Dijkgraaf-Witten Theory

THESIS

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Abstract

In this thesis we present Dijkgraaf-Witten theory. We start by considering a two-dimensional topological quantum field theory that can be used to prove Mednykh's formula along the way. Subsequently, we define the Dijkgraaf-Witten invariant as a partition function where we assign a specified weight to the 3-simplices of a compact, oriented and triangulated 3-manifold with boundary. The partition function depends on how we assign elements of a finite, discrete group G to all the oriented edges of the manifold. We prove that, whenever the triangulation of the boundary is fixed, the invariant does not depend on the triangulation of the manifold.

Finally, we define a similar invariant where we model the weight of the 3-simplices to mimic the action of Chern-Simons theory. We demonstrate that by demanding invariance, we obtain the Dijkgraaf-Witten invariant.

Contents

Chapter 1

Introduction

Some quantum mechanical systems are linked to the mathematics of topology. These systems can display interesting effects that are related to an integer number explainable by topology.
Quite recently there has been a lot of interest in topological quantum field theories. These topological quantum field theories yield topological invariants. For example three dimensional topological quantum field theories have been used for the calculation of knot invariants, such as the Jones polynomial.

Physically, these theories can be used to explain and to better understand interesting quantum mechanical phenomena (including systems that perform quantum computations).
A classic example is the Aharonov-Bohm effect. This effect demonstrates how particles are affected by electromagnetic fields, although the particles themselves do not propagate through a space where electromagnetic fields are present.

1.1 Aharonov-Bohm Effect

The following review of the Aharonov-Bohm effect is based on Steve Simon's lecture notes[1].

Let $\mathbf{x}(t)$ be the position of a particle at time t in D-dimensional space $\mathbb{R}^D$.

Starting at an initial time t_i in $\mathbf{x}_i$, the propagator

$$\langle \mathbf{x}_f | \hat{U}(t_f, t_i) | \mathbf{x}_i \rangle$$

gives the amplitude of reaching a final position $\mathbf{x}_f$ at t_f, with $\hat{U}$ the unitary time evolution operator.
The propagator acts on a wave function ψ to propagate it forward in time by

$$\langle \mathbf{x}_f | \psi(t_f) \rangle = \int d\mathbf{x}_i \langle \mathbf{x}_f | \hat{U}(t_f, t_i) | \mathbf{x}_i \rangle \langle \mathbf{x}_f | \psi(t_i) \rangle.$$

The propagator must satisfy two conditions. First of all, it must be unitary. This means that normalized wavefunctions stay normalized. Secondly, for $t_i \leq t_m \leq t_f$ it must hold that

$$\langle \mathbf{x}_f | \hat{U}(t_f, t_i) | \mathbf{x}_i \rangle = \int d\mathbf{x}_m \langle \mathbf{x}_f | \hat{U}(t_f, t_m) | \mathbf{x}_m \rangle \langle \mathbf{x}_m | \hat{U}(t_m, t_i) | \mathbf{x}_i \rangle.$$

Otherwise said, the propagator must obey composition.
It was Richard Feynman who wrote the propagator as

$$\langle \mathbf{x}_f | \hat{U}(t_f, t_i) | \mathbf{x}_i \rangle = \mathcal{N} \sum_{\text{paths } \mathbf{x}(t) \text{ from } \mathbf{x}_i \text{ to } \mathbf{x}_f} e^{iS[\mathbf{x}(t)]/\hbar},$$

where $\mathcal{N}$ is a normalization factor, $\hbar$ is the reduced Planck constant and

$$S[\mathbf{x}(t)] = \int_{t_i}^{t_f} dt L[\mathbf{x}(t), \dot{\mathbf{x}}(t), t]$$

is the action of the path with L the Lagrangian.

If we consider the two-slit experiment, we can write

$$\sum_{\text{paths}} e^{iS/\hbar} = \sum_{\text{paths, slit 1}} e^{iS/\hbar} + \sum_{\text{paths, slit 2}} e^{iS/\hbar}.$$

Now we add a magnetic field in the middle box between the two slits. We take care that this magnetic field does not leak out of the box and that it is kept constant. Then the interference pattern on the screen is changed due to the presence of the magnetic field. This is the Aharonov-Bohm effect.

To understand this phenomenon, we must look at the Lagrangian description of particle motion. First of all, the electric and magnetic field in terms of the vector potential $\mathbf{A}$ and electrostatic potential A_0 are

$$\mathbf{B} = \nabla \times \mathbf{A}$$

and

$$\mathbf{E} = -\nabla \cdot A_0 - \frac{d\mathbf{A}}{dt}.$$

The Lagrangian is given by

$$L = \frac{1}{2m}\dot{\mathbf{x}}^2 + q(\mathbf{A}(\mathbf{x}) \cdot \dot{\mathbf{x}} - A_0),$$

with q the charge of the particle.

Adding a magnetic field can be regarded as changing the action S with

$$S_0 + q\int dt\, \dot{\mathbf{x}} \cdot \mathbf{A} = S_0 + q\int d\mathbf{l} \cdot \mathbf{A},$$

with S_0 the action when there is no electric or magnetic field.
Thus we can write the amplitude as

$$\sum_{\text{paths, slit 1}} e^{iS_0/\hbar + iq/\hbar \int d\mathbf{l}\cdot\mathbf{A}} + \sum_{\text{paths, slit 2}} e^{iS_0/\hbar + iq/\hbar \int d\mathbf{l}\cdot\mathbf{A}}.$$

If we assume that the velocities are symmetric, then the phase difference of the two paths is

$$\Delta\phi = \frac{iq}{\hbar}\left(\int_{\text{slit 1}} d\mathbf{l} \cdot \mathbf{A} - \int_{\text{slit 2}} d\mathbf{l} \cdot \mathbf{A}\right).$$

If we regard the loop around the middle box and we apply Stokes' theorem, we can rewrite the phase difference as

$$\Delta\phi = \frac{iq}{\hbar}\oint d\mathbf{l} \cdot \mathbf{A} = \frac{iq}{\hbar}\int_{\text{inside loop}} d\mathbf{S} \cdot (\nabla \times \mathbf{A}) = \frac{iq}{\hbar}\Phi_{\text{enclosed}}.$$

Here Φ_{enclosed} is the flux enclosed in the loop, which is a fixed number. If Φ_{enclosed} is a multiple of the elementary flux quantum $\phi_0 = \frac{2\pi\hbar}{q}$, then the phase shift is an integer multiple of 2π, i.e. there is no phase shift.
We see that the phase difference does not depend on the distribution of the flux inside the coil.

The Aharonov-Bohm effect is an example of a single particle system in quantum mechanics. For multiple particles we shall look at Chern-Simons theory.

1.2 Chern-Simons Theory and the Quantum Hall Effect

Chern-Simons theory [2] is a topological quantum field theory (more on that in the next section) that classifies the phases in the fractional quantum Hall effect. The fractional quantum Hall effect consists of a two-dimensional system of electrons, where the Hall conductance σ takes on quantized values of $\sigma = \nu \cdot \frac{e^2}{h}$. Here, e is the elementary charge, h is Planck's constant and ν takes rational values.
To obtain an intuition of this effect and the way in which topology plays an important role in it, we can describe the particles moving through time by tubes. The braiding of these tubes can help us perform quantum computation. The topological 'part' of this system consists of the fact that small perturbations in the paths of the particles do not affect the ability to compute, as long as the way these tubes are braided does not change.
The practical advantage of this is that the system can still compute even when it is not fully insulated.
Furthermore, in mathematics Chern-Simons theory can be used to calculate knot invariants, such as the Jones polynomial.

