B. Zwetsloot

Cardinal arithmetic: The Silver and Galvin-Hajnal Theorems

Bachelor thesis

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Introduction

When introduced to university-level mathematics for the first time, one of the first subjects to come up is basic set theory, as it is a necessary basis to understanding mathematics. In particular, the concept of cardinality of sets, being a measure of their size, is learned early. But what exactly are these cardinalities for objects? They turn out to be an extension of the natural numbers, originally introduced by Cantor: The *cardinal numbers*.

These being called numbers, it is not strange to see that some standard arithmetical operations, like addition, multiplication and exponentiation have extensions to the cardinal numbers. The first two of these turn out to be rather uninteresting when generalized, as for infinite cardinals κ, λ we have $\kappa + \lambda = \kappa \cdot \lambda = \max(\kappa, \lambda)$. However, exponentiation turns out to be a lot more complex, with statements like the (Generalized) Continuum Hypothesis that are independent of ZFC.

As the body of results on the topic of cardinal exponentiation grew in the 60's and early 70's, set theorists became more and more convinced that except for a relatively basic inequality, no real grip could be gained on cardinal exponentation. Jack Silver unexpectedly reversed this trend in 1974, when he showed that some cardinals, like $\aleph_{\omega_1}$, cannot be the first cardinal where GCH fails. Building on this idea of bounding the size of a cardinal through its predecessors, in the next year Fred Galvin and András Hajnal proved that if for any cardinal $\kappa < \aleph_{\omega_1}$ implies $2^\kappa < \aleph_{\omega_1}$, then

$$2^{\aleph_{\omega_1}} < \aleph_{(2^{\omega_1})^+}$$

We shall give elementary proofs of both of these theorems in ZFC, based on those found in [1], although we will prove the more general case. Some basic knowledge of what cardinals are and the operations on them will be assumed: Readers lacking such knowledge can find the necessary details in the appendix, or any basic material on set theory, e.g. [2].

1 Prerequisites

While the more basic theory of ordinals and cardinals is explained in the appendix, to discuss and prove the theorems of cardinal exponentiation in this thesis, some further background is required. In this first chapter, we discuss this background. Further treatment of this material can be found in [1], primarily chapters 3, 5 and 8.

1.1 Cofinality

The first piece of theory we will need is the cofinality of limit ordinals, or more specifically that of cardinals.

Definition 1.1. The *cofinality* of a limit ordinal α, written $\mathrm{cf}\,\alpha$, is the minimal limit ordinal β such that there exists a (strictly) increasing β-sequence $(\alpha_\gamma)_{\gamma<\beta}$ of ordinals α_γ with supremum α. A sequence of this form is called *cofinal* in α.

In a sense, the cofinality of an ordinal measures how large it is compared to the ones before it. By replacing α_γ by $\sup_{\beta<\gamma}\alpha_\beta$ whenever γ is a limit ordinal, we can create a sequence that is still cofinal in α (The sequences agree on all successor ordinals, after all), but has the nice property of continuity:

Definition 1.2. An increasing sequence $(\alpha_\gamma)_{\gamma<\beta}$ of ordinals is called *continuous* if for any limit ordinal $\delta<\beta$ we have $\sup_{\gamma<\delta}\alpha_\gamma=\alpha_\delta$. Such sequences are also called *normal*.

The reason we use increasing sequences here, is that for increasing sequences limits are equal to suprema. Our modified cofinal sequence now is normal by definition. So what we have just stated can be summarized as 'There is a normal cofinal sequence of length $\mathrm{cf}\,\alpha$ in α.' In fact, by this argument we could redefine $\mathrm{cf}\,\alpha$ as the minimal limit ordinal such that there exists a normal cofinal sequence of length $\mathrm{cf}\,\alpha$ in α.
Clearly, as the ascending α-sequence of all elements of α is cofinal in α, we always have $\mathrm{cf}\,\alpha\le\alpha$. As all infinite cardinals in particular are limit ordinals, all infinite cardinals also have a cofinality. It turns out that splitting the equality and inequality options is interesting for cardinals:

Definition 1.3. An infinite cardinal κ is called *regular* if $\mathrm{cf}\,\kappa=\kappa$, and *singular* if $\mathrm{cf}\,\kappa<\kappa$.

We only defined this for cardinals, because of this proposition:

Proposition 1.4. *For any limit ordinal α, the following are true:*
(1): $\mathrm{cf}\,(\mathrm{cf}\,\alpha)=\mathrm{cf}\,\alpha$
(2): $\mathrm{cf}\,\alpha$ *is a regular cardinal.*
(3): A set $A\subset\alpha$ with supremum α has $|A|\ge\mathrm{cf}\,\alpha$.

Proof. (1): We know $\mathrm{cf}\,(\mathrm{cf}\,\alpha)\le\mathrm{cf}\,\alpha$. Now if $(\alpha_\gamma)_{\gamma<\mathrm{cf}\,\alpha}$ is cofinal in α and $(\xi_\beta)_{\beta<\mathrm{cf}\,(\mathrm{cf}\,\alpha)}$ is cofinal in $\mathrm{cf}\,\alpha$, the sequence $(\alpha_{\xi_\beta})_{\beta<\mathrm{cf}\,(\mathrm{cf}\,\alpha)}$ is again cofinal in $\mathrm{cf}\,\alpha$, showing $\mathrm{cf}\,(\alpha)\le\mathrm{cf}\,(\mathrm{cf}\,\alpha)$ as well, finishing the proof of (1).

(2): As a surjection $|\alpha| \to \alpha$ allows us to enumerate α increasingly with $|\alpha|$, we see that $\text{cf } \alpha \leq |\alpha|$ always holds. By (1), it follows that $|\text{cf } \alpha| \geq \text{cf } (\text{cf } \alpha) = \text{cf } \alpha$, so $\text{cf } \alpha$ is a regular cardinal.
(3): The increasing enumeration of A is cofinal in α, so $|A| \geq \text{cf } \alpha$. □

As we will be focusing on cardinals in this thesis, it is nice to have some equivalent definitions for cofinality:

Proposition 1.5. *For any cardinals κ, λ, the following are equivalent:*
(1): $\text{cf } \kappa = \lambda$.
(2): λ is the minimal cardinal such that there exists subsets $A_\alpha \subset \kappa$ with $|A_\alpha| < \kappa$ for all $\alpha < \lambda$, such that $\bigcup_{\alpha<\lambda} A_\alpha = \kappa$.
(3): λ is the minimal cardinal such that there exists a sequence $(\kappa_\alpha)_{\alpha<\lambda}$ of cardinals smaller than κ such that $\sum_{\alpha<\lambda} \kappa_\alpha = \kappa$.

Proof. Define $\lambda_1 = \text{cf } \kappa$, and λ_2 and λ_3 as the minima referenced in (2) and (3). We want to prove $\lambda_1 = \lambda_2 = \lambda_3$.

$\lambda_2 = \lambda_3$: If you have a λ_3-sequence as in (3), then the sets $A_\alpha = \sum_{\beta<\alpha} \kappa_\beta$ are cardinals with supremum κ, so their union is κ, which means $\lambda_2 \leq \lambda_3$.
Conversely, if you have a λ_2-family of sets as in (2) then take $\kappa_\alpha = |A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta|$. As the union of the sets in the definition of the κ_α is still κ, but the sets are disjoint, we see that κ is the sum of the κ_α, and as we have $\kappa_\alpha \leq |A_\alpha| < \kappa$ they fulfill the requirements, so $\lambda_3 \leq \lambda_2$. Hence $\lambda_2 = \lambda_3$.

$\lambda_1 = \lambda_2$: If (1) holds, take the A_α to be a cofinal sequence: The supremum of an increasing sequence of ordinals is the union, so this family of sets works and $\lambda_2 \leq \lambda_1$.
Now let a λ_2-family of sets be given as in (2), and take α_β to be the order type of $\bigcup_{\alpha<\beta} A_\alpha$. This sequence in particular is non-decreasing, but its range in ascending order is increasing and has cardinality at most λ_2. By minimality of λ_2 we see that the α_β are smaller than κ, so their supremum is at most κ.

