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# The Weak Homotopy Type of Non-Hausdorff Manifolds

**Bachelor thesis**

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# Introduction

For connected Hausdorff second-countable 1-manifolds, typically just called 1-manifolds or curves, a classification exists: all are homeomorphic to either the circle $S^1$ or the real line $\mathbb{R}$. Classifications in dimensions 3 and 5 and up also exist, see [2], pp. 121-134. What happens when we no longer assume Hausdorffness?

We will define a manifold as a second-countable topological space which is locally homeomorphic to $\mathbb{R}^n$. Often Hausdorffness is also assumed and often a differential structure is added as well, but we will study manifolds without this extra assumption and extra structure. The basic example of a manifold that is not Hausdorff is the line with the doubled origin, obtained by glueing two copies of the real line everywhere except at a single point: it is the pushout $\mathbb{R} \cup_{(\mathbb{R}\setminus\{0\})} \mathbb{R}$ of $\mathbb{R} \leftarrow \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$.

Considering this example one wonders if all manifolds arise as the colimit of Hausdorff manifolds. A. Hommelberg gave an example of a non-Hausdorff compact manifold which cannot be obtained in this way in her bachelor's thesis [4], shattering this hope. A result that does hold is that every $n$-manifold immerses into an $n$-manifold covered by $n$-spheres.

It was noted, by L. Taelman that some of her examples at least were weakly homotopy equivalent to finite CW complexes. Because simplicial complexes are easier to handle than CW-complexes, we hypothesised that maybe all compact manifolds are weakly homotopy equivalent to finite simplicial complexes. In chapter 4, we prove that there exists a compact non-Hausdorff manifold that is not weakly homotopy equivalent to a finite simplicial complex (Theorem 82).

To better understand the situation, after so many failed conjectures, we decided to try to classify the 1-manifolds generated by glueing $S^1$ with itself. In particular we look at all manifolds that arise when we take a subspace $U$ of $S^1$ and form the pushout of $S^1 \leftarrow U \rightarrow S^1$, where both arrows are the inclusion.

The homotopy type turns out to be stronger than practical for this classification: it differentiates between doubled point and doubled intervals for example. The weak homotopy type does not distinguish these cases, which makes it more useful. At the same time it characterises topological spaces in a finer way than just having isomorphic homotopy groups in general as we prove in Theorem 96, however in the case of .

First we looked specifically for weak homotopy equivalences between two spaces, both of which are the glueing of two circles. This gave us weak homotopy equivalences under certain assumptions (Theorem 90)) and in these cases the involved spaces were therefore weakly homotopy equivalent (Corollary 91). Next we proved that for connected 1-manifolds $X$ and $Y$ with isomorphic fundamental groups there exists a 1-manifold $Z$ with weak homotopy equivalences $Z \rightarrow X$ and $Z \rightarrow Y$ (Theorem 94). This result does not extend to general $n$-manifolds and we give a counterexample for $n = 5$ in Theorem 96.

Finally we show that if $X$ and $Y$ are topological spaces of the same weak homotopy type then there exists a CW-complex $Z$ with weak homotopy equivalences $Z \rightarrow X$ and $Z \rightarrow Y$ in Theorem 97.

# Notation

All maps between topological spaces are continuous, unless otherwise indicated. More generally every map between objects is assumed to be a morphism in the relevant category and not just a set theoretical function.

Function composition is sometimes denoted by $\circ$ to emphasise it, but also often just by juxtaposition.

The unit interval will be denoted by $I = [0, 1] \subset \mathbb{R}$. We use the convention that $0 \in \mathbb{N}$.

The word cover is used in several meanings. An open cover $\mathcal{U}$ of a topological space $X$ is a family of open subsets $\mathcal{U}$ of $X$ such that $\bigcup_{U \in \mathcal{U}} U = X$. In particular we say that opens $U_1, U_2, \ldots$ cover $X$ whenever $\{U_1, \ldots\}$ is an open cover. Usually we will call this a cover (of $X$) for short.

A covering space $p : Y \to X$ over a topological space $X$ is a continuous map with the property that around each point $x \in X$ there exists an open set $U \subseteq X$ such that for $V = p^{-1}(U)$ we have a decomposition into connected components components $V = \bigsqcup_{i \in I} V_i$ such that $p|_{V_i} : V_i \to U$ is a homeomorphism. This too will often be called a cover (of $X$).

A space without further qualification is a topological space.

# 1 Properties of Manifolds

Considering the topic of this thesis we will define manifolds. However we will first reiterate a few topological properties. Throughout this section $(X, T_X)$ is a topological space, usually referred to as $X$.

**Definition 1.** A *basis* for the topology of $X$ is a subset $S \subseteq T_X$ such that for all $U \in T_X$, there exists a family $\mathcal{U} \subseteq S$ such that $U = \bigcup_{V \in \mathcal{U}} V$.

**Definition 2.** $X$ is *second-countable* if it admits a basis $S$ such that $S$ is countable.

**Example 3.**

1. The empty space has as its unique topology $\mathcal{T}_\emptyset = \{\emptyset\}$, and so $\{\emptyset\}$ is a basis for the topology of $\emptyset$ and of course this basis is countable. $\emptyset$ is therefore a second-countable space.
2. Every countable discrete space has the set of singletons as a basis for its topology and is therefore second-countable.
3. Suppose $X = \mathbb{R}^n$ with euclidean metric. Then the balls with a rational center and rational radius form a basis for the topology that is countable. So $X$ is second-countable.

Now we can define the central object of this thesis. Note that we leave out the usual assumptions of Hausdorffness and a differentiable structure.

**Definition 4.** Let $n \in \mathbb{N}$. An *n-manifold* $M$ is a second-countable topological space such that for each $x \in M$ there exists an open subset $U \ni x$ such that $U$ is homeomorphic to $\mathbb{R}^n$. A topological space is a manifold if it is an $n$-manifold for some $n \in \mathbb{N}$.

**Lemma 5.** *An open subset of an n-manifold is an n-manifold.*

*Proof.* Let $M$ an $n$-manifold and $N$ an open subset. Let $x \in N$. First we will prove that around $x$ there exists an open $U \subseteq N$ homeomorphic to $\mathbb{R}^n$. Let $U'$ such an open in $M$ then set $U'' = U' \cap N$. We note that $U''$ is open, and homeomorphic to an open in $\mathbb{R}^n$ so there exists an open ball $U$ around $x$ contained in $U''$. This open ball is homeomorphic to $\mathbb{R}^n$.

Next we need to give a countable base for the topology of $N$. Let $\mathcal{U}$ be a countable basis for the topology of $M$, then

$$\mathcal{U}' = \{U \cap N \mid U \in \mathcal{U}\}$$

is a basis for the topology of $N$: suppose $A$ open in $N$ then there exists a $B$ open in $M$ such that $B \cap N = A$. Now write $B$ as $B = \bigcup_{u \in S \subseteq \mathcal{U}} u$ for some subset $S$, then $A = N \cap \bigcup_{u \in S} u = \bigcup u \in S(N \cap u)$, which concludes the proof. □

**Lemma 6.** *A connected component $C$ of an $n$-manifold $M$ is open in $M$.*

*Proof.* Take around each point $x \in C$ an open $U_x \cong \mathbb{R}^n$, then $U_x \cup C$ is connected and contains $x$, but $C$ is the maximal connected subset of $M$ containing $x$, so $U_x \cup C = C$ and therefore $U_x \subseteq C$. So we see that $\bigcup_{x \in C} U_x \subseteq C$. On the other hand, we know that $U_x \ni x$ for each $x \in C$ and so $C \subseteq \bigcup_{x \in C} U_x$. We conclude that $C = \bigcup_{x \in C} U_x$ and since this is a union of open sets, it is open. □

Let us now look at a consequence of the assumption that a manifold is second-countable.

**Definition 7.** A topological space is called Lindelöf if every open cover has a countable subcover.

This is a weaker version of compactness.

**Lemma 8.** *Every second-countable topological space $M$ is Lindelöf.*

*Proof.* Let $\mathcal{V}$ be an open cover of $M$. If $\mathcal{V}$ is countable then we are done, so assume it is not countable. Let $\mathcal{U}$ be a countable basis for the topology of $M$. Let $\mathcal{U}' = \{U \in \mathcal{U} \mid \exists V \in \mathcal{V} : U \subseteq V\}$ be the subset of $\mathcal{U}$ such that each open of the basis is contained in some open of the cover $\mathcal{V}$. Let $\psi : \mathcal{U}' \to \mathcal{V}$ be a function such that for all $U \in \mathcal{U}'$ we have $U \subseteq \psi(U)$. Then we set $\mathcal{V}' = \psi(\mathcal{U}')$. We will now prove that this is a countable subcover of $\mathcal{V}$.

Suppose that $x \in M$ is a point, then there exists a $V \in \mathcal{V}$ containing $x$, because $\mathcal{V}$ is a cover. Now we use that $\mathcal{U}$ is a basis, which means that we can find a subset of $\mathcal{U}$ such that its union is $V$, to find a $U \in \mathcal{U}$ such that $x \in U \subseteq V$. Indeed this means that $U \in \mathcal{U}'$, so we can apply $\psi$ to find $V' = \psi(U) \in \mathcal{V}'$. We see that $x \in V'$ and $V' \in \psi(\mathcal{U}') = \mathcal{V}'$ and since this works for all $x \in M$, that $M \subseteq \bigcup_{V \in \mathcal{V}'} V$. We conclude that $\mathcal{V}'$ is a cover.

By construction we have that $\mathcal{V}' \subseteq \mathcal{V}$ as required. We also see that $|\mathcal{V}'| \leq |\mathcal{U}'|$, because $\mathcal{V}'$ is the image of $\psi$, and $|\mathcal{U}'| \leq |\mathcal{U}|$, because $\mathcal{U}' \subseteq \mathcal{U}$, finally we see that $|\mathcal{U}| \leq \aleph_0$, because $M$ is second-countable. We conclude that $\mathcal{V}'$ is a countable subcover of $\mathcal{V}$ and so that $M$ is Lindelöf. □

**Corollary 9.** *Every manifold is Lindelöf*

In particular we find that open sets cannot be too disconnected:

**Lemma 10.** *An open set of a manifold has a countable number of connected components.*

*Proof.* We note that an open subset of a manifold is again a manifold and hence Lindelöf. Using Lemma 6 we conclude that the set of connected components forms an open cover for this manifold. This cover has no strict subcovers. By Lindelöfness then we get that it must be countable. □

## 1.1 Open Immersions

We are interested to see what happens if we identify certain parts of manifolds. We will often call this glueing and this categorically corresponds to taking colimits. Let us first construct this in the case of two spaces, which categorically is a pushout.

**Definition 11.** If $U, V$ topological spaces and $\varphi : U \to V$ is a map, then $\varphi$ is an open immersion if $\varphi$ is injective and for all open subsets $U' \subseteq U$ we have that $\varphi(U')$ is an open set.
**Example 12.** If $U$ open in $X$, then the inclusion map from $U$ to $X$ is an open immersion.

We now show that these are essentially all immersions.
**Lemma 13.** *Let $U, V$ topological spaces and $\varphi : U \to V$ an open immersion, then $\varphi$ factors as a homeomorphism followed by an inclusion. Consider $\overline{\varphi} : U \to \varphi(U), x \mapsto \varphi(x)$ and $i : \varphi(U) \to V$, the inclusion map, then we have $\varphi = i \circ \overline{\varphi}$.*

$$\begin{tikzcd} U \arrow[rr, "\varphi"] \arrow[dr, "\overline{\varphi}"'] & & V \\ & \varphi(U) \arrow[ur, "i"'] & \end{tikzcd}$$

*Proof.* We will show that $\overline{\varphi}$ is a homeomorphism. The rest is trivial. Since $\varphi$ is injective, so is $\overline{\varphi}$, while $\overline{\varphi}$ is surjective by construction. All we need to do is to show that the set theoretic inverse $\overline{\varphi}^{-1}$ is actually continuous. Suppose $U' \subseteq U$ then $(\varphi^{-1})^{-1}(U') = \varphi(U')$ by definition, and this is open since $\varphi$ is an open map. $\square$

Let us now apply this knowledge to manifolds.
**Lemma 14.** *Suppose $X$ is a topological space. Let $I$ be a countable set, $(U_i)_{i \in I}$ topological spaces and $f : U_i \to X$ open immersions such that $\{f(U_i) : i \in I\}$ is an open cover of $X$ and let $n \in \mathbb{N}$. Then $X$ is an $n$-manifold if and only if all of the $U_i$ are $n$-manifolds.*

*Proof.* Suppose first that $X$ is an $n$-manifold. Let $i \in I$ then $f(U_i)$ is an open subset of a manifold and therefore a manifold. By the previous lemma $f(U_i)$ is homeomorphic to $U_i$ so it is a manifold too.

Suppose now that all the $U_i$ are $n$-manifolds. Let $x \in X$. Since the $f_i(U_i)$ cover $X$ there is some $i \in I$ and $y \in U_i$ such that $f_i(y) = x$. Now take $V \ni y$ open such that $V \cong \mathbb{R}^n$. By the previous lemma $f_i(V) \cong \mathbb{R}^n$ and so $f_i(V) \cong \mathbb{R}^n$ is the neighbourhood of $x$ homeomorphic to $\mathbb{R}^n$ as required in the definition of an $n$-manifold.

A basis for the topology of $X$ is given by a union of bases of the $U_i$. A countable union of countable sets is countable. $\square$

## 1.2 Glueing of Manifolds

**Definition 15.** The disjoint union of two topological spaces $B, C$ is suggestively written as $B \sqcup C$. As a set it is the disjoint union of $B$ and $C$. Its topology consists of all the subsets of the form $b \sqcup c$ where $b$ is open in $B$ and $c$ is open in $C$.

For topological spaces $B, C$, the inclusions $B \to B \sqcup C$ and $C \to B \sqcup C$ are open immersions and whenever appropriate we will therefore treat them as subsets.
**Lemma 16.** *Let $U_1, U_2$ and $Z$ be topological spaces, and $\phi_i : U_i \to Z$, $i = 1, 2$ be morphisms, then there exists a unique morphism $\phi : U_1 \sqcup U_2 \to Z$ with $\phi|_{U_i} = \phi_i$, $i = 1, 2$.*

*Moreover, if $\phi_1$ and $\phi_2$ are open maps, then so is $\phi$.*

*Proof.* We set

$$\phi : U_1 \sqcup U_2 \to Z, x \mapsto \begin{cases} \phi_1(x) & \text{If } x \in U_1 \\ \phi_2(x) & \text{If } x \in U_2 \end{cases}$$

$\phi$ is well-defined since $U_1 \cap U_2 = \emptyset$ (*disjoint* union), it is continuous since for an open $V \subseteq Z$ we get $\phi^{-1}(V) = \phi_1^{-1}(V) \cup \phi_2^{-1}(V)$ and both $\phi_1, \phi_2$ are continuous and hence $\phi^{-1}(V)$ is open. $\phi$ is unique, because it is fully determined on an open cover.

For the final claim we note that if $A \in U_1 \sqcup U_2$ open, then $A = A_1 \cup A_2$ with $A_i$ open in $U_i$, $i = 1, 2$, so $\phi(A) = \phi_1(A_1) \cup \phi_2(A_2)$ both of which are open if $\phi_1, \phi_2$ are open, giving that $\phi$ is open. $\square$

Let us give the universal property of the pushout, whose construction we will give next and is central to this thesis.

**Definition 17.** If $B \xleftarrow{f} A \xrightarrow{g} C$ is a pair of morphisms in any category, then $B \xrightarrow{p} X \xleftarrow{q} C$ is called a pushout of it if $pf = qg$ and for all diagrams $B \xrightarrow{p'} X' \xleftarrow{q'} C$ with $p'f = q'g$ there exists a unique morphism $\varphi : X \to X'$ such that $\varphi p = p'$ and $\varphi q = q'$.

We will now construct general pushouts in the category of topological spaces. Intuitively it corresponds to glueing two spaces to form a new space. Consider three spaces $A, B$ and $C$ with maps $f : A \to B$ and $g : A \to C$. We now want to construct a glued space $X$.

We define a relation $\sim_0$ on $B \sqcup C$ by

$$x \sim_0 y \Leftrightarrow x = y \vee \exists a \in A : \{x, y\} = \{f(a), g(a)\}.$$

We notice that this relation is reflexive and symmetric, but not transitive. So let $\sim$ be the transitive closure of $\sim_0$ on $B \sqcup C$, then we take

$$B \cup_A C := (B \sqcup C)/\sim$$

to be the quotient space, with quotient topology. This comes with natural maps $p : B \to B \sqcup C \to B \cup_A C$ and $q : C \to B \sqcup C \to B \cup_A C$

**Theorem 18.** *The pushout of* $B \xleftarrow{f} A \xrightarrow{g} C$ *is* $B \xrightarrow{p} B \cup_A C \xleftarrow{q} C$.

*Proof.* We will prove that $(B \sqcup C)/\sim$ has the universal property. Let $Y$ be any topological space equipped with $q', p'$ making the diagram commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "p'"] \\ C \arrow[r, "q'"'] & Y \end{tikzcd}$$

By Lemma 16 we get a morphism $\varphi : B \sqcup C \to Y$ restricting to $p', q'$ on $B, C$ respectively. We identify $B$ and $C$ with their images in $B \sqcup C$ and write $[x]$ for the equivalence class in $B \cup_A C$ of an element $x \in B \sqcup C$. We write $r : B \sqcup C \to (B \sqcup C)/\sim, x \mapsto [x]$ for the reduction map and $i, j$ respectively for the inclusions of $B, C$ in $B \sqcup C$, so that $p = ri$.

Let us first prove that $\varphi$ is constant on equivalence classes. It is sufficient to prove that for all $x, y \in B \sqcup C$ we have that $x \sim_0 y$ implies $\varphi(x) = \varphi(y)$, because if $x \sim y$, then there exists a sequence $x \sim_0 z_0 \sim_0 z_1 \sim_0 \cdots \sim_0 z_n \sim_0 y$. We then see $\varphi(x) = \varphi(z_0) = \cdots = \varphi(z_n) = \varphi(y)$.