The action S_{CS} in Chern-Simons theory is given by

$$S_{CS} = \int_M \text{tr}\,(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$

where M is a 3-manifold and A is a Lie-algebra valued 1-form.
We will not delve into the specifics of this action, but mention it here since we shall define another action later on in this thesis, which will mimic this one.

1.3 Topological Quantum Field Theory

A topological quantum field theory (TQFT) can be regarded as a theory that does not change under small deformations in the metric of spacetime. That means that it depends only on topological properties, not on geometric ones.

First introduced by Edward Witten, TQFT's were axiomatically defined by Michael Atiyah in 1988 [3]. These axioms will be presented shortly hereafter, though we shall not study these as they were originally presented by Atiyah, but rather by presenting Steve Simon's interpretation of them [1]. After Atiyah's formulation of the axioms, TQFT's have been introduced to category theory. There they can be regarded as functors between the category *nCob* (where the objects are $(n-1)$-dimensional, oriented, compact, smooth manifolds without boundary and the morphisms are n-dimensional, oriented, smooth manifolds with boundary) and the category $Vect_k$ (where the objects are k-vector spaces and the morphisms are linear maps).

The basic idea behind a TQFT is that it supplies a compact orientable manifold with an invariant. Which rules hold for this invariant and of which factors the invariant is independent, shall be specified later on.

1.3.1 Atiyah's Axioms

A formal definition is given by Atiyah's axioms, which we will consider by regarding Steve Simon's interpretation of these.
Let M be a $d+1$-dimensional oriented space-time manifold, with the d-dimensional slice Σ representing physical space and the extra dimension representing time, having an orientation induced by that of M.

1. To each d-dimensional slice Σ is associated a complex Hilbert space $V(\Sigma)$. The association depends only on the topology of Σ.

2. The Hilbert space of the disjoint union of two spaces Σ and Σ' is the tensor product
$$V(\Sigma \sqcup \Sigma') = V(\Sigma) \otimes V(\Sigma').$$
This implies that $V(\emptyset) = \mathbb{C}$, since $\mathbb{C} \otimes V(\Sigma) = V(\Sigma)$.

3. If $\Sigma = \partial M$ is the boundary of M, we associate an element $Z(M) \in V(\partial M)$ to the manifold M. Again, the association depends only on the topology of M.
If we regard ∂M as the spacelike slice of M at a fixed time and $V(\partial M)$ as the Hilbert space of ground states, then $Z(M)$ is a particular wavefunction.

We remark that if M is closed (and thus $\partial M = \emptyset$), we must have that the TQFT assigns a complex number $Z(M) \in \mathbb{C}$ to M.

4. If we reverse the orientation of Σ, denoting the same manifold with opposite orientation by Σ^*, we must have
$$V(\Sigma^*) = V(\Sigma)^*,$$
where $V(\Sigma)^*$ is the dual space of $V(\Sigma)$.

1.3.2 Gluing and cobordisms

Given two manifolds M and M' with common boundary $\Sigma = \partial M = (\partial M')^*$, we can *glue* these together along the common boundary by taking inner products of the corresponding states. Otherwise stated
$$Z(M \cup_\Sigma M') = \langle \psi' | \psi \rangle,$$
for $|\psi\rangle = Z(M) \in V(\Sigma)$ and $\langle \psi'| = Z(M') \in V(\Sigma^*)$.

Cobordism theory states that if two manifolds, which are disjoint, together form the boundary of another manifold, they can be considered the same. If Σ_1 and Σ_2 are two manifolds such that $\partial M = \Sigma_1 \sqcup \Sigma_2^*$, we say that Σ_1 and Σ_2 are *cobordant*, or that M is a *cobordism* between them.
We see that $Z(M) \in V(\Sigma_1) \otimes V(\Sigma_2^*)$. This means we can write
$$Z(M) = \sum_{\alpha,\beta} U^{\alpha\beta} |\psi_{1,\alpha}\rangle \otimes \langle \psi_{2,\beta}|,$$
where $\{|\psi_{1,\alpha}\rangle\}_\alpha$ form a basis for the states of $V(\Sigma_1)$, $\{\langle\psi_{2,\beta}|\}_\beta$ form a basis for the states of $V(\Sigma_2^*)$ and $U^{\alpha\beta}$ are unitary coefficients.
We remark here a similarity to homology theory (see Chapter 3). Since the boundary of the boundary is empty, we can construct something that is analogous to a chain complex of manifolds.

1.4 Dijkgraaf-Witten Theory

For the study of quantum field theory, an oft employed tool are toy models. These toy models are used to obtain a simpler view of more advanced or more difficult situations, and also help to fully understand the basics that underlie these phenomena.

Dijkgraaf-Witten theory [4] is such a toy model. In a similar way that Chern-Simons helps us to better understand phenomena like the quantum Hall effect, Dijkgraaf-Witten helps us to better understand Chern-Simons. A key ingredient of Chern-Simons theory are groups. There are no further restrictions on the group that is used. In the case that this group is finite and discrete, we obtain Dijkgraaf-Witten theory.

Dijkgraaf-Witten assigns elements of this group to the edges of a triangulated 3-manifold. A map that sends combinations of the group elements to $U(1)$, is to be viewed as a Boltzmann weight of the parts that form the triangulation of the manifold. Ultimately a partition function of these weights is defined, which is to be the invariant of the manifold.
Of course, we see here a close similarity to the Ising model. In the Ising model a lattice consists of atomic spins which are either spin up or spin down (take values in $\mathbb{Z}/2\mathbb{Z}$). Each configuration has a certain Boltzmann weight and ultimately a partition function, depending on these weights, is defined for the whole system.

A more precise look at Dijkgraaf-Witten theory shall take place in Chapter 3.

Chapter 2

Lattice Topological Quantum Field Theory and the Mednykh Formula

Before diving into Dijkgraaf-Witten theory in three dimensions, we shall consider a two-dimensional TQFT first.
We are going to construct a topological invariant of two-dimensional surfaces attached to a semisimple algebra using an explicit triangulation. These invariants, the topological quantum field theories, can be used to easily prove Mednykh's formula.
The proof based on these invariants comes from an article of Noah Snyder [5].
Before we start constructing the invariant, we must fix a few objects.
Let G be a finite abelian group and M a two-dimensional compact, orientable manifold without boundary. Furthermore k is an algebraically closed field of characteristic 0 and A a semisimple algebra over k.*

2.1 Triangulation of the surface

Let M be the surface as above having a fixed triangulation with oriented edges. These oriented edges are not allowed to be attached to themselves, i.e. the edge to which an oriented edge is attached must have opposite

*An algebra A is *semisimple* if all non-zero A-modules are semisimple, that is to say the A-modules are a direct sum of simple modules. A non-zero module M is said to be simple, if the only submodules are 0 and M.

orientation.
We write $\#V$ for the number of vertices, $\#E$ for the number of edges and $\#F$ for the number of faces.

