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Non-measurable sets

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Contents

1 Introduction

H. Lebesgue's desire to assign a measure to every set of reals in a coherent way has been proven time and time again to be too formidable. Non-measurable sets were proven to exist soon after his 1902 paper *Intégrale, Longueur, Aire*, [Lebesgue1902] appeared. To date, many sets defying the Lebesgue measure have been found, many originating from very diverse fields of mathematics. In this text we will cover the non-measurable sets of G. Vitali, W. Sierpiński, E.B. Van Vleck and F. Bernstein, which are all constructed in a very different manner.

2 The Lebesgue measure and the Axiom of Choice

We will start with a very short summary of the most important ideas that we need to construct our sets. Most statements will be given without proof; for more information, see [Cohn2013] and [Hart2015a]. Throughout the text, we will use the Euclidean topology on $\mathbb{R}$.

Definition 1. *The Lebesgue outer measure is the function $\lambda^* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ where $\lambda^*(A)$ equals*

$$\inf \left\{ \sum_{n=0}^{\infty} (b_n - a_n) : \{(a_n, b_n)\}_{n \in \mathbb{N}} \text{ such that } A \subseteq \bigcup_{n \in \mathbb{N}} (a_n, b_n) \right\}.$$

Definition 2. *A subset A of $\mathbb{R}$ is called Lebesgue measurable if for every $E \subseteq \mathbb{R}$ we have that $\lambda^*(E) = \lambda^*(A \cap E) + \lambda^*(A^c \cap E)$.*

For abbreviation, we will regularly write *measurable* instead of Lebesgue measurable. It is clear that $\lambda^*(\varnothing) = 0$, and therefore $\mathbb{R}$ is measurable. As the definition is symmetric we know that the complement of a measurable set is also measurable. Furthermore, for any countable sequence of measurable sets its union and its intersection are again measurable. These three properties make the set of Lebesgue measurable sets $\mathcal{M}$ into a *σ-algebra* on $\mathbb{R}$. The set that contains every open subset of $\mathbb{R}$ and their complements and is closed under countable intersections and unions of its elements is a σ-algebra on $\mathbb{R}$ and is denoted by $\mathcal{B}(\mathbb{R})$. As it turns out, it is also the *minimal σ-algebra* generated by the open intervals of $\mathbb{R}$ and therefore $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{M}$.

The *Lebesgue measure* λ is the function $\lambda : \mathcal{M} \to [0, \infty]$ which satisfies $\lambda(A) = \lambda^*(A)$ for all $A \in \mathcal{M}$. Notice that for $A, B \in \mathcal{M}$ with $A \subseteq B$ we have $\lambda(A) \leq \lambda(B)$. Again, λ will generally be referred to as simply *measure*. This measure is countably additive; for every sequence $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{M}$ of pairwise disjoint sets we have $\lambda(\bigcup_{n=0}^{\infty} A_n) = \sum_{n=0}^{\infty} \lambda(A_n)$. Elements of $\mathcal{M}$ that are translations of each other or symmetric with respect to some point, have the same measure[1].

Finally, we will make great use of the Axiom of Choice:

$$(\forall X)((\varnothing \notin X) \to (\exists f : X \longrightarrow \bigcup X)(\forall A \in X)(f(A) \in A)).$$

This axiom is equivalent to the *well-ordering theorem* and *Zorn's lemma.*

[1] H. Lebesgue wanted to construct a measure on $\mathcal{P}(\mathbb{R})$ that satisfied three properties mentioned above. Namely; countable additivity, measure invariance under congruent sets and at least one set has positive measure. As we see above, his measure at least satisfies these properties for elements of $\mathcal{M}$.

3 Non-measurable Sets

Our first set is the oldest of the selection and also the most straightforward. It was constructed by G. Vitali, [Vitali1905] in 1905.

3.1 G. Vitali

Consider the set $\mathcal{A} = \{x + \mathbb{Q} : x \in \mathbb{R}\}$. As every element is a coset of $\mathbb{Q}$ in the group $(\mathbb{R}, +)$, we know that $\mathbb{R}$ is the disjoint union of elements of $\mathcal{A}$, see [Stevenhagen2016a]. Now let f be a Choice Function on $\{\alpha \cap (0, \frac{1}{2}) : \alpha \in \mathcal{A}\}$ and denote the domain of f by $\mathcal{H}$. Define for every rational number q the set $R_q = \{q + f(\alpha) : \alpha \in \mathcal{H}\}$. Suppose there is an $x \in R_q \cap R_p$ for some rationals p and q. Then $q + f(\alpha) = x = p + f(\beta)$ for some $\alpha, \beta \in H$, which would mean that $f(\alpha) - f(\beta)$ is rational. As this would imply that p is a rational translation of q, we can draw the following series of conclusions; $p + \mathbb{Q} = q + \mathbb{Q}$, hence $f(\alpha) = f(\beta)$ and finally, $p = q$. So for distinct $r, l \in \mathbb{Q}$ we have $R_r \cap R_l = \varnothing$.

Suppose R_0 is measurable. Then for every rational q the set R_q should have the same measure as R_0, by the invariance of λ under translations. Consider the countable set $\mathcal{C} := \{R_{\frac{1}{n}} : n \in \mathbb{N}_{>1}\} \cup \{R_0\}$ of disjoint sets and notice that every element is contained in the interval $(0, 1)$, so therefore the measure of its union should be at most 1. Now,

$$\lambda\left(\bigcup \mathcal{C}\right) = \lambda(R_0) + \sum_{n \in \mathbb{N} \setminus \{0\}} \lambda(R_{\frac{1}{n}}) = \lim_{n \to \infty} n \cdot \lambda(R_0)$$

so therefore $\lambda(R_0) = 0$. Consequently, for every rational q we know $\lambda(R_q) = 0$ and hence $\lambda(\bigsqcup_{q \in \mathbb{Q}} R_q) = 0$. However, $\bigsqcup_{q \in \mathbb{Q}} R_q = \mathbb{R}$ so therefore $\lambda(\bigsqcup_{q \in \mathbb{Q}} R_q) = \infty$ and hence we have arrived at a contradiction. We conclude that R_0 is not measurable.

3.2 W. Sierpiński

From now on, each set calls for some preliminaries before the actual construction can start. Sierpiński's non-measurable set requires knowledge about ultrafilters, [Mendelson2015]. Curiously enough, the word *ultrafilter* does not appear in Sierpiński's article, [Sierpiński1938]. Instead, an article of A. Tarski [Tarski1930] is referenced where the Axiom of Choice is used to show that the function f with the desired property of Proposition 6 exists. Implicitly, Tarski constructed an ultrafilter, though these were not introduced until 1937, by H. Cartan [Cartan1937a], [Cartan1937b].

3.2.1 Preliminaries

Definition 3. *A filter on a nonempty set A is a proper subset $\mathcal{F}$ of $\mathcal{P}(A)$ such that*

1. $A \in \mathcal{F}$
2. $(B \in \mathcal{F} \wedge C \in \mathcal{F}) \to (B \cap C \in \mathcal{F})$
3. $(B \in \mathcal{F} \wedge B \subseteq C \wedge C \subseteq A) \to (C \in \mathcal{F})$

Notice that for any nonempty $B \subseteq A$ the set $\mathcal{F}_B := \{C : B \subseteq C \subseteq A\}$ is a filter on A. Filters of this form are called *principal.* Property 3 guarantees that no filter can contain $\varnothing$, as filters are strict subsets of some power set.

Definition 4. *An ultrafilter $\mathcal{U}$ on a set A is a filter on A such that there is no filter $\mathcal{F}$ on A for which $\mathcal{U} \subsetneq \mathcal{F}$ holds.*

An important question is whether ultrafilters exist for any set. Luckily, the matter can be quickly resolved by Zorn's lemma [Hart2015a].

