The fiber functor and dessins d'enfants

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Bachelor thesis

July 2018

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Preface

In this bachelor's thesis an introductory study in the theory of dessins d'enfants (french for children's drawings) is exposed. At the core of this theory lies the Riemannn existence theorem, which allows for a correspondence between covers of the Riemann sphere ramified in three points (the Belyi maps) and finite graphs with some extra structure (the dessins). The correspondence will be an equivalence of categories. This is quite remarkable, as it relates two objects of seemingly different nature and complexity; the first being mostly complex analytical and more advanced, the latter mostly combinatorical and simpler. Besides assuming the Riemann existence theorem, a fairly complete exposition is given to prove this equivalence. In the proof we make use of the fiber functor, which is itself an equivalence of categories between the category of covers of a connected and locally simply connected topological space X and the category of sets equiped with a right-action of $\pi_1(X, x)$, the fundamental group of X. Analogues of the fiber functor are in general recurrent in algebraic topology and algebraic geometry. To prove that it is an equivalence of categories is the most rigorous part of the thesis, and is apart from the Galois correspondence of covers entirely proved. Furthermore we apply the correspondence between the covers of the Riemann sphere and the dessins to a combinatorical problem regarding polynomials known as Davenport's bound. At the very end we mention some more advanced material of dessins d'enfants which is founded on Belyi's theorem, to get a feeling how dessins are of more sophisticated interest, especially regarding the absolute Galois group of $\mathbf{Q}$.

1 Covers, π-sets and the fiber functor

In this section we will introduce the two categories mentioned in its title, and prove that they are equivalent. This equivalence has importance on its own, for example in providing a tool to determine properties of covers, as we will see later. Furthermore it will serve us in proving the equivalence of the category of the so-called dessins d'enfants with the category of Belyi-pairs, which we will further exploit towards the end of this report. Belyi maps will be studied in section 2 and dessins in section 3.

Our approach in this section is heavily based on the book of Szamuely [14], but hopes to clarify its material by explicitly stating certain arguments and verifications that are partially or completely omitted, or hidden. Some proofs however are more original and quite different from the approach in [14], especially 1.24, 1.26, 1.28, and 1.31. We will pass on the proof of the for our purposes important Galois correspondence for covers, which in our view is treated elaborately enough in [14] thm 2.2.10.

We begin by recalling a couple of basic notions regarding category theory, but the reader who is completely unfamiliar with categories, is referred to [12] chapter 1.

1.1. Definition. Let $\mathcal{C}_1, \mathcal{C}_2$ be two categories. A functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is called an *equivalence of categories ($\mathcal{C}_1$ and $\mathcal{C}_2$)*, when there exist a functor $G : \mathcal{C}_2 \to \mathcal{C}_1$ and isomorphisms of functors $\theta_1 : G \circ F \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}_1},\ \theta_2 : F \circ G \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}_2}$.

Two categories $\mathcal{C}_1, \mathcal{C}_2$ are said to be equivalent, when there exists an equivalence of categories $\mathcal{C}_1$ and $\mathcal{C}_2$.

Whenever two categories $\mathcal{C}_1, \mathcal{C}_2$ are equivalent, all category theoretical concepts such as mono- and epimorphism carry over from $\mathcal{C}_1$ to $\mathcal{C}_2$. See for example [1] prop. 21.2. We moreover note that the composition of two equivalences of categories with suitable domains and codomains is again an equivalance of categories.

There is another notion equivalent to the previous, which will often be of more direct use to us.

1.2. Definition. Let $\mathcal{C}_1, \mathcal{C}_2$ be two categories, and $F : \mathcal{C}_1 \to \mathcal{C}_2$ a functor.

(i) F is *faithful* resp. *full* when for all $A, B \in \mathrm{Obj}(\mathcal{C}_1)$ the map $F_{A,B} : \mathrm{Hom}_{\mathcal{C}_1}(A, B) \to \mathrm{Hom}_{\mathcal{C}_2}(F(A), F(B))$ is injective, resp. surjective. A functor that is both faithful and full is called *fully faithful.*

(ii) F is *essentially surjective* when for all $B \in \mathrm{Obj}(\mathcal{C}_2)$ there exists a $A \in \mathrm{Obj}(\mathcal{C}_1)$ and an isomorphism $F(A) \to B$.

Let us explicitly state the equivalence of 1.1 and 1.2. Proofs are usually quite tedious, and we will not give one.

1.3. Theorem. *A functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

Proof. See [14] lemma 1.4.9. □

Now we recall the notion of a cover.

1.4. Definition. A *cover* $p : Y \to X$ is a continuous map from a topological space Y to a topological space X with the property that for every $x \in X$ there exists a open neighbourhood U of x, and a possibly empty index set I, with

$$p^{-1}(U) = \bigsqcup_{i \in I} V_i \ : \ \forall i \in I,\ V_i \subset Y \text{ is open and } p|_{V_i} : V_i \xrightarrow{\sim} U \text{ is a homeomorphism.}$$

The sets V_i are called the *sheets (of U)*. When Y is moreover connected, p is called a *connected* cover.

Because we allow I to be empty, for any topological space X we have the cover $\emptyset \to X$ which we call the *empty cover.*

1.5. Examples. (i) The standard example of a cover is the following, with S^1 the unit circle with subspace topology in $\mathbf{R}^2$.

$$\begin{aligned} p : \mathbf{R} &\longrightarrow S^1 \\ \varphi &\longmapsto e^{2\pi i \varphi} \end{aligned}$$

For instance, for $U = \{(x, y) \in S^1 \mid x > 0\}$ we have $p^{-1}U = \bigsqcup_{n \in \mathbf{Z}}(-1/4 + n, 1/4 + n)$, and indeed $(-1/4 + n, 1/4 + n) \xrightarrow{\sim} U$.

(ii) It is not hard to prove that we can extend the previous example to the punctured complex plane:

$$\begin{aligned} \mathbf{C} &\longrightarrow \mathbf{C} \setminus \{0\} \\ z &\longmapsto e^z \end{aligned}$$

(iii) Also similar to (i) we have the covering of a torus:

$$\begin{aligned} \mathbf{R} \times \mathbf{R} &\longrightarrow S^1 \times S^1 \\ (\varphi, \psi) &\longmapsto (e^{2\pi i \varphi}, e^{2\pi i \psi}) \end{aligned}$$

(iv) For $n \in \mathbf{Z}_{\geq 1}$ the following is also a cover:

$$\begin{aligned} \mathbf{C} \setminus \{0\} &\longrightarrow \mathbf{C} \setminus \{0\} \\ z &\longmapsto z^n \end{aligned}$$

1.6. Proposition. *Covers are open maps.*

Proof. Exercise. □

The following two lemmas are basic but essential, and are included for completeness and reference's sake.

1.7. Lemma. *Let $p : Y \to X$ be a cover, and γ a path in X , that is, a continuous function from $[0,1]$ to X, and $y \in p^{-1}\{\gamma(0)\}$. Then there exists a unique path $\widetilde{\gamma}_y$ in Y such that $p \circ \widetilde{\gamma}_y = \gamma$, and $\widetilde{\gamma}_y(0) = y$. Moreover, whenever two paths γ, σ in X with $\gamma(0) = \sigma(0) = p(y)$ are homotopicly equivalent, then we have $\widetilde{\gamma}_y(1) = \widetilde{\sigma}_y(1)$.*

Proof. See [5] prop. 11.6 and prop. 11.8. □

1.8. Definition. $\widetilde{\gamma}_y$ as in the previous lemma is called the *lift of γ (to Y) with starting point y*. Whenever we consider a lift, corresponding to a given cover, we will always use the notation of the tilde and the subscript for the starting point.

1.9. Lemma. (The universal property of the quotient topology). *Let X, Y be topological spaces and $\sim$ an equivalence relation on X. Let $f : X \to Y$ be a continuous map such that for all $x, x' \in X : x \sim x' \Rightarrow f(x) = f(x')$. Then there exists a unique continuous map $\bar{f} : X/\sim \to Y$ with $f = \bar{f} \circ \pi$, where π denotes the natural projection from X to $X/\sim$.*

Proof. We have no choice than to define $\bar{f}([x]) = f(x)$. Let us check that it is continuous. Given an open subset $U \subset Y$, we have

$$\pi^{-1}(\bar{f}^{-1}(U)) = f^{-1}(U).$$

By definition of the quotient topology, we conclude that $\bar{f}$ is continuous. □

We will now construct the first category of this thesis.

1.10. Definition. Let X be topological space. *The category of covers over X*, denoted by $\mathbf{Cov}(X)$, is the category with objects

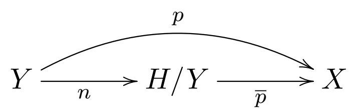
$$\mathrm{Obj}(\mathbf{Cov}(X)) := \big\{ p : Y \to X \;\big|\; Y \text{ is a topological space, } \; p \text{ a cover} \big\},$$

and given $p : Y \to X, q : Z \to X \in \mathrm{Obj}(\mathbf{Cov}(X))$ with morphisms

$$\mathrm{Hom}_{\mathbf{Cov}(X)}(p, q) := \{f : Y \to Z \text{ continuous} \mid q \circ f = p\}.$$

Furthermore, we write $\mathrm{Aut}(Y|X) := \{f \in \mathrm{Hom}_{\mathbf{Cov}(X)}(p, p) : f \text{ is a homeomorphism}\}$ for the automorphisms of p in the category $\mathbf{Cov}(X)$. They form a group together with composition as group action. One should be aware, based on the context, of which cover we are considering the automorphisms.

Let $p : Y \to X$ be a cover, $H \subset \mathrm{Aut}(Y|X)$ a subgroup and $n : Y \to H/Y$ the natural projection. Then applying 1.9 with $y \sim y' \Leftrightarrow n(y) = n(y')$ and $f = p$ gives a unique continuous map $\overline{p} : H/Y \to X$ making the following diagram commute.



1.11. Definition. A cover $p : Y \to X$ is called *Galois* when Y is connected, and for $H = \mathrm{Aut}(Y|X)$ we have that $\overline{p} : \mathrm{Aut}(Y|X)/Y \to X$ in the diagram above is a homeomorphism.

1.12. Remarks. When the action of $\mathrm{Aut}(Y|X)$ on Y is *even*, which means that for all $y \in Y$ there is a open neighbourhood U_y of y such that $fU_y \cap gU_y = \emptyset$ for all $f, g \in \mathrm{Aut}(Y|X)$, $f \neq g$, it is readily seen that n is a cover.

$\overline{p}$ is always an open map, and because $\overline{p}^{-1} = n \circ p^{-1}$, whenever sheets are relatively permuted by automorphisms of p, $\overline{p}$ inherits sheets from p and becomes a cover also. This condition is guaranteed if we assume our base topology to be locally connected, which is a reason why we make this assumption in the following theorem (we will encounter these matters again and in some more detail in 1.29 and 1.30). In this case we see that a cover is Galois when its sheets are permuted transitively by its automorphism group.

We will now state the Galois correspondence which we referred to already in the introduction of this section.

1.13. Theorem. (Galois correspondence for covers) *Let X be a locally connected topological space, Y and Z connected topological spaces, and $p : Y \to X$ a Galois cover.*

(i) *Then we have bijections inverse of each other:*

$$\Psi : \left\{ \begin{array}{c} H \subset \mathrm{Aut}(Y|X) \\ \text{a subgroup} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} q : Z \to X \\ \text{a connected cover} \end{array} \;\middle|\; \mathrm{Hom}_{\mathbf{Cov}(X)}(p,q) \neq \emptyset \right\} \Big/ \sim$$

$$H \longmapsto (H/Y \to X)$$

$$\Psi^{-1} : \qquad \mathrm{Aut}(Y|Z) \longleftarrow (Z \to X)$$

with $(q_1 : Z_1 \to X) \sim (q_2 : Z_2 \to X) \iff \exists f \in \mathrm{Hom}_{\mathbf{Cov}(X)}(q_1, q_2) :$ f is a homeomorphism.[1]

(ii) *For connected covers* $q : Z \to X$ *all elements of* $\mathrm{Hom}_{\mathbf{Cov}(X)}(p,q)$ *are Galois covers.*

(iii) *A representative of* $\Psi(\mathrm{Aut}(Y|Z)) = [(Z \to X)]$ *is Galois if and only if* $\mathrm{Aut}(Y|Z)$ *is a normal subgroup of* $\mathrm{Aut}(Y|X)$*, in which case* $\mathrm{Aut}(Z|X) \cong \mathrm{Aut}(Y|X)/\mathrm{Aut}(Y|Z)$.