To prove it is at least κ, note that we can injectively map κ to $\lambda_2 \cdot \sup_{\beta<\lambda_2} \alpha_\beta$. How do we do so? For any $\gamma \in \kappa$ we take $f(\beta) = (\xi, \delta)$ where ξ is minimal such that $\beta \in A_\xi$ and δ is the order type of $\beta \cap A_\xi$. That this is in the product is easy to see. Injectivity follows as when $f(\beta) = f(\gamma) = (\xi, \delta)$ when $\gamma > \beta$, we have $\beta, \gamma \in A_\xi$. But then $\beta \in \gamma \cap A_\xi = \delta$, while $\beta \notin \beta \cap A_\xi = \delta$, which is a contradiction. The injectivity shows that $\kappa \leq |\lambda_2 \times \sup_{\beta<\lambda_2} \alpha_\beta| = \lambda_2 \cdot \sup_{\beta<\lambda_2} \alpha_\beta$. Now as κ is the union of its singletons, we have $\lambda_2 \leq \kappa$. If equality holds here, then $\kappa \geq \lambda_1 \geq \lambda_2 = \kappa$, so $\lambda_1 = \lambda_2 = \lambda_3$ and we are done. Otherwise, we have $\kappa \leq \sup_{\beta<\lambda_2} \alpha_\beta \leq \kappa$, so the supremum is κ and we have $\lambda_1 \leq \lambda_2$, proving $\lambda_1 = \lambda_2 = \lambda_3$ as we wanted. □

A logical question to ask is: Why are regular cardinals called regular? The answer to this is that they are much more numerous, as the following proposition will make clear:

Proposition 1.6. *Every successor cardinal is regular.*

Proof. Suppose I have a sequence $(\kappa_\alpha)_{\alpha<\mu}$ of cardinals smaller than successor κ^+ with sum κ^+. Now as $\kappa_\alpha < \kappa^+$, we have $\kappa_\alpha \leq \kappa$ for all $\alpha < \mu$. Hence

$$\kappa^+ = \sum_{\alpha<\mu} \kappa_\alpha \leq \sum_{\alpha<\mu} \kappa = \mu \cdot \kappa = \max(\mu, \kappa)$$

As $\kappa^+ > \kappa$, we see that $\kappa^+ \leq \mu$ must hold. As this holds for any μ, it in particular holds for the minimal such μ, which is $\mathrm{cf}\left((\kappa)^+\right)$ by Proposition 1.5. Hence $\kappa^+ \leq \mathrm{cf}\left((\kappa)^+\right) \leq \kappa^+$, so κ^+ is regular. □

Of course, singular cardinals exist as well: Simply take any cardinal of the form $\aleph_{\alpha+\omega}$. Such a cardinal is the union of the $\aleph_{\alpha+n}$ with $n \in \mathbb{N}$, and hence has cofinality at most ω by our second definition in 1.5. And as cofinalities are limit ordinals, and ω is the least limit ordinal, we see that all of these have cofinality ω. In fact, the following holds:

Proposition 1.7. *If $\aleph_\alpha$ is a limit cardinal with $\alpha > 0$, $\mathrm{cf}\,\aleph_\alpha = \mathrm{cf}\,\alpha$.*

Proof. If $(\beta_\xi)_{\xi<\mathrm{cf}\,\alpha}$ is cofinal in α, $\aleph_{\beta_\xi}$ is cofinal in $\aleph_\alpha$, as $\aleph_\alpha$ is the union of all smaller alephs, since it is a limit cardinal, and hence the unboundedness of the β_ξ in α shows that each aleph smaller than $\aleph_\alpha$ is contained in an $\aleph_{\beta_\xi}$. Hence $\mathrm{cf}\,\aleph_\alpha \leq \mathrm{cf}\,\alpha$.
If $(\beta_\nu)_{\nu<\mathrm{cf}\,\aleph_\alpha}$ is cofinal in $\aleph_\alpha$, on the other hand, define for each $\nu < \mathrm{cf}\,\aleph_\alpha$ the ordinal $\xi(\nu)$ to be the minimal $\xi > \xi(\iota)$ for all $\iota < \nu$ for which $\aleph_\xi > \beta_\nu$. Then clearly the supremum of the $\aleph_{\xi(\nu)}$ must be $\aleph_\alpha$, which can only hold if the supremum of the $\xi(\nu)$ is α. Hence $\mathrm{cf}\,\alpha \leq \mathrm{cf}\,\aleph_\alpha$, so equality holds. □

From this it follows that any limit cardinal $\aleph_\alpha$ with $\alpha > 0$ that is not a fixed point of the aleph function is singular, as $\mathrm{cf}\,\aleph_\alpha = \mathrm{cf}\,\alpha \leq \alpha < \aleph_\alpha$ in those cases.

Cofinality also leads us to some initial cardinal exponentiation results using our third definition: If we take a $\mathrm{cf}\,\kappa$-sequence (κ_α) of cardinals smaller than κ with sum κ, König's Theorem gives us that

$$\kappa = \sum_{\alpha<\mathrm{cf}\,\kappa} \kappa_\alpha < \prod_{\alpha<\mathrm{cf}\,\kappa} \kappa = \kappa^{\mathrm{cf}\,\kappa}$$

Furthermore, while we already know thanks to Cantor that $2^\kappa > \kappa$, in fact a stronger bound holds: If we have a κ-sequence (κ_α) of cardinals smaller than 2^κ, then König gives us that

$$\sum_{\alpha<\kappa} \kappa_\alpha < \prod_{\alpha<\kappa} 2^\kappa = (2^\kappa)^\kappa = 2^{\kappa^2} = 2^\kappa$$

By our third definition of cofinality, this implies the statement:

Proposition 1.8. *For any infinite cardinal κ, $\mathrm{cf}(2^\kappa) > \kappa$.*

1.2 Stationary sets

We will now continue in a different direction: We want to look at a certain type of 'large' sets in regular uncountable cardinals.

Definition 1.9. Let κ be a regular uncountable cardinal. A set $A \subset \kappa$ is called *unbounded* when $\sup A = \kappa$. A is called *closed* when for any increasing γ-sequence $(\alpha_\beta)_{\beta<\gamma}$ with all elements in A and $\gamma < \kappa$ we also have $\sup_{\beta<\gamma} \alpha_\beta \in A$. A closed unbounded set will be shortened as a *club*. A is called *stationary* if for any club $C \subset \kappa$, $A \cap C \neq \emptyset$.

Our definition of closed is equivalent to the definition Jech gives in chapter 8 of [1], which essentially states that a set is closed when it contains all its limit points under the order topology in κ, but not for the reason Jech gives: It is easily seen to be equivalent as κ is regular. In particular, as κ is regular and $\gamma < \kappa$, we have that the supremum of any increasing γ-sequence in A can never be κ itself, and hence always is an element of κ. To get more of a feeling for these objects, some examples:

Example 1.10.

1. κ itself is always both a club and stationary: Clubs are non-empty as they are unbounded, so the clubs in κ always intersect κ in a non-empty set, κ is clearly unbounded, and κ is closed by the last remark we made before this example.
2. Any final segment (a set of the form $A_\beta = \{\alpha < \kappa : \beta < \alpha\}$ with $\beta < \kappa$) is a club. Unboundedness is trivial: We are looking at a final segment of κ. For closedness, note that any increasing sequence in A_β of $< \kappa$ elements has supremum $< \kappa$ and at least its smallest element, which as it is in A_β is larger than β. So the supremum is again in A_β.

We have already touched on why we want our κ to be regular here. The reason we want κ to be uncountable is the following:

Theorem 1.11. *The intersection of fewer than κ clubs is again a club.*

Proof. To prove this theorem, we will use transfinite induction on the number of clubs.

(Base: 2 clubs) Let C and D be clubs. $C \cap D$ is easily seen to be closed: A γ-sequence in $C \cap D$ also lives in C and D, hence the supremum is in C, D and $C \cap D$. To prove $C \cap D$ is unbounded, let $\alpha < \kappa$ be given. Now sequentially take $\beta_1, \beta_3, \ldots \in C$ and $\beta_2, \beta_4, \ldots \in D$ such that $\alpha < \beta_1 < \beta_2 < \ldots$. This is possible, as C and D are unbounded. As the β_n are an ω-sequence and κ is uncountable, by C and D being closed we know that the supremum β of the β_n is in C and D, and hence in $C \cap D$, and $\beta > \alpha$. So for any $\alpha < \kappa$ we have a $\beta \in C \cap D$ such that $\beta > \alpha$, which means $C \cap D$ is unbounded. Hence $C \cap D$ is a club.

Continuing the induction on successors is easy: Let $(C_\beta)_{\beta \leq \alpha+1}$ be clubs and suppose the intersection of α clubs is again a club. Then $\bigcap_{\beta \leq \alpha} C_\beta$ is a club by the induction hypothesis, and $\bigcap_{\beta \leq \alpha+1} C_\beta = \bigcap_{\beta \leq \alpha} C_\beta \cap C_{\alpha+1}$ is a club by our basis. So let γ be a limit ordinal and suppose any intersection of fewer than γ clubs is a club. Now let $(C_\beta)_{\beta \leq \gamma}$ be clubs. The induction hypothesis allows us to replace C_β by $\bigcap_{\xi\leq\beta} C_\xi$ and get a decreasing family of clubs with the same intersection, so we may assume without loss of generality that the C_β are decreasing already. We now essentially do the same thing as for the case with two clubs: The intersection is closed, as any sequence in the intersection is also a sequence in all the intersected clubs, which has its supremum in all the intersected clubs, so the supremum is also in the intersection.

