Ramsey theory and applications

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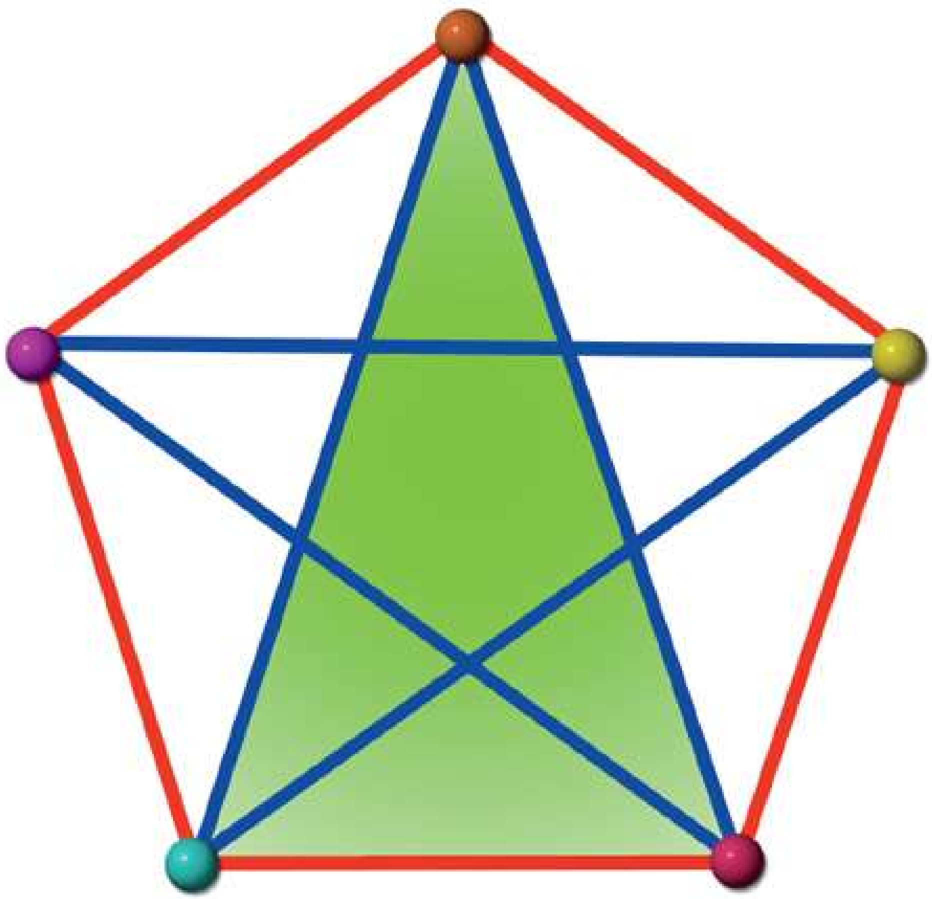

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1 Introduction

Consider a room with six people. Any two people in the room are related to each other, as follows: either they know each other or they don't know each other. We want to show that there are always three people who know each other or three people who don't know each other. Take a person, say Jim, then notice that there are five other people so there must be at least three people whom Jim knows or at least three people whom he doesn't know. Without loss of generality we can assume Jim knows at least three other people. If two of these other people know each other, then we are done, because these two people also know Jim. The alternative is that these three people do not know each other which also means we are done.
This is a simple and famous example from Ramsey theory. We may translate the example described above into mathematics by means of graphs. The six persons correspond to the vertices of a complete graph. Two vertices are connected by a red edge if the corresponding persons know each other and by a blue edge otherwise. Then the above argument says, that if we have a complete graph with six vertices and give each edge the color red or blue, then whatever the choice of the colour of each edge, there are always three vertices that are connected by edges of the same color. This is no longer true for complete graphs with fewer than six vertices, as is illustrated in Figure 1.

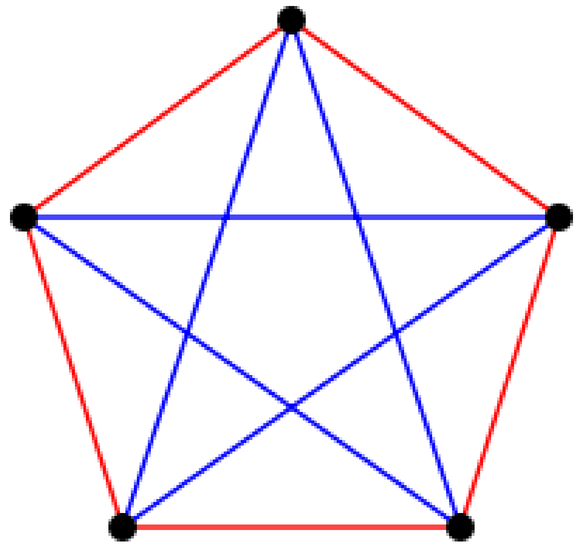

Figure 1: No red or blue triangle

So at least six people are required to be certain that among them three people know each other or three people don't know each other. A particular case of Ramsey's general theorem tells us that for each pair of integers k, l at least 3, there is a number $R(k, l)$ such that among $R(k, l)$ people there are always k knowing each other or l not knowing each other.
In section 2 we prove Ramsey's general theorem and deduce some upper and lower bounds for the Ramsey numbers $R(k, l)$. In section 3 we consider some applications of Ramsey's theorem, first to linear equations in integers, and second and more extensively, to sets of point in a plane in convex position, i.e., which are the vertices of a convex polygon. For example, we show that if we place five points in $\mathbb{R}^2$ with no three points on a line, then there are always four points in convex position. More generally, we prove the Erdős-Szekeres theorem that states that for a positive integer n_0 at least 3, there exists a smallest integer $N(n_0)$ such that for every set of $N(n_0)$ points in $\mathbb{R}^2$, without three points on a line, there are always n_0 points in convex position. Besides the existence we also look at the cardinality of this number of points.

2 Ramsey theory

2.1 Introduction

In this chapter we introduce Ramsey theory, which is an important subject in the field of combinatorics. Before we state Ramsey's main theorem we introduce some basic tools from graph theory, as a means to understand some special cases. The general theorem from Ramsey theory is stated without graphs, but we have used graphs in the statement of some special cases, because we want to visualize and understand Ramsey theory before we are going to look at the applications.

