Field line solutions of the Maxwell equations using the 3+1 formalism for general relativity

THESIS

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Abstract

We will discuss the results from [1] and construct some concrete examples of the general solutions that are proposed in that paper. To this end we will develop the necessary theory of manifolds and the basic framework of general relativity in terms of the 3+1 formalism. After which we examine the method proposed in [1] and construct concrete cases of such solutions.

Contents

Chapter 1

Introduction

We assume some introductory knowledge of manifolds in this thesis. We will mostly use the notation from [2]. We will look at the solutions of the Einstein-Maxwells equations using the 3+1 formalism on hyperbolic manifolds as proposed by [1]. We will consider some examples in chapter 6. The Maxwells equations are the following: *

$$\mathcal{L}_{\mathbf{m}} E^i - NKE^i - \epsilon^{ijk} D_j (NB_k) = 0 \tag{1.1}$$

$$\mathcal{L}_{\mathbf{m}} B^i - NKB^i - \epsilon^{ijk} D_j (NE_k) = 0 \tag{1.2}$$

$$D_i E^i = 0 \tag{1.3}$$

$$D_i B^i = 0 \tag{1.4}$$

The reader more familiar with this formalism and the Maxwell equations will easily recognize the form of these equations where the first two, 1.1 and 1.2, are of the similar form of the evolution equations, where the latter two, 1.3 and 1.4 are the constraint equations when we take the system to be devoid of sources. Here the role of the time derivative is played by the lie derivative with respect to the normal evolution vector, $\mathcal{L}_{\mathbf{m}}$, while the spacial derivative is represented as the co-variant derivative, D_j. We look at these solutions with the prospect of using the methods developed here and in previous literature to expand the theory around Hopfions as originally set out in [3]. Although the original goal of this theory of developing a theoretical framework for photons while accounting for gravity where we replace a photon with a Hopfion turned out to be unfruitful, the developed framework has other applications such as in plasma physics.

*Note that we use the Einstein summation convention here and throughout this thesis.

1.1 Outline

In this thesis we will give a construction of the 3+1 formalism similar to what is done in [2] and [4] and with this theory we will discuss some results from [1]. In chapter 2 we set up the basis of manifold theory, in section 2.2 we will then define the notion of tangent vectors on this space followed by the definitions of tensors and some derivatives in chapter 3. In chapter 4 we discuss the usage of manifold theory in general relativity. Then in chapter 5 we discuss the results of [1] and in chapter 6 we apply these results to some concrete cases.

Chapter 2

Manifolds and Tangent spaces

When discussing a manifold from a mathematical point of view, we first construct an abstract set on which we can define a topology such that for a subset we can define an homeomorphism such that the image of this set can be identified with a subset in $\mathbb{R}^n$ with the standard topology. This results in the space locally "looking" like $\mathbb{R}^n$. When we want a space on which we can define differential equations, obviously we need a concept of differentiability. In order to construct this differentiability we will use a method to transfer the differentiability form $\mathbb{R}^n$ to the space. This is done in a similar manner as we use to transport the topological properties of $\mathbb{R}^n$ to the space. This concept of differentiability gives rise to definition 3 and definition 2. Where now we consider differentiability to be smooth, $\mathcal{C}^\infty$.

We start with the notion of a manifold, which is nothing more than a space or object which we can locally describe as the familiar $\mathbb{R}^n$. To this end we first define what we mean by this topologically. This brings us to our fist definition.

Definition 1. *We call a topological space X locally Euclidean if for every point $p \in X$ there exists an open subset $U \subset X$ such that $p \in U$ and there exist an $n \in \mathbb{N}$ such that there is a $V \subset \mathbb{R}^n$ and an homeomorphism $h : U \to V$.*

Now with this notion we have an idea of what we mean by saying that we want the spaces to locally be described in a similar manner as the $\mathbb{R}^n$. The following definition gives a manner of choosing the U, h, V from definition 1 such that they will have an additional structure which makes the manifold subsequently defined a differentiable manifold.

Definition 2. *Let X be a locally Euclidean topological space then an Atlas is a set $\mathcal{A} = \{(U_i, h_i, V_i) | i \in I\}$, where I is an index set, such that for each $i \in I$ the following holds:*

- *U_i is an open subset of X and $X = \bigcup_{i \in I} U_i$.*
- *V_i is an open subset of $\mathbb{R}^n$,*
- *$h_i : U_i \to V_i$ is a homeomorphism.*
- *$\forall i, j \in I$ the gluing map*

  $$(h_j \circ h_i^{-1})|_{h_i(U_i \cap U_j)} : h_i(U_i \cap U_j) \to h_j(U_i \cap U_j)$$

  is differentiable.

Definition 3. *An n-dimensional differentiable manifold is a pair (X, M) where X is a second countable Hausdorff space, and M is a maximal * n-dimensional differentiable atlas for X.*

Throughout the rest of this thesis when ever we refer to a manifold we mean a differentiable manifold, similarly when talking about a point $p \in X$ where X is a differentiable manifold we mean $p \in X$ where $(X, \mathcal{M})$ is a differentiable manifold for a chosen maximal atlas $\mathcal{M}$.

2.1 Example: Stereographic projection

We can consider $S^n \subset \mathbb{R}^{n+1}$ under the stereographic functions, see figure 2.1. This is a relatively simple way to construct an atlas for S^n. This atlas will be the following:

$$\begin{aligned} \mathcal{A} := \{ & (S^n \setminus \{h_N^{-1}(x_1, \ldots, x_n) | x_1 = \cdots = x_{n-1}, x_n = 1\}, h_N, \mathbb{R}^n), \\ & (S^n \setminus \{h_S^{-1}(x_1, \ldots, x_n) | x_1 = \cdots = x_{n-1}, x_n = -1\}, h_S, \mathbb{R}^n)\} \end{aligned} \tag{2.1}$$

We will denote the points $h_N^{-1}(x_1, \ldots, x_n)|x_1 = \cdots = x_{n-1}, x_n = 1$ and $h_S^{-1}(x_1, \ldots, x_n)|x_1 = \cdots = x_{n-1}, x_n = -1$ by N and S respectively.

$$\begin{aligned} & h_N : S^n \setminus \{N\} \to \mathbb{R}^n, \\ & \qquad (x_1, \ldots, x_n) \mapsto \left(\frac{2x_1}{1 - x_n}, \frac{2x_2}{1 - x_n}, \ldots, \frac{2x_{n-1}}{1 - x_n}\right) \end{aligned} \tag{2.2}$$

*We call an atlas, $\mathcal{A}$ maximal if for every atlas, $\mathcal{A}'$, such that $\mathcal{A} \cup \mathcal{A}'$ is again an atlas holds that $\mathcal{A}' \subset \mathcal{A}$.

$$h_N : S^n \setminus \{S\} \to \mathbb{R}^n,$$
$$(x_1, \ldots, x_n) \mapsto (\frac{2x_1}{1+x_n}, \frac{2x_2}{1+x_n}, \ldots, \frac{2x_{n-1}}{1+x_n}) \quad (2.3)$$

Now to check that this actually constitutes a differentiable atlas we will check the transition maps, in this case we only have two of those which are:

$$h_N \circ h_S^{-1}|_{h_S(S^n \setminus \{N,S\})} : h_S(S^n \setminus \{N,S\}) \to h_N(S^n \setminus \{N,S\})$$
$$(x_1, \ldots, x_n) \mapsto (\frac{2x_1}{\sum_{i=1}^n x_i^2}, \ldots, \frac{2x_n}{\sum_{i=1}^n x_i^2}) \quad (2.4)$$

$$h_S \circ h_N^{-1}|_{h_N(S^n \setminus \{N,S\})} : h_N(S^n \setminus \{N,S\}) \to h_S(S^n \setminus \{N,S\})$$
$$(x_1, \ldots, x_n) \mapsto (\frac{2x_1}{\sum_{i=1}^n x_i^2}, \ldots, \frac{2x_n}{\sum_{i=1}^n x_i^2}) \quad (2.5)$$

We remark that

$$h_N(S^n \setminus \{N,S\}) = h_S(S^n \setminus \{N,S\}) = \mathbb{R}^n \setminus \{(0, \ldots, 0)\} \quad (2.6)$$

So these transformation maps are both differentiable on their domain. Which means $\mathcal{A}$ is indeed a differentiable atlas.

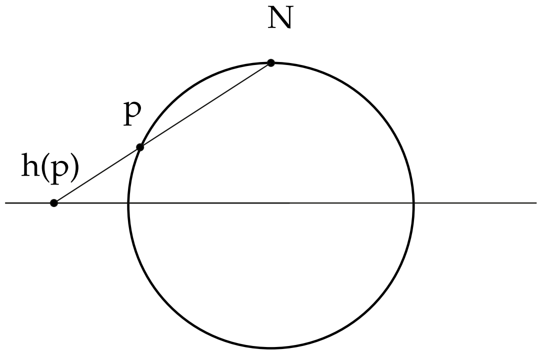

Figure 2.1: *Here we see S^1 on the real number-line where a projection via h sends point p to $h(p)$*

2.2 Tangent space

Now that we have a notion of a manifold it is natural to consider what objects we have on a manifold. Since we defined a manifold with the idea of making an object which looks like $\mathbb{R}^n$ one of the first things to look at is the notions of vectors. For this notion to make sense everywhere on the manifold we'll first need the concept of a tangent space.

Definition 4. *Let $\mathcal{M}$ be a (n-dimensional differentiable) manifold and $x \in \mathcal{M}$ a point in that manifold, then there is a coordinate chart*† *$\phi : U \to V$, such that $\mathcal{M} \supset U \ni x$, and $V \subset \mathbb{R}^n$ are open. Let $\gamma_1, \gamma_2 : I \to \mathcal{M}$, where $0 \in I \subset \mathbb{R}$ open, be two curves defined in U such that $\gamma_1(0) = x = \gamma_2(0)$. Then we define an $\gamma_1 \sim \gamma_2$ at a point r if $\frac{d}{dt}(\phi \circ \gamma_1(t))|_{t=r} = \frac{d}{dt}(\phi \circ \gamma_2(t))|_{t=r}$, which is an equivalence relation.*
Now we define the Tangent space at x, denoted by $T_x(\mathcal{M})$, to be the vector space of all equivalence classes of curves trough x ‡*. Where we define addition and scaling as point wise addition and scaling of the curves respectively, remark that this is independent of the choice of representative.*

So now a vector at a point x in a manifold $\mathcal{M}$ is a vector in a vector space formed from equivalence classes of curves. The more experienced reader might recognize the above definition as that of the geometric tangent space, since other tangent spaces (such as the algebraic) can be shown to be equivalent to the geometric we will not further discus the other ways of constructing the tangent space.

Example: Stereographic projection of S^3

Continuing the construction we did in example 2.1 we'll construct the tangent space on the 3-sphere. We'll use the chart $(S^3 \setminus \{(0,0,0,1)\}, h_N, \mathbb{R}^3)$ since the tangent space on $(0,0,0,1)$ can be constructed using the same method but another chart. We'll use h_N^{-1} to construct curves on S^3, this is certainly not the only manner to do so but is relatively easy and insight full. Let $p \in S^3$ be a point such that $h_N(p) = (x, y, z)$ then we find since

†Remark: due to the fact that we can always transform nicely from one coordinate chart to an other the definition here is not dependent on the choice of a chart.