Suppose $x \sim_0 y \in B \sqcup C$. If $x = y$ then obviously $\varphi(x) = \varphi(y)$. Suppose now that $x \neq y$ and without loss of generality that $x \in B$ and $y \in C$, instead of the other way around. Let $a \in A$ be such that $f(a) = x$ and $g(a) = y$ and therefore $\varphi(x) = p'(f(a)) = q'(g(a)) = \varphi(y)$. We conclude that $\overline{\varphi} : (B \sqcup C)/\sim \to Y, [x] \mapsto \varphi(x)$ is well-defined. Notice that for all $b \in B$ we get that $(\overline{\varphi}p)(b) = \overline{\varphi}([b]) = \varphi(b) = p'(b)$ and in the same way we have that $(\overline{\varphi}q)(c) = q'(c)$ for all $c \in C$.

Next we need to check that $\overline{\varphi}$ is continuous. By definition of the quotient topology $\overline{\varphi}$ is continuous if and only if $\overline{\varphi} \circ r = \varphi$ is continuous, which we got from Lemma 16.

Finally we need to prove uniqueness. Suppose $\psi : B \cup_A C \to Y$ such that $\psi p = p'$ and $\psi q = q'$. Let $x \in p(B)$ and $y \in B$ be such that $p(y) = x$. We then get $\overline{\varphi}(x) = (\overline{\varphi}p)(y) = p'(y) = (\psi p)(y) = \psi(x)$ A symmetric argument proves that for $x \in q(C)$ we get $\overline{\varphi}(x) = \psi(x)$ as well. Since $p(B) \cup q(C) = B \cup_A C$, we can conclude that $\overline{\varphi} = \psi$ and therefore that $\overline{\varphi}$ is uniquely determined, which means that $B \cup_A C$ is indeed the sought after pushout. □

**Corollary 19.** *We see by taking $A = \emptyset$ in Theorem 18 that $\sqcup$ is the coproduct in the category of topological spaces.*

**Lemma 20.** *Consider the pushout diagram in* Top

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "p"] \\ C \arrow[r, "q"'] & B \cup_A C \end{tikzcd}$$

1. *If $f$ is injective then $q$ is injective as well.*
2. *If $f$ is open and injective then $q$ is open as well.*
3. *If $f$ is open and $g$ is open then $q$ is open as well.*
4. *If $f$ is surjective then $q$ is surjective as well.*

*Proof.* We will freely see $B$ and $C$ as subsets of $B \sqcup C$, so that we can use the equivalence relation from Theorem 18.

1. Suppose that for $x, y \in C$ we have $q(x) = q(y)$, that is, if we interpret $C$ as a subset of $B \sqcup C$, then $x \sim y$. This means that there exist $a_0, \ldots, a_n \in A$ such that $x = g(a_0),\ f(a_0) = f(a_1),\ g(a_1) = g(a_2),\ \ldots,\ f(a_{n-1}) = f(a_n),\ g(a_n) = y$, however since $f$ is injective, this means that $a_0 = a_1,\ a_2 = a_3,\ \ldots,\ a_{n-1} = a_n$ and by combining we get that $x = g(a_0) = \cdots = g(a_n) = y$.
2. We will only apply the assumption that $f$ is injective in the last line. Let $U \subseteq C$ open, then $q(U)$ is open if and only if both $p^{-1}(q(U))$ and $q^{-1}(q(U))$ are. Write $V = p^{-1}(q(U))$.

   If $f^{-1}(V)$ is open, then $f(f^{-1}(V))$ must be open since $f$ is an open map. We see that $f(f^{-1}(V)) = f(\{a \in A \mid f(a) \in V\}) = \{f(a) \mid a \in A \wedge f(a) \in V\} = V \cap f(A)$, however we actually have that $V = p^{-1}(q(U)) = \{b \in B \mid \exists c \in C : b \sim c\} \subseteq f(A)$, because if $b \in B \setminus f(A)$, then $b$ is only equivalent under $\sim_0$ to itself, and by transitive closure also only to itself under $\sim$. We conclude that $f(f^{-1}(V)) = V$ is open if $f^{-1}(V)$ is.

   Since the diagram commutes and taking the inverse image map is functorial, we get now that $f^{-1}(V) = f^{-1}(p^{-1}(q(U))) = g^{-1}(q^{-1}(q(U))$, which is open if $q^{-1}(q(U))$ is, because $g$ is continuous.

$f$ is injective and so by (1) $q$ is as well and therefore $q^{-1}(q(U)) = U$, which is open by assumption.

3. In the previous point, we proved it is sufficient that $q^{-1}(q(U))$ is open. We note that $q^{-1}(q(U)) = \{c \in C \mid \exists u \in U : c \sim U\}$. The relation is generated by $x \sim y \Leftarrow x = y \vee \exists a \in A : \{x, y\} = \{f(a), g(a)\}$. Consider the set of points $U_1 := \{c \in C \mid \exists a_1, a_2 \in A : g(a_1) = c \wedge f(a_1) = f(a_2) \wedge g(a_2) \in U\}$. This is the set of points of $C$ which are via one point in $B$ equivalent to a point in $U$. We note:

$$\begin{aligned} U_1 &= \{c \in C \mid \exists a_1, a_2 \in A : g(a_1) = c \wedge f(a_1) = f(a_2) \wedge g(a_2) \in U\} \\ &= \{g(a_1) \mid a_1 \in A \wedge \exists a_2 \in g^{-1}(U) : f(a_1) = f(a_2)\} \\ &= g(\{a_1 \in A \mid \exists b \in f(g^{-1}(U)) : f(a_1) = b\}) \\ &= g(f^{-1}(f(g^{-1}(U)))). \end{aligned}$$

Indeed then, if we set $U_0 = U$ and $U_{i+1} = g(f^{-1}(f(g^{-1}(U_i))))$ for each $i \in \mathbb{N}_{>0}$, then $q^{-1}(q(U)) = \cup_{i=0}^{\infty} U_i$ and because both $f$ and $g$ are open and continuous, we find that each $U_i$ is open, and therefore their union must be as well.

4. Let $x \in B \cup_A C$. We want to prove $x \in q(C)$. We have that $p(B)$ and $q(C)$ together cover $B \cup_A C$ and so we either have $x \in p(B)$ or $x \in q(C)$. If the latter, then we are done so assume the former.

   Let $b \in B$ be a point such that $p(b) = x$, now since $f$ is surjective, there exists a point $a \in A$ such that $f(a) = b$, or indeed $p(f(a)) = x$, but the diagram is commutative and so $q(g(a)) = x$ as well, which proves the claim.

□

**Lemma 21.** *Let $Y$ be any topological space equipped with $q', p'$ making the diagram commute:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "p'"] \\ C \arrow[r, "q'"'] & Y \end{tikzcd}$$

*Then we get for the induced morphism $\overline{\varphi} : B \cup_A C \to Y$:*

1. *If $p, q, p'$ and $q'$ are injective and $(p' \circ f)(A) = p'(B) \cap q'(C)$, then the induced $\overline{\varphi}$ is injective.*
2. *If $p'$ and $q'$ are open, then so is $\overline{\varphi}$.*

*Proof.* We will use notations as in (the proof of) Theorem 18.

1. Suppose $x, y \in (B \sqcup C)/\sim$ such that $\overline{\varphi}(x) = \overline{\varphi}(y)$. We note that $p(B) \cup q(C) = B \cup_A C$.

   If $x, y \in p(B)$ then let $b, c \in B$ such that $p(b) = x$ and $p(c) = y$. Then we can see that $p'(b) = \overline{\varphi}(p(b)) = \overline{\varphi}(x) = \overline{\varphi}(y) = \overline{\varphi}(p(c)) = p'(c)$ so that by the injectivity of $p'$ we may conclude $b = c$, and therefore $x = p(b) = p(c) = y$. By symmetry we get the same if $x, y \in q(C)$.

   Suppose now without loss of generality that $x \in p(B)$ and $y \in q(C)$. Then let $b \in B$ and $c \in C$ be such that $p(b) = x$ and $q(c) = y$. We get that $p'(b) = (\overline{\varphi}p)(b) = (\overline{\varphi}q)(c) \in p(B) \cap q(C) = (p'f)(A)$. So take an $a \in A$ such that $q'(c) = p'(b) = p'(f(a)) = q'(g(a))$.

We now use that $p'$ and $q'$ are injective so that $c = g(a)$ and $b = f(a)$, that is, $b \sim c$ in $B \sqcup C$ and therefore $x = p(b) = q(c) = y$.

We conclude that $\overline{\varphi}$ is injective.

2. Suppose $U \subseteq (B \sqcup C)/\sim$ open, then $\rho^{-1}(U) \subseteq (B \sqcup C)$ must be open and by Lemma 16 $\varphi$ is an open man and therefore $\overline{\varphi}(U) = \varphi(\rho^{-1}(U))$ must be an open set, which proves that $\overline{\varphi}$ is an open map.

□

One might wonder if the assumption in the previous lemma that $(p'f)(A) = p'(B) \cap q'(C)$ is necessary: it is, take $A = \emptyset$ and $B, C = \mathbb{R}$, then $B \cup_A C = \mathbb{R} \sqcup \mathbb{R}$. Now take $Y = \mathbb{R} \cup_{\mathbb{R}\setminus\{0\}} \mathbb{R}$, with $p'$ injecting in the first term of $Y$ and $q'$ in the second. All the other assumptions are satisfied but the natural map $\varphi : \mathbb{R} \sqcup Y = \mathbb{R} \to \mathbb{R} \cup_{\mathbb{R}\setminus\{0\}} \mathbb{R}$ is not injective and therefore not an open immersion.

## 1.3 Connectedness Properties

Let us see how connectedness properties transfer when glueing. In the rest of this section $\{0, 1\}$ is assumed to have the discrete topology.

**Definition 22.** A topological space $X$ is called connected if there are exactly two morphisms $X \to \{0, 1\}$.

**Lemma 23.** *$X$ is connected if and only if any of:*

1. *$X \neq \emptyset$ and the only continuous maps $X \to \{0, 1\}$ are the constant maps $x \mapsto 0$ and $x \mapsto 1$.*
2. *The only partition of $X$ into exactly two disjoint open sets is $\{X, \emptyset\}$.*

*Proof.* These are just rephrasings of the definition. □

Note that this definition implies that $\emptyset$ is not connected.

**Lemma 24.** *If we have a pushout of topological spaces as in the diagram below and $A \neq \emptyset$ and $B$ and $C$ are connected, then $X$ is connected.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "p"] \\ C \arrow[r, "q"'] & X \end{tikzcd}$$

*Proof.* Let $\phi : X \to \{0, 1\}$ be any continuous map. We will prove that $\phi$ is constant, so that $X$ is connected.

Let $a \in A$ be any point. We set $\phi' : X \to \{0, 1\}, x \mapsto (\phi p f)(a)$ Now $\phi' p = x \mapsto (\phi p f)(a)$. Since $B$ is connected, $\phi p, \phi' p : B \to \{0, 1\}$ must be constant maps, so since $(\phi p)(f(a)) = (\phi' p)(f(a))$ we must have $\phi p = \phi' p$.

For $\phi q = \phi' q$ we note that by commutativity $(\phi p f)(a) = (\phi q g)(a)$ and then $(\phi q)(g(a)) = (\phi' q)(g(a))$, while $C$ is connected, so that $\phi q, \phi' q : C \to \{0, 1\}$ must be constant, hence equal.

This proves $\phi = \phi'$ is constant and therefore that $X$ is connected. □

This gives us the following nice corollary:
**Corollary 25.** *Suppose we have the following pushout diagram with $f$ and $g$ both open immersions, $B$ and $C$ connected and $A$ non-empty and all three $n$-manifolds, then $X$ is a connected $n$-manifold.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "p"] \\ C \arrow[r, "q"'] & X \end{tikzcd}$$

*Proof.* We simply use Lemmas 20.1 and 20.2 to find that $p$ and $q$ are open immersions. Then we apply Lemma 14 to find that $X$ is an $n$-manifold, using $p(B) \cup q(C) = X$. Connectedness follows from Lemma 24. $\square$

**Definition 26.** Let $M$ be a connected $n$-manifold. An atlas of $M$ ordered by connection is a sequence $(U_i)_{i\in\mathbb{N}}$ of open subsets of $M$ such that for each $i$ we have that:

1. $U_i \cong \mathbb{R}^n$
2. $U_i \cap \bigcup_{j<i} U_j \neq \emptyset$ and finally
3. $\bigcup_{j\in\mathbb{N}} U_j = M$.

**Lemma 27.** *If $M$ is a connected $n$-manifold, then there exists an atlas of $M$ ordered by connection.*

*Proof.* First we take for each $x \in M$ an open $U_x \ni x$ such that $U_x \cong \mathbb{R}^n$, then use that $M$ is Lindelöf to get a countable subcover $U_1, U_2, \dots$ of $\{U_x \mid x \in M\}$. Finally we use induction to find a reordered sequence $(V_i)$ of $(U_i)$. In spirit it is similar to Dijkstra's path finding algorithm.

We set $V_1 = U_1$. Let $i > 1$ and assume that we already defined $V_k$ for all $k < i$. Now we set $\hat{\jmath}$

$$\hat{\jmath}(i) := \min\{j \mid U_j \notin \{V_1, \dots, V_{i-1}\} \wedge U_j \cap \bigcup_{k<i} V_k \neq \emptyset\}$$

to be the index of the first $U_j$ that was not used before and that is connected to one of the earlier $V_k$ and we set $V_i = U_{\hat{\jmath}}$.

We claim that $\hat{\jmath} : \mathbb{N} \to \mathbb{N}$ is bijective. We first prove that it is an injection. Suppose there are $k, k' \in \mathbb{N}$ such that $\hat{\jmath}(k) = \hat{\jmath}(k')$. If $k = k'$ we are done, so assume without loss of generality that $k < k'$. We now get that $V_k = U_{\hat{\jmath}(k)} = U_{\hat{\jmath}(k')}$, but by construction $U_{\hat{\jmath}(k')} \notin \{V_1, \dots V_k, \dots, V_{k'-1}\}$ which is a contradiction, so we conclude that $\hat{\jmath}$ is injective.

Let $A = \mathbb{N} \setminus \hat{\jmath}(\mathbb{N})$ be the indices not in the image of $\hat{\jmath}$. For any $a \in A$, we get that if there exists some $b \in \mathbb{N}$ such that $a \in \{j \mid U_j \notin \{V_1, \dots, V_{b-1}\} \wedge U_j \cap \bigcup_{k<b} V_k\}$ then after removing the smallest element at most $a$ times, $a$ will be picked. So in fact no such $b$ can exist. This means that $U_a \cap \bigcup_k V_k = \emptyset$ and by extension

$$\left(\bigcup_{a\in A} U_a\right) \cap \left(\bigcup_k V_k\right) = \emptyset.$$

Of course the union of these two terms is $M$ and so since $M$ is connected, one of the two terms must be empty. The second term is never empty and therefore the first, $\bigcup_{a\in A} U_a$, must be. Since no $U_k$ is empty, this means that $A$ must be empty.

Now that $\hat{j}$ is bijective, we know that $(V_k)_k$ has the last property of an atlas ordered by connection, while it has the first two by construction. $\square$

**Definition 28.** A topological space $X$ is path-connected if for every pair of points $x, y \in X$ there is a continuous map $f : I \to X$ such that $f(0) = x$ and $f(1) = y$.
**Lemma 29.** *If a topological space $X$ is path-connected, then it is connected.*

*Proof.* Suppose $X$ is not connected, then take a non-constant $f : X \to \{0, 1\}$ and $x, y \in X$ such that $f(x) \neq f(y)$. If there was a path $g : I \to X$ with $g(0) = x$ and $g(1) = y$ then $f \circ g$ gives a non constant map $I \to \{0, 1\}$ but $I$ is connected so such a map $g$ does not exist and therefore $X$ is not path-connected. $\square$

**Lemma 30.** *An $n$-manifold $M$ is connected if and only if it is path-connected.*

*Proof.* We already proved that every path connected space is connected in the previous lemma, so we only need to show that every connected manifold is path-connected.

Let $(U_i)_i$ be an atlas ordered by connection of $M$. We will prove that from any point $x \in M$ there exists a path to a point in $U_1$, using path composition this gives a path between any two points.

Let us prove inductively that for all $j \in \mathbb{N}$ the subspace $\bigcup_{i \leq j} U_i$ is path-connected. Notice that $U_1 \cong \mathbb{R}^n$ is path-connected. Let $x_1 \in U_1$ be any point.

Suppose for all $j' < j$ we already have that $\bigcup_{i \leq j'} U_i$ is path-connected, then we use that $U_j \cap \bigcup_{i<j} U_i \neq \emptyset$, so for any $x \in U_j$ we take any point $x'$ in this intersection and find a path in $U_j$ between $x$ and $x'$ and using the induction hypothesis we find a path between $x'$ and $x_1$, composition yields a path between $x$ and $x_0$ as necessary. $\square$

# 2 Weak Homotopy Equivalences

In this subsection we will define weak homotopy equivalences and the weak homotopy type. To define these we will define the homotopy groups $\pi_n(X, x)$ for all spaces $X$ with a point $x \in X$ and all $n$.

The natural domain of these functors is not the category of topological spaces, but the category of pointed topological spaces. The objects of the category of pointed topological spaces are pairs of a topological space $X$ with a basepoint $x \in X$. Morphisms in the category of pointed topological spaces (pointed morphisms) are now morphisms of topological spaces that send the basepoint of the domain to the basepoint of the codomain. If $(X, x)$ and $(Y, y)$ are pointed spaces, then a homotopy $F : I \times X \to Y$ must keep the basepoint fixed throughout, that is, for all $t \in I$ we must have $F(t, x) = y$. Let $*$ the one-point space with its only point as basepoint.

If $(X, x)$ and $(Y, y)$ are pointed topological spaces then the coproduct of $(X, x)$ and $(Y, y)$ must be the pushout of $(X, x) \leftarrow * \to (Y, y)$, which is the quotient of $X \sqcup Y$ where only $x$ and $y$ are identified. We will write this coproduct as $X \vee Y$.

We will omit basepoints from notation, hopefully no confusion will arise. For pointed spaces $X, Y$ we will denote their set of morphisms as $\mathrm{Hom}_*(X, Y)$ to stress that these morphisms are continuous functions $X \to Y$ which send the basepoint of $X$ to the basepoint of $Y$.

