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Homological Algebra: completing diagrams of exact sequences

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Contents

1 Introduction

There are many theorems and lemmas about commutative diagrams and exact sequences, like the snake lemma, the horseshoe lemma and the nine lemma. In this paper we will pose and prove a new theorem in this context, with particular similarity to the nine lemma. We consider a commutative diagram of four short exact sequences, of the form of the left diagram shown beneath,

$$\begin{array}{ccccccccc} & & 0 & & & & 0 & & \\ & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & A & \hookrightarrow & E & \twoheadrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \\ & & G & & & & F & & \\ & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & C & \hookrightarrow & H & \twoheadrightarrow & D & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \\ & & 0 & & & & 0 & & \end{array} \quad , \quad \begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \hookrightarrow & E & \twoheadrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \dashrightarrow & G & \dashrightarrow & Z & \dashrightarrow & F & \dashrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C & \hookrightarrow & H & \twoheadrightarrow & D & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

and we ask ourselves: **what are the necessary and sufficient conditions on the diagram on the left, for there to exist a Z and maps that complete the diagram to the one on the right**, so that it commutes and the middle row and column are also exact?

As we will show in this paper, the answer is surprisingly explicit. We first take a look at classifying exact sequences using an equivalence relation between them, which are concepts that have been introduced by Nobuo Yoneda (1960) [4]. We will also construct an abelian group-structure on these equivalence classes. Then, returning to the situation above, we define the exact sequence

$$W := \ 0 \longrightarrow A \hookrightarrow G \coprod_A E \longrightarrow H \prod_D F \twoheadrightarrow D \longrightarrow 0$$

and we claim that **an adequate Z exists if and only if the equivalence class of W is zero**.

The core argument for proving this statement is lemma (4.1) that explicitly describes the form of all sequences of length 4 with equivalence class zero. This lemma, although made more explicit here and fitted to our situation, has been treated before in the famous *Séminaire de Géométrie Algébrique du Bois Marie* run by Alexander Gothendrieck (1968) [1]. A proof of our diagram theorem more heavily inspired by Gothendrieck has been published last year in Rakesh R. Pawar's *A generalization of Grothendieck's Extension Panachées* [6].

Finally, we will extend our theorem to diagrams of arbitrary size, and although we can't directly extend the strong statement we had for 3×3 diagrams, we will show that an analogous requirement can be made for one of the implications, that is, if we can fill in the diagram, we can explicitly describe the equivalence class of an exact sequence similar to the W above.

2 Preliminaries

Throughout this paper, all groups will be (left) R-modules for a ring R and all maps will be homomorphisms of R-modules. Most of the statements in this paper can be generalized, for example to a context of categories, but in order to make them more comprehensible and clear, we will stay in the context of R-modules.

We will first need some definitions that we will extensively use to describe and construct exact sequences.

2.1 Pushouts and pullbacks

Definition 2.1. *The **pushout** of the homomorphisms $f : A \to E$ and $f' : A \to E'$ is the quotient of $E \oplus E'$ by $N := \{(f(a), -f'(a))|a \in A\}$, and we write:*

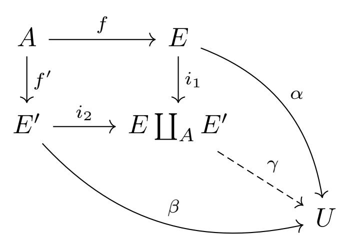
$$E \coprod_A E' := (E \oplus E')/N.$$

Note that for any $a \in A$, we have $(f(a), 0) = (0, f'(a))$ in $E \coprod_A E'$.

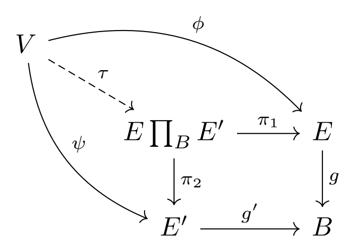
Definition 2.2. *The **pullback** of the homomorphisms $g : E \to B$ and $g' : E' \to B$ is defined as*

$$E \prod_B E' := \{(e, e') \in E \oplus E'|g(e) = g'(e')\}.$$

One defining property of pushouts and pullbacks is the **universal property**. For pushouts this means that for any $\alpha : E \to U$ and $\beta : E' \to U$ for which the following diagram commutes, there exists a unique $\gamma : E \coprod_A E' \to U$ also making the diagram commute:



The universal property for pullbacks is similar: For any $\phi : V \to E$ and $\psi : V \to E'$ for which the following diagram commutes, there exists a unique $\tau : V \to E \prod_B E'$ also making the diagram commute:



Another useful property of pushouts and pullbacks which is easy to prove is the following [2].

Lemma 2.1. *Let the diagram below be given.*

$$\begin{array}{ccc} P & \xrightarrow{f} & E \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g} \\ E' & \xrightarrow{g'} & Q \end{array}$$

Then

1) If Q is a pushout and f' is injective, then g is injective.

2) If P is a pullback and g' is surjective, then f is surjective.

2.2 Exact n-sequences

For any non-negative integer n, an **exact** (A, B)**-n-sequence** is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} E_1 \xrightarrow{g_1} \dots \xrightarrow{g_{n-1}} E_n \xrightarrow{h} B \longrightarrow 0$$

for some R-modules $E_1, \dots, E_n$ and homomorphisms $f, g_1, \dots, g_n, h$. Exact 1-sequences are also known as **short exact sequences**.

We can decompose any exact sequence into shorter exact sequences. To illustrate this, given an exact (A, B)-2-sequence, we can construct the short exact sequences

$$0 \longrightarrow A \xrightarrow{f} E_1 \xrightarrow{g} \twoheadrightarrow \mathrm{img} \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{img} \hookrightarrow E_2 \xrightarrow{h} B \longrightarrow 0$$

Similarly, given two short exact sequences:

$$\alpha := \ 0 \longrightarrow A \xrightarrow{f} E_1 \xrightarrow{\psi} C \longrightarrow 0$$

and

$$\beta := \ 0 \longrightarrow C \xhookrightarrow{\phi} E_2 \xrightarrow{h} B \longrightarrow 0$$

we can construct the exact (A, B)-2-sequence:

$$\alpha \circ \beta := \ 0 \longrightarrow A \xrightarrow{f} E_1 \xrightarrow{\phi \circ \psi} E_2 \xrightarrow{h} B \longrightarrow 0$$

It is an easy verification that these constructed sequences are indeed exact.

2.3 Short exact sequences as an abelian group

Definition 2.3. *For two short exact (A, B)-sequences α and α', we say that $\alpha \simeq \alpha'$ if and only if there exists $\phi : E \to E'$ such that the diagram*

$$\begin{array}{lccccccccc} \alpha: & 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\ & & & \| & & \downarrow{\scriptstyle \phi} & & \| & & \\ \alpha': & 0 & \longrightarrow & A & \longrightarrow & E' & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

commutes.

Definition 2.4. *Given two short exact (A, B)-sequences α and α' as above, their* ***Baer sum*** *is the short exact sequence*

$$\alpha \oplus \alpha' := \; 0 \longrightarrow A \xrightarrow{f''} E'' \xrightarrow{g''} B \longrightarrow 0$$

with $E'' := \{(e, e') \in E \oplus E' | g(e) = g'(e')\}/\{(f(a, 0) - (0, f'(a))|a \in A\}$, $f''(a) := (f(a), 0) = (0, f'(a))$ and $g''(e, e') := g(e) = g'(e')$

The relation $\simeq$ is an equivalence relation between short exact sequences, since by the five lemma the map ϕ is always an isomorphism.

We state without proof that the set of equivalence classes of short exact (A, B)-sequences under this relation together with the Baer sum forms an Abelian group, denoted $(\mathrm{Ext}^1(B, A), \oplus)$ [2]. The identity element under the Baer sum is the short exact sequence

$$0 \longrightarrow A \xrightarrow{i_1} A \oplus B \xrightarrow{\pi_2} B \longrightarrow 0$$

and for any short exact sequence α its inverse element under the Baer sum is the short exact sequence

$$-\alpha := \; 0 \longrightarrow A \xrightarrow{-f} E \xrightarrow{g} B \longrightarrow 0$$

It is easy to check that these last two definitions have all the required properties for identity resp. inverse elements under the Baer sum.

3 Exact n-sequences as an abelian group

We want to classify longer exact sequences in a similar way to short exact sequences, as we did in the previous section. In this section, n is a positive integer $n \in \mathbb{N}_{\geq 2}$. To describe an abelian group structure on exact n-sequences, we first define the following relation.

3.1 Equivalence class of exact n-sequences

Definition 3.1. *For two exact (A, B)-n-sequences E and E', we say that $E \sim E'$ if there exist an exact (A, B)-n-sequence E_q together with homomorphisms $\phi_i : E_{q,i} \to E_i$ and $\phi'_i : E_{q,i} \to E'_i$ for all $i \in \{1, \ldots, n\}$, such that the diagram*

$$
\begin{array}{lccccccccccccc}
E: & 0 & \longrightarrow & A & \longrightarrow & E_1 & \longrightarrow & \ldots & \longrightarrow & E_n & \longrightarrow & B & \longrightarrow & 0 \\
 & & & \| & & \uparrow{\scriptstyle \phi_1} & & & & \uparrow{\scriptstyle \phi_n} & & \| & & \\
E_q: & 0 & \longrightarrow & A & \longrightarrow & E_{q,1} & \longrightarrow & \ldots & \longrightarrow & E_{q,n} & \longrightarrow & B & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle \phi'_1} & & & & \downarrow{\scriptstyle \phi'_n} & & \| & & \\
E': & 0 & \longrightarrow & A & \longrightarrow & E'_1 & \longrightarrow & \ldots & \longrightarrow & E'_n & \longrightarrow & B & \longrightarrow & 0
\end{array}
$$

commutes [4].