Proof. [14] thm 2.2.10. □

1.14. Remark. We can see directly from the diagram below that the third statement makes sense, that is, when a cover q_1 is Galois, every cover q_2 with $q_1 \sim q_2$ is Galois too.

$$\begin{array}{ccccc} Z_1 & \xrightarrow{n_1} & \mathrm{Aut}(Z_1|X)/Z_1 & \xrightarrow[\sim]{\overline{q}_1} & X \\ \downarrow \psi & & \varphi \downarrow \sim & & \downarrow \mathrm{id} \\ Z_2 & \xrightarrow{n_2} & \mathrm{Aut}(Z_2|X)/Z_2 & \xrightarrow{\overline{q}_2} & X \end{array}$$

Here Z_1 and Z_2 are connected topological spaces, the map in the middle is given by $\varphi : \big(\mathrm{Aut}(Z_1|X)z\big) \mapsto \big((\psi \mathrm{Aut}(Z_1|X)\psi^{-1})\psi(z)\big) = \big(\mathrm{Aut}(Z_2|X)\psi(z)\big)$, and n_1, n_2 are the natural projections. Note that the big rectangle commutes by definition of $\sim$. The left square commutes too (this verification is straight-forward). Furthermore, let $h \in \mathrm{Aut}(Z_1|X)/Z_1$ and $z \in Z_1$ with $n_1(z) = h$, then we have

$$\overline{q}_1(h) = \overline{q}_1 \circ n_1(z) = \overline{q}_2 \circ n_2 \circ \psi(z) = \overline{q}_2 \circ \varphi \circ n_1(z) = \overline{q}_2 \circ \varphi(h)$$

so the right square commutes as well. We conclude that q_2 must be, indeed, Galois.

Next we define the second category, so that we can then introduce our functor of interest.

[1] it is already enough to pose that f is a homeomorphism; it then automatically is a morphism of covers. One proves this by using part (ii) in a diagram-chase similar to the one in 1.33.

1.15. Definition. Let G be a group. The category of G-sets, denoted by G-**Sets**, is to be the category defined as follows:

$$\mathrm{Obj}(G\text{-}\mathbf{Sets}) := \{S \text{ a set } \mid S \circlearrowleft G\};$$

in words, the objects are sets endowed with a *right* action of G.

For $S, T \in \mathrm{Obj}(G\text{-}\mathbf{Sets})$:

$$\mathrm{Hom}_{G\text{-}\mathbf{Sets}}(S,T) := \big\{f : S \to T \bigm| \forall s \in S, \forall g \in G : f(sg) = f(s)g\big\}.$$

Such a map f satisfying the defining property for G-**Sets** morphisms is called *equivariant.*

1.16. Definition. Let X be a topological space, and $x \in X$. Write $\pi := \pi_1(X,x)$ for the fundamental group of X with basepoint x. Let $p : Y \to X, q : Z \to X \in \mathrm{Obj}(\mathbf{Cov}(X))$ and $f \in \mathrm{Hom}_{\mathbf{Cov}(X)}(p,q)$. Fib_x is to denote the correspondence between $\mathbf{Cov}(X)$ and π-**Sets** as follows:

$$\begin{aligned} &\mathrm{Fib}_x : \mathbf{Cov}(X) \longrightarrow \pi\text{-}\mathbf{Sets} \\ \text{At the level of objects:}\quad &(p : Y \to X) \longmapsto p^{-1}\{x\} \\ \text{At the level of morphisms:}\quad &(f : Y \to Z) \longmapsto \big(f|_{p^{-1}\{x\}} : p^{-1}\{x\} \to q^{-1}\{x\}\big). \end{aligned}$$

1.17. Proposition. *The correspondence denoted by* Fib_x *is functorial, in other words,* Fib_x *is a functor.*

Proof. First, we realize that $p^{-1}\{x\}$ is indeed a π-set, namely by virtue of the well-known *monodromy action* (see [14] section 2.3, or for a more elaborate exposition, in Dutch, [2] section 14).

At the level of morphisms we need a slightly more subtle argument, which is based on the unicity of path-lifts of 1.7; let $[\gamma] \in \pi_1(X,x)$, $y \in p^{-1}\{x\}$, and let $\tilde{\gamma}_y$ be the unique lift of γ to Y with starting point y. By definition of f we have:

$$q \circ f \circ \tilde{\gamma}_y = p \circ \tilde{\gamma}_y = \gamma \qquad \text{and} \qquad f \circ \tilde{\gamma}_y(0) = f(y),$$

so by recalling again the unicity of path-lifts, we conclude that $f \circ \tilde{\gamma}_y = \tilde{\gamma}_{f(y)}$, with $\tilde{\gamma}_{f(y)}$ the unique path-lift of γ in Z with starting point $f(y)$. For all $[\gamma] \in \pi_1(X,x)$ and for all $y \in p^{-1}\{x\}$ it now follows that

$$f(y * [\gamma]) = f(\tilde{\gamma}_y(1)) = \tilde{\gamma}_{f(y)}(1) = f(y) * [\gamma],$$

where * denotes the monodromy action. We thus have seen that $\mathrm{Fib}_x(f)$ is indeed equivariant. Since by definition of a cover morphism we have $f(p^{-1}\{x\}) \subset$

$q^{-1}\{x\}$, Fib_x commutes with composition, and clearly $\mathrm{Fib}_x(\mathrm{id}_f) = \mathrm{id}_{\mathrm{Fib}_x(f)}$ for $f \in \mathrm{Obj}(\mathbf{Cov}(X))$. We conclude that Fib_x defines a functor. □

We call Fib_x the *fiber functor (with basepoint x).*

The universal cover

We will now construct a special cover, namely a so called *universal cover*. Such a cover satfisfies certain properties that will serve us to prove that the fiber functor is an equivalence of categories. To be able to construct a universal cover, we make some assumptions on our base topology X. After the construction, we will state and prove the relevant properties, and subsequently we will prove the equivalence of Fib_x.

1.18. Definition. A topological space X is called *locally simply connected*, when there exists a topological basis of X consisting of path-connected open sets U_i each with a trivial fundamental group when we equip U_i with the subspace topology.

Recalling that two fundamental groups of a path-connected topological space with two arbitrary basepoints are isomorphic (see [5] exc. 12.7), we see that the previous definition is independent of choice of base points.

1.19. Construction. Let X be a connected and locally simply connected space, and fix $x \in X$. Let $\sim$ denote the path-homotopy equivalence relation. Then $\widetilde{X}_x$ is to be the following set:

$$\widetilde{X}_x := \big\{\gamma : [0,1] \to X \text{ continuous, and } \gamma(0) = x\big\}/\sim .$$

$\widetilde{X}_x$ has a canonical point, namely the constant path on x; we will denote it by $\widetilde{x}$.

Next we define a map u which eventually will become our desired cover;

$$\begin{aligned} u : \widetilde{X}_x &\longrightarrow X \\ [\gamma] &\longmapsto \gamma(1) \end{aligned}$$

Here, as before, [_] denotes the homotopy-class of a path.

We endow $\widetilde{X}_x$ with the following topological basis. Let $[\gamma] = w \in \widetilde{X}_x$ and write $y = \gamma(1) = u(w)$. For every simply connected neighbourhood U_y of y in X we define

$$\widetilde{U}_w := \big\{[\gamma \odot \sigma] \;\big|\; [\gamma] = w,\ \sigma : [0,1] \to U_y \text{ continuous and } \sigma(0) = y\big\},$$

where $\odot$ denotes the composition of two paths, with the convention that the path left of $\odot$ is the first half of the composite path, i.e.,

$$\sigma \odot \gamma(t) = \begin{cases} \sigma(2t), & 0 \leq t \leq 1/2 \\ \gamma(2t-1), & 1/2 < t \leq 1 \end{cases} .$$

Indeed it is true that $\{\widetilde{U}_w : w \in \widetilde{X}_x\}$ defines a topological basis, since for any two open neighbourhoods $\widetilde{U}_w, \widetilde{V}_w$ of any w, there is a simply connected neighbourhood $W \subset U \cap V$ of y by local simple connectedness of X, and therefore $\widetilde{W}_w \subset \widetilde{U}_w \cap \widetilde{V}_w$.

1.20. Proposition. *Let X, $\widetilde{X}_x$, x, $\widetilde{x}$ and $u : \widetilde{X}_x \to X$ be as in* 1.19. *$u : \widetilde{X}_x \to X$ is a cover.*

Proof. We claim that whenever two paths σ_1, σ_2 in some simply connected neighbourhood U_y of y have the same starting and end point, they must be homotopically equivalent. To prove this claim, take a path α from $\sigma_1(1) = \sigma_2(1)$ to y, which is possible because U_y is in particular path-connected. Then by simple connectedness of U_y, we have $\sigma_1 \odot \alpha \sim y_0 \sim \sigma_2 \odot \alpha$, with y_0 the constant path at y. Then we have $\sigma_1 \sim \sigma_2$, which proves the claim.

Now $u|_{\widetilde{U}_w} : \widetilde{U}_w \to U_y$ is surjective by path-connectedness of U_y, and by the previous claim, it is injective. u is thus open on its basis, and is therefore an open map.

We note that $u^{-1}U_y = \bigcup_w \widetilde{U}_w$, with w ranging over all of $\widetilde{X}_x$, from which continuity follows at once. Futhermore, if $\gamma_1 \odot \sigma_1 \sim \gamma_2 \odot \sigma_2$, then by 1.7 those paths have the same end point, $\sigma_1(1) = \sigma_2(1)$, and therefore, by the claim, $\sigma_1 \sim \sigma_2$. It follows that in this case $\gamma_1 \sim \gamma_2$; we conclude that $\widetilde{U}_w \cap \widetilde{U}_{w'} = \emptyset$, for $w \neq w'$. We are done. □

1.21. Lemma. *Let X, $\widetilde{X}_x$, x, $\widetilde{x}$ and $u : \widetilde{X}_x \to X$ be as in* 1.19. *$\widetilde{X}_x$ is path-connected.*

Proof. We will show that there exists a path from $\widetilde{x}$ to an arbitrary point $[\gamma] \in \widetilde{X}_x$.

We write t for the parameter of paths in X, and s for the parameter of paths in $\widetilde{X}_x$, and claim that

$$\begin{aligned} \Gamma : [0,1] &\longrightarrow \widetilde{X}_x \\ s &\longmapsto [(t \mapsto \gamma(st))] \end{aligned}$$

is a path from $\widetilde{x}$ to $[\gamma]$. It is clear that $\Gamma(s=0) = \widetilde{x}$, since γ is a path starting at x, and also we have $\Gamma(s=1) = [\gamma]$. Furthermore, for any s we obtain a class of a path, since $(t \mapsto \gamma(st))$ is continuous being a composite of the continuous function γ and

$(t \mapsto st)$, so the map is well-defined. The continuity of Γ remains to be checked. This is left to the reader. □

Having proved this, we can use the following essential lemma for $\widetilde{X}_x$ in the next theorem.

