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An elementary construction of the real numbers, the p-adic numbers and the rational adele ring

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Contents

Introduction

One can construct the field $\mathbb{R}$ of real numbers directly from the group $\mathbb{Z}$ of integers without using Cauchy sequences or Dedekind cuts. The construction is due to Schanuel and amongst others A'Campo [1] and Arthan [2] describe it. Grundhöfer [3] describes the construction in a more algebraic way. The construction uses certain maps that Arthan calls almost-homomorphisms. A map $f : \mathbb{Z} \to \mathbb{Z}$ is called an *almost-homomorphism* if $|f(x+y) - f(x) - f(y)|$ is bounded. Furthermore, a map with bounded image $f(\mathbb{Z})$ is called an *almost-zero map*. It turns out that the set of all almost-homomorphisms is an abelian group and that the set of all almost-zero maps is a subgroup of it. When considering the quotient group, one finds *quasi-homomorphisms*. The quasi-homomorphisms of $\mathbb{Z}$ to $\mathbb{Z}$, denoted by $\mathrm{QHom}(\mathbb{Z}, \mathbb{Z})$, actually form a ring, which turns out to be isomorphic to the field of real numbers. Grundhöfer shows this by proving that the map

$$\begin{aligned} \mathrm{QHom}(\mathbb{Z}, \mathbb{Z}) &\longrightarrow \mathbb{R}, \\ f &\longmapsto \lim_{n\to\infty} \frac{f(n)}{n} \end{aligned}$$

is a ring isomorphism.

The construction of $\mathbb{R}$ using quasi-homomorphisms is very elementary and only uses relatively simple algebraic properties. Thus the question arises: can we use a similar construction to describe other complex objects in a simpler way? In this thesis, we will give an algebraic generalization of the definitions above. Given arbitrary abelian groups A and B, we call a map $f : A \to B$ an *almost-homomorphism* if the set $\{f(x+y) - f(x) - f(y) : x, y \in A\} \subseteq B$ is finite and we call it an *almost-zero map* if the image $f(A) \subseteq B$ is finite. Again, the set of almost-homomorphisms $\mathrm{AHom}(A, B)$ turns out to be an abelian group and the set of almost-zero maps $\mathrm{Az}(A, B)$ forms a subgroup. By a *quasi-homomorphism* $A \to B$ we mean an element of the quotient group

$$\mathrm{QHom}(A, B) = \mathrm{AHom}(A, B)/\mathrm{Az}(A, B).$$

We will give more details on these definitions in the first chapter of this thesis.

We can compose almost-homomorphisms in the natural way. One should note that although the right-distributive law holds for almost-homomorphisms, the left-distributive law does not. However, we will show that both laws hold for quasi-homomorphisms, which means that the natural composition of quasi-homomorphisms is bilinear. We will prove the following theorem, which is based on this observation:

Theorem. The class of abelian groups together with the class of quasi-homomorphisms between them and the composition law given by

$$\begin{aligned} \mathrm{QHom}(B, C) \times \mathrm{QHom}(A, B) &\to \mathrm{QHom}(A, C), \\ ([g], [f]) &\mapsto [g \circ f] \end{aligned}$$

define an additive category.

We thus find that for every abelian group A the group $\mathrm{QEnd}(A) := \mathrm{QHom}(A, A)$ of *quasi-endomorphisms* is a ring. Moreover, we can use the categorical properties of quasi-endomorphisms to investigate $\mathrm{QEnd}(A)$ in more detail for certain fixed groups A.

In the third chapter of this thesis, we will consider the quasi-endomorphism ring of the field $\mathbb{Q}$ of rational numbers. We will prove the following theorem:

Theorem. The quasi-endomorphism ring of $\mathbb{Q}$ is ring isomorphic to the rational adele ring $\mathbb{A}_{\mathbb{Q}}$.

In the construction used to prove this, we will also find that for any prime p the ring $\mathrm{QEnd}(\mathbb{Z}[1/p]/\mathbb{Z})$ is ring isomorphic to $\mathbb{Q}_p$, the field of p-adic numbers.

One might notice that $\mathbb{R}$ as well as $\mathbb{Q}_p$ and the rational adele ring $\mathbb{A}_{\mathbb{Q}}$ have a natural topology. This leads to the thought that we might be able to define a general topology on $\mathrm{QEnd}(A)$ for each abelian group A. In chapter 4 we give a suitable definition for convergence of sequences of quasi-homomorphisms and we use this to define a topology on $\mathrm{QEnd}(A)$. Moreover, we show that this topology coincides with the topologies on the rings mentioned above.

1 Definitions and categorical properties

We will first generalize the notion of quasi-homomorphisms on $\mathbb{Z}$ to quasi-homomorphisms on arbitrary abelian groups. From now on, we shall write all groups additively. To make a sensible definition of a quasi-homomorphism, we first need some other definitions. Let A and B be two abelian groups.

1.1 Definition. Let $f : A \to B$ be a map. We write

$$N_f := \{f(x+y) - f(x) - f(y) : x, y \in A\} \subseteq B.$$

Note that the set N_f can be seen as a tool to measure how "close" a map is to being a homomorphism. If N_f is finite, it means that the map f behaves like a homomorphism except for finitely many "mistakes". This justifies the following definition of an almost-homomorphism.

1.2 Definition. An *almost-homomorphism* is a map $f : A \to B$ such that the set N_f is finite. We denote the set of almost-homomorphisms from A to B by $\mathrm{AHom}(A, B)$.
An *almost-endomorphism* of A is an almost-homomorphism $A \to A$. The set of almost-endomorphisms on A is denoted by $\mathrm{AEnd}(A)$.

1.3 Example. A homomorphism $f : A \to B$ has the property that $N_f = \{0\}$. It is thus an almost-homomorphism.

1.4 Example. Consider the map $f : \mathbb{R} \to \mathbb{Z}$ given by $x \mapsto \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer $m \in \mathbb{Z}$ such that $m \leq x$. It is easy to verify that $N_f = \{0, 1\}$. We conclude that $f \in \mathrm{AHom}(\mathbb{R}, \mathbb{Z})$.

It is clear that the set $\mathrm{AHom}(A, B)$ is an abelian group under pointwise addition. The identity element is the zero homomorphism, denoted by 0, and the opposite of an almost-homomorphism f is given by $-f$. Using the idea that is used to define an almost-zero map on $\mathbb{Z}$, we can define an almost-zero map $A \to B$ for arbitrary abelian groups.

1.5 Definition. An *almost-zero map* is a map $f : A \to B$ such that the image $f(A)$ is finite. We denote the set of all almost-zero maps from A to B by $\mathrm{Az}(A, B)$.

1.6 Example. Any constant map $f : A \to B$ is an almost-zero map.

The reader should note that an almost-zero map $f : A \to B$ is automatically also an almost-homomorphism. We thus find that $\mathrm{Az}(A, B) \subseteq \mathrm{AHom}(A, B)$. In fact, the set $\mathrm{Az}(A, B)$ is a subgroup of $\mathrm{AHom}(A, B)$.

With the definitions above, we finally have enough information to define a quasi-homomorphism from A to B.

1.7 Definition. A *quasi-homomorphism* $A \to B$ is an element of the quotient group

$$\mathrm{QHom}(A, B) := \mathrm{AHom}(A, B)/\mathrm{Az}(A, B).$$

For $f \in \mathrm{AHom}(A, B)$, we shall denote the coset $f + \mathrm{Az}(A, B)$ by $[f]$. The group $\mathrm{QHom}(A, A)$ of *quasi-endomorphisms* is denoted by $\mathrm{QEnd}(A)$.

By definition, for two elements $[f], [g] \in \mathrm{QHom}(A, B)$ we have $[f] = [g]$ if and only if $[f-g] = [0]$, so if and only if the set $(f-g)(A)$ is finite. When considering quasi-endomorphisms on A, we find that $\mathrm{QEnd}(A)$ is not only an abelian group but also a ring. Its multiplication is given by $[f] \cdot [g] = [f \circ g]$ and the unit element is $[\mathrm{id}_A]$. The reader should be aware that distributivity is a bit tricky here. Although the right-distributive law holds for almost-homomorphisms, the left-distributive law does not. However, for quasi-homomorphisms both laws hold.

1.8 Lemma. Let C be an abelian group and let $f, g \in \mathrm{AHom}(A, B)$ and $h \in \mathrm{AHom}(B, C)$ be three almost-homomorphisms. Then

$$[h(f+g)] = [hf + hg].$$

Proof. Let $a \in A$. Then

$$(h(f+g) - hf - hg)(a) = h(f+g)(a) - hf(a) - hg(a) \in N_h,$$

so the image of the map $h(f+g) - hf - hg$ is contained in a finite set and is thus finite. We conclude that $h(f+g) - (hf + hg) \in \mathrm{Az}(A, C)$, which proves the Lemma. □

Note that this Lemma implies that $\mathrm{QEnd}(A)$ is a ring for any abelian group A.

Throughout this thesis, we will often consider the "sum" and "difference" of multiple sets, for which we will use the following notation.

1.9 Definition. Let X and Y be two subsets of an abelian group. We define the *sum* of X and Y to be

$$X + Y := \{x + y : x \in X, y \in Y\}$$

and we define the *difference* of X and Y to be

$$X - Y := \{x - y : x \in X, y \in Y\}.$$

Of course, these definitions extend to the sum or difference of more than two sets in a natural way.

Now that we have defined quasi-homomorphisms, one can consider them as morphisms between abelian groups and wonder if this gives a category. The following theorem states that the answer to that question is yes.

1.10 Theorem. The class of abelian groups together with the class of quasi-homomorphisms between them and the composition law given by

$$\mathrm{QHom}(B, C) \times \mathrm{QHom}(A, B) \to \mathrm{QHom}(A, C),$$
$$([g], [f]) \mapsto [g \circ f]$$

define a category.

Proof. We should first check that the given composition law is well-defined. For two almost-homomorphisms $f \in \mathrm{AHom}(A, B)$ and $g \in \mathrm{AHom}(B, C)$ the sets N_f and N_g are finite. The following Lemma shows that then N_{gf} is also finite.

1.11 Lemma. $N_{gf} \subseteq g(N_f) + N_g + N_g$.

Proof. Let $z := (gf)(x+y) - (gf)(x) - (gf)(y)$ be an element of N_{gf}. Then we have:

$$\begin{aligned} z &= g(f(x+y)) - g(f(x)) - g(f(y)) \\ &= g(f(x+y)) - g(f(x)+f(y)) + g(f(x)+f(y)) - g(f(x)) - g(f(y)). \end{aligned}$$

Note that $g(f(x)+f(y)) - g(f(x)) - g(f(y)) \in N_g$. Furthermore, we know that $f(x+y) = f(x) + f(y) + n$ for some $n \in N_f$, so we find

$$g(f(x+y)) - g(f(x)+f(y)) = g(f(x)+f(y)+n) - g(f(x)+f(y)),$$

and in the same way as before we find that this equals

$$g(f(x)+f(y)) + g(n) + m - g(f(x)+f(y)) = g(n) + m$$

for some $m \in N_g$. Note that $g(n) \in g(N_f)$. We conclude that

$$z = (gf)(x+y) - (gf)(x) - (gf)(y) \in g(N_f) + N_g + N_g,$$

which proves the Lemma. □

Since N_f is finite, we conclude that $g(N_f)$ is finite as well. It now follows directly from the Lemma that N_{gf} is finite as the sum of three finite sets. We conclude that $gf \in \mathrm{AHom}(A, C)$.