Let $\alpha < \kappa$ be given. We now construct a γ-sequence $(\beta_\xi)_{\xi<\gamma}$ such that $\beta_\xi \in C_\xi$, $\beta_\xi > \sup_{\nu<\xi} \beta_\nu$ for all $\xi < \gamma$, and $\beta_0 > \alpha$. This can be done as $\gamma < \kappa$, κ is regular, and the C_ξ are unbounded, and the supremum β of this sequence fulfills $\alpha < \beta < \kappa$. Now as $(\beta_\nu)_{\xi\leq\nu<\gamma}$ forms a sequence with at most length $\gamma < \kappa$ of elements in C_ξ, we see that as the C_ξ are closed β is in every C_ξ, so also in their intersection. Hence the intersection is also unbounded, so it is a club, finishing our induction. □

From this, we can immediately derive some more examples of stationary sets:

Corollary 1.12. *(1): Any club is a stationary set.*
(2): The intersection of a stationary set with fewer than κ clubs is stationary.

Proof. (1): The intersection of a club with any club is again a club, hence non-empty.
(2): If S is stationary and $(C_\beta)_{\beta<\gamma}, D$ are clubs with $\gamma < \kappa$, then $(S \cap \bigcap_{\beta<\gamma} C_\beta) \cap D = S \cap (\bigcap_{\beta<\gamma} C_\beta \cap D)$ is non-empty as the set in parentheses is a club, being an intersection of $\gamma + 1 < \kappa$ clubs. □

Intersection is not the only way to get more clubs:

Proposition 1.13. *The set of limit ordinals in a club is again a club.*

Proof. First note that the supremum of any sequence of limit ordinals will always be a limit ordinal. After all, if it were of the form $\alpha+1$, then as α is not an upper bound for the sequence and $\alpha+1$ is, $\alpha+1$ must be an element of our sequence, yet we had a sequence of limit ordinals, so that is not possible. By this argument, we see that if C is a club and D the set of limit ordinals in C, then the supremum of any γ-sequence of elements of D with $\gamma < \kappa$, which we know is in C as $D \subset C$, must again be in D, so D is closed. To show D is unbounded, note that for any $\alpha < \kappa$ we can find an increasing sequence $(\beta_n)_{n\in\mathbb{N}}$ such that all β_n are in C and $\beta_0 > \alpha$. Then the supremum β of this sequence must again be in C as C is closed and κ is regular and uncountable. However, it cannot be a successor, as if it were of the form $\gamma + 1$, then by the same argument as before $\gamma + 1$ must be an element of our sequence, say β_n, and then $\beta_{n+1} > \gamma + 1$, which means that $\gamma + 1$ is not an upper bound. Hence the supremum β is a limit and must be in D, so D is unbounded and hence a club. □

This of course leads to more stationary sets as well.

Corollary 1.14. *The set of limit ordinals in a stationary set is itself stationary.*

Proof. A stationary set has non-empty intersection with every club. But the set of limit ordinals in a club is again a club, so a stationary set also must have a limit ordinal in common with every club, which implies the limit ordinals in a stationary set are again stationary. □

While we are at it, let us define another operation under which the clubs are closed.

Definition 1.15. For a κ-sequence of sets $A_\alpha \subset \kappa$, we define the *diagonal intersection* $\Delta_{\alpha<\kappa} A_\alpha = \{\gamma < \kappa : \gamma \in \bigcap_{\alpha<\gamma} A_\alpha\}$.

Proposition 1.16. *The diagonal intersection of a κ-sequence $(C_\alpha)_{\alpha<\kappa}$ of clubs is a club.*

Proof. Because of how diagonal intersections work, if we replaced the C_α by $\bigcap_{\xi<\alpha} C_\xi$ we would get the same diagonal intersection, but have a decreasing sequence of clubs thanks to Theorem 1.11. Hence we can assume without loss of generality that the C_α are decreasing. Define $C = \Delta_{\alpha<\kappa} C_\alpha$. To prove C is closed, note that if we have an increasing γ-sequence $(\alpha_\xi)_{\xi<\gamma}$ in C with supremum α, then for each $\xi < \alpha$ we need to prove $\alpha \in C_\xi$. But for all ν such that $\alpha_\nu > \xi$, we have $\alpha_\nu \in C_\xi$, so C_ξ contains a final segment of our sequence. As C_ξ is closed, it then must also contain the supremum α, so $\alpha \in \bigcap_{\xi<\alpha} C_\xi$, and hence $\alpha \in C$. So C is closed.

To prove C is unbounded, let $\alpha < \kappa$ be given. We now construct an ω-sequence as follows: Take $\beta_0 \in C_0$ such that $\beta_0 > \alpha$, then for each n take $\beta_{n+1} \in C_{\beta_n}$ with $\beta_{n+1} > \beta_n$. This is possible, as all C_ξ are unbounded and our sequence has length $\omega < \kappa$. Now the supremum β of this sequence will again be in C: If $\xi < \beta$, there is an n with $\xi < \beta_n$. Then the final segment $(\beta_m)_{m>n}$ of our sequence are all elements of C_ξ as the C_ξ are decreasing, so the supremum β is in C_ξ as well, hence $\beta \in C$, and $\beta > \alpha$. So C is unbounded, making it a club. □

This all leads to an important lemma on stationary sets, which has a variety of applications. For example, using it you can prove that any stationary set splits into κ disjoint stationary sets.

Definition 1.17. An ordinal-valued function f with domain set A of ordinals is called *regressive* if for any $\alpha > 0$, $\alpha \in A$ we have $f(\alpha) < \alpha$.

Lemma 1.18 (Fodor)**.** *If $S \subset \kappa$ is a stationary set, and f is a regressive function on S, then there is a $\gamma < \kappa$ for which the set $f^{-1}\{\gamma\}$ is stationary.*

Proof. First remark that we may assume $0 \notin S$: After all, S intersected with the club $A_0 = \{\alpha < \kappa : \alpha > 0\}$ is again stationary by Corollary 1.12, so if $0 \in S$ and we can prove the statement for $A_0 \cap S$, all we might need to do is to add 0 to the found stationary set. Suppose all $f^{-1}\{\gamma\}$ are nonstationary. Then for each $\gamma < \kappa$ we can find a club C_γ such that no $\alpha \in S \cap C_\gamma$ has $f(\alpha) = \gamma$. Define $C = \Delta_{\gamma<\kappa} C_\gamma$, which we know is a club. Then $S \cap C$ is non-empty, so it contains some α. Then $\alpha \in S \cap C_\gamma$ for all $\gamma < \alpha$, so $f(\alpha) \neq \gamma$ for all $\gamma < \alpha$. Hence $f(\alpha) \geq \alpha$, but $\alpha > 0$ as $0 \notin S$, and f is regressive. This gives a contradiction, so there is some γ of which the inverse image is stationary. □

2 Silver's theorem

Around 1960-1970, we had a plethora of results around cardinal arithmetic. This started with Cohen's proof of the independence of the Generalized Continuum Hypothesis (The statement that $2^\kappa = \kappa^+$ for all infinite cardinals κ, also known as GCH) from ZFC. Consistency with ZFC had already been proven by Gödel a couple of decades before. Using the forcing techniques Cohen introduced, Solovay constructed models allowing $2^{\aleph_0}$ to assume any value consistent with Proposition 1.8 [3]. Easton then extended this by showing that the continuum function can take values for all regular κ in any way that the function remains non-decreasing and consistent with Proposition 1.8 [4]. For this reason, most set theorists in the period 1970-1974 expected the same kind of results would hold for singular cardinals, but simply could not be reached by Easton due to some weakness in the proof.

However, Silver in 1974 released a paper [5] on a new theorem that shocked this community, himself included. He proved, using more model-theoretic arguments, the following theorem:

Theorem 2.1 (Silver)**.** *For any singular cardinal κ of uncountable cofinality, if $2^\lambda = \lambda^+$ for all $\lambda < \kappa$, then $2^\kappa = \kappa^+$.*

It can also be stated as 'If GCH holds below a singular cardinal of uncountable cofinality, it also holds at that cardinal.' We will take a look at what essentially is an unwrapping of Silver's original proof, which replaces the model theory with more elementary methods. We follow the skeleton given in chapter 8 of [1], but for the general proof instead of only for $\kappa = \aleph_{\omega_1}$, and add in the arguments Jech left out.