1.22. Lemma. *Let $p : Y \to X$ be a cover, Z a connected topological space, and $f, g : Z \to Y$ two continuous maps satisfying $p \circ f = p \circ g$. If there is a point $z \in Z$ with $f(z) = g(z)$, then $f = g$.*

Proof. See [5] lemma 11.5. □

1.23. Theorem. (Existence of Universal Cover) *Let $X, \widetilde{X}_x, x, \widetilde{x}$, and $u : \widetilde{X}_x \to X$ be as in* 1.19. *Let* For *denote the forgetful functor from π-**Sets** to the category of sets. Then $u : \widetilde{X}_x \to X \in \mathrm{Obj}(\mathbf{Cov}(X))$ represents Fib_x, i.e., there exists an isomorphism of functors $\mathrm{Hom}_{\mathbf{Cov}(X)}(\widetilde{X}_x, \_) \cong \mathrm{For} \circ \mathrm{Fib}_x(\_)$.*

Proof. We will show that

$$\begin{aligned} \theta : \mathrm{Fib}_x(Y) &\longrightarrow \mathrm{Hom}_{\mathbf{Cov}(X)}(\widetilde{X}_x, Y) \\ y &\longmapsto ([\gamma] \mapsto y * [\gamma]) \end{aligned}$$

is a bijection, functorial in a cover $p : Y \to X$.

First of all, we remark that this map is indeed well-defined since we have $u([\gamma]) = \gamma(1) = p \circ \widetilde{\gamma}_y(1) = p \circ \theta(y)([\gamma])$. Moreover we have $\theta(y)(\widetilde{x}) = y$, so it directly follows that θ is injective. It is furthermore clear we obtain from θ a map that sends $\widetilde{x} \in \mathrm{Fib}_x$ to y' for every $y' \in \mathrm{Fib}_x(Y)$, which fully determines a morphism of covers by 1.22. These are also the only maps in $\mathrm{Hom}_{\mathbf{Cov}(X)}(\widetilde{X}_x, Y)$, since whenever $\widetilde{x} \mapsto y' \notin \mathrm{Fib}_x(Y)$, then $u(\widetilde{x}) = x \neq p(y')$, so we do not have a morphism of covers. We conclude that θ is bijective.

The commutative property of θ follows at once from 1.22 together with the fact that for $f \in \mathrm{Hom}_{\mathbf{Cov}(X)}(Y, Z)$ we have $f(\theta(y)(\widetilde{x})) = f(y) = \theta(f(y))(\widetilde{x})$. □

1.24. Proposition. *Let X, $\widetilde{X}_x$, x, $\widetilde{x}$ and $u : \widetilde{X}_x \to X$ be as in* 1.19. *Let $y \in \mathrm{Fib}_x(\widetilde{X}_x)$. We have a canonical equivariant bijection*

$$\mathrm{Fib}_x(\widetilde{X}_x) \cong \pi_1(X, x),$$

with right-multiplication as right-action on $\pi_1(X, x)$.

Proof. We will show that

$$\begin{aligned} \varphi : \pi_1(X,x) &\longrightarrow \mathrm{Fib}_x(\widetilde{X}_x) \\ [\gamma] &\longmapsto \widetilde{x} * [\gamma], \end{aligned}$$

which is an equivariant map because of the monodromy *action*, equals the identity map.

With the same notation for parameters as in 1.21, we have $u \circ \widetilde{\gamma}_{\widetilde{x}}(s) = \widetilde{\gamma}_{\widetilde{x}}(s)(t = 1)$, which in words wants to say that the image of a path in $\widetilde{X}_x$ under u is the path in X that we get when we evaluate a representative of $\widetilde{\gamma}_{\widetilde{x}}(s)$ in X at $t = 1$, for every $s \in [0,1]$. Γ from 1.21 meets the requirements on $\widetilde{\gamma}_{\widetilde{x}}$, so by 1.7 $\widetilde{\gamma}_{\widetilde{x}} = \Gamma$. It follows that $\varphi([\gamma]) = \widetilde{\gamma}_{\widetilde{x}}(s = 1) = \Gamma(s = 1) = [\gamma]$. □

1.25. Remarks. From the representation functor follows the existence of a canonical point $\widetilde{x}' \in u^{-1}(x)$, namely the point corresponding to the identity map in $\mathrm{Hom}_{\mathbf{Cov}(X)}(u,u)$. Note that from the proof of the previous proposition and θ in 1.23 it immediately follows that $\widetilde{x}' = \widetilde{x}$.

For an arbitrary cover $p : Y \to X$ and fixed $y \in p^{-1}\{x\}$, there exists a unique morphism of covers $\pi_y : \widetilde{X}_x \to Y$, which sends $\widetilde{x}$ to y, by commutativity of the diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Cov}(X)}(\widetilde{X}_x, \widetilde{X}_x) & \xrightarrow{\sim} & \mathrm{Fib}_x(\widetilde{X}_x) \\ \Big\downarrow{\scriptstyle \pi_y \circ} & & \Big\downarrow{\scriptstyle \pi_y|_{\mathrm{Fib}_x(\widetilde{X}_x)}} \\ \mathrm{Hom}_{\mathbf{Cov}(X)}(\widetilde{X}_x, Y) & \xrightarrow{\sim} & \mathrm{Fib}_x(Y) \end{array}$$

Some say that every cover uniquely 'factorizes through $\widetilde{X}_x$', or that $\widetilde{X}_x$ posseses 'the universal property' (cf. 1.9). This explains why we call $\widetilde{X}_x$ a *universal* cover of X. It may not be unique, but between two universal covers $\widetilde{X}_x$ and $\overline{X}$ with associated isomorphisms of functors θ_1, θ_2, there exists a unique isomorphism of covers $f : \overline{X} \to \widetilde{X}_x$ making the following diagram commutative:

$$\begin{array}{ccccc} & & \mathrm{Fib}_x & & \\ & \swarrow{\scriptstyle \theta_1} & & {\scriptstyle \theta_2}\searrow & \\ \mathrm{Hom}_{\mathbf{Cov}(X)}(\widetilde{X}_x, \_) & & \xrightarrow{\circ f} & & \mathrm{Hom}_{\mathbf{Cov}(X)}(\overline{X}, \_) \end{array}$$

This is a well-known fact from basic category theory, and it follows directly from Yoneda's lemma (see for instance [12] p.21). Using our first diagram with $Y = \overline{X}$, we see that we obtain a unique canonical isomorphism $\overline{X} \to \widetilde{X}_x$ that fixes the canonical points.

1.26. Corollary. *Let X, $\widetilde{X}_x$, x, $\widetilde{x}$ and $u : \widetilde{X}_x \to X$ be as in* 1.19. *$\widetilde{X}_x$ is simply connected.*

Proof. We have already seen in 1.21 that $\widetilde{X}_x$ is path-connected.

Let u_* be the grouphomomorphism that sends $[\widetilde{\gamma}] \in \pi_1(\widetilde{X}_x, \widetilde{x})$ to $[u \circ \widetilde{\gamma}] \in \pi_1(X, x)$. We have $u_*(\pi_1(\widetilde{X}_x, \widetilde{x})) = \mathrm{Stab}_{\pi_1(X,x)}(\widetilde{x}) = \{e\}$, where the inclusion '$\subset$' in the first equality follows from 1.7 and the second equality is implied by the proof of 1.24. By 1.7 u_* is injective, so we must have $\pi_1(\widetilde{X}_x, \widetilde{x}) = \{e\}$. □

1.27. Remark. One can see at this point that we could also have started proving that $\widetilde{X}_x$ is simply connected, after which the previous proposition follows backwards using the canonical map $\mathrm{Stab}_{\pi_1(X,x)}(\widetilde{x})/\pi_1(X, x) \to \mathrm{Fib}_x(\widetilde{X}_x)$, given by $([\gamma] \mapsto \widetilde{x} * \gamma)$. Furthermore, the representability theorem of 1.23 would then follow via a slightly more general lemma which relates the fundamental groups to factorization of covers. This is the approach of Fulton [5], and this lemma is lemma 13.5 (in last line of the statement $\widetilde{H}$ should be $\widetilde{f}$ and in the last line of the proof f should be $\widetilde{f}$). The essential point in this lemma is that the map constructed in 1.23 is well-defined, as a result of the trivial fundamental group of $\widetilde{X}_x$. But with $\widetilde{X}_x$ explicitly given, we do not need this relation, since the endpoint of a chosen path in θ is by the construction of $\widetilde{X}_x$ based on homotopy classes independent of that choice.

It might not be so much of a suprise, then, that in fact a cover of a connected and locally simply connected topological space X is simply connected if and only if it represents Fib_x for an $x \in X$. We will prove this in 1.31.

1.28. Corollary. *Let $X, \widetilde{X}_x, x, \widetilde{x}$ and $u : \widetilde{X}_x \to X$ be as in* 1.19. *Let $y \in \mathrm{Fib}_x(\widetilde{X}_x)$. We have the following canonical group isomorphisms:*

$$\mathrm{Aut}(\widetilde{X}_x|X) \overset{(1)}{\cong} \mathrm{Aut}_{\pi\text{-}\mathbf{Sets}}(\mathrm{Fib}_x(\widetilde{X}_x)) \overset{(2)}{\cong} \mathrm{Aut}_{\pi\text{-}\mathbf{Sets}}(\pi_1(X, x)) \overset{(3)}{\cong} \pi_1(X, x).$$

Proof. (1) This is a direct result of 1.23, which can be directly observed from the diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Cov}(X)}(\widetilde{X}_x, \widetilde{X}_x) & \xrightarrow{\sim} & \mathrm{Fib}_x(\widetilde{X}_x) \\ \downarrow f & & \downarrow \mathrm{Fib}_x(f) \\ \mathrm{Hom}_{\mathbf{Cov}(X)}(\widetilde{X}_x, \widetilde{X}_x) & \xrightarrow{\sim} & \mathrm{Fib}_x(\widetilde{X}_x) \end{array}$$

and the fact that both automorphisms of covers and automorphisms of π-sets form a group with composition as group action.

(2) Is induced by 1.24.

(3) By equivariance, and the right-multiplication as right-action on $\pi_1(X,x)$, it directly follows that $\mathrm{Aut}_{\pi\text{-}\mathbf{Sets}}(\pi_1(X,x)) \ni f \mapsto f(e)$ is an isomorphism. $\square$

1.29. Lemma. *Let X be a locally connected topological space, $p : Y \to X, q : Z \to X$ two covers, and $f : Y \to Z \in \mathrm{Hom}_{\mathbf{Cov}}(X)(p,q)$. Every sheet of a connected neighbourhood $U \subset X$ of p is mapped by f into a sheet of U of q.*

Proof. Write $p^{-1}U = \bigsqcup_{i\in I} U_i, q^{-1}U = \bigsqcup_{j\in J} V_j$, and suppose $f(U_i) \not\subset V_j$ for all $j \in J$, and take $k \in J$ such that $f(U_i) \cap V_k \neq \emptyset$. Remark that $f(U_i)$ is closed because it is connected, and open by local connectedness of Z, that is inherited from X via the local homeomorphisms. But then $V_k = (f(U_i) \cap V_k) \cup (V_k \cap (X \setminus f(U_j)))$, with $f(U_i) \cap V_k$ and $V_k \cap (X \setminus f(U_j))$ both open and $\emptyset \neq f(U_i) \cap V_k \neq V_k$. But V_k is connected because U is — a contradiction. $\square$

1.30. Corollary. *Let $X, \widetilde{X}_x, x, \widetilde{x}$ and $u : \widetilde{X}_x \to X$ be as in* 1.19. *$\widetilde{X}_x$ is Galois.*

Proof. We first claim that each fiber of a connected cover has the same cardinality. To prove this, we note that open sets $U \subset X$ as in 1.4 with different cardinality of sheets are disjoint, since otherwise the fiber of an element in the intersection would have an ambiguous cardinality. Therefore, taking the union of all sets U over all cardinalities would yield a decomposition of disjoint open sets of X, which is not possible since X is connected. This proves our claim.

The groupisomorphism $\mathrm{Aut}(\widetilde{X}_x|X) \cong \pi_1(X,x)$ from 1.28, and the isomorphism $\mathrm{Fib}_x(\widetilde{X}_x) \cong \pi_1(X,x)$ of π-sets from 1.24 together imply that $\mathrm{Aut}(\widetilde{X}_x|X)$ acts transitively on $\mathrm{Fib}_x(\widetilde{X}_x)$. It follows that each fiber of $\overline{u} : \mathrm{Aut}(\widetilde{X}_x|X)/\widetilde{X}_x \to X$, which is connected because $\widetilde{X}_x$ is, has cardinality equal to 1, and we conclude that $\overline{u}$ is a bijection.