Definition 2.1.1 (Flag)**.** A *flag* is a pair (edge, face) where the edge is contained in the face.

Example 2.1.1 (Torus)**.**

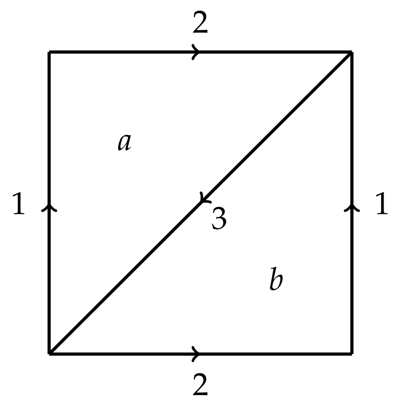


Figure 1. This triangulation of the torus has two faces a and b; three edges numbered $1, 2$ and 3; and there are six flags $(1, a), (1, b), (2, a), (2, b), (3, a)$ and $(3, b)$. Further, we orient both a and b counter-clockwise.

2.2 The Invariant

Before we can construct the invariant, we must construct the components of which it is composed.

Definition 2.2.1 (Trace map)**.** The *trace map*

$$Tr : A \to k$$

is the trace of the multiplication map $m_x : A \to A$, $a \mapsto xa$.

The map $T_n : A^{\otimes n} \to k$ given by $x_1 \otimes \cdots \otimes x_n \mapsto Tr(x_1 \cdots x_n)$ is invariant under cyclic permutations. Furthermore $T_2 : A \otimes A \to k$ is a nondegenerate symmetric bilinear form due to the semisimplicity of A, providing us an identification $A \to A^*$ of A with its dual. Again using the semisimplicity of A, we can invert this map to obtain $A^* \to A$. This in turn provides us a map $p : k \to A \otimes A$.

Definition 2.2.2 (Lattice TQFT)**.** To each edge of the surface we associate the map $p : k \to A \otimes A$, to each oriented face the map $T_3 : A^{\otimes 3} \to k$ and each flag obtains a copy of A.
In this way our surface has a map

$$I_A : k \cong k^{\otimes \#E} \xrightarrow{p} A^{\otimes \#\text{flags}} \xrightarrow{T_3} k^{\otimes \#F} \cong k.$$

This is the *lattice topological quantum field theory*.

Example 2.2.1 (Torus)**.** Since there are three edges, six flags and two faces, we will obtain a map

$$k^1 \otimes k^2 \otimes k^3 \to A^{(1,a)} \otimes A^{(1,b)} \otimes A^{(2,a)} \otimes A^{(2,b)} \otimes A^{(3,a)} \otimes A^{(3,b)} \to k^a \otimes k^b.$$

The following theorem tells us that the defined lattice TQFT is actually invariant under the triangulation of M.

Theorem 2.2.1. *The lattice topogical quantum field theory $I_A(M)$ does not depend on the triangulation of M.*

An interesting proof of this theorem can be found in [6].

2.3 Mednykh's Formula

Let $\chi(M) = \#V - \#E + \#F$ be the Euler characteristic of M, $\hat{G}$ the set of isomorphism classes of irreducible representations of G and $d(V)$ the dimension of $V \in \hat{G}$. †

Theorem 2.3.1. *(Mednykh) Mednykh's formula is given by*

$$\sum_{V \in \hat{G}} d(V)^{\chi(M)} = |G|^{\chi(M)-1} |Hom(\pi_1(M), G)|.$$

†A representation of G on a vector space V over a field k is a group homomorphism $G \to GL(V)$. A subrepresentation is a subspace $W \subset V$ that is invariant under the group action. If V has exactly two subrepresentations ($\{0\}$ and V), the representation is *irreducible*. The dimension of V is the dimension of the representation.

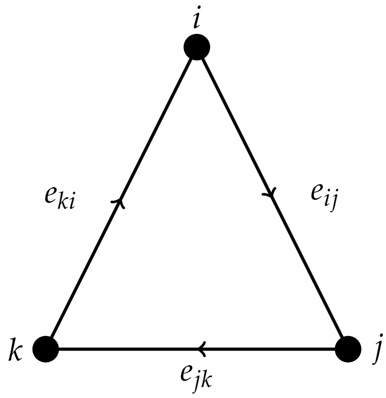
To prove this, we will compute the lattice TQFT of M in the basis of the group elements and in the basis of the matrix elements of the irreducible representations.

Since k has characteristic 0, Maschke's theorem states that the group algebra $k[G]$ is semisimple.[‡] This leads us to our first proposition:

Proposition 2.3.1. *The lattice TQFT of M attached to the group algebra $k[G]$ is*

$$I_{k[G]}(M) = \sum_{V \in \hat{G}} d(V)^{\chi(M)}.$$

Proof. Let $Mat(n)$ denote the set of $n \times n$-matrices. We will first show that $I_{Mat(n)}(M) = n^{\chi(M)}$.
The map p of each edge is defined by $1 \mapsto \frac{1}{n} \sum_{i,j} e_{ij} \otimes e_{ij}$. The map T_3 of each oriented face sends $e_{ij} \otimes e_{jk} \otimes e_{ki} \mapsto 1$ and all triples of another form to 0.
Thus we see that $I_{Mat(n)}(M) = n^{\#F - \#E} Z(Mat(n), M)$.
Here $Z(Mat(n), M)$ is the number of ways of labeling each oriented edge by a pair (i, j) such that the same edge with opposite orientation is labeled by (j, i) and the other two edges in the same face are labeled by (j, k) and (k, i).



[‡]The group algebra $k[G]$ is the free vector space on G over k, i.e. each $x \in k[G]$ can be written as $x = \sum_{g \in G} a_g g$ with $a_g \in k$.

Figure 2. This figure shows the labeling of each triangle.

As we can see in the figure above this is equivalent to labeling vertices by a number. This means that $Z(Mat(n), M) = n^{\#V}$ and thus

$$I_{Mat(n)}(M) = n^{\#F-\#E+\#V} = n^{\chi(M)}.$$

Artin-Wedderburn's theorem states that $k[G] \cong \bigoplus_{V \in \hat{G}} M_{d(V)}$. Thus we obtain that $I_{k[G]}(M) = \sum_{V \in \hat{G}} I_{M_{d(V)}}(M) = \sum_{V \in \hat{G}} (d(V))^{\chi(M)}$.[§]

□

Proposition 2.3.2. *The lattice TQFT of M attached to the group algebra $k[G]$ is*

$$I_{k[G]}(M) = |G|^{\chi(M)-1} |Hom(\pi_1(M), G)|.$$

Proof. The map p of each oriented edge is defined by $1 \mapsto \frac{1}{|G|} \sum_{g \in G} g \otimes g^{\pm 1}$ with $+1$ if the orientation of the edge agrees with the orientation of the edge induced by the face in which it lies and -1 otherwise. The map T_3 of each flag sends $a \otimes b \otimes c \mapsto \begin{cases} 1 & \text{if } abc = 1, \\ 0 & \text{else.} \end{cases}$
Thus we see that $I_{k[G]}(M) = |G|^{\#F-\#E} Z(G, M)$.
Here $Z(G, M)$ is the number of ways of labeling each oriented edge by an element $g \in G$ such that the same edge with opposite orientation is labeled by g^{-1} and the product of the elements around an oriented face is 1. Such a labeling is called *consistent*.
We will now construct a bijection between the set of consistent labelings and $G^{\#V \setminus \{v_0\}} \times \text{Hom}(\pi_1(M), G)$ (for a vertex v_0 of M) and thus show that $Z(G, M) = |G|^{\#V-1} |\text{Hom}(\pi_1(M), G)|$.