We must also prove that the given composition law is independent of the choice of representatives f and g. To show this, let $f, f' \in \mathrm{AHom}(A, B)$ and $g, g' \in \mathrm{AHom}(B, C)$ be such that $[f] = [f']$ and $[g] = [g']$. Then we have:

$$\begin{aligned} gf - g'f' &= gf - g'f + g'f - g'f' \\ &= (g - g')f + g'f - g'f'. \end{aligned}$$

Since $[g] = [g']$, we know that $(g - g')f \in \mathrm{Az}(A, C)$ and by Lemma 1.8 we know that $g'f - g'f' \in \mathrm{Az}(A, C)$. We conclude that $gf - g'f' \in \mathrm{Az}(A, C)$, thus $[gf] = [g'f']$. We have now proved that the composition is well-defined.

It is clear that the composition of quasi-homomorphisms as defined above is associative. Furthermore, for every object A there is an identity morphism $[\mathrm{id}_A] \in \mathrm{QEnd}(A)$, which is nothing more then the class of the identity map $\mathrm{id} : A \to A$. Indeed, for $[f] \in \mathrm{QHom}(A, B)$ we have

$$[\mathrm{id}_B][f] = [\mathrm{id}_B f] = [f] = [f \mathrm{id}_A] = [f][\mathrm{id}_A].$$

= □

The category discussed above is denoted by **Qab**. Now that we know that **Qab** is a category, we are interested in some categorical properties.

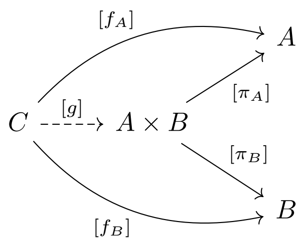
1.1 Products and coproducts

When working with quasi-homomorphisms it is very useful to know what the product and coproduct of any set of objects are. The following Proposition states that every two objects in **Qab** have a product. Moreover, it tells us that this product is the direct product of groups.

1.12 Proposition. The product of two abelian groups $A, B \in \mathrm{Ob}\ \mathbf{Qab}$ is the triple $(A \times B, [\pi_A], [\pi_B])$, where $A \times B$ is the usual direct product of A and B and $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ are the natural projection maps given by $\pi_A(a, b) = a$ and $\pi_B(a, b) = b$.

Proof. The group $A \times B$ is abelian, so it is indeed an object in **Qab**. Note that π_A and π_B are homomorphisms, so they are also almost-homomorphisms. This shows that $[\pi_A] \in \mathrm{QHom}(A \times B, A)$ and $[\pi_B] \in \mathrm{QHom}(A \times B, B)$.

We need to show the universal property. Let $(C, [f_A], [f_B])$ be any triple with C an abelian group, $[f_A] \in \mathrm{QHom}(C, A)$ and $[f_B] \in \mathrm{QHom}(C, B)$. We must show that there exists a unique $[g] \in \mathrm{QHom}(C, A \times B)$ such that the following diagram commutes:



Choose representatives f_A and f_B of $[f_A]$ respectively $[f_B]$. Consider the map $g : C \to A \times B$ given by $c \mapsto (f_A(c), f_B(c))$. Then g is an almost-homomorphism:

$$\begin{aligned} N_g &= \{g(x + y) - g(x) - g(y) : x, y \in C\} \\ &= \{(f_A(x + y), f_B(x + y)) - (f_A(x), f_B(x)) - (f_A(y), f_B(y)) : x, y \in C\} \\ &= \{(f_A(x + y) - f_A(x) - f_A(y), f_B(x + y) - f_B(x) - f_B(y)) : x, y \in C\}, \end{aligned}$$

so N_g is contained in the set $N_{f_A} \times N_{f_B}$. We know that N_{f_A} and N_{f_B} are finite, so it follows that N_g is as well. We conclude that $[g]$ is a quasi-homomorphism. It is very clear that $[\pi_B g] = [f_B]$ and that $[\pi_A g] = [f_A]$.

We will now show that the quasi-homomorphism $[g]$ does not depend on the choice of representatives f_A and f_B. Suppose that $\tilde{f}_A$ and $\tilde{f}_B$ are two other representatives of $[f_A]$ respectively $[f_B]$ and define $\tilde{g}$ by $c \mapsto (\tilde{f}_A(c), \tilde{f}_B(c))$. Then we have

$$(g - \tilde{g})(C) \subseteq (f_A - \tilde{f}_A)(A) \times (f_B - \tilde{f}_B)(B).$$

Since $(f_A - \tilde{f}_A)(A)$ and $(f_B - \tilde{f}_B)(B)$ are both finite we conclude that $(g - \tilde{g})(C)$ is too, so indeed we find $[g] = [\tilde{g}]$.

It is left to show that $[g]$ is unique. Suppose that there is another quasi-homomorphism $[h] \in \mathrm{QHom}(C, A \times B)$ such that $[\pi_A h] = [f_A]$ and $[\pi_B h] = [f_B]$. Then we have $[\pi_A h] = [\pi_A g]$ and $[\pi_B h] = [\pi_B g]$, so $\pi_A h - \pi_A g \in \mathrm{Az}(C, A)$ and $\pi_B h - \pi_B g \in \mathrm{Az}(C, B)$. With Lemma 1.8 we now find

$$\pi_A(h - g) \in \mathrm{Az}(C, A) \qquad \text{and} \qquad \pi_B(h - g) \in \mathrm{Az}(C, B).$$

We conclude that the set

$$(h - g)(C) \subseteq \pi_A(h - g)(C) \times \pi_B(h - g)(C)$$

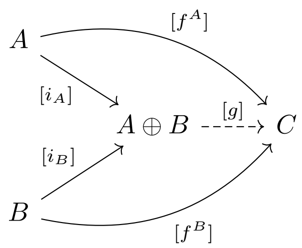
must be finite, or in other words that $[h] = [g]$. $\square$

We have a similar result for the coproduct. The Proposition below states that the coproduct of any two abelian groups always exists and that it equals their direct sum.

1.13 Proposition. The coproduct of two abelian groups $A, B \in \text{Ob } \mathbf{Qab}$ is the triple $(A \oplus B, [i_A], [i_B])$, where $A \oplus B$ is the usual direct sum of A and B and $i_A : A \to A \oplus B$ and $i_B : B \to A \oplus B$ are the natural inclusion maps given by $i_A(a) = (a, 0)$ and $i_B(b) = (0, b)$.

Proof. For two abelian groups A and B, the direct sum $A \oplus B$ equals the direct product $A \times B$. We have noted before that is indeed an abelian group. Note that i_A and i_B are homomorphisms, so they are also almost-homomorphisms, so $[i_A] \in \text{QHom}(A, A \oplus B)$ and $[i_B] \in \text{QHom}(B, A \oplus B)$.

We now need to show the universal property. Let $(C, \left[f^A\right], \left[f^B\right])$ be any triple where $C \in \text{Ob } \mathbf{Qab}$, $\left[f^A\right] \in \text{QHom}(A, C)$ and $\left[f^B\right] \in \text{QHom}(B, C)$. We must show that there exists a unique $[g] \in \text{QHom}(A \oplus B, C)$ such that the following diagram commutes:



Choose representatives f^A and f^B of $[f^A]$ respectively $[f^B]$. Consider the map $g : A \oplus B \to C$ given by $(a, b) \mapsto f^A(a) + f^B(b)$. Then g is an almost-homomorphism:

$$\begin{aligned} N_g &= \{g(a + a', b + b') - g(a, b) - g(a', b') : (a, b), (a', b') \in A \oplus B\} \\ &\subseteq N_{f^A} + N_{f^B}. \end{aligned}$$

We know that both N_{f^A} and N_{f^B} are finite, so their sum is, so N_g is as well. We thus find that $[g] \in \text{QHom}(A \oplus B, C)$. It is clear that $[gi_B] = \left[f^B\right]$ and $[gi_A] = \left[f^A\right]$.

We will show that the quasi-homomorphism $[g]$ is independent of the choice of representatives f^A and f^B. Suppose that $\tilde{f}^A$ and $\tilde{f}^B$ are two other representatives of $[f^A]$ respectively $[f^B]$ and define $\tilde{g}$ by $(a, b) \mapsto \tilde{f}^A(a) + \tilde{f}^B(b)$. Then we have

$$(g - \tilde{g})(A \oplus B) \subseteq (f^A - \tilde{f}^A)(A) + (f^B - \tilde{f}^B)(B),$$

so $(g - \tilde{g})(A \oplus B)$ is finite. We conclude that $[g] = [\tilde{g}]$.

It is left to show that $[g]$ is unique. Suppose that there is another quasi-homomorphism $[h] \in \text{QHom}(A \oplus B, C)$ such that $[hi_A] = [f^A]$ and $[hi_B] = [f^B]$. Then $[hi_A] = [gi_A]$ and $[hi_B] = [gi_B]$, so we find that

$$(g - h)i_A \in \text{Az}(A, C) \qquad \text{and} \qquad (g - h)i_B \in \text{Az}(B, C).$$

Moreover, for an element $(a, b) \in A \oplus B$ we have the following:

$$\begin{aligned}(h-g)(a,b) &= h(a,b) - f^A(a) - f^B(b) \\ &= h(a,0) + h(0,b) + m - f^A(a) - f^B(b)\end{aligned}$$

for some $m \in N_h$. Also, note that $gi_A(a) = g(a, 0) = f^A(a) + f^B(0)$ and $gi_B(b) = g(0, b) = f^A(0) + f^B(b)$. We thus find that

$$\begin{aligned}(h-g)(a,b) &= h(a,0) - f^A(a) + h(0,b) - f^B(b) + m \\ &= h(a,0) - f^A(a) - f^B(0) + h(0,b) - f^B(b) - f^A(0) + m \\ &\quad + f^B(0) + f^A(0) \\ &= ((h-g)i_A)(a) + ((h-g)i_B)(b) + m + f^A(0) + f^B(0).\end{aligned}$$

Now since the sets $((h-g)i_A)(A)$, $((h-g)i_B)(B)$ and N_h are finite, we conclude that $(h-g)(A \oplus B)$ must be finite too, so $[h] = [g]$. □

1.2 Additivity

There are some more observations on the category **Qab** to be made. First of all, one should note that any two finite groups $A, B \in \text{Ob}$ **Qab** are isomorphic in **Qab**. This can be easily seen: since A and B are finite, we find that both $\text{QHom}(A, B)$ and $\text{QHom}(B, A)$ consist of only the zero quasi-homomorphism. It is clear that this quasi-homomorphism is a **Qab**-isomorphism $A \to B$. In light of this observation, it is also clear that any finite group is a zero object of **Qab**.