We are left to check openness. For any neighbourhood V' contained in a sheet V of u of some neighbourhood $U \subset X$ we have $\overline{u}(nV') = u|_{V'} = U$, where n as before denotes the natural projection. Because $n^{-1}nV' = \bigcup_{\varphi\in\mathrm{Aut}(\widetilde{X}_x|X)} V'$, nV is clearly open – in fact n itself is open –, and we conclude that $\overline{u}$ is open. $\square$

1.31. Proposition. *Let X be a connected and locally simply connected topological space, and fix $x \in X$. A cover $u : \overline{X} \to X$ is simply connected if and only if it represents* $\mathrm{For} \circ \mathrm{Fib}_x$.

Proof. Assume $\overline{X}$ is simply connected. Then by path-connectedness of $\overline{X}$ we have an equivariant isomorphism $\mathrm{Stab}_{\pi_1(X,x)}(\overline{x})/\pi_1(X,x) \to \mathrm{Fib}_x(\overline{X})$, given by $([\gamma] \mapsto \overline{x} * \gamma)$, for an $\overline{x} \in \mathrm{Fib}_x(\overline{X})$. But as in the proof of 1.26, we have $\mathrm{Stab}_{\pi_1(X,x)}(\overline{x}) =$

$u_*(\pi_1(\overline{X}, \overline{x})) = \{e\}$. It thus follows that $\mathrm{Fib}_x(\overline{X}) \cong \pi_1(X, x) \cong \mathrm{Fib}_x(\widetilde{X}_x)$, equivariantly. Denoting the isomorphism by φ, we can put $y \mapsto ([\gamma] \mapsto \varphi(y) * \gamma)$ to obtain an isomorphism of functors from $\mathrm{Fib}_x(Y)$ to $\mathrm{Hom}_{\mathbf{Cov}(X)}(\overline{X}, Y)$, for which the proof goes as in 1.23.

If conversely $\overline{X}$ represents $\mathrm{For} \circ \mathrm{Fib}_x$, then by the remarks of 1.25 there exists a canonical isomorphism of covers $\psi : \overline{X} \to \widetilde{X}_x$, so $\overline{X}$ is path-connected, and composing ψ with φ in 1.24 yields an equivariant bijection from $\pi_1(X, x) \to \mathrm{Fib}_x(\overline{X})$ (cf. the second to last equation in the proof of 1.17). With the same reasoning of 1.26 this implies that $\mathrm{Fib}_x(\overline{X}) = \{e\}$. □

We can use the previous result together with 1.28 to determine the fundamental groups, up to isomorphism (cf. the remark after 1.18), of the path-connected base spaces in (i) and (iii) of 1.5. 1.31 confirms that these covers are universal, and with 1.28 it follows that $\pi_1(S^1, s) = \mathbf{Z}$, $\pi_1(S^1 \times S^1, (s_1, s_2)) = \mathbf{Z} \times \mathbf{Z}$.

To prove our main result of this chapter, we need one final lemma.

1.32. Lemma. *Let X be a topological space and $x \in X$ such that there exists a universal cover $u : \overline{X} \to X$ representing $\mathrm{Fib}_x(X)$. Let I be an index set, and $p_i : Y_i \to X$ connected covers of X. Then $p = \bigsqcup_{i \in I} p_i : \bigsqcup_{i \in I} Y_i \to X$ is a cover.*

Proof. Write $U \subset X$ for a neighbourhood with $u^{-1}U = \bigsqcup_j \widetilde{U}_j$, and π_i for the unique maps such that $u = p_i \circ \pi_i$. Then we have

$$p^{-1}U = \bigsqcup_i \pi_i \Big(\bigsqcup_j \widetilde{U}_j \Big),$$

and π_i is itself a cover by 1.13(ii), since $\overline{X}$ is Galois by 1.30 and the last paragraph of 1.25, so $\pi_i \widetilde{U}_j$ is open by 1.6. It follows that $p_i|_{\pi_i(\widetilde{U}_j)}$ is open and thus an homeomorphism. □

So now we are ready to state and prove our main result of this chapter.

1.33. Theorem. *Let X be a connected and locally simply connected topological space, and fix $x \in X$. Then Fib_x is an equivalence of categories. Except for the empty set, which corresponds to the empty cover, π-sets equipped with a transitive action correspond to connected covers, and Galois covers correspond to π-sets isomorphic with coset spaces of normal subgroups of $\pi_1(X, x)$.*

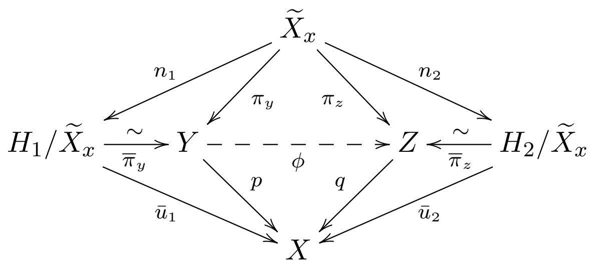
Proof. We use 1.3 and check that Fib_x is fully faithful and essentially surjective.

Let Y and Z be topological spaces, and let $(p : Y \to X), (q : Z \to X) \in \mathrm{Obj}(\mathbf{Cov}(X))$. Now let there be given an equivariant map $\varphi : \mathrm{Fib}_x(Y) \to \mathrm{Fib}_x(Z)$.

We will show there exists a unique morphism of covers $f : Y \to Z$ such that $\mathrm{Fib}_x(f) = \varphi$.

First we decompose $Y = \bigsqcup_{i \in I} U_i$ into its connected components U_i, and define a morphism of covers on each U_i seperately, after which we take them together to form a morphism of covers from the whole of Y to Z, which will be a cover by 1.32. Because a continuous map always maps connected components into connected components, with this approach we are allowed to consider in the following only the case that Y and Z are connected.

Fix a point $y \in Y$ and take, according to the observations in 1.25, the unique corresponding morphism of covers under the representation functor, $\pi_y : \widetilde{X}_x \to Y$ for which $\pi_y(\widetilde{x}) = y$, and do the same for $z := \varphi(y) \in Z$, and write $\pi_z : \widetilde{X}_x \to Z$ for this morphism. Note that we have $p \circ \pi_y = u$, and $q \circ \pi_z = u$. Write $H_1 := \mathrm{Aut}(\widetilde{X}_x|Y)$ and $H_2 := \mathrm{Aut}(\widetilde{X}_x|Z)$. Via part (ii) of the Galois correspondence for covers, 1.13, both π_y and π_z are Galois covers. It follows that we have homeomorphisms $H_1/\widetilde{X}_x \xrightarrow{\overline{\pi}_y} Y$ and $H_2/\widetilde{X}_x \xrightarrow{\overline{\pi}_z} Z$ given by the maps induced by π_y and π_z. We get the diagram:



in which we want the existence and unicity of the dotted line making the lower-middle triangle commute, to be further examined. For $i = 1, 2$, n_i are the natural projections, and $\bar{u}_i$ are the unique maps satisfying $\bar{u}_i \circ n_i = u$ (cf. 1.11). Note that the outer top triangles commute by definition of $\overline{\pi}_y$ and $\overline{\pi}_z$. Putting this together we see with $\widetilde{w} \in \widetilde{X}_x : n_1(\widetilde{w}) = w$ that we have:

$$\bar{u}_1(w) = \bar{u}_1(n_1(\widetilde{w})) = u(\widetilde{w}) = p(\pi_y(\widetilde{w})) = p(\overline{\pi}_y \circ n_1(\widetilde{w})) = p \circ \overline{\pi}_y(w).$$

$\overline{\pi}_y$ is thus a morphism of covers, and by the analogous proof so is $\overline{\pi}_z$. We apply the fiber functor on $\overline{\pi}_y, \overline{\pi}_z^{-1}$ to get two equivariant maps, between which we put φ, to obtain a equivariant map from $H_1/\mathrm{Fib}_x(\widetilde{X}_x) \to H_2/\mathrm{Fib}_x(\widetilde{X}_x)$. This shows that $H_1 \subset H_2$, so

$$\pi_y(x) = \pi_y(x') \Leftrightarrow H_1 x = H_1 x' \Longrightarrow H_2 x = H_2 x' \Leftrightarrow \pi_z(x) = \pi_z(x').$$

Applying 1.9 with $x \sim x' \Leftrightarrow \pi_y(x) = \pi_y(x')$ and $f = \pi_z$ allows us to conclude that there exists a unique continuous ϕ fitting on the dotted arrow, making the upper-middle triangle commute.

We remark that since Y is connected, it is also path-connected since Y is locally path-connected, by local path-connectedness of X via the local homeomorphisms. We therefore have transitive monodromy actions on $\mathrm{Fib}_x(Y)$. It further holds that $\phi(y) = \phi(\pi_y(\widetilde{x})) = \pi_z(\widetilde{x}) = z = \varphi(y)$, and thus by equivariance we have $\varphi = \phi|_{p^{-1}(\{x\})} = \mathrm{Fib}_x(\phi)$. It remains to check that ϕ is a morphism of covers, i.e., $p = q \circ \phi$. One can use a proof analagous to our proof of $\bar{u}_1 = p \circ \overline{\pi}_y$ to verify this. We conclude that Fib_x is fully faithful.

Now let $\emptyset \neq S \in \mathrm{Obj}(\pi\text{-}\mathbf{Sets})$ and take a $s \in S$. First assume $\pi_1(X, x)$ acts transitively on S. Write $H := \mathrm{Stab}_{\pi_1(X,x)}(s)$ for the stabilizer of some point $s \in S$, and, using the canonical correspondence $\mathrm{Aut}(\widetilde{X}_x|X) \cong \pi_1(X, x)$ from 1.28, consider H as a subgroup of $\mathrm{Aut}(\widetilde{X}_x|X)$. As in the proof of 1.31 we have an equivariant bijection $S \cong H/\pi_1(X, x)$. Using 1.13, we obtain a connected cover $\overline{u} : H/\widetilde{X}_x \to X$. Taking the image under Fib_x and using $\mathrm{Fib}_x(\widetilde{X}_x) \cong \pi_1(X, x)$ from 1.24, we get

$$\mathrm{Fib}_x(H/\widetilde{X}_x) = H/\mathrm{Fib}_x(\widetilde{X}_x) \cong H/\pi_1(X, x) \cong S. \qquad (*)$$

For S not transitive we decompose S into orbits, and define a cover $p_i : Y_i \to X$ as in the transitive case for each orbit $i \in I$ separately, which 1.32 allows us to do. Taking the fiber of the disjoint union of these covers yields an equivariant bijection from the disjoint union of these fibers to the disjoint union of orbits of S, and thus an equivariant bijection to S itself.

At last, we note that the statement about Galois covers follows directly from 1.13(iii) and $(*)$. □

1.34. Example. Let X be a connected and locally simply connected topological space. Let $p : Y \to X$ be a cover, and assume we know $H \subset \mathrm{Aut}(\widetilde{X}_x|X)$ such that Y is homeomorphic to $H/\widetilde{X}_x$. Then we have the following chain of canonical group-isomorphisms.

$$\mathrm{Aut}(Y|X) \cong \mathrm{Aut}(H/\widetilde{X}_x|X) \cong \mathrm{Aut}_{\pi\text{-}\mathbf{Sets}}(H/\mathrm{Fib}_x(\widetilde{X}_x)) \cong \mathrm{Aut}_{\pi\text{-}\mathbf{Sets}}(H/\pi_1(X, x)).$$

The second map is bijective by 1.33, and is an homomorphism because we only restrict morphisms. The last isomorphism follows from 1.24.

This shows we can use 1.33 to translate a topological problem to a problem in group theory, as to determine the expression on the right hand side is a matter of group theory only.

2 Belyi maps

We will now turn to one of the main objects of this thesis, the Belyi maps. We construct a second category whose objects are the Belyi maps, and investigate a few of their properties. Subsequently we will introduce another functor from the category of Belyi maps to the category of covers of $\mathbf{C}_* = \mathbf{C} \setminus \{0, 1\}$. After we will continue in the next section with the other main object of interst, the dessins d'enfants, and using the fiber functor we will see how they relate to the Belyi maps.