[§]For this to hold, it must be shown that I is additive under direct sums.

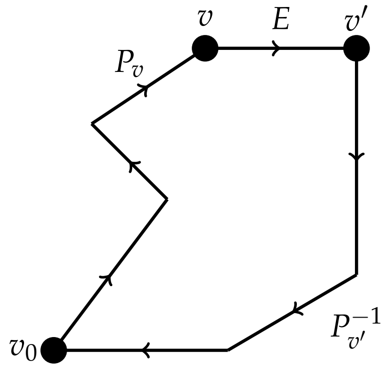


Figure 3. This figure shows the bijection we will construct.

If we fix a base vertex v_0 in M, we define P_v to be an oriented path along the edges of M from v_0 to any other vertex v.
Let f be a consistent labeling, which we will regard as a map

$$f : \{\text{oriented edges of M}\} \to G.$$

Assigning to each vertex v of M the element $\Pi_{e\in P_v} f(e)$ and to each loop L the element $\Pi_{e\in L} f(e)$, we obtain an injection from $Z(G, M)$ to $G^{\#V\setminus\{v_0\}} \times \mathrm{Hom}(\pi_1(M), G)$, since the assignment to L depends only on the class of L in $\pi_1(M)$ due to the consistency condition on the triangles.

For the converse we have assigned an element of G to each P_v and each L. In the figure above, we see for a loop $L = P_{v'}^{-1} \circ E \circ P_v$ that

$$f(E) = f(P_{v'}) \circ f(L) \circ f(P_v)^{-1}.$$

In this way we recover a consistent labeling of M.

Thus we see that

$$I_{k[G]}(M) = |G|^{\#F-\#E} Z(G, M) = |G|^{\chi(M)-1} |\mathrm{Hom}(\pi_1(M), G)|.$$

□

We conclude from Propositions 2.3.1 and 2.3.2 that we can obtain Mednykh's formula.

Example 2.3.1 (Torus)**.** We take as our group $G = \mathbb{Z}/3\mathbb{Z} = \{e, \sigma, \sigma^2\}$. Each edge obtains a map $p : 1 \mapsto \frac{1}{3}(e \otimes e + \sigma \otimes \sigma + \sigma^2 \otimes \sigma^2)$. For

$$P : k^1 \otimes k^2 \otimes k^3 \to A^{(1,a)} \otimes A^{(1,b)} \otimes A^{(2,a)} \otimes A^{(2,b)} \otimes A^{(3,a)} \otimes A^{(3,b)},$$

we see that

$$\begin{aligned}
&P(1 \otimes 1 \otimes 1) = p(1) \otimes p(1) \otimes p(1) = \\
&\frac{1}{27}\{(e \otimes e \otimes e \otimes e \otimes e \otimes e) + (e \otimes e \otimes e \otimes e \otimes \sigma \otimes \sigma^2)+ \\
&(e \otimes e \otimes e \otimes e \otimes \sigma^2 \otimes \sigma)+ \\
&(e \otimes e \otimes \sigma \otimes \sigma^2 \otimes e \otimes e) + (e \otimes e \otimes \sigma \otimes \sigma^2 \otimes \sigma \otimes \sigma^2)+ \\
&(e \otimes e \otimes \sigma \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma)+ \\
&(e \otimes e \otimes \sigma^2 \otimes \sigma \otimes e \otimes e) + (e \otimes e \otimes \sigma^2 \otimes \sigma \otimes \sigma \otimes \sigma^2)+ \\
&(e \otimes e \otimes \sigma^2 \otimes \sigma \otimes \sigma^2 \otimes \sigma)+ \\
&(\sigma \otimes \sigma^2 \otimes e \otimes e \otimes e \otimes e) + (\sigma \otimes \sigma^2 \otimes e \otimes e \otimes \sigma \otimes \sigma^2)+ \\
&(\sigma \otimes \sigma^2 \otimes e \otimes e \otimes \sigma^2 \otimes \sigma)+ \\
&(\sigma \otimes \sigma^2 \otimes \sigma \otimes \sigma^2 \otimes e \otimes e) + (\sigma \otimes \sigma^2 \otimes \sigma \otimes \sigma^2 \otimes \sigma \otimes \sigma^2)+ \\
&(\sigma \otimes \sigma^2 \otimes \sigma \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma)+ \\
&(\sigma \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma \otimes e \otimes e) + (\sigma \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma \otimes \sigma \otimes \sigma^2)+ \\
&(\sigma \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma \otimes \sigma^2 \otimes \sigma)+ \\
&(\sigma^2 \otimes \sigma \otimes e \otimes e \otimes e \otimes e) + (\sigma^2 \otimes \sigma \otimes e \otimes e \otimes \sigma \otimes \sigma^2)+ \\
&(\sigma^2 \otimes \sigma \otimes e \otimes e \otimes \sigma^2 \otimes \sigma)+ \\
&(\sigma^2 \otimes \sigma \otimes \sigma \otimes \sigma^2 \otimes e \otimes e) + (\sigma^2 \otimes \sigma \otimes \sigma \otimes \sigma^2 \otimes \sigma \otimes \sigma^2)+ \\
&(\sigma^2 \otimes \sigma \otimes \sigma \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma)+ \\
&(\sigma^2 \otimes \sigma \otimes \sigma^2 \otimes \sigma \otimes e \otimes e) + (\sigma^2 \otimes \sigma \otimes \sigma^2 \otimes \sigma \otimes \sigma \otimes \sigma^2)+ \\
&(\sigma^2 \otimes \sigma \otimes \sigma^2 \otimes \sigma \otimes \sigma^2 \otimes \sigma)\}.
\end{aligned}$$

The map belonging to each flag is given by $\alpha \otimes \beta \otimes \gamma \mapsto \begin{cases} 3 & \text{if } \alpha\beta\gamma = e, \\ 0 & \text{else.} \end{cases}$
For $T : A^{(1,a)} \otimes A^{(1,b)} \otimes A^{(2,a)} \otimes A^{(2,b)} \otimes A^{(3,a)} \otimes A^{(3,b)} \to k \otimes k$ we must be careful with regard to the order of the group elements in the terms of the sum above. The first, third and fifth element in the tensor product belong to face a; and the second, fourth and sixth to b.

The terms colored red give us non-zero contributions. Thus we see that

$$\begin{aligned} T(P(1 \otimes 1 \otimes 1)) =& \frac{1}{27}(3 \otimes 3 + 3 \otimes 3 + 3 \otimes 3 + \\ & 3 \otimes 3 + 3 \otimes 3 + 3 \otimes 3 + \\ & 3 \otimes 3 + 3 \otimes 3 + 3 \otimes 3) \\ =& \ 3. \end{aligned}$$

Chapter 3

The Dijkgraaf-Witten Invariant

In this section we will construct an invariant of 3-manifolds, first done by Dijkgraaf and Witten [4].