**Definition 31.** Let us define some spaces related to spheres central to our theory. Let $n \in \mathbb{N}_0$.

1. $D^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$
2. $B^n := \{x \in \mathbb{R}^n : \|x\| < 1\}$
3. $S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$

In the case of $D^n$ and $B^n$ we set the basepoint, when not explicitly mentioned, to be $0 \in \mathbb{R}^n$; for $S^n$ we take the basepoint $(1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$.

We will now define the homotopy groups. We will however only give a sketch of the group structure, for details, see [3], Chapter 4.1.

**Definition 32.** Let $(X, x)$ be a topological space with basepoint and $n \in \mathbb{N}_0$, then we set $\pi_n(X, x) = \mathrm{Hom}_*(S^n, X)/\sim$, where $f \sim g$ if and only if $f$ and $g$ are homotopic. For $n > 0$ we define a group structure. Let $f, g \in \mathrm{Hom}_*(S^n, X)$, then they define a unique map $f \vee g : S^n \vee S^n \to X$. Consider now $S^n \vee S^n \cong (S^n \times \{1\} \cup S^n \times \{2\})/\sim_2$, where $\sim_2$ denotes the glueing of the basepoints, then we can define a map $p : S^n \to S^n \vee S^n$ that contracts a suitable equator.

The group operation is now given for maps $f, g \in \mathrm{Hom}_*(S^n, X)$ by $[g][f] = [(f \vee g) \circ p]$. The order is chosen so that for $S^1$, if one lifts $f$ and $g$ to maps $f', g' : [0, 1] \to X$ and $[g][f]$ to a map $h' : [0, 1] \to X$, then $h'$ is obtained by first following $f'$ at twice the speed and then following $g'$ at twice the speed.

One can prove that this is a group operation with identity represented by the constant map to the basepoint of $X$. One proves the group axioms by finding suitable homotopies.

We note that $\pi_n(X, x)$ is abelian for $n > 1$. We also note that we can define functors $\pi_n : \mathrm{Top}_* \to \mathrm{Grp}$ by mapping pointed spaces $(X, x)$ to $\pi_n(X, x)$ and basepoint preserving maps $f : X \to Y$ as $\pi_n(f)([\gamma]) = [f \circ \gamma]$.

**Definition 33.** Let $X, Y$ be topological spaces. A weak homotopy equivalence $f$ from $X$ to $Y$ is a continuous map $f : X \to Y$ inducing isomorphisms on all homotopy groups.

We note that contrary to the name, which is unfortunately standard in the literature, weak equivalences actually do not define an equivalence relation, because there does not always exist a weak homotopy equivalence $Y \to X$ even when there does exist a weak homotopy equivalence $X \to Y$. To fix this problem we define the weak homotopy type.

**Definition 34.** Let $X, Y$ topological spaces. $X$ and $Y$ are of the same weak-homotopy type if they are equivalent in the equivalence relation generated by the weak homotopy equivalences, that is, if there exist spaces $X_0 := X, \ldots, X_N := Y$ and for each $i = 1, \ldots, N$ an equivalence $f_i : X_{i-1} \to X_i$ or an equivalence $f_i : X_i \to X_{i-1}$.

So to calculate the weak homotopy type it is fundamental to know the homotopy groups, since if the homotopy groups are not all isomorphic then no continuous function will induce an isomorphism on all homotopy groups.

The fundamental group is traditionally treated via covering spaces and somewhat less traditionally via the fundamental groupoid. We will quickly go over the definitions and main results.

## 2.1 Free Groups

**Definition 35.** A group $G$ is free over a basis set $S \subseteq G$ if for every group $H$ and every function $\hat{f} : S \to H$ to the underlying set of $H$, there exists a unique homomorphism $f : G \to H$ with $f(s) = \hat{f}(s)$ for all $s \in S$.

**Lemma 36.** *For each set $S$ there exists a free group $F(S)$ over $S$.*

*Proof.* To realise such a group, let $\hat{F}(S)$ be the set of finite sequences of $A := \{s, s^{-1} \mid s \in S\}$, where $s^{-1}$ will denote the formal inverse of $s$. Such sequences we will denote as $s_1 s_2 \cdots s_n$, with all the $s_i \in A$.

Denote by $\epsilon \in \hat{F}(S)$ the empty sequence. We will set $(s^{-1})^{-1} = s$. Let $\sim$ be generated by the equivalences $tss^{-1}u \sim tu$ for all $s \in A$ and $t, u \in \hat{F}(S)$. Then $F(S) := \hat{F}(S)/\sim$, with as group operation the concatenation of sequences is free over the basis $S$.

Indeed, suppose $H$ is a group and $\hat{f} : S \to H$ is a function. First we can set $f' : A \to H$ to be the extension of $\hat{f}$ such that $f'(s^{-1}) = \hat{f}(s)^{-1}$. Then we set $f : F(S) \to H, s_1 \cdots s_n \mapsto f'(s_1) \cdots f'(s_n)$, where each $s_i \in A$. One immediately sees that if $s \in A$ and $t, u \in F(S)$ that $f(tss^{-1}u) = f(tu)$, so $f$ is well defined and a homomorphism by the construction $f(s_1 \cdots s_n) = f'(s_1) \cdots f'(s_n)$, with $f(s) = \hat{f}(s)$ for each $s \in S$. $\square$

**Definition 37.** A group $G$ is generated by a subset $S \subseteq G$ if the homomorphism $f : F(S) \to G$, that is defined by sending $g \in S$ to itself, is surjective.

$G$ is finitely (respectively countably) generated if a subset $S$ of smallest cardinality generating $G$ is finite (respectively countably infinite).

Note that there exists groups which do not have a minimal generating set in the sense of inclusion. Of course we can take still take one of minimal cardinality, since the cardinals are well-ordered. An example of this phenomenon is the group $\mathbb{Z}/3\mathbb{Z}$, it is generated both by the classes $\overline{1}$ and $\overline{2}$.

**Lemma 38.** *Let $T$ be a countably infinite set and $S \subseteq F(T)$ be a finite set, then there is no surjective homomorphism from $G := F(S)$ to $H := F(T)$.*

*Proof.* We need to show that there is no surjective homomorphsim $f : F(S) \to F(T)$.

Let $G$ be the free group on $S$ and $f : G \to H$ be a surjective homomorphism. The composition $G \xrightarrow{f} H \longrightarrow H^{\text{ab}}$ is a homomorphism from $G$ to an abelian group and therefore factors via the abelianisation of $G$, so that we find an $\overline{f} : G^{\text{ab}} \to H^{\text{ab}}$ such that the following square, with the natural maps for the vertical arrows, commutes:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] & H \arrow[d] \\ G^{\text{ab}} \arrow[r, "\overline{f}"'] & H^{\text{ab}} \end{tikzcd}$$

Notice that since taking the abelianisation is just modding out the commutator subgroup that the vertical arrows are surjective, since $f$ is too, this means that $\overline{f}$ must be. We can represent this as an exact sequence

$$G^{\text{ab}} \to H^{\text{ab}} \to 0$$

Now $\overline{f}$ is a morphism of abelian groups and abelian groups are $\mathbb{Z}$-modules. Tensoring is right exact so when we get an exact sequence

$$G^{\text{ab}} \otimes \mathbb{Q} \to H^{\text{ab}} \otimes \mathbb{Q} \to 0.$$

We see that $G^{\mathrm{ab}} \otimes \mathbb{Q} \cong \mathbb{Q}^S$ and $H^{\mathrm{ab}} \otimes \mathbb{Q} \cong \mathbb{Q}^T$. However, since $\dim(\mathbb{Q}^S) = |S| < |T| = \dim(\mathbb{Q}^T)$ there is no surjective homomorphism $G^{\mathrm{ab}} \otimes \mathbb{Q} \to H^{\mathrm{ab}} \otimes \mathbb{Q}$. We conclude that $f$ cannot be surjective. □

**Definition 39.** The free product of two groups $G$ and $H$ is denoted as $G * H$, together with maps $i_G : G \to G * H$ and $i_H : H \to G * H$ and defined by the universal property that for each group $A$ with morphisms $f_G : G \to A$ and $f_H : H \to A$ there exists a unique homomorphism $G * H \to A$ such that $f \circ i_G = f_g$ and $f \circ i_H = f_H$.

It can be constructed by taking finite sequences of elements of $G \sqcup H$, that is from $F(G \sqcup H)$, modulo relations (1) $w_1 u_1 u_2 w_2 \equiv w_1 u_3 w_2$ for all $w_1, w_2 \in F(G \sqcup H)$ if $u_1, u_2, u_3 \in G$ with $u_1 u_2 = u_3$ or $u_1, u_2, u_3 \in H$ with the same relation and (2) $w_1 w_2 \equiv w_1 e w_2$ for all $w_1, w_2 \in F(G \sqcup H)$ and for $e$ the identity element of $G$ or $H$.

**Lemma 40.** *If $F(S)$ and $F(T)$ are free groups on bases $S$ and $T$ respectively then*

$$F(S) * F(T) \cong F(S \sqcup T)$$

*Proof.* $F$ is a left-adjoint functor to the forgetful functor and so preserves coproducts. A direct proof can be given via a diagram chase. □

**Lemma 41.** *If $G, H$ and $I$ are groups and $f : I \to G$ is a surjective homomorphism, then*

$$f_* : I * H \to G * H,$$

*induced by $I \xrightarrow{f} G \xrightarrow{i_G} G * H$ and $i_H$ is surjective.*

*Proof.* Suppose $x \in G * H$ then there exist $g_1, \ldots, g_n$ and $h_1, \ldots h_n$ such that $x = g_1 h_1 g_2 \cdots g_n h_n$. Now since $i_G f$ is surjective, we can find elements $f_1, \ldots, f_n$ such that for $i = 1, \ldots, n$ we have $(i_G f)(f_i) = g_i$. Now $f_*(f_1 h_1 \cdots f_n h_n) = f(f_1) h_1 \cdots f(f_n) h_n = g_1 h_1 \cdots g_n h_n = x$. □

## 2.2 The Seifert-van Kampen Theorem

**Definition 42.** A category is called small, if both its class of objects and its class of morphisms are a set.

**Definition 43.** A groupoid $G$ is a small category where every morphism is an isomorphism.

We will denote the set of morphisms in $G$ between two objects $a$ and $b$ by $\mathrm{Hom}_G(a, b)$ and in particular the group of automorphisms at an object $a$ by $\mathrm{Aut}_G(a)$

Morphisms of groupoids are functors between the involved categories.

**Example 44.** Let $G$ be a group. Now consider the category with a single object $\{x\}$ and morphisms $\mathrm{Hom}(x, x) = G$ with composition given by the group operation of $G$. Then this is a groupoid. Every group will be considered a groupoid in this way.

**Definition 45.** A groupoid $G$ is connected if between every two objects $a, b \in G$ there is a morphism $a \to b$.

**Definition 46.** A groupoid $G$ is totally disconnected if the existence of a morphism between two objects $a, b \in G$ implies that $a = b$.

**Definition 47.** Let $X$ be a topological space and $A \subseteq X$ be a subset. The fundamental groupoid of $X$ relative to $A$, is denoted by $\pi_1(X, A)$. Its objects are the elements of $A$ and its morphisms are all paths in $X$ up to homotopy between the elements of $A$. Composition is path composition.

That this defines a groupoid is completely analogous to the standard proof that the fundamental group is indeed a group.

We note that given a basepoint $x \in A$, we retrieve the fundamental group as

$$\pi_1(X, x) = \pi_1(X, \{x\}) = \mathrm{Aut}_{\pi_1(X,A)}(x)$$

The Seifert-Van Kampen theorem gives a powerful instrument to calculate the fundamental group of spaces piece by piece. If we have two open subsets $U_1, U_2 \subseteq X$ then we calculate the fundamental groupoid of $X$ from those of $U_1$, $U_2$ and $U_1 \cap U_2$. We will need to have points in each connected component of $U_1 \cap U_2$. Such a set $A \subseteq U_1 \cap U_2$ is called representative of $U_1 \cap U_2$.

**Theorem 48.** *([1], Theorem 6.7.2.) Let $X$ be a topological space, and $U_1, U_2 \subseteq X$ be open sets such that $U_1 \cup U_2 = X$. Let $A \subseteq X$ be a subset which is representative of $U_1 \cap U_2$. Then we have the following pushout squares:*

$$\begin{tikzcd} U_1 \cap U_2 \arrow[r] \arrow[d] & U_1 \arrow[d] \\ U_2 \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} \pi_1(U_1 \cap U_2, A) \arrow[r] \arrow[d] & \pi_1(U_1, A) \arrow[d] \\ \pi_1(U_2, A) \arrow[r] & \pi_1(X, A) \end{tikzcd}$$

**Corollary 49.** *Let $X$ be a topological space, and $U_1, U_2 \subseteq X$ non-empty, open and path connected such that $U_1 \cup U_2 = X$. Suppose $U_1 \cap U_2$ is simply connected. Then the fundamental group around $x \in U_1 \cap U_2$ is given by:*

$$\pi_1(X, x) \cong \pi_1(U_1, x) * \pi_1(U_2, x)$$

*Where $*$ denotes the free product of groups.*

Usually however the intersection is not simply connected. In that case the next theorem is useful.

**Theorem 50.** *(Theorem 8.4.1 of [1]) Suppose $A, B, C$ are groupoids with the same set of objects, $C$ totally disconnected and $A$ and $B$ connected. Consider their pushout $G$:*

$$\begin{tikzcd} C \arrow[r, "i"] \arrow[d, "j"'] & A \arrow[d, "u"] \\ B \arrow[r, "v"'] & G \end{tikzcd}$$

*Let $p$ be an object in $C$. Choose elements $\alpha_x \in \mathrm{Hom}_A(x, p), \beta_x \in \mathrm{Hom}_B(x, p)$ for all objects $x \in C$, with $\alpha_p = \beta_p = \mathrm{Id}$.*

*For each object $x \in C$ set $f_x = u(\alpha_x)^{-1} v(\beta_x) \in \mathrm{Aut}_G(p)$ and let $F$ be the free group generated by $\{f_x \mid x \in C \setminus p\}$*

*Let $r : A \to \mathrm{Aut}_A(p)$ mapping all objects to $p$ and every morphism $\varphi : a \to b$ to $\alpha_b^{-1} \varphi\, \alpha_a$. Define $s : B \to \mathrm{Aut}_B(p)$ in the same way using the $\beta_x$.*

*Then $\mathrm{Aut}_G(p)$ is isomorphic to the quotient of*

$$\mathrm{Aut}_A(p) * \mathrm{Aut}_B(p) * F$$

*by the subgroup*

$$H := \langle (ri\gamma) f_x (sj\gamma)^{-1} f_x^{-1} \mid x \in C, \gamma \in \mathrm{Aut}_C(x) \rangle.$$

**Definition 51.** The subgroup $H$ in the theorem above will be referred to as the subgroup of relations at $p$ of a pushout square of groupoids.

**Lemma 52.** *Suppose that $X$ is a connected $n$-manifold with any basepoint $x \in X$ and a finite open cover $\{U_1, \ldots, U_m\}$ such that for each $i = 1, \ldots, m$ we have that $U_i \cong \mathbb{R}^n$.*

*Then the fundamental group $\pi_1(X, x)$ is at most countably generated. That is, there exists a finite or countable generating set $\Gamma \subseteq \pi_1(X, x)$ such that the natural map*

$$F(\Gamma) = \mathop{\Large *}_{\gamma \in \Gamma} \mathbb{Z} \to \pi_1(X, x),$$

*induced by the inclusion of $\Gamma$ is surjective.*

*If $n = 1$ then the $\pi_1(X, x)$ is free as well.*

*Proof.* Assume without loss of generality that $U_1, \ldots, U_m$ is ordered such that $U_i \cap \bigcup_{j<i} U_j \neq \emptyset$. We set $X_1 = U_1$ and for $i = 1, \ldots, m-1$ we set $X_{i+1} = X_i \cup U_{i+1}$. We will now inductively show that each $X_i$ is connected and that for some series of basepoints $(x_i)$ that $\pi_1(X_i, x_i)$ is countably generated and free if $n = 1$. Indeed $X_m = X$ so this will prove the lemma. Let us immediately point out that this is true for $X_1 = U_1 \cong \mathbb{R}^n$, with any basepoint $x_1$.

Suppose now for some $i \in \mathbb{N}$ that $\pi_1(X_i, x_i)$ is countably generated, that $X_i$ is a connected space and if $n = 1$ that $\pi_1(X_i, x_i)$ is free.

We can see each $X_{i+1}$ as the pushout of $X_i \leftarrow X_i \cap U_{i+1} \to U_{i+1}$ where both arrows are the inclusion. Now we apply Corollary 25; We verify that $X_i \cap U_{i+1}$ is not empty because of the order of the $U_i$ we picked. By the induction hypothesis $X_i$ is a connected $n$-manifold and of course $U_{i+1} \cong \mathbb{R}^n$ is as well. Finally $X_i \cap U_{i+1}$ is an $n$-manifold, because it is an open subset of one (Lemma 5), being the intersection of two open subsets. We conclude that $X_{i+1}$ is a connected $n$-manifold.

Now we apply the Seifert-Van Kampen theorem. We let $S_{i+1}$ be a representative set of points of $X_i \cap U_{i+1}$ and $x_{i+1} \in S_{i+1}$ be a basepoint. The Seifert-Van Kampen theorem (48) now says that $\pi_1(X_{i+1}, S_{i+1})$ is the pushout of $\pi_1(X_i, S_{i+1}) \leftarrow \pi_1(X_i \cap U_{i+1}, S_{i+1}) \to \pi_1(U_{i+1}, S_{i+1})$.

By our choice of $S_{i+1}$ we have that $\pi_1(X_i \cap U_{i+1}, S_{i+1})$ is totally disconnected and both $U_{i+1}$ and $X_i$ are connected $n$-manifolds and so path connected and therefore their fundamental groupoids are connected too. Indeed we will now apply Theorem 50. In the remainder of this proof we will use all the notation of that theorem, with $B = \pi_1(X_i, S_{i+1}), C = \pi_1(X_i \cap U_{i+1}, S_{i+1}), A = \pi_1(U_{i+1}, S_{i+1})$ and $G = \pi_1(X_{i+1}, S_{i+1})$ and $p = x_{i+1}$.