2.2 Graphs

We denote by $|S|$ the cardinality of a set S. Given sets A, B we write $A \subseteq B$ if A is a subset of B and $A \subset B$ if A is a proper subset of B.

Definition 2.1. Given a positive integer k and a set N with $|N| \geq k$, a k-subset of N is a subset S of N of cardinality k.

Notice that a set N of n elements has exactly $\binom{n}{k}$ k-subsets.

Definition 2.2. A (simple, undirected) graph G is a pair $(V(G), E(G))$, consisting of a set $V(G)$ whose elements are called the vertices of G, and a collection $E(G)$ of 2-subsets of $V(G)$ called the edges of G. The graph G is called *complete* if $E(G)$ consists of all 2-subsets of $V(G)$.

A graph G is called *finite* if $V(G)$ is finite. Henceforth, we consider only finite graphs. Further $|V(G)|$ is called the *order* of G. A graph H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subset E(G)$.

2.3 Ramsey theorems

Definition 2.3. Let S be a set and $r \in \mathbb{Z}_{>0}$. An *r-colouring* of S is a function $f : S \rightarrow \{1, 2, ..., r\}$.

In Ramsey theory we consider for any pair of integers $k \geq 1, r \geq 1$, r-colourings of the k-subsets of a given set N. For $k = 1$ this means that we consider the 1-subsets of N, which are the elements of N. When $k = 2$ such a colouring may be visualized most conveniently by colouring the edges of a complete graph with set of vertices N. There is no obvious way to visualize colourings of k-subsets if $k > 2$.

An important and simple theorem in Ramsey theory is the Pigeonhole principle. There are a lot of versions of this. We describe one of them: when $n + 1$ pigeons are divided over n holes, there must be at least one hole containing at least two pigeons. This very simple principle plays a part in the proof of the main theorem of Ramsey. We first state this theorem in the special case of r-colourings of the edges of a graph.

Theorem 2.4 (Ramsey with r colours)**.** *For every tuple of integers $l_1, \dots, l_r \geq 2$, there is a positive integer R such that the following holds:*
For every complete graph G of R vertices and every r-colouring of the edges of G, there is a $i \in \{1, \dots, r\}$ and a complete subgraph of G of order l_i of which all edges have colour i.

Definition 2.5. The smallest integer R for which the above assertion is satisfied is denoted by $R_2(l_1, \dots, l_r)$. We call $R_2(l_1, \dots, l_r)$ a *Ramsey number*.

Remark 2.6. Note that if the theorem holds for a complete graph with R vertices then the theorem also holds for any complete graph with more than R vertices.

Proof of Theorem 2.4. (See [15]) We prove this by induction on $l_1+\dots+l_r$, where $l_1, \dots, l_r \geq 2$. Also we notice that for $i \in \{1, \dots, r\}$ we have $R_2(2, \dots, 2, l_i, 2, \dots, 2) = l_i$. Because if all edges have colour i, there is a complete subgraph of order l_i of which all edges have colour i . If one edge does not have colour i, then there is a complete subgraph of order 2 of which its single edge has got some colour $j \neq i$.
For $i \in \{1, \dots, r\}$ we define $R_2(l_1, \dots, l_{i-1}, 1, l_{i+1}, \dots, l_r) = 1$.
Now take $n_0 > 2r$ and assume the theorem holds for all $l_1, \dots, l_r$ with $l_1 + \dots + l_r < n_0$. Now let $l_1 + \dots + l_r = n_0$, and define

$$R := \sum_{i=1}^{r} R_2(l_1, \dots, l_{i-1}, l_i - 1, l_{i+1}, \dots, l_r) - r + 2. \tag{2.3.1}$$

Pick a vertex v from the R vertices of our graph. We partition the vertices into sets,
$L_i = \{w \in V(G)\backslash\{v\} | \{v, w\} \text{has colour } i\} \quad (i \in \{1, \dots, r\})$

We get $|V(G)| = R = (\sum_{i=1}^{r} |L_i|) + 1$. So by the Pigeonhole principle it follows that there exists a $i \in \{1, \dots, r\}$ such that $|L_i| \geq R_2(l_1 \dots, l_{i-1}, l_i - 1, l_{i+1}, \dots, l_r)$.

We distinguish two cases. The first case is that for some $j \neq i$, the set L_i contains l_j vertices such that all edges connecting them have colour j. Then we are done.
In the second case we know that L_i contains $l_i - 1$ vertices such that all edges between them have colour i. Then all edges connecting the vertices in $L_i \bigcup \{v\}$ have got colour i and we are again done.

□

Corollary 2.7. *Let $l_1, \dots, l_r$ be integers ≥ 2. Then $R_2(l_1, l_2, \dots, l_r) \leq \sum_{i=1}^{r} R_2(l_1, \dots, l_{i-1}, l_i - 1, l_{i+1}, \dots, l_r) - r + 2$.*

Proof. From the proof of Theorem 2.1 it follows that $R_2(l_1, l_2, \dots, l_r) \leq R$, with R as in (2.3.1).
□

We now state and prove the general version of Ramsey's theorem [12].

Theorem 2.8. *For any $k, r \in \mathbb{Z}_{>0}$ and $l_1, \dots, l_r \in \mathbb{Z}_{\geq k}$, there exists a positive integer R such that the following holds:*

For every set N with n elements and every r-colouring of the k-subsets of N there exists a $i \in \{1, 2, \ldots, r\}$ and a subset S of N with l_i elements such that every k-subset of S has been given colour i.

Definition 2.9. The smallest integer R for which the above assertion is satisfied is denoted by

$$R_k(l_1, \ldots, l_r). \tag{2.3.2}$$

We call $R_k(l_1, \ldots, l_r)$ a Ramsey number. Notice that k stands for the cardinality of the subsets that are coloured.

Proof of Theorem 2.8. (See [1, page 4,5]).