Remark 3.1. *From now on, if for two exact (A, B)-n-sequences E and E' there exist morphisms $\phi_i : E_i \to E'_i$ for every $i \in \{1, \ldots, n\}$ such that the diagram*

$$
\begin{array}{lccccccccccccc}
E: & 0 & \longrightarrow & A & \longrightarrow & E_1 & \longrightarrow & \ldots & \longrightarrow & E_n & \longrightarrow & B & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle \phi_1} & & & & \downarrow{\scriptstyle \phi_n} & & \| & & \\
E': & 0 & \longrightarrow & A & \longrightarrow & E'_1 & \longrightarrow & \ldots & \longrightarrow & E'_n & \longrightarrow & B & \longrightarrow & 0
\end{array}
$$

commutes, we will sometimes write $E \to E'$ instead.

Remark 3.2. *At first glance, it might make more sense to define the relation between exact 2-sequences more similar to the equivalence relation between short exact sequences; simply say that $E \sim E'$ if and only if $E \to E'$. However, this relation is not in fact an equivalence relation, since it is not symmetric. To see this, take a look at the following commutative diagram of exact $(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$-2-sequences:*

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z}/4\mathbb{Z} & \xrightarrow{i_1} & (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\pi_2} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle \cdot 2} & & \uparrow{\scriptstyle i_2} & & \| & & \\
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & = & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{0} & \mathbb{Z}/2\mathbb{Z} & = & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0
\end{array}
$$

If we call the first exact sequence X and the second Y, we have $Y \to X$, but since the map $\mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z}$ is not invertible we do not have $X \to Y$. Therefore, it is necessary to define the relation for exact n-sequences as we did.

Theorem 3.1. *The relation $\sim$ in definition* 3.1 *is an equivalence relation.*

The relation is clearly reflective and symmetric. We will prove the transitivity using the following strong lemma.

Lemma 3.2. *Let E, E', E'' be exact (A, B)-n-sequences with $E \to E' \leftarrow E''$. Then there exists an exact (A, B)-n-sequence $\tilde{E}$ with $E \leftarrow \tilde{E} \to E''$.*

Proof. We have the following commutative diagram of exact sequences:

$$
\begin{array}{lccccccccccccccc}
E: & 0 & \longrightarrow & A & \xrightarrow{f} & E_1 & \xrightarrow{g_1} & \dots & \xrightarrow{g_{n-1}} & E_n & \xrightarrow{h} & B & \longrightarrow & 0 \\
 & & & \| & & \downarrow{\scriptstyle \phi_1} & & & & \downarrow{\scriptstyle \phi_n} & & \| & & \\
E': & 0 & \longrightarrow & A & \xrightarrow{f'} & E'_1 & \xrightarrow{g'_1} & \dots & \xrightarrow{g'_{n-1}} & E'_n & \xrightarrow{h'} & B & \longrightarrow & 0 \\
 & & & \| & & {\scriptstyle \phi'_1}\uparrow & & & & {\scriptstyle \phi'_n}\uparrow & & \| & & \\
E'': & 0 & \longrightarrow & A & \xrightarrow{f''} & E''_1 & \xrightarrow{g''_1} & \dots & \xrightarrow{g''_{n-1}} & E''_n & \xrightarrow{h''} & B & \longrightarrow & 0
\end{array}
\qquad (1)
$$

We construct:

$$
\tilde{E} := \qquad 0 \longrightarrow A \xrightarrow{\tilde{f}} Z_1 \xrightarrow{\tilde{g}_1} \dots \xrightarrow{\tilde{g}_{n-1}} Z_n \xrightarrow{\tilde{h}} B \longrightarrow 0
$$

with the following definitions:

- $Z_i := \{(e'_{i-1}, e'_i, e, e'') \in E'_{i-1} \oplus E'_i \oplus E_i \oplus E''_i | g'_{i-1}(e'_{i-1}) + e'_i = \phi_i(e) - \phi'_i(e'')\}$ for $i \in \{2, \dots, n-1\}$
- $Z_1 := E_1 \oplus E''_1$, since $E_1 \oplus E''_1 \simeq \{(e', e, e'') \in E'_1 \oplus E_1 \oplus E''_1 | e' = \phi_1(e) - \phi'_1(e'')\}$
- $Z_n := \{(e', e, e'') \in E'_{n-1} \oplus E_n \oplus E''_n | g'_{n-1}(e') = \phi_n(e) - \phi'_n(e'')\}$
- $\tilde{f}(a) := (f(a), f''(a))$
- $\tilde{g}_1(e, e'') := (\phi_1(e) - \phi'_1(e''), 0, g_1(e), g''_1(e''))$
- $\tilde{g}_i(e'_{i-1}, e'_i, e, e'') := (e'_i, 0, g_i(e), g''_i(e''))$ for $i \in \{2, \dots, n-2\}$
- $\tilde{g}_{n-1}(e'_{n-2}, e'_{n-1}, e, e'') := (e'_{n-1}, g_{n-1}(e), g''_{n-1}(e''))$
- $\tilde{h}(e', e, e'') := h(e) = h''(e'')$

This construction was inspired by an e-mail from W.L.J. van der Kallen, who used a similar sequence to prove the statement in the more general setting of complexes. Note that for $i \in \{2, \dots, n-1\}$, similar to Z_1 we have $Z_i \simeq E'_{i-1} \oplus E_i \oplus E''_i$ since the second element in Z_i is fixed by the relation between the elements. Writing the Z_i in this way however does not simplify the proof in any way, since this just makes the maps $\tilde{g}_i$ more complicated.

It is easy to show that all $\tilde{g}_i$ are well defined because diagram (1) commutes. Furthermore, the equality $h(e) = h''(e'')$ in the definition of $\tilde{h}$ holds since for all $(e', e, e'') \in Z_n$, we have $\tilde{h}(e', e, e'') := h(e) = h'(\phi_n(e)) = h'(g'_{n-1}(e') + \phi'_n(e'')) = h'(\phi'_n(e'')) = h''(e'')$.

We verify that our sequence $\tilde{E}$ is exact:

- At A: We have $\ker \tilde{f} = \ker f \cap \ker f'' = \{0\}$.

- At Z_1:

  For any $(e, e'') \in \ker \tilde{g}_1$, we have $\ker g_1 \ni e = f(a) \in \text{im} f$ for some $a \in A$, $\ker g''_1 \ni e'' = f''(a'') \in \text{im} f''$ for some $a'' \in A$, and $0 = \phi_1(e) - \phi'_1(e'') = \phi_1(f(a)) - \phi'_1(f''(a'')) = f'(a) - f'(a'')$. So since f' is injective, we have $a = a''$, hence $(e, e'') \in \text{im}\tilde{f}$.

  On the other hand, for any $(e, e'') = (f(a), f''(a)) \in \text{im}\tilde{f}$ for some $a \in A$, we have $g_1(e) = g''_1(e'') = 0$ and $\phi_1(e) - \phi'_1(e'') = f'(a) - f'(a) = 0$, so $(e, e'') \in \ker \tilde{g}_1$. We conclude that $\ker \tilde{g}_1 = \text{im}\tilde{f}$.

- At Z_2: It is easy to see that $\tilde{g}_2(\tilde{g}_1(e, e'')) = (0, 0, g_2(g_1(e)), g''_2(g''_1(e''))) = (0, 0, 0, 0)$ for all $(e, e'') \in Z_1$, so $\text{im}\tilde{g}_1 \subseteq \ker \tilde{g}_2$.

  Furthermore, for any $(e'_1, e'_2, e, e'') \in \ker \tilde{g}_2$, we have $e'_2 = 0$, as well as $e = g_1(e_1)$ for some $e_1 \in E_1$ and $e'' = g''_1(e''_1)$ for some $e''_1 \in E''_1$. But since this element $(e'_1, 0, g_1(e_1), g''_1(e''_1))$ is an element of Z_2, we also have $g'_1(e'_1) = \phi_2(g_1(e_1)) - \phi_2(g''_1(e''_1)) = g'_1(\phi_1(e_1) - \phi'_1(e''_1))$. This gives us that $e'_1 - (\phi_1(e_1) - \phi'_1(e''_1)) \in \ker g'_1$, and therefore there is an element $a \in A$ such that $f'(a) = e'_1 - (\phi_1(e_1) - \phi'_1(e''_1))$. Now, we have an element $(e_1 + f(a), e''_1)$ which maps to our original element; we check:

  $\tilde{g}_1(e_1 + f(a), e''_1) = (\phi_1(e_1 + f(a)) - \phi'_1(e''_1), 0, g_1(e_1 + f(a)), g''_1(e''_1)) = (\phi_1(e_1) + f'(a) - \phi'_1(e''_1), 0, e, e'') = (e'_1, 0, e, e'')$.

  So we have $\ker \tilde{g}_2 \subseteq \text{im}\tilde{g}_1$ and we conclude that the sequence is exact at Z_2.