Proposition 1 (Ultrafilter Theorem)**.** *Every filter $\mathcal{F}$ on a set A can be extended to an ultrafilter on A*

Proof. Consider $X := \{\mathcal{G} : \mathcal{G} \text{ a filter on } A \text{ and } \mathcal{F} \subseteq \mathcal{G}\}$. Then X is partially ordered by $\subseteq$ and every $\subseteq$-chain has an upper bound in X, namely the union of the $\subseteq$-chain (for detailed proof, see [Mendelson2015]). Zorn's lemma now guarantees that there is a maximal element $\mathcal{U}$ in X, which is our required ultrafilter. $\square$

Proposition 2. *Let $\mathcal{F}$ be any filter on a nonempty set A. Assume there exists $B \subseteq A$ such that $A \setminus B \notin \mathcal{F}$. Then there is a filter $\mathcal{F}'$ on A such that $B \in \mathcal{F}'$ and $\mathcal{F} \subseteq \mathcal{F}'$.*

Proof. We will show that $\mathcal{F}' := \{H : H \subseteq A \wedge (\exists D)(D \in \mathcal{F} \wedge D \cap B \subseteq H)\}$ is the filter in question. Clearly, $B \in \mathcal{F}'$.

1. As $A \in \mathcal{F}$, it is clear that $A \in \mathcal{F}'$.

2. Let I and J be two elements of $\mathcal{F}'$ and notice that clearly $I \cap J \subseteq A$. Then there exist $\overline{I}, \overline{J} \in \mathcal{F}$ such that $\overline{I} \cap B \subseteq I$ and $\overline{J} \cap B \subseteq J$. By property 2 of filters, $\overline{I} \cap \overline{J} \in \mathcal{F}$ and this set satisfies $(\overline{I} \cap \overline{J}) \cap B \subseteq I \cap J$. Hence $I \cap J \in \mathcal{F}'$.

3. Let I and J be sets such that $I \in \mathcal{F}'$, $I \subseteq J$ and $J \subseteq A$. Then there exists a $\overline{I} \in \mathcal{F}$ such that $\overline{I} \cap B \subseteq I$. As I is a subset of J we have that $J \in \mathcal{F}'$.

Finally, property 3 of filters guarantees that for every $D \in \mathcal{F}$ holds $D \nsubseteq A \setminus B$ and hence $D \cap B \neq \varnothing$. For that reason $\mathcal{F}'$ does not contain $\varnothing$ and therefore is a filter. By property 2 of filters we now have $\mathcal{F} \subseteq \mathcal{F}'$. □

Proposition 3. *Let $\mathcal{F}$ be any filter on a nonempty set A. Then the following are equivalent.*

1. *$\mathcal{F}$ is an ultrafilter.*

2. *$(\forall B \subseteq A)(B \in \mathcal{F} \vee A \setminus B \in \mathcal{F})$.*

3. *$(\forall B, C \subseteq A)(B \notin \mathcal{F} \wedge C \notin \mathcal{F} \to B \cup C \notin \mathcal{F})$.*

Proof. We start by proving that the first and second statement are equivalent. Assume that for every $B \subseteq A$ either $B \in \mathcal{F}$ or $A \setminus B \in \mathcal{F}$. Assume there is a filter $\mathcal{F}'$ on A such that $\mathcal{F} \subsetneq \mathcal{F}'$. Let $B \in \mathcal{F}' \setminus \mathcal{F}$, then we have $A \setminus B \in \mathcal{F}$. Hence we also have $A \setminus B \in \mathcal{F}'$ and therefore by property 2 of Definition 3, $\varnothing = B \cap (A \setminus B) \in \mathcal{F}'$. Contradiction, so $\mathcal{F}$ is an ultrafilter.
If we apply Proposition 2 we can conclude the other implication.

Now we will prove that the first and third statement are equivalent. Assume that $\mathcal{F}$ is an ultrafilter and $B, C \notin \mathcal{F}$. By the first part of our proof we have $A \setminus B, A \setminus C \in \mathcal{F}$ and thus $(A \setminus B) \cap (A \setminus C) = A \setminus (B \cup C) \in \mathcal{F}$. As $\varnothing \notin \mathcal{F}$ we have $B \cup C \notin \mathcal{F}$, due to property 2 of Definition 3.
Now assume for every $B, C \subseteq A$ we have $B \notin \mathcal{F} \wedge C \notin \mathcal{F} \to B \cup C \notin \mathcal{F}$. For any $B \subseteq A$, we have $B \cup (A \setminus B) = A \in \mathcal{F}$ and therefore $B \in \mathcal{F} \vee A \setminus B \in \mathcal{F}$ and hence, by what we proved above, $\mathcal{F}$ is an ultrafilter. □

Proposition 4. *A nonprincipal ultrafilter on a set A contains no finite sets.*

Proof. First we will show that a subset of $\mathcal{P}(A)$ is a principal ultrafilter on A if and only if it is of the form $\mathcal{F}_{\{d\}}$ for some $d \in A$. Suppose that a $\mathcal{F} = \mathcal{F}_{\{d\}}$ for some $d \in A$. By definition of $\mathcal{F}_{\{d\}}$ it holds that for every subset of A either itself or its complement is in $\mathcal{F}_{\{d\}}$, but not both. By Proposition 3 it follows that $\mathcal{F}$ is an ultrafilter. Now we assume that $\mathcal{F}_B$, for some nonempty $B \subseteq A$, is a principal ultrafilter on A. Let $b \in B$ and assume $B \neq \{b\}$, then, by definition of principal filters, we have $\{b\} \notin \mathcal{F}_B$ and $B \setminus \{b\} \notin \mathcal{F}_B$. By Proposition 3 we now have $\{b\} \cup B \setminus \{b\} = B \notin \mathcal{F}_B$. Contradiction, so $B = \{b\}$.

Suppose there is a nonprincipal ultrafilter $\mathcal{U}$ that contains a finite set and let C be an element of least cardinality. By what we proved above we know that $|C| > 1$. Let $c \in C$. As both $C \setminus \{c\}$ and $\{c\}$ have lower cardinality than C we have $C \setminus \{c\} \notin \mathcal{U}$ and $\{c\} \notin \mathcal{U}$. Therefore, by Proposition 3, we conclude $\{c\} \cup C \setminus \{c\} = C \notin \mathcal{U}$. Contradiction. □

Proposition 5. *If A is an infinite set, then there exists a nonprincipal ultrafilter on A.*

Proof. We will show that the subset $\mathcal{F}$ of $\mathcal{P}(A)$ that consists of all the complements of finite subsets of A is a nonprincipal filter on A.

1. $A = A \setminus \varnothing \in \mathcal{F}$,

2. Let $B = A \setminus F_1$ and $C = A \setminus F_2$ where F_1 and F_2 are finite subsets of A. We have $B \cap C = A \setminus (F_1 \cup F_2) \in \mathcal{F}$, as $F_1 \cup F_2$ is again finite.

3. Let $B = A \setminus F$ where F is a finite subset of A. For subsets $C \subseteq A$ which contains B it holds that $C = A \setminus (F \setminus C) \in \mathcal{F}$, as $F \setminus C$ is finite.

4. Suppose $\mathcal{F} = \mathcal{F}_B$ for some nonempty $B \subseteq A$. Then $B = A \setminus F$ for some finite subset $F \subseteq A$. Let $b \in B$, then $F \cup \{b\}$ is a finite subset of A. Hence we have $C := A \setminus (F \cup \{b\}) \in \mathcal{F}$. But then $B \not\subseteq C$ as $b \notin C$. Therefore, $\mathcal{F} \neq \mathcal{F}_B$.