We prove this by induction on k. By $l_1, \ldots, l_r$ we will always denote integers $\geq k$.
In the case $k = 1$, Theorem 2.2 follows from the Pigeonhole principle. Now take $h > 1$ and assume the theorem holds for all k with $k < h$. Let $k = h$. Within our induction step, we proceed by another induction on $l_1 + \cdots + l_r$. If $l_1 = l_2 = \cdots = l_r = h$, then it follows trivially that $R_h(l_1, \ldots, l_r) = h$. Now take $n_1 > rh$ and assume the theorem holds for $k = h$ and $l_1 + l_2 + \cdots + l_r < n_1$. Define $R_h(l_1, \ldots, l_{i-1}, h-1, l_{i+1}, \ldots, l_r) = 1$.
Let $l_1, \ldots, l_r$ be integers $\geq h$ with $l_1 + l_2 + \cdots + l_r = n_1$. By our second induction hypothesis we can define $a_i := R_h(l_1, \ldots, l_{i-1}, l_i - 1, l_{i+1}, \ldots, l_r)$ for $i \in \{1, \ldots, r\}$. Further by our first induction hypothesis we can define the number $n := 1 + R_{h-1}(a_1, \ldots, a_r)$. Suppose we are given a set N with n elements and an arbitrary r-colouring c of the h-subsets of N. Take $m \in N$ and consider the set $Y := N \backslash \{m\}$.
Now we define an induced r-colouring $c*$ on the $(h-1)$-subsets of Y as follows: we give a $(h-1)$-subset S of Y the colour $i*$ if $S \bigcup \{m\}$ has colour i in the original colouring c. By our definition of n, we know that for some $i \in \{1, \ldots, r\}$ Y has a subset A_i with a_i elements all whose $(h-1)$-subsets have colour $i*$. We know by the definition of a_i that either for some $j \neq i$ the set A_i has a subset with l_j elements with all h-subsets coloured j which means we are done because $A_i \subset N$, or A_i has a subset T of $l_i - 1$ elements with all h-subsets coloured i. Let this be the case. Consider $T \bigcup \{m\}$. Notice that $|T \bigcup \{m\}| = l_i$. Take an arbitrary h-subset K of $T \bigcup \{m\}$. If $m \notin K$ then K has colour i as $K \subset T$ and all h-subsets of T have colour i. If $m \in K$, then $K \backslash \{m\} \subset A_i$ and so it has colour $i*$ in the induced colouring, but then we conclude again that K itself has colour i. This finishes the proof.

□

2.4 Ramsey numbers

What can be said about the Ramsey numbers $R_2(k, l)$? In the introduction we saw that $R_2(3, 3) = 6$. The table in Figure 2 gives what is presently known about certain Ramsey numbers $R_2(k, l)$. In a few cases, we know the precise value of $R_2(k, l)$. In many other cases, its precise value is not known, and we can give only the best known lower and upper bounds known at present. The development of Ramsey number research is maintained in the Dynamic Survey 1 of the Electronic Journal of Combinatorics[11]. It was written and is updated by Stanisaw Radziszowski. The first version was in 1994, and the latest revision was in 2017. Where no reference has been included, the numbers are from his article. Note that there is symmetry across the diagonal.

R S	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10
3		6	9	14	18	23	28	36	40-42
4			18	25[9]	36-41	49-61	59-84[5]	73-115	92-149
5				43-48	58-87	80-143	101-216	133-316	149-442[5]
6					102-165	115-298[5]	134-495[5]	183-780	204-1171
7						205-540	217-1031	252-1713	292-2826
8							282-1870	329-3583	343-6090
9								565-6588	581-12671
10									798-23556

Figure 2: An overview of the Ramsey numbers

Below we give some theorems and lemmas that give upper bounds for Ramsey numbers. We first introduce some notation.

Definition 2.10. For non-negative integers $a, r, l_1, \ldots, l_r$ with $a = l_1 + \cdots + l_r$ we define the multinomial coefficient

$$\binom{a}{l_1, l_2, \ldots, l_r} := \frac{a!}{l_1! l_2! \ldots l_r!}. \tag{2.4.1}$$

Theorem 2.11. *Let $r \in \mathbb{Z}_{>0}$, and $l_1, \ldots, l_r \in \mathbb{Z}_{>1}$. Then*

$$R_2(l_1, l_2, \ldots, l_r) \leq \binom{l_1 + l_2 + \cdots + l_r - r}{l_1 - 1, l_2 - 1, \ldots, l_r - 1}. \tag{2.4.2}$$

Proof. We prove this by induction on $l_1 + l_2 + \cdots + l_r$.
For $l_1 = \cdots = l_r = 2$ (2.4.2) is trivially true. If we allow some of the l_i to be 1 and define $R_2(l_1, \ldots, l_r) = 1$ in this case, (2.4.2) is also true.
Now assume $n_0 > 2r$ and (2.4.2) holds for all $l_1, \ldots, l_r \in \mathbb{Z}_{\geq 2}$ with $l_1 + l_2 + \cdots + l_r < n_0$.
Then by Corollary 2.7 and the induction hypothesis, we obtain for any integers $l_1, \ldots, l_r \geq 2$ with $l_1 + l_2 + \cdots + l_r = n_0$,

$$R_2(l_1, l_2, \ldots, l_r) \leq \sum_{i=1}^{r} R_2(l_1, \ldots, l_{i-1}, l_i - 1, l_{i+1}, \ldots, l_r) - r + 2 \tag{2.4.3}$$

$$\leq \sum_{i=1}^{r} \binom{l_1 + l_2 + \cdots + l_r - 1 - r}{l_1 - 1, \ldots, l_{i-1} - 1, l_i - 2, l_{i+1} - 1, \ldots, l_r - 1} \tag{2.4.4}$$

$$\leq \binom{l_1 + l_2 + \cdots + l_r - r}{l_1 - 1, l_2 - 1, \ldots, l_r - 1}. \tag{2.4.5}$$

□

Corollary 2.12. *Let again $r \in Z_{>0}$ and let $l_1, \ldots, l_r$ be integers ≥ 2. Then*

$$R_2(l_1, \ldots, l_r) \leq r^{l_1 + \cdots + l_r - r}. \tag{2.4.6}$$

Proof. Theorem 2.11 implies

$$R_2(l_1, \ldots, l_r) \leq \binom{l_1 + \cdots + l_r - r}{l_1 - 1, \ldots, l_r - 1} \leq \sum_{\substack{i_1, \ldots, i_r \geq 0 \\ i_1 + \cdots + i_r = l_1 + \cdots + l_r - r}} \binom{l_1 + \cdots + l_r - r}{i_1, \ldots, i_r} = r^{l_1 + \cdots + l_r - r}.$$

□

This establishes an upper bound for Ramsey numbers. Erdős gave a lower bound for certain Ramsey numbers using a counting argument.

