elements of* $\mathcal{P}\,\mathrm{Mor}(\mathsf{X} \times \mathsf{S})$ *such that:*

(i) For all $\varphi, \psi \in \mathcal{QP}\,\mathrm{Mor}(\mathsf{X} \times \mathsf{S})$*, if* $D\psi = C\varphi$*, then* $\psi \diamond \varphi \in \mathcal{QP}\,\mathrm{Mor}(\mathsf{X} \times \mathsf{S})$*;*

(ii) For all $\mathcal{X} \in \mathcal{P}(\mathsf{X} \times \mathsf{S})$*, property* $\mathcal{Q}$ *holds for* $\overline{\mathrm{id}}_{\mathcal{X}}$*.*

Then the pair (A, U) *with* $\mathsf{A} := (\mathcal{P}(\mathsf{X} \times \mathsf{S}), \mathcal{QP}\,\mathrm{Hom}, \overline{\mathrm{id}}, \diamond)$ *is a concrete category over* X.[42]

Proof. Requirement (i) implies the composition law $\diamond$ is well-defined on $\mathcal{QP}\,\mathrm{Mor}(\mathsf{X} \times \mathsf{S})$. The associativity of $\diamond$ and the neutrality of $\overline{\mathrm{id}}$ are both inherited from X. Moreover, the sets $\mathcal{QP}\mathrm{Hom}(\mathcal{X}, \mathcal{Y})$ are pairwise disjoint by construction. A is therefore indeed a category.

It is clear that U sends A objects to X-objects and A-morphisms $\varphi : \mathcal{X} \to \mathcal{Y}$ to X-morphisms $U\varphi : U\mathcal{X} \to U\mathcal{Y}$. Furthermore, we have $U(\psi \diamond \varphi) = U(D\varphi, U\psi \circ U\varphi, C\psi) = U\psi \circ U\varphi$ and $U\overline{\mathrm{id}}_{\mathcal{Z}} = \mathrm{id}_{U\mathcal{Z}}$ for all $\psi \diamond \varphi \in \mathcal{QP}\,\mathrm{Mor}(\mathsf{X} \times \mathsf{S})$ and $\mathcal{Z} \in \mathcal{P}(\mathsf{X} \times \mathsf{S})$. Thus, U is a functor from A to X.

Now suppose $\varphi, \psi : \mathcal{A} \to \mathcal{B}$ are A-morphisms such that $U\varphi = U\psi$. Then we have $\varphi = (\mathcal{A}, U\varphi, \mathcal{B}) = (\mathcal{A}, U\psi, \mathcal{B}) = \psi$. Therefore, for all A-objects $\mathcal{X}, \mathcal{Y}$, the hom-set restrictions $U^{\mathcal{X}\mathcal{Y}}$ from $\mathrm{Hom}_{\mathsf{A}}(\mathcal{X}, \mathcal{Y})$ to $\mathrm{Hom}_{\mathsf{X}}(U\mathcal{X}, U\mathcal{Y})$ are injective. So U is faithful, which remained to be shown. □

Notation. With data as in the lemma above, we call (A, U) a *constructed concrete category*. In the following, we denote it by $(\mathcal{QP}(\mathsf{X} \times \mathsf{S}), U)$.

Remark (Uniqueness of Constructed Concrete Categories). Let $\mathsf{X}, \mathsf{X}', \mathsf{S}, \mathsf{S}'$ be categories such that we have an equivalence $F : \mathsf{X} \to \mathsf{X}'$ and a surjective class-function G from S-objects to S'-objects (with the image of S under G written as GS). Suppose $(\mathcal{QP}(\mathsf{X} \times \mathsf{S}), U)$ and $(\mathcal{Q}'\mathcal{P}'(\mathsf{X}' \times \mathsf{S}'), U')$ are both constructed concrete categories, denoted by (A, U) resp. (A', U'). For $\mathsf{X} \times \mathsf{S}$-objects $\mathcal{X} = (X, S)$ set $H\mathcal{X} := (FX, GS)$ and for elements $\varphi = ((Y, T), f, (Z, V)) \in \mathcal{P}\,\mathrm{Mor}(\mathsf{X} \times \mathsf{S})$, define $H\varphi$ as $((FY, GT), Ff, (FZ, GV))$.

Now suppose $\mathcal{P}$ holds for $\mathcal{Y}$ if and only if $\mathcal{P}'$ holds for $H\mathcal{Y}$ and that $\mathcal{Q}$ holds for ψ if and only if $\mathcal{Q}'$ holds for $H\psi$ for all $\mathsf{X} \times \mathsf{S}$-objects $\mathcal{Y}$ and all $\psi \in \mathcal{P}\,\mathrm{Mor}(\mathsf{X} \times \mathsf{S})$. Then H is an equivalence from A to A' such that the following diagram of functors commutes:

$$\begin{array}{ccc} \mathsf{A} & \xrightarrow{H} & \mathsf{A}' \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U'} \\ \mathsf{X} & \xrightarrow{F} & \mathsf{X}' \end{array}$$

It is clear that H is a functor. For essential surjectivity, suppose (X', S') is an A'-object. Then we have an X-object X and an S-object S such that $H(X, S) = (X', S')$. Because $\mathcal{P}'$ holds for (X', S'), the property $\mathcal{P}$ holds for (X, S), so (X, S) is an A-object sent to (X', S') under H.

To see that H is fully faithful, let $\mathcal{X} = (X, S), \mathcal{Y} = (Y, T)$ be A-objects and φ, ψ morphisms $\mathcal{X} \to \mathcal{Y}$ in A such that $H\varphi = H\psi$. Then the X-morphisms $U\varphi$ and $U\psi$ are the same, so $\varphi = \psi$. Conversely, for a given A'-morphism $\tau : H\mathcal{X} \to H\mathcal{Y}$, we have some X-morphism $t : X \to Y$ such that $Ft = U'\tau$, and because $\mathcal{Q}'$ holds for $((FX, GS), U'\tau, (FY, GT)) = H(\mathcal{X}, t, \mathcal{Y})$, the triple $(\mathcal{X}, t, \mathcal{Y})$ is an A-morphism sent to τ under H. Thus $H^{\mathcal{X}\mathcal{Y}}$ is bijective, showing that H is an equivalence.

Verifying $U' \circ H = F \circ U$ is straightforward. Observe, because H is an equivalence, $U' \circ H$ is a forgetful functor, so $(\mathsf{A}, U' \circ H)$ is a concrete category over X' equal to $(\mathsf{A}, F \circ U)$.

[42] Here we use the definition of a category as given in [1], Def. 3.1.

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