Recall that a category is *preadditive* if the set of morphisms $\text{Hom}(A, B)$ is an abelian group for all objects A and B and the composition of morphisms is bilinear. An *additive category* is a preadditive category with a zero object and binary biproducts. The following result now comes very naturally:

1.14 Proposition. Qab is an additive category.

Proof. We already know that $\text{QHom}(A, B)$ is an abelian group for all abelian groups A and B, that finite products in **Qab** exist and that there is a zero object. It follows from Lemma 1.8 that the composition of quasi-homomorphisms is bilinear. □

The following trivial Lemma states a more general result about additive categories, which will later be very useful.

1.15 Lemma. Let A and B be two objects in an additive category $\mathcal{C}$. If

$$\text{Hom}(A, B) = \text{Hom}(B, A) = 0,$$

then the map

$$\begin{aligned}\text{End}(A) \times \text{End}(B) &\to \text{End}(A \oplus B) \\ (f, g) &\mapsto f \oplus g\end{aligned}$$

is a ring isomorphism.

□

1.3 The splitting criterion

One of the main goals of this thesis is to consider the ring $\mathrm{QEnd}(G)$ in more detail for abelian groups G. To this purpose, we can consider a subgroup $H \subseteq G$ and the corresponding quotient group G/H. If G is isomorphic to $(G/H) \times H$ in **Qab**, we can simplify the task of finding $\mathrm{QEnd}(G)$ by first finding $\mathrm{QEnd}(H)$ and $\mathrm{QEnd}(G/H)$. In Theorem 1.18 we will formulate a criterion for when $(G/H) \times H$ and G are isomorphic, which we will call the *splitting criterion*. We will first prove the following two Lemmas.

1.16 Lemma. Let A and B be two abelian groups and let $f : A \to B$ be an almost-homomorphism. Then the set $M_f := \{f(a) + f(-a) : a \in A\}$ is finite.

Proof. Let $a \in A$ and consider $f(0) - f(a) - f(-a) \in N_f$. Since

$$f(a) + f(-a) = f(0) - (f(0) - f(a) - f(-a)),$$

we find that $M_f \subseteq \{f(0)\} - N_f$, which is of course finite. □

1.17 Lemma. Let $A, B \in \mathrm{Ob}\ \mathbf{Qab}$ and let $f : A \to B$ be a bijective almost-homomorphism. If $g : B \to A$ is the inverse map of f, then g is also an almost-homomorphism.

Proof. Let $x, y \in B$ and consider $g(x+y) - g(x) - g(y)$. We observe the following:

$$f(g(x+y) - g(x) - g(y)) = f(g(x+y)) + f(-g(x) - g(y)) + n_1 \tag{1}$$

for some $n_1 \in N_f$. We know that $f(g(x+y)) = x + y$. Let us now consider the part $f(-g(x) - g(y))$. We have:

$$f(-g(x) - g(y)) = f(-g(x)) + f(-g(y)) + n_2$$

for some $n_2 \in N_f$. Also, since $M_f = \{f(a) + f(-a) : a \in A\}$ is finite (Lemma 1.16), we find that

$$\begin{aligned} f(-g(x)) + f(-g(y)) + n_2 &= -f(g(x)) - f(g(y)) + n_2 + m_1 + m_2 \\ &= -x - y + n_2 + m_1 + m_2 \end{aligned}$$

for some $m_1, m_2 \in M_f$. With equation 1 we now find that

$$\begin{aligned} f(g(x+y) - g(x) - g(y)) &= x + y + n_1 - x - y + n_2 + m_1 + m_2 \\ &= n_1 + n_2 + m_1 + m_2. \end{aligned}$$

Thus $f(N_g) \subseteq N_f + N_f + M_f + M_f$ and this is finite. Since f is injective, we conclude that N_g must be finite. □

1.18 Splitting Criterion. Let $G \in \mathrm{Ob}\ \mathbf{Qab}$ and let $H \subset G$ be a subgroup. Let $S \subset G$ be a set of representatives of the quotient group G/H and write s_g for the unique representative of $g \in G$. Consider the map

$$\begin{aligned} \phi : (G/H) \times H &\to G \\ (g + H, h) &\mapsto s_g + h. \end{aligned}$$

Then $[\phi]$ is a **Qab**-isomorphism if and only if the set $(S - S - S) \cap H$ is finite.

Proof. The map ϕ is injective: suppose $\phi(g+H,h) = \phi(g'+H,h')$. Then $s_g + h \in g + H$ and $s_{g'} + h' = s_g + h \in g' + H$, so g and g' are in the same class. We conclude that $s_g = s_{g'}$ and from $s_g + h = s_{g'} + h'$ we then conclude that $h = h'$.

The map ϕ is also surjective: let $g \in G$. Then $g - s_g \in H$, so we have

$$(g+H, g-s_g) \in (G/H) \times H$$

and this element maps precisely to g, since:

$$\phi(g+H, g-s_g) = s_g + g - s_g = g.$$

We conclude that ϕ is a bijection. Note that by Lemma 1.17 we know that if ϕ is an almost-homomorphism, then ϕ has an inverse that is also an almost-homomorphism. The statement that $[\phi]$ is an isomorphism is now equivalent with the statement that ϕ is an almost-homomorphism. Observe that ϕ is an almost-homomorphism if and only if N_ϕ is finite. We claim that $N_\phi = (S - S - S) \cap H$. To see this, let $(g+H,h)$ and $(g'+H,h')$ be two elements of $(G/H) \times H$. Then we have:

$$\begin{aligned}\phi(g+g'+H,h+h')-&\phi(g+H,h) - \phi(g'+H,h')\\ &= s_{g+g'} + h + h' - s_g - h - s_{g'} - h'\\ &= s_{g+g'} - s_g - s_{g'} \in S - S - S.\end{aligned}$$

There are certain $h_{g+g'}, h_g, h_{g'} \in H$ such that $s_{g+g'} = g + g' + h_{g+g'}$, $s_g = g + h_g$ and $s_{g'} = g' + h_{g'}$. We now find that

$$\begin{aligned}s_{g+g'} - s_g - s_{g'} &= g + g' + h_{g+g'} - g - h_g - g' - h_{g'}\\ &= h_{g+g'} - h_g - h_{g'} \in H.\end{aligned}$$

We thus conclude that $N_\phi \subseteq (S - S - S) \cap H$.

For the other inclusion, let $s_a - s_b - s_c \in (S - S - S) \cap H$, where $a, b, c \in G$. Again, there are certain $h_a, h_b, h_c \in H$ such that $s_a = a + h_a$, $s_b = b + h_b$ and $s_c = c + h_c$. Since $s_a - s_b - s_c \in H$, we find that $a + h_a - b - h_b - c - h_c \in H$. We conclude that $a - b - c \in H$ and thus that $a + H = b + c + H$, or in other words that $s_a = s_{b+c}$. Now consider the element

$$(b+c+H, h_b + h_c) \in (G/H) \times H.$$

Then

$$\begin{aligned}\phi((b+c+H,h_b+h_c)) - \phi((b+H,h_b)) &- \phi((c+H,h_c))\\ &= s_{b+c} + h_b + h_c - s_b - h_b - s_c - h_c\\ &= s_{b+c} - s_b - s_c\\ &= s_a - s_b - s_c.\end{aligned}$$

We conclude that $s_a - s_b - s_c \in N_\phi$, so $N_\phi \supseteq (S - S - S) \cap H$. Thus the earlier claim holds.

Since ϕ is an almost-homomorphism precisely when N_ϕ is finite, we conclude that $[\phi]$ is an isomorphism in **Qab** if and only if $(S - S - S) \cap H$ is finite. □

1.19 Corollary. The groups $(\mathbb{R}/\mathbb{Z}) \times \mathbb{Z}$ and $\mathbb{R}$ are isomorphic in **Qab**.

Proof. Apply the Splitting Criterion with $G = \mathbb{R}$, $H = \mathbb{Z}$ and S is the interval $[0, 1)$. Indeed, this is a set of representatives where every $x \in \mathbb{R}$ is represented exactly once. Furthermore, we find that $S - S - S = (-2, 1)$. We conclude that

$$(S - S - S) \cap H = (-2, 1) \cap \mathbb{Z} = \{-1, 0\}$$

and this set is most certainly finite. We have thus found an isomorphism $[\phi_{(\mathbb{R},\mathbb{Z})}]$. $\square$

1.20 Corollary. $\mathbb{Q}$ is isomorphic to $(\mathbb{Q}/\mathbb{Z}) \times \mathbb{Z}$ in **Qab**.

Proof. This follows directly from Corollary 1.19, after restricting the almost-homomorphism $\phi_{(\mathbb{R},\mathbb{Z})}$ to $\mathbb{Q}$. One can of course also obtain this result by directly applying the Splitting Criterion, using $G = \mathbb{Q}$, $H = \mathbb{Z}$ and $S = [0, 1) \cap \mathbb{Q}$. $\square$

2 The quasi-endomorphisms of $\mathbb{Q}$

We know that for any abelian group A the set of quasi-endomorphisms $\mathrm{QEnd}(A)$ is a ring. Furthermore, we know that $\mathrm{QEnd}(\mathbb{Z})$ is ring isomorphic to $\mathbb{R}$ via the isomorphism [3]

$$\begin{aligned} \mathrm{QEnd}(\mathbb{Z}) &\to \mathbb{R}, \\ f &\mapsto \lim_{n\to\infty} \frac{f(n)}{n}. \end{aligned}$$

This leads to the following question: what does the ring $\mathrm{QEnd}(\mathbb{Q})$ look like? Corollary 1.20 states that $\mathbb{Q} \cong \mathbb{Q}/\mathbb{Z} \times \mathbb{Z}$ in **Qab**. We are thus interested in finding $\mathrm{QEnd}(\mathbb{Q}/\mathbb{Z})$. To do this, we will first consider the quasi-endomorphism ring of a simpler group.