3.1 Simplices and Cohomology

Before giving the invariant we shall first delve into cohomology to obtain a firmer grasp on cocycles, which form an important part of the invariant. This introduction to cohomology is based on Hatcher's Algebraic Topology [7].

Definition 3.1.1 (Simplex)**.** An *n-simplex* is a convex hull of $n+1$ points $v_0, ..., v_n$ in $\mathbb{R}^m$ (for $m \geq n$) such that the difference vectors $v_1 - v_0, ..., v_n - v_0$ are linearly independent.
The points $v_0, ..., v_n$ are named vertices and we denote the simplex by $[v_0, ..., v_n]$.

Remark 3.1.1. We give an order to the vertices in $[v_0, ..., v_n]$ such that $v_0 < ... < v_n$. This order also determines the orientation of the edges $[v_i, v_j]$.

Example 3.1.1.

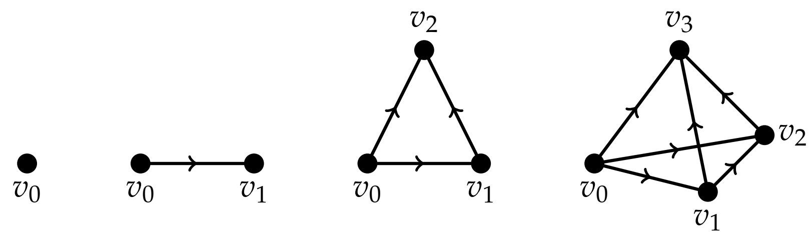


Figure 4. From left to right: the 0-, 1-, 2- and 3-simplex.

Definition 3.1.2 (Standard n-simplex)**.** The *standard n-simplex* in $\mathbb{R}^{n+1}$ is given by

$$\Delta^n = \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \geq 0\}.$$

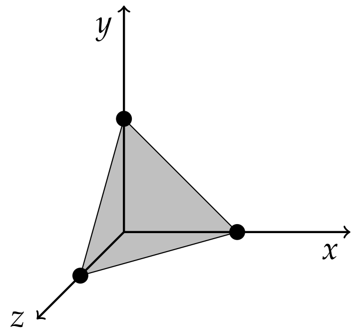


Figure 5. The standard 2-simplex in $\mathbb{R}^3$.

Definition 3.1.3 (Face)**.** Removing one vertex from $[v_0, ..., v_n]$ delivers an $(n-1)$-simplex for $n \geq 1$. This new simplex is a *face* of $[v_0, ..., v_n]$.
The 0-simplex has no faces.

Definition 3.1.4 (Boundary and open n-simplex)**.** The union of the faces of Δ^n is called the *boundary* $\partial\Delta^n$.
The *open n-simplex* is the interior $\mathring{\Delta}^n = \Delta^n \backslash \partial\Delta^n$.

Definition 3.1.5 (Δ-complex)**.** A Δ*-complex* on a topological space X is a set of maps $\sigma_\alpha : \Delta^{n(\alpha)} \to X$ such that

1. $\sigma_\alpha|_{\mathring{\Delta}^{n(\alpha)}}$ is injective and each point of X is in the image of exactly one such $\sigma_\alpha|_{\mathring{\Delta}^{n(\alpha)}}$;

2. every restriction of σ_α to a face of $\Delta^{n(\alpha)}$ is a new map $\sigma_\beta : \Delta^{n(\alpha)-1} \to X$;
3. a set $A \subset X$ is open if and only if $\sigma^{-1}(A)$ is open in $\Delta^{n(\alpha)}$ for all α.

Definition 3.1.6 (Simplicial complex)**.** A *simplicial complex* is a Δ-complex of which the simplices are uniquely determined by its vertices.

That is to say, there cannot be multiple simplices consisting of the same set of vertices.

Definition 3.1.7 (n-chain)**.** Let $\Delta_n(X) = \bigoplus_{\alpha:n(\alpha)=n} \mathbb{Z}\sigma_\alpha$. This is a free abelian group. Its elements are of the form $\sum_\alpha n_\alpha \sigma_\alpha$ (for $n_\alpha \in \mathbb{Z}$) and are called *n-chains*.

Definition 3.1.8 (Boundary operator)**.** The *boundary operator* is a map $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ given by

$$\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[v_0,...,\hat{v}_i,...,v_n]},$$

where the hat indicates the removal of vertex v_i, i.e.

$$[v_0, ..., \hat{v}_i, ..., v_n] = [v_0, ..., v_{i-1}, v_{i+1}, ..., v_n].$$

Example 3.1.2.

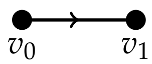


Figure 6. We see that $\partial[v_0, v_1] = [v_1] - [v_0]$.

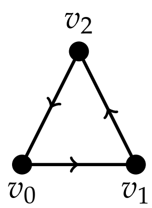


Figure 7. We see that

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1].$$

The boundary operator thus takes care to make the orientation coherent.

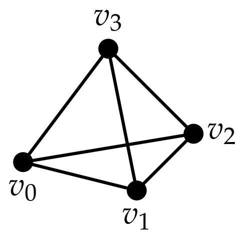


Figure 8. We see that

$$\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2].$$

Here we see that from the outside all faces are now oriented counter-clockwise.

Lemma 3.1.1. *The composition*

$$\partial_{n-1}\partial_n : \Delta_n(X) \to \Delta_{n-2}(X)$$

is zero.

Proof. We know that $\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$, so $\partial_{n-1}\partial_n(\sigma_\alpha) = \sum_i \sum_{j<i} (-1)^i(-1)^j \sigma_\alpha|_{[v_0,\ldots,\hat{v}_j,\ldots,\hat{v}_i,\ldots,v_n]} + \sum_i \sum_{j>i} (-1)^i(-1)^{j-1} \sigma_\alpha|_{[v_0,\ldots,\hat{v}_i,\ldots,\hat{v}_j,\ldots,v_n]}$. Switching i and j in the second sum gives us the negative of the first sum.

□

Definition 3.1.9. (Chain complex) Let $C_0, C_1, \ldots$ be abelian groups and $\partial_n : C_n \to C_{n-1}$ homomorphisms such that $\partial_n\partial_{n+1} = 0$ for all $n \geq 0$, where we take $\partial_0 : C_0 \to 0$. Then

$$\ldots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \ldots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

is called a *chain complex* and the groups C_n are *chain groups*.

Remark 3.1.2. The condition $\partial_n\partial_{n+1} = 0$ is equivalent with stating

$$\text{im}\, \partial_{n+1} \subset \ker \partial_n.$$

Definition 3.1.10. (Homology group) The *n-th homology group* of a chain complex is

$$H_n = \ker \partial_n / \text{ im}\, \partial_{n+1}.$$

Definition 3.1.11. (Cochain group and coboundary operator) The *cochain group* is the dual of the chain group, i.e. $C^n := C_n^* = \mathrm{Hom}(C_n, A)$ for A an abelian group.
The *coboundary operator* is the dual map $\delta = \partial^* : C^{n-1} \to C^n$.