By Proposition 1 there is an ultrafilter $\mathcal{U}$ on A such that $\mathcal{F} \subseteq \mathcal{U}$. Suppose $\mathcal{U}$ is principal. As we have seen in the previous proof, $\mathcal{U} = \mathcal{F}_d$ for some $d \in A$. By definition of $\mathcal{F}$: $A \setminus \{d\} \in \mathcal{F} \subseteq \mathcal{U}$. As we also have that $\{d\} \in \mathcal{U}$ we know that $\mathcal{U}$ is not a filter, because $(A \setminus \{d\}) \cap \{d\} = \varnothing \in \mathcal{U}$. Contradiction, so $\mathcal{U}$ is nonprincipal. □

3.2.2 Construction

Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{Z}_+ = \{1, 2, ...\}$. We define the function

$$f: \quad \mathcal{P}(\mathbb{Z}_+) \longrightarrow \{0,1\}$$
$$B \mapsto \begin{cases} 1, & B \in \mathcal{U}, \\ 0, & B^c \in \mathcal{U}. \end{cases}$$

Proposition 6. *f is finitely additive;*

$$(\forall A, B \in \mathcal{P}(\mathbb{Z}_+))(A \cap B = \varnothing \rightarrow f(A) + f(B) = f(A \cup B)).$$

Proof. Let A and B be two disjoint subsets of $\mathbb{Z}_+$. Property 2 of filters guarantees that A and B cannot both be elements of $\mathcal{U}$ which means that $f(A)+f(B)$ is at most 1. If $A \in \mathcal{U}$ or $B \in \mathcal{U}$ then so is $A \cup B$ because $A, B \subseteq A \cup B \subseteq \mathbb{Z}_+$ and if both A and B are not in $\mathcal{U}$ then $A^c, B^c \in \mathcal{U}$ and hence $A^c \cap B^c \in \mathcal{U}$ which means that $(A^c \cap B^c)^c = A \cup B \notin \mathcal{U}$. So, $f(A) + f(B) = f(A \cup B)$. $\square$

Notice that f is not countably additive. By this we mean that there is a pairwise disjoint sequence in $\mathcal{P}(\mathbb{Z}_+)$ such that the sum of the individual function values does not equal the function value of their union. Namely, the sequence $\{\{n\}\}_{n \in \mathbb{Z}_+} \subseteq \mathcal{P}(\mathbb{Z}_+)$ does the trick: $\sum_{i=1}^{\infty} f(\{i\}) = 0 \neq 1 = f(\mathbb{Z}_+) = f(\bigcup_{i=1}^{\infty} \{i\})$ by Proposition 4 and the fact that $\mathbb{Z}_+ \in \mathcal{U}$.

Let $\mathbb{Z}\left[\frac{1}{2}\right]$ denote the ring generated by $\mathbb{Z}$ and $\frac{1}{2}$. Elements of this set are called *dyadic rationals.*

Theorem 1. *Every $x \in \mathbb{R}$ has an unique representation*

$$x = G(x) + \sum_{k=1}^{\infty} \frac{b_k(x)}{2^k} \tag{1}$$

where $G(x)$ is the greatest integer smaller than x, $b_k(x) \in \{0,1\}$ and $B_x := \{k \in \mathbb{Z}_+ : b_k(x) = 1\}$ is infinite. Every $x \in \mathbb{Z}\left[\frac{1}{2}\right]$ has two representations of the form (1); one where B_x is finite and another where it is infinite.

Proof. Let $x \in \mathbb{R}_{>0}$ and $n \in \mathbb{N}$. By splitting the real line up in intervals of length 2^{-n} we can see that there exists a unique $a_n^x \in \mathbb{N}$ such that

$$a_n^x 2^{-n} \leq x < (a_n^x + 1)2^{-n}.$$

We get a sequence $\{a_n^x 2^{-n}\}_{n \in \mathbb{N}}$ that satisfies for every $n \in \mathbb{N}$:

$$a_n^x 2^{-n} \leq a_{n+1}^x 2^{-(n+1)} < (a_{n+1}^x + 1)2^{-(n+1)} \leq (a_n^x + 1)2^{-n}$$

and therefore $2a_n^x \leq a_{n+1}^x \leq 2a_n^x + 1$. This gives us, as the three numbers in the last inequality are whole, that $a_{n+1}^x - 2a_n^x \in \{0,1\}$ and therefore $a_{n+1}^x 2^{-(n+1)} - a_n^x 2^{-n} \in \{0, 2^{-(n+1)}\}$. As $0 \leq x - a_n^x 2^{-n} < 2^{-n}$ we have $\lim_{n \to \infty} a_n^x 2^{-n} = x$. Moreover, the sequence $\{a_n^x 2^{-n}\}_{n \in \mathbb{N}}$ is not strictly increasing, as that would mean that x is equal to $a_0^x + 1$, which is contradictory as $a_0^x \leq x < a_0^x + 1$. Now we have a sequence $\{b_k(x)\}_{k \in \mathbb{Z}_+}$ where $b_k(x) = a_k^x - 2a_{k-1}^x$.

Assume $x \notin \mathbb{Z}\left[\frac{1}{2}\right]$, then $G(x) = \lfloor x \rfloor$. For every $n \in \mathbb{N}$ we have

$$\lfloor x \rfloor + \sum_{k=1}^{n} b_k(x)2^{-k} = \lfloor x \rfloor + (-a_0^x) + a_n^x 2^{-n} = a_n^x 2^{-n}.$$

Taking the limit on both sides we get $\lfloor x \rfloor + \sum_{k=1}^{\infty} b_k(x)2^{-k} = x$. As $x \notin \mathbb{Z}\left[\frac{1}{2}\right]$ we know that the sequence $\{a_n^x 2^{-n}\}_{n\in\mathbb{N}}$ never becomes constant. For this reason, as we have seen before that $\{a_n^x 2^{-n}\}_{n\in\mathbb{N}}$ is also not strictly increasing, $\{b_k(x)\}_{k\in\mathbb{Z}_+}$ does not converge and hence the sets B_x and B_x^c are both infinite. Suppose there was a binary sequence $\{c_k(x)\}_{k\in\mathbb{Z}_+}$, different from $\{b_k(x)\}_{k\in\mathbb{Z}_+}$, such that $\lfloor x \rfloor + \sum_{k=1}^{\infty} c_k(x)2^{-k} = x$. Let $m \in \mathbb{N}$ be the first index where $b_m(x)$ and $c_m(x)$ are different. Without loss of generality, we can assume that $b_m(x) = 1$ and $c_m(x) = 0$. As $\{c_k(x)\}_{k\in\mathbb{Z}_+}$ can also never become constant (due to fact $x \notin \mathbb{Z}\left[\frac{1}{2}\right]$), we have that $\sum_{k=1}^{\infty} b_k(x)2^{-k} - \sum_{k=1}^{\infty} c_k(x)2^{-k} = 2^{-m} + \sum_{k=m+1}^{\infty}(b_k(x) - c_k(x))2^{-k} > 2^{-m} - \sum_{k=m+1}^{\infty} 2^{-k} = 0$, contradiction. Therefore, this representation (1) of x is unique.

If we now take any $-y \in R_{<0} \setminus \mathbb{Z}\left[\frac{1}{2}\right]$, we know that y is uniquely represented in the form (1). Define $b_k(-y) = 1 - b_k(y)$ for all $k \in \mathbb{Z}_+$ and notice $\lfloor -y \rfloor = -(\lfloor y \rfloor + 1)$ and therefore $\lfloor y \rfloor = -\lfloor -y \rfloor - 1$. We have $y = \lfloor y \rfloor + \sum_{k=1}^{\infty} b_k(y)2^{-k}$ and hence $-y = \lfloor -y \rfloor + 1 - \sum_{k=1}^{\infty} b_k(y)2^{-k} = \lfloor -y \rfloor + 1 - \sum_{k=1}^{\infty}(1 - b_k(-y))2^{-k} = \lfloor -y \rfloor + \sum_{k=1}^{\infty} b_k(-y)2^{-k}$ and also $B_{-y} = B_y^c$ is infinite.