To start off, we note that as κ is singular, by Proposition 1.6 it has to be a limit cardinal. But by our assumption that GCH holds below κ, we have that $2^\lambda = \lambda^+ < \kappa$ for all $\lambda < \kappa$, as κ is a limit. This shows that κ is a *strong limit cardinal.* Where a limit has $\lambda^+ < \kappa$ for all $\lambda < \kappa$, a strong limit has $2^\lambda < \kappa$ for all $\lambda < \kappa$. And for strong limits, we have the following proposition:

Proposition 2.2. *If κ is a strong limit cardinal, $2^\kappa = \kappa^{\operatorname{cf}\kappa}$.*

Proof. Suppose κ is a strong limit. As $\kappa < 2^\kappa$ by Cantor's Theorem, we have

$$\kappa^{\operatorname{cf}\kappa} \leq (2^\kappa)^{\operatorname{cf}\kappa} = 2^{\kappa\cdot\operatorname{cf}\kappa} = 2^\kappa$$

For the other direction, define $\tau = \sup_{\lambda<\kappa} 2^\lambda$ and a $\operatorname{cf}\kappa$-sequence $(\kappa_\alpha)_{\alpha<\operatorname{cf}\kappa}$ with $\kappa_\alpha < \kappa$ and sum equal to κ. (We can do this using Proposition 1.5.) Now clearly, as all $2^\lambda < \kappa$ for $\lambda < \kappa$, as κ is strong limit, we have $\tau \leq \kappa$. Hence we have

$$2^\kappa = 2^{\sum_{\alpha<\operatorname{cf}\kappa}\kappa_\alpha} = \prod_{\alpha<\operatorname{cf}\kappa} 2^{\kappa_\alpha} \leq \prod_{\alpha<\operatorname{cf}\kappa} \tau = \tau^{\operatorname{cf}\kappa} \leq \kappa^{\operatorname{cf}\kappa}$$

So $2^\kappa = \kappa^{\operatorname{cf}\kappa}$. □

As our κ is strong limit, we have $2^\kappa = \kappa^{\mathrm{cf}\,\kappa}$. As $2^\kappa > \kappa$ by Cantor, it suffices to prove that $2^\kappa = \kappa^{\mathrm{cf}\,\kappa} \leq \kappa^+$. So let us take the defining example of a set with cardinality $\kappa^{\mathrm{cf}\,\kappa}$: The set ${}^{\mathrm{cf}\,\kappa}\kappa$ of functions $\mathrm{cf}\,\kappa \to \kappa$. To bound this set, we will encode each of the functions in it with their 'initial segments'. Let $(\kappa_\alpha)_{\alpha<\mathrm{cf}\,\kappa}$ be a normal cofinal sequence in κ.

Define for each $f \in {}^{\mathrm{cf}\,\kappa}\kappa$ and $\alpha < \mathrm{cf}\,\kappa$ the function $f_\alpha : \mathrm{cf}\,\kappa \to \kappa_\alpha$ given by

$$f_\alpha(\beta) = \begin{cases} f(\beta) \text{ if } f(\beta) < \kappa_\alpha \\ 0 \text{ otherwise} \end{cases}$$

For a given f we see that $(f_\alpha)_{\alpha<\mathrm{cf}\,\kappa} \in \prod_{\alpha<\kappa} {}^{\mathrm{cf}\,\kappa}\kappa_\alpha$ forms a $\mathrm{cf}\,\kappa$-sequence of functions that restrict the codomain of f to the κ_α, annihilating the function values that would be outside the restricted codomain.

In particular, the family $G = \{(f_\alpha)_{\alpha<\mathrm{cf}\,\kappa} | f \in {}^{\mathrm{cf}\,\kappa}\kappa\}$ of these sequences is *almost disjoint*: For each pair $(f_\alpha) \neq (g_\alpha)$ of elements of G, we have a $\gamma < \mathrm{cf}\,\kappa$ such that for all $\alpha > \beta$, $f_\alpha \neq g_\alpha$. Why does this hold for our specific family G? Because if $f \neq g$, we have a β with $f(\beta) \neq g(\beta)$, and hence if we take γ such that $\kappa_\gamma \geq \max(f(\beta), g(\beta))$, we have for all $\alpha > \gamma$ that

$$f_\alpha(\beta) = f(\beta) \neq g(\beta) = g_\alpha(\beta)$$

as $\kappa_\alpha > \kappa_\gamma \geq \max(f(\beta), g(\beta))$, and hence $f_\alpha \neq g_\alpha$ for all $\alpha > \gamma$. This simultaneously proves that if $f \neq g$, the sequences corresponding to f and g differ, so this association is bijective. Hence $|G| = \kappa^{\mathrm{cf}\,\kappa}$.

So to bound $\kappa^{\mathrm{cf}\,\kappa}$, we need to bound almost disjoint families of functions. To do so, we will return to stationary sets to formulate and prove the following lemma:

Lemma 2.3. *Suppose $\lambda^{\mathrm{cf}\,\kappa} < \kappa$ for all $\lambda < \kappa$, and let F be an almost disjoint family of functions in $\prod_{\alpha<\mathrm{cf}\,\kappa} A_\alpha$. Assume that the set $\{\alpha < \mathrm{cf}\,\kappa : |A_\alpha| \leq \kappa_\alpha^+\}$ is stationary in* $\mathrm{cf}\,\kappa$*. Then $|F| \leq \kappa^+$.*

However, it turns out that to prove this lemma, we first need a slightly modified version of it: We just remove the two mentions of a successor to get the altered lemma.

Lemma 2.4. *Suppose $\lambda^{\mathrm{cf}\,\kappa} < \kappa$ for all $\lambda < \kappa$, and let F be an almost disjoint family of functions in $\prod_{\alpha<\mathrm{cf}\,\kappa} A_\alpha$. Assume that the set $\{\alpha < \mathrm{cf}\,\kappa : |A_\alpha| \leq \kappa_\alpha\}$ is stationary in* $\mathrm{cf}\,\kappa$*. Then $|F| \leq \kappa$.*

Now to prove Lemma 2.4, like we encoded ${}^{\mathrm{cf}\,\kappa}\kappa$ to get G, we now encode F to something we can overestimate the cardinality of:

Proof. Without loss of generality we may assume that the A_α are sets of ordinals, and that $A_\alpha \subset \kappa_\alpha$ whenever $|A_\alpha| \leq \kappa_\alpha$. So now we have a stationary set of α on which $A_\alpha \subset \kappa_\alpha$. The set S_0 of *limit ordinals* α for which this holds is again stationary by Corollary 1.14.

Now any $f \in F$ has $f(\alpha) < \kappa_\alpha$ for all $\alpha \in S_0$. For each $\alpha \in S_0$ there is some $\beta < \alpha$ such that $f(\alpha) < \kappa_\beta$ as well: After all, the κ_γ with $\gamma < \alpha$ have supremum κ_α as α is a limit and the sequence of κ_γ is normal. Hence if we take for each $\alpha \in S_0$ the minimal β for which this holds, we can define a regressive function on S_0 by taking $g(\alpha) = \beta$. Now Fodor's lemma gives us that there is some stationary $S \subset S_0$ on which g is constant, say $g|_S$ is constantly γ. Then f is bounded by κ_γ on S.

Hence for each $f \in F$ we have a pair $(S, f|_S)$ of a stationary subset of S_0 and a function on S bounded by some κ_γ. Now if $f, h \in F$ have $f|_S = h|_S$, then $f = h$, as S by being stationary is unbounded, yet if f and h are different, they are almost disjoint, so the set they agree on is bounded. This means the mapping $f \to (S, f|_S)$ is injective. Now for each S, the number of bounded functions on S is at most

$$\sum_{\gamma < \text{cf } \kappa} \kappa_\gamma^{|S|} \leq \sum_{\gamma < \text{cf } \kappa} \kappa_\gamma^{\text{cf } \kappa} = \text{cf } \kappa \cdot \sup_{\gamma < \text{cf } \kappa} \kappa_\gamma^{\text{cf } \kappa} \leq \kappa$$

as $\kappa_\gamma^{\text{cf } \kappa} < \kappa$ for all γ by the first assumption of the lemma, meaning the supremum is at most κ. Furthermore, the number of possible S is at most $2^{|S_0|} \leq 2^{\text{cf } \kappa} < \kappa$, so $|F|$, which thanks to our injective encoding is at most the product of these two, is at most κ. $\square$

The next step is to prove Lemma 2.3 from Lemma 2.4. To do so, we define a linear ordering on F, and then prove that the set of predecessors of any function in F is small by Lemma 2.4. This then turns out to be enough to prove that F itself is small enough.