‡We'll often just write a representative for such a class as tangent-vector in x

h_N is an diffeomorphism that the lines

$$L_x := \{(t \cdot x, y, z) \in \mathbb{R}^3 | t \in \mathbb{R}\}, \tag{2.7}$$

$$L_y := \{(x, t \cdot y, z) \in \mathbb{R}^3 | t \in \mathbb{R}\}, \tag{2.8}$$

$$L_z := \{(x, y, t \cdot z) \in \mathbb{R}^3 | t \in \mathbb{R}\} \tag{2.9}$$

map under h_N^{-1} to curves in S^3 trough p, we'll call these curves γ_x, γ_y and γ_z respectively. So 3 tangent vectors at p are given by:

$$\begin{aligned}
\partial_x(p) &:= \frac{\partial \gamma_x}{\partial x} h_N(p) \\
&= \left(\tfrac{2(1-x^2+y^2+z^2)}{(1+x^2+y^2+z^2)^2}, \tfrac{-4xy}{(1+x^2+y^2+z^2)^2}, \tfrac{-4xz}{(1+x^2+y^2+z^2)^2}, \tfrac{4x}{(1+x^2+y^2+z^2)^2} \right) &(2.10)\\
\partial_y(p) &:= \frac{\partial \gamma_y}{\partial y} h_N(p) \\
&= \left(\tfrac{-4xy}{(1+x^2+y^2+z^2)^2}, \tfrac{2(1+x^2-y^2+z^2)}{(1+x^2+y^2+z^2)^2}, \tfrac{-4yz}{(1+x^2+y^2+z^2)^2}, \tfrac{4y}{(1+x^2+y^2+z^2)^2} \right) &(2.11)\\
\partial_{z(p)} &:= \frac{\partial \gamma_z}{\partial z} h_N(p) \\
&= \left(\tfrac{-4xz}{(1+x^2+y^2+z^2)^2}, \tfrac{-4yz}{(1+x^2+y^2+z^2)^2}, \tfrac{2(1+x^2+y^2-z^2)}{(1+x^2+y^2+z^2)^2}, \tfrac{4z}{(1+x^2+y^2+z^2)^2} \right) &(2.12)
\end{aligned}$$

Now suppose $p = (1, 0, 0, 0)$ we fill this in and find:

$$\partial_x(1, 0, 0, 0) = (0, 0, 0, 1) \tag{2.13}$$

$$\partial_y(1, 0, 0, 0) = (0, 1, 0, 0) \tag{2.14}$$

$$\partial_z(1, 0, 0, 0) = (0, 0, 1, 0) \tag{2.15}$$

Which are clearly linearly independent, and since S^3 is a 3-manifold and thus its tangent space has dimension 3, they form a basis for the tangent space at $(1, 0, 0, 0)$

Chapter 3

Additional structure

3.1 Tensors and metrics

Now that we have the notion of tangent space we will introduce a more general object from linear algebra, namely a tensor. This we will use trough out this thesis as almost all of the physical quantities can be described by these objects.

Definition 5. *Let V be a vector space of dimension n. A (mixed) Tensor is a real valued multilinear function T*

$$T : \underbrace{V^* \times \cdots \times V^*}_{s} \times \underbrace{V \times \cdots \times V}_{r} \to \mathbb{R} \tag{3.1}$$

We call it s-times covariant and r-times contravariant, which we denote as T being a (s,r)-tensor.

The familiar examples of Tensors are (1,0)-tensors which are just contravariant vectors, or linear transformations which are (1,1)-tensors. The important property of tensors which, although not directly clear from the definition, makes them so suitable for our application is that they are independent of choice of basis or in our terms coordinates. By this we mean that given the coefficients of a tensor for one basis we can use a transformation law to determine the coefficients on an arbitrary other basis. So although the values of the coefficients might change they will change in such a way that the same vectors will still be sent to the same value. A commonly used tensor is the Levi-Civita tensor defined as follows:

Definition 6. *Let V be an n-dimensional vector space the Levi-Civita tensor is the tensor with the following property:*

$$\epsilon_{i_1 i_2 \dots i_n} = \begin{cases} +1 & \text{if } (i_1 i_2 \dots i_n) \text{ is an even permutation of } (1, \dots, n) \\ -1 & \text{if } (i_1 i_2 \dots i_n) \text{ is an odd permutation of } (1, \dots, n) \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

Now for a manifold we will define tensor fields as functions that at each point on the manifold gives a tensor from the tangent space * at that point.

Definition 7. *For a manifold $\mathcal{M}$ a (r,s)-tensor field is a function f such that*

$$f : \mathcal{M} \to \{T : \underbrace{T_x(\mathcal{M})^* \times \cdots \times T_x(\mathcal{M})^*}_{s} \times \underbrace{T_x(\mathcal{M}) \times \cdots \times T_x(\mathcal{M})}_{r} \to \mathbb{R} | T \text{ is multilinear}\} \tag{3.3}$$

We will often treat the value of a tensor field at a point as just being a tensor, meaning that we will write $f(p)(v'_1, \dots, v'_s, v_1, \dots, v_r)$ as $f_p(v'_1, \dots, v'_s, v_1, \dots, v_r)$ or $f(v'_1, \dots, v'_s, v_1, \dots, v_r)$, where $v'_i \in T_x(\mathcal{M})^*, v_j \in T_x(\mathcal{M}), i = 1, \dots, s,$ $j = 1, \dots, r$.

A specific case and often used tensor field is the metric tensor which is used as a dot product in the tangent space.

Definition 8. *g is a (pseudo-Riemann) metric tensor, or metric, on a differentiable manifold $\mathcal{M}$, if g is a tensor field obeying the following:*

- *g is a $(0,2)$-tensor field: at each point $p \in \mathcal{M}, g(p)$ is a bilinear form which acts on the vectors in the tangent space $\mathcal{T}_p(\mathcal{M})$ as follows:*

$$\begin{aligned} g(p) : \mathcal{T}_p(\mathcal{M}) \times \mathcal{T}_p(\mathcal{M}) &\to \mathbb{R} \\ (u, v) &\mapsto g(u, v) \end{aligned}$$

- *g is symmetric, i.e. $g(u, v) = g(v, u)$*
- *g is non-degenerate: $\forall p \in \mathcal{M}$, if $u \in \mathcal{T}_p(\mathcal{M})$ is a vector s.t. $\forall v \in \mathcal{T}_p(\mathcal{M}), \quad g(u, v) = 0$ means that $u = 0$*

We call vectors $v \in \mathcal{T}_p(\mathcal{M})$

*Note we mean not just vectors or forms but the tensors can be higher order by combining the vectors and forms.

- *Timelike if* $g(v, v) < 0$
- *Spacelike if* $g(v, v) > 0$
- *Null if* $g(v, v) = 0$

A metric tensor which at all points is positive definite rather than only non-degenerate is called an Riemann metric.
We call a couple $(\mathcal{M}, g)$ where $\mathcal{M}$ is a differentiable manifold and g is a non-degenerate, smooth, symmetric metric a pseudo-Riemann manifold.

Definition 9. *The signature of a metric tensor g, denoted as sign(g), is defined as the signature of the corresponding quadratic form. This is a triple (p, q, r) where p is the number of negative eigenvalues of a matrix representing the form, q the number of positive eigenvalues, and r the number of zero eigenvalues, here they are counted with their algebraic multiplicity. And Sylvester's law of inertia tells us that (p, q, r) is independent of basis choice. We some times write for a signature of $(1, 3, 0)$ that sign$(g) = (-, +, +, +)$*

We call a metric g Lorentzian if $\text{sign}(g) = (1, n-1, 0)$ where n is the dimension of the tangent space and for g a Riemann metric we find that $\text{sign}(g) = (0, n, 0)$.

Example: 3-Sphere

Using the same coordinate system as given in example 2.1 we can induce a metric on the S^3 using the standard metric on $\mathbb{R}^4$. The resulting metric is then given, depending on the chart used, by:

$$(Dh_N^{-1})^\top \cdot Dh_N^{-1} \text{ or } (Dh_S^{-1})^\top \cdot Dh_S^{-1} \tag{3.4}$$

Where Dh is the Jacobian of h. The induced metric, g, we then find to be:
In the chart $(S^3 \setminus \{(0,0,0,1)\}, h_N, \mathbb{R}^3)$

$$g = \begin{pmatrix} \frac{4}{1+x^2+y^2+z^2} & 0 & 0 \\ 0 & \frac{4}{1+x^2+y^2+z^2} & 0 \\ 0 & 0 & \frac{4}{1+x^2+y^2+z^2} \end{pmatrix} \tag{3.5}$$

Now we remark that due to symmetry between h_N and h_S and the fact that the metric g does not depend on w we find that the matrix representing the metric for both charts are the same when expressed in coordinates.[†]

[†]Note that they are written over different coordinates.

3.2 Derivatives

On an abstract manifold the notion of a derivative is not as straightforward as one might think initially, since the structure of $\mathbb{R}^n$ holds locally but not necessarily globally. As such there are multiple objects which we can see as derivatives, we will discuss some of these in this section.

3.2.1 Connection

In order to be able to consider the change of a vector field on a manifold we need to be able to compare two vectors at two points, since these vectors are in different tangent spaces the usual way of comparing the vectors by taking the difference between them does not make sense. For this we introduce the structure of an affine connection.

Definition 10. *Let $\mathcal{M}$ be a manifold and $\mathcal{C}^\infty(\mathcal{M}, T(M))$ be the space of vector fields on $\mathcal{M}$. Then an affine connection on $\mathcal{M}$ is a bilinear map:*

$$\nabla : \mathcal{C}^\infty(\mathcal{M}, T(M)) \times \mathcal{C}^\infty(\mathcal{M}, T(M)) \to \mathcal{C}^\infty(\mathcal{M}, T(M)) \tag{3.6}$$

$$(u, v) \mapsto \nabla_u v \tag{3.7}$$

Such that for all smooth functions $f \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ and all vector fields X, Y on $\mathcal{M}$ the following holds:

- *∇ is $\mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$-linear in the first variable, i.e. $\nabla_{fX} Y = f \nabla_X Y$.*
- *∇ satisfies the Leibniz rule in the second variable, i.e. $\nabla_X fY = df(X)Y + f\nabla_X Y$*

We can think of this connection $\nabla_u v$ as the change added to v when moving in the direction of u along the manifold. We then define the covariant derivative with respect to the affine connection as follows:

Definition 11. *Let ∇ be a affine connection on the manifold $\mathcal{M}$ then the covariant derivative of v with respect to the affine connection ∇ at a point $p \in \mathcal{M}$ is given by:*

$$\nabla v(p) : \mathcal{C}^\infty(\mathcal{M}, T(M))^* \times \mathcal{C}^\infty(\mathcal{M}, T(M)) \to \mathbb{R} \tag{3.8}$$

$$(\omega, v) \mapsto \langle \omega, \nabla_{u'} v(p) \rangle \tag{3.9}$$

‡ *where u' is a vector field defined on a neighbourhood of p s.t. $u'(p) = u$*

‡Remark that for this to be defined we require a metric on the manifold.