- At Z_i for $i \in \{3, \dots, n-1\}$:

  For any $(e'_{i-2}, e'_{i-1}, e, e'') \in Z_{i-1}$, we have $\tilde{g}_i(\tilde{g}_{i-1}(e'_{i-2}, e'_{i-1}, e, e'')) = (0, 0, g_i(g_{i-1}(e)), g''_i(g_{i-1}(e''))) = (0, 0, 0, 0)$, so $\text{im}\tilde{g}_{i-1} \subseteq \ker \tilde{g}_i$

  Now, suppose $(e'_{i-1}, e'_i, e, e'') \in \ker \tilde{g}_i$. Then we have $e'_i = 0$, as well as $e = g_{i-1}(e_{i-1})$ for some $e_{i-1} \in E_{i-1}$ and $e'' = g''_{i-1}(e''_{i-1})$ for some $e''_{i-1} \in E''_{i-1}$. Now, by definition of Z_i, we also have $g'_{i-1}(e'_{i-1}) = \phi_i(g_{i-1}(e_{i-1})) - \phi'_i(g''_{i-1}(e''_{i-1})) = g'_{i-1}(\phi_{i-1}(e_{i-1}) - \phi'_{i-1}(e''_{i-1}))$. This means that $(\phi_{i-1}(e_{i-1}) - \phi'_{i-1}(e''_{i-1})) - e'_{i-1} \in \ker g'_{i-1}$, so there exists an element $e'_{i-2} \in E'_{i-2}$ such that $g'_{i-2}(e'_{i-2}) = (\phi_{i-1}(e_{i-1}) - \phi'_{i-1}(e''_{i-1})) - e'_{i-1}$. Now we have an element $(e'_{i-2}, e'_{i-1}, e_{i-1}, e''_{i-1})$ in Z_{i-1}, since $g'_{i-2}(e'_{i-2}) + e'_{i-1} = \phi_{i-1}(e_{i-1}) - \phi'_{i-1}(e''_{i-1})$, and mapping this element with $\tilde{g}_{i-1}$ gives

  $\tilde{g}_{i-1}(e'_{i-2}, e'_{i-1}, e_{i-1}, e''_{i-1}) = (e'_{i-1}, 0, e, e'')$. We conclude that $\ker \tilde{g}_i \subseteq \text{im}\tilde{g}_{i-1}$ and therefore $\text{im}\tilde{g}_{i-1} = \ker \tilde{g}_i$.

(At Z_{n-1}, the proof is analogous to this general case, since $\tilde{g}_{n-1}$ has a very similar form to other $\tilde{g}_i$).

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & \dots & \xrightarrow{g_{i-2}} & E_{i-1} & \xrightarrow{g_{i-1}} & E_i & \xrightarrow{g_i} & \dots \\
 & & & & \downarrow{\scriptstyle \phi_{i-1}} & & \downarrow{\scriptstyle \phi_i} & & \\
\dots & \longrightarrow & E'_{i-2} & \xrightarrow{g'_{i-2}} & E'_{i-1} & \xrightarrow{g'_{i-1}} & E'_i & \xrightarrow{g'_i} & \dots \\
 & & & & {\scriptstyle \phi'_{i-1}}\uparrow & & {\scriptstyle \phi'_i}\uparrow & & \\
\dots & \longrightarrow & \dots & \xrightarrow{g''_{i-2}} & E''_{i-1} & \xrightarrow{g''_{i-1}} & E''_i & \xrightarrow{g''_i} & \dots
\end{array}$$

- At Z_n:

  For any $(e'_{n-2}, e'_{n-1}, e, e'') \in Z_{n-1}$ we have $\tilde{h}(\tilde{g}_{n-1}(e', e, e'')) = h(g_{n-1}(e)) = 0$ by definition, so $\mathrm{im}\tilde{g}_{n-1} \subset \ker \tilde{h}$.

  Also, for any $(e', e, e'') \in \ker \tilde{h}$, we have $\mathrm{im}g_{n-1} \ni e = g_{n-1}(e_{n-1})$ for some $e_{n-1} \in E_{n-1}$, and $\mathrm{im}g''_{n-1} \ni e'' = g''_{n-1}(e''_{n-1})$ for some $e''_{n-1} \in \mathrm{im}E''_{n-1}$. Now, by definition of Z_n we have $g'_{n-1}(e') = \phi_n(g_{n-1}(e_{n-1})) - \phi'_n(g''_{n-1}(e''_{n-1})) = g'_{n-1}(\phi_{n-1}(e_{n-1}) - \phi'_{n-1}(e''_{n-1}))$, so $e' - (\phi_{n-1}(e_{n-1}) - \phi'_{n-1}(e''_{n-1})) \in \ker g'_{n-1} = \mathrm{im}g'_{n-2}$, so we have some $e'_{n-2} \in E'_{n-2}$ such that $g'_{n-1}(e'_{n-2}) = e' - (\phi_{n-1}n(e_{n-1}) - \phi'_{n-1}(e''_{n-1}))$. Now we have an element $(e'_{n-2}, e', e_{n-1}, e''_{n-1}) \in Z_{n-1}$ that maps to (e', e, e''), similar to before. We conclude that $\mathrm{im}\tilde{g}_{n-1} = \ker \tilde{h}$.

- At B: For any $b \in B$, there exist $e \in E_n$ and $e'' \in E''_n$ such that $h(e) = h''(e'') = b$ since h and h'' are surjective. This gives us $h'(\phi_n(e) - \phi'_n(e'')) = 0$, so $\phi_n(e) - \phi'_n(e'') \in \mathrm{im}g'_{n-1}$ meaning there exists an $e' \in E'_{n-1}$ such that $g'_{n-1}(e') = \phi_n(e) - \phi'_n(e'')$. We see that (e', e, e'') is an element of Z_n which maps to b, so $\tilde{h}$ is surjective.

We conclude that $\tilde{E}$ is an exact (A, B)-n-sequence. We can now construct the following commutative diagram with exact sequences:

$$\begin{array}{cccccccccccccc}
E: & 0 & \longrightarrow & A & \xrightarrow{f} & E_1 & \xrightarrow{g_1} & \dots & \xrightarrow{g_{n-1}} & E_n & \xrightarrow{h} & B & \longrightarrow & 0 \\
 & & & \| & & \uparrow & & & & \uparrow & & \| & & \\
\tilde{E}: & 0 & \longrightarrow & A & \xrightarrow{\tilde{f}} & Z_1 & \xrightarrow{\tilde{g}_1} & \dots & \xrightarrow{\tilde{g}_{n-1}} & Z_n & \xrightarrow{\tilde{h}} & B & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & & & \downarrow & & \| & & \\
E'': & 0 & \longrightarrow & A & \xrightarrow{f''} & E''_1 & \xrightarrow{g''_1} & \dots & \xrightarrow{g''_{n-1}} & E''_n & \xrightarrow{h''} & B & \longrightarrow & 0
\end{array}$$

where the vertical maps up and down just project from Z_i onto E_i and E''_i respectively. So we have found an exact (A, B)-n-sequence $\tilde{E}$ with $E \leftarrow \tilde{E} \to E''$, proving the lemma. □

Proof of Theorem 3.1. The relation in definition 3.1 is clearly symmetric and reflexive. If we have exact (A, B)-n-sequences E, E' and E'' with $E \sim E'$ and

$E' \sim E''$, this means there exist exact (A, B)-n-sequences E_q and E'_q with $E \leftarrow E_q \rightarrow E' \leftarrow E'_q \rightarrow E''$. By lemma 3.2 we can therefore construct another exact (A, B)-n-sequence $\tilde{E}$ with $E \leftarrow E_q \leftarrow \tilde{E} \rightarrow E'_q \rightarrow E''$, so $E \sim E''$. Hence the relation is also transitive. $\square$

Now that we have established an equivalence relation on exact n-sequences, we define an operation that will be a group operation for our abelian group.

Definition 3.2. *Given two exact (A, B)-n-sequences E and E', their **Baer sum** is the exact (A, B)-n-sequence*

$$E \oplus E' := \qquad 0 \longrightarrow A \xrightarrow{\tilde{f}} Z_1 \xrightarrow{\tilde{g}_1} \dots \xrightarrow{\tilde{g}_{n-1}} Z_n \xrightarrow{\tilde{h}} B \longrightarrow 0$$

where

- $Z_1 := E_1 \coprod_A E'_1$,
- $Z_i := E_i \oplus E'_i$ *for* $i \in \{2, \dots, n-1\}$,
- $Z_n := E_n \prod_B E'_n$,
- $\tilde{f}(a) := (f(a), 0) = (0, f'(a))$,
- $\tilde{g}_i(e, e') := (g_i(e), g'_i(e'))$ *and*
- $\tilde{h}(e, e') := h(e) = h'(e')$.

Note that $\tilde{g}_{n-1}$ is well-defined, since for any pair $(e, e') \in E_{n-1} \oplus E'_{n-1}$, we have $h(g_{n-1}(e)) = h'(g'_{n-1}(e')) = 0$, so $(g_{n-1}(e), g'_{n-1}(e')) \in E_n \prod_B E'_n$.