2.1 The ring QEnd($\mathbb{Z}[1/p]/\mathbb{Z}$)

It is known that $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}[1/p]/\mathbb{Z}$.[4] It is thus useful to consider the ring $\mathrm{QEnd}(\mathbb{Z}[1/p]/\mathbb{Z})$ before we look into $\mathrm{QEnd}(\mathbb{Q}/\mathbb{Z})$. In this chapter we will use the ring $\mathbb{Z}_p$ of p-adic integers and its fraction field $\mathbb{Q}_p$ of p-adic numbers. For anyone unfamiliar with these two objects, their definitions can be found in Caruso's *Computations with p-adic numbers*.[5]

2.1 Proposition. Let p be a fixed prime. We have

$$\mathrm{Az}(\mathbb{Z}[1/p]/\mathbb{Z}, \mathbb{Z}[1/p]/\mathbb{Z}) \cap \mathrm{End}(\mathbb{Z}[1/p]/\mathbb{Z}) = 0.$$

Proof. Let $f \in \mathrm{End}(\mathbb{Z}[1/p]/\mathbb{Z})$ such that the set $f(\mathbb{Z}[1/p]/\mathbb{Z})$ is finite. Then there is some integer n such that $p^n f(\mathbb{Z}[1/p]/\mathbb{Z}) = \{0\}$. Let $\lambda_{p^n} \in \mathrm{End}(\mathbb{Z}[1/p]/\mathbb{Z})$ be the map given by $x \mapsto p^n x$. Then $\lambda_{p^n} f = 0$. Let $x \in \mathbb{Z}[1/p]/\mathbb{Z}$ be given. There exists some $y \in \mathbb{Z}[1/p]/\mathbb{Z}$ such that $x = p^n y$, so we have

$$f(x) = f(p^n y) = p^n f(y) = (\lambda_{p^n} f)(y) = 0.$$

We conclude that $f = 0$. $\square$

Note that Proposition 2.1 is equivalent with stating that the ring homomorphism

$$\begin{aligned} j : \mathrm{End}(\mathbb{Z}[1/p]/\mathbb{Z}) &\longrightarrow \mathrm{QEnd}(\mathbb{Z}[1/p]/\mathbb{Z}), \\ f &\longmapsto [f]. \end{aligned}$$

is injective.

When given a quasi-homomorphism $[f] \in \mathrm{QEnd}(\mathbb{Z}[1/p]/\mathbb{Z})$ we know that the set $N_f \subseteq \mathbb{Z}[1/p]/\mathbb{Z}$ is finite for all representatives $f \in \mathrm{AEnd}(\mathbb{Z}[1/p]/\mathbb{Z})$. This implies that there exists some integer $m \in \mathbb{Z}$ such that $p^m N_f = \{0\}$. By composing f with multiplication by p^m we can construct an endomorphism. This is shown in more detail in the proof of Proposition 2.4. Intuitively, we can thus find all quasi-endomorphisms by extending $\mathrm{End}(\mathbb{Z}[1/p]/\mathbb{Z})$ with some map that is "dividing by p". We will introduce some notation. Let $x \in \mathbb{Z}[1/p]/\mathbb{Z}$, then x is of the form $x = \frac{a}{p^k}$ for some $a \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$. We write r_{a,p^k} for the remainder of division of a by p^k. Then $0 \leq r_{a,p^k} \leq p^k - 1$ and r_{a,p^k} is unique.

2.2 Lemma. Let p be a prime number. Then there is a unique map $\lambda_{1/p} \in \mathrm{AEnd}(\mathbb{Z}[1/p]/\mathbb{Z})$ such that for all $a \in \mathbb{Z}$ and for all $k \in \mathbb{Z}_{\geq 0}$ we have

$$\lambda_{1/p}\left(\frac{a}{p^k}\right) = \frac{r_{a,p^k}}{p^{k+1}}.$$

Proof. First of all, we must show that the map $\lambda_{1/p}$ as above is well-defined. For $k \in \mathbb{Z}_{\geq 1}$ and $a \in \mathbb{Z}_{\neq 0}$ it is clear that $\lambda_{1/p}\left(\frac{a}{p^k}\right)$ is well-defined. We will now show that $\lambda_{1/p}(0)$ is well-defined too: for any element $\frac{ap^k}{p^k}$ with $a \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0}$ we have

$$\lambda_{1/p}(a) = \lambda_{1/p}\left(\frac{a \cdot p^k}{p^k}\right) = \frac{r_{ap^k,p^k}}{p^{k+1}} = \frac{0}{p^{k+1}} = 0.$$

We conclude that $\lambda_{1/p}(0) = 0$ in $\mathbb{Z}[1/p]/\mathbb{Z}$, so "dividing by p" as defined above is a well-defined map.

We will now show that the map $\lambda_{1/p}$ is an almost-homomorphism. Let $x, y \in \mathbb{Z}[1/p]/\mathbb{Z}$. Without loss of generality we can assume that x and y are of the form $x = \frac{a}{p^k}$, $y = \frac{b}{p^k}$ for some $a, b \in \{0, \ldots, p^k - 1\}$ and $k \in \mathbb{N}$. We have

$$\lambda_{1/p}\left(\frac{a}{p^k} + \frac{b}{p^k}\right) - \lambda_{1/p}\left(\frac{a}{p^k}\right) - \lambda_{1/p}\left(\frac{b}{p^k}\right) = \frac{r_{a+b,p^k}}{p^{k+1}} - \frac{a}{p^{k+1}} - \frac{b}{p^{k+1}}. \quad (2)$$

Since $a, b \leq p^k - 1$, we find that $a + b \leq 2p^k - 2 < 2p^k$. Equation 2 thus has two possible outcomes:

$$\frac{r_{a+b,p^k}}{p^{k+1}} - \frac{a}{p^{k+1}} - \frac{b}{p^{k+1}} = \begin{cases} 0, & \text{if } a + b < p^k, \\ -\frac{1}{p}, & \text{else.} \end{cases}$$

We conclude that

$$N_{\lambda_{1/p}} = \left\{-\frac{1}{p}, 0\right\}.$$

□

Write λ_p for the endomorphism on $\mathbb{Z}[1/p]/\mathbb{Z}$ that is given by $x \mapsto px$. We will confirm that the quasi-endomorphism $\left[\lambda_{1/p}\right]$ is indeed the inverse of $[\lambda_p]$.

2.3 Lemma. Let p be a prime number. Then

$$\left[\lambda_{1/p}\lambda_p\right] = [\mathrm{id}] \quad \text{and} \quad \left[\lambda_p\lambda_{1/p}\right] = [\mathrm{id}].$$

Proof. Let $x \in \mathbb{Z}[1/p]/\mathbb{Z}$. Without loss of generality we may assume that x is of the form $x = \frac{a}{p^k}$ with $k \in \mathbb{N}$, $a \in \{0, \dots, p^k - 1\}$. It is clear that $\lambda_p\lambda_{1/p}(x) = x$, so $\lambda_p\lambda_{1/p} = \mathrm{id}$.

For the other composition we will use the one above. Since $\lambda_p\lambda_{1/p} = \mathrm{id}$, we have $\lambda_p\lambda_{1/p}\lambda_p = \lambda_p$, so

$$\lambda_p\lambda_{1/p}\lambda_p - \lambda_p = 0.$$

Since

$$[\lambda_p\lambda_{1/p}\lambda_p - \lambda_p] = [\lambda_p(\lambda_{1/p}\lambda_p - 1)],$$

we now find that $\lambda_p(\lambda_{1/p}\lambda_p - 1)(\mathbb{Z}[1/p]/\mathbb{Z})$ is finite, which implies that the set $(\lambda_{1/p}\lambda_p - 1)(\mathbb{Z}[1/p]/\mathbb{Z})$ is finite too. □

Recall that $\mathrm{End}(\mathbb{Z}[1/p]/\mathbb{Z}) \cong \mathbb{Z}_p$. Write h for the canonical isomorphism $h : \mathrm{End}(\mathbb{Z}[1/p]/\mathbb{Z}) \to \mathbb{Z}_p$. Then

$$jh : \mathbb{Z}_p \hookrightarrow \mathrm{QEnd}(\mathbb{Z}[1/p]/\mathbb{Z})$$

is an injective ring homomorphism. We will now construct a ring isomorphism $\mathbb{Q}_p \to \mathrm{QEnd}(\mathbb{Z}[1/p]/\mathbb{Z})$. It is known that $\mathbb{Q}_p \cong \mathbb{Z}_p[1/p]$. Consider the unique ring homomorphism $\gamma : \mathbb{Z}_p[1/p] \to \mathrm{QHom}(\mathbb{Z}[1/p]/\mathbb{Z})$ that is given by

$$\gamma|_{\mathbb{Z}_p} = jh \qquad \text{and} \qquad \gamma\left(\frac{1}{p}\right) = \left[\lambda_{1/p}\right].$$

2.4 Proposition. Let p be a prime number. The map $\gamma : \mathbb{Q}_p \to \mathrm{QEnd}(\mathbb{Z}[1/p]/\mathbb{Z})$ as given above is a ring isomorphism.

Proof. We must show that γ is bijective. Note that the kernel of γ must be an ideal of $\mathbb{Q}_p$. Since $\mathbb{Q}_p$ is a field, we find that either $\ker\gamma = \mathbb{Q}_p$ or $\ker\gamma = (0)$. Since $\gamma|_{\mathbb{Z}_p} = jh \neq 0$, we find that $\ker\gamma \neq \mathbb{Q}_p$. We conclude that the kernel of γ must be trivial and thus that γ is injective.

It is left to show that γ is surjective. Let $[f] \in \mathrm{QEnd}(\mathbb{Z}[1/p]/\mathbb{Z})$ be given and let $f \in \mathrm{AEnd}(\mathbb{Z}[1/p]/\mathbb{Z})$ be a representative of $[f]$. The set $N_f \subseteq \mathbb{Z}[1/p]/\mathbb{Z}$ is finite, so $N_f = \{\frac{n_1}{p^{k_1}}, \dots, \frac{n_N}{p^{k_N}}\}$ for some integers $N \in \mathbb{N}$, $n_1, \dots, n_N \in \mathbb{Z}$ and $k_1, \dots, k_N \in \mathbb{N}$. Choose $m := \max\{k_1, \dots, k_N\}$. Since

$$p^m \frac{n_i}{p^{k_i}} = n_i p^{m-k_i} \in \mathbb{Z}$$

for all $i \in \{1, \dots, N\}$, we now find that $N_{f^*} = \{0\}$. We conclude that $f^* := \lambda_{p^m} f$ is an endomorphism on $\mathbb{Z}[1/p]/\mathbb{Z}$. Now since $f^* \in \mathrm{End}(\mathbb{Z}[1/p]/\mathbb{Z})$, we conclude that $f = \lambda_{1/p^m} f^* \in \mathrm{End}(\mathbb{Z}[1/p]/\mathbb{Z})[\lambda_{1/p}]$. Since

$$\mathrm{End}(\mathbb{Z}[1/p]/\mathbb{Z})[\lambda_{1/p}] \cong \mathbb{Z}_p[1/p] \cong \mathbb{Q}_p$$

we conclude that γ is surjective. □

2.2 The ring QEnd($\mathbb{Q}/\mathbb{Z}$)

For abelian groups $\mathbb{A}$ that are of the form $\mathbb{A} = \bigoplus_{i \in I} A_i$ for some infinite index set I and abelian groups A_i, there is a relation between $\mathrm{QEnd}(\mathbb{A})$ and $\mathrm{QEnd}(A_i)$. This relation uses the concept of restricted products. For the reader that is less familiar with restricted products, we will first give a definition of a restricted product.