Theorem 2.13 (Erdős). *let l, r be integers > 2. Then we have*

$$R_2(\underbrace{l, l, \ldots, l}_{r}) \geq r^{\frac{l-2}{2}}. \tag{2.4.7}$$

Proof. (See [1, page 8,9], [2]). Since $k = 2$ we can explain the proof in terms of graphs. Let G be a complete graph with n vertices and take an arbitrary r-colouring of the edges of G. Now define the following quantities: Denote by X the number of possible colourings of the edges of G. Denote by Y the number of possible colourings of the edges of G such that G has a complete subgraph of order l with all edges having the same colour.
Now notice that if $X > Y$ then there has to be at least one colouring of G such that G has no subgraph of order l with all edges having the same colour. That is, $n < R_2(l, \ldots, l)$.
Now we give estimates for X and Y respectively. First notice that $X = r^{\binom{n}{2}}$, as there are $\binom{n}{2}$ edges and there are r ways to colour each edge.
To estimate Y, we fix a complete subgraph L of G of order l and notice that there are $r^{\binom{n}{2} - \binom{l}{2}}$ ways to colour the edges of G not belonging to L. Thus, it follows that there are $r \cdot r^{\binom{n}{2} - \binom{l}{2}}$ ways to colour the edges of G such that all edges of L have the same colour. Further, because there are $\binom{n}{l}$ ways to choose L we conclude that $Y \leq \binom{n}{l} r \cdot r^{\binom{n}{2} - \binom{l}{2}}$. So $X > Y$ is certainly satisfied if $r^{\binom{n}{2}} > \binom{n}{l} \cdot r^{\binom{n}{2} - \binom{l}{2} + 1}$, that is if $\binom{n}{l} < r^{\binom{l}{2} - 1}$. Now if $n < r^{\frac{l-2}{2}}$ then indeed we have

$\binom{n}{l} \leq n^l < r^{\frac{l(l-2)}{2}} \leq r^{\binom{l}{2}-1}$, which implies $X > Y$. Thus, we can conclude that $R_2(l, l) \geq r^{\frac{l-2}{2}}$. $\square$

3 Applications of Ramsey theory

3.1 Introduction

In this section we discuss two applications of the main theorem of Ramsey: Schur's theorem on linear equations in integers and the Erdős-Szekeres Theorem on subsets of points in a plane in convex position. Also, we discuss the famous conjecture of Erdős and Szekeres.

3.2 Schur's Theorem

Theorem 3.1. *For any r-colouring of $\mathbb{Z}_{>0}$ there exist $x, y, z \in \mathbb{Z}_{>0}$, all of the same colour, with $x + y = z$.*

Proof. (See [1, page 14], [13]). We begin by defining $n = R_2(\underbrace{3, 3, \ldots, 3}_{r})$ (see Definition 2.5). Take an r-colouring

$$f : \mathbb{Z}_{>0} \to \{1, \ldots, r\} \tag{3.2.1}$$

on the elements of $\mathbb{Z}_{>0}$. We now define an induced r-colouring f' on the 2-subsets of $Z_{>0}$ as follows: $\{i, j\}$ has colour p if $f(|i - j|) = p$. By definition of n there exists a 3-subset $\{a, b, c\}$ of $\{1, \ldots, n\}$ such that all 2-subsets of $\{a, b, c\}$ have the same colour from f'. We order the triple such that $a > b > c$. Then $f'(a, b) = f'(b, c) = f'(a, c)$ which means that $f(a - b) = f(b - c) = f(a - c)$. If we define $x := a - b$, $y := b - c$ and $z := a - c$, we obtain $f(x) = f(y) = f(z)$ and $x + y = (a - b) + (b - c) = a - c = z$.

$\square$

3.3 The Erdős-Szekeres Theorem

A few definitions and lemmas have to be given before the theorem can be stated.

Definition 3.2. A set $D \subset \mathbb{R}^2$ is called *convex* if for every $x, y \in D$ the line segment connecting x, y is also in D, so formally

$$\forall x, y \in D, \forall t \in [0,1] : tx + (1-t)y \in D. \tag{3.3.1}$$

Definition 3.3. Given a subset N of $\mathbb{R}^2$, the *convex hull* $C(N)$ of N is the smallest convex subset of $\mathbb{R}^2$ containing N.

Definition 3.4. A finite subset $N \subset \mathbb{R}^2$ is said to be in *convex position*, if $C(M) \subset C(N)$ for each proper subset M of N.

From a special case of a theorem of Krein and Milman [8] from functional analysis, it follows that if N is in convex position, then $C(N)$ is a convex polygon of which N is the set of vertices.

Lemma 3.5. *Let N be a finite subset of $\mathbb{R}^2$ with at least 4 points. Then N is in convex position if and only if each 4-subset of N is in convex position.*

Proof. If N is in convex position, then clearly so is each of its 4-subsets. Now assume that each 4-subset of N is in convex position, but that N itself is not. Let M be a minimal subset of N such that $C(N) = C(M)$. Then $M \subset N$. We may assume that $M = \{A_1, \ldots, A_r\}$, where $A_1, \ldots, A_r$ are the vertices of a convex polygon, arranged clockwise. Then $C(M)$ is the union of the triangles $A_1A_2A_3, A_1A_3A_4, \ldots A_1A_{r-1}A_r$, see Figure 3.

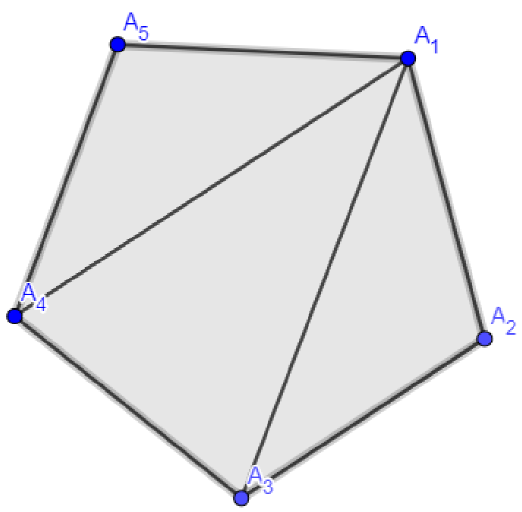

Figure 3: Union of triangles

Let $D \in N \backslash M$, so $D \in C(M)$. Then there are $A, B, C \in M$ such that D lies in triangle ABC. But then A, B, C, D are not in convex position, contrary to our assumption. Hence N is in convex position. $\square$

Erdős and Szekeres proved the following result:

Theorem 3.6. *Let $n_0 \in \mathbb{Z}_{>2}$ be given. There exists a positive integer n such that the following holds:*
Any set of n points in $\mathbb{R}^2$, no three points of which are collinear, contains n_0 points in convex position.