Now given a metric on the manifold there is an unique affine connection namely the Levi-Civita connection defined as follows:

Definition 12. *Let $\mathcal{M}$ be a manifold and g a metric on $\mathcal{M}$, the Levi-Civita connection is the unique affine connection, ∇ such that:*

- *∇ is torsion free. Meaning that for any scalar field f on $\mathcal{M}$, $\nabla\nabla f$ is a field of symmetric bilinear forms.*
- *The covariant derivative of the metric tensor vanishes identically, i.e. $\nabla g = 0$ where $\nabla g = \nabla_\gamma g \quad e_1 \otimes \cdots \otimes e_n \otimes e_\gamma$*

In order to do calculations with a connection we need the connection coefficients on the basis in which we want to do the calculation. For the Levi-Civita connection these can be calculated directly form the metric and the basis as follows:

$$\Gamma^i{}_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial e^l} + \frac{\partial g_{ml}}{\partial e^k} - \frac{\partial g_{kl}}{\partial e^m} \right) \tag{3.10}$$

Where $e_i = \partial_i$ is a local coordinate basis which corresponds to the partial derivatives with respect to the coordinates under the chosen chart. Meaning that around the point of interest, p,with a chart, (U, h, V), around p where $(x_1, \ldots, x_n)$ are coordinates in V this basis corresponds to $\frac{\partial}{\partial x_i}(p)$. The connection coefficients of the Levi-Civita connection, $\Gamma^i{}_{kl}$ are called the Christoffel symbols.[§] These are the unique coefficients such that:

$$\nabla_i e_j = \Gamma^i{}_{kl} e_k \tag{3.11}$$

Example: 3-Sphere

Using the metric for S^3 we found earlier we can apply equation 3.10. This gives us the Christoffel symbols corresponding with S^3 under the stereographic charts we constructed. We use here coordinates (x_1, x_2, x_3) rather then (x, y, z) as before this is for now just notation but we'll use this con-

[§] This convention varies some authors use the term Christoffel symbols for all connection coefficients not exclusively those corresponding with the Levi-Civita connection.

vention later when considering space-time coordinates.

$$\Gamma^1_{\alpha,\beta} = \frac{2}{1+x_1^2+x_2^2+x_3^2}\begin{pmatrix} -x_1 & -x_2 & -x_3 \\ -x_2 & x_1 & 0 \\ -x_3 & 0 & x_1 \end{pmatrix} \tag{3.12}$$

$$\Gamma^2_{\alpha,\beta} = \frac{2}{1+x_1^2+x_2^2+x_3^2}\begin{pmatrix} x_2 & -x_1 & 0 \\ -x_1 & -x_2 & -x_3 \\ 0 & -x_3 & x_2 \end{pmatrix} \tag{3.13}$$

$$\Gamma^3_{\alpha,\beta} = \frac{2}{1+x_1^2+x_2^2+x_3^2}\begin{pmatrix} x_3 & 0 & -x_1 \\ 0 & x_3 & -x_2 \\ -x_1 & -x_2 & -x_3 \end{pmatrix} \tag{3.14}$$

3.2.2 Lie derivative

This leads us to the next object we'll be discussing: the Lie derivative. As we saw in section 3.2.1 in order to define a derivative we need extra structure on the manifold, for the Lie derivative this extra structure is given by a reference vector field. For a given smooth vector field on a manifold we have a flow defined as follows:

Definition 13. *Let X be a smooth vector field on a manifold M* ¶ *then the <u>flow</u> of X is the set of 1-parameter maps, $\gamma_t(p) : M \to M$ such that for all points $p \in M$ it holds that all γ_t are diffeomorphism and $X(p) = \frac{d\gamma_t(p)}{dt}|_{t=0}$.*

With the flow of a vector field we can push a different vector field defined in a neighbourhood of a point to a different point in that neighbourhood. The difference between the second vector field pushed to that point and its value at that point is described by the Lie derivative.

Definition 14. ‖ *Let X, Y be a smooth vector fields on a manifold M and let γ_t be in the flow of X, then <u>the Lie derivative</u> of Y with respect to X, $\mathcal{L}_X Y$, is the vector field defined by:*

$$\mathcal{L}_X Y(p) := \frac{d}{dt}|_{t=0}((\gamma_t)_* Y)(p) \tag{3.15}$$

Here $(\gamma_t)_*$ is the push-forward along γ_t

¶If X is defined only locally, on an open subset $U \subset M$, then the local flow is a set of 1- parameter maps of diffeomorphisms between U and U.

‖The definition given here is a specific case of the more general definition of a Lie derivative for tensors which can be found in [2].

The Lie derivative and Levi-Civita connection turn out to satisfy the following equation, where the Levi-Civita connection is denoted as ∇:

$$\mathcal{L}_{\mathbf{u}} v^{\alpha} = u^{\mu} \nabla_{\mu} v^{\alpha} - v^{\mu} \nabla_{\mu} u^{\alpha} \tag{3.16}$$

The proof of this is relatively easy and can be found in [2].

Chapter 4

General Relativity

In this chapter we'll use the theory we discussed up until now and apply it to general relativity using the 3+1 formalism.

4.1 Foliation

For general relativity we consider space-time which is a 4-dimensional Lorentzian pseudo-Riemann manifold, we'll denote this with $\mathcal{M}$ and the metric with g, in the 3+1 formalism we consider these manifolds as consisting of a foliation. What this means is that we slice the 4-manifold in 3-dimensional slices these slices we take to be hypersurfaces which are defined as follows:

Definition 15. *A hypersurface Σ of a manifold $\mathcal{M}$ is a level set of a smooth function with a nowhere vanishing gradient i.e.*

$$\Sigma_t := \{x \in \mathcal{M} | \hat{t}(x) = t\} \quad \hat{t} \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R}) \text{ with } \nabla \hat{t}(x) \neq 0, \quad \forall x \in \mathcal{M} \tag{4.1}$$

Particularly we will take hypersurfaces so that these are Cauchy surfaces.

Definition 16. *For (X, g) a Lorentzian manifold, a Cauchy surface is an embedded submanifold $\Sigma \hookrightarrow X$ such that every timelike curve, by which we mean the integral curve of a timelike vector field, in X may be extended to a timelike curve that intersects Σ precisely in one point.*

We call a Lorentzian manifold (X, g) which admits Cauchy surfaces, globally hyperbolic.

When constructing space time we can also start with a 3-dimensional Riemann manifold, Σ, g, and add the extra dimension by simply considering the product manifold $\mathbb{R} \times \Sigma$ and reformulating the metric such that when limited to tangent vectors of Σ it is the metric g and when we consider vectors orthogonal to Σ they are timelike. In this manner the choice of a foliation becomes a choice of 3-manifold. We will only concern ourselves with foliations of this type for the rest of this thesis. We will now discuss some simple examples of such foliations.

4.1.1 Example: flat space

When we take as 3-manifold the standard Euclidean $\mathbb{R}^3$, we can define the metric as follows:

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.2}$$

Then since this 3-manifold can be regarded as a level set of the function:

$$f : \mathbb{R}^4 \to \mathbb{R} \text{ defined by, } (x_1, x_2, x_3, x_4) \mapsto x_4 \tag{4.3}$$

Then the the 'new' direction is given by

$$\mathbf{e_0} := \nabla \cdot \mathbf{f}(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4}) = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}) \tag{4.4}$$

So we can form the new space by considering the 2-tuples of the form $(x, \mathbf{x})$, where $x \in \mathbb{R}, \quad \mathbf{x} \in \mathbb{R}^{\mathbf{3}}$ and then expand the metric as

$$G(x, \mathbf{x}) := -x^{\mathbf{2}} + (\mathbf{x})^\top \mathbf{g}\mathbf{x} \tag{4.5}$$

This gives us the familiar Minkowski metric on the spacetime. Which we can represent as is done in figure 4.1.

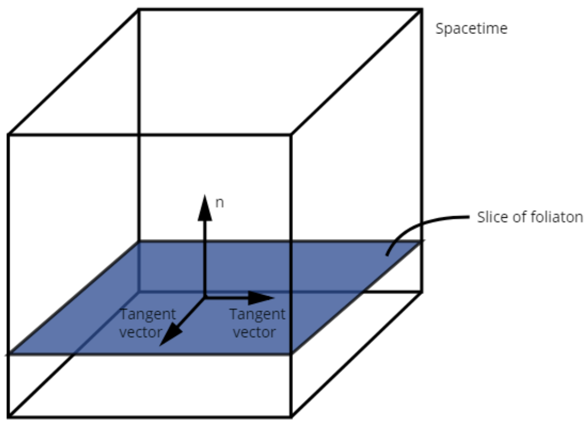

Figure 4.1: *Here we suppress a spacial dimension. The blue slice represents a moment in time for an observer moving along n through the spacetime.*

More interesting is the following example where we use S^3 as the starting point and expand this in to a foliated space-time.

4.1.2 Example: S^3 as foliation

Using the metric we found for S^3, equation 3.5, we can do a similar construction as done above for flat space and use equation 4.5. We find the metric to be the following:

$$g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{4}{1+x_1^2+x_2^2+x_3^2} & 0 & 0 \\ 0 & 0 & \frac{4}{1+x_1^2+x_2^2+x_3^2} & 0 \\ 0 & 0 & 0 & \frac{4}{1+x_1^2+x_2^2+x_3^2} \end{pmatrix} \tag{4.6}$$

Where we use the coordinates (x_0, x_1, x_2, x_3) and stereographic projections to construct the differentiable charts. With this and equation 3.10 we can calculate the corresponding Christoffel symbols which turn out to be the following:

$$\Gamma^0_{\alpha,\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.7}$$

$$\Gamma^1_{\alpha,\beta} = \frac{2}{1 + x_1^2 + x_2^2 + x_3^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -x_1 & -x_2 & -x_3 \\ 0 & -x_2 & x_1 & 0 \\ 0 & -x_3 & 0 & x_1 \end{pmatrix} \tag{4.8}$$

$$\Gamma^2_{\alpha,\beta} = \frac{2}{1 + x_1^2 + x_2^2 + x_3^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & x_2 & -x_1 & 0 \\ 0 & -x_1 & -x_2 & -x_3 \\ 0 & 0 & -x_3 & x_2 \end{pmatrix} \tag{4.9}$$

$$\Gamma^3_{\alpha,\beta} = \frac{2}{1 + x_1^2 + x_2^2 + x_3^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & x_3 & 0 & -x_1 \\ 0 & 0 & x_3 & -x_2 \\ 0 & -x_1 & -x_2 & -x_3 \end{pmatrix} \tag{4.10}$$

4.2 Normal

When we construct a foliation as described above there is a natural choice for the time direction. This is the direction that corresponds with the dual of the gradient 1-form of a smooth function. So if we have t a smooth function of which the level sets, Σ_t, form the slices of the foliation then ∇t forms a 1-form such that:

$$\forall \mathbf{v} \in \Sigma_t, \quad \langle \nabla t, v \rangle = 0 \tag{4.11}$$

The dual of this 1-form is a vector with the components:

$$\nabla^\alpha t = g^{\alpha\mu} \nabla_\mu t \tag{4.12}$$

When this vector is non-Null, (i.e. Spacelike or Timelike, see sect. 8) we can re-scale this vector to a unit vector $\mathbf{n}$, which is done via the lapse function, N.

$$\mathbf{n} := -N \overrightarrow{\nabla} t, \quad N := (\pm \overrightarrow{\nabla} t \cdot \overrightarrow{\nabla} t)^{-\frac{1}{2}} \tag{4.13}$$

Where the sign in the square root is determined by the gradient being either timelike or spacelike with a $+$ respectively a $-$ in these cases. As stated before we will only be interested in cases where $\vec{\nabla} t$ is timelike.