Remark 3.1.3. We will drop the indices in our notation of the (co)boundary operators in cases where this is clear from the context.

Remark 3.1.4. For two maps α, β, we know that $(\alpha\beta)^* = \beta^*\alpha^*$. Further we know that the dual of the zero map $0^* = 0$ is also the zero map. From these we can immediately conclude

$$\partial\partial = 0 \implies \delta\delta = 0.$$

Definition 3.1.12. (Cohomology group) For

$$... \leftarrow C^{n+1} \overset{\delta}{\leftarrow} C^n \overset{\delta}{\leftarrow} C^{n-1} \leftarrow ... \leftarrow C^1 \overset{\delta}{\leftarrow} C^0 \overset{\delta}{\leftarrow} 0$$

a cochain complex, we call $H^n(C^n; A) = \ker \delta / \operatorname{im} \delta$ the *n-th cohomology group* of C^n with coefficients in A.

Definition 3.1.13. (*i*-cocycle) Let G be a finite, discrete topological group and V a multiplicative abelian group. Let $\phi : C^i(G) \to V$ be a morphism, where $C^i(G) := \underbrace{G \times ... \times G}_{i \text{ times}}$ for $i \geq 1$, then d^i denotes an operator

$$d^i\phi : C^{i+1}(G) \to V$$

such that $d^i\phi(g_1, ..., g_{i+1}) = \phi(g_1, ..., g_i)^{(-1)^{i+1}}\phi(g_2, ..., g_{i+1})$
$\Pi_{j=1}^i \phi(g_1, ..., g_j g_{j+1}, ..., g_{i+1})^{(-1)^j}$. The map ϕ is an *i-cocycle* if $d^i\phi = 1$.

3.2 The Invariant

We now consider Wakui's [8] treatment of the Dijkgraaf-Witten invariant.

Let G be a finite group and M a compact, oriented and triangulated 3-manifold with boundary ∂M.

Definition 3.2.1 (Color)**.** A *color* of M is a map

$$\phi : \{ \text{ oriented edges of } M\} \to G,$$

satisfying two conditions:

1. for any 2-simplex F we have $\phi(\partial F) = 1$, where the notation $\phi(\partial F)$ denotes the product of the group elements along the boundary of F;

2. for any oriented edge E we have $\phi(-E) = \phi(E)^{-1}$, where $-E$ denotes the same edge E, but with opposite orientation.

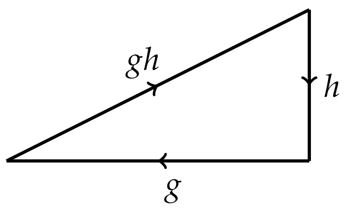


Figure 9. The color of the 2-simplex above satisfies the conditions, where g and h denote elements of the group G.

A *color of* ∂M is a map $\tau : \{\text{oriented edges of } \partial M\} \to G$ satisfying the same conditions as above.
We denote the set of all colors of M and ∂M by Col(M) and Col(∂M) respectively. For $\tau \in$ Col(∂M), we denote the set of colors of M that are equal to τ at ∂M by Col(M, τ).

Before we define the Dijkgraaf-Witten invariant, we give an order to the vertices of M. Furthermore we write each 3-simplex of M as a combination of its vertices in ascending order, so for $v_1 < v_2 < v_3 < v_4$ we write $\sigma = [v_1, v_2, v_3, v_4]$.

Definition 3.2.2 (Weight)**.** Fix a 3-cocycle $\alpha : C^3(G) \to U(1)$ and let $\phi \in \text{Col}(M)$. For $\sigma = [v_1, v_2, v_3, v_4]$ we denote

$$\phi([v_1, v_2]) = g, \phi([v_2, v_3]) = h, \phi([v_3, v_4]) = k,$$

where $[v_i, v_j]$ is the edge from v_i to v_j.
The *weight* of σ with respect to ϕ is $W(\sigma, \phi) = \alpha(g, h, k) \in U(1)$.

Definition 3.2.3 (Dijkgraaf-Witten Invariant)**.** Let $\sigma_1, ..., \sigma_n$ be the 3-simplices of M and a the number of vertices of M.
Let $\tau \in$ Col(∂M). The *Dijkgraaf-Witten invariant* is

$$Z_M(\tau) = \frac{1}{|G|^a} \sum_{\phi \in \text{Col}(M,\tau)} \prod_{i=1}^{n} W(\sigma_i, \phi)^{\epsilon_i},$$

where $\epsilon_i = \begin{cases} 1 & \text{if the orientation of } \sigma_i \text{ matches the orientation of } M, \\ -1 & \text{else.} \end{cases}$

Remark 3.2.1. In the case where G is abelian, we see that α is trivial. Then the Dijkgraaf-Witten invariant is

$$Z_M(\tau) = \frac{1}{|G|^a} |\text{Col}(M, \tau)|.$$

Theorem 3.2.1. *Fix a triangulation of ∂M and fix a color $\tau \in$ Col (∂M). Then $Z_M(\tau)$ does not depend on the order of the vertices of M and it does not depend on the triangulation of M.*

3.3 Proof of Invariance

The proof of Theorem 3.2.1 is built on a number of stages.
First of all a proof must be given for invariance under the order of the vertices. For this proof, we refer the reader to [8].
Further, the proof of invariance under triangulations consists of two parts. One must show that any two triangulations can be transformed one to another by a sequence of moves that will be specified later on; and one must demonstrate that $Z_M(\tau)$ does indeed not change under these moves. For the latter proof we again refer the reader to [8]. What follows now is a look at the former proof [10].

Definition 3.3.1 (Star)**.** Let X be a simplicial complex with triangulation T. Let E be an open simplex of X. The *star* S_E of E is the union of simplices of X containing E.

Definition 3.3.2 (Stellar subdivision or Alexander move)**.** A *stellar subdivision* (or *Alexander move*) of T along E is a transformation of T which replaces S_E by the cone over the boundary of S_E centered at a point $b \in E$.

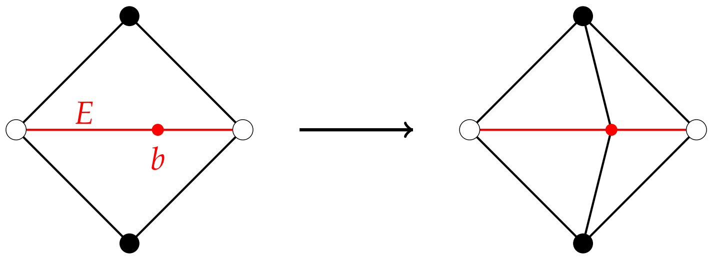


Figure 10. A stellar subdivision.

Definition 3.3.3 (Internal stellar subdivision or Alexander move)**.** A stellar subdivision of T along E is called *internal* if E does not lie in the boundary ∂X.

Theorem 3.3.1. *Let P be a simplicial complex with A the set of its simplices and let Q be a simplicial complex with simplex set $B \subset A$. Any triangulations T and T' of P that coincide on Q can be transformed one to another by a sequence of Alexander moves and transformations inverse to Alexander moves, leaving the triangulation of Q unchanged.*

Before we begin the proof we present Alexander's theorem [11], a result which we will use to prove Theorem 3.3.1.