Definition 3.7. We define $N(n_0)$ as the smallest integer n for which the above assertion is satisfied.

Definition 3.8. The *interior* of a set of three non-collinear points in $\mathbb{R}^2$ is the set of points in the interior of the triangle spanned by this set.

Proof of Theorem 3.6. (See [1, page 6,7], [3], [6]).
We prove this with $n = R_3(n_0, n_0)$. Let T be any set of n points in $\mathbb{R}^2$, no three of which are collinear. We denote the number of points from T in the interior of a 3-subset $\{A, B, C\}$ of T by $f(A, B, C)$. We define the following red-blue colouring on the 3-subsets of our set T: if for a 3-subset $\{A, B, C\}$, $f(A, B, C)$ is even then we colour this 3-subset red. If $f(A, B, C)$ is odd then we colour this 3-subset blue.
Now Ramsey's theorem tells us that there is either a subset S of T of order n_0 of which all 3-subsets have the colour red or a subset S of T of order n_0 of which all 3-subsets have the colour blue.
Now we claim that in both cases S is in convex position. This means that each 4-subset of S has to be in convex position. Suppose some 4-subset $\{A, B, C, D\}$ of S is not in convex position. Since no three points among A, B, C, D are on a line, this means that one of the four points, say D, is in the interior of $\{A, B, C\}$. We get the next equality

$$f(A, B, C) = f(A, B, D) + f(B, C, D) + f(A, C, D) + 1 \tag{3.3.2}$$

(see Figure 4). Now if all 3-subsets of S are coloured red we get a contradiction as an even

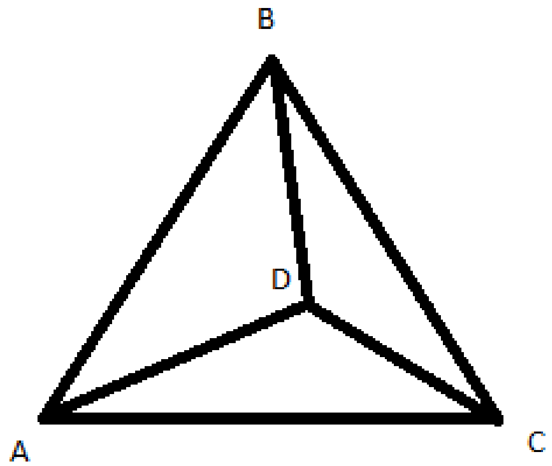

Figure 4: Triangle A, B, C

number can not be the sum of even numbers plus one. In the other case, all 3-subsets of S are coloured blue and then we get a contradiction as an odd number can not be the sum of three odd numbers plus one. We conclude that a 3-subset of S can not have an interior point from S. So S is a set of n_0 points in convex position.

□

Corollary 3.9. *For any integer $n_0 \geq 3$ from the above theorem we get the inequality $N(n_0) \leq R_3(n_0, n_0)$.*

Proof. The cardinality of the set N used in the proof contains $R_3(n_0, n_0)$ points and als contains n_0 points in convex position. □

We give another proof of Theorem 3.2 based on Ramsey theory, providing another upper bound for $N(n_0)$. We need the following lemma.

Lemma 3.10 (Happy ending problem)**.** *Every set with five points in $\mathbb{R}^2$, with no three points on a line, always contains four points in convex position.*

Proof. Let A, B, C, D, E be five points in $\mathbb{R}^2$, no three of which are collinear. Assume that A, B, C, D are not in convex position. Then we may assume without loss of generality that D is in the interior of triangle ABC. By drawing a line through each pair of points from A, B, C, D we divide the plane into eighteen regions, as shown in Figure 5.

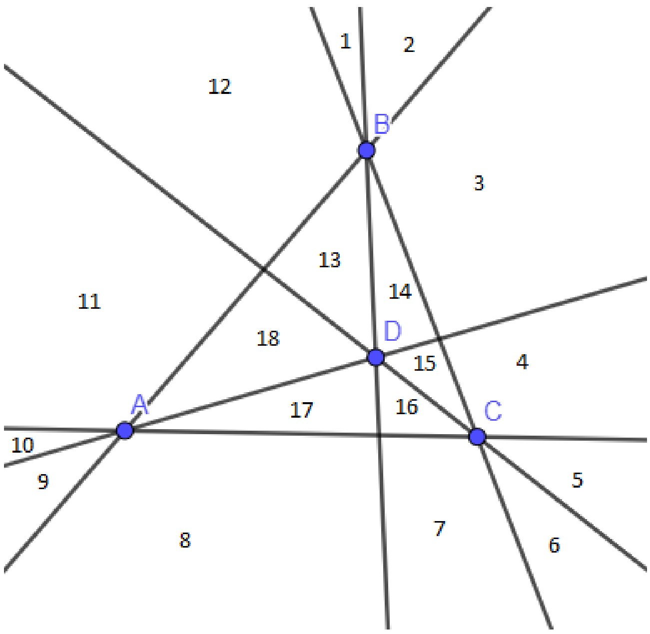

Figure 5: Possibilities for the fifth point

We consider the various cases, depending on the regions to which E belongs. First we note that by symmetry we only have to check the cases that E belongs to one of the regions $2, 3, 4, 5, 14, 15$. If E is in one of the regions $2, 3, 4, 5$ we see that the four points B, C, D, E are in convex position. If E is in region 14 then A, B, D, E are in convex position and if E is in the region 15 then A, C, D, E are in convex position. This finishes the proof because the other regions can be treated analogously. □

We now present another upper bound for $N(n_0)$.

Lemma 3.11. *For any integer $n_0 \geq 3$ we have*

$$N(n_0) \leq R_4(n_0, 5) \tag{3.3.3}$$

where $R_4(n_0, 5)$ is defined as in (2.3.2).