4.3 Normal evolution vector

Having defined the Normal vector we remark that although the vector is normalized it is not adapted to the foliation, meaning that

$$\langle \nabla t, \mathbf{n} \rangle \neq \mathbf{1}. \tag{4.14}$$

This leads us to define a new vector **m** which we'll call the normal evolution vector. We define it as follows:

$$\mathbf{m} := N\mathbf{n} = -N^{\mathbf{2}}\vec{\nabla}\mathbf{t} \tag{4.15}$$

That this vector is adapted to the foliation is easily seen as we find the following:

$$\langle \nabla t, \mathbf{m} \rangle = N\langle \nabla \mathbf{t}, \mathbf{n} \rangle = N^{\mathbf{2}}(-\langle \nabla \mathbf{t}, \vec{\nabla}\mathbf{t} \rangle) = N^{\mathbf{2}}N^{-\mathbf{2}} = \mathbf{1} \tag{4.16}$$

So this means that we have the expression

$$\begin{aligned} t(p') &= t(p + \delta t\mathbf{m}) && (4.17)\\ &= t(p) + \langle \nabla t, \delta t m \rangle && (4.18)\\ &= t(p) + \delta t \langle \nabla t, \mathbf{m} \rangle && (4.19)\\ &= t(p) + \delta t && (4.20) \end{aligned}$$

where p is in the spacetime and p' a point infinitesimally away from p, so here δ is a infinitesimal. This means that the level sets are Lie dragged into one another by **m**. This makes that the Lie derivative with respect to **m** corresponds with a time derivative.

4.4 Einstein's Equation

At this point one may wonder why we develop all this theory, how is it going to help us find and understand Maxwell's equations? As stated at the start of this chapter we will apply this theory to general relativity and thereby we will be able to discuss both electromagnetism, as described by Maxwell's equations, and gravity, as described by Einstein's equation, but for this we will first need to introduce Einstein's equation.

The Einstein equation tells us of the relation between matter* and the curvature of spacetime. Einstein's equation in general is written as follows:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{4.21}$$

Here we are already familiar with the terms $g_{\mu\nu}$ as these are the components of the metric tensor. The $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar, these tells us how volumes change in curvature with respect to Euclidean space. Λ is the famous cosmological constant, which we will take to be 0. Lastly $T_{\mu\nu}$ is the energy-stress tensor, which describes the distribution of matter through out the space. We will use units so that $G = c = 1$, this gives us a simplified version of 4.21 namely:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} \tag{4.22}$$

So if we take the trace of the left hand side of this equation we find.

$$g_{\mu\nu}8\pi T_{\mu\nu} = g^{\mu\nu}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) \tag{4.23}$$

$$g^{\mu\nu}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = R - \frac{1}{2}Rg^{\mu\nu}g_{\mu\nu} \tag{4.24}$$

$$R - 2R = -R \tag{4.25}$$

Thus substituting this in to equation 4.22 we find:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = R_{\mu\nu} + \frac{1}{2}8\pi g^{\mu\nu}T_{\mu\nu}g_{\mu\nu} \tag{4.26}$$

$$8\pi T_{\mu\nu} = R_{\mu\nu} + \frac{1}{2}8\pi T g_{\mu\nu} \tag{4.27}$$

So this leads to the equivalent equation:

$$R_{\mu\nu} = 8\pi(T_{\mu\nu} - Tg_{\mu\nu}) \tag{4.28}$$

In chapter 5 of [2] the 3+1 decomposition of T is given as follows:

$$\boldsymbol{T} = \boldsymbol{S} + \underline{\boldsymbol{n}} \otimes \boldsymbol{p} + \boldsymbol{p} \otimes \underline{\boldsymbol{n}} + E\underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{n}} \tag{4.29}$$

*We say matter but in this therm we include the energy as these two are directly linked in relativity.

Here S is the matter stress tensor , $\boldsymbol{p}$ is the matter momentum density and E the matter energy density al as measured by an Eulerian observer†. We will denote the induced metric on slice of the foliation, Σ_t, due to the metric g by γ. We define the orthogonal projection operation $\overrightarrow{\gamma}^*$ as follows:

Definition 17. *For a $(p,q)-$tensor $\mathcal{T}$ on a manifold $\overrightarrow{\gamma}^*\mathcal{T}$ is defined as follows:*

$$(\overrightarrow{\gamma}^*\mathcal{T})^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q} = \gamma^{\alpha_1}_{\mu_1}\dots\gamma^{\alpha_p}_{\mu_p}\gamma^{\nu_1}_{\beta_1}\dots\gamma^{\nu_q}_{\beta_q}\mathcal{T}^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_q} \tag{4.30}$$

We apply the projection operator $\overrightarrow{\gamma}^*$ to the equation 4.28. Then as is done in chapter 5 of [2] we can write equation 4.22 as follows:

$$\overrightarrow{\gamma}^*R_{\mu\nu} = 8\pi(\overrightarrow{\gamma}^*T_{\mu\nu} - \frac{1}{2}T\overrightarrow{\gamma}^*g_{\mu\nu}) \tag{4.31}$$

Where T is the trace of the energy-stress tensor, which equals the trace of the matter stress tensor, S, minus the matter energy density, E, as measured by an Eulerian observer (i.e. $T = S - E$). When we then define the extrinsic curvature tensor‡, $\boldsymbol{K}$, of the slice, Σ_t, to be the following:

$$\begin{aligned} \boldsymbol{K} : T_p(\Sigma_t) \times T_p(\Sigma_t) &\to \mathbb{R} \\ (u,v) &\mapsto -u\cdot\nabla_v\boldsymbol{n} \end{aligned} \tag{4.32}$$

We can write equation 4.22 as follows:

$$\begin{aligned} \mathcal{L}_{\boldsymbol{m}}\boldsymbol{K}_{\alpha\beta} = -\overrightarrow{\gamma}^*\nabla_\alpha\overrightarrow{\gamma}^*\nabla_\beta N + N(R_{\alpha\beta} + KK_{\alpha\beta} - 2K_{\alpha\mu}K^{\mu}_{\ \beta} \\ + 4\pi((S-E)\gamma_{\alpha\beta} - 2\boldsymbol{S}_{\alpha\beta}))^{\S} \end{aligned} \tag{4.33}$$

Which is the 3+1 decomposed Einstein Equation.
Since electromagnetic fields carry have an energy associated with them they can be related to the energy-stress tensor, this is relation is the following:

$$T^{\alpha\beta} = -\frac{1}{\mu_0}(F^{\alpha\phi}F_{\phi}^{\ \beta} + \frac{1}{4}g^{\alpha\beta}F_{\phi\tau}F^{\phi\tau}) \tag{4.34}$$

And as such we can see that for a given spacetime, with a choice of foliation, we can calculate a energy-stress tensor using equation 4.33 and use the relation given by 4.34 to find the corresponding Faraday tensor, F, and vice versa. Remark that this manner is not unique in the sense that there might be different Faraday tensors that correspond to the same curved

†An Eulerian observer is an observer with the 4-velocity of $\boldsymbol{n}$

‡Some times called the second fundamental form

spacetime and also that this manner of constructing Faraday tensors does not always result in physically relevant results.

Chapter 5

Field line solutions

The Maxwell equations in the 3+1 formalism we already saw in the introduction, eqs. (1.1) to (1.4) , this set of differential equations describes how EM-fields behave in empty* space. These make use of objects we defined in chapters 2 and 3. In [1] local field line solutions to these equations on arbitrary hyperbolic manifolds are proposed. By field line solutions we mean that we give solutions to the Maxwell equations in the form of vector fields where the the integral curves of these fields coincide with the field lines we are familiar with form classical electromagnetism. In [1] it is shown that locally the following fields give a solution to these equations, under certain constraints for $f(t,x),\phi(t,x)$ and $g(t,x),\theta(t,x)$ which will be given in eqs. (5.6) to (5.7):

$$B^i(t,x) = f(t,x)\epsilon^{ijk}(t,x)D^i\phi(t,x)D^i\bar{\phi}(t,x) \tag{5.1}$$

$$E^i(t,x) = \frac{f(t,x)}{N(t,x)}((\mathcal{L}_{\mathbf{m}}\bar{\phi}(t,x))D^i\phi(t,x) - (\mathcal{L}_{\mathbf{m}}\phi(t,x))D^i\bar{\phi}(t,x)) \tag{5.2}$$

Where $f(t,x)$ is a real field defined locally and $\phi(t,x)$ is a locally defined complex scalar field. This gives rise to a dual solution of the following form:

$$\tilde{E}^i(t,x) = g(t,x)\epsilon^{ijk}(t,x)D^i\theta(t,x)D^i\bar{\theta}(t,x) \tag{5.3}$$

$$\tilde{B}^i(t,x) = \frac{g(t,x)}{N(t,x)}((\mathcal{L}_{\mathbf{m}}\bar{\theta}(t,x))D^i\theta(t,x) - (\mathcal{L}_{\mathbf{m}}\theta(t,x))D^i\bar{\theta}(t,x)) \tag{5.4}$$

Where similarly $g(t,x)$ is a real field defined locally and $\theta(t,x)$ is a locally defined complex scalar field. In [1] it is shown that $g(t,x)$ and $f(t,x)$ need

*By this we mean in the absence of any charges to depend implicitly on the coordinates in order to satisfy 1.3 and 1.4 respectively. Meaning that

$$g(t,x) = g(\theta(t,x), \bar{\theta}(t,x)) \text{ and } f(t,x) = f(\phi(t,x), \bar{\phi}(t,x)) \tag{5.5}$$