We will need a couple of basic notions of complex analysis, some of which we will assume to be known. The reader who has difficulty recalling such notions can consult [4]. The main reference for this section is [6].

2.1. Definition. The *Riemann sphere* is the set $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ endowed with the topology

$$\mathcal{T}_{\overline{\mathbf{C}}} = \mathcal{T}_{\mathbf{C}} \cup \big\{\{z \in \mathbf{C} : |z| > R, R \in \mathbf{R}\} \cup \{\infty\}\big\},$$

– where $\mathcal{T}_{\mathbf{C}}$ denotes the standard Euclidean topology on $\mathbf{C}$ – together with homeomorphisms φ_1, φ_2 (the *charts*, see [6] p.1):

$$U_1 = \mathbf{C},\ \varphi_1 : U_1 \to \mathbf{C}, (z \mapsto z) \qquad \text{and} \qquad U_2 = \overline{\mathbf{C}} \setminus \{0\},\ \varphi_2 : U_2 \to \mathbf{C}, z \mapsto \begin{cases} 1/z, z \neq \infty, \\ 0, z = \infty \end{cases}$$

2.2. Definition. A *rational map* f on $\mathbf{C}$ is a map $f : \mathbf{C} \setminus S \to \mathbf{C}$ with $f(z) = g(z)/h(z)$, where g and h are coprime polynomials over $\mathbf{C}$, and $S \subset \mathbf{C}$ is the set of roots of h. The *degree* of f is defined as $\deg f = \max\{\deg g, \deg h\}$.

2.3. Definition. A *Belyi map* is

(i) a non-constant map $f : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ which is a *morphism of Riemann surfaces*, i.e., the maps

$$\varphi_i \circ f \circ \varphi_j^{-1} : f^{-1}U_i \cap \varphi_j(U_j) \to \varphi_i(f^{-1}U_i \cap \varphi_j(U_j)), \quad i, j \in \{1, 2\},$$

are holomorphic. Because the domain and codomain equal $\overline{\mathbf{C}}$ here, such an f will also be called a morphism of Riemann *spheres.*

(ii) induced by a rational map, i.e., $f|_{\mathbf{C}}$ is a rational map on $\mathbf{C}$,

(iii) satisfying $B_f = \{z \in \overline{\mathbf{C}} : f^{-1}(z) < \deg f\} \subset \{0, 1, \infty\}$.

An element $y \in B_f$ is called a *branch point* or *critical value (of f)*, and for $y \neq \infty$ a root $x \in f^{-1}\{y\}$ of $g - hy$ of multiplicity $n \in \mathbf{Z}_{\geq 2}$ is called a *ramification point* or *critical point (of f)*. Furthermore, we call the integer n the *ramifcation index* of x, and denote it by e_x.

2.4. Remark. Part (i) in the previous definiton encodes much information about f. For instance, it implies that a pole p of a Belyi map f is sent to ∞ by f, since

$$\varphi_2(f(p)) = \varphi_2 \circ f \circ \varphi_1(p) = \lim_{z \to p} \varphi_2 \circ f \circ \varphi_1(z) = \lim_{z \to p} 1/f(z) = 0.$$

We implicitly used this already in (ii).

We will not go into much more rigorous depth regarding Riemann surfaces than this, but we should remark that the theory of Belyi maps and the yet to be introduced dessins d'enfants becomes much richer when one incorporates more theory of Riemann surfaces. In fact, our notion of a Belyi map can be extended to maps $f : X \to \overline{\mathbf{C}}$, defined on an arbitrary compact Riemann surface X. The pair (f, X) is then called a Belyi-pair. We will touch upon these matters again in 3.12 and 3.14, but we will mostly restrict ourselves to the case that $X = \overline{\mathbf{C}}$.

2.5. Remark. The points $0, 1$ and ∞ seem to play a special role in the definition of a Belyi map. In fact it is only a matter of convention that we have taken B_f to be a subset of $\{0, 1, \infty\}$; by a Möbius transformation, which is an invertible morphism of Riemann spheres, we can always map three distinct points to 0,1 and ∞ (see for instance [13](p.10)). Therefore, in essence the restriction on Belyi maps is that of the cardinality of B_f, which must be smaller than or equal to 3.

The following lemma is very important for our purposes.

2.6. Lemma. *Let f be a Belyi map, and $x \in \overline{\mathbf{C}} \setminus f^{-1}\{\infty\}$. Then there exists a neighbourhood U of x in $\overline{\mathbf{C}}$ such that there is a biholomorphism $\phi : \psi(U) \to \phi(\psi(U))$ with $\psi : \overline{\mathbf{C}} \to \overline{\mathbf{C}}, (z \mapsto (z - x)^{e_x})$ and $f|_U = \phi \circ \psi$.*

Proof. For a sufficiently small neighbourhood U of x such that $\infty \notin f(U)$, Taylor's theorem says:

$$\begin{aligned} f|_U(z) &= f(x) + (z - x)^{e_x} c + \mathcal{O}((z - x)^{e_x + 1}), \ c \in \mathbf{C} \setminus \{0\} \\ &= f(x) + (z - x)^{e_x} (c + \mathcal{O}((z - x))), \end{aligned}$$

and multiplication with a term of the form $c + \mathcal{O}((z - x))$ is a biholomorphism. □

2.7. Remark. Biholomorphisms are conformal maps, which is to say that they respect angles. From the previous lemma we see that this implies that the ramification indices relating to a holomorphic function on $\mathbf{C}$ remain unchanged when we compose f with a biholomorphism ϕ. Therefore, ramification indices of the functions $\varphi_i \circ f \circ \varphi_j^{-1}$ are defined unambigiously.

As another result of the previous observation we obtain a sound definition for the ramification index of a point $\infty \neq x \in f^{-1}\{\infty\}$, as well as for the ramification index of ∞ itself. For x a pole of $f = g/h$, we simply have that e_x is the multiplicity of x in $h = 0$, which follows from considering the composition with the invertible morphism of Riemann spheres $z \mapsto 1/z$ with f, i.e., $z \mapsto 1/f(z)$.

Moreover, in case $\infty \notin f^{-1}\{\infty\}$ we compose reversely with a Möbius transformation ϕ not fixing ∞ and with $\phi(\infty) \notin f^{-1}\{\infty\}$, and obtain that e_∞ equals the ramification index of $f(\phi(\infty))$ in $f(\phi)$.

From the previous observation follows immediately:

2.8. Proposition. *For every point $z \in \overline{\mathbf{C}}$*

$$\sum_{i \in f^{-1}\{z\}} e_i = \deg f.$$

It is an easy exercise to deduce that given a rational map in $\mathbf{C}$ it extends uniquely to a morphism of Riemann spheres. Conversely we have the following proposition which will be essential in our proof of 3.22. It tells us that part (ii) in 2.3 was superfluous. The proof in the reference is technical but uses only standard theory of complex analysis.

2.9. Proposition. *Every morphism of Riemann surfaces $f : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is induced by a rational map on $\mathbf{C}$.*

Proof. [3] thm. 3.11. □

We have gathered enough results about Belyi maps, and will now construct another category.

2.10. Definition. The category of Belyi maps, denoted by **Bel**, is the category with objects:

$$\mathrm{Obj}(\mathbf{Bel}) := \{f : \overline{\mathbf{C}} \to \overline{\mathbf{C}} \mid f \text{ is Belyi}\},$$

and given $f, g \in \mathrm{Obj}(\mathbf{Bel})$ with morphisms:

$$\mathrm{Hom}_{\mathbf{Bel}}(f, g) := \{\varphi : \overline{\mathbf{C}} \to \overline{\mathbf{C}} \mid \varphi \text{ is a morphism of Riemann spheres and } f \circ \varphi = g\}.$$

Notation: In the following we will write $\mathbf{C}_*$ for $\mathbf{C} \setminus \{0,1\}$.

The next proposition will bring our work of section 1 into play.

2.11. Proposition. *Let f be a Belyi map. Then the following restriction of f is a cover.*

$$f\big|_{\overline{\mathbf{C}} \setminus f^{-1}\{0,1,\infty\}} : \overline{\mathbf{C}} \setminus f^{-1}\{0,1,\infty\} \longrightarrow \mathbf{C}_*$$

Proof. Denote the degree of the rational map that induces f by d.

Let $y \in \overline{\mathbf{C}} \setminus \{0,1,\infty\} \subset \overline{\mathbf{C}} \setminus B_f$. Note that $\#f^{-1}\{y\} = d$. Index all elements of $f^{-1}\{y\}$ from 1 to d. A Taylor expansion as in 2.6 shows that we have, in a small enough neighbourhood of $i \in f^{-1}\{y\}$ an injective function restriction:

$$f|_{U_i} : U_i \to f(U_i).$$

Take such a restriction for every $i \in f^{-1}\{y\}$.

Write $U'_{i,j}$ for a neighbourhood of i that is disjoint with a neighbourhood $U'_{j,i}$ of j. Those neighbourhoods exist because $\overline{\mathbf{C}}$ is Hausdorff. Write $U_{i,j} := U'_{i,j} \cap U_i$ and take the intersection $\bar{U}_i := \bigcap_{j=1,\ldots,d} U_{i,j}$. Then we have neighbourhoods $\bar{U}_1, \ldots, \bar{U}_d$ with $i \in \bar{U}_i$ which are pairwise disjoint. Take

$$\bigcap_{z \in f^{-1}\{y\}} f(\bar{U}_z) =: V$$

Then we have :

$$f^{-1}V = f^{-1}V \cap \Big(\bigcup_{z \in f^{-1}\{y\}} \bar{U}_z \Big) = \bigcup_{z \in f^{-1}\{y\}} (\bar{U}_z \cap f^{-1}V)$$

were we have the first equality because each element in V really has d fiber-elements in the set on the right hand side. The images of subsets of $f(\bar{U}_i)$ are open by the implicit function theorem (see [4] thm. I.5.7), so we have obtained local homeomorphisms to V. The disjoint property follows by construction of $\bar{U}_i$. □

The previous proposition enables us to define the next functor, which we call the *seperator functor*.

2.12. Definition. Let $f, g \in \mathrm{Obj}(\mathbf{Bel})$ and $\varphi \in \mathrm{Hom}_{\mathbf{Bel}}(f,g)$. Sep is to denote the following correspondence:

$$\mathrm{Sep} : \mathbf{Bel} \longrightarrow \mathbf{Cov}(\mathbf{C}_*)$$

$$\text{At the level of objects:} \quad f \longmapsto f\big|_{\overline{\mathbf{C}} \setminus f^{-1}\{0,1,\infty\}}$$

$$\text{At the level of morphisms:} \quad \varphi \longmapsto \varphi\big|_{\overline{\mathbf{C}} \setminus f^{-1}\{0,1,\infty\}}$$

2.13. Proposition. Sep *is a functor.*

Proof. What remains to prove, after 2.11, is that the restriction of a Belyi morphism as defined by Sep is indeed a morphism of covers. With the notations of 2.12, we have $f = g \circ \varphi$. It follows that:

$$\begin{aligned}\varphi(\overline{\mathbf{C}} \setminus f^{-1}\{0,1,\infty\}) = \varphi(\overline{\mathbf{C}} \setminus \varphi^{-1}g^{-1}\{0,1,\infty\}) &= \varphi(\overline{\mathbf{C}}) \setminus g^{-1}\{0,1,\infty\} \\ &\subset \overline{\mathbf{C}} \setminus g^{-1}\{0,1,\infty\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Sep}(f) = f\big|_{\overline{\mathbf{C}}\setminus f^{-1}\{0,1,\infty\}} &= g \circ \varphi\big|_{\overline{\mathbf{C}}\setminus f^{-1}\{0,1,\infty\}} \\ &= g\big|_{\overline{\mathbf{C}}\setminus g^{-1}\{0,1,\infty\}} \circ \varphi\big|_{\overline{\mathbf{C}}\setminus f^{-1}\{0,1,\infty\}} = \mathrm{Sep}(g) \circ \mathrm{Sep}(\varphi)\end{aligned}$$

so $\mathrm{Sep}(\varphi) \in \mathrm{Hom}_{\mathbf{Cov}(\mathbf{C}_*)}(\mathrm{Sep}(f), \mathrm{Sep}(g))$. Similarly by the previous inclusion it follows that $\mathrm{Sep}(\psi \circ \varphi) = \mathrm{Sep}(\psi) \circ \mathrm{Sep}(\varphi)$ with ψ a with φ composable Belyi morphism. Identity is also preserved, so we conclude that Sep is a functor. □

3 Dessins d'enfants

In this section we will introduce and study the last object of this thesis, the dessins d'enfants. The main references are [6],[15] and [10]. Our work is a mix of the rather different approaches of [6] and [10], and we will try to somewhat clarify the relation between them. Most proofs, if a reference is not given, are mostly original, especially 3.10 and 3.17. 3.22 is based on [15], but is more eleborate than the proof found there.