Proof of Lemma 2.3. As in the previous proof, assume without loss of generality that all A_α are sets of ordinals, and that for all α in a stationary set S we have $A_\alpha \subset \kappa_\alpha^+$. Now take the filter generated by the clubs in cf κ and S. We find that the intersection of S with $\bigcap_{i=1}^n C_i$, where the C_i are clubs, is non-empty as $\bigcap_{i=1}^n C_i$ is a club according to Theorem 1.11. Hence the generated filter V is proper. (After all, an intersection of the form $\bigcap_{i=1}^n C_i$ is a superset of the non-empty $S \cap \bigcap_{i=1}^n C_i$.) Now take any ultrafilter U on cf κ containing V.

Note that any set in U is not only stationary, but also has stationary intersection with S, as for any $T \in U$ and C a club we have that $(T \cap S) \cap C \in U$ is not empty since U is proper. Using our ultrafilter, we define the following relation on F:

$$f \prec g \text{ iff } \{f < g\} := \{\alpha < \text{cf } \kappa : f(\alpha) < g(\alpha)\} \in U$$

If we have $f, g, h \in F$ with $f \prec g$ and $g \prec h$, as U is a filter and $\{f < h\}$ is a superset of $\{f < g\} \cap \{g < h\}$, we see that $f \prec h$ as well, so this relation is transitive. Furthermore, as each set in U is stationary and hence unbounded, while two different functions in F agree on only a bounded set of points as they are almost disjoint, we see that $\{\alpha < \text{cf } \kappa : f(\alpha) = g(\alpha)\} \notin U$ for any $f \neq g \in F$. Hence as U is an ultrafilter, we have either $\{f < g\} \in U$ or $\{g < f\} \in U$, which means that $\prec$ is a linear ordering on F.

Now for each $f \in F$, the set of predecessors of f under this ordering is definitely contained in $F_f := \{g \in F : \{g < f\} \cap S \text{ is stationary in cf } \kappa\}$, as every set in U has stationary intersection with S. Now for each T which has stationary intersection with S, define $F_{f,T}$ as $\{g \in F : \{g < f\} \cap S = T\}$; by definition, we find $F_f = \bigcup_T F_{f,T}$. But each $F_{f,T}$ is an almost disjoint family of functions, as it is a subset of F. If we now for all $\alpha < \text{cf } \kappa$ define

$$B_{\alpha,T} = \begin{cases} f(\alpha) \text{ if } \alpha \in T \cap S \\ A_\alpha \text{ otherwise} \end{cases}$$

we see that as $f(\alpha) \in A_\alpha \subset (\kappa_\alpha)^+$ for all $\alpha \in S$, we have $|f(\alpha)| \leq \kappa_\alpha$ for all $\alpha \in S$, and hence $\{\alpha < \text{cf } \kappa : |B_{\alpha,T}| \leq \kappa_\alpha\}$ is stationary as it contains the stationary set $S \cap T$. Since $g(\alpha) < f(\alpha)$ for all $g \in F_{f,T}$ and $\alpha \in T \cap S$, we see that $F_{f,T} \subset \prod_{\alpha < \text{cf } \kappa} B_{\alpha,T}$. As the power requirement of Lemma 2.4 is the same as the one in Lemma 2.3, we see that we can apply Lemma 2.4 to find that for all T with $T \cap S$ stationary, we have $|F_{f,T}| \leq \kappa$. Hence we have

$$|\{g \in F : g \prec f\}| \leq |F_f| = \sum_T |F_{f,T}| \leq \sum_T \kappa = |\{T \subset \text{cf } \kappa : T \cap S \text{ is stationary}\}| \cdot \kappa.$$

But the number of stationary subsets of cf κ is at most the number of subsets of cf κ, which is $2^{\text{cf } \kappa} < \kappa$ by the assumptions of the lemma, so this product is in fact κ. So any element $f \in F$ has at most κ predecessors under our linear ordering. Now if we take any unbounded, increasing λ-sequence $(f_\alpha)_{\alpha<\lambda}$ in F (Which can easily be done with transfinite recursion and AC) we see that $\lambda \leq \kappa^+$, as otherwise f_{κ^++1} has at least κ^+ predecessors, but as the sequence is unbounded, $|F| \leq \lambda \cdot \sup_{\alpha<\lambda} |\{g \in F : g < f_\alpha\}| \leq \lambda \cdot \kappa^+ \leq \kappa^+$. So indeed $|F| \leq \kappa^+$. □

Now that we have our lemma, all we need to finish the proof of Silver's Theorem is checking the conditions. The first one is easy enough as we are given GCH on the relevant cardinals: If $\lambda \leq \text{cf } \kappa$, we have $\lambda^{\text{cf } \kappa} = 2^{\text{cf } \kappa} < \kappa$ as κ is a strong limit, using Proposition 4.9, and if $\lambda > \text{cf } \kappa$ we have $\lambda^{\text{cf } \kappa} \leq \lambda^\lambda = 2^\lambda < \kappa$, again as κ is a strong limit.

For the second one, we need to prove there is a stationary set of α's such that $\kappa_\alpha^{\text{cf } \kappa} \leq \kappa_\alpha^+$. Whenever $\kappa_\alpha \geq \text{cf } \kappa$, we have

$$\kappa_\alpha^{\text{cf } \kappa} \leq (\kappa^\alpha)^{\kappa_\alpha} = 2^{\kappa_\alpha} = (\kappa_\alpha)^+.$$

Hence the set $\{\alpha < \text{cf } \kappa : \kappa_\alpha \geq \text{cf } \kappa\}$, which as the κ_α are increasing is a final segment and hence stationary, contains only α's for which $\kappa_\alpha^{\text{cf } \kappa} \leq \kappa_\alpha^+$. as required.
As the conditions of the lemma have now been checked, applying the lemma indeed gives us that $\kappa < 2^\kappa = \kappa^{\text{cf } \kappa} \leq \kappa^+$, so $2^\kappa = \kappa^+$. □

3 The Galvin-Hajnal theorem

Silver's result led to a more thorough study of what limits can be posed on the behaviour of the continuum function at singular cardinals. Only a year later, Fred Galvin and András Hajnal proved the following, more general result:

Theorem 3.1 (Galvin-Hajnal, 1975)**.** *If $\aleph_\alpha$ is a singular strong limit cardinal of uncountable cofinality, we have*

$$2^{\aleph_\alpha} < \aleph_\gamma \textit{ where } \gamma = (2^{|\alpha|})^+$$

Note that this is in particular more generally applicable than Silver's theorem, as the assumptions of Silver's theorem imply those of the Galvin-Hajnal theorem. The statement itself, however, is clearly weaker. The proof below is based on Jech's proof in Chapter 24 of [1], which is similar to the original proof Galvin and Hajnal gave in [6], but again will be for the general case. To start off, we use the same encoding tricks we used in the proof of Silver's theorem to create an almost disjoint family of functions of cardinality $2^{\aleph_\alpha}$, which reduces what we need to prove to a similar statement to Lemma 2.3:

Lemma 3.2. *Assume that $\lambda^{\operatorname{cf} \aleph_\alpha} < \aleph_\alpha$ for all $\lambda < \aleph_\alpha$. Let F be an almost disjoint family of functions in*

$$\prod_{\xi < \operatorname{cf} \aleph_\alpha} A_\xi$$

If $|A_\xi| < \aleph_\alpha$ for all $\xi < \operatorname{cf} \aleph_\alpha$ then $|F| < \aleph_\gamma$ where $\gamma = (2^{|\alpha|})^+$.

Do note the difference between this lemma and the one we used for Silver: Instead of bounding $|A_\xi|$ for a stationary set of ξ with a strong bound, we bound all $|A_\xi|$ with a far weaker bound. The conditions for this lemma are easy enough to check in our case: The first one follows exactly the same as in the proof of Silver's theorem, and the second is in fact implied by the first: If the κ_ξ are the normal cofinal sequence in $\aleph_\alpha$ we use to encode our functions as sequences of functions, then $|{}^{\operatorname{cf} \aleph_\alpha}\kappa_\xi| = \kappa_\xi^{\operatorname{cf} \aleph_\alpha} < \aleph_\alpha$ by the first condition.