Now if $x \in \mathbb{Z}\left[\frac{1}{2}\right]_{>0}$ we know that B_x is finite, as there must be some $k \in \mathbb{Z}_+$ such that for every $n > k$ we have $x = a_k^x 2^{-k} = a_n^x 2^{-n}$. If we let k be the lowest value for which for every $n > k$ holds $b_n(x) = 0$, then we can define a new sequence $\{b_l(x)'\}_{l\in\mathbb{Z}_+}$ where $b_l(x)' = b_l(x)$ if $l < k$ and $b_l(x)' = 1 - b_l(x)$ if $l \geq k$. The new set B_x' will be infinite and

$$\sum_{n=k}^{\infty} b_n(x)'2^{-n} = 0 + 2^{-(k+1)} + 2^{-(k+2)} + \cdots = 2^{-(k+1)} \sum_{n=0}^{\infty} 2^{-n} = 2^{-k} = b_k(x)2^{-k}.$$

Therefore, $\sum_{n=1}^{\infty} b_n(x)'2^{-n} = \sum_{n=1}^{\infty} b_n(x)2^{-n}$ and every $x \in \mathbb{Z}\left[\frac{1}{2}\right]_{>0}$ has two representations; both look the same, as in equation (1), but one is a sum of infinitely many positive terms. Notice that if $x \in \mathbb{Z}$ we can take $B_x = \mathbb{Z}_+$. The elements left to check are of the form $-y \in \mathbb{Z}\left[\frac{1}{2}\right]_{<0}$. We know that y has a unique representation of the desired form and $\lfloor y \rfloor = -\lfloor -y \rfloor - 1$. The only diffculty is that $B_{-y} = B_y^c$ is finite. However, we can apply the trick that we used above to get an infinite B_{-y}' such that $\sum_{n=1}^{\infty} b_n(-y)'2^{-n} = \sum_{n=1}^{\infty} b_n(-y)2^{-n}$. Hence, every $x \in \mathbb{R}$ has a representation of the form (1). $\square$

Define

$$\begin{aligned} \varphi : \quad \mathbb{R} &\longrightarrow \{0,1\} \\ x &\mapsto f(B_x). \end{aligned}$$

Now define

$$S := \{x \in \mathbb{R} : \varphi(x) = 1\} \text{ and } S^c = \{x \in \mathbb{R} : \varphi(x) = 0\}.$$

Suppose S is measurable, then either S or S^c (or both) must have positive measure because $S \cup S^c = \mathbb{R}$. Let $x \in \mathbb{R}$ and let it be represented in form (1) where B_x is infinite. Define:

$$y = -G(x) + \sum_{k \in B_x^c} \frac{1}{2^k}$$

and notice that $x + y = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$, hence $y = 1 - x$. Notice that if $x \in \mathbb{R} \setminus \mathbb{Z}$ then $G(y) = \lfloor y \rfloor = \lfloor 1 - x \rfloor = 1 + \lfloor -x \rfloor = 1 + (-(\lfloor x \rfloor + 1)) = -\lfloor x \rfloor = -G(x)$. Furthermore, notice that B_x^c is finite precisely for $x \in \mathbb{Z}\left[\frac{1}{2}\right]$ represented in the form (1). Therefore, if $x \in \mathbb{R} \setminus \mathbb{Z}\left[\frac{1}{2}\right]$ we have $\varphi(1-x) = f(B_x^c)$ and because f is finitely additive and $B_x^c \cup B_x = \mathbb{Z}_+$, we get $f(B_x) + f(B_x^c) = 1$ and hence $\varphi(1-x) = 1 - \varphi(x)$. Therefore, S and S^c are, apart from countably many points, symmetric with respect to the point $\frac{1}{2}$. So they must both have positive measure.

Let $n2^{-m} \in \mathbb{Z}\left[\frac{1}{2}\right]$ and let $x \in \mathbb{R}$. As shown in the proof of Proposition 1, $n2^{-m}$ can be represented in the form (1) with $B_{n2^{-m}}$ finite. If we let x be represented in the normal form (1) we have that

$$x + n2^{-m} = G(x) + \sum_{k=1}^{\infty} b_k(x)2^{-k} + G(n2^{-m}) + \sum_{k=1}^{l} b_k(n2^{-m})2^{-k}$$

where $l = |B_{n2^{-m}}|$. Notice that this last expression can be put in the form (1) and $B_x \Delta B_{x+n2^{-m}} = (B_x \cup B_{x+n2^{-m}}) \setminus (B_x \cap B_{x+n2^{-m}})$ is finite. This means $\varphi(x) = f(B_x) = f(B_x \setminus (B_{x+n2^{-m}} \Delta B_x)) = f(B_{x+n2^{-m}} \setminus (B_{x+n2^{-m}} \Delta B_x)) = f(B_{x+n2^{-m}}) = \varphi(x + n2^{-m})$ because of Proposition 4 and 6.

We will use the following lemma to arrive at our contradiction.

Lemma 1. *Let $E \subseteq \mathbb{R}$ be measurable with positive measure. Then for every $0 < r < 1$ there is an open interval I such that*

$$\lambda(E \cap I) > r\lambda(I).$$

Proof. Let $\epsilon > 0$, then there is an open cover of intervals $\{I_n\}_{n \in \mathbb{N}}$ of E such that $\sum_{n=0}^{\infty} \lambda(I_n) < \lambda(E)(1+\epsilon)$. We can assume $\{I_n\}_{n \in \mathbb{N}}$ is pairwise disjoint as we can take the union of intervals that overlap. Now we have
$\lambda(E) = \sum_{n=0}^{\infty} \lambda(E \cap I_n) \leq \sum_{n=0}^{\infty} \lambda(I_n)$. As $\sum_{n=0}^{\infty} \lambda(I_n) < \lambda(E)(1+\epsilon)$, there is an $n \in \mathbb{N}$ such that $\lambda(I_n) < \lambda(E \cap I_n)(1+\epsilon)$. Thus, $\lambda(E \cap I_n) > \frac{1}{(1+\epsilon)}\lambda(I_n)$ and as ϵ was arbitrary we are done. □

Let $A = S \cap [0,1]$ and $n \in \mathbb{N}$. We can cover $[0,1]$ by intervals of the form $[k2^{-n}, (k+1)2^{-n}]$ where $k \in \{0, \ldots, 2^n - 1\}$ and notice

$$\lambda(A \cap [k2^{-n}, (k+1)2^{-n}]) = \lambda(A \cap [k'2^{-n}, (k'+1)2^{-n}])$$

for every $k, k' \in \{0, \ldots, 2^n - 1\}$ by the translation invariance of S under dyadic translations. Therefore $\lambda(A \cap [\frac{k}{2^n}, \frac{k+1}{2^n}]) = \frac{1}{2^n}\lambda(A)$ for every $k \in \{0, ..., 2^n - 1\}$. Let $(a,b) \subseteq [0,1]$. Let $\{e_n\}_{n\in\mathbb{N}}$, $\{f_n\}_{n\in\mathbb{N}} \subseteq (a,b)$ be two sequences of dyadic rationals such that $\lim_{n\to\infty} e_n = a$ and $\lim_{n\to\infty} f_n = b$. Existence of such sequences becomes clear if we use a similar construction to the one made at the beginning of the proof of Theorem 1. These two sequences give us a disjoint cover of (a,b) where every element is of the form $[k2^{-n}, (k+1)2^{-n})$. By the countable additivity of measures we now have $\lambda((a,b) \cap A) = (b-a)\lambda(A)$.

Let $0 < r < 1$. Using Lemma 1, there is an interval (c,d) such that

$$\lambda(A \cap (c,d)) > r(d-c).$$

By what we have just shown, we have $\lambda(A) > r$ and as r was arbitrary we conclude that $\lambda(A) = 1$. Analogously, we can show $\lambda(A^c) = 1$, which is clearly impossible. Therefore φ is not measurable, so we know that one of our sets is non-measurable and hence its complement too because the measurable sets form a σ-algebra.

3.3 E.B. Van Vleck

In 1908, E.B. Van Vleck described a construction of a non-measurable set, [Vleck1908]. Five years later, an alternative but similar construction was given by N.J. Lennes, [Lennes1913]. In this article a construction will be given that is different from the ones used by Van Vleck and Lennes. However, all three constructions make use of the same important idea; the notion of *homogeneity*. Intuitively, a subset of the unit interval is homogeneous if a copy of it can be found in any interval. Meaning; if the intersection of the set with an interval would be magnified to the unit interval, we would get our original set back. This 'magnifying' is achieved by the order preserving bijection $f : (a, b) \to (0, 1)$ where $x \mapsto \frac{x-a}{b-a}$ and (a, b) is any subinterval of $(0, 1)$.