Putting these results together we get an isomorphism:

$$\begin{aligned}
\pi_1(X_{i+1}, x_{i+1}) &= \mathrm{Aut}_G(p) \\
&\cong (\mathrm{Aut}_A(p) * \mathrm{Aut}_B(p) * F)/H \\
&= (\pi_1(U_{i+1}, x_{i+1}) * \pi_1(X_i, x_{i+1}) * F)/H,
\end{aligned}$$

where $H$ is the subgroup of relations at $p = x_{i+1}$ (Definition 51) of the pushout given by the Seifert-van Kampen theorem.

We see that $U_{i+1} \cong \mathbb{R}^n$, so that $\pi_1(U_{i+1}, x_{i+1}) \cong 0$, and of course $\pi_1(X_i, x_{i+1}) \cong \pi_1(X_i, x_i)$ and so really we have that $\pi_1(X_{i+1}, x_{i+1}) \cong (\pi_1(X_i, x_i) * F)/H$.

We notice that $F \cong F(S_{i+1} \backslash \{x_{i+1}\})$, which has a generator for each of the connected components of $X_i \cap U_{i+1}$ except the one containing $x_{i+1}$ . We know that $X_i \cap U_{i+1}$, being an open subset of

$U_{i+1} \cong \mathbb{R}^n$, has a countable number of connected components. Indeed this means that $F$ is at most countably generated. By the induction hypothesis so is $\pi_1(X_i, x_i)$ and we can now conclude that $\pi_1(X_{i+1}, x_{i+1})$ is at most countably generated as well using Lemma 41 and the fact that the natural map from a group to the quotient of this group by a normal subgroup is surjective.

If $n = 1$, then $\pi_1(X_i \cap U_{i+1}, S_{i+1})$ has only identity morphisms, because connected open subsets of $U_{i+1} \cong \mathbb{R}$ are contractible, and so the subgroup of relations $H$ at $x_{i+1}$ (Definition 51) is trivial. This means that we get an isomorphism

$$\pi_1(X_{i+1}, x_{i+1}) \cong \pi_1(X_i, x_i) * F.$$

By the induction hypothesis for $n = 1$ we have that $\pi_1(X_i, x_i)$ is free and $F$ is free as well by the Seifert-van Kampen theorem, so that by Lemma 40 the fundamental group $\pi_1(X_{i+1}, x_{i+1})$ is free too. $\square$

**Theorem 53.** *If $X$ is a connected $n$-manifold with any basepoint $x \in X$, then its fundamental group $\pi_1(X, x)$ is at most countably generated.*

*Proof.* Let $(U_i)_i$ be an atlas ordered by connection of $X$ (Definition 26 and Lemma 27). We set $X_1 = U_1$ and for $i = 1, \ldots$ we set $X_{i+1} = X_i \cup U_{i+1}$. By the previous Lemma 52 we have that $\pi_1(X_i, x_i)$ is countably generated for any series of basepoints $(x_i)$ and that each $X_i$ is connected.

Now notice that $S^1$ is compact. If we take any path $\gamma : S^1 \to X$, then $\{\gamma^{-1}(X_i)\}$ is a cover for $S^1$ and it has a finite subcover, so there exists some finite $i$ such that $\gamma(S^1) \subseteq X_i$. We conclude that the natural map in the category of sets induced by the inclusions

$$\bigsqcup_{i=1}^{\infty} \pi_1(X_i, x_i) \to \pi_1(X, x)$$

is surjective. Since each $\pi_1(X_i, x_i)$ is countable, this union is a countable union of countable sets and hence countable. We conclude that $\pi_1(X, x)$ is countable and in particular at most countably generated. $\square$

**Theorem 54.** *If $X$ is a connected 1-manifold with any basepoint $x \in X$, then its fundamental group $\pi_1(X, x)$ is free.*

*Proof.* Let $(U_i)_i$ be an atlas ordered by connection of $X$ (Definition 26 and Lemma 27). We set $X_1 = U_1$ and for $i = 1, \ldots$ we set $X_{i+1} = X_i \cup U_{i+1}$. We take $x \in U_1$ without loss of generality.

In the proof of Lemma 52 we already derived that for each $i$

$$\pi_1(X_{i+1}, x_{i+1}) \cong \pi_1(X_i, x_{i+1}) * F,$$

where we write $F$ the free group from the application of Theorem 50. For notational convenience, we identify $\pi_1(X_i, x_{i+1}) * F$ with its image in $\pi_1(X_{i+1}, x_{i+1})$. Let $\Gamma_{i+1} \subseteq \pi_1(X_{i+1}, x_{i+1})$ be a basis set for $F$.

Now let $\alpha_i : X_i \to X$ be the inclusion and $p_i$ be a path from $x_i$ to $x$ inside $X_i$. We write $p_{i*}$ for the isomomorphism $\pi_1(X_i, x_i) \cong \pi(X_i, x)$ and for each $j \geq i$ the map $u_{ij} : X_i \to X_j$ be the inclusion. Then we can apply $p_{(i+1)*}$ to find

$$\begin{aligned}\pi_1(X_{i+1}, x) &\cong \pi_1(X_i, x) * F(p_{(i+1)*}(\Gamma_{i+1})) \\ &\cong \mathop{\Huge *}_{j \leq i+1} F(u_{j(i+1)*} p_{j*}(\Gamma_j)) \\ &= F\left(\bigsqcup_{j \leq i+1} u_{j(i+1)*} p_{j*}(\Gamma_j)\right).\end{aligned}$$

We see that $\varphi_i : F\left(\bigsqcup_{j \leq i} u_{ji*} p_{j*}(\Gamma_j)\right) \to F\left(\bigsqcup_{j \leq i+1} u_{j(i+1)*} p_{j*}(\Gamma_j)\right)$ induced by the inclusion $u_{i(i+1)}$ is in fact injective, because it is injective on the basis. This suggest a limiting procedure for finding $\pi_1(X, x)$.

So we build a composed morphism $\pi_1(X_i, x_i) \xrightarrow{\alpha_{i*}} \pi_1(X, x_i) \xrightarrow{p_{i*}} \pi_1(X, x)$, we will write it as $\beta_{i*}$. We now claim that

$$\psi : F(\bigsqcup_{i=1}^{\infty} \beta_{i*}(\Gamma_i)) \to \pi_1(X, x),$$

induced by the inclusion of the base on the left in the group on the right is an isomorphism.

Let us first prove that $\psi$ is surjective. Suppose $[\gamma] \in \pi_1(X, x)$ is any class, then $\{\gamma^{-1}(X_i) \mid i \in \mathbb{N}\}$ gives an open cover for the compact space $S^1$, so we can take a finite subcover, but since the $X_i$ form a tower of subspaces, this means there exists some $i$ such that $\gamma(S^1) \subseteq X_i$ and therefore $[\gamma] \in \pi_1(X_i, x)$.

In $X_i$ we know that $[\gamma] = [\gamma_1] \cdots [\gamma_m]$, where each of the classes of loops $[\gamma_k]$ is either an element of the basis set $\bigcup_{j \leq i} u_{ji*} p_{j*}(\Gamma_j)$ or the inverse of a loop in the basis. Now we notice that $\alpha_{i*} u_{ji*} p_{j*} = (\alpha_i u_{ji})_* p_{j*} = \alpha_{j*} p_{j*} = \beta_{i*}$, so that $\alpha_{i*}(\bigcup_{j \leq i} u_{ji*} p_{j*}(\Gamma_j)) = \bigcup_{j \leq i} \beta_{j*}(\Gamma_j) \subseteq \bigcup_{j=1}^{\infty} \beta_{j*}(\Gamma_j)$ and indeed we conclude that $[\gamma] = \alpha_{i*}([\gamma]) \subseteq \mathrm{Im}(\psi)$, because for each $k$ we get $\alpha_{i*}([\gamma_k]) \in \bigcup_{i=1}^{\infty} \beta_{i*}(\Gamma_i)$.

To show that $\psi$ is injective, we have to show that $\psi$ has a trivial kernel. Let $[x]$ denote the constant path at $x$ and $[\gamma_1], \ldots, [\gamma_m] \in \bigcup_{i=1}^{\infty} \beta_{i*}(\Gamma_i \cup \Gamma_i^{-1})$ such that $[\gamma_1] \cdots [\gamma_m] = [x]$ and such that for all $i = 1, \ldots, m-1$ we have $[\gamma_i][\gamma_{i+1}] \neq [x]$.

Then there must be a homotopy $H : I \times S^1 \to X$ that homotopes the composition on the left to the constant path. Now we use that $I$ and $S^1$ are compact, as is their product, to show that $H(I \times S^1) \subseteq X_i$ for some $i$, and we used only finitely many basis elements, so there exists some $j$ sufficiently large such that all the basis elements and this homotopy exist over $X_j$, where they are basis elements again, but then we would get that $\pi_1(X_j, x)$ is not free, which we already proved. □

## 2.3 The Universal Covering Space

**Definition 55.** Let $X$ be a topological space. A covering space $Y$ of $X$ is a topological space $Y$ with a map $p : Y \to X$, such that for each $x \in X$ there exists a $U \ni x$ with $p^{-1}(U) = \bigcup_{i \in I} U_i$, where all the $U_i$ are disjoint and $p|_{U_i} : U_i \to U$ is a homeomorphism. .

We want to give a universal cover, one that covers all connected covers of $X$.
**Definition 56.** The universal cover is a connected cover $p : \tilde{X} \to X$ such that for every connected cover $q : Y \to X$, there is a connected cover $r : \tilde{X} \to Y$ such that $p = q \circ r$.

There is a Galois theory of covering spaces and this gives rise to the following lemma. See [7], Chapter 2 for a complete account.

**Lemma 57.** *If $X$ is a connected topological space with universal cover $p : \tilde{X} \to X$ then for any basepoint $x \in X$ we get that $p^{-1}(x)$ is in bijection with $\pi_1(X, x)$.*

*Proof.* We use that transitive actions of a group $G$ correspond to the actions of $G$ on $G/H$ for all subgroups $H$. We use [7], Theorem 2.3.4 to see that the action of $\pi_1(X, x)$ on $p^{-1}(x)$ corresponds to the action of $\pi_1(X, x)$ on $\pi_1(X, x)/1 = \pi_1(X, x)$. Indeed we conclude that there is an isomorphism of $\pi_1(X, x)$ sets between $\pi_1(X, x)$ and $p^{-1}(x)$ and hence a bijection. □

We want to give a criterion for the existence of $\tilde{X}$, but this requires a new connectedness property.

**Definition 58.** A space $X$ is called locally simply connected if around each point $p \in X$ and for every open $U \ni p$ around it, there exists a simply connected open $p \in V \subset U$.

**Theorem 59.** *([7], 2.3.7) If $X$ is connected and locally simply connected, then there exists a universal cover.*

**Theorem 60.** *If $X$ is a connected manifold, then there exists a universal cover.*

*Proof.* We only need to prove that $X$ is locally simply connected, but this follows, because around each point $x \in X$ there is an open $U \cong \mathbb{R}^n$ and for each open $x \in V \subseteq U$ we can take an open ball in $V$ around $x$. This ball is homeomorphic to $\mathbb{R}^n$, which is contractible and hence simply connected. □

**Theorem 61.** *The universal cover $p : \tilde{X} \to X$ of a connected $n$-manifold exists and is again a connected $n$-manifold.*

*Proof.* By the previous theorem the universal cover exists.

Consider $y \in \tilde{X}$, then around $p(y)$ there is an open neighbourhood $U \ni p(y)$ homeomorphic to $\mathbb{R}^n$, and so $p^{-1}(U) \cong U \times V$ for some discrete space $V$. This means that each connected component of $p^{-1}(U)$ is homeomorphic to $\mathbb{R}^n$ and some component must contain $p$. We conclude that $\tilde{X}$ is locally Euclidean.

Now we use Theorem 53 to get that $X$ has a countable fundamental group and Theorem 57 to get that the fibre $p^{-1}(x)$ over any point $x \in X$ is countable.

Around each point $x \in X$ choose an open set $U$ such that $p^{-1}(U) = \sqcup_i U_i$ and such that $p|U_i : U_i \to U$ is a homeomorphism. Notice that since the fiber over each point is countable, the number of $U_i$ is countable as well. Let $\mathcal{V}$ be a countable subcover of the Lindelöf space $X$ such that for each $V_i \in \mathcal{V}$ we have that $p^{-1}(V) = \bigcup_{j=1}^{\infty} V_j$ with all the $V_i^j$ disjoint for a constant $i$

Now we set $\mathcal{V}' = \{V_i^j \mid i, j \in \mathbb{N}\}$. Let $\mathcal{U}$ be a countable basis for the topology of $X$. For each $V \in \mathcal{V}'$, let $\mathcal{B}_{p(V)} = \{p(V) \cap U \subseteq \mathcal{U}\}$ be a countable basis for the open set $p(V)$. Indeed $\mathcal{B}_V = \{p^{-1}(b) \mid b \in \mathcal{B}_{p(V)}\}$ is a countable basis for $V$, because $V \cong p(V)$, and now $\bigcup_{V \in \mathcal{V}'} \mathcal{B}_V$ is a countable basis for $\tilde{X}$, because a countable union of countable sets is countable. □

# 3 The Higher Homotopy Groups of 1-Manifolds

In this section we will prove that the higher homotopy groups of 1-manifolds vanish. We will prove this in the following way: first we note that these homotopy groups are isomorphic to those of the universal cover, secondly we calculate the homology groups of the universal cover, and thirdly we apply Hurewicz theorem inductively to get our result.

## 3.1 Higher Homotopy Groups and the Universal Cover

We begin by recalling that every connected manifold has a universal cover (Lemma 60). The next theorem gives the existence of the required isomorphism between the higher homotopy groups of a space and those of its universal cover.

**Theorem 62.** *Example 4.49, [3] A path-connected covering space $B$ of a path-connected space $A$ has for $n \geq 2$ that $\pi_n(A) \cong \pi_n(B)$.*

This proof requires more theory than can be included here, see the cited example and see theorem 4.41 and onwards from [3], 4.49 is this theorem. For those well versed in homotopy theory follows a proof.

*Proof.* Note that under these premises $B \to A$ is a fibration with discrete fiber, such that for $k > 0$, the $k$-th homotopy group of the fiber is trivial. The long exact sequence of homotopy now gives the requested isomorphisms. □

In particular we can conclude that for a connected 1-manifold $M$ there exists a universal cover $\hat{M}$ and that for each $k > 1$ there are isomorphisms $\pi_k(M) \cong \pi_k(\hat{M})$.

## 3.2 Homology

Homology is in many ways similar to homotopy. Both are about giving algebraic invariants to topological spaces. But whereas homotopy uses spheres, homology uses simplices.

We encourage the reader to read a more comprehensive text on homology, [3] for example, if necessary.

Singular Homology tries to quantify "holes" in a space $X$. To detect these it looks at the maps from simplices to $X$, so let us first define the simplices.

**Definition 63.** The standard $n$-simplex $\Delta^n$ is the topological subspace of $\mathbb{R}^{n+1}$ given by the convex hull of the standard basis vectors $e_1, \ldots, e_{n+1}$.

The standard $n$-simplex comes equipped with face maps $f_i^n$ for each $i = 0, \ldots, n$, which are defined by convexly extending the map:

$$\hat{f}_i^n : \Delta^{n-1} \to \Delta^n, e_j \mapsto \begin{cases} e_j & \text{If } j < i \\ e_{j+1} & \text{If } j \geq i \end{cases}$$

An $n$-simplex in $X$ is a continuous map $\sigma : \Delta^n \to X$.

**Definition 64.** Let $X$ be a topological space, then the simplicial chain complex $C_\bullet(X)$ on $X$ is defined as having groups

$$C_k(X) = \bigoplus_{\sigma:\Delta^k \to X} \mathbb{Z}\sigma$$

with boundary map given by linearly extending

$$d_k : C_{k+1} \to C_k, \sigma \mapsto \sum_{i=0}^{k} (-1)^k (\sigma \circ f_i^k).$$

One can verify by direct computation that $d_k d_{k+1} = 0$.

**Definition 65.** We can now define the homology groups of a topological space $X$:

$$H_k(X) = \ker d_k / \operatorname{Im} d_{k+1}$$

**Lemma 66.** *(Properties of Singular Homology)*

*Singular homology satisfies the following properties:*

1. *For the empty space $\emptyset$, there are no functions $\sigma : \Delta^k \to \emptyset$ for any $k$. The empty direct sum is defined as $0$ and so $C_k(\emptyset) = 0$ for all $k$. Consequently $d_k$ is the zero map for all $k$ and $H_k(\emptyset) = 0$.*

2. *If $*$ is the one-point-space then*

$$H_k(*) \cong \begin{cases} \mathbb{Z} & \textit{If } k = 0 \\ 0 & \textit{Else} \end{cases}$$

3. *Let $X = \coprod_{\alpha\in\Omega} U_\alpha$ be a space with all $U_\alpha$ open subsets, then*

$$H_k(X) \cong \bigoplus_{\alpha\in\Omega} H_k(U_\alpha)$$

4. *If $f : X \to Y$ is a continuous map, then $H_k(f) : H_k(X) \to H_k(Y), [\sigma] \mapsto [f \circ \sigma]$ defines a map on homology and this construction is functorial for every $k$.*

5. *If $f, g : X \to Y$ are homotopic for spaces $X, Y$ then $H_k(f) = H_k(g)$. In particular if a space $X$ is contractible, then its homology is that of the point.*

6. *Homology satisfies the Mayer-Vietoris sequence. Let $U_1, U_2 \subseteq X$ open subsets of a space $X$ with inclusions $i$ and $j$ respectively, then we set $\alpha_k : H_k(U_1 \cap U_2) \to H_k(U_1) \oplus H_k(U_2), [\sigma] \mapsto [\sigma] \oplus [\sigma]$ and $\beta_k : H_k(U_1) \oplus H_k(U_2) \to H_k(X), [\sigma] \oplus [\tau] \mapsto [\sigma] - [\tau]$.*

   *Finally we can write every $x \in H_k(X)$ as $x = u + v$ where $u \in i_*(H_k(U_1))$ and $v \in j_*(H_k(U_2))$. We use this to define $\gamma_k : H_k(X) \to H_k(U_1 \cap U_2), u + v \mapsto d_{k*}(u)$. We can place this into an exact sequence as follows:*

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{k+1}(X) \arrow[dll, "\gamma_{k+1}"'] \\
H_k(U_1 \cap U_2) \arrow[r, "\alpha_k"] & H_k(U_1) \oplus H_k(U_2) \arrow[r, "\beta_k"] & H_k(X) \arrow[dll, "\gamma_k"'] \\
H_{k-1}(U \cap V) \arrow[r] & \cdots &
\end{tikzcd}$$

7. *If $f : X \to Y$ is a weak homotopy equivalence, then for all $k \in \mathbb{N}$ the map $H_k(f)$ is an isomorphism.*

8. *Suppose $X$ is a path connected space with a point $x \in X$, then $H_1(X) \cong \pi_1(X, x)^{\mathrm{Ab}}$.*

*Proof.* All of these can be found in chapter 2 of [3], except statement 7, which is Proposition 4.21 of [3]. ◻

**Example 67.** Let us calculate the homology groups $H_k(S^n)$. Let $U_1, U_2$ be all of $S^n$ except respectively the north and south poles. We apply the Mayer-Vietoris sequence

$$H_k(U_1) \oplus H_k(U_2) \longrightarrow H_k(U_1 \cup U_2) \longrightarrow H_{k-1}(U_1 \cap U_2) \longrightarrow H_{k-1}(U_1) \oplus H_{k-1}(U_2)$$

$U_1$ and $U_2$ cover $S^n$, $U_1$ and $U_2$ both contract to the pole they do contain and $U_1 \cap U_2$ contracts to the equator between the poles, which is homeomorphic to $S^{n-1}$. Indeed for $k-1>0$ the first and the last term are both zero, leading to the central morphism to be an isomorphism:

$$H_k(S^n) \cong H_{k-1}(S^{n-1})$$

For $k=0$, we note that $S^n$ is connected for $n \geq 0$ and has two discrete points for $n=0$.