As stated in the preface, the nature of these object seems to be quite different from the analytical Belyi maps, as the dessins are just graphs with some extra structure. As a consequence we first have the following definition.

3.1. Definition. We define a *graph* to be an ordered pair (V, E), where V is a set of *vertices* and E a set of ordered pairs $(\{v_0, v_1\}, n), v_0, v_1 \in V, n \in \mathbf{N}$ called *edges.* We write $\overline{e} = \{v_0, v_1\}$ with an edge $e = (\{v_0, v_1\}, n)$. A *path* on G from $a \in V$ to $b \in V$, $a \neq b$, is a finite ordered sequence of edges $(e_1, ...e_n)$ such that for $i = 1, ..., n-1$ $|\overline{e}_{i+1} \cap \overline{e}_i| = 1$ and $\{a\} = \overline{e}_1 \setminus (\overline{e}_1 \cap \overline{e}_2), \{b\} = \overline{e}_n \setminus (\overline{e}_{n-1} \cap \overline{e}_n)$.

It should be clear from this definition that we are considering undirected graphs with possibly multiple edges between two vertices.

Having formalized our intended notion of a graph, we can now define our second main object of interest.

3.2. Definition. A *dessin (d'enfant)* is a graph $G = (V, E)$ satisfying the following properties:

(i) G is finite, i.e., V and E both are finite sets;

(ii) G is bicoloured, i.e., $\exists V_0, V_1 \subset V : V = V_0 \sqcup V_1$ and $E \subset \{(\{v_0, v_1\}, n) : v_0 \in V_0, v_1 \in V_1, n \in \mathbf{N}\}$. We will call V_0 the black vertices, and V_1 the white vertices;

(iii) For every $v \in V$ the set of edges incident with v, $E_v = \{e \in E : v \in \overline{e}\}$, is equipped with a transitive $\mathbf{Z}$-action.

3.3. Definition. The category of dessins d'enfants, denoted by **Des** is the category with dessins as objects, and given two dessins $G_1 = (V = V_0 \sqcup V_1, E), G_2 = (V' = V'_0 \sqcup V'_1, E') \in \text{Obj}(\mathbf{Des})$ with morphisms

$$\text{Hom}_{\mathbf{Des}}(G_1, G_2) = \left\{ \begin{array}{c|c} \varphi : V \to V', & \text{for } i = 0, 1 : \forall v_i \in V_i \quad \varphi(v_i) \in V'_i, \\ \psi : E \to E' & \psi(E_{v_i}) \subset E'_{\varphi(v_i)}, \\ \text{maps} & \text{and } \psi|_{E_{v_i}} \text{ is equivariant} \end{array} \right\}.$$

So a morphism of dessins permutes the sets of incident edges of one colour, and moreover is equivariant under the $\mathbf{Z}$ action that is defined on such a set.

This leads us to our last functor, the *orbit functor*.

Notation. In the following we will denote the free group on 2 generators by $\mathbf{Z} * \mathbf{Z}$.

3.4. Definition. Let $\mathbf{Z} * \mathbf{Z}\text{-}\mathbf{Sets}^{\mathrm{f}}$ denote the full subcategory of $\mathbf{Z} * \mathbf{Z}\text{-}\mathbf{Sets}$ whose objects are *finite* $\mathbf{Z} * \mathbf{Z}$-sets. Let $S \in \mathrm{Obj}(\mathbf{Z} * \mathbf{Z}\text{-}\mathbf{Sets}^{\mathrm{f}})$, and write σ and τ for the two generating permutations under the action of $\mathbf{Z} * \mathbf{Z}$. Orb is to denote the following correspondence

$$\mathrm{Orb} : \mathbf{Z} * \mathbf{Z}\text{-}\mathbf{Sets}^{\mathrm{f}} \longrightarrow \mathbf{Des}$$

At the level of objects: $S \longmapsto (V = \{x\langle\sigma\rangle : x \in S\} \sqcup \{x\langle\tau\rangle : x \in S\}, E)$,

with $E = \big\{\big(\{v_0, v_1\}, n\big) : v_0 \cap v_1 \neq \emptyset \text{ and } n \in \{1, \ldots, |v_0 \cap v_1|\}\big\}$.

The idea here is that we take the two orbits under σ and τ of a point s in the set S, and color every $\langle\sigma\rangle$-orbit black, and every $\langle\tau\rangle$-orbit white. Whenever an s is part of two orbits $x\langle\sigma\rangle$ and $y\langle\tau\rangle$, we draw an edge between $x\langle\sigma\rangle$ and $y\langle\tau\rangle$.

Given $(\varphi : S \to T) \in \mathrm{Hom}_{\mathbf{Z}*\mathbf{Z}\text{-}\mathbf{Sets}^{\mathrm{f}}}(S, T)$ we have

At the level of morphism: $\varphi \longmapsto \big((V, E) \mapsto (\varphi(V), \varphi(E))\big)$,

with

$$\varphi(V) = \{\varphi(x)\langle\sigma\rangle : x \in S\} \sqcup \{\varphi(x)\langle\tau\rangle : x \in S\},$$
$$\varphi(E) = \big\{(\{\varphi(v_0), \varphi(v_1)\}, n) \bigm| (\{v_0, v_1\}, n) \in E\big\}.$$

3.5. Proposition. Orb *is a functor.*

Proof. Everything follows directly from the construction, which is well-defined because morphisms of $\mathbf{Z} * \mathbf{Z}\text{-}\mathbf{Sets}$ are equivariant. Further details are left to the reader. □

In fact, Orb is an equivalence of categories. This is not a very deep result, but requires some technical verifications. The reader can consult [8] for an elaborate and technical exposition on the orbit functor.

The orbit functor visualizes the structure of a $\mathbf{Z} * \mathbf{Z}$-set by creating a graph. The benefit of this is that we can use our knowledge of combinatorics and graphs to say something about $\mathbf{Z} * \mathbf{Z}$-sets, or, using $\mathrm{Fib}_{1/2}$, about covers of spaces with a fundamental group equal to $\mathbf{Z} * \mathbf{Z}$; here our motivation for considering the orbit functor is uncovered, as we have:

3.6. Proposition. *There is a canonical isomorphism of groups* $\pi_1(\mathbf{C}_*, 1/2) \cong \mathbf{Z} * \mathbf{Z}$. *The two generators correspond to the homotopy classes of paths* σ *and* τ *which are single loops, each around a different punctured point, about 1/2.*

Proof. This follows from the van Kampen theorem – see [5] section 14.c *&* 14.d $\square$

Intuitively the previous result can be understood when we apply the stereographically projection, which is a homeomorphism, on $\mathbf{C}_*$; its image is the real plane punctured in two points which correspond to two generators.

We now have three functors that we can compose. Because morphism of dessins are equipped with $\mathbf{Z}$-actions on the sets of incident edges, 3.5 implies:

3.7. Corollary. *Write* σ *for a single loop around* 0 *about* 1/2, *and* τ *for the single loop around* 1 *about* 1/2. *Let* f *be a Belyi map and* $\varphi \in \mathrm{Hom}_{\mathbf{Bel}}(f, g)$ *a Belyi morphism with* $\varphi(x) = y$ *for some* $x \in \mathrm{Fib}_{1/2}(\mathrm{Sep}(f)), y \in \mathrm{Fib}_{1/2}(\mathrm{Sep}(g))$. *Then* $|x\langle\sigma\rangle| \mid |y\langle\sigma\rangle|$ *and* $|x\langle\tau\rangle| \mid |y\langle\tau\rangle|$.

We will now focus on planar dessins. We need the following definitions.

3.8. Definition. A *drawing* of G is a set $D_V \subset \overline{\mathbf{C}}$ and a bijection $V \xrightarrow{\sim} D_V$, and a collection D_E of non-intersecting paths in $\overline{\mathbf{C}}$, one for each edge $e = (\{v_0, v_1\}, n)$ with begin and end point of the path equal to the to v_0 and v_1 corresponding points.

3.9. Definition. A dessin $G = (V, E)$ is called *planar* when there exists a drawing of G, and G is moreover *connected* when between every two vertices $v_0, v_1 \in V, v_0 \neq v_1$ there exists a path from v_0 to v_1.

The restriction of our attention to planar dessins is linked with our restricted attention to the Riemann sphere, as follows in the next theorem. Its reverse is also true, and will be proved in 3.20.

3.10. Theorem. *Let* $f \in \mathrm{Obj}(\mathbf{Bel})$, *then* $\mathcal{G}_f := \mathrm{Orb} \circ \mathrm{Fib}_x \circ \mathrm{Sep}(f)$ *is a connected and planar dessin, and* $\mathcal{D}_f = (D_V, D_E)$ *with* $D_V = f^{-1}\{0\} \sqcup f^{-1}\{1\}$, $D_E = f^{-1}(0,1)$ *is a drawing of* $\mathcal{G}_f$.

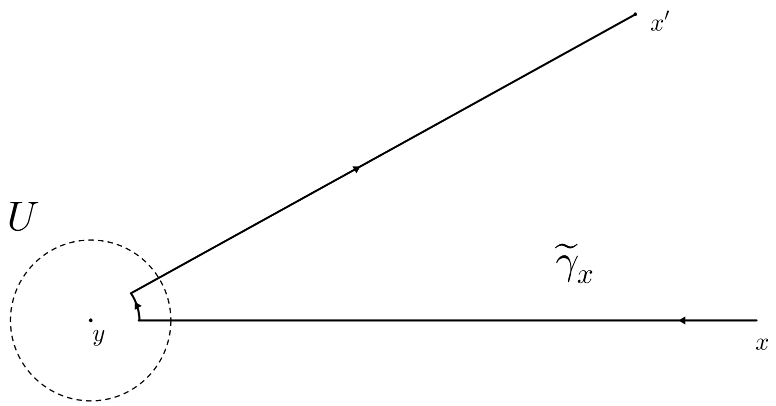
Proof. To prove connectedness, we will show that given a black vertex $x\langle\sigma\rangle$ and white vertex $y\langle\tau\rangle$ there exists a path between them. By the transitivity of $\mathrm{Fib}_x \circ \mathrm{Sep}(f)$, we have a $g \in \mathbf{Z} * \mathbf{Z}$ such that $xg = y$. Assume $g = \sigma\tau\tau\sigma$. Writing $w\langle\sigma\rangle \sim z\langle\tau\rangle$ when $(w\langle\sigma\rangle, z\langle\tau\rangle, n) \in E$ for some $n \in \mathbf{N}$, we note that

$$x\langle\sigma\rangle \sim x\sigma\langle\tau\rangle \sim x\sigma\tau\tau\langle\sigma\rangle \sim x\sigma\tau\tau\sigma\langle\tau\rangle = y\langle\tau\rangle.$$

This shows how we can construct a path from $x\langle\sigma\rangle$ to $y\langle\tau\rangle$, regardless of the form of g, and one can use induction on the length of g to complete a rigorous proof.