3.3.1 Preliminaries

Definition 5. *A subset T of $(0, 1)$ is homogeneous if for every nonempty subinterval (a, b) of $(0, 1)$ and for every $\epsilon > 0$ there exists an interval $(c, d) \subseteq (a, b)$ such that $\lambda((a, b) \setminus (c, d)) < \epsilon$ and $M^T_{(c,d)} := \{\frac{x-c}{d-c} : x \in T \cap (c, d)\} = T$.*

Sets of this character have interesting properties. For one, the aforementioned order preserving bijection f dictates that the complement of any homogeneous set is also homogeneous. Another intriguing and important quality is the following.

Theorem 2. *For any measurable and homogeneous subset of $(0, 1)$ its measure is either one or zero.*

Proof. Suppose $\lambda(T) < 1$. Because T is measurable its measure and its outer measure must coincide. Therefore, there exists a cover $\mathcal{I} = \{(a_i, b_i) : i \in I\}$ of T in $(0, 1)$ such that $\lambda(\bigcup \mathcal{I}) = 1 - \sigma$ where $\sigma \in (0, 1]$. We can assume that this cover is disjoint as we can take the union of intervals that overlap. Let $\epsilon := \frac{1}{2}\sigma$. By the homogeneity of T we have that for every $i \in I$ there is an interval $(c_i, d_i) \subseteq (a_i, b_i)$ such that $\lambda((a_i, b_i) \setminus (c_i, d_i)) < \epsilon(b_i - a_i)$ and $M^T_{(c_i,d_i)} = T$. Notice that

$$\begin{aligned}
\lambda(T \cap (a_i, b_i)) &\leq (c_i - a_i) + (b_i - d_i) + (d_i - c_i)(1 - \sigma) \\
&< \epsilon(b_i - a_i) + (d_i - c_i)(1 - \sigma) \\
&< (b_i - a_i)(1 - \sigma + \epsilon) \\
&= (b_i - a_i)(1 - \frac{1}{2}\sigma).
\end{aligned}$$

This means $\lambda(T) < (1-\sigma)(1-\frac{1}{2}\sigma) < (1-\sigma)$, as the intervals of $\mathcal{I}$ are disjoint. Let $i \in I$, then using the inverse of the order preserving bijection $f_i : (c_i, d_i) \to (0, 1)$ we can create a cover $\mathcal{I}_i$ of $T \cap (c_i, d_i)$ which becomes the cover $\mathcal{I}$ by mere magnification. Namely, $\mathcal{I}_i = \{f_i^{-1}[(a_j, b_j)] : j \in I\} = \{(a'_{ij}, b'_{ij}) : j \in I\}$.

Let $(a'_{ij}, b'_{ij}) \in \mathcal{I}_i$, then there is a $(c'_{ij}, d'_{ij}) \subseteq (a'_{ij}, b'_{ij})$ such that

$$\lambda((a'_{ij}, b'_{ij}) \setminus (c'_{ij}, d'_{ij})) < \epsilon(b'_{ij} - a'_{ij})$$

and $M^T_{(c'_{ij}, d'_{ij})} = T$. Just as before we have $\lambda(T \cap (a'_{ij}, b'_{ij})) < (b'_{ij} - a'_{ij})(1 - \frac{1}{2}\sigma)$ and therefore $\lambda(T) < (1-\sigma)(1-\frac{1}{2}\sigma)(1-\frac{1}{2}\sigma) = (1-\sigma)(1-\frac{1}{2}\sigma)^2$. This process can be carried out *ad infinitum*, each step producing a smaller positive upper bound for $\lambda(T)$, hence we can conclude that $\lambda(T) = 0$. $\square$

3.3.2 Construction

Denote by I the set of irrational numbers in $(0,1)$ and define for every $x \in I$ the sets $B_x := \{rx + s : r \in \mathbb{Q}_{>0}\ ,\ s \in \mathbb{Q}\}$, $C_x := \{rx + s : r \in \mathbb{Q}_{<0}\ ,\ s \in \mathbb{Q}\}$ and $J_x := B_x \cup C_x$.

Proposition 7. *For every $x \in I$ the following hold.*

1. $B_x \cap C_x = \varnothing$,
2. $y \in J_x \to J_y = J_x$ *and*
3. $y \in B_x \Leftrightarrow 1 - y \in C_x$.

Proof. Suppose $B_x \cap C_x \neq \varnothing$ for some $x \in I$. If a is an element of this intersection we have $r_b x + s_b = a = r_c x + s_c$ for some $r_b \in \mathbb{Q}_{>0}$, $r_c \in \mathbb{Q}_{<0}$ and $s_b, s_c \in \mathbb{Q}$. However, this would mean $\mathbb{R} \setminus \mathbb{Q} \ni (r_b - r_c)x = s_c - s_b \in \mathbb{Q}$, which is absurd. Therefore, for every irrational x the sets B_x and C_x are disjoint. Now let $x \in I$ and $y \in J_x$. If $y \in C_x$ we have $y = rx + s$ for some $r \in \mathbb{Q}_{<0}$ and $s \in \mathbb{Q}$. After some computations we get the desired results;

1. $1 - y = -rx + (1 - s) \in B_x$ (because $-r$ is positive),
2. $C_y = \{yr' + s' : r' \in \mathbb{Q}_{<0}, s' \in \mathbb{Q}\} = \{rr'x + sr' + s' : r' \in \mathbb{Q}_{<0}, s' \in \mathbb{Q}\} = B_x$ and
3. $B_y = \{yr' + s' : r' \in \mathbb{Q}_{>0}, s' \in \mathbb{Q}\} = \{rr'x + sr' + s' : r' \in \mathbb{Q}_{>0}, s' \in \mathbb{Q}\} = C_x$.

So $J_y = J_x$. Analogously, in the case that $y \in B_x$, we also determine that $J_x = J_y$ and $1 - y \in C_x$. □

Now we define $M = \{J_x : x \in I\}$ and we use a Choice Function f on M to define $N = \{f(T) : T \in M\}$. Furthermore we define $K = \{B_x \cap (0,1) : x \in N\}$ and for the sake of notation we define $V = \bigcup K$ and $C = I \setminus V$. Take $y \in V$, then $y \in B_x$ for some $x \in N$. Hence, $1 - y \in C_x \subseteq J_x$ by Proposition 7. Now suppose $1 - y \in V$, then $1 - y \in B_z \subseteq J_z$ for some $z \in N$ and therefore $1 - y \in J_z \cap J_x$ which means $J_z = J_x$. As $x, z \in N$ we know $x = f(J_x) = f(J_z) = z$, but then $1 - y \in B_z \cap C_x = B_x \cap C_x = \varnothing$. Contradiction, so $1 - y \notin V$ which means $1 - y \in C$.

Now take $z \in C$, then $z \in B_z \subseteq J_z$. Notice that because $z \notin V$ we can draw the following conclusions: $a := f(J_z) \neq z$, so $J_a = J_z$ and $z \in C_a$. Hence $z = ra + s$ for some $r \in \mathbb{Q}_{<0}$ and $s \in \mathbb{Q}$. Therefore, $1 - z = -ra + (1 - s) \in V$.

So now we can conclude that the sets V and C are symmetric with respect to the point $\frac{1}{2}$. Let us assume that V is measurable, then we know that C must be measurable as well and that both sets should have same measure by their congruence.