For $k>n$, we apply this repeatedly to get $H_k(S^n) \cong H_{k-n}(S^0) \cong H_{k-n}(* \sqcup *) \cong 0$.

For $1 \leq k \leq n$, we apply this repeatedly to get $H_k(S^n) \cong H_1(S^{n-k+1})$. Since $S^n$ is simply connected for $n>1$, we get $H_k(S^n)=0$ for $n>k$ as well, while for $n=k$ we get $H_n(S^n) \cong H_1(S^1) \cong \pi_1(S^1)^{\mathrm{Ab}} \cong \mathbb{Z}$. Collecting we see:

$$H_k(S^n) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{If } k=n=0 \\ \mathbb{Z} & \text{If } k=0, n>0 \text{ or } k=n \\ 0 & \text{Else} \end{cases}$$

Let us also recall a Künneth theorem, which relates the homology of a product to the homology of its constituents.

**Theorem 68.** *([5], Theorem V.10.1)*[1] *If $X$ and $Y$ are topological spaces, such that for each $n \in \mathbb{N}$ we have that $H_n(X)$ is a free abelian group, then we get isomorphisms:*

$$\bigoplus_{i=0}^{n} (H_i(X) \otimes H_{n-i}(Y)) \cong H_n(X \times Y)$$

Let us apply this Künneth theorem to a useful example:

**Example 69.** Let $n \in \mathbb{N}_{>0}$ be a natural number and $Y$ be a topological space, then we will calculate the homology of $S^n \times Y$.

We can apply the Künneth theorem because $H_k(S^n)$ is always a free abelian group and therefore we get an isomorphism

$$\bigoplus_{i=0}^{n} (H_i(S^n) \otimes H_{n-i}(Y)) \cong H_n(S^n \times Y).$$

We recall that $H_i(S^n) \cong \mathbb{Z}$ if $n=i$ or $i=0$ and $0$ otherwise and that for any abelian group (= $\mathbb{Z}$-module) $A$, we have that $A \otimes 0 = 0$ and $A \otimes \mathbb{Z} = A$. We can conclude then that

$$\begin{aligned} H_k(S^n \times Y) &\cong (H_0(S^n) \otimes H_k(Y)) \oplus (H_n(S^n) \otimes H_{k-n}(Y)) \\ &\cong (\mathbb{Z} \otimes H_k(Y)) \oplus (\mathbb{Z} \otimes H_{k-n}(Y)) \\ &\cong H_k(Y) \oplus H_{k-n}(Y) \end{aligned}$$

Next we will inductively calculate all the homotopy groups of the universal cover of a 1-manifold. To calculate them we will use the following classic theorem due to Hurewicz.

[1] Note that while the the reference requires that $H_n(X)$ is a projective module, we opted for it being free abelian, because it is sufficient for us and every free abelian group is projective as well.

**Theorem 70.** *(Hurewicz Theorem, [3], 4.32) Let $X$ be a path-connected topological space such that $\pi_i(X) \cong 0$ for all $i$ less than $n \in \mathbb{N}$. Then $\pi_n(X)^{\mathrm{ab}} \cong H_n(X)$.*

We note that under the assumptions of the theorem we immediately get $H_i(X) = 0$ for $1 < i < n$ and indeed that a connected space $X$ with a basepoint $x$ has that $H_n(X) = 0$ for all $n$ if and only if $\pi_n(X, x) = 0$ for all $n$.

We will now restrict $X$ to be a 1 manifold and calculate the homology of its universal cover $\tilde{M}$. We note that its first homology group must be trivial, since it is the abelianisation of the first homotopy group (the fundamental group), which must be zero for a universal cover.
**Lemma 71.** *If $X$ is a 1-manifold, then $H_l(X) \cong 0$ for $l > 1$.*

*Proof.* Find for each $x \in X$ an open $U_x \ni x$ such that $U_x \cong \mathbb{R}$. We note that these cover $X$ and that $X$ is Lindelöf such that we have a countable subcover $(U_k)_{k\in\mathbb{N}}$. Now we set $X_{k+1} = X_k \cup U_{k+1}$ with $X_1 = U_1$.

We will now use induction on $k$ to prove that $H_l(X_k) = 0$ for all $l > 1$. For $X_1$ this follows from the contractibility of $\mathbb{R}$ and Lemma 66.

For the induction step assume that for some $k \in \mathbb{N}$ we have for all $k' \leq k$ that $H_l(X_{k'}) = 0$. Then the Mayer-Vietoris sequence tells us:

$$H_l(X_k \cap U_{k+1}) \to H_l(X_k) \oplus H_l(U_{k+1}) \to H_l(X_{k+1}) \to H_{l-1}(X_k \cap U_{k+1})$$

is exact. We will now calculate $H_l(X_k \cap U_{k+1})$. We note that $X_k \cap U_{k+1} \subseteq U_{k+1} \cong \mathbb{R}$ such that we can decompose $X_k$ in connected components $X_k \cap U_k = \bigsqcup_\alpha V_\alpha$. Since all connected open subsets of $\mathbb{R}$ are intervals hence contractible, we get for $l' > 0$ that

$$H_{l'}(X_k \cap U_k) \cong 0$$

And hence that $H_l(X_k) \oplus H_l(U_{k+1}) \cong H_l(X_{k+1})$. We note that by the induction hypothesis $H_l(X_k) = 0$ and that $H_l(U_{k+1}) = 0$ since $U_{k+1}$ is contractible. Therefore $H_l(X_{k+1}) \cong 0$

Now we need to extend this result to $X$. We note that we have inclusions of chain complexes (i.e. we have inclusions at each degree which commute with the boundary maps) $C_\bullet(X_1) \subseteq C_\bullet(X_2) \subseteq \cdots$ and for all $i$ we have $C_\bullet(X_i) \subseteq C_\bullet(X)$, in particular the chain maps of the $C_\bullet(X_i)$ are the restriction of those of $C_\bullet(X)$ to their various domains.

For each $k$ let $d^k$ be the boundary map of $C_\bullet(X_k)$ and $d$ be the boundary map of $C_\bullet(X)$. For each element $x \in \mathrm{Im}(d_l^k) \subseteq C_{l-1}(X_k)$ there exists an $y \in C_l(X_k)$ with $d_l^k(y) = x$, since $C_l(X_k) \subseteq C_l(X)$ we can apply $d_l$ as well and since $d_l^k = d_l|_{C_l(X_k)}$ we get $d_l(y) = d_l^k(y) = x$ so $x \in \operatorname{Im} d_l$ and therefore $\operatorname{Im} d_l^k \subseteq \operatorname{Im} d_l$.

Let $x \in \ker(d_l) \subseteq C_l(X)$, then since the $X_k$ cover $X$, and the image of a compact is compact and $x$ is only a finite formal sum of simplices we get that it is contained in some $C_l(X_N)$. Since $H_l(X_N) = 0$ we know that $x \in \operatorname{Im} d_{l+1}^N \subseteq \operatorname{Im} d_{l+1}$ so $x$ represents 0 in $H_l(X)$ as well, that is $H_l(X) = 0$. □

Next we will prove the same statement for homotopy groups using Hurewicz theorem and the universal covering space.
**Lemma 72.** *If $X$ is a connected 1-manifold, then $\pi_k(X) \cong 0$ for $k > 1$.*

*Proof.* By Lemma 61 the universal cover of $X$ exists and is a 1-manifold and by Lemma 71 the higher homology groups vanish.

By a well known criterion for being the universal cover its fundamental group vanishes ([7], Proposition 2.4.9) and hence also its first homology group.

Now we apply induction. Suppose $\pi_n(\tilde{X}) = 0$ for all $n < N$, then by Hurewicz theorem we get $\pi_N(\tilde{X}) \cong H_N(\tilde{X})$ and we already concluded that $H_N(\tilde{X}) \cong 0$ for all $N > 1$. Indeed $\pi_1(\tilde{X}, x) \cong 0$ and so we can conclude that $\pi_n(\tilde{X}) = 0$ for all $n \in \mathbb{N}$ so that by Lemma 62 we get that $\pi_n(X) \cong 0$ for all $n > 1$ as well. ☐

# 4 Simplicial Complexes and Manifolds

We will give here an example of a compact manifold that is not weakly homotopy equivalent to a finite simplicial complex. We will first show that every finite simplicial complex has a finitely generated fundamental group. Next we construct a manifold with a free countably generated fundamental group, from which we conclude that this manifold is weakly homotopy equivalent to a finite simplicial complex.

## 4.1 Simplicial Complexes

Let us first look into simplicial complexes and homology.

**Definition 73.** A finite abstract simplicial complex $(S, K)$ is a finite set $S$, which we call the set of vertices with a collection $K \subseteq P(S)$, which we call the set of constituent simplices, such that for all $s \in K$ we also have that all its subsets $t \subseteq s$ are also in $K$. We will usually denote this simply by $S$.

When $K = P(S)$ we say that $(S, K)$ is a $(\#S - 1)$-simplex.

A complex $(T, L)$ is a subcomplex of $(S, K)$ when $T \subseteq S$ and $L \subseteq K$.

**Definition 74.** For a finite abstract simplicial complex $(S, K)$ the geometric realisation $|S|$ is for each constituent simplex the convex hull of its vertices in the free $\mathbb{R}$-vector space on the vertices of $S$.

Explicitly we take for each simplex $(\Delta, P(\Delta))$ the geometric realisation to be the topological subspace $|\Delta|$ of $\mathbb{R}^\Delta \cong \mathbb{R}^{\#\Delta}$, the topological space of real valued functions from $\Delta$ to $\mathbb{R}$, given by

$$|\Delta| = \left\{ t \in (\mathbb{R}_{\geq 0})^\Delta \;\middle|\; \sum_{s \in \Delta} t(s) = 1 \right\}.$$

This is exactly the convex hull of the standard unit vectors, so this gives $|\Delta|$ as the standard $n$-simplex for some $n \in \mathbb{N}$.

In general we take the union over all constituent simplices:

$$|S| = \bigcup_{\Delta \in K} \left\{ t \in (\mathbb{R}_{\geq 0})^S \;\middle|\; \sum_{s \in \Delta} t(s) = \sum_{s \in S} t(s) = 1 \right\}$$

From now on, whenever we talk about (abstract) simplicial complexes, we mean finite ones. In particular $S$ and therefore $K$ is always taken to be finite.

**Example 75.**

1. For any set $S$, we see that $(S, \{\{s\} \mid s \in S\})$ is an abstract simplicial complex, and its geometric realisation is the set $S$ with discrete topology.

2. Consider $S = \{0, \dots, n\}$, then $(S, P(S))$ is the standard abstract $n$-simplex, which has the standard $n$-simplex $\Delta^n$ as a geometric realisation.

**Definition 76.** A simplicial complex is a triple $(X, S, \psi)$, where $X$ is a topological space, $S$ is an abstract simplicial complex and $\psi : |S| \to X$ is a homeomorphism. Usually we will refer to it as just $X$.

**Definition 77.** Let $(X, S, \psi)$ be a simplicial complex with $S = (S, K)$ and let $n \in \mathbb{N}$ be a natural number. Then the $n$-skeleton of the abstract simplicial complex $S$ is

$$S^n := (S, \{\Delta \in K \mid \#\Delta \leq n + 1\}).$$

The $n$-skeleton of $X = (X, S, \psi)$ is the simplicial complex

$$X^n = (\psi(|S^n|), S^n, \psi|_{|S^n|}).$$

Expressed as a diagram we get:

$$\begin{tikzcd}
{|S^n|} \arrow[r, "\psi|_{|S^n|}", "\sim"'] \arrow[d, "\subseteq"'] & X^n \arrow[d, "\subseteq"] \\
{|S|} \arrow[r, "\sim", "\psi"'] & X
\end{tikzcd}$$

We want to prove that every path in a simplicial complex with end points in the 1-skeleton is homotopic to a path in the 1-skeleton. We prove this by first showing that every such path in a geometric realisation is homotopic to a path avoiding the interior of a single higher simplex, and then applying this repeatedly to get the result for geometric realisations. Finally every simplicial complex is isomorphic to the geometric realisation of some abstract complex, so we are done.

**Lemma 78.** *([3], Theorem 4.8) Let $(S, K)$ an abstract simplicial complex and $\Gamma \in K$ such that $\#\Gamma > 2$, that is, $\Gamma$ is not a 0-simplex or 1 simplex. Set $T = (S, K \setminus \{\Gamma\})$.*

*For all paths $\gamma : I \to |S|$ with $\gamma(0), \gamma(1) \in |T|$ there exists a homotopic path $\gamma' : I \to |T|$. In short we want to show that the dashed arrows exist in the diagram below:*

$$\begin{tikzcd}
\{0,1\} \arrow[r, "\gamma|_{\{0,1\}}"] \arrow[d, "\subseteq"'] & {|T|} \arrow[d, Rightarrow] \arrow[dr, "\subseteq"] & \\
I \arrow[ur, dashed, "\gamma'"] \arrow[rr, "\gamma"'] & {} & {|S|}
\end{tikzcd}$$

We will show that every path can be deformed to miss a closed ball using a homotopy. Then we use that the punctured simplex $\Delta^k \setminus \{p\}$ contracts to its boundary $\partial\Delta^k$ for every $p$ in the interior of $\Delta^k$. When we compose these two homotopies, we get the required path.

*Proof.* Let $\Delta = |S| \setminus |T|$ be the interior of $|\Gamma|$ and $p \in \Delta$ be some point. Let $U \subseteq \Delta$ be an open ball containing $p$ and $F \subseteq U$ be a closed ball containing $p$.

We see that $\{\gamma^{-1}(|S| \setminus F), \gamma^{-1}(U)\}$ is an open cover for $I = [0, 1]$. Let $\mathcal{U}$ be the set of connected components of $\gamma^{-1}(U)$, then $\{\gamma^{-1}(|S| \setminus F)\} \cup \mathcal{U}$ is an open cover of $I$ as well. Let $\{V_0 = \gamma^{-1}(|S| \setminus F), V_1, \ldots, V_n\}$ be a finite subcover.

We note that for all $i = 1, \ldots, n$ we have $\gamma(\overline{V}_i) \subseteq \overline{U}$, and $\overline{U}$ is convex so any two paths with the same endpoints are homotopic. Indeed there must exist a path $\gamma_i : \overline{V}_i \to \overline{U}$ such that $\gamma_i(V_i) \subseteq U \setminus F$ and such that $\gamma_i$ is homotopic to $\gamma|_{\overline{V}_i}$ via a homotopy $H_i : \overline{V}_i \times I \to \overline{U}$ that keeps the endpoints fixed.

Now let $H : I \times I \to |S|$ be given by $F_i$ on $\overline{V}_i \times I$ and by $H(x, t) = \gamma(x)$ elsewhere. Since $H$ is defined by continuous maps on a finite number of closed subsets, which agree on their intersection, it is continuous.

We have now found a homotopic path $\hat{\gamma}(t) = H(t, 1)$ that misses all of $F$, in particular it misses $p$. We can now apply a contraction $R : |S| \setminus \{p\} \times I \to |S| \setminus \{p\}$, which retracts $\Delta$ to $\partial\Delta$ to obtain our result. We set the requested $\gamma'$ to be $\gamma'(t) = R(\hat{\gamma}(t), 1)$. $\square$

**Lemma 79.** *Let $(X, S, \psi)$ be a finite simplicial complex and $\gamma : I \to X$ be a path such that $\gamma(0), \gamma(1) \in X^0$.*

*Then there exists a map $\gamma' : I \to X^1$ which is homotopic to $\gamma$.*

*Proof.* Note that the 1-skeleton is simply the realisation of the abstract 1-skeleton, $X^1 = \psi(S^1) \cong |S^1|$ where the isomorphism is $\psi|_{|S^1|}$. We see that $\psi^{-1}\gamma : I \to |S|$ is homotopic to some $\gamma'' : I \to |S^1|$ by applying Lemma 78 repeatedly. Now $\gamma' = \psi\gamma'' : I \to X^1$ is the required map. $\square$

**Theorem 80.** *A finite connected simplicial complex $(X, S, \psi)$ with basepoint $x$ has a finitely generated fundamental group $\pi_1(X, x)$.*

*Proof.* Since $\psi$ gives a homeomorphism, we assume that $X = |S|$.

We recall that fundamental groups around different basepoints in the same path-connected component are isomorphic, so that we may assume that $x \in X^0$.