We verify that the Baer sum of two exact (A, B)-n-sequences is indeed an exact sequence:

- At A: For any $a \in A$, $\tilde{f}(a) = 0 \implies f(a) = 0 \implies a = 0$, so $\tilde{f}$ is injective.
- At Z_1: For any $(e, e') \in E_1 \coprod_A E'_1 = Z_1$, we have $(e, e') \in \ker \tilde{g}_1$ if and only if $e \in \ker g_1 = \mathrm{im} f$ and $e' \in \ker g'_1 = \mathrm{im} f'$ which is the case if and only if $(e, e') \in \mathrm{im} \tilde{f}$, since $\tilde{f}(a + a') = (f(a), f(a'))$.
- At Z_i for $i \in \{2, \dots, n-1\}$: We trivially have $\ker \tilde{g}_i = \ker g_i \oplus \ker g'_i = \mathrm{im} g_{i-1} \oplus \mathrm{im} g_{i-1} = \mathrm{im} \tilde{g}_{i-1}$.
- At Z_n: For any $(e, e') \in E_n \prod_B E'_n = Z_n$, we have $(e, e') \in \ker \tilde{h}$ if and only if $e \in \ker h = \mathrm{im} g_{n-1}$ and $e' \in \ker h' = \mathrm{im} g'_{n-1}$, which is the case if and only if $(e, e') \in \mathrm{im} \tilde{g}_{n-1}$, since for any $(e_{n-2}, e'_{n-2}) \in Z_{n-1}$, we have $\tilde{g}_{n-1}(e + g_{n-2}(a), e' + g'_{n-2}(a')) = (g_{n-1}(e), g'_{n-1}(e'))$.
- At B: For every $b \in B$, we have some $e \in E_n$ and $e' \in E'_n$ such that $h(e) = h'(e') = b = \tilde{h}(e, e')$, so $\tilde{h}$ is surjective.

3.2 Abelian group structure

Theorem 3.3. *The set of equivalence classes of exact* (A, B)*-*n*-sequences, denoted* $\mathrm{Ext}^n(B, A)$*, together with the Baer sum* $(\oplus)$*, is an abelian group.*

Remark 3.3. *Formally, the notation* $Ext^n(B, A)$ *has another meaning, namely that of the Ext functor computed using the homology of an injective resolution. However, there is a bijective correspondence between elements of this* $Ext^n(B, A)$ *and equivalence classes of exact sequences. In the literature, the latter group is denoted* $YExt^n(B, A)$ *or the Yoneda-Ext group, introduced by Nobuo Yoneda in 1960. In this paper, we will just regard elements of* $Ext^n(B, A)$ *as these equivalence classes.*

For two exact (A, B)-n-sequences E and E', we define $[E] \oplus [E'] := [E \oplus E']$. It is easy to show that this operation is well-defined, that is, independent of the chosen representatives of $[E]$ and $[E']$. We have already shown that it is closed, and it is also a simple exercise to show that it is commutative and associative. To conclude the proof, we will define the identity element and the inverse element under the Baer sum.

Definition 3.3. *The sequence* 0_n *with* $[0_n] \in Ext^n(B, A)$ *is defined as the exact* (A, B)*-*n*-sequence*

$$0_n := \quad 0 \longrightarrow A = A \xrightarrow{0} 0 \xrightarrow{0} \dots \xrightarrow{0} 0 \xrightarrow{0} B = B \longrightarrow 0$$

This sequence is clearly exact, and we verify that it behaves as the identity element under the Baer sum, that is, $[0_n \oplus E] = [E]$ for any exact (A, B)-n-sequence E. We construct the Baer sum $0_n \oplus E$ and get:

$$0 \longrightarrow A \longrightarrow A \coprod_A E_1 \longrightarrow 0 \oplus E_2 \longrightarrow \dots \longrightarrow B \prod_B E_n \longrightarrow B \longrightarrow 0\ .$$

We will show that the pushout and pullback in this sequence are isomorphic to E_1 and E_n respectively. To see this, look at the homomorphism $\phi : E_1 \to A \coprod_A E_1$ defined by $\phi(e) := (0, e)$, which admits an inverse homomorphism $\phi^{-1} : A \coprod_A E_1 \to E_1$ by $\phi^{-1}(a, e) := e + f(a)$.

Similarly, we have a homomorphism $\psi : E_n \to B \prod_B E_n$ by $\psi(e) := (g(e), e)$ which admits an inverse $\psi^{-1} : B \prod_B E_n \to E_n$ by $\psi^{-1}(b, e) := e$.

It is easy to see that these ϕ and ψ give rise to a commutative diagram such that $E \leftarrow E \to 0_n \oplus E$, so $0_n \oplus E \sim E$ as desired.

Definition 3.4. *For any exact* (A, B)*-*n*-sequence* E*, we define its inverse* E^{-1} *to be the exact* (A, B)*-*n*-sequence*

$$E^{-1} := \qquad 0 \longrightarrow A \xrightarrow{-f} E_1 \xrightarrow{g_1} \dots \xrightarrow{g_{n-1}} E_n \xrightarrow{h} B \longrightarrow 0$$

This is clearly also an exact sequence, and we claim that this is the inverse element of E under the Baer sum, that is, $[E \oplus E^{-1}] = [0_n]$ for every exact

(A, B)-n-sequence E. To see this, we look at the following commutative diagram of exact sequences (written in two halves for readability):

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & A & \xrightarrow{\tilde{f}} & E_1 \coprod_A E_1 & \xrightarrow{\tilde{g}_1} & E_2 \oplus E_2 & \xrightarrow{\tilde{g}_2} & E_3 \oplus E_3 & \xrightarrow{\tilde{g}_3} & \dots \\
 & & \Vert & & \uparrow{\scriptstyle \phi} & & \uparrow{\scriptstyle \psi_2} & & \uparrow{\scriptstyle \psi_3} & & \\
0 & \longrightarrow & A & \xrightarrow{i_1} & A \oplus (E_1/f(A)) & \xrightarrow{g_1 \circ \pi_2} & E_2 & \xrightarrow{g_2} & E_3 & \xrightarrow{g_3} & \dots \\
 & & \Vert & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & A & = & A & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & \dots
\end{array}$$

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{\tilde{g}_{n-2}} & E_{n-1} \oplus E_{n-1} & \xrightarrow{\tilde{g}_{n-1}} & E_n \coprod_B E_n & \xrightarrow{\tilde{h}} & B & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \psi_{n-1}} & & \uparrow{\scriptstyle \psi_n} & & \Vert & & \\
\dots & \xrightarrow{g_{n-2}} & E_{n-1} & \xrightarrow{g_{n-1}} & E_n & \xrightarrow{h} & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle h} & & \Vert & & \\
\dots & \xrightarrow{0} & 0 & \xrightarrow{0} & B & = & B & \longrightarrow & 0
\end{array}$$

with:

- $\tilde{f}(a) = (f(a), 0) = (0, -f(a))$,
- $\tilde{g}_i(e, e') = (g_i(e), g_i(e'))$,
- $\tilde{h}(e, e') = h(e) = h(e')$,
- $\phi(a, e) = (e + f(a), e) = (e, e - f(a))$,
- $\psi_i(e) = (e, e)$

We conclude that $E \oplus E^{-1} \sim 0_n$, as desired.

(The middle sequence can be derived by taking the pushout $A \coprod_A E_1$ with respect to the maps $A \xrightarrow{0} A$ and $A \xrightarrow{f} E_1$ as the second element, which is isomorphic to $A \oplus (E_1/f(A))$.)

Now we have an abelian group $\text{Ext}^n(B, A)$, with the operation $(\oplus)$ which is closed, commutative, associative, has an identity element and an inverse for every element. This concludes our proof of theorem 3.3.

□

3.3 Concatenating sequences

In section 2.2, we have seen that we can concatenate short exact sequences, and this procedure trivially extends to exact n-sequences. In this way, by concatenating an exact (A, B)-n sequence and an exact (B, C)-m-sequence, we get an exact (A, C)-$(n + m - 1)$-sequence. It is easy to check that this operation induces a map $\circ : \text{Ext}^n(B, A) \times \text{Ext}^m(C, B) \to \text{Ext}^{n+m-1}(C, A)$ on the equivalence classes of sequences. The following lemma shows that this map is bilinear, which gives it some properties that we will need later.

Lemma 3.4. *The map* $\circ : Ext^n(B,A) \times Ext^m(C,B) \to Ext^{n+m-1}(C,A)$ *is bilinear.*

Proof. We want to show that given any exact (A,B)-n-sequence α, two exact (B,C)-m-sequences β and β' and two exact (D,A)-k-sequences γ and γ', the following equalities hold:

$$[\alpha] \circ [\beta \oplus \beta'] = [(\alpha \circ \beta) \oplus (\alpha \circ \beta')]$$

$$[\gamma \oplus \gamma'] \circ [\alpha] = [(\gamma \circ \alpha) \oplus (\gamma' \circ \alpha)]$$

We write our sequences as:

$$\alpha := \ 0 \longrightarrow A \longrightarrow E_1 \longrightarrow \dots \longrightarrow E_n \longrightarrow B \longrightarrow 0$$

$$\beta := \ 0 \longrightarrow A \longrightarrow F_1 \longrightarrow \dots \longrightarrow F_m \longrightarrow B \longrightarrow 0$$

$$\gamma := \ 0 \longrightarrow A \longrightarrow G_1 \longrightarrow \dots \longrightarrow G_k \longrightarrow B \longrightarrow 0$$

and β' and γ' as β and γ respectively, with accents on the F_i and G_i. For the first equality we want to show that $(\alpha\circ\beta)\oplus(\alpha\circ\beta') \to \alpha\circ(\beta\oplus\beta')$, which implies that the two sequences are in the same equivalence class. We can compute both these sequences explicitly with the definition of the Baer sum. The following homomorphisms from $(\alpha \circ \beta) \oplus (\alpha \circ \beta')$ to $\alpha \circ (\beta \oplus \beta'$ give rise to a commuting diagram as in remark (3.1). This is an easy verification, and proves the first equality.

$\phi_1 : E_1 \coprod_A E_1 \to E_1$
$\phi_1(a,b) := a + b$

$\phi_{n+1} : F_1 \oplus F_1' \to F_1$
$\phi_{n+1}(a,b) := (a,b)$

$\phi_i : E_i \oplus E_i \to E_i, i \in \{2,\dots,n\}$
$\phi_i(a,b) := a + b$

$\mathrm{id} : F_j \oplus F_j' \to F_j \oplus F_j', j \in \{2,\dots,m-1\}$
$\mathrm{id} : F_m \coprod_C F_m' \to F_m \coprod_C F_m'$

For the second equality, we take a similar approach and give homomorphisms such that $(\gamma \oplus \gamma') \circ \alpha \to (\gamma \circ \alpha) \oplus (\gamma' \circ \alpha)$.