Proposition 8. *The set V is homogeneous.*

Proof. Let $(a, b) \subseteq (0, 1)$ be any nonempty interval and let $\epsilon > 0$. Choose any rational $c \geq a$ and $d \leq b$ such that $\lambda((a, b) \setminus (c, d)) < \epsilon$ and $c \neq d$. Let $\frac{x-c}{d-c} \in M^V_{(c,d)}$. Then $x = gr + s$ for some $g \in N$ and rational r and s with $r > 0$. Then

$$\frac{x-c}{d-c} = \frac{gr+s-c}{d-c} = g\frac{r}{d-c} + \frac{s-c}{d-c} \in B_g$$

because $d - c > 0$. Also, $d - c > x - c$ so $0 < \frac{x-c}{d-c} < 1$ and hence $\frac{x-c}{d-c} \in V$. So $M^V_{(c,d)} \subseteq V$. Now let $h \in V$, then $h = ry + s$ for some $y \in N$ and rationals r and s with $r > 0$. Let $x := h(d - c) + c$. Assume $d \leq x = hd - hc + c$, then $d(h-1)+c(1-h) \geq 0$ and hence $c-d \geq 0$. Contradiction, so $x \in (c, d) \subseteq (0, 1)$. As $x = h(d - c) + c = (ry + s)(d - c) + c = (rd - rc)y + sd - sc + c \in B_y$ we conclude $x \in V \cap (c, d)$ and $h = \frac{x-c}{d-c} \in M^V_{(c,d)}$. So $V \subseteq M^V_{(c,d)}$. Now we have $V = M^V_{(c,d)}$ and therefore V is homogeneous. $\square$

Now we know that the measures of V and C are both either zero or one by Proposition 2. However, now we have $1 = \lambda(I) = \lambda(V) + \lambda(C) \in \{0, 2\}$. Contradiction, so the sets V and C are non-measurable.

3.4 F. Bernstein

Our last non-measurable set in our selection was constructed by F. Bernstein in 1908, [Bernstein1908]; it is a set with the property that it and its complement intersect every uncountable closed subset of $\mathbb{R}$. Such sets are now called Bernstein sets. In this section we follow the presentation of these results as given in [Ma2018a], [Ma2018b], [Ma2018c] and [Ma2018d].

3.4.1 Preliminaries

The set in this section uses quite a bit of set theory. Below is a (very) short summary of ideas from this field that we need, for more details see [Hart2015a].

From now on, the cardinality of the set $\mathbb{R}$ will be denoted by $\mathfrak{c}$ for *continuum* and the cardinality of the set $\mathbb{N}$ by $\aleph_0$. These cardinalities or *cardinal numbers* are well-ordered sets, and we use $<$ to denote this well-ordering, [Hart2015a]. Sets have the same cardinal number if and only if there exists a bijection between them. Furthermore, we say that a set A is *countable*, which is denoted by $|A| \leq \aleph_0$, if it is finite (denoted by $|A| < \aleph_0$) or if it is *countably infinite*, which means that there exists a bijection from A to $\mathbb{N}$. As expected, this last case is denoted by $|A| = \aleph_0$. The fact that the cartesian product of two countable sets is countable is a consequence of the unique prime factorisation of elements of $\mathbb{N}$, [Stevenhagen2016b]. For any set A we have that $|\mathcal{P}(A)| = 2^{|A|}$ and an important equality is $2^{\aleph_0} = \mathfrak{c}$. Moreover, if for two sets A and B we denote by ${}^{A}B$ the set of functions from A to B, then we have $|{}^{A}B| = |A|^{|B|}$. Finite sets have as cardinal number an element of $\mathbb{N}$, which represents the number of elements in it. This next definition is instrumental for this discussion.

Definition 6. *For any sets X and Y we say $|X| \leq |Y|$ if there exists an injection from X to Y.*

As a corollary of the Cantor-Bernstein Theorem [Hart2015b], we have the following result.

Proposition 9. *For any sets X and Y we have $|X| \leq |Y|$ and $|Y| \leq |X|$ if and only if $|X| = |Y|$.*

With the use of the Axiom of Choice the following proposition follows inmediately from the previous one.

Proposition 10. *Let X and Y be any sets. If there exists a surjection from X to Y then $|Y| \leq |X|$.*

Proof. By surjectivity of f we have $\varnothing \notin C := \{f^{-1}(y) : y \in Y\}$. By the Axiom of Choice there exists a function $g : C \to \bigcup C$ such that for every $y \in Y : g(f^{-1}(y)) \in f^{-1}(y)$. Notice $\bigcup C = X$. Define $h : Y \to X$ where $y \mapsto g(f^{-1}(y))$, and suppose that $h(y_1) = h(y_2)$ for $y_1, y_2 \in Y$. Then $y_1 = f(h(y_1)) = f(h(y_2)) = y_2$, making h injective and hence $|Y| \leq |X|$. $\square$

Proposition 11. *The set $\mathcal{B} = \{(a, b) : a, b \in \mathbb{Q}\}$ is countably infinite.*

Proof. Note that the map from $f : \mathcal{B} \to \mathbb{Q}^2$ where $(a, b) \mapsto \langle a, b\rangle$ is injective. Therefore, as $\mathbb{Q}$ is countable, we can say $|\mathcal{B}| \le |\mathbb{Q}^2| = \aleph_0$. Also, $g : \mathbb{N} \to \mathcal{B}$ where $n \mapsto (n, n + 6282)$ is injective. By Proposition 9 we have $|\mathcal{B}| = \aleph_0$. $\square$

Proposition 12. *A countable union of countable sets is countable.*

Proof. Let $\{A_i\}_{i\in I}$ be a set of countable sets. We will start by assuming that $I = \mathbb{N}$ and $(\forall i \in I)(|A_i| = \aleph_0)$. We know that for every $i \in \mathbb{N}$ there is a bijection $f_i : A_i \to \mathbb{N}$, and therefore we can denote every $a \in A_i$ in the following way: $a = a_{if(a)}$. Equipped with this representation we can put all the elements of $\bigcup_{i\in I} A_i$ in a diagram as follows.

$$\begin{matrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ \dots & \dots & \dots & \dots \end{matrix}$$

By applying the diagonal argument on this diagram, neglecting elements that we have already come across (if any), we conclude that $|\bigcup_{i\in I} A_i| = \aleph_0$. If either I or a positive number of elements of $\{A_i\}_{i\in I}$ are finite then either the number of rows in our diagram is finite or some rows in our diagram have finitely many elements. However, by proceeding in a similar fashion to the diagonal argument we can still find an injection from $\bigcup_{i\in I} A_i$ to $\mathbb{N}$. $\square$

Topological concepts are also important for this section. For more information, see [Bruin2017].

Proposition 13. *For any sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ of nonempty compact subsets of $\mathbb{R}$ such that $C_0 \supseteq C_1 \supseteq C_2 \supseteq \dots$ we have $\bigcap_{i=0}^{\infty} C_i \neq \varnothing$.*

Proof. See [Bruin2017]. $\square$

Definition 7. *A subset A of $\mathbb{R}$ is called perfect if it is closed and every point is a limit point of A. We call x a limit point of A if for every $\epsilon > 0$ we have $A \cap (B_\epsilon(x) \setminus \{x\}) \neq \varnothing$. Notice that x need not be an element of A.*

Proposition 14. *Every nonempty perfect set is uncountable.*

Proof. Suppose $|F| \le \aleph_0$. If F is finite and we choose any point $x \in F$ we can find another point $y \in F$ which is closest to x, however by the special property of F we have that $F \cap B_{|x-y|}(x) \neq \varnothing$ and thus any element of this set is closer to x than y is. Contradiction. If $|F| = \aleph_0$ we can enumerate F as $\mathbb{N}$: $F = \{f_0, f_1, ...\}$. Let A_0 be a bounded open interval such that $f_0 \in A_0$. By the property of F we can find an open interval A_1 such that $\overline{A_1} \subseteq A_0$, $f_0, f_1 \notin A_1$ and $A_1 \cap F \neq \varnothing$. In fact, for all $n \in \mathbb{Z}_+$ we can choose an open interval A_n such that $\overline{A_n} \subseteq A_{n-1}$, $f_n \notin A_n$ and $A_n \cap F \neq \varnothing$. Define $C_n = \overline{A_n} \cap F$ for all $n \in \mathbb{Z}_+$ and notice it is compact as $\overline{A_n}$ is compact and F is closed. By Proposition 13 we have $\bigcap_{i=0}^{\infty} C_i \neq \varnothing$, so there exists a $j \in \mathbb{N}$ such that $f_j \in \bigcap_{i=0}^{\infty} C_i$. However, by construction it holds that $f_j \notin \overline{A_j} \subseteq C_j$. Contradiction, so $|F| > \aleph_0$. $\square$

Every uncountable subset of $\mathbb{R}$ has a limit point and also contains one of them. The proof of Proposition 16 shows that for any uncountable subset of $\mathbb{R}$ all but countably many points of the set are limit points of it.