We will first show that $|S^1|$ has a finitely generated fundamental group, from which we will derive the same for $X$.

Notice that $|S^1|$ is just a finite connected graph. Let us call the elements of $S$ with two elements themselves edges and the singletons vertices. We can take a spanning tree $T \subseteq S^1$, where each element of $T$ is an edge and form the abstract subcomplex $T' = T \cup S^0$. We remind the reader that $T'$ is a tree as defined in graph theory; it is *not* the geometric realisation of such a tree.

Let $e_1, \ldots, e_n$ the edges of $S^1$ not in $T$ and form the series of subspaces of $|S|$, given by $X_1 = |T'|$ the spanning tree and adjoining the remaining edges one by one. We write $(e_i, P(e_i))$ for the abstract subcomplex of a single edge, so that for the sequence of subspaces of $X$ we get $X_{i+1} = X_i \cup |(e_i, P(e_i))|$.

Now let $U \subset X_{n+1}$ be an open neighbourhood of $X_n$ given by $X_n$ together with in each edge if exactly one of the vertices is in $X_n$ then the open third of the edge around the vertex in $X_n$. By construction $U$ is contractible to $X_n$.

Remember that $|T'| \subset Y_n$ is a spanning tree which means that there exists a unique path between the vertices of $e_i$ in $|T|$ up to homotopy since trees are contractible. Form the loop $L = T \cup \{e_i\}$ and the subcomplex $L' = (\bigcup_{e \in L} e, \bigcup_{e \in L} P(e))$ of $S$. Now we take an open neighbourhood $V$ of the loop $|L'|$ consisting of the union of $|L'|$ and in each edge connecting to a vertex of $L$ the open third of the edge around the vertex in $L$. We note that $V$ contracts to the loop $|L'|$: any edge not in $L$ contains an open set outside of $V$.

Let $y \in U \cap V$ be a basepoint. Note that $U \cap V$ is contractible to the subspace $|T'| \cap |L'|$, which looks like a path between the vertices of $e_i$ and contracts to a point. We conclude that $U \cap V$ is simply connected. Also note that $U \cup V = X_{n+1}$ by construction. By applying the classic Seifert-van Kampen theorem, Corollary 49, we have

$$\pi_1(X_{i+1}, y) \cong \pi_1(U \cup V, y) \cong \pi_1(U, y) * \pi_1(V, y) \cong \pi_1(X_i, y) * \pi_1(S^1) \cong \pi_1(X_i, y) * \mathbb{Z},$$

so $\pi_1(X^1, y)$ is free in $n$ variables, hence finitely generated. Let $i : X^1 \to X$ the inclusion map. From Theorem 79 we get that every path is homotopic to a path in the 1-skeleton $X^1$, such that $i_* : \pi_1(X^1, x) \to \pi_1(X, x)$ is indeed surjective, which proves the theorem. □

We remark that the induced homomorphism $i_* : \pi_1(X^1, x) \to \pi_1(X, x)$, induced by the inclusion of the 1-skeleton in a finite simplicial complex is not always injective. Consider $X = |\Delta^2|$.

## 4.2 A 1-Manifold Which is not a Simplicial Complex

Now we will give a compact 1-manifold with a fundamental group, freely generated by a countably infinite set.

As we saw before in Lemma 38, this implies it is not a simplicial complex.

**Theorem 81.** *Suppose $U \subseteq S^1$ is open and has an infinite number of connected components. Let $Y$ be the pushout of $S^1 \leftarrow U \to S^1$ with both arrows the inclusion and let $y \in Y$, then $\pi_1(Y, y)$ is the free group on a countably infinite number of generators.*

*Proof.* Let $A \subseteq U$ be a representative set of $U$. From Lemma 10 we know that $A$ is countable.

First we apply the Seifert-van Kampen Theorem 48 to get the following pushout square in the category of groupoids:

$$\begin{tikzcd} \pi_1(U, A) \arrow[r, "i"] \arrow[d, "j"'] & \pi_1(S^1, A) \arrow[d, "u"] \\ \pi_1(S^1, A) \arrow[r, "v"'] & \pi_1(Y, A) \end{tikzcd}$$

Now we want to apply Theorem 50 to calculate the fundamental group of $Y$.

We note that for $a, b \in A$, if there is a path from $a$ to $b$ in $U$, then they must lie in the same path-connected component of $U$ and therefore in the same component of $U$, which implies that $a = b$. So $\pi_1(U, A)$ is totally disconnected.

We note that $S^1$ is path-connected such that $\pi_1(S^1, A)$ is connected, so Theorem 50 applies.

Let $p \in A$ be an object. We note that connected open strict subsets of $S^1$ are open intervals on $S^1$ and therefore contractible. We conclude that any path in $U$ is homotopic to a constant path, that is all morphisms in $\pi_1(U, A)$ are identity morphisms.

We note that $i, j$ and $r, s$ as in Theorem 50 are functors such that they map identity maps to identity maps and therefore that $ri\gamma$ and $si\gamma$ must be identity maps. Therefore the subgroup of relations at $p$ (using the notation of Theorem 50) is

$$\begin{aligned} Q :&= \langle (ri\gamma) f_x (si\gamma)^{-1} f_x^{-1} \mid x \in A, \gamma \in \mathrm{Aut}_{\pi_1(U,A)}(x) \rangle \\ &= \langle \mathrm{Id}_p \, f_x \, \mathrm{Id}_p^{-1} \, f_x^{-1} \mid x \in A, \gamma \in \mathrm{Aut}_{\pi_1(U,A)}(x) \rangle \\ &= \langle 1 \rangle = 0 \end{aligned}$$

We choose elements $\alpha_x \in \mathrm{Hom}_{\pi_1(S^1,p)}(x, p)$ for all objects $x \in A$, with $\alpha_p = \mathrm{Id}_p$.

For each object $x \in A$ we set $f_x = u(\alpha_x)^{-1} v(\beta_x) \in \mathrm{Aut}_{\pi_1(Y,A)}(p)$ and let $F$ be the free group generated by $\{f_x \mid x \in A \setminus p\}$

We conclude that

$$\pi_1(Y, p) \cong \pi_1(S^1, p) * \pi_1(S^1, p) * F$$

We note that since $F$ is freely generated by a countable number generators and $\pi_1(S^1, p) \cong \mathbb{Z}$, so is $\pi_1(Y, p)$. Since $Y$ is path-connected (Lemma 30) $\pi_1(Y, p) \cong \pi_1(Y, y)$. □

**Theorem 82.** *Let $(Y, y)$ be as in Theorem 81, then $Y$ is not weakly homotopy equivalent to a finite simplicial complex.*

*Proof.* By Theorem 81 we know that $\pi_1(Y, y)$ is free on a countably infinite basis.

For any pointed simplicial complex $(C, c)$, the fundamental group is finitely generated by Theorem 80.

In Lemma 38 we proved that this is means that $\pi_1(Y, y) \ncong \pi_1(C, c)$. We see that no map will induce an isomorphism on $\pi_1$ and therefore that $Y$ and $C$ are not weakly homotopy equivalent. □

# 5 Compact Manifolds Immerse in Glued Spheres

In this section we prove that every compact manifold can also be seen as a subspace of a space obtained by glueing spheres. This was an early result that didn't really bear fruit later on.

**Theorem 83.** *For every compact n-manifold $M$ there exists an n-manifold $M'$ with an open immersion $\varphi : M \to M'$ and an open cover $\mathcal{U}$ of $M'$ such that for every $U \in \mathcal{U}$ there is a homeomorphism $U \cong S^n$.*

*Proof.* Let $(U_x \subseteq M)_{x \in M}$ be open in $M$, such that for all $x \in M$ we have $x \in U_x$ and a homeomorphism $f_x : U_x \xrightarrow{\sim} B^n \subset \mathbb{R}^n$ to the open unit ball.

Notice that $\{U_x\}$ is an open cover for $M$ such that by compactness we can select a finite subcover $\{U_1, \ldots, U_m\}$ with corresponding homeomorphisms $f_i : B^n \xrightarrow{\sim} U_i$ and inclusions $u_i : U_i \to M$.

Now let $1 \leq i \leq m$ be an integer. We will inductively glue a second half to each $U_i$ such that the final space is covered by spheres and $M$ immerses into it. Set $W_0 = M$. We define $W_i$ recursively.

Take $w_0 : M \to W_0 = M$ to be the identity morphism, which is an open immersion. For $i \geq 1$ we construct an open immersion recursively. Let us assume that we have an open immersion $w_{i-1} : M \to W_{i-1}$.

Let us set $C = (0,1) \times S^{n-1}$. Remember that $S^{n-1} = \partial B^n \subseteq \mathbb{R}^n$

Now let $s : C \to B^n, (t,x) \mapsto tx$ and $r : C \to B^n, (t,x) \mapsto (1-t)x)$ be open immersions. If we were to glue $B^n$ to $B^n$ along $C$ using $r$ and $s$, we would get a space homeomorphic to $S^n$. Indeed we attach $B^n$ to $W_{i-1}$ along $C$ to get an open around $U_i$ homeomorphic to $S^n$. The attaching map $c_{i-1} : C \to W_{i-1}$ we use is the composition

$$C \xrightarrow{r} B^n \xrightarrow{f_i} U_i \xrightarrow{u_i} M \xrightarrow{w_{i-1}} W_{i-1}.$$

Note that $c_{i-1}$ is a composition of open immersions and therefore an open immersion itself.

Let $W_i$ be defined by the pushout square below.

$$\begin{tikzcd} C \arrow[r, "s"] \arrow[d, "c_{i-1}"'] & B^n \arrow[d, "v_i"] \\ W_{i-1} \arrow[r, "\hat{w}_i"'] & W_i \end{tikzcd}$$

Because $s$ is an open immersion, so is $\hat{w}_i$ by Lemma 18. We set $w_i = \hat{w}_i w_{i-1}$, which is an open immersion, because it is a composition of open immersions.

Indeed to ease notation we assume without loss of generality that $W_0 \subset W_1 \subset \cdots \subset W_m$.

We find that

$$\begin{aligned} W_m &= W_{m-1} \cup v_m(B^n) && (1) \\ &= \cdots && (2) \\ &= W_0 \cup v_1(V) \cup \cdots \cup v_m(V) && (3) \\ &= U_1 \cup \cdots U_m \cup v_1(V) \cup \cdots \cup v_m(V) && (4) \\ &= (U_1 \cup v_1(V)) \cup \cdots \cup (U_m \cup v_m(V)). && (5) \end{aligned}$$

Each of the terms $(U_i \cup v_i(V))$ is just the glueing of $B^n$ to itself along $C$, which is homeomorphic to $S^n$. $\square$

# 6 The Weak Homotopy Type, Structure and Classification

Let $X$ and $Y$ be two $n$-manifolds. We want to know whether in the case of manifolds, there are apparently weaker conditions for $X$ and $Y$ to have the same weak homotopy type. We recall that in general $X$ and $Y$ have the same weak homotopy type if there exists spaces $Z_0 = X$, $Z_1, \ldots, Z_{m-1}$, $Z_m = Y$ and for each $i = 0, \ldots, m-1$ weak homotopy equivalences $f_i : Z_i \to Z_{i+1}$ or $f_i : Z_{i+1} \to Z_i$. Specifically we will investigate:

1. If it is sufficient for two spaces to have the same weak homotopy type, that all the homotopy groups of $X$ and $Y$ are abstractly isomorphic.
2. If, whenever $X$ and $Y$ are of the same weak homotopy type, there exists some topological space $Z$ with weak homotopy equivalences $f : Z \to X$ and $g : Z \to Y$ (or potentially with both arrows reversed). If such a space can always be found, we can do away with general zig-zags in the definition of the weak homotopy type and instead just require this special form.

These questions seem quite hard so we will first study them for a specific kind of 1-manifolds.

## 6.1 Classification in a Special Case

The 1-manifolds we will now study are constructed by glueing two circles along an open. Let for every open subset $U \subseteq S^1$ the space $S(U)$ be the pushout of $S^1 \leftarrow U \rightarrow S^1$, where the maps are the inclusion map.

Suppose we have open subsets $U, V \subset S^1$, when do we get that $S(U)$ is of the same weak homotopy type as $S(V)$? One way of proving that they have the same weak homotopy type is giving a weak equivalence. It turns out that is helpful to first study the behaviour when glueing intervals, and then use these results for when we glue circles.

Let us first state this general topological theorem due to Tietze.

**Theorem 84.** *([6], Theorem 4.1.13) Suppose $X$ is a normal space and $A$ is a closed subspace. Let $f : A \to \mathbb{R}$ be continuous, then there exists a continuous function $F : X \to \mathbb{R}$, such that $F|A = f$.*

**Lemma 85.** *Suppose we have open subsets $U, V \subseteq I = [0,1]$, where $0, 1 \notin \overline{U}$, the closure of $U$ in $I$, with a map $\psi : \overline{U} \to \overline{V}$ such that $\psi(U) = V$ and $\psi|U : U \to V$ is a homeomorphism. Then there exists a map $f : I \to I$ with $f(0) = 0, f(1) = 1$ and $f|\overline{U} = \psi$.*

*Proof.* This is just a basic application of Tietze's Theorem. Set $A = \overline{U} \cup \{0,1\}$, clearly $A$ is closed, since it is the union of two closed sets. Also note that since $I$ is normal, there are pairwise disjoint opens around each of $\{0\}$, $\{1\}$ and $\overline{U}$. Indeed this means that $\{\{0\}, \{1\}, \overline{U}\}$ forms an open cover of $A$.

Set $f_0 : A \to I$, given by

$$f_0(x) = \begin{cases} 0 & \text{If } x = 0 \\ 1 & \text{If } x = 1 \\ \psi(x) & \text{Else} \end{cases}.$$

We note that $f_0$ is continuous since it is on an open cover of its domain. Tietze's theorem (84) now yields the required map. □

We will now show that circles are intervals glued at the ends, which will allow us to apply Tietze's theorem on circles.

**Lemma 86.** *The circle $S^1$ is the pushout of $\{0\} \leftarrow \{0,1\} \rightarrow [0,1]$, where the left arrow is the unique one $u$ and the right arrow is the inclusion $i$. Let us denote the associated map $I \to S^1$ by $p$. We will write $s_0 = p(0) = p(1)$ for its basepoint.*

*Proof.* We set $p : I \to S^1, t \mapsto (\cos(2\pi t), \sin(2\pi t))$ and we set $q : \{0\} \to S^1, 1 \mapsto (1,0)$. The trigonometric functions sin and cos have period $2\pi$ so that $pi = qu$.

For any other space $Y$ with maps $q' : \{0\} \to Y$ and $p' : I \to Y$ such that $q'u = p'i$, we set $\psi : S^1 \to Y, x \mapsto p'(p^{-1}(x))$. A diagram chase verifies that this is well defined and unique. □

**Lemma 87.** *If $f : I \to I$, with $f(0) = 0$ and $f(1) = 1$, then there exists a map $\gamma : S^1 \to S^1$, with $\gamma \circ p = p \circ f$. Furthermore $\gamma$ is homotopic to the identity on $S^1$.*

*Proof.* The pushout property of $S^1$ tells us that such a map exists when $(p \circ f)(0) = (p \circ f)(1)$. We note $p(f(0)) = p(0) = p(1) = p(f(1))$, where the middle equality comes from the pushout property of $S^1$ again.

For the second property, unfortunately the Seifert-Van Kampen Theorem's (48) assumptions are not satisfied, but its conclusion still holds using [1], Corollary 9.1.5. It tells us that if we choose a representing set $B \subseteq \{0,1\}$, that is $B = \{0,1\}$ then $\pi_1(-, B)$ preserves pushouts. We use an $*$-subscript to denote application of this functor. This yields the following pushout diagram:

$$\begin{tikzcd}
\pi_1(\{0,1\}, B) \arrow[r] \arrow[d] & \pi_1(I, B) \arrow[r, "f_* = \mathrm{Id}_*"] \arrow[d, "p_*"] & \pi_1(I, B) \arrow[dd, "p_*"] \\
\pi_1(\{0\}, B) \arrow[r] \arrow[drr] & \pi_1(S^1, B) \arrow[dr, "\gamma_*"] & \\
 & & \pi_1(S^1, B)
\end{tikzcd}$$

As it happens, $\pi_1(S^1, B) \cong \mathbb{Z}$ has only a single object, while $\pi_1(\{0\}, B)$ has only a single object with only identity morphisms. We conclude then that the morphisms $\pi_1(\{0\}, B) \to \pi_1(S^1, B)$ in the diagram must be equal. Since $p_* \circ f_* = p_* \circ \mathrm{Id}_* = p_*$ we conclude that $\mathrm{Id} : \pi_1(S^1, B) \to \pi_1(S^1, B)$ makes the diagram commute, and by the pushout property it is the unique such map. We conclude that $\gamma_* = \mathrm{Id}$. □

Let us now combine Lemmas 85 and 87 in the following theorem.

**Theorem 88.** *Suppose $U, V \subseteq S^1$ are open subsets such that $s_0 \notin \overline{U}, \overline{V}$ and $\psi : \overline{U} \to \overline{V}$ is a map such that $\psi(U) = V$ and $\psi|U : U \to V$ is a homeomorphism.*

*Then there exists a map $\gamma : S^1 \to S^1$, homotopic to the identity, with $\gamma(s_0) = s_0$ (i.e. it fixes the basepoint) and $\gamma|U = \psi$.*

*Proof.* This is fairly straightforward. See the diagram at the end of the proof for help. Let $U' = p^{-1}(U)$ and $V' = p^{-1}(V)$, note $0, 1 \notin U', V'$. Since the restriction $p|I \setminus \{0,1\} : I \setminus \{0,1\} \to S^1 \setminus \{s_0\}$ is a homeomorphism, we get a map $\phi : \overline{U'} \to \overline{V'}$, such that $p \circ \phi = \psi \circ p$ Indeed $\phi|U' : U' \to V'$ must be a homeomorphism as well.

Now we apply Lemma 85 to obtain a map $f : I \to I$ with $f|U' = \phi$ and $f(0) = 0$ and $f(1) = 1$.

Next we apply Lemma 87 to obtain a map $\gamma : S^1 \to S^1$, homotopic to the identity with $\gamma \circ p = p \circ f$.