Theorem 3.3.2 (Alexander)**.** *Any triangulated simplicial complex can be transformed into the cone over its (triangulated) boundary by a sequence of Alexander moves and transformations inverse to Alexander moves.*

Proof of 3.3.1. It is sufficient to prove the theorem for the case that one of the triangulations is a subdivision of the other. For if this is not the case, we can consider the subdivision that is formed by taking the intersection of the triangulations.

We will use induction on the dimension of P.
For $\dim P = 0$ the theorem clearly holds.
Assume the theorem holds for dimension less than n for a certain $n \in \mathbb{N}_{>0}$. Let X and Y be two triangulations of P (with $\dim P < n$) that coincide on Q and suppose Y is finer than X.
For each n-simplex A of X, the n-simplices of Y that lie in A form a triangulation of A. By Alexander's theorem, we can transform A into the cone over ∂A. Doing this for all n-simplices of X, we obtain a new triangulation Z that is finer than X and on all n-simplices A of X is the cone over ∂A.
Let X_{n-1} be the $(n-1)$-skeleton of X. This is the union of all simplices of X of dimension $\leq n-1$. We see that the triangulation of X_{n-1} induced by X is identical to the subdivision of Z on $X_{n-1} \cap Q$. So by the induction hypothesis we can Alexander transform the triangulation induced by Z to that induced by X. The cone structure of Z on n-simplices of X allows us to extend these transformations to Alexander transformations that convert Z to X and are identical on Q. $\square$

3.4 Relation to Snyder's Invariant

Recall that Snyder's invariant was defined for two-dimensional manifolds without boundary, and the group was abelian. To observe the similarity between this invariant and the Dijkgraaf-Witten invariant, we should define an analogous Dijkgraaf-Witten invariant for 2-manifolds without boundary.
We saw for an abelian group G that the Dijkgraaf-Witten invariant of a 3-manifold was given by

$$Z_M(\tau) = \frac{1}{|G|^a} |\mathrm{Col}(M, \tau)|.$$

Adapting this slightly to account for 2-manifolds without boundary (which means that we will not fix a color τ of the boundary anymore), we obtain

$$Z_M = \frac{1}{|G|^a} |\mathrm{Col}(M)|.$$

We refer to the proof of Proposition 2.3.2 to remark that Snyder's invariant is given by

$$I_{k[G]}(M) = |G|^{\chi(M) - \#V} Z(G, M),$$

where $\chi(M) = \#V - \#E + \#F$ is the Euler characteristic of M.
We would like to remind the reader that $Z(G, M)$ was the number of consistent labelings. Per definition a consistent labeling is a way of labeling each oriented edge by an element $g \in G$ such that the same edge with opposite orientation is labeled by g^{-1} and the product of the elements around a face is 1. Thus it is clearly the same as a color of M. So we find

$$Z_M = |G|^{-\chi(M)} I_{k[G]}(M).$$

We see that the two invariants are not equal, but they differ only by a factor of $|G|^{-\chi(M)}$.

Chapter 4

Connection to Chern-Simons Theory

In this section we will construct a model of the action of Chern-Simons theory and show that this leads to the Dijkgraaf-Witten invariant.
The following discussion is based on an article by Danny Birmingham and Mark Rakowski [12].

4.1 Formalism

Let $V = \{v_i\}_{i \in I}$ be the set of vertices in our simplicial complex. We denote an ordered k-simplex of $k+1$ vertices by $[v_0, ..., v_k]$, or shorter by $[0, ..., k]$. The boundary ∂ on the simplex $\sigma = [0, ..., k]$ acts as

$$\partial\sigma = \sum_{i=0}^{k} (-1)^i [0, ..., \hat{i}, ..., k].$$

Here $\hat{i}$ means that we omit the i-th vertex.

We then assign elements of $\mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{Z}_{>0}$ to simplices of a certain dimension. For k-simplices such an assignment is called a k-color B^k. Evaluated at the k-simplex $[0, ..., k]$, we obtain the element

$$\langle B^k, [0, ..., k] \rangle =: B^k_{0 \cdots k} \in \mathbb{Z}/n\mathbb{Z}.$$

Furthermore, we assume $B^1_{01} = -B^1_{10} \mod n$. Next, we have the coboundary operator δ, which acts on a $k-1$-color and is evaluated at a k-simplex by

$$\langle \delta B^{k-1}, [0, ..., k] \rangle = \sum_{i=0}^{k} (-1)^i B_{0\cdots\hat{i}\cdots k}.$$

The last operator we define, is the cup product operator $\cup$. This operates on a k-color B^k and a l-color C^l to be evaluated at a $k+l$-simplex by

$$\langle B^k \cup C^l, [0, ..., k+l] \rangle = B_{0\cdots k} \cdot C_{k\cdots k+l}.$$

4.2 Partition Function

Let K be the simplicial complex representing a manifold of dimension n. Giving an order to the set of vertices will determine the orientation of the n-simplices of K.
Let $K^n := \sum_i \epsilon_i \sigma_i$ be the ordered set of n-simplices σ_i, where ϵ_i indicates whether the simplex is positively or negatively oriented with respect to the orientation of K.

Now we assign a certain weight $W[\sigma_i]$ to each n-simplex σ_i. The value of $W[\sigma_i]$ is a non-zero complex number for each σ_i and is a function of the colors. In the next section we will see which conditions we must impose on this weight in order to obtain triangulation invariance for the partition function we define next. Further, K^n obtains a weight

$$W[K^n] = \Pi_i W[\sigma_i]^{\epsilon_i}$$

and the partition function is defined by

$$Z = \frac{1}{|G|^a} \sum_{\text{colors}} W[K^n],$$

where G is the group where the colors take their values and a is the number of 0-simplices.

4.3 Obtaining Dijkgraaf-Witten

The model we study in this section consists of a 3-manifold and uses a 1-color A with values in $\mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{Z}_{>0}$. The weight assigned to an ordered 3-simplex $[0,1,2,3]$ is

$$\begin{aligned} W[[0,1,2,3]] &:= \exp\{\beta\langle A\cup\delta A,[0,1,2,3]\rangle\} \\ &= \exp\{\beta\langle A,[0,1]\rangle\langle\delta A,[1,2,3]\rangle\} \\ &= \exp\{\beta A_{01}\langle A,\partial[1,2,3]\rangle\} \\ &= \exp\{\beta A_{01}\langle A,[2,3]-[1,3]+[1,2]\rangle\} \\ &= \exp\{\beta A_{01}(A_{23}-A_{13}+A_{12})\}. \end{aligned}$$

Here β is a complex number upon which we shall impose certain conditions later on.
Remark the similarity between the definition of the weight defined here and the term $A\wedge dA$ in the Chern-Simons action that we saw earlier.