As stated before these solutions hold under certain constraints for the scalar fields used to construct the solutions. These are equations (18) and (19) form [1] which are the following:

$$\begin{aligned} f(t,x)\epsilon^{ijk}(t,x)D^i\phi(t,x)D^i\bar{\phi}(t,x) &= \frac{g(t,x)}{N(t,x)}((\mathcal{L}_{\mathbf{m}}\bar{\theta}(t,x))D^i\theta(t,x) \\ &\quad - (\mathcal{L}_{\mathbf{m}}\theta(t,x))D^i\bar{\theta}(t,x)) \qquad (5.6) \\ g(t,x)\epsilon^{ijk}(t,x)D^i\theta(t,x)D^i\bar{\theta}(t,x) &= \frac{f(t,x)}{N(t,x)}((\mathcal{L}_{\mathbf{m}}\bar{\phi}(t,x))D^i\phi(t,x) \\ &\quad - (\mathcal{L}_{\mathbf{m}}\phi(t,x))D^i\bar{\phi}(t,x)) \qquad (5.7) \end{aligned}$$

Which we recognize as:

$$B^i(t,x) = \tilde{B}^i(t,x) \tag{5.8}$$

$$E^i(t,x) = \tilde{E}^i(t,x) \tag{5.9}$$

This means that using these two representations we can describe the same fields. Where level curves ϕ are the field lines of the magnetic field, B, similarly the level curves of θ are the field lines of the electric field, E. This allows us for a solution of this form to easily switch between these different field lines and as such this gives an easier way to gain insight in to how such fields behave. As pointed out in [1] this method is the same as is done by Rañada in the case of flat spacetime as in [3] and [5]. This has the benefit that these solutions under the appropriate limit reduce to the flat spacetime solutions, meaning that for a choice of gravitational gauge where $(N = 1, \beta = 0)$ the solutions as presented here are the same as in [3] and [5]. Furthermore in [1] the general form of the Einstein's equations in the context of the 3+1-formalism as is given in [2] is used to calculate the trace of the energy density, the momentum density and the stress tensor corresponding to the fields given earlier. We will not discuss this further in this thesis except for in the context of the 3-torus given in chapter 6 namely section 6.2.

Chapter 6

Spacetimes and solutions of Maxwell's equations

In this chapter we'll take a look at some concrete cases of different dimensional manifolds in order to get a better understanding of the formalism and notation we use. Most of the calculations of the large expressions involved will be done using Wolfram Mathematica 11.0 and the package [6]. The code used for the S^3 example can be found in the appendix, similar code is used for the other examples.

6.1 Example 2-Torus

We'll now discuss an example of a Riemann manifold in $\mathbb{R}^3$ with the metric.

$$\mathbf{g} = \mathrm{Diag}(1,1,1) \tag{6.1}$$

With this we'll consider the hypersurface

$$\Sigma = \{(x,y,z) \in \mathbb{R}^3 | t(x,y,z) = 0\} \tag{6.2}$$

and t given by

$$t(x,y,z) = (\sqrt{(x)^2 + (y)^2} - R)^2 + (z)^2 - r^2 \tag{6.3}$$

for a given $R, r \in \mathbb{R}_{>0}$. We define a new set of coordinates (θ, ϕ) such that

$$x(\theta,\phi) = (R + r\cos\theta)\cos\phi \tag{6.4}$$
$$y(\theta,\phi) = (R + r\cos\theta)\sin\phi \tag{6.5}$$
$$z(\theta,\phi) = r\sin\theta \tag{6.6}$$

Substituting in the function $t(x, y, z)$ gives

$$\begin{aligned} t(x(\theta,\phi), y(\theta,\phi), z(\theta,\phi)) &:= t((R + r\cos\theta)\cos\phi, (R + r\cos\theta)\sin\phi, r\sin\theta) \\ &= 2R(R + r\cos\theta - \sqrt{(R + r\cos\theta)^2}) \end{aligned} \tag{6.7}$$

We then find that in this coordinate system the metric becomes

$$\mathbf{g}(\theta,\phi) = \begin{pmatrix} 1 & 0 \\ 0 & (2+\cos\theta)^2 \end{pmatrix} \tag{6.8}$$

For this case we have that

$$\overrightarrow{\nabla} t(x(\theta,\phi), y(\theta,\phi), z(\theta,\phi)) = \begin{pmatrix} \frac{2(R+r\cos\theta)(-R+\sqrt{(R+r\cos\theta)^2})\cos\phi}{\sqrt{(R+r\cos\theta)^2}} & \frac{2(R+r\cos\theta)(-R+\sqrt{(R+r\cos\theta)^2})\sin\phi}{\sqrt{(R+r\cos\theta)^2}} & 2r\sin\theta \end{pmatrix} \tag{6.9}$$

Which makes that

$$\mathbf{n} := (\langle \overrightarrow{\nabla} t, \overrightarrow{\nabla} t \rangle)^{\frac{-1}{2}} \overrightarrow{\nabla} t \tag{6.10}$$

becomes

$$\mathbf{n} = \frac{1}{2\sqrt{r^2 + 2rR\cos\theta + 2R(R - \sqrt{(R + r\cos\theta)^2}}} \cdot \begin{pmatrix} \frac{2(R+r\cos\theta)(-R+\sqrt{(R+r\cos\theta)^2})\cos\phi}{\sqrt{(R+r\cos\theta)^2}} & \frac{2(R+r\cos\theta)(-R+\sqrt{(R+r\cos\theta)^2})\sin\phi}{\sqrt{(R+r\cos\theta)^2}} & 2r\sin\theta \end{pmatrix} \tag{6.11}$$

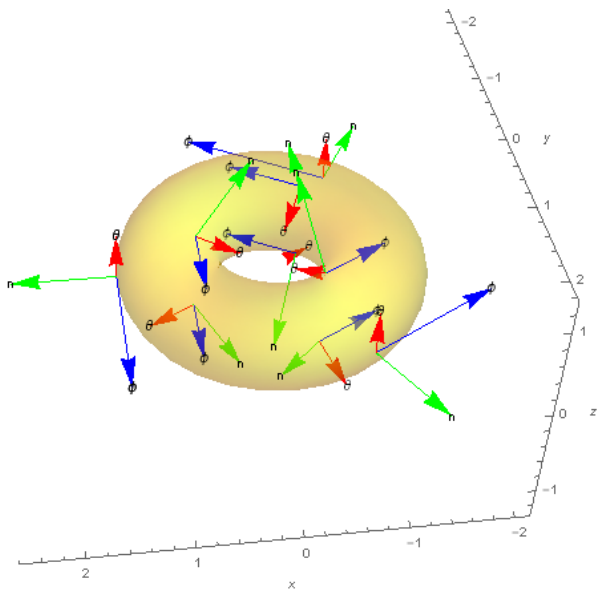

Figure 6.1: *Here we take $R = 1, r = 0.5$. Using the coordinates (θ, ϕ) we construct a frame everywhere on the 2-Torus where the third direction is that of the normal* **n**. *These are represented by giving a basis on the torus, red corresponds with $\boldsymbol{\partial\theta}$, blue with $\boldsymbol{\partial\phi}$ and green with* **n**

6.2 3-Torus

We consider, for a given $r \in \mathbb{R}_{>0}$, the function:

$$t_1(x_1, x_2) = x_1^2 + x_2^2 - r^2 \tag{6.12}$$

The level sets of this function form circles around the origin.
Let $R_i \in \mathbb{R}_{>0}, \forall i \in 1, \dots, n-1$ be given, then we define the following functions:

$$t_i(x_1, \dots, x_{i+1}) = t_{i-1}(\sqrt{x_1^2 + x_{i+1}} - R_{i-1}, x_2, \dots, x_i) \tag{6.13}$$

Then the set

$$\{x = (x_1, \dots, x_{i+1}) \in \mathbb{R}^{i+1} | t_i(x) = 0\} \tag{6.14}$$

forms i-torus in $\mathbb{R}^{i+1}$.
We will now be interested in the 3-manifold given by $t_3(x) = 0$, this will be a 3-torus given as a level set in $\mathbb{R}^4$. So these are the points $(x, y, z, w) \in \mathbb{R}^4$ that satisfy:

$$t_3(x,y,z,w) = (\sqrt{(\sqrt{w^2+x^2} - R_2)^2 + z^2} - R_1)^2 + y^2 - R_0^2 = 0 \quad (6.15)$$

6.2.1 An Atlas

We consider $\mathbb{T}^3 \cong (S^1)^3$. For all $p \in \mathbb{T}^3$ we can identify it with a 3-tuple $(\phi_1, \phi_2, \phi_3) \in (S^1)^3$. We can then, using this identification, embed $\mathbb{T}^3$ in $\mathbb{R}^4$ via the following function:

$$\Phi : \mathbb{T}^3 \to \mathbb{R}^4, (\phi_1, \phi_2, \phi_3) \mapsto (x_1, x_2, x_3, x_4)$$

Where $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ are given by:

$$\begin{aligned}
x_1 &:= ((R_0 \cos\phi_1 + R_1)\cos\phi_2 + R_2)\cos\phi_3 \\
x_2 &:= R_0 \sin\phi_1 \\
x_3 &:= (R_0 \cos\phi_1 + R_1)\sin\phi_2 \\
x_4 &:= ((R_0 \cos\phi_1 + R_1)\cos\phi_2 + R_2)\sin\phi_3
\end{aligned}$$

Now we define two subsets U, U' of $\mathbb{T}^3 \subset \mathbb{R}^4$ using the relations given above, as

$$U := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | \phi_1, \phi_2, \phi_3 \in (-\pi, \pi)\} \cong (S^1 \setminus \{(-1, 0)\})^3$$

$$U' := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | \phi_1, \phi_2, \phi_3 \in (0, 2\pi)\} \cong (S^1 \setminus \{(1, 0)\})^3$$

Remark that these subsets are open under the product topology. We then define $V, V' \subset \mathbb{R}^3$ open, given by $V := (-\pi, \pi)^3, V' := (0, 2\pi)^3$ and the maps:

$$\begin{aligned}
m : U \to V, &\qquad (x_1, x_2, x_3, x_4) \mapsto (\phi_1, \phi_2, \phi_3) \\
m' : U' \to V', &\qquad (x_1, x_2, x_3, x_4) \mapsto (\phi_1, \phi_2, \phi_3)
\end{aligned}$$

We remark that the two glue maps $(m \circ m'^{-1}) : U \cap U' \to (0, \pi)^3$ and $(m' \circ m^{-1}) : U \cap U' \to (0, \pi)^3$ are differentiable, so
$\mathcal{A} := \{(U, m, V), (U', m', V')\}$ is a differentiable atlas for $\mathbb{T}^3$. Now we can use m, m' as coordinate maps for the 3-Torus.