Proof. (See [3] and [10]) Let T be any set in $\mathbb{R}^2$ containing $R_4(n_0, 5)$ points, no three of which are collinear. We define the following red-blue colouring on the 4-subsets of T: we give a 4-subset colour red if the 4 points are in convex position, otherwise, we give the 4-subset the colour blue. Ramsey's Theorem guarantees us the existence of an n_0-subset S of T with all 4-subsets coloured red or a 5-subset with all 4-subsets coloured blue. By Lemma 3.10 the last case can not hold, so the set T must contain a n_0-subset S with all 4-subsets coloured red. Hence all 4-subsets of S are in convex position, which implies that S itself must be too. □

Erdős and Szekeres posed the following conjecture.

Conjecture 3.12. *([3])*

$$N(n_0) = 2^{n_0-2} + 1 \text{ for all } n_0 \geq 3 \tag{3.3.4}$$

Despite many attempts this conjecture remains unsolved. For small values of n_0 this equality indeed holds. Erdős and Szekeres proved that $N(n_0) \geq 2^{n_0-2} + 1$ for all $n_0 \geq 3$. A proof of this will be given below. Before that however, we introduce a caps and cups system which will be needed both to deduce another upper bound for $N(n_0)$ as well in a proof of the lower bound of $N(n_0)$ just mentioned.

Definition 3.13. A *k-cup* is a set $\{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\} \subset \mathbb{R}^2$ with $x_1 < x_2 < \cdots < x_k$ and

$$\frac{y_1 - y_2}{x_1 - x_2} < \frac{y_2 - y_3}{x_2 - x_3} < \cdots < \frac{y_{k-1} - y_k}{x_{k-1} - x_k}. \tag{3.3.5}$$

Analogously, a *k-cap* is a set $\{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\} \subset \mathbb{R}^2$ with $x_1 < x_2 < \cdots < x_k$ and

$$\frac{y_1 - y_2}{x_1 - x_2} > \frac{y_2 - y_3}{x_2 - x_3} > \cdots > \frac{y_{k-1} - y_k}{x_{k-1} - x_k}. \tag{3.3.6}$$

In order to make sense of cups and caps in a subset X of $\mathbb{R}^2$ we have to require that the points in X have different x-coordinates, that is, no line through any two points of X is parallel to the y-axis. This motivates the following definition.

Definition 3.14. A set X in $\mathbb{R}^2$ is called *admissible* if no three points of X are on a line and if no line through any two points of X is parallel to the y-axis.

Any finite set X of which no three points are on a line can be made admissible by a suitable rotation. Indeed, there are finitely many lines going through any two points of X, and after a suitable rotation around the origin, we can achieve that none of these lines is parallel to the y-axis.

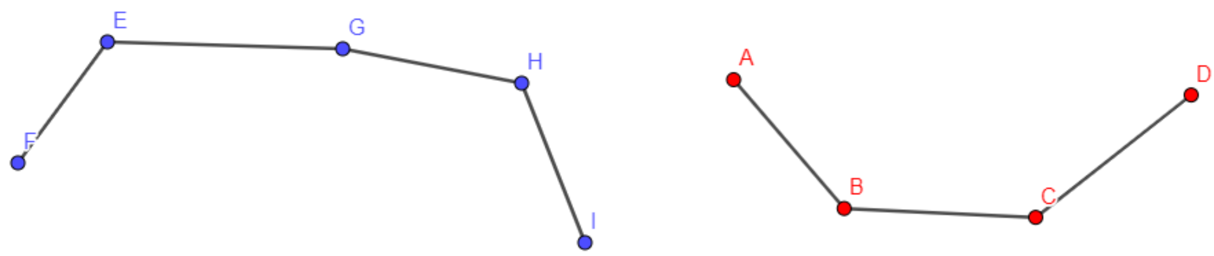

Figure 6: Examples of a 5-cap and a 4-cup

Lemma 3.15. *For any positive integers $k, l \geq 2$, there exists a positive integer F such that for all $n \geq F$ the following holds:*
Every admissible set N in $\mathbb{R}^2$ with n points contains a k-cup or a l-cap.

Definition 3.16. The smallest integer F for which the above assertion is satisfied is denoted by *f(k,l)*.

Proof of Lemma 3.15. It follows from Theorem 3.17 given below, which gives the precise value of $f(k, l)$.

$\square$

Theorem 3.17. *([3]) For any two integers $k, l \geq 3$, we have*

$$f(k,l) = \binom{k+l-4}{k-2} + 1. \tag{3.3.7}$$

Proof. (See [10]). First we prove

$$f(k,l) \leq \binom{k+l-4}{k-2} + 1. \tag{3.3.8}$$

We prove this by induction on $k+l$. As $f(k,3) = k$ we can assume $k, l > 3$. We now show that $f(k,l) \leq f(k-1,l) + f(k,l-1) - 1$.

Take an admissible set $X = \{(x_1, y_1), \ldots, (x_r, y_r)\} \subset \mathbb{R}^2$ with $r := f(k-1,l) + f(k,l-1) + 1$ points where $x_1 < \cdots < x_r$. Define Y to be the subset of X consisting of the left endpoints of the $(k-1)$-cups of X. Now if $X \backslash Y$ contains $f(k-1,l)$ points then it contains a l-cap as it cannot contain a $(k-1)$-cup which again means that we are done. So Y must contain at least $f(k,l-1)$ points. Hence it contains a k-cup or a $(l-1)$-cap. If it contains a k-cup we are done, so we assume it contains a $(l-1)$-cap, $\{(x_{i_1}, y_{i_1}), (x_{i_2}, y_{i_2}), \ldots, (x_{i_{l-1}}, y_{i_{l-1}})\}$ say, with $x_{i_1} < x_{i_2} < \cdots < x_{i_{l-1}}$. As these elements are also left endpoints of $(k-1)$-cups in X, we conclude that there exists a $(k-1)$-cup $\{(x_{i_{l-1}}, y_{i_{l-1}}), (x_{j_2}, y_{j_2}), \ldots, (x_{j_{k-1}}, y_{j_{k-1}})\}$ with $x_{i_{l-1}} < x_{j_2} < \cdots < x_{j_{k-1}}$. So it can be easily seen that either our $(k-1)$-cup can be extended in X to a k-cup by adding the point $(x_{i_{l-2}}, y_{i_{l-2}})$ or our $(l-1)$-cap can be extended to a l-cap by adding the point (x_{j_2}, y_{j_2}) which means that X contains a k-cup or a l-cap.
So $f(k,l) \leq f(k-1,l) + f(k,l-1) - 1$, which by the induction hypothesis is at most $\binom{k+l-5}{k-3} + \binom{k+l-5}{k-2} + 1 = \binom{k+l-4}{k-2} + 1$.