Next we prove planarity. First we remark that every single loop around 0 is homotopic to a loop of the form as shown below

On a local coordinate U about y, f is of the form $z \mapsto z^{e_y}$ in the sense of 2.6. Recalling that by the open mapping theorem biholomorphisms are homeomorphisms, we conclude that the lift of γ starting at a point $x \in f^{-1}\{1/2\}$ is homotopic to the path that by continuity of f approaches some $y \in f^{-1}\{0\}$ which is moreover unique by 1.7, and turns $1/e_y$ of a circle around y and then runs directly, that is, without approaching other points in $f^{-1}\{0, 1, \infty\}$ towards an $x' \in f^{-1}\{1/2\}$:



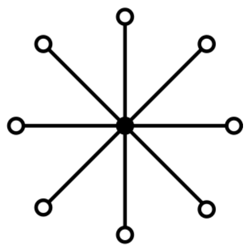
Now we can see what the orbit functor actually does: the point y is represented by $x\langle\sigma\rangle$, since in $\mathcal{G}_f$ we exactly get $e_y = |x\langle\sigma\rangle|$ with $x\langle\sigma\rangle$ incident edges here, one for each $x' \in x\langle\sigma\rangle$, with other end at $x'\langle\tau\rangle$. In terms of lifts this means that every 'half-edge', i.e., a lift of $(0, 1/2]$ extends uniquely to a lift of $(0, 1)$, which is an immediate consequence of 1.7 and the fact that $[1/2, 1) \subset \mathbf{C} \setminus \{0, 1\}$. From these observations it is clear that we have a one-to-one correspondence between lifts of $(0, 1)$ under f and edges in $\mathcal{G}_f$, and points in $f^{-1}\{0\}$ to black vertices V_0 in $\mathcal{G}_f$. At last we remark that the lifts of $(0, 1)$ do not intersect because this would contradict 1.7, so we have obtained a planar drawing of $\mathcal{G}_f$. $\square$

Notation. In the following we will continue to denote the drawing and the dessin induced by f as in the previous theorem by $\mathcal{D}_f$ and $\mathcal{G}_f$ respectively.

3.11. Examples. We will give three examples of drawings/dessins corresponding to Belyi maps.

(i)
$$f(z) = \frac{z^2}{z - 1/4}$$
is a Belyi map corresponding to a so-called leaf; see 3.16 and 3.17. The ramifications points are 0 and 1/2, and we have $\{f(0), f(1/2)\} \subset \{0, 1\}$. To see that these points are indeed the ramification points of f, one can look at the zeroes of the derivative of f. Indeed, from 2.6 it follows that the derivative of f vanishes at a ramification point. Therefore, if $f(\infty) = \infty$ and $f'(z_0) = 0$ implies $f(z_0) \in \{0, 1\}$ it follows f is Belyi. Here we have $f(\infty) = \infty$ (cf. the proof 3.21). Moreover $f'(z) = \frac{z^2 - 1/2z}{(z-1/4)^2}$ and indeed $f'(z) = 0 \Rightarrow z \in \{0, 1/2\} \wedge f(z) \in \{0, 1\}$.

(ii) Let $n \in \mathbf{Z}_{\geq 0}$ and $f(z) = z^n$. Then f is Belyi because $B_f = \{0, \infty\} \subset \{0, 1, \infty\}$. $\mathcal{D}_f$ is a star with n edges. For $n = 8$ this looks like:



A star with 8 edges.

(iii)
$$f(z) = \frac{4}{27} \frac{(z^2 - z + 1)^3}{z^2(z - 1)^2}$$
is a Belyi map. The corresponding dessin is the complete bipartite graph $K_{2,3}$. See [6] example 4.60 for more details.

The crucial theorem that combines our previous work is Riemann's existence theorem.

3.12. Theorem. (Riemann's existence theorem) *For a sequence $y_1, y_2, ..., y_k \in \overline{\mathbf{C}}$ and $g_1, g_2, ..., g_k \in S_n$ satisfying $g_1 g_2 ... g_k = e$, there exists a compact Riemann surface X and a morphism of Riemann surfaces $f : X \to \overline{\mathbf{C}}$ such that $y_1, y_2, ..., y_k$ are the branch points of f and g_i is the permutation of the fibers of a point $x \in \overline{\mathbf{C}} \backslash \{y_1, ..., y_k\}$ obtained from the lift of a single loop around y_i about x. For another morphism of*

Riemann surfaces $g : X' \to \overline{\mathbf{C}}$ satisfying the properties of f there exist an invertible morphism of Riemann surfaces $\varphi : X \to X'$ such that $f = g \circ \varphi$.

Proof. See [10] thm 1.8.14. □

3.13. Remarks. Such a sequence $g_1, g_2, ..., g_k$ corresponds to a cover of the punctured sphere in k points, and uniquely up to isomorphism of covers; [10] prop. 1.2.15 *&* prop. 1.2.16 proves this. It is indeed true that the product of the elements of such a sequence equals the identity element because it is the lift of the composition of single loops running around one branch point, and this product is homotopic to a loop going around all branch points in a single loop, and can therefore be retracted — see [10] p.19,20 for figures picturing this idea.

We can extend the Belyi category **Bel** to the category of Belyi-pairs: we replace $\overline{\mathbf{C}}$ as the domain of a Belyi map by an arbitrary *compact* Riemann surface X. The compactness ensures we have finite covers and thus finite $\mathbf{Z} * \mathbf{Z}$-sets, and thus finite graphs. Consequently, for our case where $k = 3$, we can summarize by saying that every finite cover of $\mathbf{C}_*$ corresponds to a Belyi-pair, and uniquely up to an invertible morphism of Belyi-pairs. Extending Sep in the natural way yields then, as won't come as a big surprise, another equivalence of categories. It is however the deepest equivalence of categories of the three we have by now seen, as it relies on the deepest theorem in this thesis, 3.12. Letting **Bel-pairs** denote the category of Belyi-pairs, and $\mathbf{Cov}(\mathbf{C}_*)^{\mathrm{f}}, \mathbf{Z} * \mathbf{Z}\text{-}\mathbf{Sets}^{\mathrm{f}}$ the full subcategories with covers with finite fibers, and finite sets as objects respectively, we can summarize as follows.

3.14. Theorem. *We have a functorial diagram, with all arrows equivalences of categories.*

$$\mathbf{Bel\text{-}pairs} \xrightarrow{\mathrm{Sep}} \mathbf{Cov}(\mathbf{C}_*)^{\mathrm{f}} \xrightarrow{\mathrm{Fib}_{1/2}} \mathbf{Z} * \mathbf{Z}\text{-}\mathbf{Sets}^{\mathrm{f}} \xrightarrow{\mathrm{Orb}} \mathbf{Des}$$

3.15. Remarks. Replacing $\overline{\mathbf{C}}$ in 3.8 by any compact Riemann surface X the drawings from 3.10 generalize to general Riemann surfaces, because 2.6 generalizes; this is quite clear from the observations in 2.7 — see [6] thm. 1.74, or [14] prop. 3.2.1 for details.

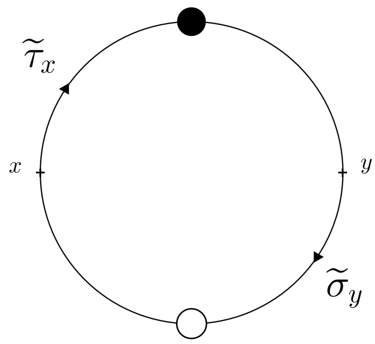
One can construct a category of drawings, with morphisms of Riemann surfaces mapping one drawing homeomorphically onto the other. Because for $\varphi \in \mathrm{Hom}_{\mathbf{Bel}}(f, g)$ $\varphi \circ f^{-1} = g^{-1}$, one gets a functor from **Bel-pairs** to this category of drawings.

For every $f \in \mathrm{Obj}(\mathbf{Bel})$ $\mathcal{D}_f$ induces a dessin by considering the incidence relations and using the natural **Z**-action on a set of incident edges induced by the monodromy action that we have seen in the proof of 3.10. $\mathrm{Fib}_{1/2} \circ \mathrm{Sep}(\varphi)$ is equivariant, so φ considered as a morphism of drawings induces a morphism of dessins. It should also be clear from the proof of 3.10 that this induced morphism is $\mathrm{Orb} \circ \mathrm{Fib}_{1/2} \circ \mathrm{Sep}(\varphi)$; it already might seem we are going in circles here, and it would be because – at least intuitively – it is clear that $\mathcal{D}_f$ and $\mathcal{G}_f$ carry the same information. After some verifications one arrives at another equivalence of the category of drawings and **Des**. The main point for considering the abstract dessins as we defined them is that we can use the fiber functor to prove the equivalence of **Bel-pairs** and **Des**.

Alternatively, one can define a dessin as a drawing, like in [6], which at first sight can seem rather different from the one we use, also because certain other properties that are implied by the theory of Riemann surfaces are sometimes included in the definition of a dessin. But given 3.14 and the observations regarding the drawings it can be understood that these defining properties are no restriction compared to our definition of dessins.

We are interested what happens in the category of **Bel-pairs** when we only consider planar and connected graphs under the equivalence of categories of 3.14. The following lemma about leafs will help to clarify this question, and is also relevant for the problem at the end of this section known as Davenport's bound. First a definition.

3.16. Definition. Let a dessin $\mathcal{G}_f$ be given, and write $\overline{\mathbf{C}} \setminus \mathcal{D}_f = \bigsqcup_{i=1}^n F_i$ with F_i the connected components of $\overline{\mathbf{C}} \setminus \mathcal{D}_f$. F_i are called the *faces* of $\mathcal{G}_f$. If two edges enclose a face we say that they form a *leaf.*



A leaf.

3.17. Lemma. *For every* $f \in \mathrm{Obj}(\mathbf{Bel})$*, every face of* $\mathcal{G}_f$ *contains exactly 1 point in the fiber of* ∞*. The ramification index of* e_x *for an* $x \in f^{-1}\{\infty\}$ *corresponding to the face of a leaf equals 1.*

Proof. For a rigorous proof, see [6] thm. 1.74. We will only give a sketch, to get some intuition for what happens here, which is easiest when we consider a leaf.

Denote two separate single loops around 0 and 1 about 1/2 as before by σ and τ respectively. If there would not be a pole inside the face of a leaf, we would have that $\widetilde{\sigma \odot \tau_y} = \widetilde{\sigma}_y \odot \widetilde{\tau}_x$ would be retractable and therefore so would be $\sigma_y \odot \tau_x$, which is homotopic to a loop around ∞ by the first paragraph of the remarks in 3.13, and so can not be retractable.

By continuity of f and 1.7 it is clear that a lift homotopic to $\widetilde{\sigma \odot \tau_y}$ only can approach a unique point in $f^{-1}\{\infty\}$, so this point must be the pole inside the face of the leaf, since this pole can be approached. Lets call this point p. We conclude that $\widetilde{\sigma}_y \odot \widetilde{\tau}_x$ is homotopic to the lift of a loop around ∞ that approaches p. Since $\widetilde{\sigma}_y \odot \widetilde{\tau}_x$ is a loop, by 3.10 this means that $e_p = 1$.

At last, we remark that if there would be more than one pole inside a leaf, then one pole would never be approached, because the paths can not intersect by 1.7. Every pole can be approached, so we always have exactly one pole for each face. □

Of its own interest, and ingredient of our solution too, is the following proposition stating the so-called Riemann-Hurwitz formula.

3.18. Proposition. *Let* $f : X \to \overline{\mathbf{C}}$ *be a morphism of compact connected Riemann surfaces. Then*

$$2g - 2 = -2\deg(f) + \sum_{z \in \overline{\mathbf{C}}} (e_z - 1),$$

with e_z *the ramification index of* z *and* g *the genus of* X*.*

Proof. See [7] IV §2. □

3.19. Remark. The genus of a Riemann surface can be thought of as the number of holes in it. For instance, the torus, which is a Riemann surface, has genus 1. The Riemann sphere has genus 0. See [5] for more details.

We can now prove the reverse statement of 3.10.