To prove this lemma, we will again introduce a secondary lemma. To do so, we first need to define the following relation on functions $\phi : \operatorname{cf} \aleph_\alpha \to \operatorname{cf} \aleph_\alpha$:

$$\phi \prec \psi \text{ iff } \{\phi \geq \psi\} \text{ is nonstationary.}$$

First note that there is no infinite descending sequence of ϕ_n: If there were, for each $n \in \mathbb{N}$ we could find a club C_n such that $\phi_n(\xi) > \phi_{n+1}(\xi)$ for all $\xi \in C_n$. But then for each $\xi \in \bigcap_{n\in\mathbb{N}} C_n$, which is non-empty as it is a club, we would have

$$\phi_0(\xi) > \phi_1(\xi) > \dots$$

which is impossible, since there are no infinite descending sequences of ordinals as they are well-ordered by $\in$. This proves that $\prec$ is a *well-founded relation.* Because of this, it is possible to define the *norm* $\|.\|$ on ${}^{\mathrm{cf}\,\aleph_\alpha}\mathrm{cf}\,\aleph_\alpha$ as the unique function from this set to the ordinals satisfying

$$\|\phi\| := \sup\{\|\psi\| + 1 : \psi \prec \phi\}$$

for all $\phi \in {}^{\mathrm{cf}\,\aleph_\alpha}\mathrm{cf}\,\aleph_\alpha$. To prove existence of such a function, define $A_0 = \emptyset$, $A_{\xi+1}$ as the set of functions with all of their predecessors in A_ξ, and $A_\xi = \bigcup_{\beta<\xi} A_\beta$ whenever ξ is a limit ordinal. Now clearly the A_ξ form an increasing sequence of sets, and as they can only contain up to $\mathrm{cf}\,\aleph_\alpha^{\mathrm{cf}\,\aleph_\alpha}$ elements, there is a ξ such that $A_\xi = A_{\xi+1}$. Suppose $\phi_0 \notin A_\xi$. As $A_{\xi+1} = A_\xi$, we see that ϕ must have a predecessor $\phi_1 \notin A_\xi$. Repeating this argument, we find a sequence $\phi_0 \succ \phi_1 \succ \phi_2 \succ \ldots$ of functions not in A_ξ. But that is an infinite descending sequence, which does not exist. So $A_\xi = {}^{\mathrm{cf}\,\aleph_\alpha}\mathrm{cf}\,\aleph_\alpha$. Now define $\|\phi\|$ as $\beta + 1$ where β is the minimal ordinal such that $\phi \in A_\beta$, and we can easily check that this $\|.\|$ works. Uniqueness follows by noting that if we have another function $\|.\|_2$ satisfying the requirements, there must exist a ϕ with minimal $\|\phi\|$ such that $\|\phi\| \neq \|\phi\|_2$.

In particular, $\|\phi\| = 0$ if and only if no function preceeds it, which happens precisely when ϕ is 0 on a stationary set of ordinals. Now that we know this, we can formulate our secondary lemma:

Lemma 3.3. *Assume that $\lambda^{\mathrm{cf}\,\aleph_\alpha} < \aleph_\alpha$ for all $\lambda < \aleph_\alpha$. Let F be an almost disjoint family of functions in*

$$\prod_{\xi<\mathrm{cf}\,\aleph_\alpha} A_\xi$$

Let $(\kappa_\xi)_{\xi<\mathrm{cf}\,\aleph_\alpha}$ be a normal cofinal sequence in $\aleph_\alpha$, and let $\phi : \mathrm{cf}\,\aleph_\alpha \to \mathrm{cf}\,\aleph_\alpha$ be given with $|A_\xi| \leq \kappa_{\xi+\phi(\xi)}$ for all $\xi < \mathrm{cf}\,\aleph_\alpha$. Then $|F| \leq \aleph_{\alpha+\|\phi\|}$.

To prove Lemma 3.2 from this lemma, note that if $|A_\xi| < \aleph_\alpha$ for all $\xi < \mathrm{cf}\,\aleph_\alpha$, there must be some $\nu < \mathrm{cf}\,\aleph_\alpha$ such that $|A_\xi| \leq \kappa_{\xi+\nu}$: Now define $\phi(\xi)$ as the minimal such ν, and we have a ϕ as in Lemma 3.3. Now as $\|\phi\|$ is clearly at most the number of functions $\mathrm{cf}\,\aleph_\alpha \to \mathrm{cf}\,\aleph_\alpha$, which is by definition

$$(\mathrm{cf}\,\aleph_\alpha)^{\mathrm{cf}\,\aleph_\alpha} = 2^{\mathrm{cf}\,\aleph_\alpha} = 2^{\mathrm{cf}\,\alpha} \leq 2^{|\alpha|} < (2^{|\alpha|})^+ = \gamma.$$

Hence $\alpha + \|\phi\| < \gamma$ as well, so $|F| \leq \aleph_{\alpha+\|\phi\|} < \aleph_\gamma$.

Proof of Lemma 3.3. We will use transfinite induction on $\|\phi\|$. For $\|\phi\| = 0$, as this means $\phi(\beta) = 0$ for a stationary set of β, we actually get Lemma 2.3, so the base case we have already done in the proof of Silver's theorem. Now to do the case $\|\phi\| > 0$, we first need to generalize $\prec$: If S is a stationary set in $\mathrm{cf}\,\aleph_\alpha$, we define the relation

$$\phi \prec_S \psi \text{ iff } \{\phi \geq \psi\} \cap S \text{ is nonstationary}$$

Similar to before we can prove that $\prec_S$ does not admit any infinite descending sequences, so $\prec_S$ is again well-founded. This means that for each stationary S we can define the corresponding norm $\|.\|_S$. Note that if $T \subset S$ is also stationary, we have $\|\phi\|_S \leq \|\phi\|_T$:

Any set that has nonstationary intersection with S also has nonstationary intersection with T, so $\prec_S \subset \prec_T$ and the T-norm of ϕ is at least the S-norm of ϕ. In particular we have $\|\phi\| \leq \|\phi\|_S$ for any stationary S. Another thing we can deduce is

$$\|\phi\|_{S\cup T} = \min(\|\phi\|_S, \|\phi\|_T).$$

This follows easily using the fact that the union of two nonstationary sets is nonstationary. For each $\phi : \text{cf } \aleph_\alpha \to \text{cf } \aleph_\alpha$ we can now define I_ϕ as the family of stationary sets S such that $\|\phi\| < \|\phi\|_S$, as well as all nonstationary sets. As for any nonstationary X and stationary S, we have $\|\phi\|_{S\cup X} = \|\phi\|_S$, since $Y \subset S \cup X$ is nonstationary exactly when $Y \cap S$ is, as $Y \cap S \subset Y \subset (Y \cap S) \cup X$, implying that $\prec_S = \prec_{S\cup X}$, using everything shown in the previous paragraph we see that I_ϕ is a proper ideal.

Now define the following sets:

$$\begin{aligned} S &:= \{\beta < \text{cf } \aleph_\alpha : \phi(\beta) \text{ is a successor ordinal}\} \\ L &:= \{\beta < \text{cf } \aleph_\alpha : \phi(\beta) \text{ is a limit ordinal}\} \\ Z &:= \{\beta < \text{cf } \aleph_\alpha : \phi(\beta) = 0\} \end{aligned}$$

Clearly, as any ordinal is zero, a limit, or a successor, we have $S \cup L \cup Z = \text{cf } \aleph_\alpha$. Furthermore, whenever $\|\phi\| > 0$, we know that Z is nonstationary, which implies $Z \in I_\phi$. As I_ϕ is a proper ideal, this implies at most one of S and L can be an element of I_ϕ.
We will now prove that if $\|\phi\|$ is a limit ordinal, $L \notin I_\phi$. As we have just shown, it is enough to prove $S \in I_\phi$. If $S \notin I_\phi$, we see that $\|\phi\| = \|\phi\|_S = \|\psi\|_S + 1$, where $\psi(\beta) = \phi(\beta) - 1$ for all $\beta \in S$, and $\phi(\beta)$ otherwise. Hence if $\|\phi\|$ is a limit, we must have $S \in I_\phi$ and thus $L \notin I_\phi$.

Now we will prove that if $\|\phi\|$ is a successor ordinal, $S \notin I_\phi$. In this case, there exists some $\psi \prec \phi$ such that $\|\phi\| = \|\psi\| + 1$. It suffices to prove $L \in I_\phi$. If L is nonstationary, we already know $L \in I_\phi$, so suppose L is stationary. Define $\chi : \text{cf } \aleph_\alpha \to \text{cf } \aleph_\alpha$ by

$$\chi(\beta) = \begin{cases} \psi(\beta) \text{ when } \beta \in \{\psi \geq \phi\} \\ \psi(\beta) + 1 \text{ otherwise} \end{cases}$$

Then $\{\psi \geq \chi\} \cap L = \{\psi \geq \phi\} \cap L$ is nonstationary, so $\psi \prec_L \chi$, and as whenever $\phi(\beta)$ is a limit ordinal and $\psi(\beta) < \phi(\beta)$, also $\chi(\beta) = \psi(\beta) + 1 < \phi(\beta)$, we see that $\{\chi \geq \phi\} \cap L = \{\psi \geq \phi\} \cap L$ as well, so $\chi \prec_L \phi$. This shows that

$$\|\phi\|_L \geq \|\chi\|_L + 1 \geq \|\psi\|_L + 2 \geq \|\psi\| + 2 = \|\phi\| + 1,$$

hence $L \in I_\phi$, and thus $S \notin I_\phi$ whenever $\|\phi\|$ is a successor ordinal.