We now prove

$$f(k,l) \geq \binom{k+l-4}{k-2} + 1. \tag{3.3.9}$$

We construct a set of $\binom{k+l-4}{k-2}$ points in $\mathbb{R}^2$ with no k-cup or l-cap.
For $k = l = 3$ we already proved (3.3.9). We proceed by induction. Assume (3.3.9) holds for all integers l, k with $k, l \geq 3$ and $l + k < N_0$ and consider $l + k = N_0$.
Take an admissible set A with $\binom{k+l-5}{k-3}$ points. By our induction hypothesis A does not contain a $(k-1)$-cup or a l-cap. Take another admissible set B with $\binom{k+l-5}{k-2}$ elements. Again, by our induction hypothesis B does not contain a k-cup or a $(l-1)$-cap. We assume the following. The x-coordinates of the points of B are all larger than the x-coordinates of A, and the slopes of the line segments connecting a point from A to a point from B are all larger than the slopes of the line segments connecting any two points of A or any two points of B. This can be achieved by translating B sufficiently far to the right and then sufficiently upwards.
Now consider $X := A \bigcup B$. Any cup of X that contains elements of A and B has at most one element of B by our second condition. Also, any cap of X that contains elements of both A and B contains at most one element of A.
So as A does not have a $(k-1)$-cup, X does not have a k-cup. Analogously, as B does not have a $(l-1)$-cap, X does not have a l-cap. It follows that

$$f(k,l) \geq \binom{k+l-5}{k-3} + \binom{k+l-5}{k-2} + 1 = \binom{k+l-4}{k-2} + 1. \tag{3.3.10}$$

□

Theorem 3.17 gives the following inequality:

Corollary 3.18. *For any integer $n_0 \geq 3$, we have*

$$N(n_0) \leq \binom{2n_0-4}{n_0-2} + 1. \tag{3.3.11}$$

Proof. If a set contains a n_0-cup or a n_0-cap it contains always n_0 points in convex position. So $N(n_0) \leq f(n_0, n_0) = \binom{2n_0-4}{n_0-2} + 1$. □

Now we have collected enough tools for the proof of our final and important Theorem, which gives the lower bound in Conjecture 3.3.4.

Theorem 3.19. *([4] and [7]) Let n_0 be a positive integer ≥ 3. Then*

$$N(n_0) \geq 2^{n_0-2} + 1. \tag{3.3.12}$$

Proof. (See [10]). For this proof we construct an admissible set X in $\mathbb{R}^2$ of 2^{n_0-2} points without n_0 points in convex position. First consider for $i = 0, 1, \ldots, n_0 - 2$ an admissible set T_i in $\mathbb{R}^2$ of $\binom{n_0-2}{i}$ points without a $(i+2)$-cap or a (n_0-i)-cup. Theorem 3.17 tells us that such a set T_i exists since $|T_i| < f(n_0 - i, i + 2)$.
Now we apply various linear transformations to T_i, which have no effect on the lengths of the caps and the cups in T_i. First, by scaling the y-coordinates, we make it so that the slopes of the lines through any two points of T_i have absolute value < 1. Then, by shrinking and then translating T_i, we arrange that T_i is contained in a circle with radius ϵ and with center $(\cos(\frac{\pi}{4} - \frac{\pi i}{4(n_0-2)}), \sin(\frac{\pi}{4} - \frac{\pi i}{4(n_0-2)}))$, where ϵ will be chosen sufficiently small, see Figure 7. Let

$X = \bigcup_{i=0}^{n_0-2} T_i$. In fact we may choose ϵ such that:
(1) X is admissible;
(2) the sets T_i are pairwise disjoint;
(3) the slope of a line through a point of T_i and a point of $T_j, i \neq j$ is < -1.
(4) if $h < i < j$, then T_i lies on the right of any line through a point of T_h and a point of T_j.
In addition to (1)-(4) we recall
(5) the slope of a line through any two points of the same set T_i has absolute value < 1;
(6) T_i does not contain a $(i+2)$-cap or a (n_0-k)-cup, for $i = 0, \ldots, n_0-2$.