6.2.2 Spacetime

Since we consider $\mathbb{T}^3$ as a subset of $\mathbb{R}^4$, we inherit the Euclidean metric from $\mathbb{R}^4$ which is given by ${}^4\mathbf{g} = \text{Diag}(1,1,1,1)$. Note that this is in Cartesian coordinates. So the metric on $\mathbb{T}^3$ becomes $\boldsymbol{\gamma}(u,v) = {}^4\mathbf{g}(\Phi^* u, \Phi^* v), \forall u, v \in T_{\Phi(p)}(\mathbb{R}^4)$.
Now we have a $\Sigma = \mathbb{T}^3$ such that we can construct a spacetime $(\mathcal{M}, \mathbf{g})$. Where $\mathcal{M} \simeq \mathbb{R} \times \Sigma$ and $\mathbf{g}$ satisfies $\mathbf{g}(u,v) = \gamma(u,v), \forall u,v \in T(\mathbb{T}^3)$ and $\text{sign}(\mathbf{g}) = (-,+,+,+)$. This means that $g_{0i} = -\delta_0^i$ and $g_{\mu\nu} = \gamma_{\mu\nu}$ where $i \in \{0,1,2,3\}, \mu, \nu \in \{1,2,3\}$. So this gives us the following metric on $\mathcal{M}$:

$$g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & R_0^2 & 0 & 0 \\ 0 & 0 & (R_1 + R_0 \cos\phi_1)^2 & 0 \\ 0 & 0 & 0 & (R_2 + (R_1 + R_0\cos\phi_1)\cos\phi_2)^2 \end{pmatrix}$$

Since this metric is not dependent on ϕ_3 we expect there to be a conserved quantity that corresponds to this symmetry. Thus we can calculate the Christoffel symbols, which we find to be the following:

$$\Gamma^0_{\alpha,\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^1_{\alpha,\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{(R_1+R_0\cos\phi_1)\sin\phi_1}{R_0} & 0 \\ 0 & 0 & 0 & \frac{(R_2+(R_1+R_0\cos\phi_1)\cos\phi_2)\sin\phi_1\cos\phi_2}{R_0} \end{pmatrix}$$

$$\Gamma^2_{\alpha,\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{R_0\sin\phi_1}{R_1+R_0\cos\phi_1} & 0 \\ 0 & -\frac{R_0\sin\phi_1}{R_1+R_0\cos\phi_1} & 0 & 0 \\ 0 & 0 & 0 & \frac{(R_2+(R_1+R_0\cos\phi_1)\cos\phi_2)\sin\phi_2}{R_1+R_0\cos\phi_1} \end{pmatrix}$$

$$\Gamma^3_{\alpha,\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{R_0\cos\phi_2\sin\phi_1}{(R_2+(R_1+R_0\cos\phi_1)\cos\phi_2)} \\ 0 & 0 & 0 & -\frac{(R_1+R_0\cos\phi_1)\sin\phi_2}{(R_2+(R_1+R_0\cos\phi_1)\cos\phi_2)} \\ 0 & -\frac{R_0\cos\phi_2\sin\phi_1}{(R_2+(R_1+R_0\cos\phi_1)\cos\phi_2)} & -\frac{(R_1+R_0\cos\phi_1)\sin\phi_2}{(R_2+(R_1+R_0\cos\phi_1)\cos\phi_2)} & 0 \end{pmatrix}$$

We find that using the coordinates $(\phi_0, \phi_1, \phi_2, \phi_3)$ we can plot the T^{00}

element for the entire range of coordinates, ϕ_1, ϕ_2, for fixed $\phi_0 = 0, \phi_3 = 0$. This can be seen in figure 6.2. Where it is interesting to note that in certain area's it is negative or seems to diverge to infinity, since this element correspond with the energy density in Minkowski space this is somewhat unexpected.

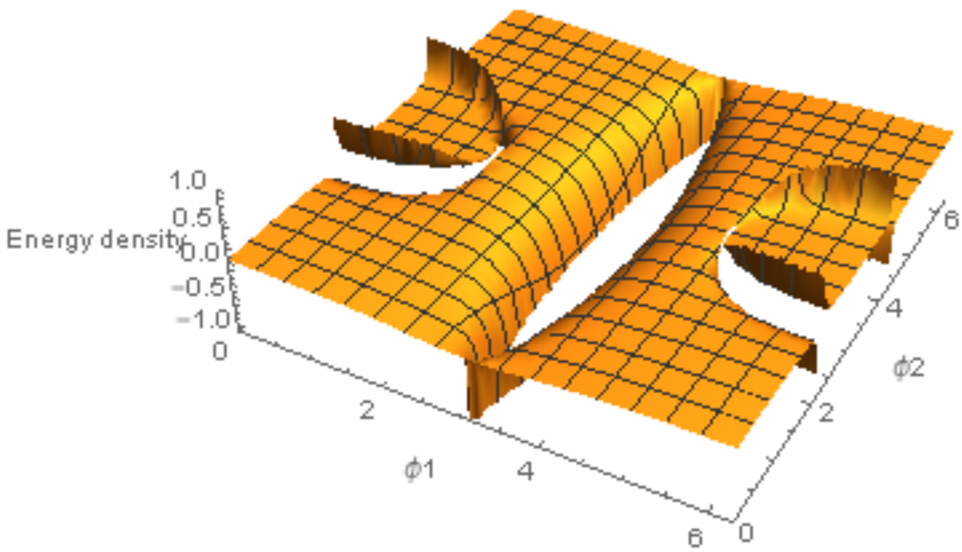

Figure 6.2: *Here we plot using the coordinates ϕ_1, ϕ_2 since this element of the tensor doesn't depend on ϕ_3 the only interesting change happens in these directions. Since the manifold we consider here is a torus this means these coordinates are periodic so we can identify the edges of this plot with the one across.*

6.3 S^3

Another relatively simple 3-manifold which we can use to form a foliation is S^3 which we already discussed in earlier examples. Using this foliation where we use stereographic projections to construct maps, see example 2.1, we have constructed a metric, in example 2.2. Here we will now discuss field line solutions of eqs. (1.1) to (1.4) as proposed by [1] and explained in chapter 5. We consider the fields of the same form as given in section 3.3 of [1] where we take the fields of the form as in eq 5.1 and choose the scalar fields as follows:

$$f(t,x) = \frac{1}{2\pi i(1+|\phi|^2)^2} \tag{6.16}$$

$$g(t,x) = \frac{1}{2\pi i(1+|\theta|^2)^2} \tag{6.17}$$

$$\phi = |\phi|e^{i\Phi} \tag{6.18}$$

$$\theta = |\theta|e^{i\Theta} \tag{6.19}$$

This choice comes from the fact that in Minkowski space these are used to construct the electromagnetic Hopf knots, so here they represent locally the generalization of those solutions to the space. With this choice we can run through the calculations and find that the fields are of the following form:

$$B^i(\phi_0,\phi_1,\phi_2,\phi_3) = \epsilon^{ijk}D_j\alpha_1(\phi_0,\phi_1,\phi_2,\phi_3)D_k\alpha_2(\phi_0,\phi_1,\phi_2,\phi_3) \tag{6.20}$$

$$E^i(\phi_0,\phi_1,\phi_2,\phi_3) = \epsilon^{ijk}D_j\beta_1(\phi_0,\phi_1,\phi_2,\phi_3)D_k\beta_2(\phi_0,\phi_1,\phi_2,\phi_3) \tag{6.21}$$

Where

$$\alpha_1 = \frac{1}{1+|\phi|^2} \qquad \alpha_2 = \frac{\Phi}{2\pi} \tag{6.22}$$

$$\beta_1 = \frac{1}{1+|\theta|^2} \qquad \beta_2 = \frac{\Theta}{2\pi} \tag{6.23}$$

We remark that thus far all the geometry of the foliation is still hidden in the covariant derivatives and thus the form of the equations 6.20 is not specific to this foliation but holds more generally. This results in the fact that we get the following vector fields:

$$B((\phi_0,\phi_1,\phi_2,\phi_3)) = \frac{|\phi|}{\pi(1+|\phi|^2)^2}\begin{pmatrix} 0 \\ (-\partial_3\Phi\partial_2|\phi| + \partial_3|\phi|\partial_2\Phi) \\ (\partial_3\Phi\partial_1|\phi| - \partial_3|\phi|\partial_1\Phi) \\ (-\partial_2\Phi\partial_1|\phi| + \partial_2|\phi|\partial_1\Phi) \end{pmatrix} \tag{6.24}$$

$$= \frac{|\phi|}{\pi(1+|\phi|^2)^2}\begin{pmatrix} 0 \\ \begin{pmatrix}\partial_1\Phi \\ \partial_2\Phi \\ \partial_3\Phi\end{pmatrix} \times \begin{pmatrix}\partial_1|\phi| \\ \partial_2|\phi| \\ \partial_3|\phi|\end{pmatrix} \end{pmatrix} \tag{6.25}$$

$$E((\phi_0,\phi_1,\phi_2,\phi_3)) = \frac{|\theta|}{\pi(1+|\theta|^2)^2}\begin{pmatrix} 0 \\ (-\partial_3\Theta\partial_2|\theta| + \partial_3|\theta|\partial_2\Theta) \\ (\partial_3\Theta\partial_1|\theta| - \partial_3|\theta|\partial_1\Theta) \\ (-\partial_2\Theta\partial_1|\theta| + \partial_2|\theta|\partial_1\Theta) \end{pmatrix} \tag{6.26}$$

$$= \frac{|\theta|}{\pi(1+|\theta|^2)^2}\begin{pmatrix} 0 \\ \begin{pmatrix}\partial_1\Theta \\ \partial_2\Theta \\ \partial_3\Theta\end{pmatrix} \times \begin{pmatrix}\partial_1|\theta| \\ \partial_2|\theta| \\ \partial_3|\theta|\end{pmatrix} \end{pmatrix} \tag{6.27}$$

$$\tag{6.28}$$

These forms are defined locally and as such it is not surprising that the solutions for Minkowski space, which we will see in section 6.4, have the same form since locally the foliation has the structure of flat space. Now the next step would be to find functions $|\phi|, \Phi, |\theta|, \Theta$ such that these satisfy the constraints as given in equations 5.6 and 5.7 where we make the identifications as done in eqs. (6.16) to (6.19). This gives 8 non-linear partial differential equations, two of which turn out to be trivially satisfied. * This leaves us with a system of 6 non-linear partial differential equations for 4 scalar fields so this is a over-determined system and as such does not necessarily have a solution.† In [7] a different approach is taken to finding such a solution, which in [1] is claimed results in the same solution, which gives rise to the suspicion that this system does have a solution.

*These are the ones where $i = 0$ in 5.6 and 5.7 as these simplify to $0 = 0$ for any field.

†Due to the complexity of the equations we where not able to find solutions or even show or exclude their existence.

6.4 Minkowski space

As remarked in section 6.3 the solutions we find there are of the general form. Since for Minkowski space it is possible to define global solutions, as we can map the entire space using a single chart, we can find the global solutions from the form as proposed by [1]. So for Minkowski space we find that the global solutions for the scalar fields as given by eqs. (6.16) to (6.19) are the following:

$$B((\phi_0, \phi_1, \phi_2, \phi_3)) = \frac{|\phi|}{\pi(1+|\phi|^2)^2} \begin{pmatrix} 0 \\ \begin{pmatrix} \partial_1 \Phi \\ \partial_2 \Phi \\ \partial_3 \Phi \end{pmatrix} \times \begin{pmatrix} \partial_1 |\phi| \\ \partial_2 |\phi| \\ \partial_3 |\phi| \end{pmatrix} \end{pmatrix} \tag{6.29}$$

$$E((\phi_0, \phi_1, \phi_2, \phi_3)) = \frac{|\theta|}{\pi(1+|\theta|^2)^2} \begin{pmatrix} 0 \\ \begin{pmatrix} \partial_1 \Theta \\ \partial_2 \Theta \\ \partial_3 \Theta \end{pmatrix} \times \begin{pmatrix} \partial_1 |\theta| \\ \partial_2 |\theta| \\ \partial_3 |\theta| \end{pmatrix} \end{pmatrix} \tag{6.30}$$

$$\tag{6.31}$$

Similarly as in the case for the S^3 discussed in section 6.3 this leaves us with a system of 6 differential equations. For which a solution is proposed in [8] which has the same form as we see here.