With this information, we can proceed with the induction.
First note that we can assume without loss of generality that $A_\xi \subset \aleph_{\xi+\phi(\xi)}$ for all ξ.
Suppose $\|\phi\| > 0$ is a limit ordinal and our lemma holds for all ψ of smaller norm. By

our assumption on the A_ξ we have for any $f \in F$ that $f(\xi) < \aleph_{\xi+\phi(\xi)}$ for all $\xi < \operatorname{cf} \aleph_\alpha$. Now for all $\xi \in L$ we can find some $\beta_\xi < \phi(\xi)$ such that $f(\xi) < \aleph_{\xi+\beta_\xi}$: Define $\psi(\xi) = \beta_\xi$ for all $\xi \in L$, and $\psi(\xi) = \phi(\xi)$ on all $\xi \notin L$. As we have just proven $L \notin I_\phi$, and $\{\psi \geq \phi\} \cap L = \emptyset$, we have $\|\psi\| \leq \|\psi\|_L < \|\phi\|_L = \|\phi\|$. Furthermore, we have $f \in F_\psi := \{g \in F : g(\xi) \in \aleph_{\xi+\psi(\xi)} \text{ for all } \xi < \operatorname{cf} \aleph_\alpha\}$. So every $f \in F$ is in some F_ψ with $\|\psi\| < \|\phi\|$, which means that

$$F = \bigcup_{\psi:\|\psi\|<\|\phi\|} F_\psi$$

By the induction hypothesis, each of the F_ψ has cardinality at most $\aleph_{\alpha+\|\psi\|}$. As the number of functions ψ is at most $2^{\operatorname{cf}\alpha} < \aleph_\alpha$ by the assumptions of the lemma, we see that

$$|F| \leq \sum_{\psi:\|\psi\|<\|\phi\|} |F_\psi| \leq \aleph_\alpha \cdot \sup_{|\psi|<|\phi|} \aleph_{\alpha+\|\psi\|} = \aleph_{\alpha+\|\phi\|}$$

as we wanted.

Let $\|\phi\| > 0$ be a successor ordinal, say $\beta + 1$, and assume our lemma holds for all ψ of smaller norm. We have already proven $S \notin I_\phi$.
For $f \in F$, define the set

$$F_f := \{g \in F : (\exists S_0 \subset S), S_0 \notin I_\phi, (\forall \xi \in S_0)\, g(\xi) < f(\xi)\}$$

To prove F is small enough, we write F as a union of F_f. So we first prove the F_f have cardinality at most $\aleph_{\alpha+\beta}$. To do so, we first split them up in the sets
$F_{f,S_0} := \{g \in F : (\forall \xi \in S_0), g(\xi) < f(\xi)\}$ for each $S_0 \subset S$, $S_0 \notin I_\phi$. Now define for each such S_0 the function $\psi : \operatorname{cf} \aleph_\alpha \to \operatorname{cf} \aleph_\alpha$ given by $\psi(\xi) = \phi(\xi) - 1$ whenever $\xi \in S_0$, and $\psi(\xi) = \phi(\xi)$ otherwise. Because $S_0 \notin I_\phi$, we see that as $\{\psi \geq \phi\} \cap S_0 = \emptyset$, we have

$$\|\psi\| \leq \|\psi\|_{S_0} < \|\phi\|_{S_0} = \|\phi\| = \beta + 1$$

so $\|\psi\| \leq \beta$. Now define for all $\xi \in S_0$ the sets $B_\xi = f(\xi)$, and define $B_\xi = A_\xi$ for all other $\xi < \operatorname{cf} \aleph_\alpha$. Then each F_{f,S_0} is an almost disjoint function family in $\prod_{\xi<\operatorname{cf}\aleph_\alpha} B_\xi$, and $|B_\xi| \leq \aleph_{\alpha+\psi(\xi)}$ for all ξ, so by the induction hypothesis $|F_{f,S_0}| \leq \aleph_{\alpha+\|\psi\|} \leq \aleph_{\alpha+\beta}$ for each F_{f,S_0}. But clearly the number of S_0 we find is at most $2^{|S|} \leq 2^{\operatorname{cf}\aleph_\alpha} < \aleph_\alpha \leq \aleph_{\alpha+\beta}$, so the union F_f of the F_{f,S_0} also has cardinality at most $\aleph_{\alpha+\beta}$.

To finish our proof, we construct a sequence $(f_\xi)_{\xi<\delta}$ of length $\delta \leq \aleph_{\alpha+\beta+1} = \aleph_{\alpha+\|\phi\|}$ such that $F = \bigcup_{\xi<\delta} F_{f_\xi}$. By our bound on the $|F_f|$, we see that this shows that $|F| \leq \aleph_{\alpha+\|\phi\|}$. To do so, given f_ν for $\nu < \xi$, take f_ξ to be any function in $F \setminus \bigcup_{\nu<\xi} F_{f_\nu}$. If this set were empty, take $\delta = \xi$, and we are done creating our sequence. If not, continue with the transfinite recursion.

Now note that by definition for all $\nu < \xi < \delta$ we have $\{\beta \in S : f_\xi(\beta) < f_\nu(\beta\} \in I_\phi$.

Furthermore, as F is almost disjoint, $\{\beta \in S : f_\xi(\beta) = f_\nu(\beta)\}$ is nonstationary as it is bounded, so this set is also in I_ϕ. As $S \notin I_\phi$, we see that $\{\beta \in S : f_\nu(\beta) < f_\xi(\beta)\} \notin I_\phi$, which means that for all $\nu < \xi$, $f_\nu \in F_{f_\xi}$. As $|F_{f_\xi}| \leq \aleph_{\alpha+\beta}$, we see that any ξ for which f_ξ exists fulfills $\xi < \aleph_{\alpha+\beta+1}$, which implies $\delta \leq \aleph_{\alpha+\beta+1}$, and as discussed earlier this finishes our proof. Hence our transfinite induction works, and the lemma holds for all $\|\phi\|$. As we have now finished the proof of our lemma, we have now also proven the Galvin-Hajnal theorem. $\square$

4 Appendix: Ordinal and cardinal numbers

This appendix contains all the information about ordinal and cardinal numbers necessary to understand everything in the other chapters. However, some basic familiarity with the axiom system ZFC for set theory and the concept of (ultra)filters will be assumed. If you need more information on these subjects: The first can be found in many works on set theory, for example [2], chapter 2 or [1], chapter 1. In both of these sources, you could also find a more general introduction to the theory of ordinals and cardinals: We will primarily follow [1], chapters 2,3 and 5 for this appendix. For the second, the relevant details can be found in [1], pages 73-75.

Ordinal and cardinal numbers are essentially extensions of the natural numbers in two different directions. Cardinal numbers generalize the way the natural numbers allow us to describe the sizes of finite sets to all sets. Ordinal numbers, on the other hand, extend the way the natural numbers allow us to count and order objects as 'zeroth, first, second, ...'.

To start things off, we will define the ordinal numbers. These are the archetypical well-ordered sets, in the sense that any well-ordered set has a unique order-preserving bijection with a unique ordinal number. Of course, we need the ordinals themselves to have an ordering as well. For this, as we are working in pure set theory, the logical choice is $\in$. To make that work, we will need this:

Definition 4.1. A set S is called *transitive* if for any $A \in S$, $A \subset S$ holds as well.

The reason for this definition is to allow transitivity of $\in$, which we need for it to be our ordering. Now that we have this definition, we can easily define ordinal numbers.

Definition 4.2. An *ordinal number* is a transitive set that is well-ordered by $\in$.

We generally write ordinal numbers as lowercase Greek letters, particularly using α, β, γ and ξ.

Definition 4.3. For ordinal numbers α, β we write $\alpha < \beta$ if $\alpha \in \beta$. $\leq$, $>$ and $\geq$ are defined similarly.

The ordinals turn out to be a proper class, which under $<$ is well-ordered in the sense that it is linearly ordered and any non-empty set of ordinals has a minimum. By our choice of ordering, an ordinal is the set of all smaller ordinals. Furthermore, any set of ordinals has a supremum: The union of all ordinals in the set. To get our copy of the natural numbers in the ordinals, we start by taking $0 = \emptyset$, then defining the successor operation:

Definition 4.4. For any ordinal α, $\alpha + 1 := \alpha \cup \{\alpha\}$ is the *successor* of α.