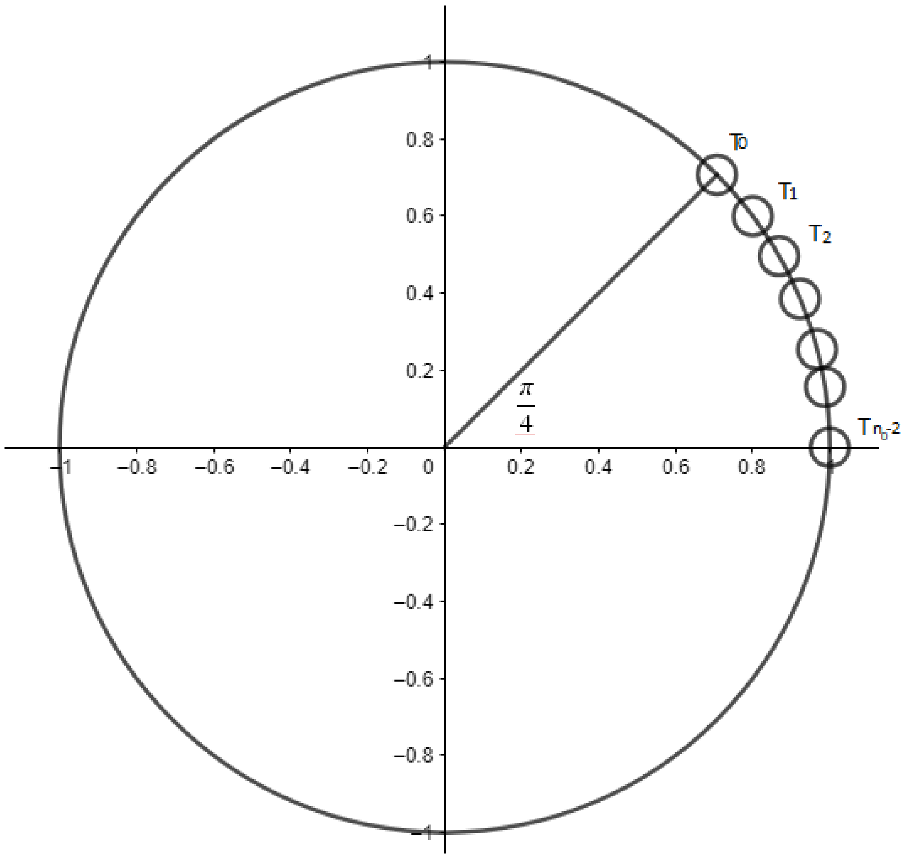

Figure 7: All T_is translated

Notice that because all the T_is are disjoint we have $|X| = \sum_{i=0}^{n_0-2} \binom{n_0-2}{i} = 2^{n_0-2}$.

Now we have sufficiently adjusted our set X to prove that X does not contain n_0 points in convex position.
Let Y be a subset of X in convex position with a maximal number of elements. Since X is admissible, Y is the union of a cap and a cup. We may write $Y = \{n_1, \ldots, n_r, n_{r+1}, \ldots, n_{r+s}\}$ where $n_1, \ldots, n_{r+s}$ are arranged clockwise, $n_1, \ldots, n_r$ is a cap and $n_1, n_{r+s}, \ldots, n_{r+1}, n_r$ is a cup, see Figure 8.

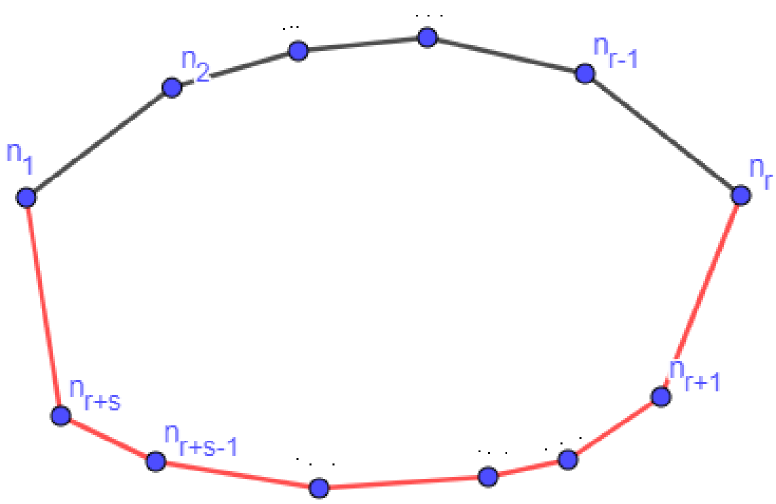

Figure 8: Union of a black cap and a red cup

Let k be the smallest value $i \in \{1, \dots, n_0 - 2\}$ such that $Y \bigcap T_i \neq \emptyset$ and l the largest value $i \in \{1, \dots, n_0 - 2\}$ such that $Y \bigcap T_i \neq \emptyset$.
Now if $k = l$, then $Y \subset T_k$, and by our choice of T_k, we have $r \leq n_0 - k - 1$ and $s + 2 \leq k + 1$. So $|Y| \leq r + s \leq n_0 - 2$.
Next assume $l > k$. We claim the following statements:
(a) $|Y \bigcap T_k| \leq k + 1$;
(b) $|Y \bigcap T_i| \leq 1$ for $i = k + 1, k + 2, \dots, l - 1$;
(c) $|Y \bigcap T_l| \leq n_0 - l - 1$.
From (3),(5) it follows that $Y \bigcap T_k$ is a cap and $Y \bigcap T_l$ is a cup, and then from (6), statements (a) and (c) follow. Statement (b) holds because if not, then then there exists a $i \in \{k+1, \dots, l-1\}$ such that $|Y \bigcap T_i| \geq 2$.

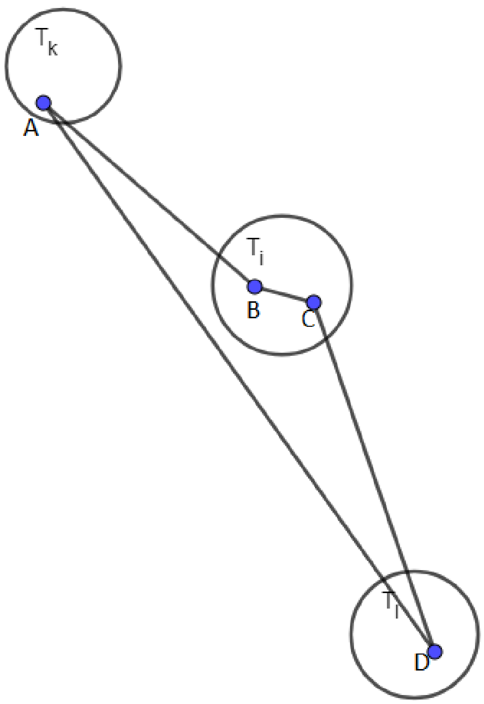

Figure 9: Example when $|Y \bigcap T_i| \geq 2$

Consider four points $A, B, C, D \in Y$ with $A \in T_k, B, C \in T_i$, and $D \in T_l$, where $k < i < l$. By (3), the slopes of the line segments AB, AC, BD, CD are all < -1, while by (5), the slope of BC is > -1. But then A, B, C, D are not in convex position, contradicting that Y is in convex position, see Figure 9. So $|Y \bigcap T_i| \leq 1$ for $i \in \{k+1, \ldots, l-1\}$.

Now combining (a)-(c) we find $|Y| \leq (k+1) + (l-k-1) + (n_0 - l - 1) = n_0 - 1$.
So Y can not contain n_0 points in convex position which finishes the proof. □

As said before, an upper bound of this conjecture has not been proved yet. The best known upper bound for the Erdős and Szekeres number has been obtained in 2016 by Andrew Suk.

Theorem 3.20. *([14]) For $n_0 \geq 3$ we have*

$$N(n_0) \leq 2^{n_0 + 6n_0^{2/3} \log(n_0)}. \tag{3.3.13}$$

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