2.5 Definition. Let I be an index set. Given a collection of rings R_i and a collection of subrings $S_i \subseteq R_i$, we define the *restricted product over* $i \in I$ *of* R_i *with respect to* $(S_i)_{i \in I}$, denoted by $\prod'_{i \in I} R_i$ *w.r.t.* $(S_i)_{i \in I}$, to be the ring

$$\left\{ (x_i)_{i \in I} \in \prod_{i \in I} R_i : \ x_i \in S_i \text{ for all but finitely many } i \in I \right\}.$$

Whenever it is clear from the context which subrings S_i we mean, we will just denote the restricted product with $\prod'_{i \in I} R_i$ for the purpose of readability.

In the situation that $\mathbb{A} = \bigoplus_{i \in I} A_i$, we want to use the quasi-endomorphisms of each A_i to construct a quasi-endomorphism $[f] \in \mathrm{QEnd}(\mathbb{A})$.

2.6 Proposition. If $\mathrm{Az}(A_i, A_i) \cap \mathrm{End}(A_i) = 0$ for all $i \in I$, then there is a natural ring homomorphism

$$\begin{aligned} \omega : \prod\nolimits'_{i \in I} \mathrm{QEnd}(A_i) \ \textit{w.r.t} \ (\mathrm{End}(A_i))_{i \in I} &\longrightarrow \mathrm{QEnd}(\mathbb{A}), \\ ([f_i])_{i \in I} &\longmapsto [f] := [\oplus_{i \in I} f_i], \end{aligned}$$

where by f_i we mean some chosen representative of the quasi-homomorphism $[f_i]$.

Proof. Once it is verified that the map given above is well-defined, it can actually easily be seen that ω is a ring homomorphism. We will thus only show that ω is well-defined. Note that N_{f_i} is finite for all $i \in I$. We have

$$N_f \subseteq \prod_{i \in I} N_{f_i}.$$

Since $f_i \in \mathrm{End}(A_i)$ for all but finitely many $i \in I$, we find that $N_{f_i} = \{0\}$ for almost all $i \in I$. Let J be the finite index set such that for all $j \in J$ we have $f_j \notin \mathrm{End}(A_j)$ and for all $i \in I \backslash J$ we have $f_i \in \mathrm{End}(A_i)$. We conclude that

$$N_f \subseteq \prod_{j \in J} N_{f_j} \times \prod_{i \in I \backslash J} \{0\},$$

so N_f is finite. It follows that $f \in \mathrm{AHom}(\mathbb{A})$.

Now we will show that the image of an element $([f_i])_{i \in I}$ under ω is independent of the choice of representatives. To this end, let $([f_i])_{i \in I} \in \prod'_{i \in I} \mathrm{QEnd}(A_i)$ and let $(f_i)_{i \in I}$ and $(g_i)_{i \in I}$ be two sets of representatives and define

$$f := [\oplus_{i \in I} f_i] \qquad \text{and} \qquad g := [\oplus_{i \in I} g_i].$$

We will show that $[f] = [g]$. Note that

$$(f - g)(\mathbb{A}) \subseteq \prod_{i \in I} (f_i - g_i)(A_i).$$

Since f_i is an endomorphism for almost all $i \in I$ and g_j is an endomorphism for almost all $j \in I$, we find that $f_i - g_i$ is an endomorphism for almost all $i \in I$. Since $[f_i] = [g_i]$ for all $i \in I$, we know that $(f_i - g_i)(A_i)$ is finite for all $i \in I$. The assumption that $\mathrm{Az}(A_i, A_i) \cap \mathrm{End}(A_i) = 0$ for all $i \in I$ now implies that $f_i - g_i = 0$ for all but finitely many $i \in I$. We conclude that $(f - g)(\mathbb{A})$ must be finite, so $[f] = [g]$, which proves that ω is well-defined. □

We can see Proposition 2.6 as a recipe to "cook up" a quasi-endomorphism $\mathbb{A} \to \mathbb{A}$. We will use this to understand $\mathrm{QEnd}(\mathbb{Q}/\mathbb{Z})$. Write $\mathcal{P}$ for the set of all primes. Recall that $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p\in\mathcal{P}} \mathbb{Z}[1/p]/\mathbb{Z}$. Note that in Proposition 2.1 we have already seen that for any prime p we have

$$\mathrm{Az}(\mathbb{Z}[1/p]/\mathbb{Z}, \mathbb{Z}[1/p]/\mathbb{Z}) \cap \mathrm{End}(\mathbb{Z}[1/p]/\mathbb{Z}) = 0.$$

2.7 Theorem. The ring homomorphism

$$\omega : {\prod}'_{p\in\mathcal{P}} \mathrm{QEnd}(\mathbb{Z}[1/p]/\mathbb{Z}) \ \textit{w.r.t.}\ (\mathrm{End}(\mathbb{Z}[1/p]/\mathbb{Z}))_{p\in\mathcal{P}} \longrightarrow \mathrm{QEnd}(\mathbb{Q}/\mathbb{Z}),$$
$$([f_p])_{p\in\mathcal{P}} \longmapsto [f] := [\oplus_{p\in\mathcal{P}} f_p]$$

is an isomorphism.

Proof. We will prove this Theorem by giving an inverse map of ω. Note that for each $p \in \mathcal{P}$ there is a natural embedding

$$i_p : \mathbb{Z}[1/p]/\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z} = \bigoplus_{p\in\mathcal{P}} \mathbb{Z}[1/p]/\mathbb{Z}$$

which just sends an element $x \in \mathbb{Z}[1/p]/\mathbb{Z}$ to the element in $\bigoplus_{p\in\mathcal{P}} \mathbb{Z}[1/p]/\mathbb{Z}$ with zeros on all coordinates and x on the coordinate that coincides with $\mathbb{Z}[1/p]/\mathbb{Z}$. There is also a natural projection

$$\pi_p : \mathbb{Q}/\mathbb{Z} \twoheadrightarrow \mathbb{Z}[1/p]/\mathbb{Z}$$

that sends an element in $\bigoplus_{p\in\mathcal{P}} \mathbb{Z}[1/p]/\mathbb{Z}$ to the coordinate corresponding to $\mathbb{Z}[1/p]/\mathbb{Z}$. Given a quasi-homomorphism $[f] \in \mathrm{QEnd}(\mathbb{Q}/\mathbb{Z})$, we can choose any representative f of $[f]$ and compose it with i_p and π_p. We then get a map

$$\pi_p f i_p : \mathbb{Z}[1/p]/\mathbb{Z} \xrightarrow{i_p} \mathbb{Q}/\mathbb{Z} \xrightarrow{f} \mathbb{Q}/\mathbb{Z} \xrightarrow{\pi_p} \mathbb{Z}[1/p]/\mathbb{Z},$$

that we from now on will denote by f_p. We will now define the map

$$\pi : \mathrm{QEnd}(\mathbb{Q}/\mathbb{Z}) \longrightarrow {\prod}'_{p \text{ prime}} \mathrm{QEnd}(\mathbb{Z}[1/p]/\mathbb{Z})$$

by

$$\pi([f]) = ([f_p])_{p\in\mathcal{P}},$$

where $[f_p] \in \mathrm{QEnd}(\mathbb{Z}[1/p]/\mathbb{Z})$ is the class of $f_p = \pi_p f i_p$ for some chosen representative f of $[f]$.

2.8 Lemma. The map

$$\pi : \mathrm{QEnd}(\mathbb{Q}/\mathbb{Z}) \longrightarrow {\prod}'_{p \text{ prime}} \mathrm{QEnd}(\mathbb{Z}[1/p]/\mathbb{Z})$$

as given above is well-defined.

Proof. We need to show that $f_p \in \text{AEnd}(\mathbb{Z}[1/p]/\mathbb{Z})$ for all $p \in \mathcal{P}$, that f_p is an endomorphism for all but finitely many $p \in \mathcal{P}$ and that the image of $[f]$ under π is independent of the chosen representative f.

To show that $f_p \in \text{QEnd}(\mathbb{Z}[1/p]/\mathbb{Z})$ for all p, let $p \in \mathcal{P}$ be given and consider N_{f_p}. Since $f \in \text{AEnd}(\mathbb{Q}/\mathbb{Z})$, we know that N_f is finite. We can find an injection $N_{f_p} \hookrightarrow N_f$ by sending an element $f_p(x+y) - f_p(x) - f_p(y)$, $x, y \in \mathbb{Z}[1/p]/\mathbb{Z}$, to $f(i_p(x+y)) - f(i_p(x)) - f(i_p(y)) \in N_f$. We now find that N_{f_p} has at most as many elements as N_f, so N_{f_p} is finite.

We are again going to use the fact that N_f is finite to show that π actually maps into the restricted product above, or in other words: that f_p is an endomorphism for all but finitely many p. Since $N_f \subseteq \mathbb{Q}/\mathbb{Z}$ is finite, there exists an integer $n \in \mathbb{N}$ such that $n \cdot N_f = \{0\}$. Let $\lambda_n f$ be the map given by $x \mapsto nf(x)$. Then $\lambda_n f$ is an endomorphism. It is known that

$$\text{Hom}(\mathbb{Z}[1/p]/\mathbb{Z}, \mathbb{Z}[1/p']/\mathbb{Z}) = 0$$

for any two primes $p \neq p'$, so we conclude that $\lambda_n f$ can be written as $\oplus_{p \in \mathcal{P}} \phi_p$, where $\phi_p : \mathbb{Z}[1/p]/\mathbb{Z} \to \mathbb{Z}[1/p]/\mathbb{Z}$ is an endomorphism. Now let $p \in \mathcal{P}$ be given and consider $(\lambda_n f)_p = \pi_p(\lambda_n f) i_p$. We find:

$$\begin{aligned} \pi_p(\lambda_n f) i_p &= \pi_p(\oplus_{p \in \mathcal{P}} \phi_p) i_p \\ &= \phi_p \\ &= \lambda_n(\pi_p f i_p) \\ &= \lambda_n f_p, \end{aligned}$$

so we conclude that $\lambda_n f_p \in \text{End}(\mathbb{Z}[1/p]/\mathbb{Z})$. The endomorphism λ_n is a unit in $\text{End}(\mathbb{Z}[1/p]/\mathbb{Z})$ whenever $p \nmid n$. So for all primes p with $p \nmid n$, we find that $f_p \in \text{End}(\mathbb{Z}[1/p]/\mathbb{Z})$. Since every integer has only finitely many prime divisors, we conclude that π indeed maps in the restricted product over p of $\text{QEnd}(\mathbb{Z}[1/p]/\mathbb{Z})$.