$\mathrm{id} : G_1 \coprod_D G_1' \to G_1 \coprod_D G_1'$
$\mathrm{id} : G_i \oplus G_i' \to G_i \oplus G_i', i \in \{2,\dots,k-1\}$

$\psi_{k+j} : E_j \to E_j \oplus E_j, j \in \{1,\dots,n-1\}$
$\psi_{k+j}(a) := (a,a)$

$\psi_k : G_k \prod_A G_k' \to G_k \oplus G_k'$
$\psi_k(a,b) := (a,b)$

$\psi_{k+n} : E_n \to E_n \prod_B E_n$
$\psi_{k+n}(a) := (a,a)$

□

In particular, this lemma implies that $[\alpha] \circ [0_m] = [0_{n+m}]$, since $[\alpha] \circ [0_m] = [\alpha] \circ [0_m \oplus 0_m] = 2([\alpha] \circ [0_m])$, and similarly $[0_k] \circ [\alpha] = [0_{k+n}]$. It is also easy to check that $[\alpha] \circ [-\beta] = [-\alpha] \circ [\beta] = [-(\alpha \circ \beta)]$.

4 Filtration theorem

The following theorem, which has been discussed by Alexander Grothendieck in his famous seminar [1], will explicitly describe the form of exact (A, B)-2-sequences with class $[0_2]$, which we will extensively use in the proof of our main theorem on commutative diagrams.

Theorem 4.1. (Filtration theorem) *For an exact (A, B)-2-sequence E, the following are equivalent:*

1) $[E] = [0_2] \in Ext^2(B, A)$.

2) There exist X_2, X_1 and X, with $X_2 \subseteq X_1 \subseteq X$ and isomorphisms Φ, Ψ, Θ and Δ, such that the diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & B & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \Psi} & & \Big\downarrow{\scriptstyle \Theta} & & \Big\downarrow{\scriptstyle \Delta} & & \\ 0 & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X/X_2 & \longrightarrow & X/X_1 & \longrightarrow & 0 \end{array} \tag{2}$$

commutes.

Proof. 2) $\Longrightarrow$ 1): We want to find an exact $(X_2, X/X_1)$-2-sequence

$$0 \longrightarrow X_2 \hookrightarrow U \longrightarrow V \twoheadrightarrow X/X_1 \longrightarrow 0$$

such that the diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & X_2 & \hookrightarrow & X_1 & \longrightarrow & X/X_2 & \twoheadrightarrow & X/X_1 & \longrightarrow & 0 \\ & & \| & & \Big\uparrow & & \Big\uparrow & & \| & & \\ 0 & \longrightarrow & X_2 & \hookrightarrow & U & \longrightarrow & V & \twoheadrightarrow & X/X_1 & \longrightarrow & 0 \\ & & \| & & \Big\downarrow & & \Big\downarrow & & \| & & \\ 0 & \longrightarrow & X_2 & = & X_2 & \xrightarrow{0} & X/X_1 & = & X/X_1 & \longrightarrow & 0 \end{array}$$

commutes, since clearly the top sequence has equivalence class $[0_2] \in \mathrm{Ext}^2(X/X_1, X_2)$ if and only if $[E] = [0_2] \in \mathrm{Ext}^2(B, A)$.

We try the following:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & X_2 & \hookrightarrow & X_1 & \longrightarrow & X/X_2 & \twoheadrightarrow & X/X_1 & \longrightarrow & 0 \\ & & \| & & {\scriptstyle \pi_1+\pi_2}\Big\uparrow & & \Big\uparrow & & \| & & \\ 0 & \longrightarrow & X_2 & \xhookrightarrow{i_1} & X_2 \oplus X_1 & \xrightarrow{\pi_2} & X & \twoheadrightarrow & X/X_1 & \longrightarrow & 0 \\ & & \| & & \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow & & \| & & \\ 0 & \longrightarrow & X_2 & = & X_2 & \xrightarrow{0} & X/X_1 & = & X/X_1 & \longrightarrow & 0 \end{array}$$

Here, it's an easy verification that the middle row is indeed exact, and that the diagram commutes.

1) $\implies$ *2):*

Since we have $[E] = [0_2]$, there exists a commutative diagram of exact sequences of the following form:

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & A & \overset{f}{\hookrightarrow} & E_1 & \overset{g}{\longrightarrow} & E_2 & \longrightarrow\!\!\!\!\rightarrow & B & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \rho} & & \| & & \\
0 & \longrightarrow & A & \overset{f'}{\hookrightarrow} & U & \overset{g'}{\longrightarrow} & V & \overset{h'}{\longrightarrow\!\!\!\!\rightarrow} & B & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \phi} & & \downarrow & & \| & & \\
0 & \longrightarrow & A & = & A & \overset{0}{\longrightarrow} & B & = & B & \longrightarrow & 0
\end{array}
\tag{3}
$$

We see that $\mathrm{im} f \subseteq E_1$. It now makes sense to try to find a X with $\mathrm{im} f \subseteq E_1 \subseteq X$ and $E_2 \simeq X/\mathrm{im} f$.

We try $X := E_1 \coprod_{\ker\phi} V$ (recall: this means $X = (E_1 \oplus V)/\{(\psi(u), -g'(u)) | u \in \ker\phi\}$). This pushout can be visualized in the following diagram.

$$
\begin{array}{ccc}
E_1 & \overset{i_1}{\longrightarrow} & X \\
\uparrow{\scriptstyle \psi} & & \uparrow \\
\ker\phi & \overset{g'}{\longrightarrow} & V
\end{array}
$$

In diagram (3), we see that for $x \in \ker\phi$ we have $x \notin \mathrm{im} f' = \ker g'$, since $\phi \circ f' = \mathrm{id}_A$. This means that $g' : \ker\phi \to V$ is injective, so recalling lemma (2.1), this means that $i_1 : E_1 \to X$ is also injective. Now, since X is a quotient module of $E_1 \oplus V$, this injectivity means that $E_1 \oplus \{0\} \subseteq X$. This gives us $\mathrm{im} f \oplus \{0\} \subseteq E_1 \oplus \{0\} \subset X$, so we set $X_2 := \mathrm{im} f \oplus \{0\}$ and $X_1 := E_1 \oplus \{0\}$.

To verify that $E_2 \simeq X/X_2$, we construct the exact (A, B)-2-sequence

$$
0 \longrightarrow A \overset{f}{\hookrightarrow} E_1 \overset{\tilde{i}_1}{\longrightarrow} X/X_2 \overset{h' \circ \pi_2}{\longrightarrow\!\!\!\!\rightarrow} B \longrightarrow 0
$$

We verify its exactness:

- At A we already know the sequence is exact.
- At E_1: We have $\ker \tilde{i}_1 = \mathrm{im} f$ by construction of X/X_2.
- At X/X_2: For any $(x, y) \in X/X_2$, we have $(x, y) \in \ker(h' \circ \pi_2)$ if and only if $y \in \ker h' = \mathrm{im} g'$, in other words if and only if there exists a $z \in \ker\phi$ with $g'(z) = y$. This means that $(x, y) = (x, g'(z)) = (x + \psi(z), 0)$, which is exactly the form of all elements in $\mathrm{im}\tilde{i}_1$.
- At B: We have $\mathrm{im} h' \circ \pi_2 = \mathrm{im} h' = B$.

With this exact 2-sequence, we have the following commutative diagram:

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & A & \hookrightarrow & E_1 & \overset{i_1}{\longrightarrow} & X/X_2 & \overset{h' \circ \pi_2}{\longrightarrow\!\!\!\!\rightarrow} & B & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow{\scriptstyle \tau} & & \| & & \\
0 & \longrightarrow & A & \hookrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow\!\!\!\!\rightarrow & B & \longrightarrow & 0
\end{array}
$$

where $\tau(x, y) := g(x) + \rho(y)$. By the five lemma τ is an isomorphism, and so we have $E_2 \simeq X/X_2$.

The last thing we need to verify is that our isomorphisms actually give rise to a commutative diagram of the form of (2). We have the commutative diagram (2) with the following definitions.

$$\begin{array}{ll} X_2 := \mathrm{im} f \oplus \{0\} & \Phi(a) := (f(a), 0) \\ X_1 := E_1 \oplus \{0\} & \Psi(e) := (e, 0) \\ X := E_1 \coprod_{\ker \phi} V & \Theta(e) := \tau^{-1}(e) \\ & \Delta(b) := \tau^{-1}(\xi(b)) \end{array}$$

Here, $\xi : B \to X/X_1$ is constructed as follows: Use the first isomorphism theorem on the short exact sequence

$$0 \longrightarrow E_1/\mathrm{im} f \xrightarrow{k} E_2 \xrightarrow{h} B \longrightarrow 0 \ ,$$

which can be derived from E immediately as in (0.2), to get an isomorphism $\alpha : B \to E_2/(\mathrm{im} k)$. We then get the following isomorphisms:

$$B \xrightarrow{\alpha} E_2/(\mathrm{im} k) \xrightarrow{\tau^{-1}} (X/X_2)/(X_1/X_2) \xrightarrow{\beta} X/X_1$$

where $\beta : (X/X_2)/(X_1/X_2)$ exists by the third isomorphism theorem and $\mathrm{im} k \simeq X_1/X_2$ since k is injective. We then define $\xi := \beta \circ \tau^{-1} \circ \alpha$, which is a composition of isomorphisms and therefore also an isomorphisms. We see that for $b = h(e) \in B$, $\xi(b) = \tau^{-1}(e) + E_1 \oplus \{0\}$.