Chapter 7

Code

Here by included is an example of a Mathematica notebook with the code used to calculate the examples throughout this thesis. This notebook is the code used to calculate the cases of S^3 using the stereographic coordinates, at the end the code calculates the differential equations that describe the scalar fields as seen in section 6.3. In order to apply this code to other foliations one only needs to change the coordinate functions and mapping functions defined in at the top of the code, here named ”x,y,z, πN3Inverse and πN3.

In[617]:=
```
Needs["ccgrg`"];
```

In[618]:=
```
(*Defining coordinate charts using stereographic projections for the S3,
we assume that the radius is 1 and constant*)
R0 = 1;
myCoords = {ϕ1, ϕ2, ϕ3};
x[ϕ1_, ϕ2_, ϕ3_] := 2 ϕ1 / (ϕ1^2 + ϕ2^2 + ϕ3^2 + 1)
y[ϕ1_, ϕ2_, ϕ3_] := 2 ϕ2 / (ϕ1^2 + ϕ2^2 + ϕ3^2 + 1)
z[ϕ1_, ϕ2_, ϕ3_] := 2 ϕ3 / (ϕ1^2 + ϕ2^2 + ϕ3^2 + 1)
w[ϕ1_, ϕ2_, ϕ3_] := (ϕ1^2 + ϕ2^2 + ϕ3^2 - 1) / (ϕ1^2 + ϕ2^2 + ϕ3^2 + 1)
πN3Inverse[ϕ1_, ϕ2_, ϕ3_] := {x[ϕ1, ϕ2, ϕ3], y[ϕ1, ϕ2, ϕ3], z[ϕ1, ϕ2, ϕ3], w[ϕ1, ϕ2, ϕ3]}
πN3[x_, y_, z_, w_] := {(x) / (1 - w), (y) / (1 - w), (z) / (1 - w)}
```

In[626]:=
```
(*Showing that indeed the funcions are eachothers inverse*)
πN3[πN3Inverse[d, e, f][[1]], πN3Inverse[d, e, f][[2]],
  πN3Inverse[d, e, f][[3]], πN3Inverse[d, e, f][[4]]] // Simplify
Assuming[xtemp^2 + ytemp^2 + ztemp^2 + wtemp^2 == 1 ,
 Simplify[πN3Inverse[πN3[xtemp, ytemp, ztemp, wtemp][[1]],
   πN3[xtemp, ytemp, ztemp, wtemp][[2]], πN3[xtemp, ytemp, ztemp, wtemp][[3]]]]]
(*For remark that -(-1+wtemp+xtemp^2+ytemp^2+ztemp^2)/(-1+wtemp) == wtemp using that we work on S3*)
```

Out[626]= $\{d, e, f\}$

Out[627]= $\left\{\text{xtemp}, \text{ytemp}, \text{ztemp}, -\frac{-1+\text{wtemp}+\text{xtemp}^2+\text{ytemp}^2+\text{ztemp}^2}{-1+\text{wtemp}}\right\}$

In[628]:=
```
(*Calulating the induced metric*)
a := FullSimplify[Transpose[∇{ϕ1,ϕ2,ϕ3} πN3Inverse[ϕ1, ϕ2, ϕ3]].∇{ϕ1,ϕ2,ϕ3} πN3Inverse[ϕ1, ϕ2, ϕ3]]
a // MatrixForm
```

Out[629]//MatrixForm=

$$\begin{pmatrix} \frac{4}{(1+\phi1^2+\phi2^2+\phi3^2)^2} & 0 & 0 \\ 0 & \frac{4}{(1+\phi1^2+\phi2^2+\phi3^2)^2} & 0 \\ 0 & 0 & \frac{4}{(1+\phi1^2+\phi2^2+\phi3^2)^2} \end{pmatrix}$$

In[630]:=
```
(*then g becomes our metric on the spacetime*)
g := {{-1, 0, 0, 0}, {0, a[[1, 1]], a[[1, 2]], a[[1, 3]]},
  {0, a[[2, 1]], a[[2, 2]], a[[2, 3]]}, {0, a[[3, 1]], a[[3, 2]], a[[3, 3]]}}
```

In[631]:=
```
t[ϕ0_, ϕ1_, ϕ2_, ϕ3_] := ϕ0
```

In[632]:=
```
myCoord = {ϕ0, ϕ1, ϕ2, ϕ3}; (* Name coordinates
 (note: myCoord is space-time coordinates while myCoords is coordinates on the S3)*)
(*Open ccgrg with the defined Coordinates and Metric*)
open[myCoord, g]
```

continuation:

Out[633]=

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{4}{(1+\phi1^2+\phi2^2+\phi3^2)^2} & 0 & 0 \\ 0 & 0 & \frac{4}{(1+\phi1^2+\phi2^2+\phi3^2)^2} & 0 \\ 0 & 0 & 0 & \frac{4}{(1+\phi1^2+\phi2^2+\phi3^2)^2} \end{pmatrix}$$

simplification method: Together

In[634]:=
```
(*Adjusting the Lie-Derivative function from ccgrg to work for scalar functions aswell*)
lieD[U_][T_][i___Integer] :=
  If[rank[T] == 0, D[T, {myCoord}].U, LieD[U][T][i]];
```

In[635]:=
```
(*Calculate the Christoffel Symbols*)
MatrixForm[Table[Γ[-1, m, n], {m, 1, 4}, {n, 1, 4}]]
MatrixForm[Table[Γ[-2, m, n], {m, 1, 4}, {n, 1, 4}]]
MatrixForm[Table[Γ[-3, m, n], {m, 1, 4}, {n, 1, 4}]]
MatrixForm[Table[Γ[-4, m, n], {m, 1, 4}, {n, 1, 4}]]
```

Out[635]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[636]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{2\phi1}{1+\phi1^2+\phi2^2+\phi3^2} & -\frac{2\phi2}{1+\phi1^2+\phi2^2+\phi3^2} & -\frac{2\phi3}{1+\phi1^2+\phi2^2+\phi3^2} \\ 0 & -\frac{2\phi2}{1+\phi1^2+\phi2^2+\phi3^2} & \frac{2\phi1}{1+\phi1^2+\phi2^2+\phi3^2} & 0 \\ 0 & -\frac{2\phi3}{1+\phi1^2+\phi2^2+\phi3^2} & 0 & \frac{2\phi1}{1+\phi1^2+\phi2^2+\phi3^2} \end{pmatrix}$$

Out[637]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2\phi2}{1+\phi1^2+\phi2^2+\phi3^2} & -\frac{2\phi1}{1+\phi1^2+\phi2^2+\phi3^2} & 0 \\ 0 & -\frac{2\phi1}{1+\phi1^2+\phi2^2+\phi3^2} & -\frac{2\phi2}{1+\phi1^2+\phi2^2+\phi3^2} & -\frac{2\phi3}{1+\phi1^2+\phi2^2+\phi3^2} \\ 0 & 0 & -\frac{2\phi3}{1+\phi1^2+\phi2^2+\phi3^2} & \frac{2\phi2}{1+\phi1^2+\phi2^2+\phi3^2} \end{pmatrix}$$

Out[638]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2\phi3}{1+\phi1^2+\phi2^2+\phi3^2} & 0 & -\frac{2\phi1}{1+\phi1^2+\phi2^2+\phi3^2} \\ 0 & 0 & \frac{2\phi3}{1+\phi1^2+\phi2^2+\phi3^2} & -\frac{2\phi2}{1+\phi1^2+\phi2^2+\phi3^2} \\ 0 & -\frac{2\phi1}{1+\phi1^2+\phi2^2+\phi3^2} & -\frac{2\phi2}{1+\phi1^2+\phi2^2+\phi3^2} & -\frac{2\phi3}{1+\phi1^2+\phi2^2+\phi3^2} \end{pmatrix}$$

In[639]:=
```
(*Calculate the T tensor form the Einstein Equation
   where tEinsteinG gives the lhs of the eq.: R-(1/2) R*g=8πT*)
MF[Simplify[Table[(1/(8 Pi)) * tEinsteinG[-m, -n], {m, 1, 4}, {n, 1, 4}]]]
```

Out[639]//MatrixForm=

$$\begin{pmatrix} \frac{3}{8\pi} & 0 & 0 & 0 \\ 0 & -\frac{(1+\phi1^2+\phi2^2+\phi3^2)^2}{32\pi} & 0 & 0 \\ 0 & 0 & -\frac{(1+\phi1^2+\phi2^2+\phi3^2)^2}{32\pi} & 0 \\ 0 & 0 & 0 & -\frac{(1+\phi1^2+\phi2^2+\phi3^2)^2}{32\pi} \end{pmatrix}$$

In[640]:=
```
(*Defining the function that gives us the time
 gradient vector and form (and makes it into a ccgrg-tensor)*)
dtcov[i_] := {partialD[t[ϕ0, ϕ1, ϕ2, ϕ3]][1], partialD[t[ϕ0, ϕ1, ϕ2, ϕ3]][2],
   partialD[t[ϕ0, ϕ1, ϕ2, ϕ3]][3], partialD[t[ϕ0, ϕ1, ϕ2, ϕ3]][4]}[[i]]
dt[i_] := tensorExt[dtcov][i]
rank[dt]
tabular[dt][all] // MF
tabular[dt][-all] // MF
```

Out[642]= 1

Out[643]//MatrixForm=

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Out[644]//MatrixForm=

$$\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In[645]:=
```
(*Now we will calculate the B field via the methode given in 3.3 of Vancea,
in our earlier defined space*)
(*Calculating some the normal, n, to spacial slices the lapsefuncion,
LapseN, and the normal evolution vector, m. Notatie: dt = ∇t*)
LapseN := Simplify[(Sum[-g‡[μ, ν] * dt[-μ] * dt[-ν], {μ, 1, 4}, {ν, 1, 4}]) ^ (-1/2)]
LapseN
ncov[i_] := FullSimplify[-LapseN * (dt[i])]
n[i_] := tensorExt[ncov][i]
Table[n[i], {i, 1, dim}] (*covariant n*)
Table[n[-i], {i, 1, dim}] (*contravariant n*)
mcov[i_] := LapseN * n [i];
m[i_] := tensorExt[mcov][i]
Table[m[i], {i, 1, dim}] (*covariant m*)
Table[m[-i], {i, 1, dim}] (*contravariant m*)
```