Definition 8. *Let $A \subseteq \mathbb{R}$ and $x \in A$. The point x is called a left-sided limit point of A if for every $a \in \mathbb{R}$ such that $x > a$ we have $(a, x) \cap A \neq \varnothing$. The point x is called a right-sided limit point of A if for every $b \in \mathbb{R}$ such that $x < b$ we have $(x, b) \cap A \neq \varnothing$. This x is called a two-sided limit point of A if it is both a right-sided limit point of A and a left-sided limit point of A.*

Proposition 15. *Every uncountable subset A of $\mathbb{R}$ contains a two-sided limit point of A.*

Proof. Suppose there is an uncountable $A \subseteq \mathbb{R}$ such that it has no two-sided limit points of A. Then there are uncountably many points of A that are right-sided limit points of A or there are uncountably many points of A that are left-sided limit points of A.
Assume the last case and define $B = \{x \in A : x \text{ is a left-sided limit point of } A\}$. Then for every $x \in B$ there is a rational number r such that $(x, x+r) \cap A = \varnothing$. For every $x \in B$, define $A_x = \{r \in \mathbb{Q} : (x, x+r) \cap A = \varnothing\}$ and use a Choice Function f on $\{A_x : x \in B\}$ to get the set $R := \{f(A_x) : x \in B\}$. We will show that for the map $g : B \to R$ where $x \mapsto f(A)$ there must be an element q in the codomain such that $|g^{-1}(q)| > \aleph_0$. Suppose for every $q \in R$ the set $g^{-1}(q)$ is countable. Then $\bigcup_{q \in R} g^{-1}(q)$ is countable by Proposition 12. However, this union is equal to B, which was uncountable. Therefore, there is a $p \in \mathbb{Q}$ such that $g^{-1}(p)$ is uncountable. Now we will show that for distinct $y, z \in g^{-1}(p)$ we have $(y, y+p) \cap (z, z+p) = \varnothing$. Suppose $y > z$ and the interval is not disjoint. Then $y \in (z, z+p) \cap (y, y+p)$ but $(z, z+p) \cap A$ was empty. Analogously, we come to the same contradiction if $z > y$.
As $\mathbb{Q}$ is countably infinite, there exists a bijection $b : \mathbb{N} \to \mathbb{Q}$. With this bijection, we can define an injection $i : \{(x, x+p) : x \in g^{-1}(p)\} \to \mathbb{Q}$ where $(x, x+p)$ maps to $b(j)$ where $j = \min\{k \in \mathbb{N} : b(k) \in (x, x+p)\}$. Therefore, $|g^{-1}(p)| = |\{(x, x+p) : x \in g^{-1}(p)\}| \leq \aleph_0$. Contradiction. To get a contradiction in the other case we work analogously. $\square$

Thus all but countably many elements of an uncountable subset of $\mathbb{R}$ are two-sided limit points of the set.

3.4.2 Construction

To start with the construction of the Bernstein set we need to take the statement of Proposition 14 a step further. Namely, we will prove that every nonempty perfect subset of $\mathbb{R}$ has cardinality $\mathfrak{c}$. Notice that for sets that contain an interval we are inmediately done; the cardinlaity of an interval is $\mathfrak{c}$. We will need some auxiliary lemmas to show that also nonempty perfect sets which do not contain an interval, have the desired cardinality.

Lemma 2. *Let $A \subseteq \mathbb{R}$ be perfect such that A contains no interval. Suppose there is a $[a, b] \subseteq \mathbb{R}$ such that*

1. *$(a, b) \cap A \neq \varnothing$,*
2. *a is a right-sided limit point of A and*
3. *b is a left-sided limit point of A.*

Then there are $c, d \in \mathbb{R}$ such that

1. *$a < c < d < b$,*
2. *$(c, d) \cap A = \varnothing$,*
3. *c is a left-sided limit point of A and*
4. *d is a right-sided limit point of A.*

Proof. As A contains no interval there is a $x \in [a, b]$ such that $x \notin A$. For this x there is an open interval (e, f) such that $a < e < f < b$, $x \in (e, f)$ and $(e, f) \cap A = \varnothing$ (otherwise x would be a limit-point of A). Notice that $[a, e) \cap A \neq \varnothing$ is bounded from above by all the elements of (e, f). By the Least Upper Bound Property of $\mathbb{R}$ (see [Coplakova et al.2013]) there exists a supremum c of $[a, e) \cap A$. We apply the Greatest Lower Bound Property on $(f, b] \cap A$ to get the infimum d. Now we have that $a < e \leq c < d \leq f < b$, $(c, d) \cap A \subseteq (e, f) \cap A = \varnothing$ and c and d are respectively left-sided and right-sided limit points of A. $\square$

Lemma 3. *Let $A \subseteq \mathbb{R}$ be nonempty and perfect. Suppose there is $[a, b] \subseteq \mathbb{R}$ such that a is a right-sided limit point of A and b is a left-sided limit point of A. Then there are $c, d \in \mathbb{R}$ such that*

1. *$a < c < d < b$,*
2. *$(c, d) \cap A \neq \varnothing$,*
3. *c and d are two-sided limit points of A and*
4. *$\lambda((c, d)) < \frac{1}{2}\lambda((a, b))$.*

Proof. As a and b are limit points of A in a specific direction, the set $[a,b] \cap A$ is nonempty and perfect. By Proposition 14 we know $[a,b] \cap A$ is uncountable and thus, by Proposition 15, it contains a two-sided limit point $t \in (a,b)$. Let $e, f \in \mathbb{R}$ such that $a < e < t < f < b$ and $\lambda((e,f)) < \frac{1}{2}\lambda((a,b))$. Let $c \in (e,t)$ and $d \in (t,f)$ be two-sided limit points of $[e,t] \cap A$ and $[t,f] \cap A$ respectively. Now we have $t \in (c,d)$ and thus $(c,d) \cap A \neq \varnothing$ and $\lambda((c,d)) < \lambda((e,f)) < \frac{1}{2}\lambda((a,b))$. □

Lemma 4. *Suppose that $A \subseteq \mathbb{R}$ is nonempty, perfect and contains no interval. If there is a $[a,b] \subseteq \mathbb{R}$ such that a and b are both two-sided limit points of A, then there are disjoint $[c_0,d_0],[c_1,d_1] \subseteq \mathbb{R}$ such that*

1. $[c_0,d_0],[c_1,d_1] \subseteq [a,b]$,
2. $\lambda([c_0,d_0]),\lambda([c_1,d_1]) < \frac{1}{2}\lambda([a,b])$ *and*
3. c_0,d_0,c_1,d_1 *are all two-sided limit points of A.*

Proof. First we apply Lemma 2 on the interval $[a,b]$ to get the interval (c,d) where c and d are left and right-sided limit points of A respectively and $(c,d) \cap A = \varnothing$. Now $[a,c]$ and $[d,b]$ are two disjoint intervals where the left boundaries are right-sided limit points of A and the right boundaries are left-sided limit points of A. Hence, we can apply Lemma 3 to both of these intervals to get $[c_0,d_0] \subseteq [a,c]$ and $[c_1,d_1] \subseteq [d,b]$ where every endpoint is a two-sided limitpoint of A. We also have $\lambda([c_i,d_i]) < \frac{1}{2}\max\{\lambda([a,c]),\lambda([d,b])\} < \frac{1}{2}\lambda([a,b])$. □

Theorem 3. *Every nonempty perfect set $F \subseteq \mathbb{R}$ has cardinality $\mathfrak{c}$.*