For these definitions, diagram (2) commutes, concluding our proof of our filtration theorem. □

5 Diagram theorem

We will now formulate the main theorem of this paper. We start with four short exact sequences in a commutative diagram of the following form

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & A & \overset{e}{\hookrightarrow} & E & \overset{b}{\twoheadrightarrow} & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle f} & & \\
 & & G & & & & F & & \\
 & & \downarrow{\scriptstyle c} & & & & \downarrow{\scriptstyle d_f} & & \\
0 & \longrightarrow & C & \overset{h}{\hookrightarrow} & H & \overset{d_h}{\twoheadrightarrow} & D & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

Now we ask ourselves: what are the necessary conditions to complete this diagram with a Z, such that the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \vdots & & \downarrow & & \\
0 & \longrightarrow & A & \overset{e}{\hookrightarrow} & E & \overset{b}{\twoheadrightarrow} & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \vdots & & \downarrow{\scriptstyle f} & & \\
0 & \dashrightarrow & G & \dashrightarrow & Z & \dashrightarrow & F & \dashrightarrow & 0 \\
 & & \downarrow{\scriptstyle c} & & \vdots & & \downarrow{\scriptstyle d_f} & & \\
0 & \longrightarrow & C & \overset{h}{\hookrightarrow} & H & \overset{d_h}{\twoheadrightarrow} & D & \longrightarrow & 0 \\
 & & \downarrow & & \vdots & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{4}$$

commutes and all rows and columns are short exact sequences?

To answer this question, we first look at the sequences in the top row and the right column, and see that we can connect these two short exact sequences as described in 2.2. We can do the same thing for the sequences in the left column and the bottom row. Now we take the Baer sum of the resulting exact 2-sequences, and define:

$$W := \ 0 \longrightarrow A \hookrightarrow G \coprod\nolimits_A E \longrightarrow H \prod\nolimits_D F \twoheadrightarrow D \longrightarrow 0$$

Theorem 5.1. (Diagram theorem) *With definitions as above, the following are equivalent:*

1) $[W] = [0_2]$

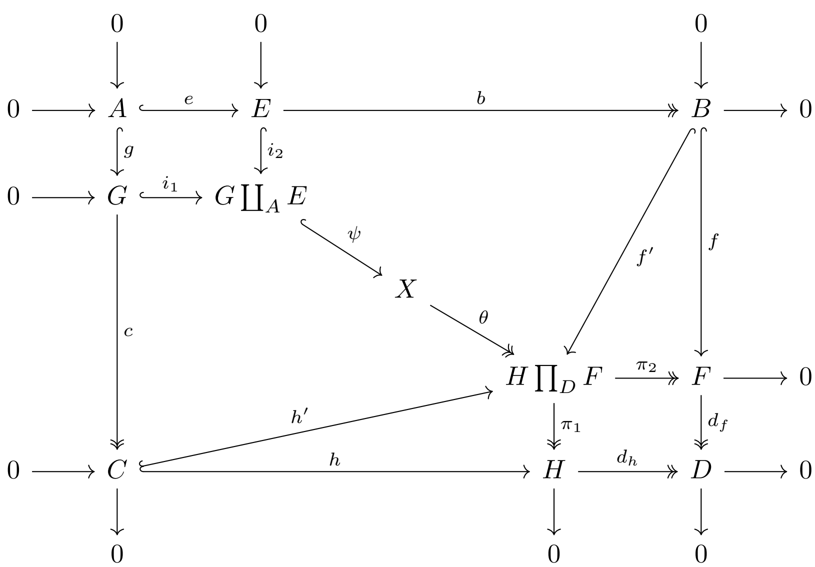
2) There exists a Z *with which we can complete diagram* (4)*, so that all sequences are exact and the diagram commutes.*

Proof. 1) $\Longrightarrow$ *2):*

We immediately apply our filtration theorem 4.1 to W, and we get a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A & \hookrightarrow & G\coprod_A E & \longrightarrow & H\prod_D F & \twoheadrightarrow & D & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Theta} & & \downarrow{\scriptstyle\Delta} & & \\ 0 & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X/X_2 & \longrightarrow & X/X_1 & \longrightarrow & 0 \end{array} \tag{5}$$

where all vertical maps are isomorphisms. We now try to complete our diagram (4) with $Z := X$. We need to verify that the resulting diagram commutes and that the middle column and row are exact. We have the following diagram:



Here, $\psi : G\coprod_A E \to X$ is defined as $\psi = i \circ \Psi$, where $i : X_1 \to X$ is the inclusion map, so ψ is clearly injective. On the other hand, $\theta : X \to H\prod_D F$ is defined as $\theta = \Theta^{-1} \circ \pi$, where $\pi : X \to X/X_2$ is the projection map, and this is clearly surjective. The injectivity of i_1 and i_2, as well as the surjectivity of π_1 and π_2, follow from lemma (2.1).

We first check that the diagram commutes. The top left and bottom right squares commute by the definitions of pushouts and pullbacks. Furthermore, by the universal property of the pushout $G\coprod_A E$, using the maps $f' \circ b : E \to H\prod_D F$ and $h' \circ c : G \to H\prod_D F$, the map $\theta \circ \psi : G\coprod_A E \to H\prod_D F$ must be the unique map that makes all these maps commute. This means the top right and bottom left squares also commute.

All that remains now is to check the exactness of the middle column and row at X. These are completely symmetric cases. We look at the middle column, that is, we check that the sequence

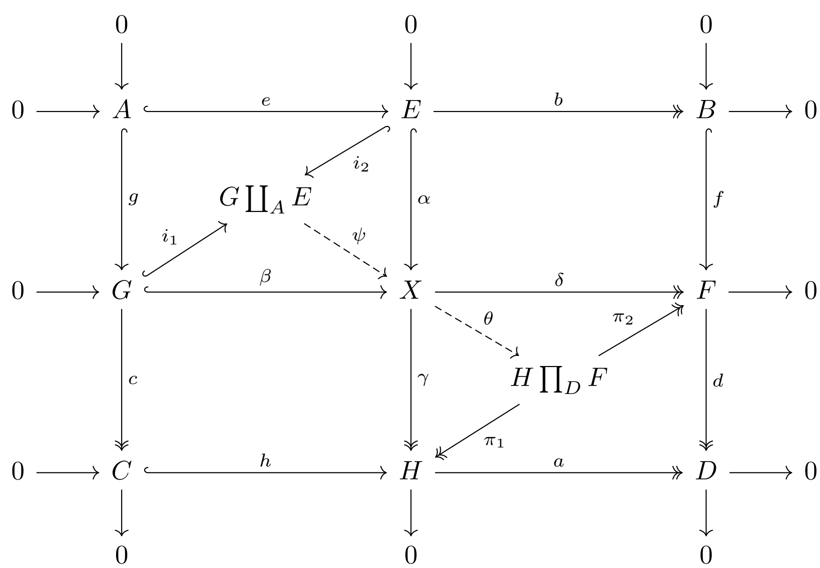
$$0 \longrightarrow E \xhookrightarrow{\psi\circ i_2} X \xrightarrow{\pi_1\circ\theta} H \longrightarrow 0$$

is exact.

- We have $\text{im}(\psi \circ i_2) \subseteq \ker(\pi_1 \circ \theta)$ since $\pi_1(\theta(\psi(i_2(e)))) = \pi_1(f'(b(e))) = 0$.
- Suppose $\pi_1(\theta(x)) = 0$. This means $\theta(x) \in \ker \pi_1$, so $\exists b \in B$ with $f'(b) = \theta(x)$, so $\exists e \in E$ with $\theta(\psi(i_2(e))) = \theta(x)$. This means that $\theta(x - \psi(i_2(e))) = 0$, so $\exists a \in A$ that maps to $x - \psi(i_2(e))$ in X. So $e + e(a)$ maps to x in X, which means that $x \in \text{im}(\psi \circ i_2)$.

So all rows and columns are exact and the diagram commutes, and we are done.

2) $\implies$ *1):*



We are given the above commutative diagram of exact sequences, where the maps with dashed arrows are induced by the universal properties of the pushout and pullback. We now want to show $[W] = [0_2]$. We take $X_2 := \text{im}(\alpha \circ e)$, $X_1 := \text{im}\psi$ and X to see if the conditions in our filtration theorem hold.

Since $\text{im}(\alpha \circ e) = \text{im}(\psi \circ i_2 \circ e) \subseteq \text{im}\psi$, we have $X_2 \subseteq X_1$, and obviously $\text{im}\psi = X_1 \subseteq X$. Next, we verify that all four isomorphisms in the filtration theorem exist.

- Since $\alpha \circ e$ is injective, we have $X_2 \simeq A$.
- For X_1, we now show that ψ is also injective. Suppose $\psi(g, e) = 0 = \beta(g) + \alpha(e)$. This means that $\beta(g) = -\alpha(e)$. We then have $\gamma(\beta(g)) = 0 = h(c(g))$, and since h is injective this means $c(g) = 0$. Therefore $g \in \text{im}g$, so we have an $a \in A$ with $g(a) = g$. Similarly, $e(a') = e$ for some $a' \in A$. In $G \coprod_A E$, by construction we have the equality $(g(a), e(a')) = (0, e(a+a'))$.