To show that π is independent of the choice of the representative, let $[f] \in \text{QEnd}(\mathbb{Q}/\mathbb{Z})$ be given and let $f, g \in \text{AEnd}(\mathbb{Q}/\mathbb{Z})$ be such that $f \neq g$ and $[f] = [g]$. For all $p \in \mathcal{P}$, the following holds:

$$\begin{aligned} f_p - g_p &= \pi_p f i_p - \pi_p g i_p \\ &= \pi_p(f - g) i_p, \end{aligned}$$

so we find that

$$\begin{aligned} (f_p - g_p)(\mathbb{Z}[1/p]/\mathbb{Z}) &= (\pi_p(f - g) i_p)(\mathbb{Z}[1/p]/\mathbb{Z}) \\ &= (\pi_p(f - g))(i_p(\mathbb{Z}[1/p]/\mathbb{Z})) \subseteq \pi_p(f - g)(\mathbb{Q}/\mathbb{Z}). \end{aligned}$$

Now since $[f] = [g]$, we know that $(f - g)(\mathbb{Q}/\mathbb{Z})$ is finite. We conclude that $(f_p - g_p)(\mathbb{Z}[1/p]/\mathbb{Z})$ must be finite too. This implies that $[f_p] = [g_p]$ for all $p \in \mathcal{P}$, so π is indeed independent of the choice of the representative.
We conclude that the map π is well-defined, which proves our Lemma. □

To complete the proof of 2.7, it is left to show that π is the inverse map of ω. It is easy to verify that π is the left inverse of ω: let $([f_p])_{p \in \mathcal{P}} \in \prod'_{p \text{ prime}} \text{QEnd}(\mathbb{Z}[1/p]/\mathbb{Z})$. We have

$$(\pi\omega)\left(([f_p])_{p \in \mathcal{P}}\right) = \pi\left([\oplus_{p \in \mathcal{P}} f_p]\right),$$

and since

$$\pi_p(\oplus_{p\in\mathcal{P}} f_p) i_p = f_p$$

for all p, we have

$$\pi\left([\oplus_{p\in\mathcal{P}} f_p]\right) = ([f_p])_{p\in\mathcal{P}}.$$

To show that π is also the right inverse of ω, let $[f] \in \mathrm{QEnd}(\mathbb{Q}/\mathbb{Z})$. Choose a representative $f \in \mathrm{AEnd}(\mathbb{Q}/\mathbb{Z})$. Note that there is an $n \in \mathbb{Z}$ such that $nN_f = \{0\}$. Let λ_n be multiplication with n, then $\lambda_n f \in \mathrm{End}(\mathbb{Q}/\mathbb{Z})$. Since $\lambda_n f$ is an endomorphism, it can be written as the sum of endomorphisms

$$\phi_p : \mathbb{Z}[1/p]/\mathbb{Z} \to \mathbb{Z}[1/p]/\mathbb{Z},$$

so we get $\lambda_n f = \oplus_{p\in\mathcal{P}} \phi_p$. Of course we have

$$(\lambda_n f)_p = \pi_p(\lambda_n f) i_p = \phi_p$$

for all $p \in \mathcal{P}$. Also, since λ_n operates coordinate-wise, it is clear that

$$\pi_p(\lambda_n f) i_p = \lambda_n(\pi_p f i_p) = \lambda_n f_p.$$

We may thus conclude that

$$\begin{aligned}\lambda_n f &= \oplus_{p\in\mathcal{P}}\ \phi_p \\ &= \oplus_{p\in\mathcal{P}}\ (\lambda_n f)_p \\ &= \oplus_{p\in\mathcal{P}}\ \lambda_n f_p \\ &= \lambda_n \oplus_{p\in\mathcal{P}}\ f_p.\end{aligned}$$

Note that

$$\omega\pi([f]) = [\oplus_{p\in\mathcal{P}} f_p] =: [f'],$$

so to show that ω is the left inverse of π we need to show that $[f] = [f']$. From our findings above, we get

$$\begin{aligned}n \cdot f(\mathbb{Q}/\mathbb{Z}) &= (\lambda_n f)(\mathbb{Q}/\mathbb{Z}) \\ &= \left(\lambda_n \oplus_{p\in\mathcal{P}} f_p\right)(\mathbb{Q}/\mathbb{Z}) \\ &= n \cdot f'(\mathbb{Q}/\mathbb{Z}),\end{aligned}$$

so

$$n \cdot (f(\mathbb{Q}/\mathbb{Z}) - f'(\mathbb{Q}/\mathbb{Z})) = \{0\},$$

or in other words

$$n \cdot (f - f')(\mathbb{Q}/\mathbb{Z}) = \{0\}.$$

Since the set $(f - f')(\mathbb{Q}/\mathbb{Z})$ is annihilated by some integer n, we conclude that it must be finite. This proves that $[f] = [f']$, which concludes the proof that $\pi = \omega^{-1}$.

We find that ω is an isomorphism, which proves the theorem. □

Using the facts that $\mathrm{QEnd}(\mathbb{Z}[1/p]/\mathbb{Z}) \cong \mathbb{Q}_p$ and $\mathrm{End}(\mathbb{Z}[1/p]/\mathbb{Z}) = \mathbb{Z}_p$ we can now conclude that $\mathrm{QEnd}(\mathbb{Q}/\mathbb{Z})$ is isomorphic to *the finite adele ring of the rational numbers*:

$$\mathrm{QEnd}(\mathbb{Q}/\mathbb{Z}) \cong \prod\nolimits'_{p \text{ prime}} \mathbb{Q}_p \ w.r.t.\ (\mathbb{Z}_p)_{p\in\mathcal{P}} =: \mathbb{A}_{\mathbb{Q}}^{\mathrm{fin}}.$$

2.3 Rational adele ring

Write $\mathbb{A}_{\mathbb{Q}} := \mathbb{A}_{\mathbb{Q}}^{\mathrm{fin}} \times \mathbb{R}$ for the rational adele ring. We will show that QEnd($\mathbb{Q}$) is ring isomorphic to $\mathbb{A}_{\mathbb{Q}}$. We first need the two following propositions.

2.9 Proposition. $\mathrm{QHom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) = \{0\}$.

Proof. Suppose $[f] \in \mathrm{QHom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z})$. We want to show that $[f] = [0]$, or in other words that the image $f(\mathbb{Q}/\mathbb{Z})$ is finite. We know that $N_f \subseteq \mathbb{Z}$ is finite. This implies that there is some constant $B \in \mathbb{N}$ such that $|m| \leq B$ for all $m \in N_f$.
Assume that there exists an element $y \in \mathbb{Q}/\mathbb{Z}$ such that $f(y) > B$. Let $x \in \mathbb{Q}/\mathbb{Z}$. Then there is some $m_1 \in N_f$ such that

$$f(x + y) = f(x) + f(y) + m_1.$$

Using the bound on the elements of N_f, we find that

$$f(x + y) > f(x) + f(y) - B.$$

By assumption we have $y > B$, so we find

$$f(x + y) > f(x).$$

Likewise, we find that there is some element $m_2 \in N_f$ such that

$$f(x + 2y) = f(x + y) + f(y) + m_2,$$

and we find

$$\begin{aligned} f(x + 2y) &> f(x + y) + f(y) - B \\ f(x + 2y) &> f(x + y). \end{aligned}$$

Continuing in this way, we find an infinite sequence of integers

$$\cdots > f(x + ny) > f(x + (n-1)y) > \cdots > f(x).$$

However, since $y \in \mathbb{Q}/\mathbb{Z}$ we know that there exists a $k \in \mathbb{Z}_{\geq 1}$ such that $ky = 0$. For this k, we find

$$f(x) = f(x + ky) > f(x),$$

which is a contradiction. We conclude that there does not exist a $y \in \mathbb{Q}/\mathbb{Z}$ such that $f(y) > B$.
By replacing the assumption above that there exists a $y \in \mathbb{Q}/\mathbb{Z}$ such that $f(y) > B$ with the assumption that there exists a y with $f(y) < -B$, we can use a similar argument to the one above to find an infinite sequence

$$\cdots < f(x + ny) < f(x + (n-1)y) < \cdots < f(x).$$

Again, considering that y has finite order, we find a contradiction. We conclude that there do not exist any $y \in \mathbb{Q}/\mathbb{Z}$ such that $f(y) < -B$. This proves that $f(\mathbb{Q}/\mathbb{Z}) \subseteq \mathbb{Z}$ is bounded, hence $[f] = [0]$. □

2.10 Proposition. $\mathrm{QHom}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \{0\}$.

Proof. First of all, note that $\text{Hom}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$. Let $[f] \in \text{QHom}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$. We know that N_f is finite, so there is some $n \in \mathbb{Z}$ such that $nN_f = \{0\}$. This implies that $\lambda_n f \in \text{Hom}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$, so $(\lambda_n f)(1)$ corresponds with some unique element $x \in \mathbb{Q}/\mathbb{Z}$. We know that x has finite order. This implies that the image $(\lambda_n f)(\mathbb{Z})$ is finite. Since $f(\mathbb{Z}) \subseteq \frac{1}{n}(\lambda_n f)(\mathbb{Z})$, we conclude that the image $f(\mathbb{Z})$ of f is finite and thus that $[f] = [0]$. □

2.11 Theorem. The quasi-endomorphism ring of $\mathbb{Q}$ is isomorphic to the rational adele ring $\mathbb{A}_\mathbb{Q}$.

Proof. Using Lemma 1.15 and the result of Theorem 2.7, we find that

$$\begin{aligned} \text{QEnd}(\mathbb{Q}) &\cong \text{QEnd}(\mathbb{Q}/\mathbb{Z}) \times \text{QEnd}(\mathbb{Z}) \\ &\cong \mathbb{A}_\mathbb{Q}^{\text{fin}} \times \mathbb{R} \\ &= \mathbb{A}_\mathbb{Q}. \end{aligned}$$

□

3 Convergence of quasi-endomorphisms

Up until now we know the following quasi-endomorphism rings:

- $\text{QEnd}(\mathbb{Z}) \cong \mathbb{R}$, via the isomorphism $f \mapsto \lim_{n\to\infty} \frac{f(n)}{n}$ with inverse map $x \mapsto (f : n \mapsto \lfloor nx \rfloor)$,
- $\text{QEnd}(\mathbb{Z}[1/p]/\mathbb{Z}) \cong \mathbb{Q}_p$,
- $\text{QEnd}(\mathbb{Q}/\mathbb{Z}) \cong \mathbb{A}_\mathbb{Q}^{\text{fin}}$,
- $\text{QEnd}(\mathbb{Q}) \cong \mathbb{A}_\mathbb{Q}$.

One might notice that on each of the rings above there is a natural topology. On $\mathbb{R}$, we have the Euclidean topology, on $\mathbb{Q}_p$ the p-adic topology [6] and on the rational adele ring we have the restricted product topology [7]. Each of these four topological spaces is what S. P. Franklin calls a sequential space [8], in fact their topologies are first-countable. In these cases we can thus describe the open sets with converging sequences. Since the four spaces above are also topological groups [9], it suffices to know when a sequence converges to zero.

In this chapter we will define a topology on $\text{QEnd}(A)$ by giving a definition of convergence to zero of quasi-endomorphisms. We will show that a sequence of quasi-endomorphisms on $\mathbb{Z}$, $\mathbb{Z}[1/p]/\mathbb{Z}$, $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Q}$ converges under this definition if and only if the corresponding sequence in respectively $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{A}_\mathbb{Q}^{\text{fin}}$ and $\mathbb{A}_\mathbb{Q}$ converges.