So now we can create the rest of our copy of $\mathbb{N}$ in the ordinals as $1 = 0 + 1 = \{\emptyset\}$, $2 = 1 + 1 = \{\emptyset, \{\emptyset\}\}$, et cetera. However, while each ordinal has a successor, not every ordinal is a successor itself. Ordinals bigger than 0 that are not the successor of some other ordinal are known as *limit ordinals*, the smallest of which is $\omega = \bigcup_{n\in\mathbb{N}} n$, the order type of $\mathbb{N}$. These in particular have the property that if α is a limit and $\beta < \alpha$, then $\beta+1 < \alpha$ as well.

Two important concepts that generalize to the ordinal numbers are induction and recursion.

Proposition 4.5 (Transfinite Induction)**.** *Let S be a class of ordinals. If the conditions*

1. $0 \in S$
2. *If $\alpha \in S$, then $\alpha + 1 \in S$.*
3. *If α is a limit ordinal, and for all $\beta < \alpha$, $\beta \in S$, then also $\alpha \in S$*

all hold, then S contains all ordinals.

Proof. The proof is easy: If S fulfills these conditions, but does not contain all ordinals, there is a smallest ordinal $\alpha \notin S$. However, our conditions make sure that α can neither be 0, nor a successor, nor a limit, which is impossible. □

As we can see, transfinite induction is the same proof technique on the ordinals as induction is on the natural numbers, but requires the stronger assumption on limit ordinals. Recursion, on the other hand, generalizes directly: It is possible to create sequences (a_α) with an element for each ordinal number, in such a way that each a_α only depends on the a_ξ with $\xi < \alpha$. In particular, you can also create sequences like this where the indices only run up to a specific ordinal θ.
Of course, the '+1' in the notation for a successor $\alpha + 1$ is not coincidental. We can define an addition on the ordinals using transfinite recursion.

Definition 4.6. We define the operation + so that for any ordinal α:

- $\alpha + 0 = \alpha$
- $\alpha + (\beta + 1) = (\alpha + \beta) + 1$ for any ordinal β
- $\alpha + \beta = \sup_{\xi<\beta} \alpha + \xi$ for any limit ordinal β.

It turns out that under this addition, $\alpha + \beta$ is also the order type of the disjoint union $\alpha \sqcup \beta = \alpha \times \{1\} \cup \beta \times \{2\}$, ordered so that α and β keep their internal ordering, and any element $(\gamma, 2)$ is bigger than any $(\xi, 1)$. However, this shows that our addition is non-commutative as $(0, 2)$ is the largest element of $\omega + 1$, while $1 + \omega$ has no largest element, and in fact is order isomorphic to ω as can be seen by sending $(1, 1)$ to 0 and $(n, 2)$ to $n+1$.

Now that we have discussed some basic things about ordinals, let us talk about cardinals.

Definition 4.7. The *cardinality* $|\alpha|$ of ordinal number α is the minimal ordinal β for which there exists a bijection $\alpha \to \beta$. A *cardinal number* is an ordinal α for which $|\alpha| = \alpha$.

Note that as the identity is always a bijection of α with itself, $|\alpha| \leq \alpha$, so for non-cardinals we have $|\alpha| < \alpha$. For any well-ordered set S, we define the cardinality $|S|$ as $|\alpha|$, where α is the ordinal with which S has an order isomorphism. Hence one of the reasons we will assume AC when working with cardinals is to have a cardinality for every set. In fact, the cardinality of a set is independent of the choice of well-ordering, as the identity on S combined with the order isomorphisms and bijections with the cardinality give a bijection between any two cardinalities we might find, while by definition no bijection exists between any pair of different cardinals. We will generally write cardinals, especially infinite ones, with lowercase Greek letters as κ, λ and μ.

We continue by defining some standard operations on cardinals:

Definition 4.8. For cardinals κ, λ and $(\kappa_i)_{i\in I}$ for some index set I, we define

$$\begin{aligned} \kappa + \lambda &= |\kappa \sqcup \lambda| \\ \kappa \cdot \lambda &= |\kappa \times \lambda| \\ \kappa^\lambda &= |{}^\lambda\kappa| \\ \sum_{i\in I} \kappa_i &= |\bigcup_{i\in I} \kappa_i \times \{i\}| \\ \prod_{i\in I} \kappa_i &= |\bigtimes_{i\in I} \kappa_i| \end{aligned}$$

Here ${}^\lambda\kappa$ is the set of functions $\lambda \to \kappa$, and the $\bigtimes$ is the Cartesian product consisting of all functions $f : I \to \bigcup_{i\in I} \kappa_i$ with $f(i) \in \kappa_i$ for all $i \in I$, written like this to prevent confusion with the product of cardinal numbers on the left.

These operations, unlike ordinal addition, are commutative, as well as associative and distributive. They obey many of the arithmetic rules we would expect, but not all of them. For example, while $\kappa \leq \mu$ does imply $\kappa^\lambda \leq \mu^\lambda$, the same does not always hold when the inequalities are made to be strict. Furthermore, sums and products are easily calculated: Whenever κ or λ is infinite, we have $\kappa + \lambda = \kappa \cdot \lambda = \max(\kappa, \lambda)$. Another simplification can be made for infinite sums and products: For any cardinal λ and sequence $(\kappa_\xi)_{\xi<\lambda}$ of non-zero cardinals we have

$$\sum_{\xi<\lambda} \kappa_\xi = \lambda \cdot \sup_{\xi<\lambda} \kappa_\xi$$

If the sequence is also non-decreasing, we have

$$\prod_{\xi<\lambda} \kappa_\xi = (\sup_{\xi<\lambda} \kappa_\xi)^\lambda$$

The following proposition will show why in particular the function $\kappa \to 2^\kappa (= |\mathcal{P}(\kappa)|)$, known as the continuum function, is rather important in cardinal arithmetic.

Proposition 4.9. *If κ, λ are cardinals with λ infinite and $2 \leq \kappa \leq 2^\lambda$, then $\kappa^\lambda = 2^\lambda$.*

Proof. By the given assumptions on κ, we have

$$2^\lambda \leq \kappa^\lambda \leq (2^\lambda)^\lambda = 2^{\lambda \cdot \lambda} = 2^\lambda$$

Hence $\kappa^\lambda = 2^\lambda$. □

As we see, infinite cardinals behave rather differently from the finite ones. Because of this, we shall primarily study infinite cardinals. For this, some notation is useful. First define for any cardinal κ the cardinal κ^+ to be the smallest cardinal larger than κ. This cardinal is called the cardinal successor of κ.

Definition 4.10. Using transfinite recursion, we define for each ordinal α a cardinal $\aleph_\alpha$:

- $\aleph_0 = |\omega| = \omega$ is the smallest infinite cardinal.
- For any ordinal α, $\aleph_{\alpha+1} = (\aleph_\alpha)^+$.
- For any limit ordinal α, $\aleph_\alpha = \bigcup_{\xi<\alpha} \aleph_\xi$.

It is easy to see that the sequence of alephs is in fact increasing, and it turns out that it enumerates all infinite cardinals. As cardinals are also ordinals, we will write ω_α instead of $\aleph_\alpha$ when we want to emphasize this. In the case of ω_0 we sometimes leave out the 0 and just use ω. As any infinite ordinal has a bijection with its successor, all infinite cardinals are limit ordinals. However, we still make a distinction between successor cardinals and limit cardinals, by defining $\aleph_\alpha$ to be a successor/limit cardinal whenever α is a successor/limit ordinal.

Now, while addition and multiplication are to an extent easy with cardinals, exponentiation tends to be difficult. In particular, this is true with regard to the alephs. For example, when Cantor originally introduced the theory of cardinals, he hypothesized that $2^{\aleph_0} = \aleph_1$, which would mean that every infinite subset of $\mathbb{R}$ (which have cardinality $2^{\aleph_0}$) is either countable or has a bijection with $\mathbb{R}$. This result is known as the Continuum Hypothesis, and has been proven to be independent of ZFC by Cohen in 1963. Even the Generalized Continuum Hypothesis, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all ordinals α, is consistent with ZFC. However, there are some things we can say using the following theorem:

Theorem 4.11 (König's theorem)**.** *If $\kappa_i < \lambda_i$ are cardinals for every $i \in I$, then*

$$\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i$$

For example, as an easy corollary we find a familiar theorem of Cantor, noting again that $|\mathcal{P}(X)| = 2^{|X|}$:

Corollary 4.12 (Cantor)**.** *For any cardinal κ, $2^\kappa > \kappa$*

Proof. Use König's theorem with $I = \kappa$, $\kappa_i = 1$ for all $i \in \kappa$, and $\lambda_i = 2$ for all $i \in \lambda$. □

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