Let us finally prove that $\gamma|U = \psi$. We recall that $p|U' : U' \to U$ is a homeomorphism, so we consider $(\gamma|U) \circ (p|U') = (p \circ f)|U' = p \circ \phi = \psi \circ p|U'$. So indeed $\gamma|U = \psi$.

$$\begin{tikzcd}
S^1 \arrow[rrr, bend left, "\gamma"] & \overline{U} \arrow[l, hook', "\supset"] \arrow[r, "\psi"] & \overline{V} \arrow[r, hook, "\subset"] & S^1 \\
I \arrow[u, "p"] \arrow[rrr, bend right, "f"'] & \overline{U'} \arrow[l, hook', "\supset"'] \arrow[r, "\phi"] \arrow[u, "p", "\cong"'] & \overline{V'} \arrow[r, hook, "\subset"'] \arrow[u, "p"', "\cong"] & I \arrow[u, "p"']
\end{tikzcd}$$

□

This will allow us to prove this fairly general result:

**Theorem 89.** *Suppose $U, V \subseteq S^1$ are open, but not dense subsets. Furthermore, assume that $\psi : \overline{U} \to \overline{V}$ is a continuous map such that $\psi(U) = V$ and $\psi|_U : U \to V$ is a homeomorphism.*

*Then there exists a weak equivalence $h : S(U) \to S(V)$.*

*Proof.* First we need to pick a point $s_1$ outside $\overline{U}$ and a point $s_2$ outside $\overline{V}$. Let $r_1 : S^1 \to S^1$ be the rotation that sends $s_1$ to $s_0 = (1, 0)$, the default basepoint of $S^1$, and $r_2 : S^1 \to S^1$ be the rotation that sends $s_2$ to $s_0$. Since these are homeomorphisms we also get homeomorphisms $\varphi : S(U) \to S(r_1(U))$ and $\psi : S(V) \to S(r_2(V))$.

If we find a weak homotopy equivalence $h' : S(r_1(U)) \to S(r_2(V))$, then $h = \psi^{-1} \circ h' \circ \varphi$ is a weak homotopy equivalence from $S(U)$ to $S(V)$. So assume from now on that $s_0 \notin \overline{U}, \overline{V}$.

The previous Theorem 88 yields a map $\gamma : S^1 \to S^1$ making the following diagram commute:

$$\begin{tikzcd}
S^1 \arrow[d, "\gamma"'] & U \arrow[l] \arrow[r] \arrow[d, "\psi"'] & S^1 \arrow[d, "\gamma"] \\
S^1 & V \arrow[l] \arrow[r] & S^1
\end{tikzcd}$$

Indeed this $\gamma$ induces a map $h : S(U) \to S(V)$.

Let $A \subseteq U$ be a representative subset. After applying the fundamental groupoid functor $\pi_1(-, A)$, the vertical morphisms $\gamma_*$ and $\psi_*$ are isomorphisms. The Seifert-Van Kampen theorem (48) guarantees that the two pushout squares remain pushout squares, so that after the application of the fundamental groupoid functor the squares must be isomorphic, with $h$ inducing an isomorphism $h_* : \pi_1(S(U), A) \to \pi_1(S(V), A)$.

Finally let us note that $S(U)$ and $S(V)$ are 1-manifolds, so that their higher homotopy groups are trivial by Theorem 72. We conclude that $h : S(U) \to S(V)$ is a weak homotopy equivalence. $\square$

**Theorem 90.** *Suppose $U \subseteq S^1$ and $V \subseteq S^1$ are open, but not dense and that for each connected component $U_i \subseteq U$ there exists an open $U_i' \supset \overline{U_i}$ such that $U_i' \cap \overline{U} = \overline{U_i}$.*

*There exists a weak homotopy equivalence $h : S(U) \to S(V)$ if any of:*

1. *$U$ and $V$ have the same finite number of connected components.*
2. *$U$ and $V$ have a countably infinite number of connected components and $\partial V$ has exactly one limit point.*

*Proof.* We only have to find $\psi : \overline{U} \to \overline{V}$ as required for Theorem 89.

Using rotations if necessary, we may assume that $s_0 \notin \overline{U}, \overline{V}$. Clearly this means that $\overline{U}, \overline{V} \subseteq S^1 \setminus \{s_0\}$ Let $p : I \to S^1$ be the natural projection, then the restriction $p|_{(0,1)} : (0, 1) \to S^1 \setminus \{s_0\}$ is a homeomorphism. We can conclude then that $\overline{p^{-1}(U)} = p^{-1}(\overline{U})$ and that $p|_{\overline{U}} : \overline{U} \to p(\overline{U})$ is a homeomorphism as well. The last sentence applies with $U$ replaced by $V$ as well.

We will first prove the result using just the first assumption and then again using just the second.

1. We simply use linear maps on the closures of each of the components.

   We can decompose $p^{-1}(U)$ into its connected components as $p^{-1}(U) = (a_1, b_1) \cup \cdots \cup (a_n, b_n)$ for some $a_i, b_i \in I$, with for each $i = 1, \ldots, n-1$ the inequalities $a_i < b_i < a_{i+1} < b_{i+1}$.

When we decompose $V$ in the same manner we get that $p^{-1}(V) = (c_1, d_1) \cup \cdots \cup (c_n, d_n)$ for some $c_i, d_i \in I$, with for each $i = 1, \ldots, n-1$ the inequalities $c_i < d_i \leq c_{i+1} < d_{i+1}$.

We can now define $\hat{\psi} : \overline{p^{-1}(U)} \to \overline{p^{-1}(V)}$ by setting its value for each $\epsilon \in [0,1]$ and $i = 1, \ldots, n$ to

$$\hat{\psi}(\epsilon a_i + (1-\epsilon) b_i) = \epsilon c_i + (1-\epsilon) d_i.$$

This is well-defined because $U^c$ has no isolated points and therefore $u : [0,1] \times \{1, \ldots, n\} \to \overline{p^{-1}(U)}, (\epsilon, i) \mapsto \epsilon a_i + (1-\epsilon) b_i$ is a continuous bijection between compact Hausdorff spaces, which means that it is a homeomorphism. Of course $v : [0,1] \times \{1, \ldots, n\} \to \overline{p^{-1}(V)}, (\epsilon, i) \mapsto \epsilon c_i + (1-\epsilon) d_i$ is continuous as well, such that $\hat{\psi} = v \circ u^{-1}$ is continuous too.

Note that $v$ is not necessarily bijective, since the endpoints of two connected components of $p^{-1}(V)$ may be the same, however $v|_{(0,1) \times \{1,\ldots,n\}} : (0,1) \times \{1, \ldots, n\} \to p^{-1}(V)$ does give a bijection and hence a homeomorphism.

Now $\psi = p|_{\overline{V}} \circ \hat{\psi} \circ p|_{\overline{U}}^{-1}$ gives the required map. We see that $\psi|_U : U \to V$ is indeed a homeomorphism, exactly because $\hat{\psi}|_{p^{-1}(U)} : p^{-1}(V)$ is.

2. We again use linear maps on the closure of each component of $U$, and we send all limit points of $\partial U$ to the limit point of $\partial V$. We will now construct this.

   We can decompose $p^{-1}(U)$ into its connected components as $p^{-1}(U) = \bigcup_{i=1}^{\infty} U_i$, where we take the $U_i$ connected and disjoint. For each $i \neq j \in \mathbb{N}$ we also get that $\overline{U}_i \cap \overline{U}_j = \emptyset$, because $U^c$ has no isolated points. Let $a_i, b_i \in [0,1]$ be such that for all $i = 1, \ldots$ we have $(a_i, b_i) = U_i$ .

   When we decompose $V$ in the same manner we get that $p^{-1}(V) = \bigcup_{i=1}^{\infty} V_i$. Let $c_i, d_i \in [0,1]$ be such that for all $i = 1, \ldots$ we have $(c_i, d_i) = V_i$.

   We can now define $\hat{\psi} : \overline{p^{-1}(U)} \to \overline{p^{-1}(V)}$ by setting its value for each $\epsilon \in [0,1]$ and $i \in \mathbb{N}$ to

$$\hat{\psi}(\epsilon a_i + (1-\epsilon) b_i) = \epsilon c_i + (1-\epsilon) d_i$$

   and for all other points $l$ of $\overline{U}$ we set their image to be be $\hat{\psi}(l) = l_V$, where $l_V$ is the single limit point of $p^{-1}(\partial V)$.

   This is well-defined because for all $i$ and $\epsilon$ we have $\epsilon a_i + (1-\epsilon) b_i \in \overline{U_i}$ and all the $\overline{U_i}$ are disjoint. Notice that $\hat{\psi}|_{p^{-1}(U)} : p^{-1}(U) \to p^{-1}(V)$ is a homeomorphism, because $u : (0,1) \times \mathbb{N} \to p^{-1}(U), (\epsilon, i) \mapsto \epsilon a_i + (1-\epsilon) b_i$ is bijective and and open and therefore a homeomorphism. The same is true for the similarly defined $v : (0,1) \times \mathbb{N} \to p^{-1}(V), (\epsilon, i) \mapsto \epsilon c_i + (1-\epsilon) d_i$ and $\hat{\psi}|_{p^{-1}(U)} : p^{-1}(U) \to p^{-1}(V)$ is given by $\hat{\psi}|_{p^{-1}(U)} = v \circ u^{-1}$ and hence a homeomorphism.

   Let us prove that $\hat{\psi}$ is continuous. We will do this by proving it is sequentially continuous. This is sufficient, because $p^{-1}(\overline{U}) \subseteq [0,1] \subseteq \mathbb{R}$ is metrisable.

   Suppose $(x_n)_n$ is a converging sequence in $p^{-1}(\overline{U})$ and let $x_\infty$ be its limit.

   If there is an $i \in \mathbb{N}$ such that $x_n \in \overline{U_i}$ for infinitely many $n$, then we use that $\overline{U}_i^{\,c}$ is open so that if $x_\infty \in \overline{U}_i^{\,c}$ were the case then there would exist some $N$ such that for all $n > N$ we had that $x_n \in \overline{U_i}^{c}$, but then there would be only finitely many $n$ with $x_n \in \overline{U_i}$, which is a contradiction and so $x_\infty \in \overline{U}_i$.

Notice that there exists an open $U_i' \supset \overline{U_i}$ such that $U_i' \cap \overline{U} = \overline{U_i}$. This means that for all $n$ sufficiently large, $x_n \in U_i'$ and hence $x_n \in \overline{U_i}$. Since $\hat{\psi}|_{\overline{U_i}}$ is linear, it is continuous and so we get

$$\hat{\psi}(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} \hat{\psi}(x_n).$$

Suppose now no such $i$ exists. Let $F = \bigcup_{k=1}^{\infty} \overline{U_k}$ be the componentwise closure of $U$ and let $L = \overline{U} \setminus F$ be the final part of the closure $\overline{U}$. We remark immediately that $x_\infty \in L$, since if it lay in any $\overline{U_i}$, then $(x_n)$ would eventually only take values in the open $U_i' \supset \overline{U_i}$ and $U_i \cap \overline{U} = \overline{U_i}$, which would contradict the assumption that for no $\overline{U}_j$ there were infinitely many $n$ such that $x_n \in \overline{U}_j$.

Indeed we conclude then that $\hat{\psi}(x_\infty) = l_V$, the unique limit point of $\partial V$, so that we want to show that

$$\lim_{n\to\infty} \hat{\psi}(x_n) = l_V.$$

Let $(y_n)$ be the subsequence of $(x_n)$ of points in $L$ and $(z_n)$ the subsequence of $(x_n)$ of points in $\bigcup_k \overline{U}_k$. We need to show that both $(\hat{\psi}(y_n))_n$ and $(\hat{\psi}(z_n))_n$ converge to $l_V$ whenever these subsequences contain infinitely many elements. Notice that $\hat{\psi}$ only takes the value $l_V$ on $L$, so $(\psi(y_n))_n$ is constantly $l_V$ and hence convergent with limit $l_V$.

If $(z_n)$ is infinite, then it must take values in infinitely many $\overline{U_i}$. We see that $\hat{\psi}(\overline{U_i}) = \overline{V_i}$ and therefore that $(\hat{\psi}(z_n))_n$ takes values in infinitely many $\overline{V_i}$ and therefore must converge to $l_V$ using the squeeze theorem from real analysis applied to the sequences of lower bounds and upper bounds of the respective intervals $\overline{U_i}$ these $z_n$ are in. Both must converge to the single limit point $l_V$ by the premise that $\partial V$ has a single limit point.

We conclude that $(\hat{\psi}(x_n))_n$ also converges to $l_V$ in this case as required.

Now $\psi = p|_{\overline{V}} \circ \hat{\psi} \circ p|_{\overline{U}}^{-1}$ gives the required map. Note that $\psi|_U : U \to V$ is indeed a homeomorphism, exactly because $\hat{\psi}|_{p^{-1}(U)} : p^{-1}(U) \to p^{-1}(V)$ is.

□

The assumption that $\partial V$ has only a single limit point can be relaxed, to an assumption on the order types of the linearly ordered sequences of connected components of $U$ and $V$. It doesn't seem worthwile to persue this strengthening however in the light of Theorem 94. For example, if both $\partial U$ and $\partial V$ have a finite number of limit points and $\partial U$ has more than $\partial V$, then this is sufficient as well.

Let us call a subset $U \subseteq S^1$ tame if $\overline{U} \neq S^1$ and either $\partial U$ has a single limit point or for each connected component $U_i \subseteq U$ there exists an open $U_i' \supset \overline{U_i}$ such that $U_i' \cap \overline{U} = \overline{U_i}$.

**Corollary 91.** *Suppose $U \subseteq S^1$ and $V \subseteq S^1$ are open, but not dense, then $S(U)$ and $S(V)$ are of the same weak homotopy type if any of:*

1. *$U$ and $V$ have the same finite number of connected components.*
2. *$U$ and $V$ both have a countably infinite number of connected components and are both tame.*

*Proof.* In both cases we will show how to choose an open $W \subseteq S^1$ such that the previous theorem gives weak homotopy equivalences among $S(W)$, $S(U)$ and $S(V)$ to show they are all of the same weak homotopy type. Let us first prove the result under the first assumption and then under the second.

1. We just need to show there exists a tame subset $W \subseteq S^1$ with the same finite number of connected components as $U$. Let $p : [0,1] \to S^1$ be the natural projection again. Suppose $U$ has $n$ connected components, then we set

$$W = p\left(\bigcup_{i=1}^{n}\left(\frac{2i-1}{2n+1}, \frac{2i}{2n+1}\right)\right)$$

   These intervals all have the same length, with the same length in between each time, so opens around each component of the closure of $W$ can easily be found, e.g.

$$\left[\frac{2i-1}{2n+1}, \frac{2i}{2n+1}\right] \subset \left(\frac{2(i-1)}{2n+1}, \frac{2i+1}{2n+1}\right).$$

   Notice that the closure of $W$ is simply the union of the closure of the connected components, so that these opens indeed verify the necessary property.

   The previous theorem now guarantees that there are weak homotopy equivalences $S(W) \to S(U)$ and $S(W) \to S(V)$ and so all must be of the same weak homotopy type.

2. We will exhibit a subset $W \subseteq S^1$ which verifies both tameness conditions. We take

$$W = p\left(\bigcup_{i=1}^{\infty}(3^{-i}, 2 \cdot 3^{-i})\right)$$

   Let us take $W_i' = p((2 \cdot 3^{-(i+1)}, 3^{-(i-1)}))$ for the component $W_i = p((3^{-i}, 2 \cdot 3^{-i}))$. Since the closure of $W$ is just the union of the closures of its components and $s_0$, we see that $W_i'$ indeed shows that $W$ is tame.

   On the other hand $\partial W = \{p(a3^{-i}) \mid a = 0, 1, 2 \wedge i = 1, 2, \ldots\}$; the only limit point of this is 0, since around all the other points $x \in \partial W$, there is an open $W_x \subseteq S^1$ such that $W_x \cap \partial W$.

   Now the previous theorem gives that there must be either be weak homotopy equivalence $S(U) \to S(W)$, if $U$ verifies the second tameness property, or $S(W) \to S(U)$, if $U$ verifies the first. In any case $S(U)$ and $S(W)$ are of the same weak homotopy type. The same is true for $V$.

□

### 6.1.1 A General Construction for 1-Manifolds.

We will now give a more general construction which will prove more elegantly that $S(U)$ and $S(V)$ are of the same weak homotopy type under certain conditions, which we have already shown in Corollary 91. However, it does not prove Theorem 90, which actually gives a homotopy equivalence $S(U) \to S(V)$. Instead we construct a space $X(\tau)$ with weak homotopy equivalences $X(\tau) \to S(U)$ and $X(\tau) \to S(V)$.

Let us quickly describe this proof. For any open $U \subseteq S^1$, we get that $S(U)$ is a compact 1-manifold and hence has a free fundamental group. Now we find a basis for this group and glue as many circles together in an open neighbourhood of the basepoint $s_0 \in S^1$ as there are elements in this basis, yielding a space $X(\tau)$ only depending on the cardinality of $\tau$. This construction now naturally allows for a map inducing isomorphism on $\pi_1$, which then must actually be a weak homotopy equivalence since both $Y$ and $X(\tau)$ are connected 1-manifolds and hence have trivial $\pi_n$ for $n > 1$ by Theorem 72.

Let us first construct the spaces $X(\tau)$ for cardinals $\tau \leq |\mathbb{N}|$. We put the discrete topology on $\tau$ for this construction. We need a connected open subset $W \subseteq S^1$ around the basepoint $s_0 \in S^1$ such that the complement is not empty, that is $W^c \neq \emptyset$. We will now glue $\tau$ circles $S^1$ along their respective $W$ subsets to obtain the space $X(\tau)$. Let $\mathrm{pr}_W : \tau \times W \to W$ be the projection and $i : \tau \times W \to \tau \times S^1$ be induced by the inclusion $W \subseteq S^1$. Then $X(\tau)$ is defined by the following pushout diagram:

$$\begin{tikzcd}
\tau \times W \arrow[r, "i"] \arrow[d, "\mathrm{pr}_W"'] & \tau \times S^1 \arrow[d, "r"] \\
W \arrow[r, "q"'] & X(\tau)
\end{tikzcd}$$

We take $a = q(s_0)$ to be the basepoint of $X(\tau)$.