3.1 Definition. Let $A \in \text{Ob } \mathbf{Qab}$ and let $([f_i])_{i\in\mathbb{N}} \subseteq \text{QEnd}(A)$ be a sequence of quasi-endomorphisms. We say that the sequence $([f_i])_{i\in\mathbb{N}}$ *converges to zero* if there is a sequence of almost-endomorphisms $(g_i)_{i\in\mathbb{N}}$ with g_i a representative of $[f_i]$ for all $i \in \mathbb{N}$ such that there is a finite subset $X \subseteq A$ with the following properties:

1. $N_{g_i} \subseteq X$ for all $i \in \mathbb{N}$, and
2. for all $a \in A$ there is a $B \in \mathbb{N}$ such that for all $i > B$ we have $g_i(a) \in X$.

Notation: for such a sequence $([f_i])_{i\in\mathbb{N}}$ we write $[f_i] \to 0$.

We will first show that this definition of convergence of sequences is invariant under **Qab**-isomorphisms.

3.2 Proposition. Let $A, B \in$ **Qab** and let $[f] \in \mathrm{QHom}(A, B)$ be an isomorphism in **Qab** and write $[g] := [f]^{-1}$. Let $([\phi_i])_{i\in\mathbb{N}} \subseteq \mathrm{QEnd}(A)$ be a sequence such that $[\phi_i] \to 0$ and define $[\psi_i] := [f\phi_i g]$. Then the sequence $([\psi_i])_{i\in\mathbb{N}} \subseteq \mathrm{QEnd}(B)$ converges to zero as well.

Proof. Write h_i for the representatives of $[\phi_i]$ as in definition 3.1 and X for a finite set that satisfies conditions 1 and 2 of the definition of convergence. There are representatives of $[\psi_i]$ given by fh_ig. We will first show that there is some finite set Z such that

$$N_{fh_ig} \subseteq Z \quad \text{for all } i \in \mathbb{N}.$$

Observe that by Lemma 1.11 we have

$$N_{h_ig} \subseteq h_i(N_g) + N_{h_i} + N_{h_i}.$$

Note that $N_{h_i} \subseteq X$ for all $i \in \mathbb{N}$ and that for all $x \in N_g \subseteq A$ there is some B_x such that for all $i > B_x$ we have $h_i(x) \in X$. Since N_g is finite, we can define

$$B := \max_{x\in N_g} B_x.$$

Then $h_i(N_g) \subseteq X$ for all $i > B$. Write

$$Y := \bigcup_{1\leq i\leq B} h_i(N_g)$$

and note that Y is finite. We now conclude that

$$N_{h_ig} \subseteq Y \cup X + X + X,$$

so again using Lemma 1.11 we find that

$$N_{fh_ig} \subseteq f(Y \cup X + X + X) + N_f + N_f.$$

Write $Z := f(Y \cup X + X + X) + N_f + N_f$ and note that Z is independent of the index i, so we indeed find that $N_{fh_ig} \subseteq Z$ for all $i \in \mathbb{N}$.

We will now show that the second condition of convergence is satisfied. Let $x \in B$ be given and note that $g(x) \in A$, so there is some integer $B \in \mathbb{N}$ such that $h_i(g(x)) \in X$ for all $i > B$. We conclude that $f(h_i(g(x)))$ is an element of $f(X)$ for all $i > B$.

Combining the above, we have found representatives fh_ig of $[\psi_i]$ and a finite set $\widetilde{X} := Z \cup f(X)$ that satisfy conditions 3.1(1) and 3.1(2) and conclude that $[\psi_i] \to 0$. □

We will show that the given definition of convergence coincides with the definitions of convergence in $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{A}_{\mathbb{Q}}^{\mathrm{fin}}$ and $\mathbb{A}_{\mathbb{Q}}$.

3.3 Proposition. Let $([f_i])_{i\in\mathbb{N}} \subseteq \mathrm{QEnd}(\mathbb{Z})$ be a sequence of quasi-endomorphisms and let $(a_i)_{i\in\mathbb{N}} \subseteq \mathbb{R}$ be the corresponding sequence in $\mathbb{R}$. Then $[f_i] \to 0$ if and only if $a_i \to 0$.

Proof. (⇐). Suppose $(a_i)_{i\in\mathbb{N}} \subseteq \mathbb{R}$ is a sequence such that $a_i \to 0$ and for each $i \in \mathbb{N}$, write $[f_i]$ for the corresponding quasi-homomorphism of a_i. We can choose representatives g_i of $[f_i]$ given by

$$g_i : x \mapsto \lfloor a_i x \rfloor.$$

We then have $N_{g_i} \subseteq \{0, 1\}$ for all $i \in \mathbb{N}$.

Let $x \in \mathbb{Z}$ be given and choose $\epsilon = \frac{1}{x}$. Since $a_i \to 0$, there is an integer $n_0 \in \mathbb{N}$ such that $|a_i| < \epsilon = \frac{1}{x}$ for all $i > n_0$. Choose $B = n_0$. Then

$$g_i(x) = \lfloor a_i x \rfloor = \begin{cases} 0 & \text{if } a_i \geq 0, \\ -1 & \text{if } a_i < 0, \end{cases}$$

so $g_i(x) \in \{0, -1\}$ for all $i > B$. Choose $X := \{-1, 0, 1\}$. We conclude that $[f_i] \to 0$.

(⇒). Suppose $([f_i])_{i\in\mathbb{N}} \subseteq \mathrm{QEnd}(\mathbb{Z})$ is a sequence such that $[f_i] \to 0$, let $(g_i)_{i\in\mathbb{N}}$ be the representatives as mentioned in definition 3.1 and for each $i \in \mathbb{N}$ write $a_i = \lim_{n\to\infty} \frac{g_i(n)}{n}$ for the corresponding real number. We know that every quasi-homomorphism $[f_i]$ has a representative h_i that is given by $n \mapsto \lfloor a_i n \rfloor$, where $\lfloor\cdot\rfloor$ denotes rounding down. However, we cannot assume that g_i equals h_i for all $i \in \mathbb{N}$. We will first prove the following Lemma:

3.4 Lemma. Suppose $f : \mathbb{Z} \to \mathbb{Z}$ is a bounded almost-homomorphism. Then the image $f(\mathbb{Z})$ is contained in the interval $[-\max(N_f), -\min(N_f)]$.

Proof. Suppose n is an integer such that $f(x) \leq f(n)$ for all $x \in \mathbb{Z}$. Then we have

$$f(n + n) = f(n) + f(n) + m$$

for some element $m \in N_f$. Since $f(n)$ is the maximum element of $f(\mathbb{Z})$, we conclude that $f(n) + m \leq 0$. It follows that $f(n) \leq -\min(N_f)$.
Now suppose n' is an integer such that $f(x) \geq f(n')$ for all $x \in \mathbb{Z}$. Then we have

$$f(n' + n') = f(n') + f(n') + m'$$

for some element $m' \in N_f$. Since $f(n')$ is the minimum element of $f(\mathbb{Z})$, we conclude that $f(n') + m' \geq 0$. It follows that $f(n') \geq -\max(N_f)$. □

Since for all $i \in \mathbb{N}$ the almost-homomorphisms g_i and h_i are of the same equivalence class, we conclude that $g_i - h_i$ is bounded for all $i \in \mathbb{N}$. Moreover, $N_{h_i} = \{0, 1\}$ for all positive integers i, so $N_{g_i - h_i} \subseteq X - \{0, 1\} =: \tilde{X}$ for all $i \in \mathbb{N}$. With the lemma above, we now find that

$$(g_i - h_i)(\mathbb{Z}) \subset [-\max(N_{g_i-h_i}), -\min(N_{g_i-h_i})] \subseteq [-\max(\tilde{X}), -\min(\tilde{X})]$$

for all $i \in \mathbb{N}$. We thus find that there is some integer $C_1 \in \mathbb{Z}_{\geq 0}$ such that for all $i \in \mathbb{N}$ and for all $n \in \mathbb{Z}$ we have

$$|h_i(n)| < |g_i(n)| + C_1.$$

Furthermore, for all $n \in \mathbb{Z}$ there is an integer $B_n \in \mathbb{N}$ such that for all $i > B_n$ we have $g_i(n) \in X$. Since X is bounded, this implies that there is some integer

$C_2 \in \mathbb{Z}_{\geq 0}$ such that $|h_i(n)| < C_2$ for all $i > B_n$. Observe that $|a_i n| - 1 \leq |\lfloor a_i n \rfloor|$ for all $n \in \mathbb{Z}$. Let $\epsilon > 0$ be given and choose $n := \lceil \frac{C+1}{\epsilon} \rceil$, where $\lceil \cdot \rceil$ denotes rounding up. Then we find that there is an integer B_n such that $|h_i(n)| < C_2$ for all $i > B_n$ and we get

$$|a_i n| - 1 \leq |\lfloor a_i n \rfloor| = |h_i(n)| < C_2 \quad \text{for all } i > B_n,$$

so

$$|a_i| < \frac{C+1}{|n|} \leq \epsilon \quad \text{for all } i > B_n$$

and we conclude that $a_i \to 0$.

□

3.5 Proposition. Let p be a prime, let $([f_i])_{i\in\mathbb{N}} \subseteq \mathrm{QEnd}(\mathbb{Z}[1/p]/\mathbb{Z})$ be a sequence of quasi-endomorphisms and let $(\alpha_i)_{i\in\mathbb{N}} \subseteq \mathbb{Q}_p$ be the corresponding sequence in $\mathbb{Q}_p$. Then $[f_i] \to 0$ if and only if $\alpha_i \to 0$ under the p-adic norm.

Proof. ($\Leftarrow$). Suppose $(\alpha_i)_{i\in\mathbb{N}} \subseteq \mathbb{Q}_p$ is a sequence such that $\alpha_i \to 0$ and for each $i \in \mathbb{N}$, write $[f_i]$ for the corresponding quasi-endomorphism of α_i. Note that a basis around zero for the p-adic topology on $\mathbb{Z}_p \subset \mathbb{Q}_p$ is given by $\{p^m\mathbb{Z}_p\}_{m\in\mathbb{Z}_{\geq 0}}$. Since $\alpha_i \to 0$, for all $m \in \mathbb{Z}_{\geq 0}$ there is an integer B_m such that for all $i > B_m$ we have $\alpha_i \in p^m\mathbb{Z}_p$. Consider $m = 0$. Then we find that for all $i > B_0$ there are representatives g_i of $[f_i]$ such that g_i is an endomorphism and such that g_i is given by $g_i(x) = a_i x$. We find that $N_{g_i} = \{0\}$ for all $i > B_0$. Choose representatives g_j of $[f_j]$ for $1 \leq j \leq B_0$. Write

$$Y := \bigcup_{1\leq j\leq B_0} N_{g_j}$$

and note that Y is finite. We find that

$$N_{g_i} \subseteq Y \cup \{0\} = Y \quad \text{for all } i \in \mathbb{N}.$$

We will now show the second property of convergence. Let $x \in \mathbb{Z}[1/p]/\mathbb{Z}$ be given, then x is of the form $x = \frac{a}{p^b}$ for certain $a \in \mathbb{Z}$, $b \in \mathbb{Z}_{\geq 0}$. There is an integer B_{1/p^b} such that for all $i > B_{1/p^b}$ we have that $a_i \in p^b\mathbb{Z}_p$. We find that

$$g_i(x) = a_i \frac{1}{p^b} = 0 \quad \text{for all } i > \max\{B_{1/p^b}, B_0\},$$

and we conclude that indeed $[f_i] \to 0$.