3.20. Proposition. *Let* $\mathbf{Des}_{\mathrm{pc}}$ *denote the full subcategory of* **Des** *whose objects are* planar *and* connected *dessins. Then* **Bel** *and* $\mathbf{Des}_{\mathrm{pc}}$ *are equivalent categories.*

Proof. We need to prove essential surjectivity, because fully-faithfulness is obtained by 3.14. By the same theorem it suffices to show that (X, f) for $X \neq \overline{\mathbf{C}}$ corresponds to a non-planar dessin.

Let $(f : X \to \overline{\mathbf{C}}) \in \mathrm{Obj}(\mathbf{Bel\text{-}pairs})$ and suppose $\mathcal{G}_f$ is planar and connected. Note first that with 2.8 we have

$$\sum_{z \in f^{-1}\{x\}} (e_z - 1) = \deg(f) - \# f^{-1}\{x\}$$

Furthermore, we have

- $\# f^{-1}\{\infty\} = \#$ faces $= F$, by 3.17,
- $\# f^{-1}\{0\} = \#$ black vertices $= B$,
- $\# f^{-1}\{1\} = \#$ whites vertices $= W$,
- $\#$ edges $= \deg f = E$, which follows from 3.10 and 2.8 with the fact that $1/2$ is not a branch point.
- $W + Z - E + F = 2$, by the Euler characteristic for planar graphs.

Noting that for a Belyi map we have $\sum_{z \in f^{-1}\{x\}} (e_z - 1) = 0$ when $x \neq 0, 1, \infty$, we obtain with 3.18

$$\begin{aligned} 2g - 2 &= -2\deg(f) + 3\deg(f) - (F + W + Z) \\ &= E - (F + W + Z) = E - (2 + E) = -2, \end{aligned}$$

hence g is zero. By a classical classification theorem based on the Riemann-Roch theorem we know that every connected compact Riemann surface with genus zero is isomorphic to the Riemann sphere. In the a priori possible case that X is a non-connected Riemann surface, we can restrict f to the connected component C in which $\mathcal{D}_f$ lies. The monodromy permutations g_1, g_2, g_3 are thus also achieved by this restriction of f to C. [10] prop. 1.2.16 then implies that $X \cong C$. Moreover, the genus of C is by the previous argument equal to zero, so once more $C \cong \overline{\mathbf{C}}$. We obtain also in this case that $X \cong \overline{\mathbf{C}}$. We conclude that $f \in \mathbf{Bel}$ and we are done. □

Davenport's bound on $f^3 - g^2 \in \mathbf{C}[X]$

3.20 has a nice and perhaps surprising application. Suppose we wonder what is the minimal degree of a non-zero polynomial $h = f^3 - g^2 \in \mathbf{C}[X]$. To minimize the degree of h, in any case we have to make sure the leading coefficients cancel, so we set $\deg(f) = 2m, \deg(g) = 3m, m \in \mathbf{Z}_{\geq 1}$. Common factors of f and g are not of any interest, so we consider only relatively prime triples of polynomials.

The following theorem is known as the ABC theorem for polynomials. Note that because $a = -b - c$ the assumption on the polynomials being relatively prime is natural, but not a real restriction. As we will see shortly, the ABC theorem settles a lower bound on the degree of $f^3 - g^2$. The dessins come into play to prove that this bound is sharp.

3.21. Theorem. *Write $N(f)$ for the number of roots of a complex polynomial $0 \neq f \in \mathbf{C}[X]$. If $a+b+c = 0$ for $a, b, c \in \mathbf{C}[X]$ relatively prime and not all constant, then we have* $\max\{\deg(a), \deg(b), \deg(c)\} \leq N(abc) - 1$.

Proof. By symmetry on a, b, c we have two cases: (i) $\deg(a) = \deg(b) > \deg(c)$; or (ii) $\deg(a) = \deg(b) = \deg(c)$. Write $f = -a/c$, and extend $f : \mathbf{C} \to \mathbf{C}$ to a morphism of Riemann spheres. Since $\deg(a) > \deg(c)$, from

$$0 = \lim_{z \to 0} 1/f(1/z) = \varphi_2 \circ f \circ \varphi_2^{-1}(0) = \varphi_2 f(\infty)$$

we see that $f(\infty) = \infty$. Write r_1, r_2, r_3 for the number of roots of respectively a, b, c, write $n = \deg a$ and as before denote the ramification indices corresponding to f of a point x by e_x. We have $N(abc) = r_1 + r_2 + r_3$ and by 2.8 $e_\infty = n - \deg(c)$. Noting that $n = \deg(f)$, 3.18 gives with $g = 0$ because f is a morphism of Riemann spheres:

$$-2 = -2n + \sum_{i=1}^{r_1}(e_i - 1) + \sum_{j=1}^{r_2}(e_j - 1) + \sum_{k=1}^{r_3}(e_k - 1) + (n - \deg(c) - 1) + M,$$

where $M \in \mathbf{Z}_{\geq 0}$ is the term representing all ramifications above other points than $0, 1, \infty$. Because $f(z) = -a/c = 1 \Leftrightarrow b(z) = 0$, since a, b and c are relatively prime by assumption, $M = 0$ if and only if f is Belyi. Using 2.8 to rewrite the previous equation we get our result:

$$\begin{aligned} 2n - 2 &= (n - r_1) + (n - r_2) + (\deg(c) - r_3) + (n - \deg(c) - 1) + M \\ &\geq 3n - 1 - (r_1 + r_2 + r_3) = 3n - 1 - N(abc) \\ &\Longrightarrow n \leq N(abc) - 1 \end{aligned}$$

In case (ii) everything goes analogously, except that $f(\infty) \in \mathbf{C}$, which again follows from a limit argument. We directly obtain

$$\begin{aligned}2n - 2 &= (n - r_1) + (n - r_2) + (n - r_3) + M \\ &\geq 3n - N(abc) \\ &\Longrightarrow n \leq N(abc) - 2\end{aligned}$$

□

Now we can solve the problem.

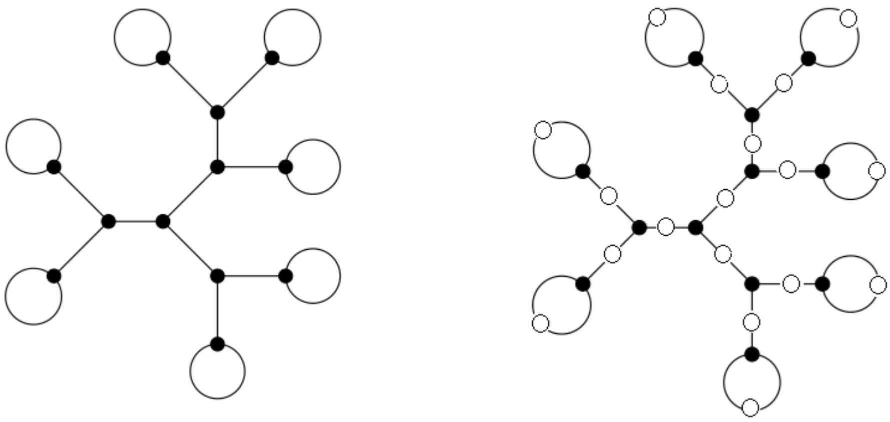
3.22. Theorem. *Let $f, g, h \in \mathbf{C}[X]$ be three relatively prime polynomials, not all constant and such that $f^3 - g^2 = h$ and with $\deg(f) = 2m$, $\deg(g) = 3m$ for $m \in \mathbf{Z}_{\geq 1}$. Then $\deg(h) \geq m + 1$. Moreover, this bound is sharp for all m, i.e., for all $m \in \mathbf{Z}_{\geq 1}$ there exist polynomials f, g, h as above with $\deg(h) = m + 1$.*

Proof. Take $a = f^3, b = -g^2, c = -h$ in 3.21. With $6m = \max\{\deg(f^3), \deg(g^2), \deg(h)\}$ we obtain

$$\begin{aligned}6m \leq N(f^3 g^2 h) - 1 = N(fgh) - 1 &\leq \deg(f) + \deg(g) + \deg(h) - 1 \\ &= 2m + 3m + \deg(h) - 1 \\ \Longrightarrow \deg(h) \geq m + 1 &\end{aligned}$$

This proves the lower-bound.

For the sharpness, we will use 3.20. Let $m \in \mathbf{Z}_{\geq 1}$ and consider the following graph: a tree with $2m$ vertices with $m + 1$ loops. An example for $m = 6$ is given below (left-side).



Constructed graph for $m = 6$, with on the right-side its 'bicolourfication'.

Now we put one white vertex on every edge, to obtain a planar bicoloured graph which we denote by $G = (E, V)$; for $m = 6$ we get the graph on the right-side. Using the fact that the Riemann sphere is orientable, we equip the set of incident edges of a vertex with the counter-clockwise permutation, so that G becomes a dessin. 3.20 assures us that we have $G \cong \mathcal{G}_\varphi$, for some $\varphi \in \mathrm{Obj}(\mathbf{Bel})$. Applying a Möbius transformation to make sure that the point $x \in \varphi^{-1}\{\infty\}$ corresponding to the outer face, that is, the face that is not enclosed by a leaf, equals ∞, we claim that

$$\varphi = \frac{f^3}{h} \quad \text{and} \quad g^2 = f^3 - h$$

for some $f, g, h \in \mathbf{C}[X]$ with $\deg f = 2m$, $\deg g = 3m$, and $\deg h = m + 1$. Indeed, by 2.9 φ is induced by a rational map. Moreover, it must be that in the numerator of φ we have a polynomial coprime with the polynomial in the denominator – which we denote by h – that is a third power, since every black vertex has degree equal to 3, and by 3.10 the degree of a vertex must equal the ramification index of the corresponding ramification point of φ.

The same reasoning shows that $\varphi - 1 = (f^3 - h)/h$ must be a rational map with a polynomial coprime with h in the numerator that equals the square of some polynomial in $\mathbf{C}[X]$, which we call g. We conclude that $f^3 - h = g^2$.

Finally by 3.17 all poles of φ, that is, all roots of h, have ramification index 1, since these points correspond to the faces of the leafs. Together with 2.8 this affirms that $\deg h = m + 1$. We are done. □

3.23. Remark. In fact, the use of lemma 3.17 is superfluous in the concluding paragraph of the proof of 3.22; we already saw in 3.21 that we have equality in $\max\{\deg(a), \deg(b), \deg(c)\} \leq N(abc) - 1$ if and only if $-a/c$ is Belyi and once it is proved that $f^3 - g^2 = h$ it is clear that $\deg(h) \leq 6m$. However this argument is based on the Riemann-Hurwitz formula, and the perhaps more intuitive lemma also provides some extra insight in the relations between the lifts and ramification indices between points in the fibers of 0 and 1 and point in the fiber of ∞.

3.24. Remark. As we have reached the end of our exposition, a last remark on what lies ahead of this material.

Like hinted at before, the theory of Riemann surfaces is strongly related to the material of this thesis. One fact of Riemann surfaces is of particular importance in this context, namely the fact that a compact Riemann surface can be represented as an algebraic curve (proved in [11] thm 5.8.4). The mathematician Belyi proved there exists a curve representing a Riemann surface X with coefficients contained inside

$\overline{\mathbf{Q}}$ (the algebraic closure of $\mathbf{Q}$) if and only if there exists a morphism of Riemann surfaces $f : X \to \overline{\mathbf{C}}$ unramified outside $\{0, 1, \infty\}$, in other words, if and only if $(X, f) \in \mathrm{Obj}(\textbf{Bel-pairs})$ (of course, this is the reason why we call such maps Belyi maps); a proof can be found in [10] section 2.6. This means that drawing a dessin, one specifies a Belyi-pair consisting of a Riemann surface X 'defined over $\overline{\mathbf{Q}}$' and a morphism f which in the same sense then also can be defined over $\overline{\mathbf{Q}}$. This opens the door to the study of $\mathrm{Gal}(\overline{\mathbf{Q}}|\mathbf{Q})$, the absolute Galois group of $\mathbf{Q}$, remarkably through the use of objects as simple as 'children's drawings'. Much surprised by this fact, Grothendieck, who developed the correspondence of 3.14 and much of the relating theory, more or less mockingly coined the term dessin d'enfant.

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