**Lemma 92.** *Assume that $\tau$ is at most countably infinite. Then:*

1. *$X(\tau)$ is a connected 1-manifold. And*
2. *Its fundamental group is isomorphic to the free group on $\tau$, that is $\pi_1(X(\tau), a) \cong F(\tau)$.*

*Proof.* We will first prove the first assertion and then the second.

1. Let $\psi : X(\tau) \to \{0, 1\}$ be a continuous function, then $\psi \circ q$ must be a constant function since $W$ is connected and therefore $\psi \circ q \circ \mathrm{pr}_W = \psi \circ r \circ i$ must be constant. We notice that $i$ meets all connected components of $\tau \times S^1$ and so $\psi \circ r$ must be constant as well. If $\tau$ is not empty, $\tau \times W$ is not empty and therefore $\psi \circ q$ and $\psi \circ r$ are constant maps mapping to the same value, which implies that $\psi$ is constant as well, mapping again to this same value. If $\tau = \emptyset$ then $\tau \times W = \tau \times S^1 = \emptyset$ and so $X(\tau) = W$, which is connected.

   $\tau$ is at most countably infinite by assumption, so $\tau$ is a second-countable topological space. Since $S^1$ is too, their product must be a second-countable topological space as well. For any point $(x, t) \in \tau \times S^1$, we can find an open set $t \in U \subseteq S^1$ with $u \cong \mathbb{R}$, and so $(x, t) \in \{x\} \times U \cong U \cong \mathbb{R}$ gives an open neighbourhood around $(x, t)$ isomorphic to $\mathbb{R}$. We conclude that $\tau \times S^1$ is a 1-manifold.

   Now let us prove that $X(\tau)$ is a 1-manifold as well. We will show that around each point in $X(\tau)$ there is a neighbourhood homeomorphic to $S^!$. This circle comes from some subspace $\{x\} \times S^1 \subset \tau \times S^1$.

   For each $x \in \tau$ we have an open immersion $i_x : S^1 \to \tau \times S^1, t \mapsto (x, t)$, indeed $\{x\}$ is open in $\tau$, because we chose the discrete topology. We will now prove that $r \circ i_x$ is an open immersion as well and that their images $\{(r \circ i_x)(S^1) \mid x \in \tau\}$ form a cover of $X(\tau)$. From Lemma 20.3 we get that $r$ is open, because both $\mathrm{pr}_W$ and $i$ are and from 20.4 that $r$ is surjective since $\mathrm{pr}_W$ is. Since $\{i_x(S^1) \mid x \in \tau\}$ is an open cover of $\tau \times S^1$, the images $\{(r \circ i_x)(S^1) \mid x \in \tau\}$ under the surjective open map $r$ must form a cover of $X(\tau)$.

   Suppose we have $z_1, z_2 \in S^1$ such that $(r \circ i_x)(z_1) = (r \circ i_x)(z_2)$, that is $r((x, z_1)) = r((x, z_2))$, then there must be $\alpha_1, \ldots, \alpha_n \in \tau \times W$ such that $i(\alpha_1) = z_1, \mathrm{pr}_W(\alpha_1) = \mathrm{pr}_W(\alpha_2), i(\alpha_2) = i(\alpha_3), \ldots, i(\alpha_n) = z_2$. But $i$ is injective, so for each $k = 1, 2, \ldots, n/2$ (indeed $n$ must be even), we get $\alpha_{2k} = \alpha_{2k+1}$ and therefore $\mathrm{pr}_W(\alpha_1) = \mathrm{pr}_W(\alpha_2) = \cdots = \mathrm{pr}_W(\alpha_{n-1})$ and so it is sufficient to consider just two $\alpha_1, \alpha_2$. Necessarily we must have that $z_1 = z_2$ or that $i_x(z_1)$ and $i_x(z_2)$ are in the image of $i$ and that $\alpha_1 = i_x(z_1) = (x, z_1)$ and $\alpha_2 = i_x(z_2) = (x, z_2)$. Now we apply $z_1 = \mathrm{pr}_W(z_1) = \mathrm{pr}_W(z_2) = z_2$.

We conclude that each $r \circ i_x$ is injective. We already proved that $r$ is open and so is $i_x$ so we can conclude that their composition is open and therefore that all $r \circ i_x : S^1 \to X(\tau)$ are open immersions.

For any point $y \in X(\tau)$ we can now take an $x \in \tau$ such that $y$ lies in the image of $r \circ i_x$, we find an open $\hat{U}$ around $(r \circ i_x)^{-1}(x)$ in $S^1$ such that $\hat{U} \cong \mathbb{R}$, which is possible since $S^1$ is a 1-manifold, and then push it back to $X(\tau)$ as $U = (r \circ i_x)(\hat{U})$, which must contain $x$ and be homeomorphic to $\mathbb{R}$ as well because this map was an open immersion.

2. For the fundamental group we apply the Seifert-van Kampen Theorem 48. We take connected opens $\hat{U}_1, \hat{U}_2$ of $S^1$ such that neither covers $W$ nor $W^c$, but $\hat{U}_1 \cup \hat{U}_2 = S^1$. This guarantees that both $U_1 = r(\tau \times \hat{U}_1)$ and $U_2 = r(\tau \times \hat{U}_2)$ get a shape like a fork with $\tau$ tines and one handle, where it meets $W$, whose intersection consists of an interval for each tine and an interval on the handle. $\hat{U}_1 \cap \hat{U}_2$ must have two connected components. Note that one of the components is contained in $W$ and the other is disjoint from $W$, let $a_2$ be a point in $\hat{U}_1 \cap \hat{U}_2 \cap W^c$ while $a_1$ is a point in the other component, then $A = r(\tau \times a_2) \cup \{r(i_1(a_1))\}$ gives a suitable set of basepoints for $X(\tau)$, set $a = r(i_1(a_1))$ to be the basepoint we will use later.

   We will now calculate $\pi_1(X(\tau), A)$ using Theorem 48 and Theorem 50. For $\pi_1(U_1 \cap U_2, A)$ we note that this has one point in each connected component and no non-trivial paths up to homotopy. For $\pi_1(U_1, A) \cong \pi_1(U_2, A)$, we note that $U_1$ is contractible, so that there is exactly one path up to homotopy between each pair of points in $A$.

   Now we can apply Theorem 50, which tells us that

   $$\pi_1(X(\tau), a) \cong (\pi_1(U_1, a) * \pi_1(U_2, a) * F(A \setminus a))/H,$$

   here by $F(-)$ we mean the free group with basis $S$ and $H$ is the subgroup of relations at $a$ (Definition 51). In fact since for all $b \in A$ we have $\pi_1(U_1 \cap U_2, b) \cong 0$, this subgroup is trivial too. The first two terms of the free product are trivial too, so we are left with

   $$\pi_1(X(\tau), a) \cong F(A \setminus a) \cong F(\tau),$$

   where the last isomorphism is obtained by noticing that $A \setminus a = r(\tau \times \{a_2\})$ is in bijection with $\tau$.

□

**Lemma 93.** *Let $Y$ be a connected 1-manifold and $y \in Y$ be a basepoint. By Theorems 53 and 54 we have that $\pi_1(Y, y)$ is a free group so we can choose a basis set $\Gamma \subseteq \pi_1(Y, y)$.*

*There exists a weak homotopy equivalence $f : X(|\Gamma|) \to Y$.*

*Proof.* We set $\tau = |\Gamma|$ with $\varphi : \tau \to |\Gamma|$ a bijection, as above we take $\tau$ to have the discrete topology. Notice that $\Gamma$ is countable due to Theorem 53.

By the previous lemma $X(\tau)$ is a connected 1-manifold, so using Theorem 72 all that is left to check is that there exists a continuous function $f : X(\tau) \to Y$ inducing isomorphism on $\pi_1$. The pushout property tells us we need to find maps $f_W : W \to Y$ and $f_{\tau \times S^1} : \tau \times S^1 \to Y$ such that $f_W \circ \mathrm{pr}_W = f_{\tau \times S^1} \circ i$. We take $f_W : W \to Y, w \mapsto y$ to be the constant map to the basepoint of $Y$.

For the other we find a set of representatives $\hat{\Gamma} \subseteq \mathrm{Hom}_*(S^1, Y)$ for $\Gamma \subseteq \pi_1(Y, y)$ such that for all $\gamma \in \hat{\Gamma}$ and $w \in W$, $\gamma(w) = y$.

Such representatives exist by the following argument. Let $\gamma_y : x \mapsto y$ be the constant path. Take any representave $\gamma' : S^1 \to Y$ of a class in $\pi_1(Y, y)$. Notice that $\gamma_y$ is a representative of the unit element in $\pi_1(Y, y)$, so $[\gamma'] = [\gamma_y][\gamma'][\gamma_y]$. The path composition on the right gives a path that is constant on an open set around the basepoint of the circle $s_0$. We can now a apply a stretching homotopy equivalence to make this neighbourhood at least $W$.

Let $\psi : \Gamma \to \hat{\Gamma}$ be the inverse of $\gamma \mapsto [\gamma]$, that is the bijection mapping an equivalence class to a representative. Now we set $f_{\tau\times S^1} : \tau \times S^1 \to Y, (x, t) \mapsto \psi(\varphi(x))(t)$. We need to show that for all $(x, t) \in \tau \times S^1$ we have $(f_W \circ \mathrm{pr}_W)(x, t) = (f_{\tau\times S^1} \circ i)(x, t)$. Indeed we have $f_W(\mathrm{pr}_W(x, t)) = f_W(t) = y$, while for each $x \in \tau$ we get that $\psi(\varphi(x)) \in \hat{\Gamma}$ and so by construction this path is constant on $W$, taking the value $y$, so $f_w \circ \mathrm{Pr}_W = f_{\tau\times S^1} \circ i$.

We will now prove that the resulting $f$ indeed induces an isomorphism on $\pi_1$.

$X(\tau)$ has as as its fundamental group the free group on $\tau$, it is free on the basis $G = \{[g_x] = [r \circ i_x] \mid x \in \tau\}$, where $i_x : S^1 \to \tau \times S^1, z \mapsto (x, z)$ is the inclusion as above. This can be verified using the Seifert-Van Kampen theorem, Theorem 48 as applied in the previous lemma. Notice that this exhibits a bijection $\psi : \tau \to G, x \mapsto [g_x]$. Now we calculate $f_*([g_x]) = [f \circ p \circ i_x] = [\psi(\varphi(x))] = \varphi(x)$, so $f_* \circ \psi = \varphi$, or indeed $f_* = \varphi \circ \psi^{-1}$. Indeed both functions on the right are bijective, and therefore we find that $f_*$ is bijective on generators of free groups and hence an isomorphism. □

It must be noted that the product being only with a countable discrete space is crucial, since otherwise the topology would not be second-countable and therefore $X(|\Gamma|)$ would not be a manifold.

Now we prove this stronger version of Corollary 91.

**Theorem 94.** *If $Y$ and $Z$ are connected 1-manifolds with $\pi_1(Y) \cong \pi_1(Z)$, then there exists a connected 1-manifold $X$ with weak homotopy equivalences $f_Y : X \to Y$ and $f_Z : X \to Z$.*

*Proof.* Let $\Gamma_Y$ and $\Gamma_Z$ be generating sets for the respective fundamental groups and form $X(|\Gamma_Y|)$ and $X(|\Gamma_Z|)$ as in the previous Lemma 93, which come with weak homotopy equivalences $f_Y : X(|\Gamma_Y|) \to Y$ and $f_Z : X(|\Gamma_Z|) \to Z$.

Because $\pi_1(Y) \cong \pi_1(Z)$ we must have that $\Gamma_Y$ and $\Gamma_Z$ have the same cardinality, that is, $|\Gamma_Y| = |\Gamma_Z|$. Indeed we conclude then that $X(|\Gamma_Y|) = X(|\Gamma_Z|)$ and so we set $X = X(|\Gamma_Y|)$. □

**Corollary 95.** *If $U, V \subseteq S^1$ are homeomorphic subsets then $X := S^1 \cup_U S^1 = S(U)$ and $Y := S^1 \cup_V S^1 = S(V)$ are of the same weak homotopy type.*

*Proof.* We simply apply Theorem 94 to find weak homotopy equivalences equivalences and hence that $X$ and $Y$ are of the same weak homotopy type. □

We thus find that both questions from the start of this section have a positive answer in this special case.

## 6.2 Are Isomorphic Homotopy Groups Sufficient to Guarantee the Same Weak Homotopy Type?

Considering Theorem 94 one wonders whether a version for all $n$-manifolds $Y$ and $Z$ could be true. We will treat two versions, one by a counterexample and one by a proof. We will show that in general we can not take $X$ to be an $n$-manifold, but that it is possible to take $X$ to be a CW-complex.

One might guess from the previous subsection that for manifolds at least, two spaces are in the same weak homotopy type, if and only if all their homotopy groups are isomorphic, as we proved this for 1-manifolds already. One would be mistaken as we show now, using an example we learned of at MathOverflow from Daniel Groves and Steven Sivek[2].

**Theorem 96.** *There exist compact Hausdorff connected* $5$*-manifolds* $Y$ *and* $Z$ *with for all* $k \in \mathbb{N}$ *isomorphisms* $\pi_k(Y) \cong \pi_k(Z)$ *of homotopy groups, such that* $Y$ *and* $Z$ *are not of the same weak homotopy type.*

*Proof.* Consider the space $X = S^2 \times S^3$. We notice that $X$ is simply connected since $\pi_1(X) = \pi_1(S^2) \times \pi_1(S^3) \cong 0 \times 0 = 0$ and that it is a 5-manifold.

Consider the actions by $\{\pm 1\} = C_2$ on $S^2$ and on $S^3$ given by mulitplication $g \cdot x = gx$. We notice that this action is even so that $S^k$ is a covering space of the orbit space $S^k/C_2$ and consequently $S^k/C_2$ is a compact connected $k$-manifold. In fact it is homeomorphic to the real projective spece $\mathbb{R}P^k$.

We observe that $X$ covers both $Y := (S^2/C_2) \times S^3$ and $Z := S^2 \times (S^3/C_2)$ and therefore both are 5-manifolds as well. Since $X$ is simply connected, it must be the universal cover of both $Y$ and $Z$, which yields that their higher homotopy groups are isomorphic. By construction $\pi_1(Y) \cong C_2 \cong \pi_1(Z)$ and for $k > 1$ we use that the higher homotopy groups of a space are isomorphic to the higher homotopy groups of the universal cover so that $\pi_k(Y) \cong \pi_k(X) \cong \pi_k(Y)$.

We recall that a weak homotopy equivalence induces isomorphisms on all homology groups as well (see 66, property 7 and [3], Proposition 4.21), such that all spaces of the same weak homotopy type have isomorphic homology groups.

We will now apply the Künneth formula (Theorem 68) to the spaces $Y \cong S^2 \times \mathbb{R}P^3$ and $Z \cong S^3 \times \mathbb{R}P^2$ to show that not all of their homology groups are isomorphic.

We recall that the homology of real projective space is ([3] Example, 2.42):

$$H_k(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & \text{If } k = 0 \text{ or } (n \text{ is odd and } k = n) \\ \mathbb{Z}/2\mathbb{Z} & \text{If } k \text{ odd and } 0 < k < n \\ 0 & \text{Else} \end{cases} .$$

In Example 69 we calculated the homology of the product of a space $Y$ and a sphere, yielding

$$H_k(S^n \times Y) \cong H_k(Y) \oplus H_{k-n}(Y).$$

We can easily combine this to find the homology of $Y$ and $Z$

[2] `https://mathoverflow.net/questions/3540`

| $k$ | $H_k(\mathbb{R}P^2)$ | $H_k(S^3 \times \mathbb{R}P^2)$ | $H_k(\mathbb{R}P^3)$ | $H_k(S^2 \times \mathbb{R}P^3)$ |
|---|---|---|---|---|
| 0 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ |
| 2 | 0 | 0 | 0 | $\mathbb{Z}$ |
| 3 | 0 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ |
| 4 | 0 | $\mathbb{Z}/2\mathbb{Z}$ | 0 | 0 |
| 5 | 0 | 0 | 0 | $\mathbb{Z}$ |

All higher homology groups vanish. We see that the homology groups of $Y$ and $Z$ are not all isomorphic, and hence not of the same weak homotopy type. $\square$

### 6.3 Does The Weak Homotopy Type Admit a 'Universal' Element?

Our intuition from the 1-manifold case suggests that this might be true. Indeed let us prove this.
**Theorem 97.** *If $M$ and $N$ are topological spaces of the same weak homotopy type, then there exists a CW-complex $C$ and weak equivalences $\psi_M : C \to M$ and $\psi_N : C \to N$.*

To prove this we recall the CW-approximation theorem:
**Lemma 98.** *([3], Proposition 4.13, corollary 4.19) Let $X$ be a topological space, then there exist a CW-complex $C$ with a weak equivalence $f : C \to X$. Furthermore, if $D$ is also a CW-complex with $g : D \to X$ a weak equivalence, then there exists a homotopy equivalence $h : C \to D$ such that $g \circ h$ and $f$ are homotopic.*

Now we will proof the previous Theorem 97.

*Proof.* (Theorem 97) Let $N$ be a topological space, of the same homotopy type as $M$, as evidenced by the chain of spaces $X_0 = M, \ldots, X_n = N$, with a weak equivalence in either direction between each successive pair. Then we find CW-approximations $f_i : C_i \to X_i$.

We obtain a diagram like this:

$$\begin{tikzcd}
X_0 & X_1 \arrow[l] & X_2 \arrow[l] \arrow[r] & \cdots & X_n \arrow[l] \\
C_0 \arrow[u] & C_1 \arrow[u] & C_2 \arrow[u] & \cdots & C_n \arrow[u]
\end{tikzcd}$$

Since the composition of weak equivalences is a weak equivalence, this means that for all $i = 0, \ldots, n-1$ either $C_i$ also approximates $X_{i+1}$ or $C_{i+1}$ approximates $X_i$, in both cases yielding that $C_i$ and $C_{i+1}$ are homotopy equivalent. We conclude that there is a homotopy equivalence between $C_0$ and $C_n$ and therefore $C = C_0$ has the claimed property. $\square$

# References


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