If we add a new vertex c in the middle of $[0,1,2,3]$, link it to all vertices and order the vertices such that c is first, we obtain the new simplices

$$[0,1,2,3]\to[c,1,2,3]-[c,0,2,3]+[c,0,1,3]-[c,0,1,2].$$

We see that

$$\begin{aligned} &W[[0,1,2,3]]\exp\{-\beta\langle\delta A\cup\delta A,[c,0,1,2,3]\rangle\} \\ =&W[[0,1,2,3]]\exp\{-\beta\langle\delta A,[c,0,1]\rangle\langle\delta A,[1,2,3]\rangle\} \\ =&W[[0,1,2,3]]\exp\{-\beta(A_{01}-A_{c1}+A_{c0})(A_{23}-A_{13}+A_{12})\} \\ =&W[[c,1,2,3]]\exp\{-\beta A_{c0}(A_{23}-A_{13}+A_{12})\} \\ =&W[[c,1,2,3]] \\ &\exp\{-\beta A_{c0}(A_{23}-A_{13}+A_{12}+A_{02}-A_{02}+A_{03}-A_{03}+A_{01}-A_{01})\} \\ =&W[[c,1,2,3]]\exp\{-\beta A_{c0}((A_{23}-A_{03}+A_{02}) \\ &+(-A_{13}+A_{03}-A_{01})+(A_{12}-A_{02}+A_{01}))\} \\ =&W[[c,1,2,3]]W[[c,0,2,3]]^{-1}W[[c,0,1,3]]W[[c,0,1,2]]^{-1}. \end{aligned}$$

We see that the weight is not yet invariant under subdivisions. To obtain invariance we must trivialize the term $\exp\{-\beta\langle\delta A\cup\delta A,[c,0,1,2,3]\rangle\}$. Therefore we impose two restrictions:
• the factor e^β must be a n^2 root of unity;
• the colors A must be restricted such that

$$\delta A=0 \bmod n.$$

The latter restriction applied to a 2-simplex $[0,1,2]$ becomes

$$[A_{12} - A_{02} + A_{01}] = 0,$$

where the notation of square brackets indicates the remainder of division by n.
Otherwise written, this means

$$[A_{12} + A_{01}] = A_{02}.$$

The second of the two restrictions clearly causes the term $\exp\{-\beta\langle \delta A \cup \delta A, [c,0,1,2,3]\rangle\}$ to become trivial. Furthermore we see that the weight defined earlier on $[0,1,2,3]$ can be written as

$$W[[0,1,2,3]] = \exp\left\{\frac{2\pi i k}{n^2} A_{01}(A_{12} + A_{23} - [A_{12} + A_{23}])\right\},$$

with $k \in \{0,...,n-1\}$.
Lemma 4.3.1. *The function $w : C^3(\mathbb{Z}/n\mathbb{Z}) \to U(1)$ given by*

$$w(a,b,c) = \exp\left\{\frac{2\pi i k}{n^2} a(b + c - [b+c])\right\}$$

with $k \in \{0,...,n-1\}$ is a 3-cocycle of $\mathbb{Z}/n\mathbb{Z}$ with coefficients in $U(1)$.

Proof. We must check whether the cocycle condition is being respected. Therefore we need to prove

$$w(a,b,c)w(b,c,d)w(a+b,c,d)^{-1}w(a,b+c,d)w(a,b,c+d)^{-1} = 1$$

for all $a,b,c,d \in \mathbb{Z}/n\mathbb{Z}$.
Before we show this, we must mention a few things about the remainder of the division by n.

For $\alpha, \beta \in \{0,...,n-1\}$, we have $\alpha + \beta = x_{\alpha\beta} n + r$ for a unique $x_{\alpha\beta} \in \mathbb{Z}_{\geq 0}$ and $0 \leq r < n$. Thus, if we identify elements of $\mathbb{Z}/n\mathbb{Z}$ with their corresponding elements in $\{0,...,n-1\}$, we can write for all $a, b \in \mathbb{Z}/n\mathbb{Z}$ that

$$[a+b] = a + b - x_{ab} n.$$

Furthermore, for all $a,b,c \in \mathbb{Z}/n\mathbb{Z}$ we see

$$a + b + c = x_{abc} n + [a+b+c].$$

Thus

$$[a+b]+c = a+b+c-x_{ab}n = (x_{abc}-x_{ab})n + [a+b+c]$$

and similarly

$$a+[b+c] = (x_{abc}-x_{bc})n + [a+b+c].$$

This gives us that $[[a+b]+c] = [a+[b+c]]$.

Now we can compute

$$w(a,b,c)w(b,c,d)w(a+b,c,d)^{-1}w(a,b+c,d)w(a,b,c+d)^{-1} =$$
$$w(a,b,c)w(b,c,d)w([a+b],c,d)^{-1}w(a,[b+c],d)w(a,b,[c+d])^{-1} =$$
$$\exp\left\{\frac{2\pi ik}{n^2}a(b+c-[b+c])\right\}\times$$
$$\exp\left\{\frac{2\pi ik}{n^2}b(c+d-[c+d])\right\}\times$$
$$\exp\left\{\frac{2\pi ik}{n^2}[a+b](c+d-[c+d])\right\}^{-1}\times$$
$$\exp\left\{\frac{2\pi ik}{n^2}a([b+c]+d-[[b+c]+d])\right\}\times$$
$$\exp\left\{\frac{2\pi ik}{n^2}a(b+[c+d]-[b+[c+d]])\right\}^{-1}.$$

We see that

$$\begin{aligned}
&a(b+c-[b+c]) \\
&+b(c+d-[c+d]) \\
&-[a+b](c+d-[c+d]) \\
&+a([b+c]+d-[[b+c]+d]) \\
&-a(b+[c+d]-[b+[c+d]]) \\
=&a(x_{bc}n) \\
&+b(x_{cd}n) \\
&-(a+b-x_{ab}n)(x_{cd}n) \\
&+a(b+c+d-x_{bc}n-[[b+c]+d]) \\
&-a(b+c+d-x_{cd}n-[b+[c+d]]) \\
=&x_{ab}x_{cd}n^2 \\
&+a(x_{bc}n-x_{cd}n-x_{bc}n+x_{cd}n-[[b+c]+d]+[b+[c+d]]) \\
=&x_{ab}x_{cd}n^2.
\end{aligned}$$

Since $x_{ab}x_{cd}n^2$ is a multiple of n^2, it follows that

$$\exp\left\{\frac{2\pi ik}{n^2}x_{ab}x_{cd}n^2\right\} = 1.$$

□

Chapter 5

Conclusion

There exist many topological quantum field theories that have varying applications. In this thesis, we focused on Dijkgraaf-Witten theory. First we briefly discussed its analogies with the Ising model. Then we could start our more detailed mathematical consideration of the theory, by initially studying a two-dimensional topological invariant that would later on prove to closely resemble the invariant of Dijkgraaf and Witten.
Subsequently, we carefully formulated the Dijkgraaf-Witten invariant $Z_M(\tau)$, after discussing a few necessary tools, such as simplices and cohomology.

We formulated the theorem that the invariant $Z_M(\tau)$ is independent of the order of the vertices of M and of the triangulation of M, as long as τ and the triangulation of ∂M are fixed. Furthermore, we sketched the proof of this theorem and described the relation between Snyder's invariant and Dijkgraaf-Witten's.
Lastly, we reconsidered the weights placed on the 3-simplices, defining them in such a way to remind us of the Chern-Simons action, and we demonstrated that demanding invariance under subdivision returned us the familiar Dijkgraaf-Witten invariant.

As a further suggestion, we could try to resolve what would happen if we let the order of the group G tend to infinity. Another object of study could be a more detailed investigation of Chern-Simons theory.

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