We now have $\beta(g(0)) = 0 = -\alpha(e(a+a'))$, and because α is injective we have $e(a+a') = 0$, so $(g,e) = (0,0)$.

This means that ψ is injective, and therefore $X_1 \simeq G \coprod_A E$.

- For X/X_2, we start off with our map $\theta : X \to H \prod_D F$, given by $\theta(x) = (\gamma(x), \delta(x))$. We prove that $\tilde{\theta} : X/X_2 \to H \prod_D F$ given by $\tilde{\theta} = \pi \circ \theta$ is bijective.

  If for some $x \in X$, $\tilde{\theta}(x) = 0$, then $x \in \ker \gamma$ and $x \in \ker \delta$ so for some $g \in G$ and $e \in E$, we have $x = \beta(g) = \alpha(e)$. Then $(g, 0) = (0, e) \in G \coprod_A E$ because ψ is injective, and also $g \notin \text{im}g$, $e \notin \text{im}e$ because $x \in X/X_2$. So this equality implies $g = 0$ and $e = 0$, so $x = 0$, meaning $\tilde{\theta}$ is injective.

  As for the surjectivity, for any $a \in A$ we have $\theta(\alpha(e(a))) = 0$ so $\text{im}\theta = \text{im}\tilde{\theta}$. We verify that θ is surjective.

  Take any $(h, f) \in H \prod_D F$. By definition of the pullback there exists a $d \in D$ with $a(h) = d(f) = d$. Since $a \circ \gamma$ and $d \circ \delta$ are surjective, there also exsits an $x \in X$ with $a(\gamma(x)) = d(\delta(x)) = d$. We now have $a(\gamma(x) - h) = 0$, so there exists a $c \in C$ with $h(c) = \gamma(x) - h$. Since c is surjective, there exists a $g \in G$ such that $h(c(g)) = \gamma(\beta(g)) = \gamma(x) - h$. We now find $h = \gamma(x - \beta(g))$. If we do the exact same procedure on the other side of the diagram, we analogously find some $e \in E$ such that $f = \delta(x - \alpha(e))$. Now we have the element $x - \beta(g) - \alpha(e) \in X$, with the property that $\theta(x - \beta(g) - \alpha(e)) = (\gamma(x - \beta(g)) - \gamma(\alpha(e)), \delta(x - \alpha(e)) - \delta(\beta(g))) = (h, f)$. So θ is surjective.

  So we have an isomorphism $\tilde{\theta} : X/X_2 \simeq H \prod_D F$.

- Finally, to show that $X/X_1 \simeq D$, we first show that the sequence

$$0 \longrightarrow X_1 \hookrightarrow X \xrightarrow{d\circ\delta = a\circ\gamma} D \longrightarrow 0$$

  is exact. We have $X_1 \subseteq \ker \alpha \circ \gamma$, since $\text{im}\psi \subseteq (\text{im}\alpha \cup \text{im}\beta) \subseteq \ker(a \circ \gamma) = \ker(d \circ \delta)$.

  Now, suppose for some $x \in X$, $\delta(d(x)) = a(\gamma(x)) = 0$. Then, $\gamma(x) \in \ker a$, so there exists a $g \in G$ with $\gamma(\beta(g)) = \gamma(x)$. Now we have $x - \beta(g) \in \ker \gamma$, so there exists an $e \in E$ with $\alpha(e) = x - \beta(g)$. Therefore, for these elements we have $\psi(g, e) = x$, so $x \in \text{im}\psi = X_1$.

  So our sequence is indeed exact, and by applying the first isomorphism theorem, this short exact sequence now gives us $D \simeq X/X_1$, with the explicit isomorphism $\nu : X/X_1 \to D$ given by $\nu(x) := a(\gamma(x)) = d(\delta(x))$.

With these four isomorphisms, we now get diagram (5) back, this time with:

$$\begin{array}{cc} X_2 := \text{im}(\alpha \circ e) & X_1 := \text{im}\psi \\ X := X & \Phi := \alpha \circ e \\ \Psi := \psi & \Theta := \tilde{\theta}^{-1} \\ & \Delta := \nu \end{array}$$

Looking at our constructed 3×3-diagram, it is easy to see that for these definitions our diagram (5) commutes. Now we have satisfied all conditions for the filtration theorem, and we conclude that $[W] = [0_2]$. $\square$

6 Extended diagram theorem

Our diagram theorem raises the question if any similar statement can be made for larger diagrams, that is, if the four short exact sequences forming the edges of diagram (4) were two exact n-sequences and two exact m-sequences, for $n, m \geq 3$, can we construct an analogous requirement on these sequences for the diagram to be able to be completed?

The primary problem here is that in these larger diagrams, the sequence W will no longer be an exact 2-sequence, and for $n > 2$ we cannot describe the form of exact n-sequences with class $[0_n]$ as precisely as we did for exact 2-sequences in our filtration theorem. Therefore, assumptions on $[W]$ will not give us as much to work with in the general case as in the previous chapter. However, assuming that a diagram can be completed does give us a lot of information, and we will actually use the diagram theorem from the previous chapter to prove something to similar to one implication (specifically (2) $\implies$ (1)) of the diagram theorem.

Definition 6.1. *Let $n, m \in \mathbb{N}_{\geq 3}$. An* ***exact*** (n, m)***-diagram*** *is a commutative diagram of the form:*

$$
\begin{array}{ccccccccccccc}
 & & 0 & & & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_{1,1} & \longrightarrow & \dots & \longrightarrow & E_{n-2,1} & \xrightarrow{f_{n-1,1}} & E_{n-1,1} & \xrightarrow{f_{n,1}} & E_{n,1} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow{\scriptstyle g_{n-2,2}} & & \downarrow{\scriptstyle g_{n-1,2}} & & \downarrow{\scriptstyle g_{n,2}} & & \\
0 & \longrightarrow & E_{1,2} & \longrightarrow & \dots & \longrightarrow & E_{n-2,2} & \xrightarrow{f_{n-1,2}} & E_{n-1,2} & \xrightarrow{f_{n,2}} & E_{n,2} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow{\scriptstyle g_{n-2,3}} & & \downarrow{\scriptstyle g_{n-1,3}} & & \downarrow{\scriptstyle g_{n,3}} & & \\
 & & \dots & & \dots & & \dots & & \dots & & \dots & & \\
 & & \downarrow & & & & \downarrow{\scriptstyle g_{n-2,m-1}} & & \downarrow{\scriptstyle g_{n-1,m-1}} & & \downarrow{\scriptstyle g_{n,m-1}} & & \\
0 & \longrightarrow & E_{1,m-1} & \longrightarrow & \dots & \longrightarrow & E_{n-2,m-1} & \xrightarrow{f_{n-1,m-1}} & E_{n-1,m-1} & \xrightarrow{f_{n,m-1}} & E_{n,m-1} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow{\scriptstyle g_{n-2,m}} & & \downarrow{\scriptstyle g_{n-1,m}} & & \downarrow{\scriptstyle g_{n,m}} & & \\
0 & \longrightarrow & E_{1,m} & \longrightarrow & \dots & \longrightarrow & E_{n-2,m} & \xrightarrow{f_{n-1,m}} & E_{n-1,m} & \xrightarrow{f_{n,m}} & E_{n,m} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & & & 0 & & 0 & & 0 & &
\end{array}
$$

where all rows and columns are exact sequences.

In an exact (n, m)-diagram, we call:

- the (upper) $(E_{1,1}, E_{n,1})$-$(n-2)$-sequence α;
- the (left) $(E_{1,1}, E_{1,m})$-$(m-2)$-sequence β;
- the (right) $(E_{n,1}, E_{n,m})$-$(m-2)$-sequence γ;

- the (bottom) $(E_{1,m}, E_{n,m})$-$(n-2)$-sequence δ.

Theorem 6.1. (Extended diagram theorem) *Given an exact (n,m)-diagram with definitions as above, the following equality holds:*

$$[\alpha \circ \gamma] = (-1)^{nm} \cdot [\beta \circ \delta] \in Ext^{n+m-4}(E_{n,m}, E_{1,1})$$

The proof of this theorem relies heavily on the following theorem, which lets us use our diagram theorem from the previous chapter in this more general context.

Theorem 6.2. *Let $n, m \in \mathbb{N}_{\geq 3}$. If $n > 3$, then we can split any exact (n,m)-diagram into an exact $(n-1,m)$-diagram(of the form (7)) and an exact $(3,m)$-diagram (of the form (6)).*

Proof. We first construct the $(3,m)$-diagram. For an exact (n,m)-diagram as above, we define $F_i := \mathrm{im} f_{n-1,i}$ for $i \in \{1, \dots, m\}$.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1 & \xrightarrow{i_1} & E_{n-1,1} & \xrightarrow{f_{n,1}} & E_{n,1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tilde{g}_{n-1,2}} & & \downarrow{\scriptstyle g_{n-1,2}} & & \downarrow{\scriptstyle g_{n,2}} & & \\
0 & \longrightarrow & F_2 & \xrightarrow{i_2} & E_{n-1,2} & \xrightarrow{f_{n,2}} & E_{n,2} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tilde{g}_{n-1,3}} & & \downarrow{\scriptstyle g_{n-1,3}} & & \downarrow{\scriptstyle g_{n,3}} & & \\
 & & \cdots & & \cdots & & \cdots & & \\
 & & \downarrow{\scriptstyle \tilde{g}_{n-1,m-1}} & & \downarrow{\scriptstyle g_{n-1,m-1}} & & \downarrow{\scriptstyle g_{n,m-1}} & & \\
0 & \longrightarrow & F_{m-1} & \xrightarrow{i_{m-1}} & E_{n-1,m-1} & \xrightarrow{f_{n,m-1}} & E_{n,m-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tilde{g}_{n-1,m}} & & \downarrow{\scriptstyle g_{n-1,m}} & & \downarrow{\scriptstyle g_{n,m}} & & \\
0 & \longrightarrow & F_m & \xrightarrow{i_m} & E_{n-1,m} & \xrightarrow{f_{n,m}} & E_{n,m} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{6}$$

where $\tilde{g}_{n-1,i} := g_{n-1,i}\big|_{F_i}$. These are well defined since $\mathrm{im}\tilde{g}_{n-1,i} = \mathrm{im}(g_{n-1,i} \circ f_{n-1,i-1}) = \mathrm{im}(f_{n-1,i} \circ g_{n-2,i}) \subseteq F_i$. The horizontal maps i_i are all embeddings of F_i into $E_{n-1,i}$ for all $i \in \{1, \dots, m\}$, and therefore injective. The horizontal short sequences are exact, since $\mathrm{im} i_i = F_i = \mathrm{im} f_{n-1,i} = \ker f_{n,i}$. Also, since per definition $\tilde{g}_{n-1,i} = g_{n-1,i}\big|_{F_i}$, the diagram commutes. We now have to show that the left (F_1, F_m)-$(m-2)$-sequence is exact.