Proof. By Proposition 14 we have $|F| > \aleph_0$. As discussed at the beginning of the construction, we only have to treat the case where F does not contain an interval. Let $a,b \in F$ be two distinct two-sided limit points of F. After applying Proposition 4 to the interval $[a,b]$ we get two new disjoint intervals which we can denote by B_0 and B_1. We define A_1 to be their union. If we apply the lemma again on these two new intervals respectively we get B_{00} and B_{01} from B_0 and B_{10} and B_{11} from B_1. Let $A_2 = B_{00} \cup B_{01} \cup B_{10} \cup B_{11}$. Using recursion, we get a monotone decreasing sequence of compact sets $\langle A_n \rangle_{n\in\mathbb{N}}$. Therefore we again have $C := \bigcap_{n=1}^{\infty} A_n \neq \varnothing$ by Proposition 13. This intersection will be our desired subset of cardinality $\mathfrak{c}$. To show that this is indeed a subset of F, let $y \in C$. Notice that there is a binary sequence g^y such that, for all $n \in \mathbb{N}$, $g^y_n = i$ if and only if $y \in B_{s(n)}$ where $s(n)$ is a binary sequence of length $n+1$ and the last entry is i. Hence, $\{y\} = \bigcap_{n=1}^{\infty} B_{g^y_n}$. This means that y is a limit point of F and therefore $y \in F$. Finally, consider the function $f : C \to {}^{\mathbb{N}}\{0,1\}$, that sends an element y to its corresponding sequence g^y such that $\{y\} = \bigcap_{n=1}^{\infty} B_{g^y_n}$. It is clear that two different elements of C must have different images, as for every $n \in N$ the set A_n is the union of disjoint intervals. So f is injective. Also, any sequence of ${}^{\mathbb{N}}\{0,1\}$ gives rise to an element of C and therefore f is surjective. Therefore, $\mathfrak{c} = 2^{\aleph_0} = |{}^{\mathbb{N}}\{0,1\}| = |C| \leq |F|$ as f is bijective. As noted before, we can now conclude that $|F| = \mathfrak{c}$. □

Theorem 4. *The set $\mathcal{L}$ of nonempty perfect subsets of $\mathbb{R}$ has cardinality continuum.*

Proof. Let $\mathcal{B}$ be as in Proposition 11. Define $\mathcal{O}$ and $\mathcal{C}$ as the sets of open and closed subsets of $\mathbb{R}$ respectively. Then $|\mathcal{O}| = |\mathcal{C}|$ follows from the natural bijection $S \mapsto \mathbb{R} \setminus S$. Now define $g : \mathcal{P}(\mathcal{B}) \to \mathcal{O}$ by $A \mapsto \bigcup A$ and notice that it is surjective as $\mathcal{B}$ is a base for the topology of $\mathbb{R}$. Finally, by Proposition 10 we have that $|\mathcal{O}| \leq |\mathcal{P}(\mathcal{B})| = 2^{\aleph_0} = \mathfrak{c}$. Hence, $|\mathcal{L}| \leq |\mathcal{C}| \leq \mathfrak{c}$. On the other hand, $\{[-b, b] : b \in \mathbb{R}_{>0}\} \subseteq \mathcal{L}$ has cardinality $\mathfrak{c}$ because there is an obvious bijection to $\mathbb{R}_{>0}$, so therefore $|\mathcal{L}| \geq \mathfrak{c}$. So $|\mathcal{L}| = \mathfrak{c}$. □

By assumption of the Axiom of Choice we have the well-ordering theorem at our disposal and we thus can well-order $\mathcal{L}$. Let $\langle F_\alpha \rangle_{\alpha<\mathfrak{c}} = \mathcal{L}$ be this well-ordering. We also well-order $\mathbb{R}$ and use the symbol $\prec$ for this relation. Define $x_0 := \min_\prec F_0$ and y_0 to be its successor in F_0 with respect to $\prec$. In the next step, define $x_1 := \min_\prec(F_1 \setminus \{x_0, y_0\})$ and y_1 to be its successor in $F_1 \setminus \{x_0, y_0\}$ with respect to $\prec$. To continue with this process, we will need to use *transfinite recursion*. Let $\alpha < \mathfrak{c}$ and suppose for every $\beta < \alpha$ the distinct points x_β and y_β have been selected in F_β. Then the set $F_\alpha \setminus \bigcup_{\beta<\alpha}\{x_\beta, y_\beta\}$ is nonempty because $|F_\alpha| = \mathfrak{c}$ and $|\bigcup_{\beta<\alpha}\{x_\beta, y_\beta\}| \leq 2{\cdot}\alpha$. We have $\alpha < \mathfrak{c}$ and thus $\alpha \leq 2{\cdot}\alpha < 2{\cdot}\mathfrak{c} = \mathfrak{c}$, (see [Hart2015a]). Therefore $F_\alpha \setminus \bigcup_{\beta<\alpha}\{x_\beta, y_\beta\}$ has cardinality $\mathfrak{c}$ and hence our two desired points, $x_\alpha := \min_\prec(F_\alpha \setminus \bigcup_{\beta<\alpha}\{x_\beta, y_\beta\})$ and y_α its successor in the same set with respect to $\prec$, exist. Therefore, by transfinite recursion, x_α and y_α exist for each $\alpha < \mathfrak{c}$. Our non-measurable set will be

$$B := \{x_\alpha : \alpha < \mathfrak{c}\}.$$

Proposition 16. *Every uncountable compact subset of $\mathbb{R}$ has a nonempty perfect subset.*

Proof. Let $C \subseteq \mathbb{R}$ be compact and uncountable. Define $\mathcal{O} = \{(p, q) : p, q \in \mathbb{Q}\ |(p, q) \cap C| \leq \aleph_0\}$ and notice that $O := \bigcup \mathcal{O}$ is open. Furthermore, $O \cap C$ is a countable union of countable sets and hence, by Proposition 12, is also countable. Let $x \in C \setminus O$, then for every rational interval (p, q) with $x \in (p, q)$ we have $|(p, q) \cap C| > \aleph_0$ and therefore, as $C \cap O$ is countable, $|(p, q) \cap (C \setminus O)| > \aleph_0$ which means x is a limit point of $C \setminus O$. We also have $C \setminus O = C \cap O^c$, which makes it closed, and therefore perfect. Also notice $C \setminus O \neq \varnothing$ as $C \cap O$ is countable. □

Suppose B is measurable, then so is B^c. Without loss of generality, we can assume that $B \cap (0, 1)$ has positive measure, because its sum with the measure of $B^c \cap (0, 1)$ must equal one. Suppose $\lambda(B \cap (0, 1)) < 1$, then there is a sequence $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ of subintervals of $(0, 1)$ such that $B \cap (0, 1) \subseteq \bigcup_{n\in\mathbb{N}}(a_n, b_n)$ and $\sum_{n=0}^{\infty}(b_n - a_n) < 1$. The set $(0, 1) \setminus \bigcup_{n\in\mathbb{N}}(a_n, b_n)$ must be uncountable, otherwise the cover would have measure one. As $(\bigcup_{n\in\mathbb{N}}(a_n, b_n))^c$ is also compact it contains by Proposition 16 a nonempty perfect subset P. However, this gives us a contradiction as $P \in \mathcal{L}$, which means $P = F_\alpha$ for some $\alpha < \mathfrak{c}$ and hence we have $x_\alpha \in P \cap B$ and $x_\alpha \notin \bigcup_{n\in\mathbb{N}}(a_n, b_n)$. If $\lambda(B \cap (0, 1)) = 1$ we can analogously derive a contradiction by showing that there is a y_α for some $\alpha < \mathfrak{c}$ such that y_α does not belong to a cover of $B^c \cap (0,1)$. So B and B^c are non-measurable.

4 Discussion

An attentive reader will have noticed that in all four constructions the Axiom of Choice was used in some form. A natural question is whether *every* construction of a non-measurable set uses this axiom. This question was answered affirmatively in 1970 when R. Solovay constructed a model in ZF (the standard Zermelo-Fraenkel axiomatic system for set theory) in which every subset of $\mathbb{R}$ is Lebesgue measurable, [Solovay1970]. Ergo, the statetement "every subset of reals is Lebesgue measurable" does not lead to a contradiction in ZF. However, as we have seen, the addition of the Axiom of Choice to ZF (or ZFC) surely causes the assumption of the statement to lead to a contradiction.

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