($\Rightarrow$). Suppose $([f_i])_{i\in\mathbb{N}} \subseteq \mathrm{QEnd}(\mathbb{Z}[1/p]/\mathbb{Z})$ is such that $[f_i] \to 0$ and write $(\alpha_i)_{i\in\mathbb{N}}$ for the corresponding sequence in $\mathbb{Q}_p$. Let $(g_i)_{i\in\mathbb{N}}$ be representatives and X a finite set as mentioned in definition 3.1. Then there is an integer k such that $p^k X = \{0\}$, so $h_i := \lambda_{p^k} g_i \in \mathrm{End}(\mathbb{Z}[1/p]/\mathbb{Z})$ for all $i \in \mathbb{N}$ and for all $x \in \mathbb{Z}[1/p]/\mathbb{Z}$ there is a B_x such that for all $i > B_x$ we have $h_i(x) = 0$. Consider the sequence $(\beta_i)_{i\in\mathbb{N}} \subseteq \mathbb{Z}_p$ corresponding with h_i. Let $m \in \mathbb{Z}_{\geq 0}$ and let $p^m\mathbb{Z}_p$ be an open around zero. Choose $x := \frac{1}{p^m}$, then there is an integer B_x such that for all $i > B_x$ we have

$$h_i(x) = \beta_i \frac{1}{p^m} = 0,$$

so $\beta_i \in p^m\mathbb{Z}_p$ and we conclude that $\beta_i \to 0$. Furthermore, we have $(\beta_i)_{i\in\mathbb{N}} = p^k(\alpha_i)_{i\in\mathbb{N}}$, so we conclude that $\alpha_i \to 0$. □

3.6 Proposition. Let $([f_i])_{i\in\mathbb{N}} \subseteq \mathrm{QEnd}(\mathbb{Q}/\mathbb{Z})$ be a sequence of quasi-endomorphisms and let $(\alpha_i)_{i\in\mathbb{N}} \subseteq \mathbb{A}_\mathbb{Q}$ be the corresponding sequence in $\mathbb{A}_\mathbb{Q}^{\mathrm{fin}}$. Then $[f_i] \to 0$ if and only if $\alpha_i \to 0$ in the finite adele ring under the restricted product topology.

Proof. ($\Leftarrow$). Suppose $(\alpha_i)_{i\in\mathbb{N}} \subseteq \mathbb{A}_\mathbb{Q}^{\mathrm{fin}}$ is a sequence such that $\alpha_i \to 0$ under the restricted product topology. Write $([f_i])_{i\in\mathbb{N}}$ for the corresponding sequence of quasi-endomorphisms on $\mathbb{Q}/\mathbb{Z}$. Note that a basis around zero for the restricted product topology on the finite adele ring is given by

$$\{m \cdot \prod_{p\in\mathcal{P}} \mathbb{Z}_p\}_{m\in\mathbb{Z}_{\geq 1}},$$

where $\mathcal{P}$ is the set of prime numbers. Since $(\alpha_i)_{i\in\mathbb{N}}$ converges to zero, for all $m \in \mathbb{Z}_{\geq 1}$ there is an integer B_m such that for all $i > B_m$ we have $\alpha_i \in m\prod_{p\in\mathcal{P}} \mathbb{Z}_p$. Consider $m = 1$, then we find that there are representatives g_i of $[f_i]$ that are endomorphisms and that are given by $g_i(x) = \alpha_i x$ for all $i > B_1$, so $N_{g_i} = 0$ for all $i > B_1$. Choose representatives $g_j \in \mathrm{AEnd}(\mathbb{Q}/\mathbb{Z})$ of $[f_j]$ for $1 \leq j \leq B_1$. We now find:

$$N_{g_i} \subseteq \bigcup_{1\leq j\leq B_1} N_{g_j} \quad \text{for all } i \in \mathbb{N}.$$

Write $X := \bigcup_{1\leq i\leq B_1} N_{g_i}$.

To show the second property of convergence, let $\frac{a}{b} \in \mathbb{Q}/\mathbb{Z}$ be given arbitrarily, where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geq 1}$. Then there is an integer B_b such that for all $i > B_b$ we have $\alpha_i \in b\prod_{p\in\mathcal{P}} \mathbb{Z}_p$, so for all $i > \max\{B_b, B_1\}$ we have

$$g_i\left(\frac{a}{b}\right) = \alpha_i \frac{a}{b} = 0.$$

Since $0 \in X$, we conclude that $[f_i] \to 0$.

($\Rightarrow$). Suppose $([f_i])_{i\in\mathbb{N}} \subseteq \mathrm{QEnd}(\mathbb{Q}/\mathbb{Z})$ is such that $[f_i] \to 0$ and write $(\alpha_i)_{i\in\mathbb{N}}$ for the corresponding sequence in $\mathbb{A}_\mathbb{Q}^{\mathrm{fin}}$. Write $(g_i)_{i\in\mathbb{N}}$ for representatives as in definition 3.1. Let $m \in \mathbb{Z}_{\geq 1}$ be given and consider the open set $m\prod_{p\in\mathcal{P}} \mathbb{Z}_p$. Let X be a finite set as in the definition of convergence. We know that there is an integer $k \in \mathbb{Z}$ such that $kX = \{0\}$, so $kg_i \in \mathrm{End}(\mathbb{Q}/\mathbb{Z})$ for all $i \in \mathbb{N}$. By the second property of convergence, we find that for all $x \in \mathbb{Q}/\mathbb{Z}$ there is an integer $B_x \in \mathbb{Z}_{\geq 1}$ such that for all $i > B_x$ we have $kg_i(x) = 0$. Choose $x := \frac{1}{m}$, then there is an integer $B_{1/m}$ such that for all $i > B_{1/m}$ we have

$$kg_i\left(\frac{1}{m}\right) = k\alpha_i \frac{1}{m} = 0,$$

so we find that $k\alpha_i \in m\prod_{p \text{ prime}} \mathbb{Z}_p$ for all $i > B_{1/m}$. We conclude that $k\alpha_i \to 0$ in $\mathbb{A}_\mathbb{Q}^{\mathrm{fin}}$. It follows that $\alpha_i \to 0$ in $\mathbb{A}_\mathbb{Q}^{\mathrm{fin}}$. □

We will combine the findings above to prove that the given definition of convergence also coincides with convergence of sequences in the adele ring.

3.7 Proposition. Let $([f_i])_{i\in\mathbb{N}} \subseteq \mathrm{QEnd}(\mathbb{Q})$ be a sequence of quasi-endomorphisms and let $(\alpha_i)_{i\in\mathbb{N}} \subseteq \mathbb{A}_\mathbb{Q}$ be the corresponding sequence in $\mathbb{A}_\mathbb{Q}$. Then $[f_i] \to 0$ if and only if $\alpha_i \to 0$ in the adele ring.

Proof. By Corollary 1.20, we have a **Qab**-isomorphism $\phi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \times \mathbb{Z}$. It follows from proposition 3.2 that the sequence $([f_i])_{i\in\mathbb{N}} \subseteq \mathrm{QEnd}(\mathbb{Q})$ converges to zero if and only if the associated sequence $[\phi f_i \phi^{-1}]_{i\in\mathbb{N}} \subseteq \mathrm{QEnd}(\mathbb{Q}/\mathbb{Z} \times \mathbb{Z})$ converges to zero.

Recall that Lemma 1.15 provides us with a ring isomorphism

$$\mathrm{QEnd}(\mathbb{Q}/\mathbb{Z}) \times \mathrm{QEnd}(\mathbb{Z}) \to \mathrm{QEnd}(\mathbb{Q}/\mathbb{Z} \times \mathbb{Z})$$

given by

$$([g], [h]) \mapsto [k] := [g \oplus h].$$

It is very clear that a sequence $([g_i], [h_i])_{i\in\mathbb{N}} \subseteq \mathrm{QEnd}(\mathbb{Q}/\mathbb{Z}) \times \mathrm{QEnd}(\mathbb{Z})$ converges to zero if and only if the sequence $([g_i \oplus h_i])_{i\in\mathbb{N}} \subseteq \mathrm{QEnd}(\mathbb{Q}/\mathbb{Z} \times \mathbb{Z})$ converges to zero. We can thus associate a sequence $([k_i])_{i\in\mathbb{N}} \subseteq \mathrm{QEnd}(\mathbb{Q}/\mathbb{Z} \times \mathbb{Z})$ with a sequence $([g_i], [h_i])_{i\in\mathbb{N}} \subseteq \mathrm{QEnd}(\mathbb{Q}/\mathbb{Z}) \times \mathrm{QEnd}(\mathbb{Z})$ and with Propositions 3.3 and 3.6 we know that this sequence converges to zero if and only if the associated sequence $(x_i, y_i)_{i\in\mathbb{N}} \subseteq \mathbb{A}_{\mathbb{Q}}^{\mathrm{fin}} \times \mathbb{R}$ converges to zero.

Combining the two observations above, we find that a sequence $([f_i])_{i\in\mathbb{N}} \subseteq \mathrm{QEnd}(\mathbb{Q})$ converges to zero if and only if the associated sequence

$$(\alpha_i)_{i\in\mathbb{N}} = (x_i, y_i)_{i\in\mathbb{N}} \subseteq \mathbb{A}_{\mathbb{Q}}^{\mathrm{fin}} \times \mathbb{R} = \mathbb{A}_{\mathbb{Q}}$$

converges to zero. □

For arbitrary abelian groups A we can define a topology on $\mathrm{QEnd}(A)$ by using the definition of convergence in the following way: we call a set $O \subseteq A$ *open* if and only if for all $x \in O$ and for all sequences $(x_i)_{i\in\mathbb{N}} \subseteq \mathrm{QEnd}(A)$ such that $x_i \to x$ there is an integer $B \in \mathbb{N}$ such that for all $i > B$ we have $x_i \in O$. Since $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{A}_{\mathbb{Q}}^{\mathrm{fin}}$ and $\mathbb{A}_{\mathbb{Q}}$ are sequential topological groups, it follows from Propositions 3.3 - 3.7 that the topology as mentioned above coincides with the topologies on these four spaces in the cases $A = \mathbb{Z}$, $A = \mathbb{Z}[1/p]/\mathbb{Z}$, $A = \mathbb{Q}/\mathbb{Z}$ and $A = \mathbb{Q}$, respectively.

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