First, we have $\ker \tilde{g}_{n-1,2} \subseteq \ker g_{n-1,2} = 0$, so $\tilde{g}_{n-1,2}$ is injective. It is also clear that $\mathrm{im}\tilde{g}_{n-1,m} = \mathrm{im}(\tilde{g}_{n-1,m} \circ f_{n-1,m-1}) = \mathrm{im}(f_{n-1,m} \circ g_{n-2,m}) = F_m$ so $\tilde{g}_{n-1,m}$ is surjective. Now, for all $i \in \{2, \dots, m-1\}$, we should verify that $\mathrm{im}\tilde{g}_{n-1,i} = \ker \tilde{g}_{n-1,i+1}$.

Right inclusion: We have $\mathrm{im}\tilde{g}_{n-1,i} = \mathrm{im}(g_{n-1,i} \circ f_{n-1,i-1}) = \mathrm{im}(f_{n-1,i} \circ g_{n-2,i})$. But for any $x \in \mathrm{im}(f_{n-1,i} \circ g_{n-2,i})$, say $x = f_{n-1,i}(g_{n-2,i}(y))$ for some $y \in E_{n-2,i-1}$, we have $\tilde{g}_{n-1,i+1}(x) = f_{n-1,i+1}(g_{n-2,i+1}(g_{n-2,i}(y))) = 0$. So $\mathrm{im}\tilde{g}_{n-1,i} \subseteq \ker \tilde{g}_{n-1,i+1}$.

Left inclusion: Suppose we have an element $x \in \ker \tilde{g}_{n-1,i+1} \subseteq F_i$. Then we have $i_{i+1}(\tilde{g}_{n-1,i+1}(x)) = g_{n-1,i+1}(i_i(x)) = 0$, so $i_i(x) \in \mathrm{im}g_{n-1,i}$. So there exists an element $y \in E_{n-1,i-1}$ with $g_{n-1,i}(y) = i_i(x)$; we can think of this as $(x,y) \in F_i \prod_{E_{n-1,i}} E_{n-1,i-1}$. Now, since $f_{n,i}(i_i(x)) = 0$, we also have $f_{n,i}(g_{n-1,i}(y)) = g_{n,i}(f_{n,i-1}(y)) = 0$.

We can now again say that because $f_{n,i-1}(y) \in \ker g_{n,i} = \mathrm{im}g_{n,i-1}$, there is an element $z \in E_{n,i-2}$ with $(y,z) \in E_{n-1,i-1} \prod_{E_{n,i-1}} E_{n,i-2}$. Now, however, since $f_{n,i-2}$ is surjective, we also know that there is an element $z' \in E_{n-1,i-2}$ such that $f_{n,i-2}(z') = z$. Since y and z have the same image in $E_{n,i-1}$, we know that $f_{n,i-1}(y - g_{n-1,i-1}(z')) = f_{n,i-1}(y) - g_{n,i-1}(f_{n,i-2}(z')) = 0$, so there exists an element $y' \in F_{i-1}$ such that $i_{i-1}(y') = y - g_{n-1,i-1}(z')$.

For the same element in $E_{n-1,i-1}$ we know that $g_{n-1,i}(y - g_{n-1,i-1}(z')) = g_{n-1,i}(y) = i_i(x)$. Therefore, $i_i(x - \tilde{g}_{n-1,i}(y')) = i_i(x) - g_{n-1,i}(i_{i-1}(y')) = 0$, but since i_i is injective, this implies that $x - \tilde{g}_{n-1,i}(y') = 0$ and therefore x is the image of y'. We conclude that $\ker \tilde{g}_{n-1,i+1} \subseteq \mathrm{im}\tilde{g}_{n-1,i}$, and thus that the left column of above diagram is an exact (F_1, F_m)-$(m-2)$-sequence.

We now know that we can construct an exact $(3,m)$-diagram. For the $(n-1,m)$ diagram, consider the following:

$$
\begin{array}{ccccccccccc}
 & & 0 & & & & 0 & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_{1,1} & \longrightarrow & \dots & \longrightarrow & E_{n-2,1} & \xrightarrow{f_{n-1,1}} & F_1 & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow g_{n-2,2} & & \downarrow \tilde{g}_{n-1,2} & & \\
0 & \longrightarrow & E_{1,2} & \longrightarrow & \dots & \longrightarrow & E_{n-2,2} & \xrightarrow{f_{n-1,2}} & F_2 & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow g_{n-2,3} & & \downarrow \tilde{g}_{n-1,3} & & \\
 & & \dots & & \dots & & \dots & & \dots & & \\
 & & \downarrow & & & & \downarrow g_{n-2,m-1} & & \downarrow \tilde{g}_{n-1,m-1} & & \\
0 & \longrightarrow & E_{1,m-1} & \longrightarrow & \dots & \longrightarrow & E_{n-2,m-1} & \xrightarrow{f_{n-1,m-1}} & F_{m-1} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow g_{n-2,m} & & \downarrow \tilde{g}_{n-1,m} & & \\
0 & \longrightarrow & E_{1,m} & \longrightarrow & \dots & \longrightarrow & E_{n-2,m} & \xrightarrow{f_{n-1,m}} & F_m & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 & & & & 0 & & 0 & &
\end{array}
\tag{7}
$$

We claim that this is an exact $(n-1,m)$-diagram. It clearly commutes since our original diagram does, and the maps $f_{n-1,i}$ are now surjective onto the F_i. The horizontal sequences are all (still) exact, and we have just shown that the right column is indeed an exact (F_1, F_m)-$(m-2)$-sequence. This is therefore an exact $(n-1,m)$-diagram. $\square$

Proof of theorem (6.1). We will prove the theorem with induction on n and m. First, fix $m \in \mathbb{N}_{\geq 3}$ and suppose that the theorem holds for all exact (m, n)-diagrams with $3 \leq n \leq N$ for some $N \in \mathbb{N}_{\geq 3}$. We will now prove that this implies that the theorem holds for all exact $(m, N+1)$-diagrams.

Suppose we have an exact $(m, N+1)$-diagram. We apply theorem (6.2) to split it into an exact $(m, 3)$-diagram as in (6) and an exact (m, N)-diagram as in (7).

In the (m, N)-diagram we name the top exact $(E_{1,1}, F_1)$-$(N-2)$-sequence α_N, the bottom exact $(E_{1,m}, F_m)$-$(N-2)$-sequence δ_N and the right exact (F_1, F_m)-$(m-2)$-sequence η. The left sequence is the same β as in our original $(m, N+1)$-diagram. In the $(m, 3)$-diagram, the same sequence η is on the left edge, the sequence γ from our original diagram is on the right, and here we name the top short exact $(F_1, E_{N+1,1})$-sequence α_3 and the bottom short exact $(F_m, E_{n,m})$-sequence δ_3.

Now, since by assumption the theorem holds for these smaller diagrams, we have $[\alpha_N \circ \eta] = (-1)^{Nm}[\beta \circ \delta_N]$ and $[\alpha_3 \circ \gamma] = (-1)^{3m}[\eta \circ \delta_3]$. It is also easy to show that $\alpha = \alpha_N \circ \alpha_3$ and $\delta = \delta_N \circ \delta_3$. We can therefore now compute (using the properties of both the abelian group structure of the equivalence classes and of the composition; see section (3.3))

$$[\alpha \circ \gamma] = [\alpha_N \circ \alpha_3 \circ \gamma] = (-1)^{3m}[\alpha_N \circ \eta \circ \delta_3] = (-1)^{3m+Nm}[\beta \circ \delta_N \circ \delta_3] = (-1)^{(N+1)m}[\beta \circ \delta]$$

and therefore we have shown that the theorem holds for all exact $(m, N+1)$-diagrams.

Since we know by the previous chapter that the theorem holds for exact $(3, 3)$-diagrams, we have just shown with induction that the theorem holds for all $(3, n)$-diagrams. If we fix an $n \in \mathbb{N}_{\geq 3}$, we can prove completely analogously that if the theorem holds for all exact (m, n)-diagrams with $3 \leq m \leq M$ for some M, it also holds for exact $(M+1, n)$-diagrams. Using induction again, we have therefore proven that the theorem holds for all exact (m, n)-diagrams, and therefore we are done. □

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