Out[646]= 1

Out[649]= {-1, 0, 0, 0}

Out[650]= {1, 0, 0, 0}

Out[653]= {-1, 0, 0, 0}

Out[654]= {1, 0, 0, 0}

In[655]:=
```
(*Calculating the contracted LeviCivitaTensor ϵ_{μνσ}=n^{ρ}ϵ_{ρμνσ}*)
ϵ :=
 Table[Sum[n[-ρ] * LeviCivitaTensor[4][[ρ, i, j, k]], {ρ, 1, 4}], {i, 1, 4}, {j, 1, 4}, {k, 1, 4}]
```

```
In[656]:= Simplify[ϵ[[1, All, All]], TimeConstraint → Infinity] // MF
          Simplify[ϵ[[2, All, All]], TimeConstraint → Infinity] // MF
          Simplify[ϵ[[3, All, All]], TimeConstraint → Infinity] // MF
          Simplify[ϵ[[4, All, All]], TimeConstraint → Infinity] // MF
```

Out[656]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[657]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Out[658]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Out[659]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
In[660]:= (*Now we define the fields as given by Ranada and fill these in as is done in Vancea 3.3*)
          (*ψ[ϕ0_,ϕ1_,ϕ2_,ϕ3_]:= (2(Φ[ϕ1,ϕ2,ϕ3][[2]]+ⅈ Φ[ϕ1,ϕ2,ϕ3][[3]]))/
             (2Φ[ϕ1,ϕ2,ϕ3][[4]]+ⅈ(Φ[ϕ1,ϕ2,ϕ3][[2]]^2+Φ[ϕ1,ϕ2,ϕ3][[3]]^2+Φ[ϕ1,ϕ2,ϕ3][[4]]^2-1));*)
          (*Ψ[ϕ0_,ϕ1_,ϕ2_,ϕ3_]:= Abs[ψ[ϕ0,ϕ1,ϕ2,ϕ3]]*Exp[ⅈ ψ[ϕ0,ϕ1,ϕ2,ϕ3]];*)
          α1[ϕ0_, ϕ1_, ϕ2_, ϕ3_] := 1/(1 + γ[ϕ0, ϕ1, ϕ2, ϕ3] * γ̄[ϕ0, ϕ1, ϕ2, ϕ3]);
          α2[ϕ0_, ϕ1_, ϕ2_, ϕ3_] := Q[ϕ0, ϕ1, ϕ2, ϕ3] / (2 * Pi);
          γ[ϕ0_, ϕ1_, ϕ2_, ϕ3_] := q[ϕ0, ϕ1, ϕ2, ϕ3] * Exp[ⅈ * Q[ϕ0, ϕ1, ϕ2, ϕ3]];
          γ̄[ϕ0_, ϕ1_, ϕ2_, ϕ3_] := q[ϕ0, ϕ1, ϕ2, ϕ3] * Exp[-ⅈ * Q[ϕ0, ϕ1, ϕ2, ϕ3]];
          B[ϕ0_, ϕ1_, ϕ2_, ϕ3_] :=
           Table[Sum[ϵ[[i, j, k]] * covariantD[α1[ϕ0, ϕ1, ϕ2, ϕ3]][{j}] * covariantD[α2[ϕ0, ϕ1, ϕ2, ϕ3]][{k}],
              {j, 1, 4}, {k, 1, 4}], {i, 1, 4}]

In[665]:= Block[{$RecursionLimit = Infinity}, Simplify[B[ϕ0, ϕ1, ϕ2, ϕ3], TimeConstraint → Infinity]]
```

Out[665]=

$$\Big\{0,\ \big(q[\phi0,\phi1,\phi2,\phi3]\,\big(-Q^{(0,0,0,1)}[\phi0,\phi1,\phi2,\phi3]\,q^{(0,0,1,0)}[\phi0,\phi1,\phi2,\phi3] + q^{(0,0,0,1)}[\phi0,\phi1,\phi2,\phi3]\,Q^{(0,0,1,0)}[\phi0,\phi1,\phi2,\phi3]\big)\big) \Big/ \big(\pi\,\big(1+q[\phi0,\phi1,\phi2,\phi3]^2\big)^2\big),$$
$$\big(q[\phi0,\phi1,\phi2,\phi3]\,\big(Q^{(0,0,0,1)}[\phi0,\phi1,\phi2,\phi3]\,q^{(0,1,0,0)}[\phi0,\phi1,\phi2,\phi3] - q^{(0,0,0,1)}[\phi0,\phi1,\phi2,\phi3]\,Q^{(0,1,0,0)}[\phi0,\phi1,\phi2,\phi3]\big)\big) \Big/ \big(\pi\,\big(1+q[\phi0,\phi1,\phi2,\phi3]^2\big)^2\big),$$
$$\big(q[\phi0,\phi1,\phi2,\phi3]\,\big(-Q^{(0,0,1,0)}[\phi0,\phi1,\phi2,\phi3]\,q^{(0,1,0,0)}[\phi0,\phi1,\phi2,\phi3] + q^{(0,0,1,0)}[\phi0,\phi1,\phi2,\phi3]\,Q^{(0,1,0,0)}[\phi0,\phi1,\phi2,\phi3]\big)\big) \Big/ \big(\pi\,\big(1+q[\phi0,\phi1,\phi2,\phi3]^2\big)^2\big)\Big\}$$

```
In[666]:= β1[ϕ0_, ϕ1_, ϕ2_, ϕ3_] := 1/(1 + λ[ϕ0, ϕ1, ϕ2, ϕ3] * λ̄[ϕ0, ϕ1, ϕ2, ϕ3]);
          β2[ϕ0_, ϕ1_, ϕ2_, ϕ3_] := P[ϕ0, ϕ1, ϕ2, ϕ3] / (2 * Pi);
          λ[ϕ0_, ϕ1_, ϕ2_, ϕ3_] := p[ϕ0, ϕ1, ϕ2, ϕ3] * Exp[ⅈ * P[ϕ0, ϕ1, ϕ2, ϕ3]];
          λ̄[ϕ0_, ϕ1_, ϕ2_, ϕ3_] := p[ϕ0, ϕ1, ϕ2, ϕ3] * Exp[-ⅈ * P[ϕ0, ϕ1, ϕ2, ϕ3]];
          H[ϕ0_, ϕ1_, ϕ2_, ϕ3_] :=
           Table[Sum[ϵ[[i, j, k]] * covariantD[β1[ϕ0, ϕ1, ϕ2, ϕ3]][{j}] * covariantD[β2[ϕ0, ϕ1, ϕ2, ϕ3]][{k}],
              {j, 1, 4}, {k, 1, 4}], {i, 1, 4}]
```

```
Block[{$RecursionLimit = Infinity}, Simplify[H[ϕ0, ϕ1, ϕ2, ϕ3], TimeConstraint → Infinity]]
```

```
{0, (p[ϕ0, ϕ1, ϕ2, ϕ3] (-P^(0,0,0,1)[ϕ0, ϕ1, ϕ2, ϕ3] p^(0,0,1,0)[ϕ0, ϕ1, ϕ2, ϕ3] +
      p^(0,0,0,1)[ϕ0, ϕ1, ϕ2, ϕ3] P^(0,0,1,0)[ϕ0, ϕ1, ϕ2, ϕ3])) / (π (1 + p[ϕ0, ϕ1, ϕ2, ϕ3]^2)^2),
  (p[ϕ0, ϕ1, ϕ2, ϕ3] (P^(0,0,0,1)[ϕ0, ϕ1, ϕ2, ϕ3] p^(0,1,0,0)[ϕ0, ϕ1, ϕ2, ϕ3] -
      p^(0,0,0,1)[ϕ0, ϕ1, ϕ2, ϕ3] P^(0,1,0,0)[ϕ0, ϕ1, ϕ2, ϕ3])) / (π (1 + p[ϕ0, ϕ1, ϕ2, ϕ3]^2)^2),
  (p[ϕ0, ϕ1, ϕ2, ϕ3] (-P^(0,0,1,0)[ϕ0, ϕ1, ϕ2, ϕ3] p^(0,1,0,0)[ϕ0, ϕ1, ϕ2, ϕ3] +
      p^(0,0,1,0)[ϕ0, ϕ1, ϕ2, ϕ3] P^(0,1,0,0)[ϕ0, ϕ1, ϕ2, ϕ3])) / (π (1 + p[ϕ0, ϕ1, ϕ2, ϕ3]^2)^2)}
```

```
In[671]:= (*Applying the equations (20) and (21) from Vancea*)
f[ϕ0, ϕ1, ϕ2, ϕ3] := (1/(2 Pi ⅈ)) * (1/(1 + (q[ϕ0, ϕ1, ϕ2, ϕ3])^2)^2)
h[ϕ0, ϕ1, ϕ2, ϕ3] := (1/(2 Pi ⅈ)) * (1/(1 + (p[ϕ0, ϕ1, ϕ2, ϕ3])^2)^2)

(*Calculating the constraind equations (18),
(19) using the hopf fields as given in 3.3 van Vancea*)
DiffEquations1 = Table[
   Sum[f[ϕ0, ϕ1, ϕ2, ϕ3] ϵ[[i, j, k]] covariantD[γ[ϕ0, ϕ1, ϕ2, ϕ3]][{j}]
      covariantD[γ̅[ϕ0, ϕ1, ϕ2, ϕ3]][{k}], {j, 1, dim}, {k, 1, dim}]
    == (1/LapseN) h[ϕ0, ϕ1, ϕ2, ϕ3] ((lieD[{m[1], m[2], m[3], m[4]}][λ̅[ϕ0, ϕ1, ϕ2, ϕ3]][])
        covariantD[λ[ϕ0, ϕ1, ϕ2, ϕ3]][-i] - (lieD[{m[1], m[2], m[3], m[4]}][λ[ϕ0, ϕ1, ϕ2, ϕ3]][])
        covariantD[λ̅[ϕ0, ϕ1, ϕ2, ϕ3]][-i]), {i, 1, dim}];
DiffEquations2 = Table[
   Sum[h[ϕ0, ϕ1, ϕ2, ϕ3] ϵ[[i, j, k]] covariantD[λ[ϕ0, ϕ1, ϕ2, ϕ3]][{j}]
      covariantD[λ̅[ϕ0, ϕ1, ϕ2, ϕ3]][{k}], {j, 1, dim}, {k, 1, dim}]
    == (1/LapseN) h[ϕ0, ϕ1, ϕ2, ϕ3] ((lieD[{m[1], m[2], m[3], m[4]}][γ̅[ϕ0, ϕ1, ϕ2, ϕ3]][])
        covariantD[γ[ϕ0, ϕ1, ϕ2, ϕ3]][-i] -
       (lieD[{m[1], m[2], m[3], m[4]}][γ[ϕ0, ϕ1, ϕ2, ϕ3]][]) covariantD[γ̅[ϕ0, ϕ1, ϕ2, ϕ3]][-i]),
   {i, 1, dim}];(*g from vancea is named h since g is the metric*)

DiffEquations = Join[DiffEquations1, DiffEquations2] ;
```

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